Cambridge International AS and A Level Mathematics Pure Mathematics 2 and 3 Practice Book

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Answers

Chapter 1 Algebra

Exercise 1.1

1

Expression	Polynomial Yes / No?	Order
$2x - x^2$	Yes	2
$\frac{x}{2} - \frac{2}{x}$	No	
0	Yes	0
$x^{23} + 2x^{15} + 1$	Yes	23
$x^3 - 2x^2 + \sqrt{x}$	No	
$x^2 + \sqrt{2}$	Yes	2
$5 + x + \pi x^{45}$	Yes	45
$1 - 3x$	Yes	1

2 **(i)** $A = 1, B = 1, C = 1, D = -3$

(ii) $A = 2, B = -7, C = 15, D = -32$

3 **(i)** $4x^3 + 2x^2 + 5x$

(ii) $x^4 - x^3 - x^2 + 10x + 1$

(iii) $x^6 - 2x^4 - 14x^3 + 10x - 5$

(iv) $3x^6 + 3x^5 - 6x^4 - 18x^3 + 8x^2 + 24x$

(v) $2x^2 - 3$

(vi) $x^7 + 2x^6 - 3x^4 - 4x^3 + 2x + 2$

4 **(i)** Quotient $x + 1$ Remainder -3

(ii) Quotient $x^2 + x - 3$ Remainder 0

(iii) Quotient $x^3 - x^2 + 8x$ Reminder $12 - 8x$

(iv) Quotient $3x^2 + 2x + 7$ Remainder 22

(v) Quotient 3 Remainder $5x - 4$

(vi) Quotient $4x + 1$ Remainder $6x + 13$

5 Quotient $x^2 + 2x - 1$ Remainder $x - 4$

6 $a = 2, b = -5$

Exercise 1.2

1 **(i)** **(a)** 10

(b) -10

(c) $\frac{81}{8}$

(d) 0

(ii) $f(x) = (x - 1)(x - 4)(x + 2)$

(iii) $x = -2$ or 1 or 4

2 **(i)** $h(-1) = 2(-1)^3 - 5(-1)^2 - 4(-1) + 3$

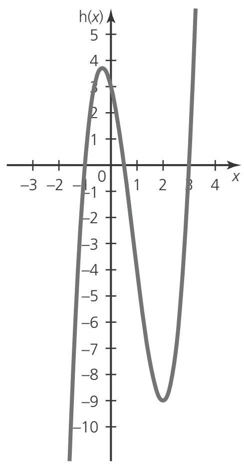
$= -2 - 5 + 4 + 3$

$= 0$

(ii) $h(x) = (x+1)(2x-1)(x-3)$

(iii) $x = -1$ or $\frac{1}{2}$ or 3

(iv)



3 $p = -4$

4 $m = -12$ and $f(x) = (2x - 1)(x + 1)^2 (x - 4)$

5 $p = 10, q = -4$

6 $p = -6, q = 4$

7 $k = \frac{3}{2}$

8 **(i)** $a = 8, b = -1$

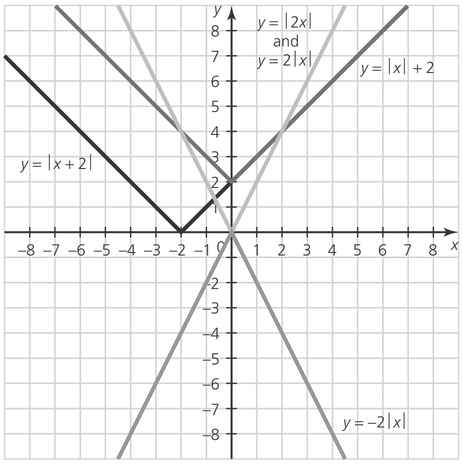
(ii) Quotient $2x + 6$ Remainder $6x - 7$

9 28

10 $a = 5, b = -4, c = -1$

Exercise 1.3

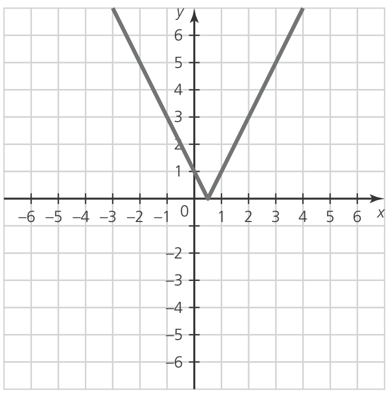
1 (i)



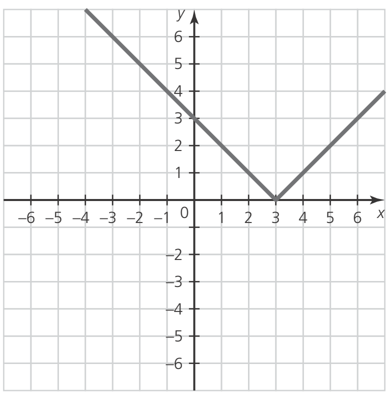
(ii)

Equation	Description
$y = \|x + a\|$	Shift *a* units to the left or right
$y = \|x\| + b$	Shift *b* units up or down
$y = \|cx\|$	Makes the line steeper or shallower
$y = d\|x\|$	Makes the line steeper or shallower
$y = -e\|x\|$	Makes the line steeper or shallower and turns it upside-down

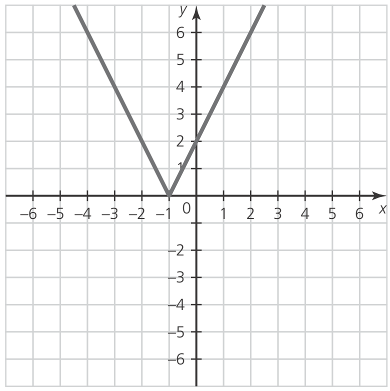
2 (i)



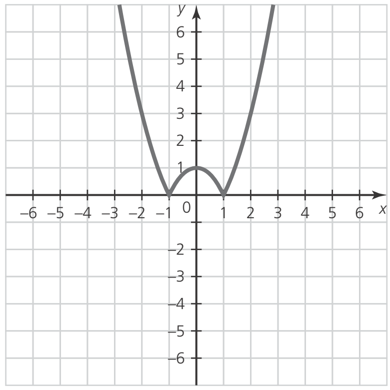
(ii)



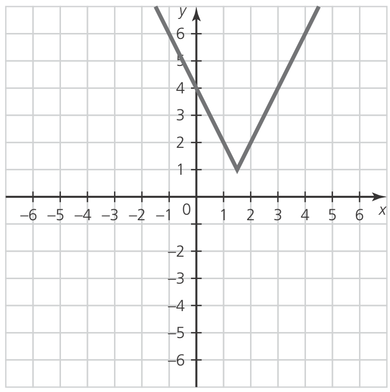
(iii)



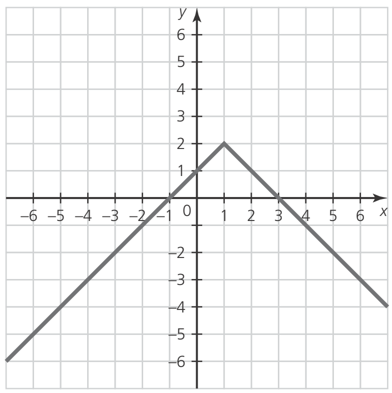
(iv)



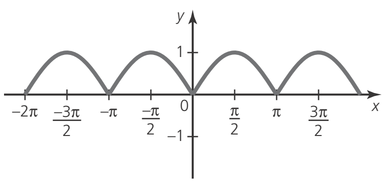
(v)



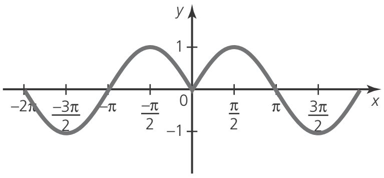
(vi)



3 (i)



(ii)



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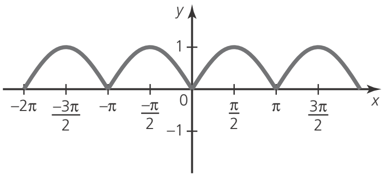


4 (i) $x=3$ or $x=-1$

(ii) $x=2$ or $x=-5$

(iii) $x=\frac{1}{2}$ or -2

(iv) $x=-1$

(v) $x=4,\ x=-\frac{2}{3}$

(vi) $x=-4$ or $x=0$

5 (i) $-7<x<3$

(ii) $x\geqslant 3$ or $x\leqslant -\frac{7}{3}$

(iii) $x\leqslant \frac{1}{2}$

(iv) $x<0$ or $x>6$

(v) $-1\leqslant x\leqslant \frac{3}{5}$

(vi) $-\frac{7}{2}<x<-\frac{5}{4}$

6 (i) $-4\leqslant x\leqslant \frac{2}{3}$

(ii) $2\frac{2}{3}$

7 $x=-3a$ or $\frac{1}{3}a$

8 $x>\frac{1}{2}a$

Stretch and challenge

1 $a=-15,\ b=-40,\ c=-38$

2 One solution is $a=3,\ b=4,\ c=1,\ d=4$

3 $x\leqslant \frac{1}{4}$

4 (i)

$$\begin{array}{r} ax^2+(b+ar)x+(c+br+ar^2) \\ x-r\ \big)\ ax^3 + bx^2 + cx + d \\ -(ax^3 - arx^2) \\ \hline (b+ar)x^2 + cx \\ -((b+ar)x^2 - r(b+ar)x) \\ \hline (c+br+ar^2)x+d \\ -((c+br+ar^2)x-r(c+br+ar^2)) \\ \hline d+rc+br^2+ar^3 \end{array}$$

(ii) $x=\dfrac{-(b+ar)\pm\sqrt{b^2-2abr-3a^2r^2-4ac}}{2a}$

5 (i) $ab+ac+bc=8,\ abc=4$

(ii) If a, b and c are the roots then

$(x-a)(x-b)(x-c)=0$

$x^3-(a+b+c)x^2+(ab+ac+bc)x-abc=0$

Hence by comparing coefficients,

$a+b+c=5,\ ab+ac+bc=8$ and $abc=4$

(as above)

(iii) $a=1,\ b=2,\ c=2$

6 $2x^2-3x+5$

Exam focus

1 (i) $a=-3$

(ii) (a) $\mathrm{p}(x)=(x+2)(2-3x)(x-1)$

(b) -70

2 (i) $a=3$

(ii) $3x^2+2x-6$

3 $\frac{2}{5}\leqslant x\leqslant 2$

4 (i) $2x-1$

(ii) $6x^3+13x^2-14x+3=(3x-1)(x+3)(2x-1)$

5 $x<-7a$ or $x>-\frac{5}{3}a$

Chapter 2 Logarithms and exponentials

Exercise 2.1

1 **(i)** $\log_2 16 = 4$

(ii) $\log_3 27 = 3$

(iii) $\log_4 \frac{1}{16} = -2$

2 **(i)** $2^{-1} = \frac{1}{2}$

(ii) $3^2 = 9$

(iii) $4^{\frac{1}{2}} = 2$

3 **(i)** 3

(ii) –3

(iii) –3

(iv) 2

(v) –1

(vi) 0

(vii) $\frac{1}{2}$

(viii) $\frac{3}{2}$

(ix) 1

(x) 4

(xi) $\frac{1}{6}$

(xii) $\frac{1}{4}$

(xiii) 3

(xiv) $\frac{2}{3}$

(xv) $-\frac{1}{2}$

4 **(i)** $a = 32$

(ii) $b = \frac{1}{9}$

(iii) $c = 3$

(iv) $d = 64$

(v) $e = -2$

(vi) $f = \frac{1}{2}$

5 **(i)** 1

(ii) 2

(iii) 0

(iv) $\frac{1}{3}$

(v) –1

(vi) $-\frac{1}{2}$

6 **(i)** $\log 20$

(ii) $\log 7$

(iii) $\log 64$

(iv) $\log 6$

(v) $\log 72$

(vi) $\log \frac{2}{5}$

(vii) $\log 20$

(viii) $\log\left(\frac{100}{11}\right)$

7 **(i)** $a + b$

(ii) $2a + b$

(iii) $2b - 2a$

(iv) $3a$

(v) $a - 2b$

(vi) $2a - \frac{1}{2}b$

8 **(i)** $\frac{A}{b^2} = 10$ or $A = 10b^2$

(ii) $D^2 = 3E - 3$

9 **(i)** 3.58

(ii) 1.89

(iii) –0.631

(iv) 1.934

(v) $\frac{3}{999}$ or $\frac{1}{333}$

(vi) $\frac{5}{4}$

(vii) 8

(viii) 2

(ix) 0 or $\frac{1}{2}$

(x) 0.0106

10 **(i)** $x < 0.161$

(ii) $x \leqslant 0$ or $x \geqslant 1$

11 **(i)** $a = 3, b = 2$

(ii) $a = -2, b = 1$

12 **(i)** 16

(ii) 8

(iii) 7.4

13 5.62

14 $x = 5, y = 4$

Exercise 2.2

1 (i) $y = kx^p$

$\log y = \log\left(kx^p\right)$

$\log y = \log k + \log x^p$

$\log y = p\log x + \log k$

(ii) Gradient is p, y-intercept is $\log k$.

2 $y = 3160(5.01)^x$ to 3 sf

3 (i)

m	W	$\log_{10}m$	$\log_{10}W$
1	8.00	0	0.903
2	5.66	0.301	0.753
5	3.58	0.699	0.554
10	2.53	1	0.403

(ii) $A = 8$, $b = -0.5$

(iii) (a) 2.31 kg

(b) $m > 64$

4 (i)

t	L	$\log_{10}L$
0	50	1.70
5	57	1.76
10	66	1.82
15	77	1.89
20	87	1.94
36	128	2.11

(ii) $L = Ab^t$

$\log L = \log(Ab^t)$

$\log L = \log A + \log b^t$

$\log L = t\log b + \log A$

$\log L = (\log b)t + \log A$

$y = m\,x + c$

(iii) $A = 50$, $b = 1.026$

(iv) 128 m. The answer is clearly unreasonable because the model does not account for the fact that the rate of growth changes as we age. This model may only be useful from birth to 36 months.

5 (i) $A = 10$, $b = -2$

(ii) $y = 0.025$

6 $k = 40770.39$, $b = 8.53$

7 (i) Since the graph of $\log D$ against L is a straight line, the appropriate model is $D = k\left(p\right)^L$.

(ii) $k \approx 377$, $p \approx 0.968$

(iii) $L = 35.2°$

8 $m = 10$, $n = \frac{19}{4}$

Exercise 2.3

1 (i) $x = \dfrac{\ln 4 - 1}{2}$

(ii) $x = 8$

(iii) $x = e^2 - 1$

(iv) $x = \ln 5$

(v) $x = \frac{16}{3}$

(vi) $x = \ln\frac{3}{2}$

(vii) $x = \dfrac{e}{1-e}$

(viii) $x = \ln 2$

2 (i) $A = 9B^3$

(ii) $\dfrac{\sqrt{P}}{Q} = eR^3$ or equivalent

3 (i) $\ln\left(e^{x+y}\right)^2 = 2x + 2y$

(ii) $e^{2\ln x + 3\ln y} = x^2y^3$

(iii) $2\ln\sqrt{e^{x-y}} = x - y$

4 $A = 0.212$ (3sf), $b = 1.71$ (3sf)

5 (i) $N_o = 1000$

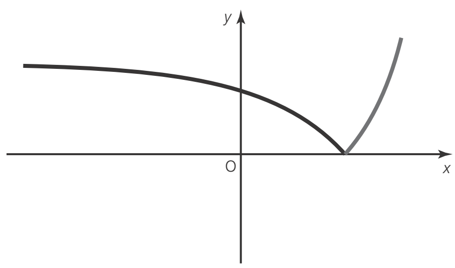
(ii) $N = 7390$ (3sf)

(iii) 1 hr 44 mins

(iv) 17 hr 16 mins

6 $x = \frac{1}{2}$

7 (i)



(ii) $a = 14$, $k = 3$

8 (i) $p = e^{280} \Rightarrow \ln p = 280$

$q = e^{300} \Rightarrow \ln q = 300$

$\ln\left(\dfrac{ep^2}{q}\right) = \ln e + 2\ln p - \ln q$

$= 1 + 2 \times 280 - 300$

$= 261$

(ii) Smallest integer is 361

Stretch and challenge

1 **(i)** $-\frac{1}{2}$ **(ii)** -16

(iii) π **(iv)** 2

2 **(i)** Sometimes, e.g. if $a = b$ or when $b = \frac{1}{a}$ (provided both a and b are positive)

(ii) Sometimes, e.g. when $a = 1$

(iii) Never

(iv) Sometimes, e.g. when $a = 1$

(v) Sometimes e.g. if the logarithm is base n then a solution is $a = n^4$, $b = n^2$.

$$\frac{\log_n n^4}{\log_n n^2} = \frac{4\log_n n}{2\log_n n} = 2$$

$$\log_n n^4 - \log_n n^2 = 4\log_n n - 2\log_n n = 2$$

(vi) Never

(vii) Sometimes e.g. when $x = 1$

3 $x = 0$ or 1

4 $x = 2.69$ (3sf)

5 $3\log_x y + 3\log_y x = 10$

Let $t = \log_x y \Rightarrow \frac{1}{t} = \log_y x$

$$3t + \frac{3}{t} = 10$$

$$3t^2 + 3 = 10t$$

$$3t^2 - 10t + 3 = 0$$

$$(3t - 1)(t - 3) = 0$$

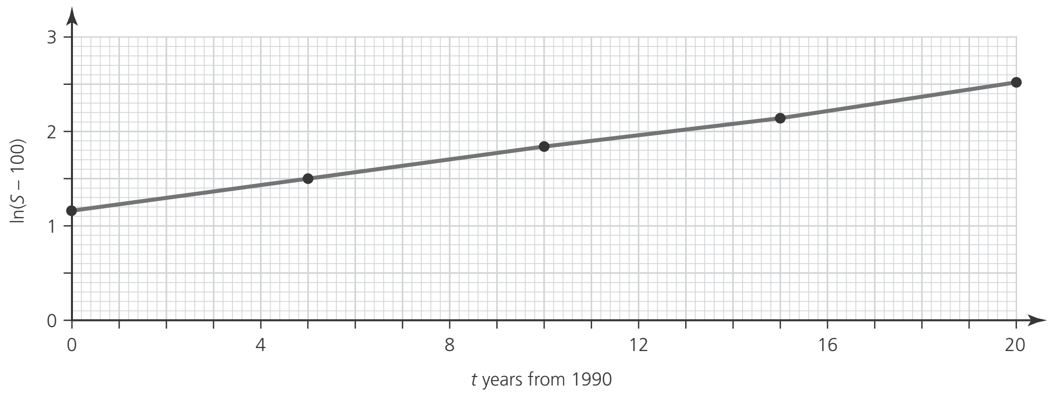
$$t = \tfrac{1}{3} \text{ or } 3$$

$$\log_x y = \tfrac{1}{3} \Rightarrow y = x^{\frac{1}{3}} \Rightarrow y = \sqrt[3]{x} \Rightarrow x = y^3$$

$$\log_x y = 3 \Rightarrow y = x^3$$

6 $\frac{x}{y} = 1$ or $\frac{x}{y} = 4$

7 **(i)**



$a = 3.2$, $b = 1.07$

(ii) 147.9 km/h

(iii) The model predicts that the speeds will keep increasing whereas they will probably taper off.

8 **(i)** $y = 17$

(ii) $x = -\frac{2}{3}$ or 3

9 $x = 100$

10 $x > \ln\frac{1}{2}$ or $x > -\ln(2 - a)$, $0 < a < 2$

Exam focus

1 $A = 0.00128$, $b = 15.2$

2 $x = 1.10$ (3sf)

3 **(i)** $\log_4(x - 4) = 2 - \log_4 x$

$$\log_4(x - 4) + \log_4 x = 2$$

$$\log_4 x(x - 4) = 2$$

$$x(x - 4) = 4^2$$

$$x^2 - 4x - 16 = 0$$

(ii) $x = 6.47$

4 $x = 2.85$

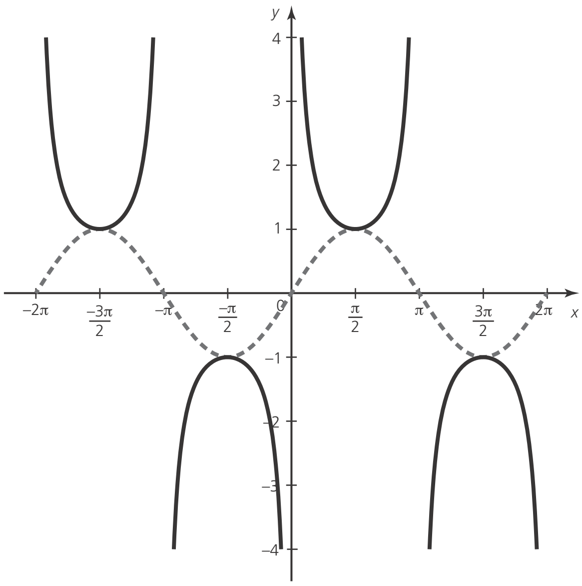
5 $x = -\frac{1}{2}$ or 2, but since $\ln x$ is only defined for positive values of x, the only solution is $x = 2$.

6 $x = 0.147$ (3sf)

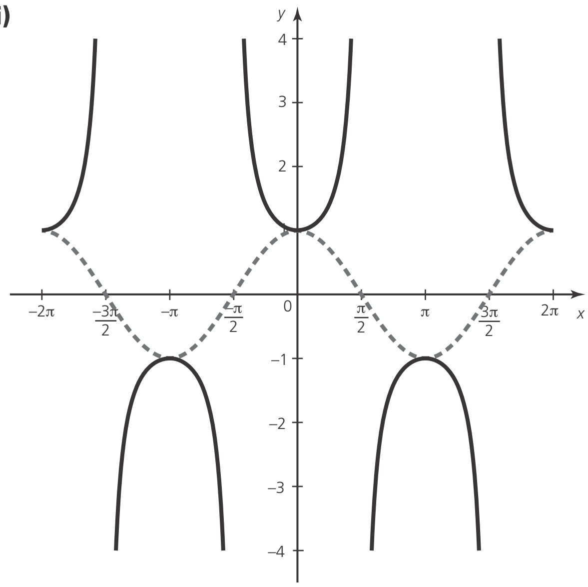
Chapter 3 Trigonometry

Exercise 3.1

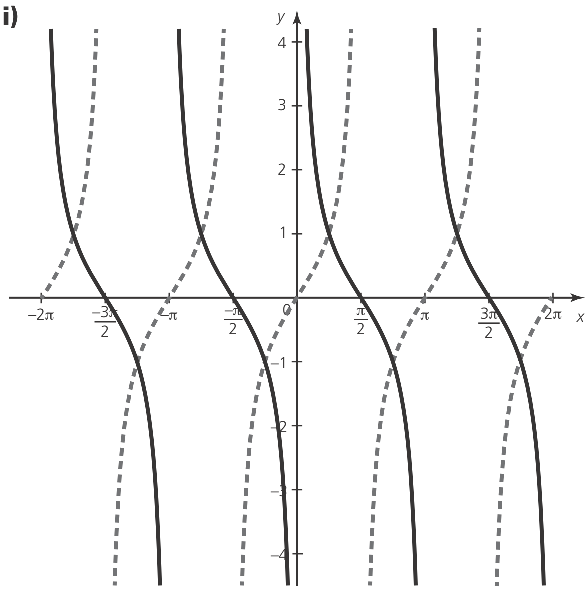
1 (i)



(ii)



(iii)



2 (i) $\operatorname{cosec} 150° = 2$

(ii) $\sec \frac{\pi}{4} = \sqrt{2}$

(iii) $\cot 300° = -\frac{1}{\sqrt{3}}$

(iv) $\operatorname{cosec} \frac{4\pi}{3} = -\frac{2\sqrt{3}}{3}$

(v) $\sec 120° = -2$

(vi) $\cot \frac{3\pi}{4} = -1$

3 (i) $\sin^2\theta + \cos^2\theta \equiv 1$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 \equiv \sec^2\theta$$

(ii) $\sin^2\theta + \cos^2\theta \equiv 1$

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} \equiv \frac{1}{\sin^2\theta}$$

$$1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$$

4 (i) $9x^2 - 4y^2 = 36$

(ii) $x^2 + y^2 = 2$

5 (i) $\theta = 14.5°$ or $165.5°$

(ii) $\theta = 151.0°$

(iii) $\beta = 69.3°$ or $110.7°$

(iv) $x = 39.2°$ or $140.8°$

(v) $\theta = -\frac{\pi}{12}, -\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$

6 (i) $\cos\theta = \frac{1}{3}$

(ii) $\sin\theta = \frac{\sqrt{8}}{3}$

(iii) $\operatorname{cosec}\theta = \frac{3\sqrt{8}}{8}$

(iv) $\cot\theta = \frac{\sqrt{8}}{8}$

7 (i) $\frac{1}{\tan\theta + \cot\theta} \equiv \sin\theta\cos\theta$

LHS:

$$= \frac{1}{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}}$$

$$= \frac{1}{\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}}$$

$$= \frac{1}{\frac{1}{\sin\theta\cos\theta}}$$

$$= \sin\theta\cos\theta$$

$$= RHS$$

(ii) $\sec^2\theta + \operatorname{cosec}^2\theta \equiv \sec^2\theta\operatorname{cosec}^2\theta$

LHS:

$$= \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta\cos^2\theta}$$

$$= \frac{1}{\sin^2\theta\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta} \times \frac{1}{\sin^2\theta}$$

$$= \sec^2\theta\operatorname{cosec}^2\theta$$

$= RHS$

(iii) $\sec^4\theta - \tan^4\theta \equiv \sec^2\theta + \tan^2\theta$

LHS:

$$= (\sec^2\theta - \tan^2\theta)(\sec^2\theta + \tan^2\theta)$$

$$= (\tan^2\theta + 1 - \tan^2\theta)(\sec^2\theta + \tan^2\theta)$$

$$= (1)(\sec^2\theta + \tan^2\theta)$$

$$= \sec^2\theta + \tan^2\theta$$

$= RHS$

(iv) $(\tan\theta - \sin\theta)^2 + (1 - \cos\theta)^2 = (1 - \sec\theta)^2$

LHS:

$$= \tan^2\theta - 2\tan\theta\sin\theta + \sin^2\theta + 1 - 2\cos\theta + \cos^2\theta$$

$$= \frac{\sin^2\theta}{\cos^2\theta} - \frac{2\sin^2\theta}{\cos\theta} - 2\cos\theta + 2$$

$$= \frac{1-\cos^2\theta}{\cos^2\theta} - \frac{2(1-\cos^2\theta)}{\cos\theta} - 2\cos\theta + 2$$

$$= \frac{1}{\cos^2\theta} - 1 - \frac{2}{\cos\theta} + 2\cos\theta - 2\cos\theta + 2$$

$$= 1 - \frac{2}{\cos\theta} + \frac{1}{\cos^2\theta}$$

$$= \left(1 - \frac{1}{\cos\theta}\right)^2$$

$$= (1 - \sec\theta)^2$$

$= RHS$

(v) $(\operatorname{cosec}^2\theta - 1)(\tan^2\theta + 1) \equiv \operatorname{cosec}^2\theta$

LHS:

$$= (1 + \cot^2\theta - 1)(\sec^2\theta)$$

$$= \cot^2\theta\sec^2\theta$$

$$= \frac{\cos^2\theta}{\sin^2\theta} \times \frac{1}{\cos^2\theta}$$

$$= \frac{1}{\sin^2\theta}$$

$$= \operatorname{cosec}^2\theta$$

$= RHS$

(vi) $\dfrac{\cos\theta}{1-\tan\theta} - \dfrac{\sin\theta}{1-\cot\theta} \equiv \dfrac{1}{\cos\theta - \sin\theta}$

LHS:

$$= \frac{\cos\theta}{1 - \frac{\sin\theta}{\cos\theta}} - \frac{\sin\theta}{1 - \frac{\cos\theta}{\sin\theta}}$$

$$= \frac{\cos\theta}{\frac{\cos\theta - \sin\theta}{\cos\theta}} - \frac{\sin\theta}{\frac{\sin\theta - \cos\theta}{\sin\theta}}$$

$$= \frac{\cos^2\theta}{\cos\theta - \sin\theta} - \frac{\sin^2\theta}{\sin\theta - \cos\theta}$$

$$= \frac{\cos^2\theta}{\cos\theta - \sin\theta} + \frac{\sin^2\theta}{\cos\theta - \sin\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\cos\theta - \sin\theta}$$

$$= \frac{1}{\cos\theta - \sin\theta}$$

$= RHS$

8 (i) $m = \dfrac{1+\cos\theta}{\sin\theta}$

$$\frac{1}{m} = \frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{\sin\theta}{1+\cos\theta} \times \frac{1-\cos\theta}{1-\cos\theta}$$

$$= \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta}$$

$$= \frac{\sin\theta(1-\cos\theta)}{\sin^2\theta}$$

$$= \frac{1-\cos\theta}{\sin\theta}$$

(ii) $\cos\theta = \dfrac{m^2-1}{m^2+1}$

Exercise 3.2

1 (i) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$

(ii) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$

(iii) $-2-\sqrt{3}$

(iv) $\sqrt{6}-\sqrt{2}$

2 (i) $\sin(\theta - 30°) = \dfrac{\sqrt{3}}{2}\sin\theta - \dfrac{1}{2}\cos\theta$

(ii) $\cos\left(\dfrac{\pi}{4} - \theta\right) = \dfrac{\sqrt{2}}{2}\cos\theta + \dfrac{\sqrt{2}}{2}\sin\theta$

(iii) $\tan(\theta + 60°) = \dfrac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta}$

(iv) $\operatorname{cosec}(2\theta + 120°) = \dfrac{2}{\sqrt{3}\cos 2\theta - \sin 2\theta}$

(v) $\dfrac{\cos A + \cos(-A)}{\sin A - \sin(-A)} = \cot A$

(vi) $\dfrac{\cos(30° + A) - \cos(30° - A)}{\sin(30° + A) - \sin(30° - A)} = -\dfrac{1}{\sqrt{3}}$

3 (i) $\sin\theta\cos 2\beta + \sin 2\beta\cos\theta = \sin(\theta + 2\beta$

(ii) $\cos 3\theta\cos\theta + \sin 3\theta\sin\theta = \cos 2\theta$

(iii) $\sin\dfrac{4\pi}{3}\cos\dfrac{7\pi}{6} - \sin\dfrac{7\pi}{6}\cos\dfrac{4\pi}{3} = \sin\dfrac{\pi}{6}$

(iv) $\cos 280°\cos 20° - \sin 280°\sin 20° = \cos 300°$

4 $\sin(x + y) = \frac{56}{65}$

5 (i) $\cos(P - Q) = \dfrac{6 + 7\sqrt{21}}{5\sqrt{58}}$

(ii) $\sin(P + Q) = \dfrac{3\sqrt{21} + 14}{5\sqrt{58}}$

6 (i) $\sin(A - B) = \sin A\cos B - \cos A\sin B = \dfrac{1 + 2\sqrt{30}}{12}$

(ii) $\cot(B - A) = \dfrac{\sqrt{8} - \sqrt{15}}{\sqrt{120} - 1}$

7 (i) $\theta = -37.5°$ or $142.5°$

(ii) $\theta = 1.14$ or 3.05

(iii) $\theta = 0°$ or $63.4°$

8 $\cos(A + B)\cos(A - B) \equiv \cos^2 A - \sin^2 B$

LHS:
$= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$
$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$
$= \cos^2 A(1 - \sin^2 B) - (1 - \cos^2 A)\sin^2 B$
$= \cos^2 A - \cos^2 A\sin^2 B - \sin^2 B + \cos^2 A\sin^2 B$
$= \cos^2 A - \sin^2 B$
$= RHS$

9 (i) $\tan(A + B) = \frac{7}{4}$

(ii) $\cos(A - B) = \dfrac{8\sqrt{65}}{65}$

10 (i) $\tan B = -\frac{7}{4}$

(ii) $\sin(A + B)$
$= \sin A\cos B + \cos A\sin B$
$= \dfrac{1}{\sqrt{10}} \times -\dfrac{4}{\sqrt{65}} + \dfrac{3}{\sqrt{10}} \times \dfrac{7}{\sqrt{65}}$
$= -\dfrac{4}{\sqrt{650}} + \dfrac{21}{\sqrt{650}}$
$= \dfrac{17}{\sqrt{650}}$

Exercise 3.3

1 (i) $\sin\dfrac{2\pi}{3} = \dfrac{\sqrt{3}}{2}$

(ii) $\cos\dfrac{2\pi}{3} = -\dfrac{1}{2}$

(iii) $\tan\dfrac{2\pi}{3} = -\sqrt{3}$

2 (i) $\sin 2\theta = \frac{24}{25}$

(ii) $\cos 2\theta = \frac{7}{25}$

3 (i) $\sin 2\theta = -\dfrac{4\sqrt{5}}{9}$

(ii) $\cos 2\theta = \dfrac{1}{9}$

(iii) $\sin 4\theta = -\dfrac{8\sqrt{5}}{81}$

4 $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

5 $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

6 (i) $\tan\theta = \frac{3}{2}$

(ii) (a) $\tan(\theta + 45) = -5$

(b) $\tan 2\theta = -\frac{12}{5}$

7 (i) $\theta = 0, \dfrac{\pi}{3}, \dfrac{5\pi}{3}, 2\pi$

(ii) $\theta = 210°, 330°$

(iii) $\theta = 0°, 63.4°, -116.6°, -63.4°, 116.6°$

(iv) $\theta = 138.6°, 221.4°$

(v) $\theta = 135°, 315°$

(vi) $\theta = 0.322, -2.82, \dfrac{3\pi}{4}, -\dfrac{\pi}{4}$

8 (i) $\sin(45° + \theta)\sin(45° - \theta) \equiv \frac{1}{2}\cos 2\theta$

LHS:
$= (\sin 45°\cos\theta + \cos 45°\sin\theta)(\sin 45°\cos\theta - \cos 45°\sin\theta)$
$= \left(\dfrac{\sqrt{2}}{2}\cos\theta + \dfrac{\sqrt{2}}{2}\sin\theta\right)\left(\dfrac{\sqrt{2}}{2}\cos\theta - \dfrac{\sqrt{2}}{2}\sin\theta\right)$
$= \frac{1}{2}\cos^2\theta - \frac{1}{2}\sin^2\theta$
$= \frac{1}{2}(\cos^2\theta - \sin^2\theta)$
$= \frac{1}{2}\cos 2\theta$
$= RHS$

(ii) $\dfrac{2\sin\theta\cos\theta}{\cos^4\theta - \sin^4\theta} \equiv \tan 2\theta$

LHS:
$= \dfrac{\sin 2\theta}{(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)}$
$= \dfrac{\sin 2\theta}{(\cos^2\theta - \sin^2\theta)(1)}$
$= \dfrac{\sin 2\theta}{\cos 2\theta}$
$= \tan 2\theta$
$= RHS$

(iii) $\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 2$

$LHS:$

$= \frac{3\sin\theta - 4\sin^3\theta}{\sin\theta} - \frac{4\cos^3\theta - 3\cos\theta}{\cos\theta}$

$= 3 - 4\sin^2\theta - 4\cos^2\theta + 3$

$= 3 - 4(\sin^2\theta + \cos^2\theta) + 3$

$= 3 - 4 + 3$

$= 2$

$= RHS$

9 (i) $\cot\theta + \tan\theta \equiv 2\operatorname{cosec} 2\theta$

$LHS:$

$= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$

$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$

$= \frac{1}{\sin\theta\cos\theta}$

$= \frac{2}{2\sin\theta\cos\theta}$

$= \frac{2}{\sin 2\theta}$

$= 2\operatorname{cosec} 2\theta$

$= RHS$

(ii) $\theta = 0.126, 1.44, 3.27, 4.59$

10 (i) $\sin 2\theta + 2\tan 2\theta \sin^2\theta \equiv \tan 2\theta$

$LHS:$

$= \sin 2\theta + 2\tan 2\theta \sin^2\theta$

$= 2\sin\theta\cos\theta + \frac{2\sin 2\theta}{\cos 2\theta} \times \sin^2\theta$

$= 2\sin\theta\cos\theta + \frac{2(2\sin\theta\cos\theta)}{1 - 2\sin^2\theta} \times \sin^2\theta$

$= 2\sin\theta\cos\theta + \frac{4\sin\theta\cos\theta}{1 - 2\sin^2\theta} \times \sin^2\theta$

$= \frac{2\sin\theta\cos\theta(1 - 2\sin^2\theta) + 4\sin^3\theta\cos\theta}{1 - 2\sin^2\theta}$

$= \frac{2\sin\theta\cos\theta - 4\sin^3\theta\cos\theta + 4\sin^3\theta\cos\theta}{1 - 2\sin^2\theta}$

$= \frac{2\sin\theta\cos\theta}{1 - 2\sin^2\theta}$

$= \frac{\sin 2\theta}{\cos 2\theta}$

$= \tan 2\theta$

$= RHS$

(ii) $\theta = 40.9°, 130.9°$

11 (i) $\sin 50° = 2k\sqrt{1 - k^2}$

(ii) $\cos 50° = 1 - 2k^2$

(iii) $\tan 155° = -\frac{k}{\sqrt{1 - k^2}}$

12 (i) $\frac{1 + p}{1 - p}$

(ii) $\frac{-1 + \sqrt{1 + p^2}}{p}$

(iii) $\frac{7 - 3p}{3 - 7p}$

13 $\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

14 (i) $\sec\theta\sin\theta = 36\cot\theta$

$\frac{1}{\cos\theta}\sin\theta = 36\frac{\cos\theta}{\sin\theta}$

$\sin^2\theta = 36\cos^2\theta$

$\frac{\sin^2\theta}{\cos^2\theta} = 36$

$\tan^2\theta = 36$

$\tan\theta = 6$

(ignore $\tan\theta = -6$ as θ is acute)

(ii) **(a)** $\frac{5}{7}$

(b) $\frac{-12}{35}$

Exercise 3.4

1 (i) $\sin\theta - 3\cos\theta = \sqrt{10}\sin(\theta - 71.6°)$

(ii) $12\cos\theta + 5\sin\theta \equiv 13\cos(\theta - 22.6°)$

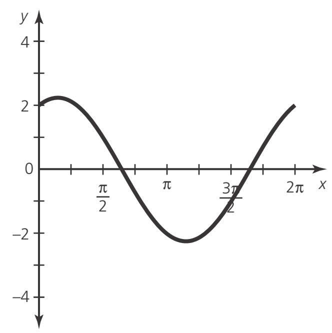
(iii) $6\sin\theta + 8\cos\theta \equiv 10\sin(\theta + 53.1°)$

(iv) $7\cos\theta - 24\sin\theta \equiv 25\cos(\theta + 73.7°)$

2 (i) $2\cos\theta + \sin\theta \equiv \sqrt{5}\cos(\theta - 26.6°)$

(ii) $\theta = 90.0°, 323.2°$

(iii)



(iv) $-\sqrt{5} + 5 \leqslant 2\cos\theta + \sin\theta + 5 \leqslant \sqrt{5} + 5$

3 (i) $4\sin\theta - 3\cos\theta \equiv 5\sin(\theta - 36.9°)$

(ii) $\theta = 73.8°, 180.0°$

(iii) Minimum value: 1 at $\theta = 306.9°$

Maximum value: 11 at $\theta = 126.9°$

4 (i) $T(\theta) = \frac{5}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta$

$A = \frac{\sqrt{3}}{2}, B = \frac{5}{2}$

(ii) $T(\theta) = \frac{\sqrt{3}}{2}\sin\theta + \frac{5}{2}\cos\theta \equiv \sqrt{7}\sin(\theta + 70.9°)$

(iii) Smallest positive angle is 131.3°.

5 (i) $3\sin\theta + 4\cos\theta \equiv 5\sin(\theta + 53.1°)$

(ii) (a) $\theta = -64.6°, 138.4°$

(b) $c = 3, k = 8$

6 (i) $\sqrt{2}\cos\theta + \sqrt{7}\sin\theta \equiv 3\cos(\theta - 61.87°)$

(ii) (a) $\theta = 171.4°$

(b) $\theta = 27.4°$

Stretch and Challenge

1 (i) $R = \frac{2V^2}{5}\sin B\cos\left(B + \frac{\pi}{4}\right)$

(ii) $B = 0.272 = 15.6°$

2 (i) $\theta = 17.1°, w = 8.31$ cm

(ii) 17 cm

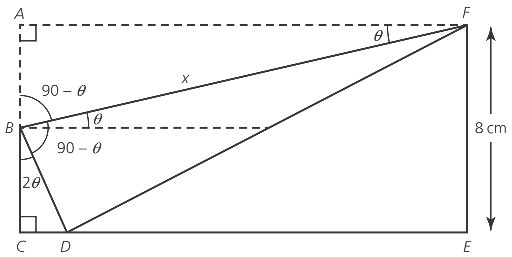
3 $k = \frac{7}{8}$

4 $\sin^3 A + \cos^3 A = \frac{9}{16}$

5 1

6 $h(t) = 35 + 20\sin\left(\frac{\pi}{2}t\right) + 15\sin\left(\frac{2\pi}{3}t\right)$

7



$\sin\theta = \frac{AB}{x} \Rightarrow AB = x\sin\theta$ and $BD = x\sin\theta$

$\angle ABF = \angle DBF = 90° - \theta$

$\angle CBD = 180° - 2(90° - \theta) = 2\theta$

$\cos 2\theta = \frac{BC}{BD} = \frac{BC}{x\sin\theta} \Rightarrow BC = x\sin\theta\cos 2\theta$

$AC = AB + BC$

$8 = x\sin\theta + x\sin\theta\cos 2\theta$

$8 = x\sin\theta(1 + \cos 2\theta)$

$8 = x\sin\theta(1 + 2\cos^2\theta - 1)$

$8 = x\sin\theta \times 2\cos^2\theta$

$4 = x\sin\theta\cos^2\theta$

$x = \frac{4}{\sin\theta\cos^2\theta}$

8 $\tan\frac{\theta}{4} = \frac{1}{\sqrt{6}}$

Exam focus

1 $x = 135°, 225°$

2 $x = 164.1°$

3 (i) $2\text{cosec}2\theta \equiv \sec\theta\text{cosec}\theta$

LHS:

$= \frac{2}{\sin 2\theta}$

$= \frac{2}{2\sin\theta\cos\theta}$

$= \frac{1}{\sin\theta\cos\theta}$

$= \frac{1}{\cos\theta} \times \frac{1}{\sin\theta}$

$= \sec\theta\text{cosec}\theta$

$= RHS$

(ii) $\theta = 15°, 75°$

4 $\theta = 33.7°, 116.6°$

5 (i) $8\sin\theta + 15\cos\theta \equiv 17\sin(\theta + 61.9°)$

(ii) $\theta = 353.5°, 62.7°$

(iii) $k < -17$ or $k > 17$

6 (i) $\tan(\theta + 60°)\tan(\theta - 60°) \equiv \frac{\tan^2\theta - 3}{1 - 3\tan^2\theta}$

LHS:

$\tan(\theta + 60°)\tan(\theta - 60°)$

$= \left(\frac{\tan\theta + \tan 60°}{1 - \tan\theta\tan 60°}\right)\left(\frac{\tan\theta - \tan 60°}{1 + \tan\theta\tan 60°}\right)$

$= \left(\frac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta}\right)\left(\frac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta}\right)$

$= \frac{\tan^2\theta - 3}{1 - 3\tan^2\theta}$

$= RHS$

(ii) $\theta = 37.2°, 142.8°$

(iii) $\frac{\tan^2\theta - 3}{1 - 3\tan^2\theta} = k^2$

$\tan^2\theta - 3 = k^2(1 - 3\tan^2\theta)$

$\tan^2\theta - 3 = k^2 - 3k^2\tan^2\theta$

$(1 + 3k^2)\tan^2\theta = 3 + k^2$

$\tan^2\theta = \frac{3 + k^2}{1 + 3k^2}$

$\tan\theta = \pm\sqrt{\frac{3 + k^2}{1 + 3k^2}}$

Since $k^2 \geqslant 0$, $\frac{3 + k^2}{1 + 3k^2} \geqslant 0$

so there are two roots of the equation.

7 (i) $\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$

$\alpha = 41.8°, 138.2°$

(ii) (a) $\text{cosec}\beta = \frac{7}{6}$

(b) $\cot^2\beta = \frac{13}{36}$

Chapter 4 Differentiation

Exercise 4.1

1 (i) $3x^2 + 8x - 3$

(ii) $\dfrac{x^2(2x-3)}{(x-1)^2}$

(iii) $x^2(1-2x)^3(3-14x)$

(iv) $-\dfrac{(x+4)}{3x^3}$

(v) $\dfrac{6x(1+5x)}{\sqrt{1+4x}}$

(vi) $\dfrac{6x-2}{\sqrt{(6x-1)^3}}$

2 $y = 84x - 320$

3 $y = -\frac{3}{2}x + 2$

4 $0, -3, -\frac{6}{5}$

5 (i) 0.8

(ii) No, since $\dfrac{dy}{dx} = \dfrac{5}{(x+2)^2}$, which is always greater than zero, hence no stationary value.

6 (i) $x = \dfrac{1-k}{2k+1}$

(ii) $k = 4$

7 (i) $\dfrac{(x-1)(x+3)}{(x+1)^2}$

(ii) $(-3, -7)$ or $(1, 1)$

Exercise 4.2

1 (i) $\dfrac{1}{x+4}$

(ii) $\dfrac{2}{x}$

(iii) $-e^{1-x}$

(iv) $9x^2e^{x^3+1}$

(v) $-\dfrac{3}{x(x-1)}$

(vi) $\dfrac{10}{2\sqrt{x}+2x} = \dfrac{5}{\sqrt{x}+x}$

(vii) $e^{x-2}(1+x)$

(viii) $\dfrac{x^2e^x(3-x)}{e^{2x}} = \dfrac{x^2(3-x)}{e^x}$

(ix) $\dfrac{e^{3x}-2e^{2x}-e^x}{e^{2x}-2e^x+1} = \dfrac{e^x-e^{-x}-2}{(1-e^{-x})^2}$

(x) $e^{\frac{1}{2}x}\left(\ln x + \dfrac{2}{x}\right)$

(xi) $\dfrac{3e^{3x}\ln(3x-1) - \dfrac{3(e^{3x}-1)}{3x-1}}{[\ln(3x-1)]^2}$

(xii) $\dfrac{2e^{7x}}{1+e^{2x}} + 5e^{5x}\ln(1+e^{2x})$

2 (i) $\dfrac{dy}{dx} = \dfrac{\ln x - 1}{(\ln x)^2}, \dfrac{d^2y}{dx^2} = \dfrac{(2-\ln x)}{x(\ln x)^3}$

(ii) (a) (e, e)

(b) $\left(e^2, \frac{1}{2}e^2\right)$

3 (i) $\dfrac{dy}{dx} = xe^{-x}(2-x), \dfrac{d^2y}{dx^2} = -e^{-x}(x^2+4x+2)$

(ii) $\left(2, \dfrac{4}{e^2}\right)$

(iii) $x = -2 \pm \sqrt{2}$. These points represent the x values where the gradient is a maximum or minimum.

4 $\left(e^{-\frac{1}{2}}, \frac{1}{2}e^{-\frac{1}{2}}\right)$ or $\left(\dfrac{1}{\sqrt{e}}, \dfrac{1}{2\sqrt{e}}\right)$

5 $\left(e^{-\frac{1}{3}}, -\dfrac{1}{3e}\right)$ or $\left(\dfrac{1}{\sqrt[3]{e}}, -\dfrac{1}{3e}\right)$

6 (i) $x = e^2$

(ii) (e, e)

(iii) $\dfrac{dy}{dx} = 1 - \ln x$

at A, $\dfrac{dy}{dx} = 1 - \ln e^2 = -1$

at B, $\dfrac{dy}{dx} = 1 - \ln 1 = 1$

Since $m_1 \times m_2 = -1$ tangents are perpendicular.

7 (i) $\frac{1}{2}e^{\frac{1}{2}x}(3+x)$

(ii) $\left(-3, -2e^{-\frac{3}{2}}\right)$ or $\left(-3, -\dfrac{2}{\sqrt{e^3}}\right)$

Exercise 4.3

1 (i) $6\cos 2x$

(ii) $-3\sin(1+3x)$

(iii) $2x\sec^2(x^2)$

(iv) $x^2(3\sin 2x + 2x\cos 2x)$

(v) $\dfrac{-5x^2\sin 5x - 2x\cos 5x}{x^4} = \dfrac{-5x\sin 5x - 2\cos 5x}{x^3}$

(vi) $\dfrac{\sin^2 x(3x\cos x - 2\sin x)}{x^3}$

(vii) $\cos x\, e^{\sin x + 1}$

(viii) $\dfrac{4}{\sin 4x}$

(ix) $\dfrac{\cos(\ln 3x)}{x}$

(x) $2xe^{x^2}\sec^2(e^{x^2})$

(xi) $-\dfrac{\sin 2x}{\sqrt{\cos 2x}}$

(xii) $6e^{x-1}\sin^2(2e^{x-1})\cos(2e^{x-1})$

(xiii) $-4e^x \cot e^x \cos^3[\ln(\sin e^x)]\sin[\ln(\sin e^x)]$

(xiv) $\dfrac{2\sin(\ln x)\cos(\ln x)e^{\sin^2(\ln x)}}{x}$ or can be simplified further to $\dfrac{\sin(\ln x^2)\cdot e^{\sin^2}(\ln x)}{x}$

2 (i) $x(2\sin x + x\cos x)$

(ii) $y = -\pi^2 x + \pi^3$

3 $3\sqrt{3}$

4 (i) $\dfrac{e^{2x}(2\cos x + \sin x)}{\cos^2 x}$

(ii) 2.03 (3sf)

5 0.685

6 $x = \frac{3}{8}\pi, \frac{7}{8}\pi$

7 (i) $\dfrac{\pi}{4} \leqslant x \leqslant \dfrac{3\pi}{4}$ and $\dfrac{5\pi}{4} \leqslant x \leqslant \dfrac{7\pi}{4}$

(ii) $x = \dfrac{\pi}{8}, \dfrac{5\pi}{8}, \dfrac{9\pi}{8}, \dfrac{13\pi}{8}$

(iii) All of the points are maximum points.

Exercise 4.4

1 (i) $6y^2\dfrac{dy}{dx}$

(ii) $6x - 20y^3\dfrac{dy}{dx}$

(iii) $2\cos 2x - 2\sin 2y\dfrac{dy}{dx}$

(iv) $3e^{3y}\dfrac{dy}{dx}$

(v) $8xy + 4x^2\dfrac{dy}{dx}$

(vi) $\dfrac{1}{x} + \dfrac{1}{y}\dfrac{dy}{dx}$

(vii) $\sec^2(xy^2)\times\left(y^2 + 2xy\dfrac{dy}{dx}\right) - e^y\dfrac{dy}{dx}$

(viii) $e^{x\sin y}\times\left(\sin y + \cos y\dfrac{dy}{dx}\times x\right)$

2 (i) $\dfrac{dy}{dx} = \dfrac{6x^2}{2y} = \dfrac{3x^2}{y}$

(ii) $\dfrac{dy}{dx} = \dfrac{2xy}{\cos y - x^2}$

(iii) $\dfrac{dy}{dx} = \dfrac{4y - 1}{6y^2 - 4x}$

(iv) $\dfrac{dy}{dx} = \dfrac{2}{xe^{xy}} - \dfrac{y}{x}$

3 $y = \frac{3}{2}x + \frac{1}{4}$

4 (1, −2) and (−1, 2)

5 $\left(1, -\frac{10}{3}\right)$ or (1, 2)

6 (i) (2, 2) and (−2, −2)

(ii)
$$2x = \left(2y + 2x\frac{dy}{dx}\right) + 2y\frac{dy}{dx}$$
$$2x - 2y = (2x + 2y)\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{2x - 2y}{2x + 2y} = \frac{x - y}{x + y}$$

Slope parallel to the y axis $\Rightarrow x + y = 0 = y = -x$

$$8 + x^2 = 2x(-x) + (-x)^2$$
$$8 + x^2 = -2x^2 + x^2$$
$$8 = -2x^2$$
$$x^2 = -4$$

There are no real numbers that satisfy this equation, hence the slope of the curve is never parallel to the y axis.

7 (−2, −2) and (4, −2)

8 (i)
$$5x^2 - 2xy + 3y^2 - 70 = 0$$
$$10x - \left(2y + 2x\frac{dy}{dx}\right) + 6y\frac{dy}{dx} = 0$$
$$10x - 2y - 2x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$$
$$(6y - 2x)\frac{dy}{dx} = 2y - 10x$$
$$\frac{dy}{dx} = \frac{2y - 10x}{6y - 2x} = \frac{y - 5x}{3y - x}$$

(ii) (1, 5) or (−1, −5)

(iii) $\left(\sqrt{15}, \frac{\sqrt{15}}{3}\right)$ or $\left(-\sqrt{15}, -\frac{\sqrt{15}}{3}\right)$

9 (i) $(2, -4)$ or $(2, 3)$

(ii) $\frac{dy}{dx} = \frac{3x^2+2}{2y+1}$

at $(2,-4)$ $\frac{dy}{dx} = \frac{3\times2^2+2}{2\times-4+1} = -\frac{14}{7} = -2$

at $(2,3)$ $\frac{dy}{dx} = \frac{3\times2^2+2}{2\times3+1} = \frac{14}{7} = 2$

10 $7x - 11y + 4 = 0$

11 (i) $\frac{2y-x^2}{y^2-2x}$

(ii) $\left(2^{\frac{4}{3}}\right)^3 + \left(2^{\frac{5}{3}}\right)^3 = 6\left(2^{\frac{4}{3}}\right)\left(2^{\frac{5}{3}}\right)$

$2^4 + 2^5 = 6\times2^3$

$LHS: 16+32 = 48$

$RHS: 6\times2^3 = 48$

So the point lies on the curve.

$$\frac{dy}{dx} = \frac{2y-x^2}{y^2-2x} = \frac{2\left(2^{\frac{5}{3}}\right) - \left(2^{\frac{4}{3}}\right)^2}{\left(2^{\frac{5}{3}}\right)^2 - 2\left(2^{\frac{4}{3}}\right)} = \frac{2^{\frac{8}{3}} - 2^{\frac{8}{3}}}{2^{\frac{10}{3}} - 2^{\frac{7}{3}}} = 0$$

(iii) $a = 3$

At the point $(3,3)$, $\frac{dy}{dx} = \frac{2\times3-3^2}{3^2-2\times3} = \frac{-3}{3} = -1$

Exercise 4.5

1 (i) t

(ii) $-\frac{3}{2}\cot\theta$

(iii) $\frac{1+2\sin2\theta}{1+\cos\theta}$

(iv) $\frac{2(2t+1)}{3t^2}$

(v) $\frac{1-2e^{2t}}{3e^t}$

(vi) $\frac{5\frac{1}{\cos\theta}}{2\frac{\sin\theta}{\cos\theta}} = \frac{5}{2\sin\theta}$ after cancelling by $\cos\theta$

2 (i) $(2t+t^2), \left(\frac{1-t}{1+t}\right)^2; t \neq -1$

(ii) $\left(-\frac{2}{3}, -4\right)$, $(0, 0)$

3 (i) $\frac{dy}{dt} = -\frac{6}{t^2}, \frac{dx}{dt} = \frac{-3}{2-3t}$

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = -\frac{6}{t^2}\times\frac{2-3t}{-3}$$
$$= \frac{2(2-3t)}{t^2}$$
$$= \frac{4-6t}{t^2}$$

(ii) $(\ln 8, -3)$

(iii) $y = \frac{6}{t} \Rightarrow t = \frac{6}{y}$

$x = \ln\left(2 - 3\times\frac{6}{y}\right)$

$x = \ln\left(2 - \frac{18}{y}\right)$

$e^x = 2 - \frac{18}{y}$

$\frac{18}{y} = 2 - e^x$

$\frac{y}{18} = \frac{1}{2-e^x}$

$y = \frac{18}{2-e^x}$

4 1

5 (i) $y = 2x - 3$

(ii) (a) $x^2 = (2\sin\theta + \cos\theta)^2$

$= 4\sin^2\theta + 4\sin\theta\cos\theta + \cos^2\theta$

$y^2 = (\sin\theta + 2\cos\theta)^2$

$= \sin^2\theta + 4\sin\theta\cos\theta + 4\cos^2\theta$

$x^2 + y^2 = 4\sin^2\theta + 4\sin\theta\cos\theta + \cos^2\theta$

$+ \sin^2\theta + 4\sin\theta\cos\theta + 4\cos^2\theta$

$= 5\sin^2\theta + 8\sin\theta\cos\theta + 5\cos^2\theta$

$= 5(\sin^2\theta + \cos^2\theta) + 8\sin\theta\cos\theta$

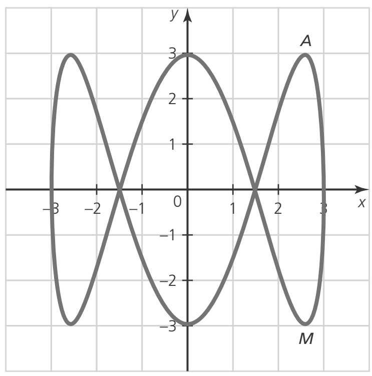
$= 5 + 4(2\sin\theta\cos\theta)$

$= 5 + 4\sin2\theta$

(b) Least value is 1, greatest value is 3

6 (i) $(3\sin2, 3\cos6)$

(ii)



(iii) $2\pi = 6.28$ seconds

(iv) -0.784 (3sf)

(v) $\left(\frac{3\sqrt{3}}{2}, 3\right)$

(vi) 1.05 s

7 (i) $\frac{dy}{dx} = \frac{2t^2-2}{2t-1}$

$(2-\ln 2,\ 1)$

(ii) $(2\sqrt{2}-\ln 2\sqrt{2},\ 2-\ln 2)$

8 (i) At A, $\theta = \frac{\pi}{2}$

At B, $\theta = 2\pi$

(ii) $\frac{dx}{d\theta} = 2+2\cos 2\theta,\ \frac{dy}{d\theta} = 4\cos\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx}$$
$$= 4\cos\theta \times \frac{1}{2+2\cos 2\theta}$$
$$= \frac{4\cos\theta}{2+2\cos 2\theta}$$
$$= \frac{4\cos\theta}{2(1+\cos 2\theta)}$$
$$= \frac{4\cos\theta}{2(1+(2\cos^2\theta-1))}$$
$$= \frac{4\cos\theta}{2(2\cos^2\theta)}$$
$$= \frac{4\cos\theta}{4\cos^2\theta}$$
$$= \frac{1}{\cos\theta}$$
$$= \sec\theta$$

(iii) $\left(-\frac{2\pi}{3}-\frac{\sqrt{3}}{2}, -2\sqrt{3}\right)$

9 (i) $x = 10\cos\theta + 5\cos 2\theta, \quad y = 10\sin\theta + 5\sin 2\theta$

$\frac{dx}{d\theta} = -10\sin\theta - 10\sin 2\theta,\ \frac{dy}{d\theta} = 10\cos\theta + 10\cos 2\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx}$$
$$= 10\cos\theta + 10\cos 2\theta \times \frac{1}{-10\sin\theta - 10\sin 2\theta}$$
$$= \frac{10\cos\theta + 10\cos 2\theta}{-10\sin\theta - 10\sin 2\theta}$$
$$= \frac{\cos\theta + \cos 2\theta}{-\sin\theta - \sin 2\theta}$$
$$= \frac{\cos\theta + \cos 2\theta}{-(\sin\theta + \sin 2\theta)}$$
$$= -\frac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta}$$

(ii) A is $\left(\frac{5}{2}, \frac{15\sqrt{3}}{2}\right)$

(iii) $x^2 + y^2 = 100\cos^2\theta + 100\cos\theta\cos 2\theta + 25\cos^2 2\theta$

$$+100\sin^2\theta + 100\sin\theta\sin 2\theta + 25\sin^2 2\theta$$
$$= 100(\cos^2\theta + \sin^2\theta) + 100(\cos\theta\cos 2\theta + \sin\theta\sin 2\theta)$$
$$+25(\cos^2 2\theta + \sin^2 2\theta)$$
$$= 125 + 100(\cos(2\theta - \theta))$$
$$= 125 + 100\cos\theta$$

(iv) Least distance is 25 m, greatest distance is 225 m.

Stretch and challenge

1 (i) The x co-ordinate of any point on the curve

$= OD - AB$
$= r\theta - r\sin\theta$
$= r(\theta - \sin\theta)$

The y co-ordinate of any point on the curve

$= CD - CB$
$= r - r\cos\theta$
$= r(1-\cos\theta)$

(ii) $\frac{dx}{d\theta} = r - r\cos\theta = r(1-\cos\theta),\ \frac{dy}{d\theta} = r\sin\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx}$$
$$= r\sin\theta \times \frac{1}{r(1-\cos\theta)}$$
$$= \frac{r\sin\theta}{r(1-\cos\theta)}$$
$$= \frac{\sin\theta}{(1-\cos\theta)}$$
$$= \frac{\sin\theta}{(1-\cos\theta)}\times\frac{1+\cos\theta}{1+\cos\theta}$$
$$= \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta}$$
$$= \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta}$$
$$= \frac{(1+\cos\theta)}{\sin\theta}$$

$$\cos 2\theta = 2\cos^2\theta - 1 \Rightarrow \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$$
$$\cos\theta + 1 = 2\cos^2\left(\frac{\theta}{2}\right)$$
$$\sin 2\theta = 2\sin\theta\cos\theta \Rightarrow \sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$
$$\therefore \frac{(1+\cos\theta)}{\sin\theta} = \frac{2\cos^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$

$$= \frac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

$$= \cot\left(\frac{\theta}{2}\right)$$

(iii) $y = \sqrt{3}x + r\left(2 - \frac{\sqrt{3}\pi}{3}\right)$

(iv) $s = r\sqrt{2(1-\cos\theta)}$

(v) $a = \frac{r\sin\theta}{\sqrt{2(1-\cos\theta)}}$

2 (i) $x = t^3, y = t^2$

$\frac{dx}{dt} = 3t^2, \frac{dy}{dt} = 2t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2t \times \frac{1}{3t^2} = \frac{2}{3t}$

Equation of the tangent at $t = p$:

$y = mx + c \Rightarrow p^2 = \frac{2}{3p} \times p^3 + c$

$c = p^2 - \tfrac{2}{3}p^2 = \tfrac{1}{3}p^2$

$y = \frac{2}{3p}x + \frac{1}{3}p^2$

$3py = 2x + p^3$

$3py - 2x = p^3$

(ii) (–64, 16), (–1, 1), (125, 25)

3 (i) $l = \frac{2}{\cos\theta} + \frac{3}{\sin\theta}$

(ii) 8.13 m

4 $x = 32.7$ m

5 The second term of $\frac{dT}{dt}$ is always positive, so the only critical point is at $k = \frac{1}{2}$

When $k = 0, \frac{dT}{dk} = -n_A - n_B\frac{\sqrt{3}}{2} < 0$ and when $k = 1$

$\frac{dT}{dk} = n_A + n_B\frac{\sqrt{3}}{2} > 0$, so we have a minimum.

6 (i) $\frac{(k\sin\theta + \cos\theta)}{(k\cos\theta - \sin\theta)} = [\tan(\theta + 2)]$

(ii) $\cos 2 = k$ and $\sin 2 = 1$

$\Rightarrow \tan\alpha = \frac{1}{k}$

$\alpha = \tan^{-1}\left(\frac{1}{k}\right)$

7 (i) $x = 0.920$ or 2.22 or 3.82 or 5.60 (3sf) for $0 \leqslant x \leqslant 2\pi$

(ii) $a = 2.23, b = \pi, c = 2$

(iii) $1.68 < k < 4.4887$

Exam focus

1 $x = 1$ or 3

2 $\frac{dy}{dx} = \frac{\ln x(2 - \ln x)}{x^2}$

M is $\left(e^2, \frac{4}{e^2}\right)$

3 Stationary points at $x = \frac{\pi}{3}, \frac{5\pi}{6}$

$x = \frac{\pi}{3}$ is a maximum.

$x = \frac{5\pi}{6}$ is a minimum.

4 (i) $x^2 + y^2 - xy - 48 = 0$

$2x + 2y\frac{dy}{dx} - \left(y + x\frac{dy}{dx}\right) = 0$

$(2y - x)\frac{dy}{dx} = y - 2x$

$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$

$= \frac{2x - y}{x - 2y}$

(ii) $(4, 8)$ and $(-4, -8)$

(iii) $(8, 4)$ and $(-8, -4)$

5 $\frac{dy}{dx} = \frac{-4x - y}{x + 2y}$

Stationary points where $\frac{dy}{dx} = 0$

$-4x - y = 0 \Rightarrow y = -4x$

$2x^2 + x(-4x) + (-4x)^2 = 14$

$2x^2 - 4x^2 + 16x^2 = 14$

$14x^2 = 14$

$x^2 = 1$

$x = \pm 1$

$y = \mp 4$

Points are $(1, -4)$ and $(-1, 4)$

6 (i) $\frac{dy}{dx} = \frac{27t + 36}{8}$

(ii) $-\frac{9}{4}$

Chapter 5 Integration

Exercise 5.1

1 (i) $\frac{1}{2}e^{2x}+c$

(ii) $-\frac{1}{3}e^{1-3x}+c$

(iii) $\frac{1}{2}\ln|x|+c$

(iv) $2\ln|x|+c$

(v) $-\dfrac{2}{e^{2x+3}}+c$

(vi) $\frac{3}{2}\ln|2x+1|+c$

(vii) $\dfrac{1}{3}e^{3x}-\dfrac{1}{e^{x}}+c$

(viii) $-27e^{-\frac{1}{3}x}+c$

(ix) $\frac{1}{2}e^{2x}-2e^{x}+x+c$

(x) $\ln\left|\dfrac{x^{\frac{1}{3}}}{(1-2x)^{\frac{1}{2}}}\right|+c$

2 (i) $\frac{3}{4}e^{4}(e^{4}-1)$

(ii) $\ln\frac{25}{9}$ or $2\ln\left(\frac{5}{3}\right)$

3 $\displaystyle\int_1^{\infty}\frac{e^x+2}{e^{2x}}\,dx=\int_1^{\infty}\frac{e^x}{e^{2x}}+\frac{2}{e^{2x}}\,dx$

$$=\int_1^{\infty}\left(e^{-x}+2e^{-2x}\right)dx$$

$$=\left[-e^{-x}-e^{-2x}\right]_1^{\infty}$$

$$=\left[-\frac{1}{e^{x}}-\frac{1}{e^{2x}}\right]_1^{\infty}$$

$$=\left[\left(-\frac{1}{e^{\infty}}-\frac{1}{e^{2\infty}}\right)-\left(-\frac{1}{e^{1}}-\frac{1}{e^{2}}\right)\right]$$

$$=\left[(0)-\left(-\frac{1}{e^{1}}-\frac{1}{e^{2}}\right)\right]$$

$$=\left[\frac{1}{e}+\frac{1}{e^{2}}\right]$$

$$=\frac{e+1}{e^{2}}$$

4 $\displaystyle\int_{-1}^{1}\frac{9}{1-3x}\,dx=\left[-3\ln|1-3x|\right]_{-1}^{1}$

$$=\left[(-3\ln|1-3\times1|)-(-3\ln|1-3\times-1|)\right]$$

$$=\left[(-3\ln|-2|)-(-3\ln|4|)\right]$$

$$=\left[-3\ln2+3\ln4\right]$$

$$=\left[\ln2^{-3}+\ln4^{3}\right]$$

$$=\left[\ln\tfrac{1}{8}+\ln64\right]$$

$$=\ln\left(\tfrac{1}{8}\times64\right)$$

$$=\ln8$$

5 $\ln|x+1|+c$

6 $\dfrac{x^2}{2}-3x+2\ln|3x+2|+c$

7 $k=\dfrac{2e+1}{3}$

8 $k=\frac{17}{2}$

Exercise 5.2

1 (i) $-\frac{1}{4}\cos4x+c$

(ii) $\frac{1}{3}\sin(3x-1)+c$

(iii) $\frac{1}{2}\tan2x+c$

(iv) $-6\cos\frac{1}{3}x+c$

(v) $8\sin\frac{1}{2}x+x+c$

(vi) $\frac{1}{2}\sin2x-\frac{1}{3}\tan3x+4x+c$

2 (i) $\frac{1}{2}$

(ii) $\dfrac{\pi}{16}-\dfrac{\sqrt{2}}{8}$

(iii) $2-\frac{2}{\sqrt{3}}$

(iv) $1-\frac{\pi}{4}$

3 0.342 or $\left(\frac{\sqrt{3}}{2}-\frac{\pi}{4}\right)$ exactly

4 $1-\frac{\pi}{4}$

5 (i) $\frac{1}{2}x-\frac{1}{4}\sin 2x+c$

(ii) $\frac{1}{8}\sin 4x+\frac{1}{2}x+c$

(iii) $-\cos 2x+c$

(iv) $\frac{1}{4}\sin 4x+c$

(v) $\frac{3x}{8}-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+c$

(vi) $x+\frac{1}{2}\cos 2x+c$

(vii) $-\frac{1}{3}\cos 3x+c$

(viii) $\frac{1}{32}\sin 4x+\frac{1}{4}\sin 2x+\frac{3}{8}x+c$

6 $\int_0^{\frac{\pi}{4}}\sqrt{1+\cos 4x}\,dx$

$=\int_0^{\frac{\pi}{4}}\sqrt{1+(2\cos^2 2x-1)}\,dx$

$=\int_0^{\frac{\pi}{4}}\sqrt{2\cos^2 2x}\,dx$

$=\int_0^{\frac{\pi}{4}}\sqrt{2}\cos 2x\,dx$

$=\sqrt{2}\times\left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$=\sqrt{2}\times\left[\frac{1}{2}\sin 2\times\frac{\pi}{4}-\frac{1}{2}\sin 0\right]$

$=\sqrt{2}\times\left[\frac{1}{2}-0\right]$

$=\frac{\sqrt{2}}{2}$

7 (i) $2\sqrt{3}-\frac{2\pi}{3}$

(ii) 12.0 or $\frac{2\pi^2}{3}+\sqrt{3}\,\pi$ exactly

8 (i) $\cos 3x=\cos(2x+x)$

$=\cos 2x\cos x-\sin 2x\sin x$

$=(2\cos^2 x-1)\cos x-2\sin x\cos x\sin x$

$=2\cos^3 x-\cos x-2\sin^2 x\cos x$

$=2\cos^3 x-\cos x-2(1-\cos^2 x)\cos x$

$=2\cos^3 x-\cos x-2\cos x+2\cos^3 x$

$=4\cos^3 x-3\cos x$

$=RHS$

(ii) $\frac{2}{3}-\frac{3\sqrt{3}}{8}$

9 (i) $A=1, B=4$

(ii) $\frac{\pi}{2}$

10 $\frac{1}{3}\pi-\frac{2\sqrt{3}}{3}$

Exercise 5.3

1 3.575

2 (i)

x	0	0.25	0.5	0.75	1
$f(x)$	1	0.9846	0.8888	0.7033	0.5

(ii) 0.832 (3sf)

(iii) Underestimate

3 (i) 5.9 m^2 (1dp)

(ii) 295 m^3/min

4 4.13 (3sf). This is an overestimate.

5 0.25 (2dp)

6 (i) 3.28 (3sf)

(ii) Chris is correct.

The answer from **(i)** is an overestimate of the true area.

Increasing the number of strips would decrease the value found.

Stretch and challenge

1 If $y=\frac{x^3}{6}+\frac{1}{2x}, \frac{dy}{dx}=\frac{1}{2}x^2-\frac{1}{2x^2}$

$$L=\int_1^2\sqrt{1+\left[\frac{1}{2}x^2-\frac{1}{2x^2}\right]^2}\,dx$$
$$=\int_1^2\sqrt{1+\left(\frac{1}{4}x^4-\frac{1}{2}+\frac{1}{4x^4}\right)}\,dx$$
$$=\int_1^2\sqrt{\left(\frac{1}{4}x^4+\frac{1}{2}+\frac{1}{4x^4}\right)}\,dx$$
$$=\int_1^2\sqrt{\left(\frac{1}{2}x^2+\frac{1}{2x^2}\right)^2}\,dx$$
$$=\int_1^2\left(\frac{1}{2}x^2+\frac{1}{2x^2}\right)dx$$
$$=\left[\frac{x^3}{6}-\frac{1}{2x}\right]_1^2$$
$$=\left[\left(\frac{2^3}{6}-\frac{1}{2\times2}\right)-\left(\frac{1^3}{6}-\frac{1}{2\times1}\right)\right]$$
$$=\left[\left(\frac{8}{6}-\frac{1}{4}\right)-\left(\frac{1}{6}-\frac{1}{2}\right)\right]$$
$$=\frac{13}{12}-\left(-\frac{1}{3}\right)$$
$$=\frac{17}{12}$$

2 $\int_0^{\frac{\pi}{4}}\sin^2x\cos^2x\,dx=\int_0^{\frac{\pi}{4}}\sin^2x(1-\sin^2x)\,dx$

$=\int_0^{\frac{\pi}{4}}(\sin^2x-\sin^4x)\,dx$

$\cos2x=1-2\sin^2x=2\cos^2x-1$

$\Rightarrow\sin^2x=\frac{1}{2}(1-\cos2x)$

$\Rightarrow\sin^4x=\frac{1}{4}(1-\cos2x)^2$

$=\frac{1}{4}(1-2\cos2x+\cos^22x)$

$=\frac{1}{4}(1-2\cos2x+\frac{1}{2}(\cos4x+1))$

$=\frac{1}{4}-\frac{1}{2}\cos2x+\frac{1}{8}\cos4x+\frac{1}{8}$

$=\frac{3}{8}-\frac{1}{2}\cos2x+\frac{1}{8}\cos4x$

$\therefore\int_0^{\frac{\pi}{4}}(\sin^2x-\sin^4x)\,dx$

$=\int_0^{\frac{\pi}{4}}\left(\frac{1}{2}(1-\cos2x)-\left(\frac{3}{8}-\frac{1}{2}\cos2x+\frac{1}{8}\cos4x\right)\right)dx$

$=\int_0^{\frac{\pi}{4}}\left(\frac{1}{8}-\frac{1}{8}\cos4x\right)dx$

$=\frac{1}{8}\int_0^{\frac{\pi}{4}}(1-\cos4x)\,dx$

$=\frac{1}{8}\left[x-\frac{1}{4}\sin4x\right]_0^{\frac{\pi}{4}}$

$=\frac{1}{8}\left[\left(\frac{\pi}{4}-\frac{1}{4}\sin4\times\frac{\pi}{4}\right)-0\right]$

$=\frac{1}{8}\left[\left(\frac{\pi}{4}\right)-0\right]$

$=\frac{\pi}{32}$

3 **(i)** $k=2,\ m=\frac{3\pi}{4},\ t=\pi$

(ii) 8

(iii) $8r$

4 $\frac{1}{4}(\cos^3\phi-3\cos\phi+2)$

Exam focus

1 **(i)** $\int_0^2\frac{4}{2x+1}dx=[2\ln|2x+1|]_0^2$

$=[2\ln|2\times2+1|-2\ln|2\times0+1|]$

$=2[\ln5-\ln1]$

$=2\ln5-0$

$=\ln5^2$

$=\ln25$

(ii) $k=-3$

2 **(i)** $(2\cos x+\sin x)^2$

$=4\cos^2x+4\sin x\cos x+\sin^2x$

$=3\cos^2x+4\sin x\cos x+\sin^2x+\cos^2x$

$=3\cos^2x+4\sin x\cos x+1$

$=3\left(\frac{1}{2}(\cos2x+1)\right)+2(2\sin x\cos x)+1$

$=\frac{3}{2}\cos2x+\frac{3}{2}+2\sin2x+1$

$=2\sin2x+\frac{3}{2}\cos2x+\frac{5}{2}$

$a=2,\ b=\frac{3}{2},\ c=\frac{5}{2}$

(ii) $\frac{5\pi}{4}+2$

3 $LHS: \tan^2 x + \sin^2 x$

$= \sec^2 x - 1 + \tfrac{1}{2}(1 - \cos 2x)$

$= \sec^2 x - 1 + \tfrac{1}{2} - \tfrac{1}{2}\cos 2x$

$= \sec^2 x - \tfrac{1}{2}\cos 2x - \tfrac{1}{2}$

$= RHS$

$\int_0^{\frac{\pi}{6}} \left(\tan^2 x + \sin^2 x\right) dx = \frac{5\sqrt{3}}{24} - \frac{\pi}{12}$

4 (i) 6.26 (3sf)

(ii) The first trapezium will overestimate the area from $0 \leqslant x \leqslant \frac{\pi}{2}$ but the second trapezium will underestimate the area from $\frac{\pi}{2} < x \leqslant \pi$, so taken together the trapezium rule will give a good estimate of the true value of the integral.

5 (i) $e^x(5\sin 2x)$

(ii) $\tfrac{1}{5}e^{\frac{1}{4}\pi} + \tfrac{2}{5}$

6 (i) $\cos\theta + \sqrt{3}\sin\theta = 2\cos\left(\theta - \frac{\pi}{3}\right)$

(ii) $\int_0^{\frac{1}{3}\pi} \frac{1}{(\cos\theta + \sqrt{3}\sin\theta)^2} d\theta$

$= \int_0^{\frac{1}{3}\pi} \frac{1}{\left(2\cos\left(\theta - \frac{\pi}{3}\right)\right)^2} d\theta$

$= \int_0^{\frac{1}{3}\pi} \frac{1}{4\cos^2\left(\theta - \frac{\pi}{3}\right)} d\theta$

$= \frac{1}{4}\int_0^{\frac{1}{3}\pi} \sec^2\left(\theta - \frac{\pi}{3}\right) d\theta$

$= \frac{1}{4}\left[\tan\left(\theta - \frac{\pi}{3}\right)\right]_0^{\frac{1}{3}\pi}$

$= \frac{1}{4}\left[\tan\left(\frac{1}{3}\pi - \frac{\pi}{3}\right) - \tan\left(0 - \frac{\pi}{3}\right)\right]$

$= \frac{1}{4}\left[\tan 0 - \tan\left(-\frac{\pi}{3}\right)\right]$

$= \frac{1}{4}\left[0 - -\sqrt{3}\right]$

$= \frac{\sqrt{3}}{4}$

Chapter 6 Numerical solution of equations

Exercise 6.1

1 (i) $f(x) = e^{x-3} - x^3$

(ii) $f(0) = e^{0-3} - 0^3 = e^{-3} > 0$

$f(1) = e^{1-3} - 1^3 = e^{-2} - 1 < 0$

Since there is a change in sign, the root must lie between 0 and 1.

(iii) $f(0) = e^{0-3} - 0^3 = e^{-3} > 0$

$f(0.5) = e^{0.5-3} - 0.5^3 = -0.0429 < 0$

Since there is a change in sign, the root must lie between 0 and 0.5.

(iv) The root lies between $x = 0.4$ and $x = 0.5$

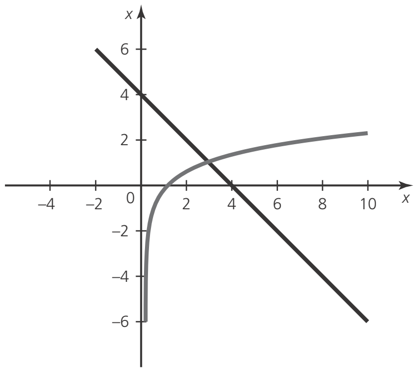
2 The first root is in the interval $[-2, -1]$

The second root is in the interval $[1, 2]$

The third root is in the interval $[2, 3]$

3 The statement is false because the curve is not continuous between $x = 0$ and $x = 2$.

4 (i)



(ii) Either $4 - x - \ln x = 0$ or $\ln x + x - 4 = 0$

(iii) The integer bounds are 2 and 3.

(iv) $f(2.9) = \ln 2.9 + 2.9 - 4 = -0.0353$

$f(3.0) = \ln 3 + 3 - 4 = 0.0986$

Change in sign

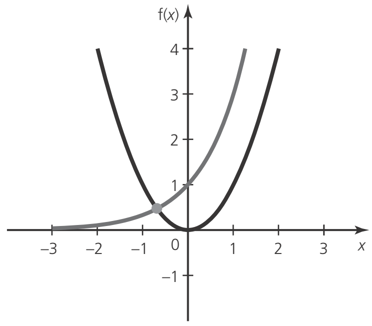
$\Rightarrow$ Root lies between 2.9 and 3.0

5 (i) $f(-0.7) = (-0.7)^2 - 3^{-0.7} = 0.0265$

$f(-0.6) = (-0.6)^2 - 3^{-0.6} = -0.157$

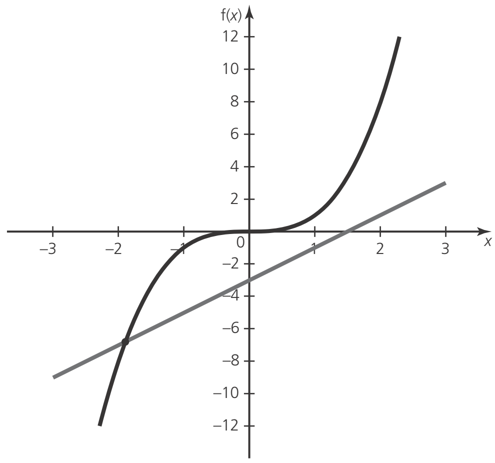
Sign change $\Rightarrow$ Root in the interval $[-0.7, -0.6]$

(ii)



Exercise 6.2

1 (i)



(ii) The root lies in the interval $[-2, -1]$.

(iii) $x_n = -1.8933$

(iv) $x_{n+1} = \frac{1}{2}(x_n{}^3 + 3)$ or $\frac{2}{x_n} - \frac{3}{x_n{}^2}$

2 2.849 (3dp)

3 (i) $x^3 - x^2 = 15 \Rightarrow x^3 - x^2 - 15 = 0$

$f(x) = x^3 - x^2 - 15$

$f(2) = 2^3 - 2^2 - 15 = -11 < 0$

$f(3) = 3^3 - 3^2 - 15 = 3 > 0$

Change in sign

$\Rightarrow$ Root lies between 2 and 3.

(ii) $x_n = 2.770$ (3dp)

4 (i)

$$x_{n+1} = \sqrt{\frac{3 - x_n}{x_n}}$$

$$x = \sqrt{\frac{3 - x}{x}}$$

$$x^2 = \frac{3 - x}{x}$$

$$x^3 = 3 - x$$

$$x^3 + x - 3 = 0$$

(ii) $x_n = 1.21$

5 (i) Area of sector = $\frac{1}{2}r^2\theta$

Area of triangle $OAB = \frac{1}{2}r^2 \sin\theta$

Area of segment

$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin\theta$

$= \frac{1}{2}r^2(\theta - \sin\theta)$

(ii) Area of segment = $\frac{1}{6}\pi r^2$

$$\frac{1}{2}r^2(\theta - \sin\theta) = \frac{1}{6}\pi r^2$$
$$3r^2(\theta - \sin\theta) = \pi r^2$$
$$3(\theta - \sin\theta) = \pi$$
$$(\theta - \sin\theta) = \frac{1}{3}\pi$$
$$\theta = \frac{1}{3}\pi + \sin\theta$$

(iii) $\theta_n = 1.97$

6 (i) Iterations diverge.

(ii) Iterations do not converge to a single value.

(iii) Iterations do not converge.

7 (i)
$$\int_0^a (6e^{2x} + x)dx = 42$$
$$\left[3e^{2x} + \frac{x^2}{2}\right]_0^a = 42$$
$$\left[\left(3e^{2a} + \frac{a^2}{2}\right) - \left(3e^0 + \frac{0^2}{2}\right)\right] = 42$$
$$3e^{2a} + \frac{a^2}{2} - 3 = 42$$
$$3e^{2a} + \frac{a^2}{2} = 45$$
$$e^{2a} + \frac{a^2}{6} = 15$$
$$e^{2a} = 15 - \frac{a^2}{6}$$
$$\ln(e^{2a}) = \ln\left(15 - \frac{a^2}{6}\right)$$
$$2a = \ln\left(15 - \frac{a^2}{6}\right)$$
$$a = \frac{1}{2}\ln\left(15 - \frac{1}{6}a^2\right)$$

(ii) $a_n = 1.344$ (3dp)

8 (i) $x_n = 2.877$ (3dp)

(ii) $2x^3 + 5x - 62 = 0$

Stretch and challenge

1 (i) In general, $x_{n+1} = \dfrac{f(x_{n-1})x_n - f(x_n)x_{n-1}}{f(x_{n-1}) - f(x_n)}$

It can also be shown that

$$x_{n+1} = x_n - \frac{f(x_n)[x_n - x_{n-1}]}{f(x_n) - f(x_{n-1})}$$

(ii) $x_3 = \dfrac{2}{e^2 - 1}$

Exam focus

1 2.13 (2dp)

2 (i) An equation satisfied by α is

$\alpha = 2\ln(48 + 16\alpha^2)$

Starting with the original equation,

$$y = e^{-\frac{1}{4}x}\sqrt{3 + x^2}$$
$$\frac{1}{4} = e^{-\frac{1}{4}x}\sqrt{3 + x^2}$$
$$\left(\frac{1}{4}\right)^2 = \left(e^{-\frac{1}{4}x}\sqrt{3 + x^2}\right)^2$$
$$\frac{1}{16} = e^{-\frac{1}{2}x}(3 + x^2)$$
$$\frac{1}{e^{-\frac{1}{2}x}} = 16(3 + x^2)$$
$$e^{\frac{1}{2}x} = 48 + 16x^2$$
$$\frac{1}{2}x = \ln(48 + 16x^2)$$
$$x = 2\ln(48 + 16x^2)$$

(ii) 16.87 (2dp)

3 (i) $\alpha = -1$

(ii) If $\alpha = -1$ is a root of the equation $f(x) = 0$, then $(x + 1)$ is a factor of $f(x)$.

Using polynomial division,

$$\frac{x^4 - 4x^3 + 4x^2 + 2x - 7}{x + 1} = x^3 - 5x^2 + 9x - 7$$

$\therefore x^4 - 4x^3 + 4x^2 + 2x - 7$

$= (x + 1)(x^3 - 5x^2 + 9x - 7)$

To find β we need to solve

$(x+1)(x^3 - 5x^2 + 9x - 7) = 0$

$x = -1$ or

$x^3 - 5x^2 + 9x - 7 = 0$

$$x^3 = 5x^2 - 9x + 7$$
$$x = \sqrt[3]{5x^2 - 9x + 7}$$

(iii) 2.54 (2dp)

4 (i) $\dfrac{dy}{dx}=\dfrac{-2\sin 2x+2x\sin 2x+\cos 2x}{(1-x)^2}$

Maximum when $\dfrac{dy}{dx}=0$

$$-2\sin 2x+2x\sin 2x+\cos 2x=0$$

$$\sin 2x(2x-2)+\cos 2x=0$$

$$\frac{\sin 2x(2x-2)}{\cos 2x}+\frac{\cos 2x}{\cos 2x}=\frac{0}{\cos 2x}$$

$$\tan 2x(2x-2)+1=0$$

$$\tan 2x(2x-2)=-1$$

$$\tan 2x=\frac{-1}{2x-2}=\frac{1}{2-2x}$$

(ii) $f(0.6)=(2-2\times 0.3)\tan 2\times 0.3-1=-0.042$

$f(0.8)=(2-2\times 0.4)\tan 2\times 0.4-1=0.236$

Change in sign $\Rightarrow$ Root lies between 0.3 and 0.4

(iii) 0.315 (3dp)

5 (i) $\frac{1}{2}r^2\theta=\frac{1}{2}r\times r\tan\theta-\frac{1}{2}r^2\theta$

$$\tfrac{1}{2}r^2\theta=\tfrac{1}{2}r^2\tan\theta-\tfrac{1}{2}r^2\theta$$

$$\theta=\tan\theta-\theta$$

$$2\theta=\tan\theta$$

(ii) 1.17 (2dp)

Chapter 7 Further algebra

Exercise 7.1

1 (i) $1-2x+3x^2-4x^3+\cdots$

Valid for $|x|<1$

(ii) $1+2x+4x^2+8x^3+\cdots$

Valid for $|2x|<1 \Rightarrow |x|<\frac{1}{2}$

(iii) $1+2x-2x^2+4x^3-\cdots$

Valid for $|4x|<1 \Rightarrow |x|<\frac{1}{4}$

(iv) $1+3x+18x^2+126x^3+\cdots$

Valid for $|9x|<1 \Rightarrow |x|<\frac{1}{9}$

2 (i) $\frac{1}{16}-\frac{1}{8}x+\frac{5}{32}x^2-\cdots$

Valid for $\left|\frac{1}{2}x\right|<1 \Rightarrow |x|<2$

(ii) $3-\frac{1}{2}x-\frac{1}{24}x^2-\cdots$

Valid for $\left|\frac{1}{3}x\right|<1 \Rightarrow |x|<3$

(iii) $\frac{1}{2}+\frac{1}{16}x+\frac{3}{256}x^2+\cdots+$

Valid for $\left|\frac{1}{4}x\right|<1 \Rightarrow |x|<4$

(iv) $\frac{3}{2}x-\frac{3}{4}x^2+\cdots$

Valid for $\left|\frac{1}{2}x\right|<1 \Rightarrow |x|<2$

3 (i) $a=5,\ b=-2$

(ii) Valid for $|5x|<1 \Rightarrow |x|<\frac{1}{5}$

4 (i) $1+x^2+\frac{3}{2}x^4+\cdots$

Valid for $|2x^2|<1 \Rightarrow |x|<\sqrt{\frac{1}{2}}$

(ii) $1+x+x^2+x^3+\frac{3}{2}x^4+\frac{3}{2}x^5$

5 (i) $n=\frac{1}{3}$

(ii) $1-x-x^2-\frac{5}{3}x^3-\cdots$

6 (i) $a=-2$

(ii) $-4x^3$

7 (i) $1+x+2x^2+\frac{14}{3}x^3+\cdots$

(ii) $\frac{17}{3}$

8 (i) $1-4ax+10a^2x^2-\cdots$

(ii) $a=\frac{1}{3},\ b=\frac{7}{3}$

9 (i) $\frac{1}{a^2}-\frac{2}{a^3}x+\frac{3}{a^4}x^2+\cdots$

(ii) $a=-\frac{3}{2}$

10 (i)
$$y=\frac{1}{\sqrt{1-2x}-\sqrt{1-x}}$$
$$=\frac{1}{\sqrt{1-2x}-\sqrt{1-x}}\times\frac{\left(\sqrt{1-2x}+\sqrt{1-x}\right)}{\left(\sqrt{1-2x}+\sqrt{1-x}\right)}$$
$$=\frac{\left(\sqrt{1-2x}+\sqrt{1-x}\right)}{1-2x-(1-x)}$$
$$=\frac{\left(\sqrt{1-2x}+\sqrt{1-x}\right)}{-x}$$
$$=-\frac{1}{x}\left(\sqrt{1-2x}+\sqrt{1-x}\right)$$

(ii) $\frac{5}{8}$

Exercise 7.2

1 (i) $\frac{2b}{3a}$

(ii) $\frac{5c^2e}{3d}$

(iii) $\frac{f+4}{f-2}$

(iv) $\frac{1}{4}$

(v) $\frac{2h(h+2)}{h-2}$

(vi) $(j+k)^2$

2 (i) $\frac{11}{4m}$

(ii) $\frac{17n-3}{12}$

(iii) $\frac{p^2+5p-6}{3p}$

(iv) $\frac{2q^2-6q-4}{(q+1)(q-1)}$

(v) $\frac{17r^2+25r-40}{20r^3}$

(vi) $\frac{4-3s}{(s-3)(s+3)}$

Exercise 7.3

1 (i) $\frac{2}{x+1}+\frac{3}{x+2}$

(ii) $\frac{4}{x-1}-\frac{2}{x+3}$

(iii) $\frac{1}{x-2}+\frac{2}{x+2}$

(iv) $\frac{2}{x-2}-\frac{5}{x+3}$

(v) $-\frac{4}{x}+\frac{4}{x-1}$

(vi) $\frac{3}{2x+1}-\frac{1}{x-3}$

(vii) $\frac{4}{x+1}-\frac{1}{x-2}-\frac{3}{x-4}$

(viii) $\frac{2}{x}+\frac{4}{x-4}-\frac{1}{x+4}$

(ix) $2x+\frac{3}{x-3}+\frac{2}{x+1}$

(x) $3+\frac{3}{x+2}-\frac{1}{x-2}$

2 (i) $-\frac{2}{x}+\frac{1}{x^2}+\frac{3}{x-1}$

(ii) $\frac{1}{2(x-1)}+\frac{5}{2(x+1)}+\frac{1}{(x+1)^2}$

(iii) $\frac{1}{x}+\frac{1}{(x-1)^2}$

(iv) $-\frac{1}{(x+1)^2}-\frac{2}{(x+1)^3}$

3 (i) $\frac{1}{2(x+1)}+\frac{1-x}{2(x^2+1)}$

(ii) $\frac{2}{x-1}+\frac{3}{x^2+2}$

(iii) $\frac{1}{2(x-4)}-\frac{x+4}{2(x^2+4)}$

(iv) $\frac{2x+1}{x^2+1}+\frac{1}{(x-1)^2}-\frac{2}{(x-1)}$

Exercise 7.4

1 (i) $A = 3, B = -4$

(ii) $1-2x+4x^2-\cdots$

Valid for $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$.

(iii) $\frac{1}{2}+\frac{1}{4}x+\frac{1}{8}x^2+\cdots$

Valid for $|x|<2$.

(iv) $1-7x+\frac{23}{2}x^2$

Valid for $|x|<\frac{1}{2}$.

2 (i) $\frac{2}{x-2}-\frac{3}{x+1}$

$-4+\frac{5}{2}x-\frac{13}{4}x^2$

Valid for $|x|<1$

(ii) $\frac{3}{1-2x}+\frac{1}{1+x}-\frac{2}{(1+x)^2}$

$2+9x+7x^2+\cdots$

Valid for $|2x|<1\Rightarrow|x|<\frac{1}{2}$

(iii) $\frac{4}{1+x}-\frac{2x+1}{1+x^2}$

$3-6x+5x^2\cdots$

Valid for $|x^2|<1\Rightarrow|x|<1$

3 (i) $\frac{3}{1-x}-\frac{2}{2+x}+\frac{5}{1+x}$

(ii) $7-\frac{3}{2}x+\frac{31}{4}x^2-\cdots$

4 (i) $B = 1, C = 2$

x^2 terms $\Rightarrow 2=-4A+C\Rightarrow A=0$

(ii) $3+6x+35x^2+\cdots$

Stretch and challenge

1 (i) $2-\frac{3}{4}x-\frac{9}{64}x^2-\cdots$

(ii) $a=-\frac{5}{2}$

2 $k=-4$ or 8

3 $n>7$

4 (i) $a=3, b=-\frac{1}{3}$

(ii) Valid for $|3x|<1\Rightarrow|x|<\frac{1}{3}$

Exam focus

1 (i) $a=-\frac{1}{2}$

(ii) $1+\frac{3}{2}x+\frac{3}{2}x^2+\cdots$

2 (i) $\frac{4}{2+x^2}-\frac{3}{2-x}$

(ii) $\frac{1}{2}-\frac{3}{4}x-\frac{11}{8}x^2-\frac{3}{16}x^3+\cdots$

Chapter 8 Further integration

Exercise 8.1

1 (i) $\frac{1}{4}(2x+1)^2+c$

(ii) $\frac{(x^2+1)^5}{5}+c$

(iii) $-\frac{1}{3}\sqrt{(1-x^2)^3}+c$

(iv) $\frac{1}{6}\ln|3+2x^3|+c$

(v) $\frac{4\sqrt{(x+4)^5}}{5}-\frac{16\sqrt{(x+4)^3}}{3}+c$

(vi) $-\frac{2}{3(4-x)^3}+\frac{2}{(4-x)^4}+c$

(vii) $\frac{2\sqrt{(x-1)^7}}{7}+\frac{4\sqrt{(x-1)^5}}{5}+\frac{2\sqrt{(x-1)^3}}{3}+c$

2 (i) 0.305 (3sf)

(ii) $\frac{5}{144}$

(iii) 42

(iv) 0.001 54 or $\frac{1}{648}$

(v) 0.134 or $\frac{15}{112}$

(vi) 14.28

3 2.73

4 (i) (−4, 0)

(ii) 8.53

Exercise 8.2

1 (i) $4e^{x^2+3}+c$

(ii) $\ln|x-x^2|+c$

(iii) $\ln|e^x-1|+c$

(iv) $-\frac{1}{6(1+e^{3x})^2}+c$

(v) $-e^{\frac{1}{x}}+c$

(vi) $-\frac{2}{3}\ln|1-3x^3|+c$

(vii) $-e^{1-\cos x}+c$

2 $u=2x+1 \Rightarrow du=2dx$

$x=1 \Rightarrow u=3,\ x=0 \Rightarrow u=1$

$$\int_0^1 \frac{x}{2x+1}\,dx=\int_{u=1}^{u=3}\frac{\frac{u-1}{2}}{u}\frac{du}{2}$$
$$=\frac{1}{4}\int_1^3\frac{u-1}{u}du$$
$$=\frac{1}{4}\int_1^3\left(1-\frac{1}{u}\right)du$$
$$=\frac{1}{4}\left[u-\ln u\right]_1^3$$
$$=\frac{1}{4}\left[(3-\ln 3)-(1-\ln 1)\right]$$
$$=\frac{1}{4}\left[(3-\ln 3)-1\right]$$
$$=\frac{1}{4}(2-\ln 3)$$

3 9897.6

4 $\frac{1}{6}$

5 (i) $\left(0,\frac{1}{2}\right)$

(ii) $\frac{dy}{dx}=\frac{2e^{2x}}{(1+e^{2x})^2}$

When $x=0$, $\frac{dy}{dx}=\frac{1}{2}$

(iii) Area = $\int_0^1\frac{e^{2x}}{1+e^{2x}}\,dx$

Let $u=1+e^{2x} \Rightarrow du=2e^{2x}dx$

$x=1 \Rightarrow u=1+e^2,\ x=0 \Rightarrow u=2$

$$\int_0^1\frac{e^{2x}}{1+e^{2x}}dx=\int_{u=2}^{u=1+e^2}\frac{e^{2x}}{u}\frac{du}{2e^{2x}}$$
$$=\frac{1}{2}\int_2^{1+e^2}\frac{1}{u}du$$
$$=\frac{1}{2}\left[\ln|u|\right]_2^{1+e^2}$$
$$=\frac{1}{2}\left[\ln|1+e^2|-\ln|2|\right]$$
$$=\frac{1}{2}\ln\left(\frac{1+e^2}{2}\right)$$

Exercise 8.3

1 (i) $-\frac{1}{2}\cos(x^2)+c$

(ii) $\frac{(1+\sin 3x)^4}{12}+c$

(iii) $-e^{\cos x}+c$

(iv) $\ln|\sin x|+c$

(v) $-\frac{1}{4}\ln|\cos 4x|+c$

(vi) $-\frac{\cos^6 x}{6}+c$

(vii) $\frac{\sin^4 3x}{6}+c$

(viii) $\frac{1}{2}\tan 2\theta+c$

2 $\frac{(1-\cos x)^4}{4}-\frac{(1-\cos x)^5}{5}+c$

3 $\frac{\sqrt{3}}{2}+\frac{\pi}{3}$

4 $\frac{\pi}{4}$

5 (i) $x=\sin^2\theta \Rightarrow dx=2\sin\theta\cos\theta\,d\theta$

$x=1\Rightarrow \sin^2\theta=1\Rightarrow\theta=\frac{\pi}{2}$,

$x=0\Rightarrow\sin^2\theta=0\Rightarrow\theta=0$

$$\int_0^1\sqrt{\frac{1-x}{x}}\,dx$$

$$=\int_{\theta=0}^{\theta=\frac{\pi}{2}}\sqrt{\frac{1-\sin^2\theta}{\sin^2\theta}}\,2\sin\theta\cos\theta\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\sqrt{\frac{\cos^2\theta}{\sin^2\theta}}\,2\sin\theta\cos\theta\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta}\,2\sin\theta\cos\theta\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}2\cos^2\theta\,d\theta$$

(ii) $\frac{\pi}{2}$

6 $\frac{1}{4}\sin\left(\cos^{-1}\frac{2}{x}\right)+c$

7 $\frac{\pi}{6}+1-\sqrt{3}$

8 (i) $x=\frac{1}{y}\Rightarrow dx=-\frac{1}{y^2}dy$

$$\int\frac{1}{x\sqrt{x^2-1}}\,dx$$

$$=\int\frac{1}{\frac{1}{y}\sqrt{\left(\frac{1}{y^2}-1\right)}}\left(-\frac{1}{y^2}dy\right)$$

$$=-\int\frac{1}{y\sqrt{\left(\frac{1}{y^2}-1\right)}}\,dy$$

$$=-\int\frac{1}{y\sqrt{\left(\frac{1-y^2}{y^2}\right)}}\,dy$$

$$=-\int\frac{1}{y\frac{\sqrt{1-y^2}}{y}}\,dy$$

$$=-\int\frac{1}{\sqrt{1-y^2}}\,dy$$

(ii) $-\sin^{-1}\left(\frac{1}{x}\right)+c$

Exercise 8.4

1 (i) $2\ln|x+2|+5\ln|x-1|+c$

(ii) $\ln|x+3|+2\ln(x^2+1)+c$

2 (i) $\frac{2}{x-3}+\frac{7}{(x-3)^2}$

(ii) $2\ln 7+6$

3 (i) $2x+3+\frac{x}{x^2+4}$

$A=2, B=3, C=1, D=0$

(ii) $14+\ln\sqrt{\frac{13}{5}}$

4 $\ln\frac{4}{3}$

5 (i) $u=\sqrt{x} \Rightarrow du=\frac{1}{2\sqrt{x}}dx \Rightarrow dx = 2\sqrt{x}du$

$$\int \frac{1}{x(1+\sqrt{x})}dx = \int \frac{1}{u^2(1+u)}2\sqrt{x}\,du$$
$$= \int \frac{1}{u^2(1+u)}2u\,du$$
$$= \int \frac{2}{u(1+u)}du$$

(ii) $\ln\frac{9}{4}$

6 (i) $u=\sqrt{2-x} \Rightarrow du = -\frac{1}{2\sqrt{2-x}}dx$

$$dx = -2\sqrt{2-x}\,du$$

$u^2 = 2-x \Rightarrow x = 2-u^2$

$x=2 \Rightarrow u=0, x=1 \Rightarrow u=1$

$$I=\int_1^2 \frac{3}{x+\sqrt{2-x}}dx$$
$$=\int_{u=1}^{u=0} \frac{3}{2-u^2+u}\left(-2\sqrt{2-x}\,du\right)$$
$$=-\int_1^0 \frac{6u}{2-u^2+u}du$$
$$=-\int_1^0 \frac{6u}{2-u^2+u}du$$
$$=\int_0^1 \frac{6u}{(2-u)(1+u)}du$$

(ii) $\int_0^1 \frac{6u}{(2-u)(1+u)}du$

$$=\int_0^1\left(\frac{4}{2-u}-\frac{2}{1+u}\right)du$$
$$=\left[-4\ln(2-u)-2\ln(1+u)\right]_0^1$$
$$=[(-4\ln(2-1)-2\ln(1+1))$$
$$-(-4\ln(2-0)-2\ln(1+0))]$$
$$=\left[(-4\ln 1-2\ln 2)-(-4\ln 2-2\ln 1)\right]$$
$$=\left[(0-\ln 4)-(-\ln 16)\right]$$
$$=\ln 16-\ln 4$$
$$=\ln 4$$
$$=2\ln 2$$

Exercise 8.5

1 (i) $-x\cos x+\sin x+c$

(ii) $4xe^x-4e^x+c$

(iii) $\frac{x^3}{3}\ln x-\frac{x^3}{9}+c$

2 (i) e^4

(ii) $\frac{\sqrt{3}\pi}{24}-\frac{1}{8}$

(iii) $9\ln 3-4$

3 (i) P is $\left(\frac{\pi}{6},0\right)$ Q is $\left(\frac{\pi}{2},0\right)$

(ii) $-\frac{\pi}{2}$

(iii) $\frac{\pi-2}{18}$

4 $\frac{4e^5+1}{25}$

5 (i) $(-\sin x)e^{\cos x}$ **(ii)** 1

6 $u=x^2+5x+7 \quad v'=\sin x$

$u'=2x+5 \quad v=-\cos x$

$$\int_0^\pi (x^2+5x+7)\sin x\,dx$$
$$=-(x^2+5x+7)\cos x-\int_0^\pi (2x+5)\times -\cos x\,dx$$
$$=-(x^2+5x+7)\cos x+\int_0^\pi (2x+5)\cos x\,dx$$

$u=2x+5 \quad v'=\cos x$

$u'=2 \quad v=\sin x$

$$=-(x^2+5x+7)\cos x+\left((2x+5)\sin x-\int_0^\pi 2\sin x\,dx\right)$$
$$=\left[-(x^2+5x+7)\cos x+(2x+5)\sin x+2\cos x\right]_0^\pi$$
$$=[-(\pi^2+5\pi+7)\cos\pi+(2\pi+5)\sin\pi+2\cos\pi$$
$$-(-(0^2+5\times 0+7)\cos 0+(2\times 0+5)\sin 0+2\cos 0)]$$
$$=\left[(\pi^2+5\pi+7)-2-(-7+2)\right]$$
$$=\pi^2+5\pi+10$$

7 (i) $\frac{1-2\ln x}{x^3}$

(ii) $u=\ln x \quad v'=\frac{1}{x^2}$

$u'=\frac{1}{x} \quad v=-\frac{1}{x}$

$$\int\frac{\ln x}{x^2}dx$$
$$=-\frac{1}{x}\ln x-\int\left(\frac{1}{x}\times-\frac{1}{x}\right)dx$$
$$=-\frac{1}{x}\ln x+\int\left(\frac{1}{x^2}\right)dx$$
$$=-\frac{1}{x}\ln x-\frac{1}{x}+c$$
$$=-\frac{1}{x}(1+\ln x)+c$$

8 $u = e^x \quad v' = \sin x$

$u' = e^x \quad v = -\cos x$

$$I = \int e^x \sin x \, dx$$
$$= -e^x \cos x - \int -e^x \cos x \, dx$$
$$= -e^x \cos x + \int e^x \cos x \, dx$$

$u = e^x \quad v' = \cos x$

$u' = e^x \quad v = \sin x$

$$= -e^x \cos x + (e^x \sin x - \int e^x \sin x \, dx)$$
$$= -e^x \cos x + e^x \sin x - I + c$$
$$2I = -e^x \cos x + e^x \sin x + c$$
$$2I = e^x(\sin x - \cos x) + c$$
$$I = \frac{e^x(\sin x - \cos x)}{2} + c$$

Stretch and challenge

1 (i) $\frac{5}{6} - \ln 2$

(ii) (a) $2x\ln(1+x) + \frac{x^2}{1+x}$

(b) $\frac{2}{3}\ln 2 - \frac{5}{18}$

2 (i) $u = x^n, \quad v' = \cos x$

$u' = nx^{n-1}, \quad v = \sin x$

$$\int_0^{\frac{1}{2}\pi} x^n \cos x \, dx$$
$$= x^n \sin x - \int nx^{n-1} \sin x$$
$$= x^n \sin x - n\left[(x^{n-1} \times -\cos x) - \int (n-1)x^{n-2} \times -\cos x\right]$$
$$= \left[x^n \sin x + nx^{n-1}\cos x - n(n-1)\int x^{n-2}\cos x \, dx\right]_0^{\frac{1}{2}\pi}$$
$$= \left(\tfrac{1}{2}\pi\right)^n + 0 - n(n-1)\int_0^{\frac{1}{2}\pi} x^{n-2}\cos x \, dx$$
$$= \left(\tfrac{1}{2}\pi\right)^n - n(n-1)I_{n-2}$$

(ii) $\frac{1}{16}\pi^4 - 3\pi^2 + 24$

3 (i) Consider $I = \int_0^a f(a-x)\,dx$.

Let $u = a - x$ then $\frac{du}{dx} = -1$,

and when $x = 0$, $u = a$; when $x = a$, $u = 0$, so

$$I = \int_a^0 -f(u)\,du$$
$$I = \int_0^a f(u)\,du = \int_0^a f(x)\,dx$$

OR

Let $F(x)$ be an antiderivative of f

$$\int_0^a f(x)\,dx = [F(x)]_0^a = F(a) - F(0)$$
$$\int_0^a f(a-x)\,dx = [-F(a-x)]_0^a = -F(0) + F(a)$$

and hence result.

(ii) $\frac{\pi}{4}$

4 (i) $\frac{dy}{dx} = \frac{2x(1+x^2) - 2x^3}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$

When $\frac{dy}{dx} = \frac{1}{2}$

$$\frac{2x}{(1+x^2)^2} = \frac{1}{2}$$

$x^4 + 2x^2 - 4x + 1 = 0$

but $x = 1$ is a solution, so by division or otherwise

$(x-1)(x^3 + x^2 + 3x - 1) = 0$

and any other solutions are from $x^3 + x^2 + 3x - 1 = 0$

Let $g(x) = x^3 + x^2 + 3x - 1$, then

$$g\left(\frac{1}{4}\right) = \frac{1 + 4 + 48 - 64}{64} = -\frac{11}{64} < 0$$
$$g\left(\frac{1}{2}\right) = \frac{1 + 2 + 12 - 8}{8} = \frac{7}{8} > 0$$

Hence there is a root in the interval $\left[\frac{1}{4}, \frac{1}{2}\right]$

$g(x) = 0$ for some $\frac{1}{4} < x < \frac{1}{2}$.

(ii) $\pi\left(\ln(2) - \frac{1}{2}\right)$

Exam focus

1 $\frac{4x^2 + 4x - 17}{2x^2 + 5x - 3} = 2 - \frac{6x + 11}{2x^2 + 5x - 3}$

$$= 2 - \frac{6x + 11}{(2x-1)(x+3)}$$
$$= 2 - \left(\frac{4}{2x-1} + \frac{1}{x+3}\right)$$
$$= 2 - \frac{4}{2x-1} - \frac{1}{x+3}$$

$$\int_1^2 \frac{4x^2 + 4x - 17}{2x^2 + 5x - 3}\,dx$$
$$= \int_1^2 \left(2 - \frac{4}{2x-1} - \frac{1}{x+3}\right)dx$$
$$= \left[2x - 2\ln(2x-1) - \ln(x+3)\right]_1^2$$
$$= \left[2x - (\ln(2x-1)^2 + \ln(x+3))\right]_1^2$$
$$= \left[2x - (\ln(2x-1)^2(x+3))\right]_1^2$$
$$= \left[4 - \ln(9 \times 5) - (2 - \ln(1 \times 4))\right]$$
$$= \left[4 - \ln 45 - 2 + \ln 4\right]$$
$$= 2 + \ln\tfrac{4}{45}$$
$$= 2 - \ln\tfrac{45}{4}$$

2 (i) $\dfrac{d}{dx}(\sec x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right)$

$= \dfrac{0 \times \cos x - (-\sin x) \times 1}{\cos^2 x}$

$= \dfrac{\sin x}{\cos^2 x}$

$= \dfrac{1}{\cos x} \times \dfrac{\sin x}{\cos x}$

$= \sec x \tan x$

(ii) $\frac{1}{3}\sec x^3 + c$

(iii) $\int_0^{\frac{\pi}{3}} (\sec x + \tan x)^2 \, dx$

$= \int_0^{\frac{\pi}{3}} (\sec^2 x + 2\sec x \tan x + \tan^2 x) dx$

$= \int_0^{\frac{\pi}{3}} (\sec^2 x + 2\sec x \tan x + \sec^2 x - 1) dx$

$= \int_0^{\frac{\pi}{3}} (2\sec^2 x + 2\sec x \tan x - 1) dx$

$= [2\tan x + 2\sec x - x]_0^{\frac{\pi}{3}}$

$= \left[\left(2\tan\frac{\pi}{3} + 2\sec\frac{\pi}{3} - \frac{\pi}{3}\right) - (2\tan 0 + 2\sec 0 - 0)\right]$

$= \left(2\sqrt{3} + 2 \times 2 - \frac{\pi}{3}\right) - (2)$

$= 2\sqrt{3} + 2 - \frac{\pi}{3}$

$= 2(\sqrt{3} + 1) - \frac{\pi}{3}$

3 (i) $x = 0.685$ (3sf)

(ii) $\frac{8}{15}$

4 (i) $A = 2, B = -2$

(ii) $\frac{1}{2}\pi - \ln 2$

5 $\dfrac{1}{\sqrt{3}}$

6 (i) $y = \dfrac{2\ln 2 + 1}{2\sqrt{\ln 2}} x - \dfrac{1}{\sqrt{\ln 2}}$

(ii) $\pi\left(\dfrac{2e^3 + 1}{9}\right)$

Chapter 9 Differential equations

Exercise 9.1

1 (i) $\frac{dv}{dt}=k$

(ii) $\frac{dB}{dt}=kB$

(iii) $\frac{dh}{dt}=k\sqrt[3]{t}$

(iv) $\frac{dV}{dt}=k\sqrt{V}$

(v) $\frac{dr}{dt}=\frac{k}{\sqrt{r}}$

(vi) $\frac{dP}{dt}=k(1-P)$

(vii) $\frac{dV}{dt}=kS$

(viii) $\frac{dA}{dt}=kr^2$

Exercise 9.2

1 (i) $y=x-x^2+c$

(ii) $y=\pm\sqrt{4x^2+c}$

(iii) $x=Ae^{\frac{1}{2}e^{2t}}$

(iv) $A=Ke^{0.01t}$

(v) $y=Ae^{x+\cos x}$

(vi) $y=\tan^{-1}(-e^{-t}+c)$

2 (i) $y=-\ln(-e^x+e^{-2}+1)$

(ii) $y=2(1+x^2)$

3 (i) $\tan y=\frac{1}{4}\sin 4x+x+c$

(ii) $y=\tan^{-1}\left(\frac{1}{4}\sin 4x+x+1\right)$; $\tan^{-1}\left(\frac{\sqrt{3}}{8}+\frac{2\pi}{3}+1\right)$

4 (i) $t=-20\ln\left|\frac{6-h}{5}\right|$

(ii) 4.46 years

(iii) 2.97 m

(iv) 6 m

5 109°C

6 (i) $y=\frac{x}{1-cx}$

(ii) $\frac{8}{5}$

7 (i) $(x-4)+\frac{2}{x-1}$

(ii) **(a)** $y=A(x-1)^2e^{\frac{x^2}{2}-4x}+5$

(b) 5

8 (i) 2.625 mins

(ii) 4.5 mins

9 (i) $\frac{dx}{dt}=k\sqrt{x}$

(ii) 3150 (3*sf*)

10 (i) $\frac{2}{2x+1}-\frac{1}{x+1}$

(ii) $\frac{dy}{dx}=\frac{y}{(2x+1)(x+1)}$

$$\int\frac{1}{y}\,dy=\int\frac{1}{(2x+1)(x+1)}\,dx$$

$$\int\frac{1}{y}\,dy=\int\left(\frac{2}{2x+1}-\frac{1}{x+1}\right)dx$$

$$\ln|y|=\ln|2x+1|-\ln|x+1|+c$$

$$\ln|y|=\ln\frac{|2x+1|}{|x+1|}+c$$

$$x=0,y=2\Rightarrow\ln 2=\ln 1+c\Rightarrow c=\ln 2$$

$$\ln|y|=\ln\left|\frac{2x+1}{x+1}\right|+\ln 2$$

$$e^{\ln|y|}=e^{\ln\left|\frac{2x+1}{x+1}\right|+\ln 2}$$

$$y=e^{\ln\left|\frac{2x+1}{x+1}\right|}\times e^{\ln 2}$$

$$y=2\left(\frac{2x+1}{x+1}\right)=\frac{4x+2}{x+1}$$

11 (i) $v=20-20e^{-\frac{1}{2}t}$

(ii) 20 m/s

(iii) $\frac{1}{9(w-4)}-\frac{1}{9(w+5)}$

(iv) $\frac{dw}{dt}=-\frac{1}{2}(w-4)(w+5)$

$$\Rightarrow\int\frac{dw}{(w-4)(w+5)}=\int-\frac{1}{2}dt$$

$$\Rightarrow\int\left[\frac{1}{9(w-4)}-\frac{1}{9(w+5)}\right]dw=\int-\frac{1}{2}dt$$

$$\Rightarrow\tfrac{1}{9}\ln(w-4)-\tfrac{1}{9}\ln(w+5)=-\tfrac{1}{2}t+c$$

$$\Rightarrow\tfrac{1}{9}\ln\left|\frac{w-4}{w+5}\right|=-\tfrac{1}{2}t+c$$

When $t=0, w=10\Rightarrow c=\frac{1}{9}\ln\frac{6}{15}=\frac{1}{9}\ln\frac{2}{5}$

$$\Rightarrow\ln\frac{w-4}{w+5}=-\frac{9}{2}t+\ln\frac{2}{5}$$

$$\Rightarrow\frac{w-4}{w+5}=e^{-\frac{9}{2}t+\ln\frac{2}{5}}=\frac{2}{5}e^{-\frac{9}{2}t}=0.4e^{-4.5t}$$

(v) 4 m/s

12 42 minutes ago = 3:18 pm

13 30 days

14 5.45 g

Stretch and challenge

1 (i) $x = a(1+kt)^{-1}$

$$\frac{dx}{dt} = -ka(1+kt)^{-2}$$
$$= -ka\left(\frac{x}{a}\right)^2$$
$$= -\frac{kx^2}{a}$$

(ii) $a = 2.5,\ k = 0.5625$

(iii) 0

(iv) $\frac{1}{2y} + \frac{1}{2(2-y)}$

(v) $\int \frac{1}{2y-y^2}dy = \int dt$

$$\Rightarrow \int\left[\frac{1}{2y} + \frac{1}{2(2-y)}dy\right] = \int dt$$
$$\Rightarrow \tfrac{1}{2}\ln y - \tfrac{1}{2}\ln(2-y) = t + c$$
When $t = 0, y = 1 \Rightarrow 0 - 0 = 0 + c \Rightarrow c = 0$
$$\Rightarrow \ln y - \ln(2-y) = 2t$$
$$\Rightarrow \ln\frac{y}{2-y} = 2t$$
$$\frac{y}{2-y} = e^{2t}$$
$$\Rightarrow y = 2e^{2t} - ye^{2t}$$
$$\Rightarrow y + ye^{2t} = 2e^{2t}$$
$$\Rightarrow y(1-e^{2t}) = 2e^{2t}$$
$$\Rightarrow y = \frac{2e^{2t}}{1+e^{2t}} = \frac{2}{1+e^{-2t}}$$

(vi) 2000

2 (i) 100

(ii) 140

(iii) $P = \frac{700}{5+2e^{-\frac{t}{2}}} \Rightarrow 5 + 2e^{-\frac{t}{2}} = \frac{700}{P}$

$$e^{-\frac{t}{2}} = \frac{350}{P} - \frac{5}{2}$$

Differentiating wrt t,

$$-\frac{1}{2}e^{-\frac{t}{2}} = -\frac{350}{P^2}\frac{dP}{dt}$$
$$\frac{1}{2}\left(\frac{350}{P} - \frac{5}{2}\right) = \frac{350}{P^2}\frac{dP}{dt}$$
$$\frac{P^2}{700}\left(\frac{350}{P} - \frac{5}{2}\right) = \frac{dP}{dt}$$
$$\frac{dP}{dt} = \frac{P}{2} - \frac{P^2}{280}$$
$$\frac{dP}{dt} = \frac{P}{2}\left(1 - \frac{P}{140}\right)$$

(iv) 14.3

3 $y = \frac{1}{\sqrt[m]{m(k-x)}}$

Using the chain rule,

$$\frac{d}{dx}(y^n) = \frac{d}{dy}(y^n)\frac{dy}{dx} = ny^{n-1}y^{m+1} = ny^{n+m},$$ as required.

4 (i) $y = x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha$ or

$$y = x\tan\alpha - \frac{gx^2}{2V^2}(1+\tan^2\alpha)$$

(ii) (a) If the centre passes through the point (kh, h), then from the above

$$h = kh\tan\alpha - \frac{g(kh)^2}{2V^2}(1+\tan^2\alpha)$$

which is a quadratic in $\tan\alpha$, so rearranging

$$g(kh)^2\tan^2\alpha - 2V^2kh\tan\alpha + 2V^2h + g(kh)^2 = 0 \text{ and, } h \neq 0$$

$$ghk^2\tan^2\alpha - 2V^2k\tan\alpha + 2V^2 + ghk^2 = 0$$

This has two distinct real solutions if and only if $b^2 - 4ac > 0$

$$(2V^2k)^2 - 4ghk^2(2V^2 + ghk^2) > 0$$
$$4V^4k^2 - 4ghk^2(2V^2 + ghk^2) > 0$$
$$V^4 - 2ghV^2 - g^2h^2k^2 > 0 \quad (k^2 > 0)$$

The LHS of which is a quadratic in V^2, and has the form $(V^2 - a)(V^2 - b)$ where a, b arise from

$$V^2 = \frac{2gh \pm \sqrt{(2gh)^2 + 4g^2h^2k^2}}{2}$$
$$V^2 = \frac{gh \pm gh\sqrt{1+k^2}}{1}$$
$$V^2 = gh\left(1 \pm \sqrt{1+k^2}\right) \text{ but } 1 - \sqrt{1+k^2} < 0$$

so $4V^2 - 8ghV^2 - 4g^2h^2k^2 > 0$ when

$$V^2 > gh\left(1 + \sqrt{1+k^2}\right)$$

(b) Use

$$g(kh)^2\tan^2\alpha - 2V^2kh\tan\alpha + 2V^2h + g(kh)^2 = 0$$

Let $\tan\alpha_1, \tan\alpha_2$ be the two roots, then by the sum and the product of the roots:

$$\tan\alpha_1 + \tan\alpha_2 = \frac{2V^2kh}{gk^2h^2} = \frac{2V^2}{gkh}$$

and

$$\tan\alpha_1 \times \tan\alpha_2 = \frac{2V^2h + gk^2h^2}{gk^2h^2} = \frac{2V^2 + gk^2h}{gk^2h}$$

and, since

$$\tan(\alpha_1 + \alpha_2) = \frac{\tan\alpha_1 + \tan\alpha_2}{1 - \tan\alpha_1 \times \tan\alpha_2}$$

$$\tan(\alpha_1 + \alpha_2) = \frac{\frac{2V^2}{gkh}}{1 - \frac{2V^2 + gk^2h}{gk^2h}} = \frac{2kV^2}{gk^2h - 2V^2 - gk^2h} = -k$$

and $\alpha_1 + \alpha_2 = \tan^{-1}(-k)$.

5 117

6 17.3 litres

Exam focus

1 (i) $y = \operatorname{cosec} x = \dfrac{1}{\sin x}$

$$\frac{dy}{dx} = \frac{-\cos x}{\sin^2 x} = -\frac{1}{\sin x} \times \frac{\cos x}{\sin x} = -\operatorname{cosec} x \cot x$$

(ii) $x = \sin^{-1}\left(\dfrac{1}{\ln|\sin t| + 2}\right)$

2 (i) $\dfrac{1}{3(3-x)} - \dfrac{1}{3(6-x)}$

(ii) (a) $\dfrac{dx}{dt} = k(3-x)(6-x)$

$$\int \frac{1}{(3-x)(6-x)}\,dx = \int k\,dt$$

$$\int\left(\frac{1}{3(3-x)} - \frac{1}{3(6-x)}\right)dx = \int k\,dt$$

$$-\tfrac{1}{3}\ln(3-x) + \tfrac{1}{3}\ln(6-x) = kt + c$$

$$\tfrac{1}{3}[\ln(6-x) - \ln(3-x)] = kt + c$$

$$\frac{1}{3}\left[\ln\frac{(6-x)}{(3-x)}\right] = kt + c$$

$$t = 0, x = 0 \Rightarrow \tfrac{1}{3}\left[\ln\tfrac{6}{3}\right] = k \times 0 + c$$

$$\Rightarrow c = \tfrac{1}{3}\ln 2$$

$$t = 1, x = 1 \Rightarrow \tfrac{1}{3}\left[\ln\tfrac{5}{2}\right] = k + \tfrac{1}{3}\ln 2$$

$$\Rightarrow k = \tfrac{1}{3}\left[\ln\tfrac{5}{2}\right] - \tfrac{1}{3}\ln 2$$

$$\Rightarrow k = \tfrac{1}{3}\left[\ln\tfrac{5}{2} - \ln 2\right]$$

$$\Rightarrow k = \tfrac{1}{3}\ln\tfrac{5}{4}$$

(b) 1.59

3 $y = \frac{1}{2}\ln(\ln(\cos^2 x) + 1)$

4 (i) $\dfrac{dA}{dt} = k\sqrt{3A - 2}$

(ii) 34 m^2

5 $y^3 = \dfrac{x^3}{8} + 1$

Chapter 10 Vectors

Exercise 10.1

1 (i) $r=\begin{pmatrix}3\\-1\end{pmatrix}+t\begin{pmatrix}-2\\5\end{pmatrix}$

(ii) $r=\begin{pmatrix}-2\\1\end{pmatrix}+t\begin{pmatrix}2\\7\end{pmatrix}$

(iii) $r=\begin{pmatrix}4\\-3\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\4\end{pmatrix}$

(iv) $r=\begin{pmatrix}2\\0\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\-9\\4\end{pmatrix}$

(v) $r=\begin{pmatrix}2\\3\\0\end{pmatrix}+\lambda\begin{pmatrix}-5\\-7\\5\end{pmatrix}$

(vi) $r=\begin{pmatrix}2\\3\\-2\end{pmatrix}+\lambda\begin{pmatrix}8\\-2\\1\end{pmatrix}$

(vii) $r=\begin{pmatrix}1\\9\\-5\end{pmatrix}+\lambda\begin{pmatrix}5\\7\\-5\end{pmatrix}$

2 (i) The point lies on the line.

(ii) The point does NOT lie on the line.

3 $a = 5, b = 2$

Exercise 10.2

1 (i) (8, –5, 7)

(ii) The lines do not intersect.
Since the direction vectors are not multiples, the vectors are skew.

(iii) (–7, –9, –1)

(iv) There is no point of intersection.
The direction vectors are multiples of each other so the lines are parallel.

(v) (–8, 4, 7)

2 (i) $(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}).(2\mathbf{i} - \mathbf{j} - 5\mathbf{k})$

$= 3 \times 2 + -4 \times -1 + 2 \times -5$

$= 0$

(ii) Equating **i**, **j**, and **k** components

$5 + 3s = 2 + 2t$ ①

$-2 - 4s = -2 - t$ ②

$-2 + 2s = 7 - 5t$ ③

Solving ① and ② simultaneously gives

$s=\frac{3}{5},\ t=\frac{12}{5}$

Substituting into ③,

$-2+2\times\frac{3}{5}=7-5\times\frac{12}{5}$

$-\frac{4}{5}\neq-5$

Hence there is no point of intersection.
The lines are not parallel so the lines are skew.

3 (–1, 6, 5)

4 Line through (9, 7, 5) and (7, 8, 2):

$\text{Direction}=\begin{pmatrix}7\\8\\2\end{pmatrix}-\begin{pmatrix}9\\7\\5\end{pmatrix}=\begin{pmatrix}-2\\1\\-3\end{pmatrix}$

$\mathbf{r}=\begin{pmatrix}7\\8\\2\end{pmatrix}+s\begin{pmatrix}-2\\1\\-3\end{pmatrix}$

Equating **i**, **j**, and **k** components

$2 + t = 7 - 2s$ ①

$-3 + 4t = 8 + s$ ②

$5 - 2t = 2 - 3s$ ③

Solving ① and ② simultaneously gives

$t = 3, s = 1$

Substituting into ③,

$5 - 2 \times 3 = 2 - 3 \times 1$

$-1 = -1$

So the lines intersect.

$r=\begin{pmatrix}7\\8\\2\end{pmatrix}+1\begin{pmatrix}-2\\1\\-3\end{pmatrix}=\begin{pmatrix}5\\9\\-1\end{pmatrix}$

Point of intersection is (5, 9, –1)

6 (i) $\mathbf{r}=\begin{pmatrix}5\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$

(ii) P is (1, 3, 1)

7 (i) $k = 8$

(ii) (7, 6, 5)

8 (i) $a = 1$

(ii) $a = -2$

9 (i) Equating **i**, **j**, and **k** components

$-3 + \lambda = 7 + a\mu$ ①

$2 + 2\lambda = 3 + b\mu$ ②

$3 + \lambda = 3 - 2\mu$ ③

① − ③ gives

$$-6 = 4 + (a+2)\mu$$
$$-10 = (a+2)\mu$$
$$\mu = -\frac{10}{a+2}$$

② − 2 × ③ gives

$$-4 = -3 + (b+4)\mu$$
$$-1 = (b+4)\mu$$
$$\mu = -\frac{1}{b+4}$$

So $-\frac{10}{a+2} = -\frac{1}{b+4}$

$$10(b+4) = a+2$$
$$10b + 40 = a + 2$$
$$a - 10b = 38$$

(ii) $a = 8, b = -3$

(iii) $\mathbf{r} = -\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}$

Exercise 10.3

1 33.2°

2 (i) $b = 4, c = -12$

(ii) 36°

3 42.5°

4 $a = -4$ or 6

5 54.7°

Exercise 10.4

1 (i) $(4, -5, -1)$ Shortest distance $= \sqrt{24}$

(ii) $(0, 0, 5)$ Shortest distance $= \sqrt{41}$

(iii) $(2.5, -0.5, 2)$ Shortest distance $= \sqrt{\frac{37}{2}}$

2 (i) 35.3°

(ii) Q is $\frac{1}{3}\begin{pmatrix} 7 \\ 5 \\ -1 \end{pmatrix}$

(iii) $\frac{5\sqrt{3}}{3}$

3 (i) $\mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ -9 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$

(ii) OT is $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0 \Rightarrow$ Lines are perpendicular

(iii) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix}$

(iv) $\sqrt{54}$

4 (i) 0.611 km

(ii) 32 seconds

5 (i) 10 seconds

(ii) The eagle catches the rabbit.

Exercise 10.5

1 (i) $2x + 4y - z = 4$

(ii) $3x - 2y - z = 1$

(iii) $2x + y - 4z = 11$

2 (i) $\mathbf{r}.(5\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 10$

(ii) $\mathbf{r}.(3\mathbf{i} - \mathbf{j} - 7\mathbf{k}) = 15$

3 (i) $\sqrt{140} = 11.83$

(ii) $\sqrt{9.5} = 3.08$

4 (i) $2x + y - 3z = 5;\ \mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = 5$

(ii) 0.53 km

(iii) $3x - y - z = -8;\ \mathbf{r} \cdot \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = -8$

(iv) $4x + y - 4z = 10;\ \mathbf{r} \cdot \begin{pmatrix} 4 \\ 1 \\ -4 \end{pmatrix} = 10$

(v) $4x - 5y + 3z = 17;\ \mathbf{r} \cdot \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix} = 17$

(vi) $-27x + 68y + 35z = 115;\ \mathbf{r} \cdot \begin{pmatrix} -27 \\ 68 \\ 35 \end{pmatrix} = 115$

(vii) $2x - 11y - 5z = 3$; $\mathbf{r}\cdot\begin{pmatrix}2\\-11\\-5\end{pmatrix} = 3$

5 $\mathbf{r} = \begin{pmatrix}2+2\lambda\\-\lambda\\2+\lambda\end{pmatrix}$, (0, 1, 1)

6 (i) The line is parallel to the plane.

(ii) Point of intersection is = (–1, 0, 2)

(iii) The line lies in the plane.

(iv) Point of intersection is (2, 3, 0)

7 (i) 7.8°

(ii) 10.9°

(iii) 64.6°

8 (i) Point is (3, –1, 6); distance is 3.

(ii) Point is (4, 5, –3); distance is 6.

(iii) Point is $\left(-\frac{3}{2}, 0, \frac{1}{2}\right)$; distance is 1.87.

9 (i) $k = -24$

(ii) Point is (7, 3, –8); distance is 8.19.

(iii) $-62x + 33y + 5z = 27$

Exercise 10.6

1 (i) $\mathbf{r} = \begin{pmatrix}3\\2\\-2\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\2\end{pmatrix}$; Acute angle = 78.5°

(ii) $\mathbf{r} = \begin{pmatrix}0\\2\\-5\end{pmatrix} + \lambda\begin{pmatrix}13\\11\\7\end{pmatrix}$; Acute angle = 64.0°

(iii) $\mathbf{r} = \begin{pmatrix}3\\3\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\2\\1\end{pmatrix}$; Acute angle = 65.9°

2 (i) $n_p = \begin{pmatrix}1\\3\\-2\end{pmatrix}$ $n_\pi = \begin{pmatrix}-1\\-3\\2\end{pmatrix}$

Since $n_p = n_\pi$ the normals to the planes are parallel, hence the planes are parallel.

(ii) $p: 5 + 3 \times 2 - 2 \times 3 = 5$

$\pi: -4 - 3 \times -3 + 2 \times 2 = 1$

(iii) $\overrightarrow{AB} = \begin{pmatrix}-1\\-5\\-5\end{pmatrix}$; $|\overrightarrow{AB}| = \sqrt{51}$

(iv) 103° or 77°

(v) 1.60

(vi) $\left(\frac{32}{7}, \frac{5}{7}, \frac{27}{7}\right)$

(vii) 1.60

3 (i) $k = 7$

(ii) 57.1°

(iii) 1.56

(iv) (2, 0, –6)

(v) $x + 2y - z = 8$

4 $\mathbf{r} = \begin{pmatrix}0\\8\\6\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\2\end{pmatrix}$

5 4; $2x - y + 2z + 10 = 0$

6 $a = 4$, $b = -3$, $c = -15$, $d = 5$, $k = 5$

7 (i) 25

(ii) BD: $\mathbf{r} = \begin{pmatrix}-1\\-7\\11\end{pmatrix} + \lambda\begin{pmatrix}15\\-20\\0\end{pmatrix}$; D is (8, –19, 11)

(iii) At A: $-3(0) + 4(0) + 5(6) = 30$

At B: $-3(-1) + 4(-7) + 5(11) = 30$

At C: $-3(-8) + 4(-6) + 5(6) = 30$

Normal vector is $\begin{pmatrix}-3\\4\\5\end{pmatrix}$

(iv) $\begin{pmatrix}4\\3\\5\end{pmatrix}\cdot\overrightarrow{AE} = \begin{pmatrix}4\\3\\5\end{pmatrix}\cdot\begin{pmatrix}15\\-20\\0\end{pmatrix} = 0$

$\begin{pmatrix}4\\3\\5\end{pmatrix}\cdot\overrightarrow{AB} = \begin{pmatrix}4\\3\\5\end{pmatrix}\cdot\begin{pmatrix}-1\\-7\\5\end{pmatrix} = 0$

$\therefore \begin{pmatrix}4\\3\\5\end{pmatrix}$ is normal to the plane

(v) 60°

8 (i) $\mathbf{r} = \begin{pmatrix}2\\0\\-1\end{pmatrix} + \lambda\begin{pmatrix}-2\\-2\\1\end{pmatrix}$

(ii) 15

(iii) $k = 50$; D is (16, 8, 4)

Stretch and challenge

1 (i) (–1, 0, 1) and (–1, –1, 2)

(ii) 1.41 km

2 (i) P(0, 10, 30); Q(0, 20, 15); R(–15, 20, 30)

$$\overrightarrow{PQ}=\begin{pmatrix}0\\20\\15\end{pmatrix}-\begin{pmatrix}0\\10\\30\end{pmatrix}=\begin{pmatrix}0\\10\\-15\end{pmatrix}$$

$$\overrightarrow{PR}=\begin{pmatrix}-15\\20\\30\end{pmatrix}-\begin{pmatrix}0\\10\\30\end{pmatrix}=\begin{pmatrix}-15\\10\\0\end{pmatrix}$$

(ii) $$\begin{pmatrix}0\\10\\-15\end{pmatrix}\times\begin{pmatrix}-15\\10\\0\end{pmatrix}=\begin{pmatrix}150\\225\\150\end{pmatrix}=75\begin{pmatrix}2\\3\\2\end{pmatrix}$$

$$\therefore 2x+3y+2z=k$$

$$20\times0+3\times10+2\times30=90=k$$

$$\therefore 2x+3y+2z=90$$

(iii) $$\overrightarrow{OS}=\begin{pmatrix}-7.5\\20\\22.5\end{pmatrix}$$

$$\overrightarrow{PS}=\begin{pmatrix}-7.5\\20\\22.5\end{pmatrix}-\begin{pmatrix}0\\10\\30\end{pmatrix}=\begin{pmatrix}-7.5\\10\\-7.5\end{pmatrix}$$

$$\overrightarrow{OT}=\overrightarrow{OP}+\overrightarrow{PT}$$

$$=\overrightarrow{OP}+\frac{2}{3}\overrightarrow{PS}$$

$$=\begin{pmatrix}0\\10\\30\end{pmatrix}+\frac{2}{3}\begin{pmatrix}-7.5\\10\\-7.5\end{pmatrix}$$

$$=\begin{pmatrix}-5\\16\frac{2}{3}\\25\end{pmatrix}$$

(iv) $$\mathbf{r}=\begin{pmatrix}-5\\16\frac{2}{3}\\25\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\2\end{pmatrix}$$

At $C(-30,0,0)$

$-5+2\lambda=-30\Rightarrow\lambda=\frac{-25}{2}$ inconsistent

$16\frac{2}{3}+3\lambda=0\Rightarrow\lambda=\frac{-50}{9}$

So the line does not pass through C.

3 (i) $$\overrightarrow{AB}=\begin{pmatrix}0\\0\\2\end{pmatrix}-\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}-1\\-2\\0\end{pmatrix}$$

$$\overrightarrow{AB}\text{ is }\mathbf{r}=\begin{pmatrix}0\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}$$

(ii) $\mathbf{n}=\begin{pmatrix}1\\0\\1\end{pmatrix}$; $\theta=71.6°$

(iii) $$\begin{pmatrix}-2\\-2\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}=-2+0+-1=-3$$

$$-3=\sqrt{9}\times\sqrt{2}\times\cos\alpha$$

$$\therefore\alpha=135°$$

$$\therefore\phi=180-135=45°$$

(iv) k = 1.34

(v) Point is (–2, –2, 1); Distance is 3 cm.

4 a is any real number.

Exam focus

1 (i) $$\overrightarrow{PQ}=\begin{pmatrix}1\\2\\4\end{pmatrix}-\begin{pmatrix}-5\\-1\\3\end{pmatrix}=\begin{pmatrix}6\\3\\1\end{pmatrix}$$

$$PQ\text{ is }\mathbf{r}=\begin{pmatrix}1\\2\\4\end{pmatrix}+s\begin{pmatrix}6\\3\\1\end{pmatrix}$$

$$l\text{ is }\mathbf{r}=\begin{pmatrix}1\\3\\5\end{pmatrix}+t\begin{pmatrix}1\\-2\\2\end{pmatrix}$$

To intersect,

$1+6s=1+t$ ①

$2+3s=3-2t$ ②

$4+s=5+2t$ ③

$6+4s=8$ ②+③

$s=\frac{1}{2}$

$t=\frac{-1}{4}$

Substitute into ①

$\Rightarrow 1+6\times\frac{1}{2}=1+\left(-\frac{1}{4}\right)$

$4\neq\frac{3}{4}$

∴ Lines don't intersect.

(ii) (–1, 7, 1)

2 (i) A is $\begin{pmatrix} 5 \\ -6 \\ -1 \end{pmatrix}$

(ii) 23.2°

(iii) $\mathbf{r} = \begin{pmatrix} 5 \\ -6 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix}$

3 (i) $14x + 24y - 3z = 71$

(ii) 9.59

4 (i) $k = -5$; point is (8, –1, 5)

(ii) $2x - 8y + 5z = 49$

Chapter 11 Complex numbers

Exercise 11.1

1 (i) $-i$

(ii) 4

(iii) 2

(iv) 5i

(v) -9

(vi) 0

2 (i) $9i + 2$

(ii) $8 - 4i - 6 - 3i = 2 - 7i$

(iii) $12i^2 = -12$

(iv) $-1 + 7$

(v) $17 + i$

(vi) $-5 - 12i$

(vii) 61

(viii) $-1 + 3i$

(ix) $2\sqrt{3} - 2 + (2\sqrt{3} - 2)i$

3 (i) $z = \pm 3i$

(ii) $z = 1 \pm i$

(iii) $z = \frac{3}{2} \pm \frac{1}{2}i$

(iv) $z = 2 - i$ or $-2 - i$

4 (i) $11 - 29i$

(ii) $1 + 41i$

(iii) 29

(iv) 0

(v) 58

(vi) $-5 + 8i$

(vii) 6

(viii) -42

(ix) $-1682i$

(x) $-17 - 144i$

5 (i) Let $z = a + bi, \quad a, b \in \mathbb{R}$

$$zz^* = (a + bi)(a - bi)$$
$$= a^2 - abi + abi - b^2i^2$$
$$= a^2 + b^2$$

(ii) $z + z^* = a + bi + a - bi$

$$= 2a$$

Exercise 11.2

1 (i) $-2i$

(ii) -225

(iii) $\frac{3}{2} + \frac{3}{2}i$

(iv) $\frac{2}{13} - \frac{3}{13}i$

(v) $-\frac{1}{5} + \frac{3}{5}i$

(vi) $\frac{29}{25} - \frac{3}{25}i$

(vii) $\frac{-27}{10} - \frac{31}{10}i$

(viii) $-\frac{6}{25} + \frac{17}{25}i$

(ix) $\frac{-6}{5} - \frac{23i}{5}$

2 (i) $(a, b) = (2, -3)$ or $(-2, 3)$

(ii) $(a, b) = (1, 5)$ or $(-1, -5)$

3 (i) $z = 3 + 2i$

(ii) $z = -3 + 5i$

(iii) $z = 5 + 6i$

(iv) $z = -4 - 4$

4 $z = \sqrt{12} - 2i$ *or* $-\sqrt{12} - 2i$

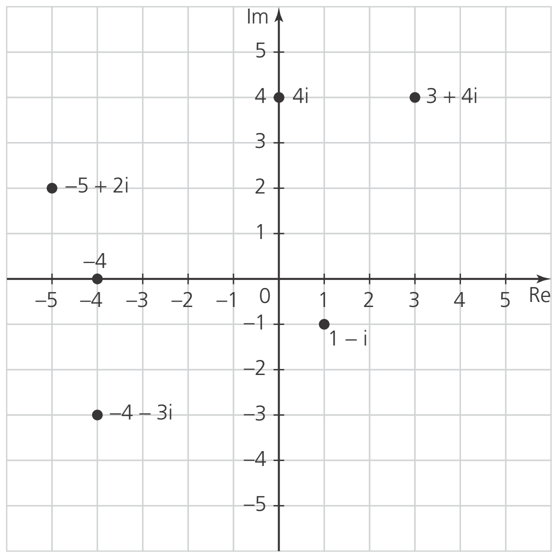
$z = 0 + 0i$ or $4i$

5 $a = 3, b = -1$

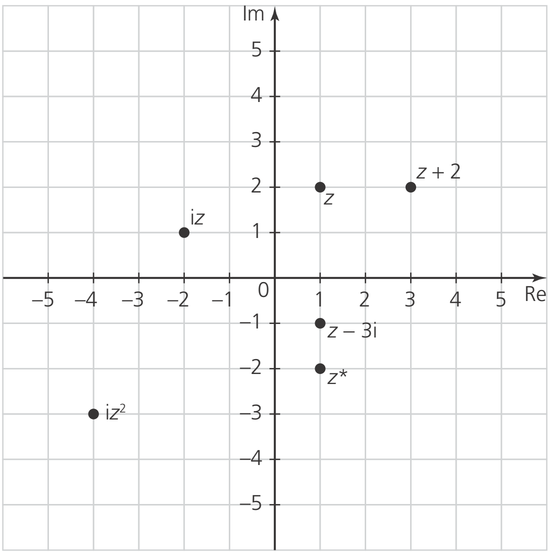
6 $u = \dfrac{-12}{a^2 + 9} + \dfrac{4a}{a^2 + 9}i$

Exercise 11.3

1



2



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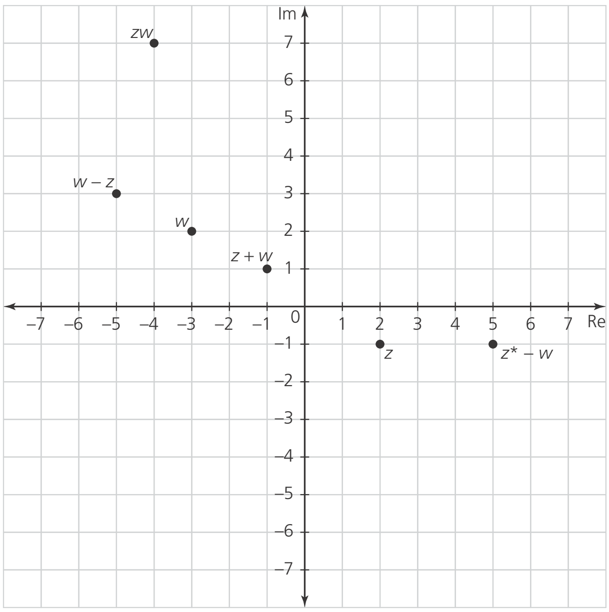


4 **(i)** 2

(ii) 13

(iii) 4

(iv) $\sqrt{7}$

5 **(i)** $\sqrt{2}$

(ii) 5

(iii) $\sqrt{13}$

(iv) $\sqrt{2}$

(v) 2

(vi) $\sqrt{45}$

(vii) $\dfrac{\sqrt{2}}{2}$

(viii) $\sqrt{50}$ or $5\sqrt{2}$

6 **(i)** 5

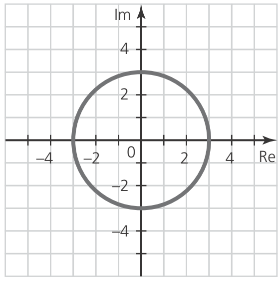
(ii) $\sqrt{x^2 - 2x + 2}$

(iii) $\sqrt{x^2 + y^2 + 2x + 2y + 2}$

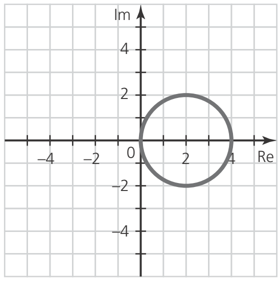
(iv) 1

Exercise 11.4

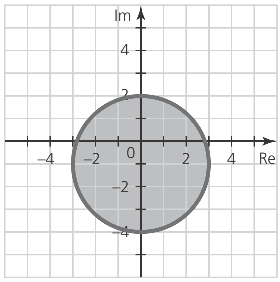
1 **(i)**



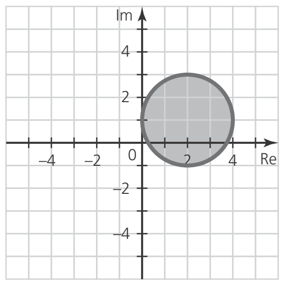
(ii)



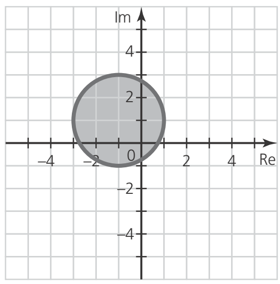
(iii)



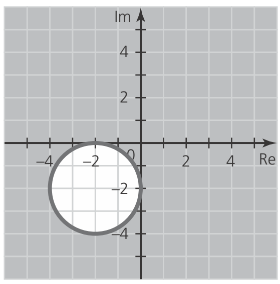
(iv)



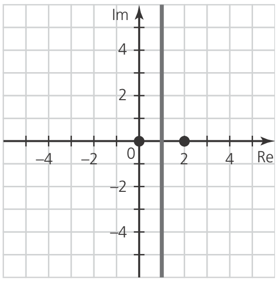
(v)



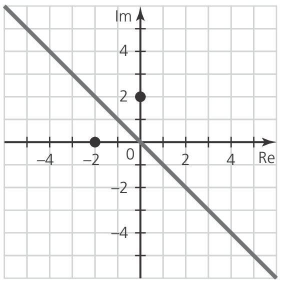
(vi)



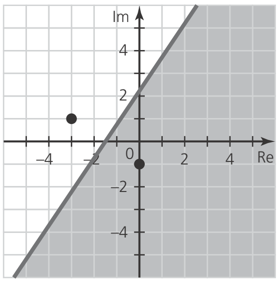
2 (i)



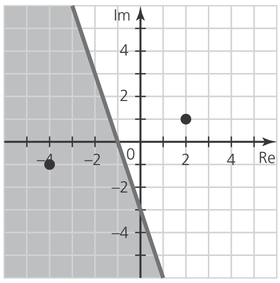
(ii)



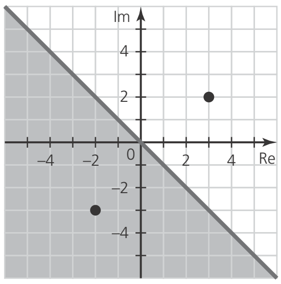
(iii)



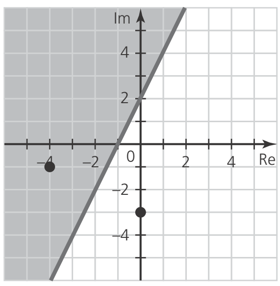
(iv)



(v)



(vi)

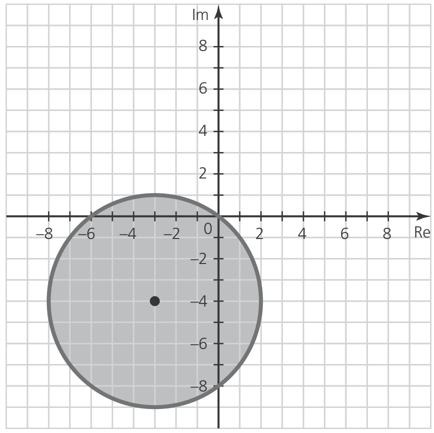


3 If the < or > inequality signs had been used, the boundary curves or lines would need to be dashed, representing the fact that the actual boundary curve or line is not included in the region.

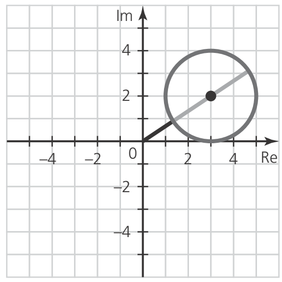
4 (i) $w^2 = -3 - 4\text{i}$

$$\left|w^2\right| = \sqrt{(-3)^2 + (-4)^2} = 5$$

(ii)



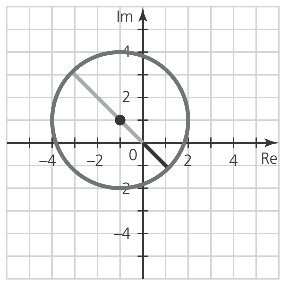
5 (i)



Minimum value of $|z|$ is $\sqrt{13} - 2$

Maximum value of $|z|$ is $\sqrt{13} + 2$

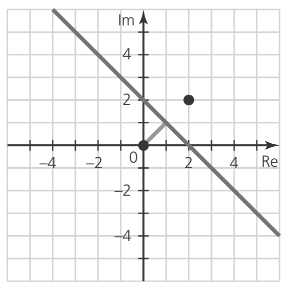
(ii)



Minimum value of $|z|$ is $3 - \sqrt{2}$

Maximum value of $|z|$ is $3 + \sqrt{2}$

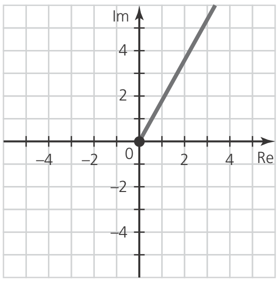
6



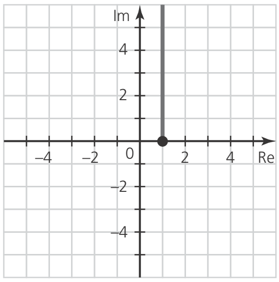
$|z| = |z - (2 + 2\text{i})|$

Minimum value of $|z|$ is $\sqrt{2}$

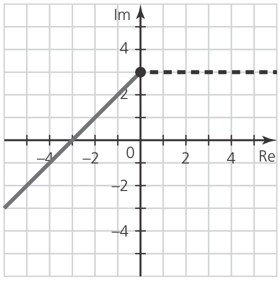
7 (i)



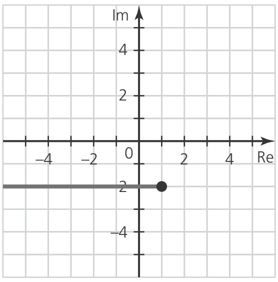
(ii)



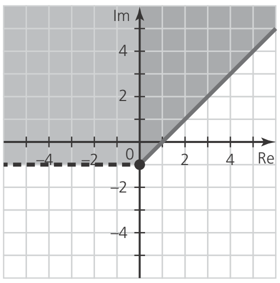
(iii)



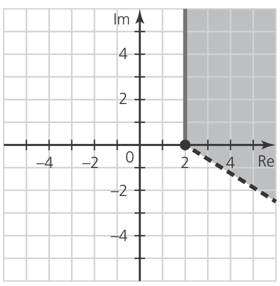
(iv)



(v)



(vi)



8 $\arg(z-2+2\text{i})=\dfrac{\pi}{4}$

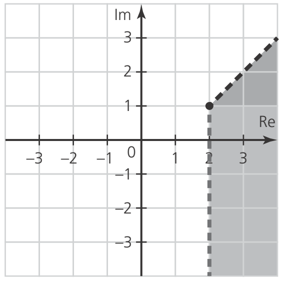
9 (i) $|z-4-2\text{i}|=2$

(ii) $\arg(z-4-2\text{i})=0$

(iii) $\text{P}=4-\sqrt{2}+(2+\sqrt{2})\text{i}$

(iv) $0<\arg(z-4-2\text{i})<\dfrac{3\pi}{4}$ and $|z-4-2\text{i}|<2$

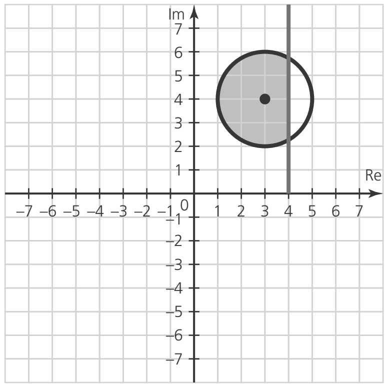
10 (i)



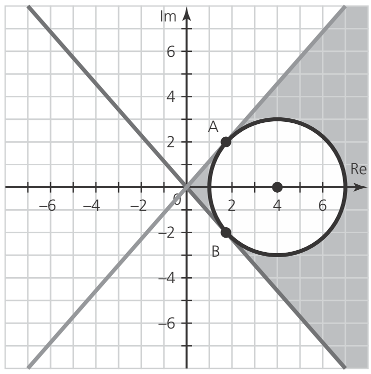
(ii) The equation of the half-line is $y = x - 1$.

When $x = 43$, $y = 42$ so the point 43 + 47i would be outside the region.

11



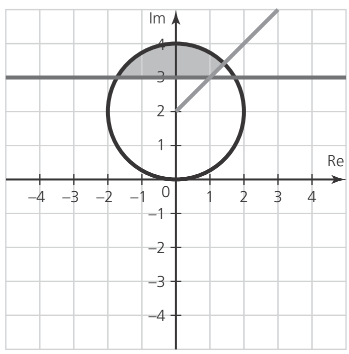
12 (i) (ii) (iii)



(iv) $\alpha=48.6^{\circ}$

$\beta=-48.6^{\circ}$

13 (i)

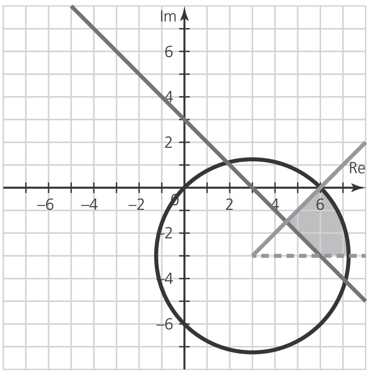


(ii) $\sqrt{2}$

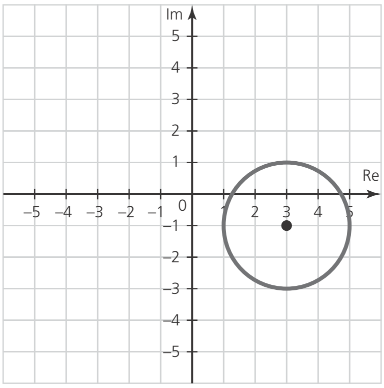
14 (i) $|a|=\sqrt{18}$ or $3\sqrt{2}$

$\arg\ a=-\dfrac{\pi}{4}$

(ii)



15 (i)



(ii) **(a)** Maximum value of $|z|$ is $\sqrt{10}+2$

Minimum value of $|z|$ is $\sqrt{10}-2$

(b) Maximum value of $|z-3|$ is 3

Minimum value of $|z-3|$ is 1

(c) $\sin\theta = \dfrac{2}{\sqrt{10}} \Rightarrow \theta = 39.23^\circ$

$\tan\alpha = \frac{1}{3} \Rightarrow \alpha = 18.43^\circ$

Maximum value of arg(z) is $39.23^\circ - 18.43^\circ = 20.8^\circ$

Minimum value of arg(z) is $-(39.23^\circ + 18.43^\circ) = -57.7^\circ$

Exercise 11.5

1 (i) $2\left(\cos-\dfrac{\pi}{2} + i\sin-\dfrac{\pi}{2}\right)$

(ii) $5(\cos-0.644 + i\sin-0.644)$

(iii) $13(\cos 1.97 + i\sin 1.97)$

(iv) $2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

(v) $8(\cos 0 + i\sin 0)$

(vi) $\sqrt{2}\left(\cos\dfrac{-3\pi}{4} + i\sin\dfrac{-3\pi}{4}\right)$

2 (i) $\sqrt{2} + \sqrt{2}i$

(ii) $\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$

(iii) $-6 + 6\sqrt{3}\,i$

(iv) $-0.5\,i$

(v) $2.30 + 5.54i$

(vi) $-1 + i$

3 (i) $u^* = 4\text{cis}(-30^\circ)$

(ii) $u \times v = 8 \text{ cis } 90^\circ$

(iii) $v \times w = 2\text{cis}(-30^\circ)$

(iv) $u \times w = 4\text{cis}(-60)$

(v) $u \times w^* = 4\text{cis}120^\circ$

(vi) $\dfrac{v}{u} = \dfrac{1}{2}\text{cis}30^\circ$

(vii) $\dfrac{u}{v^*} = 2\text{cis}90^\circ$

(viii) $u^2 = 16\text{cis}60^\circ; v^3 = 8(150 + i\sin 15); \dfrac{u^2}{v^3} = 2\text{cis}(-120^\circ)$

(ix) $w^3 = \text{cis}90^\circ$

(x) $iu = 4\text{cis}120^\circ$

(xi) $\dfrac{1}{v} = \dfrac{1}{2}\text{cis} - 60^\circ$

(xii) $\dfrac{iv}{u} = \dfrac{1}{2}\text{cis}120^\circ$

4 (i) T

(ii) F

(iii) T

(iv) T

(v) T

(vi) T

5 (i) $|\alpha| = 2$

$\arg\alpha = \dfrac{-5\pi}{6}$

(ii) $|\beta| = 4$

$\arg\beta = \frac{1}{2}\pi$

(iii) $|\alpha\beta| = 8$

$\arg|\alpha\beta| = -\frac{1}{3}\pi$

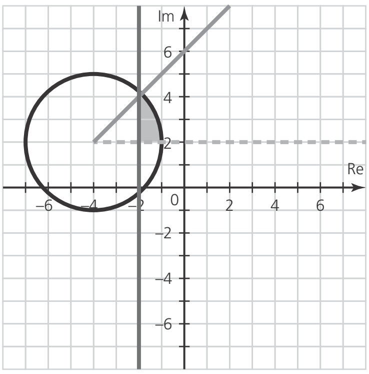
(iv) $\left|\dfrac{\alpha}{\beta}\right| = \dfrac{1}{2}$

$\arg\left(\dfrac{\alpha}{\beta}\right) = \dfrac{2\pi}{3}$

6 (i) $\frac{1}{m} = -\frac{1}{5} - \frac{1}{10}\mathrm{i}$

(ii) $m = \sqrt{20}\,\mathrm{cis}\,153.4^{\circ}$ or $\sqrt{20}\,\mathrm{cis}\,2.68$

(iii) (iv)



7 Let $z = x + \mathrm{i}y$

$$\frac{1}{z} = \frac{1}{x+\mathrm{i}y} \times \frac{x-\mathrm{i}y}{x-\mathrm{i}y}$$

$$= \frac{x-\mathrm{i}y}{x^2 - \mathrm{i}^2 y^2}$$

$$= \frac{x-\mathrm{i}y}{x^2 + y^2}$$

$$= \frac{z^*}{|z|^2}$$

8 (i) $z^2 = 13\mathrm{cis}(-1.176)$ *or* $13\mathrm{cis}(-67.4^{\circ})$

(ii) $z = (13\mathrm{cis} - 1.176)^{\frac{1}{2}}$

(iii) $z = \sqrt{13}$ cis (-0.588)

(iv) $z = \sqrt{13}$ cis (2.554)

(v) $z = -3 + 2\mathrm{i}$ *or* $3 - 2\mathrm{i}$

9 $5 - 2\mathrm{i}$ *or* $-5 + 2\mathrm{i}$

10 (i) $\frac{6}{4+a^2} - \frac{3a}{4+a^2}\mathrm{i}$

(ii) (a) $a = 2$

(b) $a = \pm\frac{1}{\sqrt{2}}$

11 (i) $2e^{\frac{\pi}{2}\mathrm{i}}$

(ii) $\sqrt{2}e^{\frac{3\pi}{4}\mathrm{i}}$

(iii) $\sqrt{13}e^{-0.983\mathrm{i}}$

(iv) $5e^{\frac{\pi}{2}\mathrm{i}}$

12 (i) $\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$

(ii) $1.08 - 1.68\mathrm{i}$

(iii) -4

(iv) $\frac{3\sqrt{3}}{2} - \frac{3}{2}\mathrm{i}$

13 (i) $12e^{\frac{\pi}{2}\mathrm{i}}$

(ii) (a) $\sqrt{3} - \mathrm{i}$

(b) $2\ \mathrm{cis}\left(\frac{-\pi}{6}\right)$

Exercise 11.6

1 $1 - 2\mathrm{i}$; $k = 15$, $k = 3$

2 $p = -78$

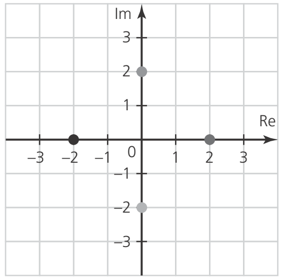
Other roots are $2 + 3\mathrm{i}$ and $2 - 3\mathrm{i}$.

3 (i) $z = -2\mathrm{i}$ or i

(ii) The coefficients of the polynomial are not real numbers.

4 (i) $z = 2, -2, 2\mathrm{i}, -2\mathrm{i}$

(ii)



The solutions are all 90° apart.

(iii) $k = -4$; other solutions are $-1 + \mathrm{i}, -1 - \mathrm{i}, 1 - \mathrm{i}$

5 (i) Roots $2 + \mathrm{i}$, $2 - \mathrm{i}$, $-2\mathrm{i}$, $2\mathrm{i}$

(ii) $A = -4,\ B = 9,\ C = -16,\ D = 20$

6 (i) $\alpha^2 = 2\mathrm{i}$ $\qquad$ $\alpha^3 = -2 + 2\mathrm{i}$

$$z^3 + 3z^2 + pz + q = 0$$

$$-2 + 2\mathrm{i} + 3(2\mathrm{i}) + p(1+\mathrm{i}) + q = 0$$

$$-2 + 2\mathrm{i} + 6\mathrm{i} + p + p\mathrm{i} + q = 0$$

$$(-2 + p + q) + (8 + p)\mathrm{i} = 0$$

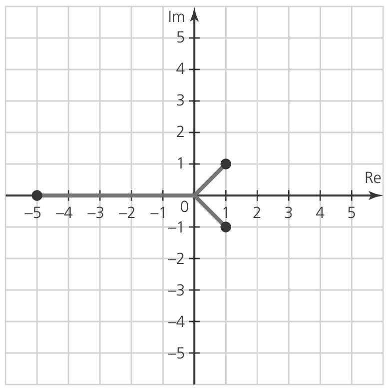
$$-2 + p + q = 0 \qquad 8 + p = 0$$

$$-2 - 8 + q = 0 \qquad p = -8$$

$$q = 10$$

(ii) $1 - \mathrm{i}$ and -5

(iii)



7 (i) $1-2\mathrm{i}$

(ii) The equation has 3 roots. Since roots occur in conjugate pairs, we have two roots so the other must be a real number.

(iii) $x=-3;\ A=1\ ,\ B=-1$

8 (i) $3^3+3^2-7\times3-15$

$= 27+9-21-15$

$=0$

$\therefore\ z=3$ is a root

$$\begin{array}{r|l} & z^2+4z+5 \\ z-3 & z^3+z^2-7z-15 \end{array}$$

$z^2+4z+5=0$

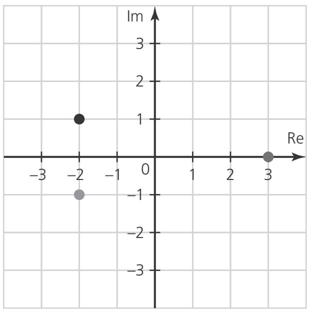
$(z+2)^2=-1$

$z+2=\pm\sqrt{-1}$

$z=-2+\mathrm{i}$

$\therefore$ Other roots are $-2+\mathrm{i}$ and $-2-\mathrm{i}$

(ii)



9 (i) $\alpha+\beta=3$

$\alpha\alpha^*=(1+\mathrm{i})(1-\mathrm{i})=1-\mathrm{i}^2=1-(-1)=2$

$\alpha\beta\ =(1+\mathrm{i})(2-\mathrm{i})=2-\mathrm{i}+2\mathrm{i}-\mathrm{i}^2=3+\mathrm{i}$

(ii) $x^2-2x+2=0$

(iii) $\alpha=1+\mathrm{i}$ is a root so $1-\mathrm{i}$ is a root

$\beta=2-\mathrm{i}$ is a root so $2+\mathrm{i}$ is a root

$z^4-6z^3+15z^2-18z+10=0$

10 $n=4,\ a=\dfrac{\sqrt{3}}{2},\ b=\dfrac{1}{2}$

Stretch and challenge

1 If $y=\ldots-3\pi,-\pi,\pi,3\pi,5\pi\ldots$

then $(\cos y=-1)(\mathrm{e}^x)=3$ has no solutions

If $y=\ldots-4\pi,-2\pi,0,2\pi,4\pi,6\pi\ldots$

then $\cos y=1\mathrm{e}^x=3\ \Rightarrow x=\ln3$

2 (i) $z^n+\dfrac{1}{z^n}=z^n+z^{-n}$

$=(\mathrm{cis}\theta)^n+(\mathrm{cis}\theta)^{-n}$

$=\mathrm{cis}\,n\theta+\mathrm{cis}(-n\theta)$

$=\cos n\theta+\mathrm{i}\sin n\theta+\cos(-n\theta)+\mathrm{i}\sin(-n\theta)$

$=\cos n\theta+\mathrm{i}\sin n\theta+\cos n\theta-\mathrm{i}\sin n\theta$

$=2\cos n\theta$

$z^n-\dfrac{1}{z^n}=z^n-z^{-n}$

$=\mathrm{cis}\,n\theta-\mathrm{cis}(-n\theta)$

$=\cos n\theta+\mathrm{i}\sin n\theta-(\cos n\theta-\mathrm{i}\sin n\theta)$

$=2\mathrm{i}\sin(n\theta)$

(ii) $\left(z+\dfrac{1}{z}\right)^4=z^4+\dfrac{1}{z^4}+4(z^2+\dfrac{1}{z^2})+6$

$(2\cos\theta)^4=\ 2\cos4\theta+4(2\cos2\theta)+6$

$16\cos^4\theta=2\cos4\theta+8\cos2\theta+6$

$\cos^4\theta\ =\frac{1}{8}\cos4\theta+\frac{1}{2}\cos2\theta+\frac{3}{8}$

$=\frac{1}{8}(\cos4\theta+4\cos2\theta+3)$

(iii) (a) $\dfrac{z-\dfrac{1}{z}}{\mathrm{i}\left(z+\dfrac{1}{z}\right)}=\dfrac{2\mathrm{i}\sin\theta}{\mathrm{i}(2\cos\theta)}$

$=\dfrac{2\sin\theta}{2\cos\theta}$

$=\tan\theta$

(b) $\dfrac{1-\tan^2\theta}{1+\tan^2\theta}=\dfrac{1-\left(\dfrac{z-z^{-1}}{\mathrm{i}(z+z^{-1})}\right)^2}{1+\left(\dfrac{z-z^{-1}}{\mathrm{i}(z+z^{-1})}\right)^2}$

$=\dfrac{1-\left(\dfrac{z^2+\dfrac{1}{z^2}-2}{\mathrm{i}^2(z^2+\dfrac{1}{z^2}+2)}\right)}{1+\left(\dfrac{z^2+\dfrac{1}{z^2}-2}{\mathrm{i}^2(z^2+\dfrac{1}{z^2}+2)}\right)}$

$=\dfrac{1+\left(\dfrac{2\cos2\theta-2}{2\cos2\theta+2}\right)}{1-\left(\dfrac{2\cos2\theta-2}{2\cos2\theta+2}\right)}$

$=\dfrac{\dfrac{2\cos2\theta+2+2\cos2\theta-2}{2\cos2\theta+2}}{\dfrac{2\cos2\theta+2-(2\cos2\theta-2)}{2\cos2\theta+2}}$

$=\dfrac{4\cos2\theta}{4}$

$=\cos2\theta$

3 (i) 4096

(ii) $\frac{1}{2}\text{cis}\left(\frac{\pi}{10}\right), \frac{1}{2}\text{cis}\left(\frac{1}{2}\pi\right), \frac{1}{2}\text{cis}\left(\frac{9\pi}{10}\right), \frac{1}{2}\text{cis}\left(-\frac{3\pi}{10}\right), \frac{1}{2}\text{cis}\left(-\frac{7\pi}{10}\right)$

(iii) (a) $(\cos\theta+\text{i}\sin\theta)^5$

$$= \cos^5\theta + 5\text{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10\text{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \text{i}\sin^5\theta$$

(b) $(\text{cis}\theta)^5 = \text{cis}5\theta = \cos 5\theta + \text{i}\sin 5\theta$

$$\therefore \sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$$

$$= 5(1-2\sin^2\theta + \sin^4\theta)\sin\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$

$$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$

$$= 16\sin^5\theta - 20\sin^3\theta + 5\sin^5\theta$$

(c) $x = \sin\frac{\pi}{10}, \sin\frac{\pi}{2}, \sin\frac{9\pi}{10}, \sin\left(\frac{-7\pi}{10}\right), \sin\left(\frac{-3\pi}{10}\right)$

4 (i) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$

(ii) $\overline{c}, -c, -\overline{c}, \sqrt{k}$ and $-\sqrt{k}$

(iii) $\text{p}(z) = (z-(a+b\text{i}))^8 + r^8$

$$= \left(z - \left(1+\frac{1}{\sqrt{2}}\right) - \left(1+\frac{1}{\sqrt{2}}\right)\text{i}\right)^8 + (2+\sqrt{2})^4$$

5 (i) $z^2 = \frac{a}{2} + \frac{a}{2} + 2\text{i}\frac{|b|}{2} = a + \text{i}|b|$

(ii) (a) $\text{f}(0) = c = 1 + \frac{\text{i}}{2}$

$$\text{f}^2(0) = c^2 + c = \left(1+\frac{\text{i}}{2}\right)^2 + \left(1+\frac{\text{i}}{2}\right)$$

$$\text{f}^2(0) = \frac{7}{4} + \frac{3\text{i}}{2}$$

and

$$|\text{f}^2(0)| = \frac{1}{4}\left(\sqrt{49+36}\right) = \frac{\sqrt{85}}{4} > 2,$$

so $z = 1 + \frac{\text{i}}{2}$ is not part of the set.

$\text{f}(0) = c = \text{i}$

$\text{f}^2(0) = c^2 + c = (\text{i})^2 + \text{i} = -1 + \text{i}$

and $|\text{f}^2(0)| = \sqrt{2} < 2$, so OK so far.

$\text{f}^3(0) = (-1+\text{i})^2 + \text{i} = -2\text{i} + \text{i} = -1$

and $|\text{f}^3(0)| = 1 < 2$, so OK so far.

$\text{f}^4(0) = (-\text{i})^2 + \text{i} = -1 + \text{i}$ and so the process will loop continually with $|\text{f}^n(0)| < 2$ for all n, and $z = \text{i}$ is part of the set.

(b) $\text{f}^2(0) = a + \sqrt{\frac{3a}{2}} + \text{i}\left(a\sqrt{3} + \sqrt{\frac{a}{2}}\right)$

and when $a = \frac{1}{8}$, $\text{f}^2(0) = \frac{1}{8} + \sqrt{\frac{3}{16}} + \text{i}\left(\frac{\sqrt{3}}{8} + \frac{1}{4}\right)$

$$\frac{1+2\sqrt{3}}{8} + \text{i}\left(\frac{\sqrt{3}+2}{8}\right)$$

so $|\text{f}^2(0)| = \frac{1}{8}\sqrt{(1+2\sqrt{3})^2 + (\sqrt{3}+2)^2}$

$$= \frac{1}{8}\sqrt{(13+4\sqrt{3}) + (7+4\sqrt{3})}$$

$$= \frac{1}{8}\sqrt{(20+8\sqrt{3})} = \frac{1}{4}\sqrt{(5+2\sqrt{3})}$$

Exam focus

1 (i) (a) $-8+\text{i}$

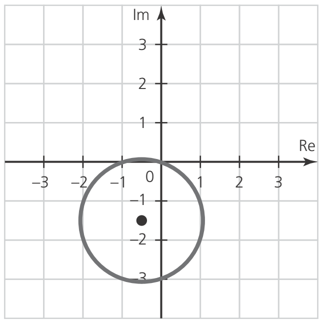
(b) $1-7\text{i}$

(c) $\frac{3}{10} + \frac{1}{10}\text{i}$

(d) $\frac{1}{5} - \frac{7}{5}\text{i}$

(ii) (a) $-\frac{1}{2} - \frac{3}{2}\text{i}$

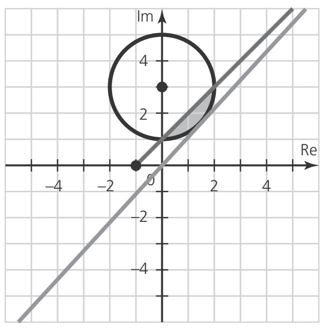
(b)



2 $u = 2+3\text{i}, \quad w = 2-3\text{i}$

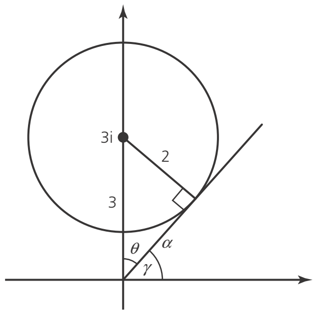
$u = -2+3\text{i}, \ w = -2-3\text{i}$

3 (i), (ii) and (iii) (a)



(iii) (b) $\sin\theta = \frac{2}{3} \Rightarrow \theta = 41.8°$

$\therefore \alpha = 90° - 41.8° = 48.2°$

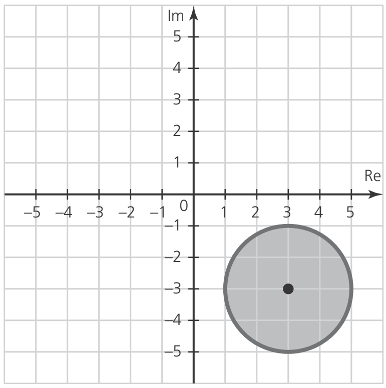


4 (i) $1 - 2i$ is another root

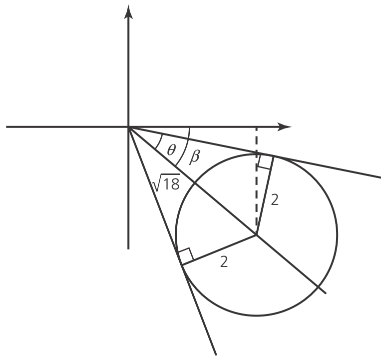
(ii) $x = -1$ and $x = 2$

5 (i) $w = \pm\sqrt{18}i$

(ii) (a) $|z - (3 - 3i)| \leqslant 2$



(b)



$p = \sqrt{18} - 2$ or $3\sqrt{2} - 2$

$q = \sqrt{18} + 2$ or $3\sqrt{2} + 2$

$\alpha = -73.1°$

$\beta = -16.9°$

6 (i) $k = 3$

(ii) (a) $(z - 3)(z^2 + 2z + 9) = 0$

if $(z^2 + 2z + 9) = 0$

$$\Rightarrow \quad z = \frac{-2 \pm \sqrt{4 - 4(1)(9)}}{2}$$

$$= \frac{-2 \pm \sqrt{4 - 36}}{2} = \frac{-2 \pm \sqrt{-32}}{2}$$

$$= \frac{-2 \pm \sqrt{32i^2}}{2}$$

$$= \frac{-2 \pm 4i\sqrt{2}}{2}$$

$z = -1 + 2\sqrt{2}i$ or $-1 - 2\sqrt{2}i$

$z = 3, \ -1 + 2\sqrt{2}i, \ -1 - 2\sqrt{2}i$

(b) Roots are $\sqrt{3}, \ -\sqrt{3}, \ 1 + \sqrt{2}i, \ -1 - \sqrt{2}i,$
$1 - \sqrt{2}\ i, \ -1 + \sqrt{2}i$

Past examination questions

1 Algebra

1 $a = 4$. Other factor is $x^2 - x + 2$

2 $-1 < x < -\frac{1}{7}$

3 (i) $a = 3$

(ii) $x < -\frac{1}{2}$

4 $-3 < x < 1$

5 $-7 < x < 1$

6 (i) $a = -7$

(ii) Quotient is $x^2 + 5x - 2$

7 (i) $a = -1, b = -21$

(ii) $p(x) = (x+2)(3x-5)(2x-1)$

8 $x < -8$ and $x > 0$

9 (i) $a = 2, b = 6$

(ii) $p(x) = (x+2)(2x-1)(x-3)$

2 Logarithms and exponentials

1 (i) $4^x = y^2$

(ii) $x = -1.58$ or $x = 1.58$ (2dp)

2 0.313 (3sf)

3 −1.68 (2dp)

4 0.107

5 (i)

$$2^x + 3(2^{-x}) = 4$$
$$2^x + \frac{3}{2^x} = 4$$
$$y + \frac{3}{y} = 4$$
$$y^2 + 3 = 4y$$
$$y^2 - 4y + 3 = 0$$

(ii) $x = 0$ or $x = 1.58$

6 0.763

7 $A = 1.11$, $b = 1.65$

8 3.42

3 Trigonometry

1 (i) $\cos 4\theta + 4\cos 2\theta \equiv 8\cos^4\theta - 3$

LHS

$$= (2\cos^2 2\theta - 1) + 4(2\cos^2\theta - 1)$$
$$= (2(2\cos^2\theta - 1)^2 - 1) + 8\cos^2\theta - 4$$
$$= 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1 + 8\cos^2\theta - 4$$
$$= 8\cos^4\theta - 8\cos^2\theta + 2 - 1 + 8\cos^2\theta - 4$$
$$= 8\cos^4\theta - 3$$
$$= RHS$$

(ii) $\theta = 27.2°$ or $332.8°$ or $152.8°$ or $207.2°$

2 (i) $\tan(x+45) - \tan(45-x) \equiv 2\tan 2x$

LHS

$$= \frac{\tan x + \tan 45}{1 - \tan x \tan 45} - \left(\frac{\tan 45 - \tan x}{1 + \tan 45 \tan x}\right)$$
$$= \frac{\tan x + 1}{1 - \tan x} - \frac{1 - \tan x}{1 + \tan x}$$
$$= \frac{(\tan x + 1)^2 - (1 - \tan x)^2}{(1 - \tan x)(1 + \tan x)}$$
$$= \frac{\tan^2 x + 2\tan x + 1 - (1 - 2\tan x + \tan^2 x)}{(1 - \tan x)(1 + \tan x)}$$
$$= \frac{4\tan x}{1 - \tan^2 x}$$
$$= 2\left(\frac{2\tan x}{1 - \tan^2 x}\right)$$
$$= 2\tan 2x$$
$$= RHS$$

(ii) $x = 22.5°$ or $112.5°$

3 (i) $5\sin x + 12\cos x = 13\sin(x + 67.38°)$

(ii) $\theta = 27.4°$ or $175.2°$

4 $x = 48.2°$ or $120°$

5 (i) $3\cos x + 4\sin x = 5\cos(x - 53.13°)$

(ii) $x = 78.9°, 27.3°$

6 (i)

$$\tan(x + 45°) = 6\tan x$$
$$\frac{\tan x + \tan 45°}{1 - \tan x \tan 45°} = 6\tan x$$
$$\frac{\tan x + 1}{1 - \tan x} = 6\tan x$$
$$\tan x + 1 = 6\tan x(1 - \tan x)$$
$$\tan x + 1 = 6\tan x - 6\tan^2 x$$
$$6\tan^2 x - 5\tan x + 1 = 0$$

(ii) $x = 18.4°$ or $26.6°$

7 $\theta = 48.6°$ or $131.4°$ or $270°$

8 $\theta = 9.9°$ or $189.9°$

9 (i) *LHS*

$$\sin^2 2\theta(\mathrm{cosec}^2\theta - \sec^2\theta)$$
$$= \sin^2 2\theta\left(\frac{1}{\sin^2\theta} - \frac{1}{\cos^2\theta}\right)$$
$$= \sin^2 2\theta\left(\frac{\cos^2\theta - \sin^2\theta}{\sin^2\theta\cos^2\theta}\right)$$
$$= \sin^2 2\theta\left(\frac{4(\cos^2\theta - \sin^2\theta)}{4\sin^2\theta\cos^2\theta}\right)$$
$$= \sin^2 2\theta\left(\frac{4\cos 2\theta}{\sin^2 2\theta}\right)$$
$$= 4\cos 2\theta$$
$$= RHS$$

(ii) **(a)** $\theta = 20.7°$ or $159.3°$

(b) $8\sqrt{3}$

4 Differentiation

1 $y = x$

2 $x = a(2\theta - \sin 2\theta)$ $\qquad y = a(1 - \cos 2\theta)$

$$\frac{dx}{d\theta} = a(2 - 2\cos 2\theta) \qquad \frac{dy}{d\theta} = a(2\sin 2\theta)$$
$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{2a\sin 2\theta}{a(2-2\cos 2\theta)}$$
$$= \frac{2\sin 2\theta}{2(1-\cos 2\theta)}$$
$$= \frac{2\sin\theta\cos\theta}{(1-(1-2\sin^2\theta))}$$
$$= \frac{\sin\theta\cos\theta}{\sin^2\theta}$$
$$= \frac{\cos\theta}{\sin\theta}$$
$$= \cot\theta$$

3 $x = \sqrt{\frac{2}{3}}$

4 $x = 1.206$ or 0.365

5 (i)
$$x^2y + y^2 = 6$$
$$2xy + x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 6$$
$$(x^2 + 2y)\frac{dy}{dx} = 6 - 2xy$$
$$\frac{dy}{dx} = \frac{6 - 2xy}{x^2 + 2y}$$

(ii) $2x - 5y + 8 = 0$

6 (i)
$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-\frac{1}{2}}\cdot\left(\frac{-2}{(1+x)^2}\right)$$
$$= -\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\cdot\frac{1}{(1+x)^2}$$
$$= -\frac{\sqrt{1+x}}{\sqrt{1-x}}\cdot\frac{1}{(1+x)^2}$$
$$= -\frac{1}{\sqrt{1-x}\sqrt{(1+x)^3}}$$
$$= -\frac{1}{\sqrt{(1-x)(1+x)^3}}$$
$$= -\frac{1}{\sqrt{(1-x)(1+x)(1+x)^2}}$$
$$= -\frac{1}{\sqrt{(1-x^2)(1+x)^2}}$$
$$= -\frac{1}{\sqrt{(1-x^2)}(1+x)}$$

$\therefore$ Gradient of normal $= (1+x)\sqrt{1-x^2}$

(ii) $x = \frac{1}{2}$

7 $\left(e^{-\frac{1}{3}}, -\frac{1}{3}e^{-1}\right)$

8 (i) $2x + 2y + 2x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$

At (–2, 2)

$$2(-2) + 2(2) + 2(-2)\frac{dy}{dx} - 2(2)\frac{dy}{dx} = 0$$
$$-4 + 4 - 4\frac{dy}{dx} - 4\frac{dy}{dx} = 0$$
$$-8\frac{dy}{dx} = 0$$
$$\therefore \frac{dy}{dx} = 0$$

So tangent is parallel to the x axis.

(ii) $y = 2x - 2$

9 (i) $\frac{dy}{dx} = 2\sin^2 t\cos^2 t$

(ii) $y = \frac{1}{2}x + \frac{1}{2}$

10 $9x + 2y - 16 = 0$

11 (i) $\frac{dx}{d\theta} = 4\sin\theta\cos\theta \qquad \frac{dy}{d\theta} = 4\sec^2\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$
$$= 4\sec^2\theta\cdot\frac{1}{4\sin\theta\cos\theta}$$
$$= \frac{4}{\cos^2\theta}\cdot\frac{1}{4\sin\theta\cos\theta}$$
$$= \frac{1}{\sin\theta\cos^3\theta}$$

(ii) $y = 4x - 4$

12 $y = \frac{e^{2x}}{1+e^{2x}}$

$$\frac{dy}{dx} = \frac{2e^{2x}(1+e^{2x}) - 2e^{2x}(e^{2x})}{(1+e^{2x})^2}$$
$$= \frac{2e^{2x}}{(1+e^{2x})^2}$$

When $x = \ln 3$

$$\frac{dy}{dx} = \frac{2e^{2\ln 3}}{(1+e^{2\ln 3})^2}$$
$$= \frac{2e^{\ln 3^2}}{(1+e^{\ln 3^2})^2}$$
$$= \frac{2(9)}{(1+9)^2}$$
$$= \tfrac{18}{100}$$
$$= \tfrac{9}{50}$$

5 Integration

1 (i) $\frac{dy}{dx} = 2\sec^2 2x$

(ii) $\int_0^{\frac{1}{6}\pi}\sec^2 2x\,dx$

$$= \tfrac{1}{2}\left[\tan 2x\right]_0^{\frac{1}{6}\pi}$$
$$= \tfrac{1}{2}\left[\tan\frac{\pi}{3}\right]$$
$$= \tfrac{1}{2}\sqrt{3}$$

$$\int_0^{\frac{1}{6}\pi}\tan^2 2x\,dx$$
$$= \int_0^{\frac{1}{6}\pi}(\sec^2 2x - 1)\,dx$$
$$= \left[\tan 2x - x\right]_0^{\frac{1}{6}\pi}$$
$$= \tfrac{1}{2}\sqrt{3} - \tfrac{1}{6}\pi$$

(iii) $\frac{1}{4}\sqrt{3}$

2 (i) **(a)** *LHS*:

$$\sec^2 x + \sec x\tan x$$
$$= \frac{1}{\cos^2 x} + \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}$$
$$= \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}$$
$$= \frac{1+\sin x}{\cos^2 x}$$
$$= RHS$$

(b) $\sec^2 x + \sec x\tan x$

$$= \frac{1+\sin x}{\cos^2 x}$$
$$= \frac{1+\sin x}{(1-\sin^2 x)}$$
$$= \frac{1+\sin x}{(1-\sin x)(1+\sin x)}$$
$$= \frac{1}{1-\sin x}$$

(ii) $\frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{0\cdot\cos x - (-\sin x)\cdot 1}{\cos^2 x}$

$$= \frac{\sin x}{\cos^2 x}$$
$$= \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}$$
$$= \sec x\tan x$$

(iii) $\int_0^{\frac{\pi}{4}}\frac{1}{1-\sin x}\,dx$

$$= \int_0^{\frac{\pi}{4}}(\sec^2 x + \sec x\tan x)\,dx$$
$$= \left[\tan x + \sec x\right]_0^{\frac{\pi}{4}}$$
$$= \left(\tan\frac{\pi}{4} + \sec\frac{\pi}{4}\right) - (\tan 0 + \sec 0)$$
$$= 1 + \frac{1}{\frac{1}{\sqrt{2}}} - 1$$
$$= \sqrt{2}$$

3 (i) 0.98

(ii) Using 6 intervals would be closer to the actual value of the integral. Since the value from **(i)** overestimates the true value, the value using 6 intervals would be less than *E*.

4 (i) *LHS*

$= 1 - 2\sin^2 2\theta - 4(1 - 2\sin^2\theta) + 3$

$= 1 - 2(2\sin\theta\cos\theta)^2 - 4 + 8\sin^2\theta + 3$

$= 1 - 2(4\sin^2\theta\cos^2\theta) - 4 + 8\sin^2\theta + 3$

$= 1 - 8\sin^2\theta(1 - \sin^2\theta) - 4 + 8\sin^2\theta + 3$

$= 1 - 8\sin^2\theta + 8\sin^4\theta - 4 + 8\sin^2\theta + 3$

$= 8\sin^4\theta$

$= RHS$

(ii) $\frac{1}{32}(2\pi - \sqrt{3})$

5 (i) $y = \cot x = \dfrac{\cos x}{\sin x}$

$$\frac{dy}{dx} = \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= -\frac{1}{\sin^2 x}$$

$$= -\operatorname{cosec}^2 x$$

(ii) $\cot^2 x = \operatorname{cosec}^2 x - 1$

$$\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot^2 x \, dx$$

$$= \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} (\operatorname{cosec}^2 x - 1)\, dx$$

$$= \left[-\cot x - x\right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}$$

$$= \left(-\cot\tfrac{1}{2}\pi - \tfrac{1}{2}\pi\right) - \left(-\cot\tfrac{1}{4}\pi - \tfrac{1}{4}\pi\right)$$

$$= \left(0 - \tfrac{1}{2}\pi\right) - \left(-1 - \tfrac{1}{4}\pi\right)$$

$$= 1 - \tfrac{1}{4}\pi$$

(iii) $\cos 2x = 1 - 2\sin^2 x$

$$\therefore \quad \frac{1}{1 - \cos 2x} = \frac{1}{1 - (1 - 2\sin^2 x)}$$

$$= \frac{1}{2\sin^2 x}$$

$$= \tfrac{1}{2}\operatorname{cosec}^2 x$$

$$\int \frac{1}{1 - \cos 2x}\, dx = \int \tfrac{1}{2}\operatorname{cosec}^2 x \, dx$$

$$= -\tfrac{1}{2}\cot x + c$$

6 (i) $\cos(3x - x) = \cos 3x\cos x + \sin 3x \sin x$

$\cos(3x + x) = \cos 3x\cos x - \sin 3x \sin x$

$\frac{1}{2}(\cos 2x - \cos 4x)$

$= \frac{1}{2}\left[\cos(3x - x) - \cos(3x + x)\right]$

$= \frac{1}{2}[(\cos 3x\cos x + \sin 3x\sin x) - (\cos 3x\cos x - \sin 3x\sin x)]$

$= \frac{1}{2}\left[2\sin 3x\sin x\right]$

$= \sin 3x \sin x$

$= RHS$

(ii)
$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sin 3x \sin x \, dx$$

$$= \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \tfrac{1}{2}(\cos 2x - \cos 4x)\, dx$$

$$= \tfrac{1}{2}\left[\tfrac{1}{2}\sin 2x - \tfrac{1}{4}\sin 4x\right]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}$$

$$= \left(\tfrac{1}{4}\sin\tfrac{2}{3}\pi - \tfrac{1}{8}\sin\tfrac{4}{3}\pi\right) - \left(\tfrac{1}{4}\sin\tfrac{1}{3}\pi - \tfrac{1}{8}\sin\tfrac{2}{3}\pi\right)$$

$$= \left(\frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{16}\right) - \left(\frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{16}\right)$$

$$= \frac{\sqrt{3}}{8}$$

7 (i) 3.41 (2dp)

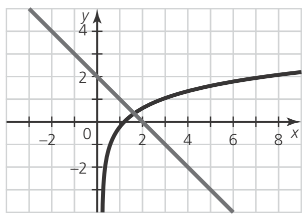
(ii) Area B $\approx$ 2.59 (2dp)

Since the calculation of Area *A* overestimates the true value of *A*, this approximation will be an underestimate of the true value of *B*.

8 $k = \dfrac{\ln 3}{2}$

6 Numerical solution of equations

1 (i)



(ii) $f(1.4) = -0.264 \qquad f(1.7) = 0.231$

where $f(x) = \ln x + x - 2$

Change of sign so root is between 1.4 and 1.7.

(iii) $x = \frac{1}{3}(4 + x - 2\ln x)$

$3x = 4 + x - 2\ln x$

$2x = 4 - 2\ln x$

$x = 2 - \ln x$

$\ln x = 2 - x$

(iv) 1.56 (2dp)

2 (i) $y = x^2 \cos x$

$$\frac{dy}{dx} = 2x\cos x + x^2(-\sin x)$$

$\therefore$ M satisfies

$$2x\cos x - x^2 \sin x = 0$$
$$2x\cos x = x^2 \sin x$$
$$2\cos x = x\,\text{sinx}$$
$$\frac{2}{x} = \frac{\sin x}{\cos x}$$
$$\frac{2}{x} = \tan x$$

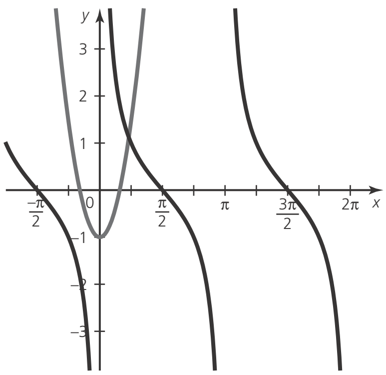
(ii) $f(1) = -0.443 \quad f(1.2) = 0.905$

where $f(x) = \tan x + \frac{-2}{x}$

Change of sign so root is between 1 and 1.2.

(iii) 1.08 (2dp)

3 (i)



(ii) $f(0.6) = 1.022 \quad f(1) = -2.36$

where $f(x) = \cos x + 1 - 4x^2$

Change of sign so root is between 0.6 and 1.

(iii) 0.73 (2dp)

4 (i) 1.82 (2dp)

(ii) Equation is $x = \frac{7x}{8} + \frac{5}{2x^4}$

$$8x^5 = 7x^5 + 20$$
$$x^5 = 20$$
$$x = \sqrt[5]{20}$$

5 (i) Area shaded region

= Area triangle OCT – Area sector OBC

$= \frac{1}{2} \cdot r \cdot r \tan x - \frac{1}{2} r^2 x$

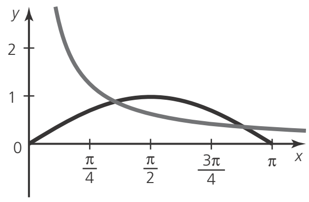
$= \frac{1}{2} r^2 (\tan x - x)$

$\therefore \frac{1}{2} r^2 (\tan x - x) = \frac{1}{2} \pi r^2$

$$\tan x - x = \pi$$
$$\tan x = x + \pi$$

(ii) 1.35 (2dp)

6 (i)



(ii) $f(1.1) = -0.018 \quad f(1.2) = 0.0987$

Change of sign so root between 1.1 and 1.2.

(iii) 1.11 (2dp)

7 Further algebra

1 (i) $f(x)=1-\dfrac{1}{x-1}+\dfrac{2x}{x^2+1}$

(ii) $\displaystyle\int_2^3\left(1-\frac{1}{x-1}+\frac{2x}{x^2+1}\right)dx$

$=\left[x-\ln|x-1|+\ln|x^2+1|\right]_2^3$

$=\left[(3-\ln 2+\ln 10)-(2-\ln 1+\ln 5)\right]$

$=1+\ln 10-\ln 2-\ln 5$

$=1+\ln\left(\frac{10}{2}\right)-\ln 5$

$=1+\ln 5-\ln 5$

$=1$

2 (i) $f(x)=-\dfrac{1}{x-1}+\dfrac{4}{x-2}-\dfrac{2}{x+1}$

(ii) $f(x)=-(x-1)^{-1}+4(x-2)^{-1}-2(x+1)^{-1}$

$=-(-1+x)^{-1}+4(-2+x)^{-1}-2(1+x)^{-1}$

$=-(-1)^{-1}(1-x)^{-1}+4\times(-2)^{-1}\left(1-\frac{1}{2}x\right)^{-1}-2(1+x)^{-1}$

$=(1-x)^{-1}-2\left(1-\frac{1}{2}x\right)^{-1}-2(1+x)^{-1}$

$(1-x)^{-1}=1+(-1)(-x)+\dfrac{(-1)(-2)}{2!}(-x)^2+\dfrac{(-1)(-2)(-3)}{3!}(-x)^3+\ldots$

$=1+x+x^2+x^3+\ldots$

$-2\left(1-\frac{1}{2}x\right)^{-1}=-2\left(1+(-1)\left(-\frac{1}{2}x\right)+\dfrac{(-1)(-2)}{2!}\left(-\frac{1}{2}x\right)^2+\dfrac{(-1)(-2)(-3)}{3!}\left(-\frac{1}{2}x\right)^3+\ldots\right)$

$=-2\left(1+\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3+\ldots\right)$

$=-2-x-\frac{1}{2}x^2-\frac{1}{4}x^3-\ldots$

$-2(1+x)^{-1}=-2\left(1+(-1)(x)+\dfrac{(-1)(-2)}{2!}x^2+\dfrac{(-1)(-2)(-3)}{3!}x^3+\ldots\right)$

$=-2(1-x+x^2-x^3+\ldots)$

$=-2+2x-2x^2+2x^3+\ldots$

$f(x)=-3+2x-\frac{3}{2}x^2+\frac{11}{4}x^3$

3 (a) (i) $\dfrac{A}{x+4}+\dfrac{Bx+C}{x^2+3}$

(ii) $\dfrac{A}{x-2}+\dfrac{B}{x+2}+\dfrac{C}{(x+2)^2}$

(b) $\dfrac{3x}{(x+1)(x-2)}\equiv\dfrac{A}{x+1}+\dfrac{B}{x-2}$

$3x\equiv A(x-2)+B(x+1)$

$x=2\Rightarrow 6\equiv 3B\Rightarrow B=2$

$x=-1\Rightarrow -3\equiv -3A\Rightarrow A=1$

$\displaystyle\int_3^4\frac{3x}{(x+1)(x-2)}\,dx=\int_3^4\left(\frac{1}{x+1}+\frac{2}{x-2}\right)dx$

$=[\ln|x+1|+2\ln|x-2|]_3^4$

$=[(\ln 5+2\ln 2)-(\ln 4+2\ln 1)]$

$=\ln 5+\ln 2^2-\ln 4$

$=\ln 5+\ln 4-\ln 4$

$=\ln 5$

4 $\frac{1}{8}-\frac{3}{16}x+\frac{3}{16}x^2+\ldots$

5 $1-2x+6x^2-20x^3+\ldots$

6 (i) $\dfrac{5x-x^2}{(1+x)(2+x^2)}\equiv\dfrac{-2}{1+x}+\dfrac{x+4}{2+x^2}$

(ii) $\frac{5}{2}x-3x^2+\frac{7}{4}x^3\ldots$

7 (i) $a=2$

$p(x)=(2x-1)(x^2+2)$

(ii) $\dfrac{8x-13}{p(x)}\equiv\dfrac{-4}{2x-1}+\dfrac{2x+5}{x^2+2}$

8 Further integration

1 (i) $x^2\sqrt{1-x^2}=0$, where $x=0, 1, -1$

$x=\sin\theta$

$\dfrac{dx}{d\theta}=\cos\theta\Rightarrow dx=\cos\theta\,d\theta$

$1=\sin\theta\Rightarrow\theta=\dfrac{\pi}{2}$

$0=\sin\theta\Rightarrow\theta=0$

$A=\displaystyle\int_0^1 x^2\sqrt{1-x^2}\,dx$

$=\displaystyle\int_0^{\frac{\pi}{2}}\sin^2\theta\sqrt{1-\sin^2\theta}\cos\theta\,d\theta$

$=\displaystyle\int_0^{\frac{\pi}{2}}\sin^2\theta\sqrt{\cos^2\theta}\cos\theta\,d\theta$

$=\displaystyle\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^2\theta\,d\theta$

$=\displaystyle\int_0^{\frac{\pi}{2}}\frac{1}{4}(2\sin\theta\cos\theta)^2\,d\theta$

$=\displaystyle\frac{1}{4}\int_0^{\frac{\pi}{2}}\sin^2 2\theta\,d\theta$

(ii) $\dfrac{\pi}{16}$

2 (i) $\dfrac{2}{(x+1)(x+3)} = \dfrac{1}{x+1} - \dfrac{1}{x+3}$

(ii) $\left\{\dfrac{2}{(x+1)(x+3)}\right\}^2$

$= \left(\dfrac{1}{x+1} - \dfrac{1}{x+3}\right)^2$

$= \dfrac{1}{(x+1)^2} - \dfrac{2}{(x+1)(x+3)} + \dfrac{1}{(x+3)^2}$

$= \dfrac{1}{(x+1)^2} - \left(\dfrac{1}{x+1} - \dfrac{1}{x+3}\right) + \dfrac{1}{(x+3)^2}$

$= \dfrac{1}{(x+1)^2} - \dfrac{1}{x+1} + \dfrac{1}{x+3} + \dfrac{1}{(x+3)^2}$

(iii) $\displaystyle\int_0^1 \frac{4}{(x+1)^2(x+3)^2}\,dx$

$= \displaystyle\int_0^1 \left[(x+1)^{-2} - \frac{1}{x+1} + \frac{1}{x+3} + (x+3)^{-2}\right]dx$

$= \left[-\dfrac{1}{x+1} - \ln(x+1) + \ln(x+3)\left(-\dfrac{1}{x+3}\right)\right]_0^1$

$= \left[\left(-\tfrac{1}{2} - \ln 2 + \ln 4 - \tfrac{1}{4}\right) - \left(-1 - \ln 1 + \ln 3 - \tfrac{1}{3}\right)\right]$

$= \left[\left(-\tfrac{3}{4} + \ln 2\right) - \left(-\tfrac{4}{3} + \ln 3\right)\right]$

$= \tfrac{7}{12} + \ln\tfrac{2}{3}$

$= \tfrac{7}{12} - \ln\tfrac{3}{2}$

3 $4\ln 2 - \frac{15}{16}$

4 (i) $x = 2\sin\theta \qquad 1 = 2\sin\theta \Rightarrow \theta = \dfrac{\pi}{6}$

$0 = 2\sin\theta => \theta = 0$

$\dfrac{dx}{d\theta} = 2\cos\theta$

$dx = 2\cos\theta\, d\theta$

$\therefore \displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$

$= \displaystyle\int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}\, 2\cos\theta\, d\theta$

$= \displaystyle\int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta}{\sqrt{4(1-\sin^2\theta)}}\, 2\cos\theta\, d\theta$

$= \displaystyle\int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta}{\sqrt{4(\cos^2\theta)}}\, 2\cos\theta\, d\theta$

$= \displaystyle\int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta}{2\cos\theta}\, 2\cos\theta\, d\theta$

$= \displaystyle\int_0^{\frac{\pi}{6}} 4\sin^2\theta\, d\theta$

(ii) $\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{2}$

5 (i) $A = \displaystyle\int_0^3 x^2 e^{-x}\,dx$ $\quad \left|\begin{matrix} u = x^2 \\ u' = 2x \end{matrix}\right| \left|\begin{matrix} v' = e^{-x} \\ v = -e^{-x} \end{matrix}\right|$

$= -x^2e^{-x} - \displaystyle\int -2xe^{-x}\,dx$ $\quad \left|\begin{matrix} u = 2x \\ u' = 2 \end{matrix}\right| \left|\begin{matrix} v' = e^{-x} \\ v = -e^{-x} \end{matrix}\right|$

$= -x^2e^{-x} + \displaystyle\int 2xe^{-x}\,dx$

$= -x^2e^{-x} + \left[-2xe^{-x} - \displaystyle\int -2e^{-x}\,dx\right]$

$= \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^3$

$= \left[-e^{-x}(x^2 + 2x + 2)\right]_0^3$

$= [-e^{-3}(3^2 + 2\times 3 + 2) - \{-e^0(0^2 + 2\times 0 + 2)\}]$

$= [-e^{-3}(17) + 2]$

$= 2 - \dfrac{17}{e^3}$

(ii) $x = 2$

(iii) $x = 1$

6 (i) $u = \tan x$

$u = \tan\dfrac{\pi}{4} = 1 \qquad u = \tan 0 = 0$

$du = \sec^2 x\, dx$

$dx = \dfrac{du}{\sec^2 x} \Rightarrow dx = \cos^2 x\, du$

$\therefore \displaystyle\int_0^{\frac{\pi}{4}} (\tan^{n+2} x + \tan^n x)\,dx$

$= \displaystyle\int_0^{\frac{\pi}{4}} \tan^n x\,(\tan^2 x + 1)\,dx$

$= \displaystyle\int_0^1 u^n(\sec^2 x)\cos^2 x\, du$

$= \displaystyle\int_0^1 u^n\, du$

$= \left[\dfrac{u^{u+1}}{n+1}\right]_0^1$

$= \dfrac{1^{n+1}}{n+1} - \dfrac{0^{n+1}}{n+1}$

$= \dfrac{1}{n+1}$

(ii) **(a)** $\frac{1}{3}$

(b) $\frac{25}{24}$

9 Differential equations

1 (i) $1000h = 30 => h = 0.03$

$\dfrac{dh}{dt} = 0.03 - k\sqrt{h}$

$0.02 = 0.03 - k \times \sqrt{1}$

$k = 0.01$

$\therefore \dfrac{dh}{dt} = 0.03 - 0.01\sqrt{h}$

$= 0.01(3 - \sqrt{h})$

(ii) $200\left(\ln\left(\dfrac{3}{x}\right)^3 + x - 3\right)$

(iii) 4 min 19 s

2 (i) $\frac{dx}{dt} = kx - 25$

$75 = k \times 1000 - 25$

$100 = k \times 1000$

$k = 0.1$

$\frac{dx}{dt} = 0.1x - 25$

$\frac{dx}{dt} = 0.1(x - 250)$

(ii) $x = 250(1 + 3e^{0.1t})$

3 (i) $\frac{dh}{dt} = k(9-h)^{\frac{1}{3}}$

$0.2 = k(9-1)^{\frac{1}{3}}$

$0.2 = k \times \sqrt[3]{8}$

$0.2 = k \times 2$

$k = 0.1$

$\therefore \frac{dh}{dt} = 0.1(9-h)^{\frac{1}{3}}$

(ii) $h = 9 - \sqrt{\left(4 - \frac{1}{15}t\right)^3}$

(iii) 9 metres after 60 years

(iv) 19.1 years

4 (i) $N = 125e^{50k\sin(0.02t)}$

(ii) $k = 0.010$

(iii) $N = 125e^{0.502\sin(0.02t)}$, least value of N is 75.7

5 (i) Area $\Delta PTN = \frac{1}{2}TN \times PN$

$\tan x = \frac{1}{2}TN \times PN$

$\frac{2\tan x}{PN} = TN$

$\frac{dy}{dx} = \frac{y}{TN} = \frac{y}{\frac{2\tan x}{PN}}$

$= \frac{y}{\frac{2\tan x}{y}}$

$= \frac{y^2}{2\tan x}$

$= \frac{1}{2}y^2\cot x$

(ii) $y = \frac{2}{1 - \ln(2\sin x)}$

6 (i) $\frac{dA}{dt} \propto V \Rightarrow \frac{dA}{dt} = kV$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$ $\quad A = 4\pi r^2 \Rightarrow \frac{dA}{dr} = 8\pi r$

$kV = 8\pi r \times \frac{dr}{dt}$

$k \times \frac{4}{3}\pi r^3 = 8\pi r \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{4k\pi r^3}{24\pi r}$

$\frac{dr}{dt} = \frac{k}{6}r^2$

$r = 5, \frac{dr}{dt} = 2$

$\Rightarrow 2 = \frac{k}{6} \times 5^2$

$k = \frac{12}{25}$

$\therefore \frac{dr}{dt} = \frac{\frac{12}{25}}{6}r^2$

$\frac{dr}{dt} = 0.08r^2$

(ii) $r = \frac{5}{1 - 0.4t}$

(iii) $0 \leqslant t < 2.5$

7 (i) $\frac{dy}{dx} = kxy$

$4 = k \times 1 \times 2 \Rightarrow k = 2$

$\therefore \frac{dy}{dx} = 2xy$

$\int \frac{1}{y}\,dy = \int 2x\,dx$

$\ln|y| = x^2 + C$

$\ln|2| = 1^2 + C$

$C = \ln 2 - 1$

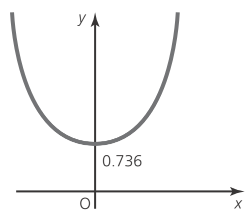
$\therefore \ln y = x^2 + \ln 2 - 1$

$y = e^{(x^2 + \ln 2 - 1)}$

$y = e^{x^2} \times e^{\ln 2} \times e^{-1}$

$y = 2e^{x^2 - 1}$

(ii) -4



8 (i) $N = (40 - 30e^{-0.02t})^2$

(ii) $N = 1600$

10 Vectors

1 (i) To intersect,

$1 + 2s = 6 + t$ ①

$s = -5 - 2t$ ②

$-2 + 3s = 4 + t$ ③

$-3 + s = -2$ ③ − ①

$\Rightarrow s = 1, t = -3$

Check in ②, $1 = -5 - 2(-3)$

$1 = 1$

$\therefore$ The lines intersect at

$$\mathbf{r} = \begin{pmatrix}1\\0\\-2\end{pmatrix} + 1\begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}3\\1\\1\end{pmatrix} \text{ or } 3\mathbf{i}+\mathbf{j}+\mathbf{k}$$

(ii) $7x + y - 5z = 17$

2 (i) $2x + 3y - 6z = 8$

(ii) N is $(1, 2, 0)$

$$|\overrightarrow{SN}| = \sqrt{(3-1)^2 + (5-2)^2 + (-6-0)^2}$$
$$= \sqrt{49}$$
$$= 7$$

3 (i) To intersect,

$2 + s = -2 - 2t$ ①

$-1 + s = 2 + t$ ②

$4 - s = 1 + t$ ③

$3 = 3 + 2t$ ② + ③

$\therefore t = 0, s = 3$

Substitute into ①

$2 + 3 = -2 - 2(0)$

$5 \neq -2$

$\Rightarrow$ the lines don't intersect

(ii) P is $\begin{pmatrix}2+2\\-1+2\\4-2\end{pmatrix} = \begin{pmatrix}4\\1\\2\end{pmatrix}$ or $4\mathbf{i}+\mathbf{j}+2\mathbf{k}$

(iii) If Q lies on m, $\begin{pmatrix}-2-2t\\2+t\\1+t\end{pmatrix} = \begin{pmatrix}2\\0\\-1\end{pmatrix}$

When $t = -2$

$-2 - 2(-2) = 2$

$2 + -2 = 0$

$1 + -2 = -1$

$\therefore Q$ lies on m

$$\overline{PQ} = \begin{pmatrix}-2\\1-2\\-5+2\end{pmatrix} = \begin{pmatrix}-2\\-1\\-3\end{pmatrix}$$

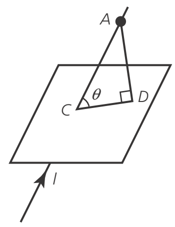
$$\begin{pmatrix}-2\\-1\\-3\end{pmatrix} \cdot \begin{pmatrix}-2\\1\\1\end{pmatrix} = -2(-2) + -1(1) + -3(1)$$
$$= 4 - 1 - 3$$
$$= 0$$

$\therefore PQ$ is perpendicular to m.

4 (i) C is $\begin{pmatrix}2+-2(-1)\\2+-2(2)\\1+-2(1)\end{pmatrix} = \begin{pmatrix}4\\-2\\-1\end{pmatrix}$ or $4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

(ii) $\theta = 65.9°$

(iii)



$$|\overrightarrow{AC}| = \sqrt{(4-2)^2 + (-2-2)^2 + (-1-1)^2}$$
$$= \sqrt{24}$$

$\therefore$ In ΔACD,

$$\sin 24.1° = \frac{AD}{\sqrt{24}}$$

$\therefore AD = \sqrt{24}\sin 24.1°$

$AD = 2$

OR

Using the formula for distance from a point $x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ to plane $ax + by + cz = d$

$$\text{Distance} = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$
$$= \frac{|1(2) + -2(2) + 2(1) - 6|}{\sqrt{1^2 + (-2)^2 + 2^2}}$$
$$= \frac{|2 - 4 + 2 - 6|}{\sqrt{9}}$$
$$= \frac{|-6|}{\sqrt{9}}$$
$$= \frac{6}{3}$$
$$= 2$$

5 (i) $\mathbf{r}=\begin{pmatrix}2\\1\\-2\end{pmatrix}+\lambda\begin{pmatrix}1\\-3\\3\end{pmatrix}$

or $\mathbf{r}=2\mathbf{i}+\mathbf{j}-2\mathbf{k}+\lambda(\mathbf{i}-3\mathbf{j}+3\mathbf{k})$

(ii) P is $\begin{pmatrix}3\\2\\-3\end{pmatrix}+2\begin{pmatrix}1\\-5\\5\end{pmatrix}=\begin{pmatrix}5\\-8\\7\end{pmatrix}$ or $5\mathbf{i}-8\mathbf{j}+7\mathbf{k}$

6 (i) To intersect,

$1+s=4+2t$ ①

$1-s=6+2t$ ②

$1+2s=1+t$ ③

$2=10+4t$ ① + ②

$-8=4t$

$t=-2, s=-1$

Check in ③

$1+2(-1)=1+-2$

$-1=-1$

∴ The lines intersect

(ii) $\theta=74.2°$

(iii) $5x-3y-4z=-2$

7 (i) $\mathbf{r}=\begin{pmatrix}1\\2\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\2\\-2\end{pmatrix}$

(ii) P is $\frac{1}{3}(2\mathbf{i}+5\mathbf{j}+7\mathbf{k})$

(iii) $2x+5y+7z=26$

8 (i) l is parallel to p if normal to plane and vector in the direction of the line are perpendicular

ie $\begin{pmatrix}4\\3\\-2\end{pmatrix}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}=4(2)+3(-2)+-2(1)$
$=8-6-2$
$=0$

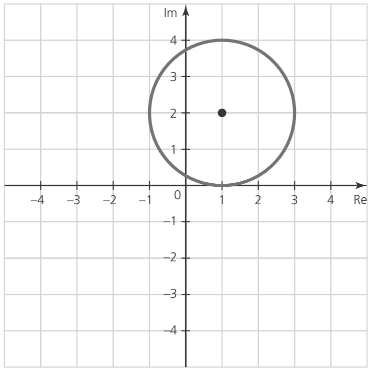
(ii) $a=4$

(iii) $a=13$ or $a=-5$

11 Complex numbers

1 (i) $1+2\text{i}$

(ii)



(iii) 126.9° or 2.21

2 (i) $\frac{1}{2}+\frac{\sqrt{3}}{2}\text{i}$ or $\frac{1}{2}-\frac{\sqrt{3}}{2}\text{i}$

(ii) $|z_1|=1\quad \arg(z_1)=60°$ or $\frac{\pi}{3}$

$|z_2|=1\quad \arg(z_2)=-60°$ or $-\frac{\pi}{3}$

(iii) $z_1{}^3=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3$ $\qquad z_2{}^3=\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^3$

$=\left(1\text{cis}\frac{\pi}{3}\right)^3$ $\qquad =\left(1\text{cis}\left(-\frac{\pi}{3}\right)\right)^3$

$=\text{cis}\pi$ $\qquad =\text{cis}(-\pi)$

$=-1$ $\qquad =-1$

3 (i) $u-v=-3+\text{i}$

$\frac{u}{v}=\frac{1}{2}+\frac{1}{2}\text{i}$

(ii) 45° or $\frac{\pi}{4}$

(iii) OC and BA are equal in length and are parallel

(iv) $\angle AOB=\arg u-\arg v$

$=\arg\left(\frac{u}{v}\right)$

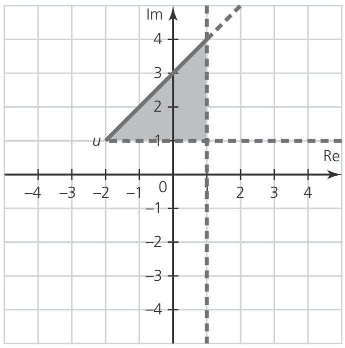
$=\frac{\pi}{4}$

4 (i) $k = 20$

(ii) $-2 - \mathrm{i}$

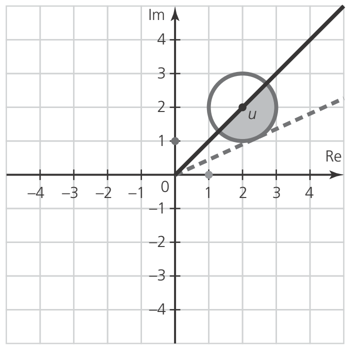
(iii) $|u| = \sqrt{5}$ $\arg(u) = 2.68$ or $153.4°$

(iv)



5 (i) $|u| = 2\sqrt{2}$; $\arg u = \frac{\pi}{4}$

(ii)



(iii) $\sqrt{7}$

6 (i) $|w^2| = 2$ $\arg w^2 = -\frac{\pi}{2}$

$|w^3| = 2\sqrt{2}$ $\arg w^3 = \frac{\pi}{4}$

(ii) $\left|z - \left(-\frac{1}{2} - \frac{1}{2}\mathrm{i}\right)\right| = \frac{\sqrt{10}}{2}$