Chapter 2

Sections 2-1 to 2-4: Transmission-Line Model

Problem 2.1 A transmission line of length l connects a load to a sinusoidal voltage source with an oscillation frequency f. Assuming the velocity of wave propagation on the line is c, for which of the following situations is it reasonable to ignore the presence of the transmission line in the solution of the circuit:

(a) $l = 20$ cm, $f = 10$ kHz,
(b) $l = 50$ km, $f = 60$ Hz,
(c) $l = 20$ cm, $f = 300$ MHz,
(d) $l = 1$ mm, $f = 100$ GHz.

Solution: A transmission line is negligible when $l/\lambda \leq 0.01$.

(a) $$\frac{l}{\lambda} = \frac{lf}{u_p} = \frac{(20 \times 10^{-2}\text{ m}) \times (10 \times 10^{3}\text{ Hz})}{3 \times 10^{8}\text{ m/s}} = 6.67 \times 10^{-6} \text{ (negligible)}.$$

(b) $$\frac{l}{\lambda} = \frac{lf}{u_p} = \frac{(50 \times 10^{3}\text{ m}) \times (60 \times 10^{0}\text{ Hz})}{3 \times 10^{8}\text{ m/s}} = 0.01 \text{ (borderline)}.$$

(c) $$\frac{l}{\lambda} = \frac{lf}{u_p} = \frac{(20 \times 10^{-2}\text{ m}) \times (300 \times 10^{6}\text{ Hz})}{3 \times 10^{8}\text{ m/s}} = 0.20 \text{ (nonnegligible)}.$$

(d) $$\frac{l}{\lambda} = \frac{lf}{u_p} = \frac{(1 \times 10^{-3}\text{ m}) \times (100 \times 10^{9}\text{ Hz})}{3 \times 10^{8}\text{ m/s}} = 0.33 \text{ (nonnegligible)}.$$

Problem 2.2 Calculate the line parameters R', L', G', and C' for a coaxial line with an inner conductor diameter of 0.5 cm and an outer conductor diameter of 1 cm, filled with an insulating material where $\mu = \mu_0$, $\varepsilon_r = 2.25$, and $\sigma = 10^{-3}$ S/m. The conductors are made of copper with $\mu_c = \mu_0$ and $\sigma_c = 5.8 \times 10^{7}$ S/m. The operating frequency is 1 GHz.

Solution: Given

$$a = (0.5/2)\text{ cm} = 0.25 \times 10^{-2}\text{ m},$$
$$b = (1.0/2)\text{ cm} = 0.50 \times 10^{-2}\text{ m},$$

combining Eqs. (2.5) and (2.6) gives

$$\begin{aligned} R' &= \frac{1}{2\pi}\sqrt{\frac{\pi f \mu_c}{\sigma_c}}\left(\frac{1}{a} + \frac{1}{b}\right) \\ &= \frac{1}{2\pi}\sqrt{\frac{\pi(10^{9}\text{ Hz})(4\pi \times 10^{-7}\text{ H/m})}{5.8 \times 10^{7}\text{ S/m}}}\left(\frac{1}{0.25 \times 10^{-2}\text{ m}} + \frac{1}{0.50 \times 10^{-2}\text{ m}}\right) \\ &= 0.788\ \Omega/\text{m}. \end{aligned}$$

From Eq. (2.7),

$$L' = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) = \frac{4\pi\times10^{-7}\text{ H/m}}{2\pi}\ln 2 = 139\text{ nH/m}.$$

From Eq. (2.8),

$$G' = \frac{2\pi\sigma}{\ln(b/a)} = \frac{2\pi\times10^{-3}\text{ S/m}}{\ln 2} = 9.1\text{ mS/m}.$$

From Eq. (2.9),

$$C' = \frac{2\pi\varepsilon}{\ln(b/a)} = \frac{2\pi\varepsilon_r\varepsilon_0}{\ln(b/a)} = \frac{2\pi\times2.25\times(8.854\times10^{-12}\text{ F/m})}{\ln 2} = 181\text{ pF/m}.$$

Problem 2.3 A 1-GHz parallel-plate transmission line consists of 1.5-cm-wide copper strips separated by a 0.2-cm-thick layer of polystyrene. Appendix B gives $\mu_c = \mu_0 = 4\pi\times10^{-7}$ (H/m) and $\sigma_c = 5.8\times10^7$ (S/m) for copper, and $\varepsilon_r = 2.6$ for polystyrene. Use Table 2-1 to determine the line parameters of the transmission line. Assume $\mu = \mu_0$ and $\sigma \simeq 0$ for polystyrene.

Solution:

$$R' = \frac{2R_s}{w} = \frac{2}{w}\sqrt{\pi f\mu_c\sigma_c} = \frac{2}{1.5\times10^{-2}}\left(\frac{\pi\times10^9\times4\pi\times10^{-7}}{5.8\times10^7}\right)^{1/2} = 1.0 \quad (\Omega\text{/m}),$$

$$L' = \frac{\mu d}{w} = \frac{4\pi\times10^{-7}\times2\times10^{-3}}{1.5\times10^{-2}} = 1.67\times10^{-7} \quad \text{(H/m)},$$

$$G' = 0 \qquad \text{because } \sigma = 0,$$

$$C' = \frac{\varepsilon w}{d} = \varepsilon_0\varepsilon_r\frac{w}{d} = \frac{10^{-9}}{36\pi}\times2.6\times\frac{1.5\times10^{-2}}{2\times10^{-3}} = 1.72\times10^{-10} \quad \text{(F/m)}.$$

Problem 2.4 Show that the transmission line model shown in Fig. 2-37 (P2.4) yields the same telegrapher's equations given by Eqs. (2.14) and (2.16).

Solution: The voltage at the central upper node is the same whether it is calculated from the left port or the right port:

$$\begin{aligned} v(z+\tfrac{1}{2}\Delta z,t) &= v(z,t) - \tfrac{1}{2}R'\Delta z\, i(z,t) - \tfrac{1}{2}L'\Delta z\frac{\partial}{\partial t}i(z,t) \\ &= v(z+\Delta z,t) + \tfrac{1}{2}R'\Delta z\, i(z+\Delta z,t) + \tfrac{1}{2}L'\Delta z\frac{\partial}{\partial t}i(z+\Delta z,t). \end{aligned}$$

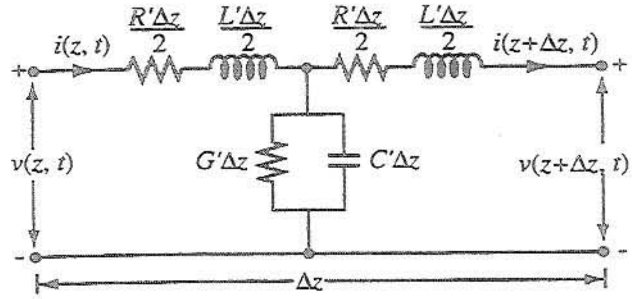

Figure P2.4: Transmission line model.

Recognizing that the current through the $G' \parallel C'$ branch is $i(z,t) - i(z+\Delta z,t)$ (from Kirchhoff's current law), we can conclude that

$$i(z,t) - i(z+\Delta z,t) = G'\Delta z\, v(z+\tfrac{1}{2}\Delta z,t) + C'\Delta z \frac{\partial}{\partial t} v(z+\tfrac{1}{2}\Delta z,t).$$

From both of these equations, the proof is completed by following the steps outlined in the text, ie. rearranging terms, dividing by Δz, and taking the limit as $\Delta z \to 0$.

Problem 2.5 Find α, β, u_p, and Z_0 for the coaxial line of Problem 2.2.

Solution: From Eq. (2.22),

$$\begin{aligned}\gamma &= \sqrt{(R' + j\omega L')(G' + j\omega C')}\\ &= \sqrt{(0.788\ \Omega/\text{m}) + j(2\pi \times 10^9\ \text{s}^{-1})(139 \times 10^{-9}\ \text{H/m})}\\ &\quad \times \sqrt{(9.1 \times 10^{-3}\ \text{S/m}) + j(2\pi \times 10^9\ \text{s}^{-1})(181 \times 10^{-12}\ \text{F/m})}\\ &= (140 \times 10^{-3} + j31.5)\ \text{m}^{-1}.\end{aligned}$$

Thus, from Eqs. (2.25a) and (2.25b), $\alpha = 0.140$ Np/m and $\beta = 31.5$ rad/m.

From Eq. (2.29),

$$\begin{aligned}Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} &= \sqrt{\frac{(0.788\ \Omega/\text{m}) + j(2\pi \times 10^9\ \text{s}^{-1})(139 \times 10^{-9}\ \text{H/m})}{(9.1 \times 10^{-3}\ \text{S/m}) + j(2\pi \times 10^9\ \text{s}^{-1})(181 \times 10^{-12}\ \text{F/m})}}.\\ &= (27.7 + j0.098)\ \Omega.\end{aligned}$$

From Eq. (2.33),

$$u_p = \frac{\omega}{\beta} = \frac{2\pi \times 10^9}{31.5} = 2 \times 10^8\ \text{m/s}.$$

Section 2-5: The Lossless Line

Problem 2.6 In addition to not dissipating power, a lossless line has two important features: (1) it is dispertionless (μ_p is independent of frequency) and (2) its characteristic impedance Z_0 is purely real. Sometimes, it is not possible to design a transmission line such that $R' \ll \omega L'$ and $G' \ll \omega C'$, but it is possible to choose the dimensions of the line and its material properties so as to satisfy the condition

$$R'C' = L'G' \quad \text{(distortionless line)}.$$

Such a line is called a *distortionless* line because despite the fact that it is not lossless, it does nonetheless possess the previously mentioned features of the loss line. Show that for a distortionless line,

$$\alpha = R'\sqrt{\frac{C'}{L'}} = \sqrt{R'G'}, \qquad \beta = \omega\sqrt{L'C'}, \qquad Z_0 = \sqrt{\frac{L'}{C'}}.$$

Solution: Using the distortionless condition in Eq. (2.22) gives

$$\begin{aligned}
\gamma = \alpha + j\beta &= \sqrt{(R' + j\omega L')(G' + j\omega C')} \\
&= \sqrt{L'C'}\sqrt{\left(\frac{R'}{L'} + j\omega\right)\left(\frac{G'}{C'} + j\omega\right)} \\
&= \sqrt{L'C'}\sqrt{\left(\frac{R'}{L'} + j\omega\right)\left(\frac{R'}{L'} + j\omega\right)} \\
&= \sqrt{L'C'}\left(\frac{R'}{L'} + j\omega\right) = R'\sqrt{\frac{C'}{L'}} + j\omega\sqrt{L'C'}.
\end{aligned}$$

Hence,

$$\alpha = \Re\mathfrak{e}(\gamma) = R'\sqrt{\frac{C'}{L'}}, \qquad \beta = \Im\mathfrak{m}(\gamma) = \omega\sqrt{L'C'}, \qquad u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}.$$

Similarly, using the distortionless condition in Eq. (2.29) gives

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} = \sqrt{\frac{L'}{C'}}\sqrt{\frac{R'/L' + j\omega}{G'/C' + j\omega}} = \sqrt{\frac{L'}{C'}}.$$

Problem 2.7 For a distortionless line with $Z_0 = 50\ \Omega$, $\alpha = 40$ (mNp/m), $u_p = 2.5 \times 10^8$ (m/s), find the line parameters and λ at 250 MHz.

Solution: The product of the expressions for α and Z_0 given in Problem 2.6 gives

$$R' = \alpha Z_0 = 40 \times 10^{-3} \times 50 = 2 \quad (\Omega/\text{m}),$$

and taking the ratio of the expression for Z_0 to that for $u_p = \omega/\beta = 1/\sqrt{L'C'}$ gives

$$L' = \frac{Z_0}{u_p} = \frac{50}{2.5 \times 10^8} = 2 \times 10^{-7} \text{ (H/m)} = 200 \quad \text{(nH/m)}.$$

With L' known, we use the expression for Z_0 to find C':

$$C' = \frac{L'}{Z_0^2} = \frac{2 \times 10^{-7}}{(50)^2} = 8 \times 10^{-11} \text{ (F/m)} = 80 \quad \text{(pF/m)}.$$

The distortionless condition given in Problem 2.6 is then used to find G'.

$$G' = \frac{R'C'}{L'} = \frac{2 \times 80 \times 10^{-12}}{2 \times 10^{-7}} = 8 \times 10^{-4} \text{ (S/m)} = 800 \quad (\mu\text{S/m}),$$

and the wavelength is obtained by applying the relation

$$\lambda = \frac{\mu_p}{f} = \frac{2.5 \times 10^8}{250 \times 10^6} = 1 \text{ m}.$$

Problem 2.8 Find α and Z_0 of a distortionless line whose $R' = 4\ \Omega/\text{m}$ and $G' = 4 \times 10^{-4}$ S/m.

Solution: From the equations given in Problem 2.6,

$$\alpha = \sqrt{R'G'} = [4 \times 4 \times 10^{-4}]^{1/2} = 4 \times 10^{-2} \quad \text{(Np/m)},$$

$$Z_0 = \sqrt{\frac{L'}{C'}} = \sqrt{\frac{R'}{G'}} = \left(\frac{4}{4 \times 10^{-4}}\right)^{1/2} = 100\ \Omega.$$

Problem 2.9 A transmission line operating at 125 MHz has $Z_0 = 40\ \Omega$, $\alpha = 0.02$ (Np/m), and $\beta = 0.75$ rad/m. Find the line parameters R', L', G', and C'.

Solution: Given an arbitrary transmission line, $f = 125$ MHz, $Z_0 = 40\ \Omega$, $\alpha = 0.02$ Np/m, and $\beta = 0.75$ rad/m. Since Z_0 is real and $\alpha \neq 0$, the line is distortionless. From Problem 2.6, $\beta = \omega\sqrt{L'C'}$ and $Z_0 = \sqrt{L'/C'}$, therefore,

$$L' = \frac{\beta Z_0}{\omega} = \frac{0.75 \times 40}{2\pi \times 125 \times 10^6} = 38.2 \text{ nH/m}.$$

Then, from $Z_0 = \sqrt{L'/C'}$,

$$C' = \frac{L'}{Z_0^2} = \frac{38.2 \text{ nH/m}}{40^2} = 23.9 \text{ pF/m}.$$

From $\alpha = \sqrt{R'G'}$ and $R'C' = L'G'$,

$$R' = \sqrt{R'G'}\sqrt{\frac{R'}{G'}} = \sqrt{R'G'}\sqrt{\frac{L'}{C'}} = \alpha Z_0 = 0.02 \text{ Np/m} \times 40\ \Omega = 0.8\ \Omega\text{/m}$$

and

$$G' = \frac{\alpha^2}{R'} = \frac{(0.02 \text{ Np/m})^2}{0.8\ \Omega\text{/m}} = 0.5 \text{ mS/m}.$$

Problem 2.10 Using a slotted line, the voltage on a lossless transmission line was found to have a maximum magnitude of 1.5 V and a minimum magnitude of 0.8 V. Find the magnitude of the load's reflection coefficient.

Solution: From the definition of the Standing Wave Ratio given by Eq. (2.59),

$$S = \frac{|\widetilde{V}|_{\max}}{|\widetilde{V}|_{\min}} = \frac{1.5}{0.8} = 1.9.$$

Solving for the magnitude of the reflection coefficient in terms of S, as in Example 2-4,

$$|\Gamma| = \frac{S-1}{S+1} = \frac{1.9-1}{1.9+1} = 0.3.$$

Problem 2.11 Polyethylene with $\varepsilon_r = 2.25$ is used as the insulating material in a lossless coaxial line with characteristic impedance of 50 Ω. The radius of the inner conductor is 1 mm.

(a) What is the radius of the outer conductor?

(b) What is the phase velocity of the line?

Solution: Given a lossless coaxial line, $Z_0 = 50\ \Omega$, $\varepsilon_r = 2.3$, $a = 1$ mm:

(a) From Table 2-2, $Z_0 = (60/\sqrt{\varepsilon_r})\ln(b/a)$ which can be rearranged to give

$$b = ae^{Z_0\sqrt{\varepsilon_r}/60} = (1 \text{ mm})e^{50\sqrt{2.3}/60} = 3.5 \text{ mm}.$$

(b) Also from Table 2-2,

$$u_p = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3 \times 10^8 \text{ m/s}}{\sqrt{2.3}} = 1.98 \times 10^8 \text{ m/s}.$$

Problem 2.12 A 50-Ω lossless transmission line is terminated in a load with impedance $Z_L = (30 - j60)$ Ω. The wavelength is 5 cm. Find:

(a) the reflection coefficient at the load,
(b) the standing-wave ratio on the line,
(c) the position of the voltage maximum nearest the load,
(d) the position of the current maximum nearest the load.

Solution:

(a) From Eq. (2.49a),

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(30 - j60) - 50}{(30 - j60) + 50} = 0.632e^{-j71.6^\circ}.$$

(b) From Eq. (2.59),

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.632}{1 - 0.632} = 4.44.$$

(c) From Eq. (2.56)

$$l_{\max} = \frac{\theta_r \lambda}{4\pi} + \frac{n\lambda}{2} = \frac{-71.6^\circ \times 5 \text{ cm}}{4\pi} \frac{\pi \text{ rad}}{180^\circ} + \frac{n \times 5 \text{ cm}}{2}$$
$$= -0.50 \text{ cm} + 2.50 \text{ cm} = 2.00 \text{ cm}.$$

(d) A current maximum occurs at a voltage minimum, and from Eq. (2.58),

$$l_{\min} = l_{\max} - \lambda/4 = 2.00 \text{ cm} - 5 \text{ cm}/4 = 0.75 \text{ cm}.$$

Problem 2.13 On a 150-Ω lossless transmission line, the following observations were noted: distance of first voltage minimum from the load = 3 cm; distance of first voltage maximum from the load = 9 cm; $S = 3$. Find Z_L.

Solution: Distance between a minimum and an adjacent maximum = $\lambda/4$. Hence,

$$9 \text{ cm} - 3 \text{ cm} = 6 \text{ cm} = \lambda/4,$$

or $\lambda = 24$ cm. Accordingly, the first voltage minimum is at $\ell_{min} = 3 \text{ cm} = \frac{\lambda}{8}$. Application of Eq. (2.57) with $n = 0$ gives

$$\theta_r - 2 \times \frac{2\pi}{\lambda} \times \frac{\lambda}{8} = -\pi,$$

which gives $\theta_r = -\pi/2$.

$$|\Gamma| = \frac{S-1}{S+1} = \frac{3-1}{3+1} = \frac{2}{4} = 0.5.$$

Hence, $\Gamma = 0.5e^{-j\pi/2} = -j0.5$.

Finally,

$$Z_L = Z_0 \left[\frac{1+\Gamma}{1-\Gamma}\right] = 150 \left[\frac{1-j0.5}{1+j0.5}\right] = (90 - j120)\ \Omega.$$

Problem 2.14 Using a slotted line, the following results were obtained: distance of first minimum from the load = 4 cm; distance of second minimum from the load = 14 cm, voltage standing-wave ratio = 2.5. If the line is lossless and $Z_0 = 50\ \Omega$, find the load impedance.

Solution: Following Example 2.5: Given a lossless line with $Z_0 = 50\ \Omega$, $S = 2.5$, $l_{min(0)} = 4$ cm, $l_{min(1)} = 14$ cm. Then

$$l_{min(1)} - l_{min(0)} = \frac{\lambda}{2}$$

or

$$\lambda = 2 \times (l_{min(1)} - l_{min(0)}) = 20 \text{ cm}$$

and

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi \text{ rad/cycle}}{20 \text{ cm/cycle}} = 10\pi \text{ rad/m}.$$

From this we obtain

$$\begin{aligned}\theta_r = 2\beta l_{min(n)} - (2n+1)\pi \text{ rad} &= 2 \times 10\pi \text{ rad/m} \times 0.04 \text{ m} - \pi \text{ rad} \\ &= -0.2\pi \text{ rad} = -36.0^\circ.\end{aligned}$$

Also,

$$|\Gamma| = \frac{S-1}{S+1} = \frac{2.5-1}{2.5+1} = 0.429.$$

So

$$Z_L = Z_0\left(\frac{1+\Gamma}{1-\Gamma}\right) = 50\left(\frac{1+0.429e^{-j36.0^\circ}}{1-0.429e^{-j36.0^\circ}}\right) = (83.3 - j51.4)\ \Omega.$$

Problem 2.15 A load with impedance $Z_L = (25 - j50)\ \Omega$ is to be connected to a lossless transmission line with characteristic impedance Z_0, with Z_0 chosen such that the standing-wave ratio is the smallest possible. What should Z_0 be?

Solution: Since S is monotonic with $|\Gamma|$ (i.e., a plot of S *vs.* $|\Gamma|$ is always increasing), the value of Z_0 which gives the minimum possible S also gives the minimum possible $|\Gamma|$, and, for that matter, the minimum possible $|\Gamma|^2$. A necessary condition for a minimum is that its derivative be equal to zero:

$$0 = \frac{\partial}{\partial Z_0}|\Gamma|^2 = \frac{\partial}{\partial Z_0}\frac{|R_L + jX_L - Z_0|^2}{|R_L + jX_L + Z_0|^2}$$

$$= \frac{\partial}{\partial Z_0}\frac{(R_L - Z_0)^2 + X_L^2}{(R_L + Z_0)^2 + X_L^2} = \frac{4R_L(Z_0^2 - (R_L^2 + X_L^2))}{((R_L + Z_0)^2 + X_L^2)^2}.$$

Therefore, $Z_0^2 = R_L^2 + X_L^2$ or

$$Z_0 = |Z_L| = \sqrt{(25^2 + (-50)^2)} = 55.9\ \Omega.$$

A mathematically precise solution will also demonstrate that this point is a minimum (by calculating the second derivative, for example). Since the endpoints of the range may be local minima or maxima without the derivative being zero there, the endpoints (namely $Z_0 = 0\ \Omega$ and $Z_0 = \infty\ \Omega$) should be checked also.

Problem 2.16 A 50-Ω lossless line terminated in a purely resistive load has a voltage standing wave ratio of 4. Find all possible values of Z_L.

Solution:

$$|\Gamma| = \frac{S-1}{S+1} = \frac{4-1}{4+1} = 0.6.$$

For a purely resistive load, $\theta_r = 0$ or π. For $\theta_r = 0$,

$$Z_L = Z_0\left[\frac{1+\Gamma}{1-\Gamma}\right] = 50\left[\frac{1+0.6}{1-0.6}\right] = 200\ \Omega.$$

For $\theta_r = \pi$, $\Gamma = -0.6$ and

$$Z_L = 50\left[\frac{1-0.6}{1+0.6}\right] = 12.5\ \Omega.$$

Section 2-6: Input Impedance

Problem 2.17 At an operating frequency of 300 MHz, a lossless 50-Ω air-spaced transmission line 2.5 m in length is terminated with an impedance $Z_L = (60 + j20)$ Ω. Find the input impedance.

Solution: Given a lossless transmission line, $Z_0 = 50$ Ω, $f = 300$ MHz, $l = 2.5$ m, and $Z_L = (60 + j20)$ Ω. Since the line is air filled, $u_p = c$ and therefore, from Eq. (2.38),

$$\beta = \frac{\omega}{u_p} = \frac{2\pi \times 300 \times 10^6}{3 \times 10^8} = 2\pi \text{ rad/m}.$$

Since the line is lossless, Eq. (2.69) is valid:

$$Z_{in} = Z_0 \left(\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \right) = 50 \left(\frac{(60 + j20) + j50 \tan(2\pi \text{ rad/m} \times 2.5 \text{ m})}{50 + j(60 + j20) \tan(2\pi \text{ rad/m} \times 2.5 \text{ m})} \right)$$
$$= 50 \left(\frac{(60 + j20) + j50 \times 0}{50 + j(60 + j20) \times 0} \right) = (60 + j20) \text{ Ω}.$$

Problem 2.18 A lossless transmission line of electrical length $l = 0.35\lambda$ is terminated in a load impedance as shown in Fig. 2-38 (P2.18). Find Γ, S, and Z_{in}.

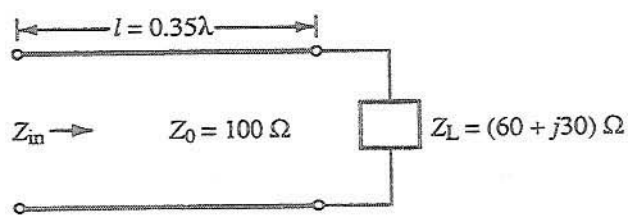

Figure P2.18: Loaded transmission line.

Solution: From Eq. (2.49a),

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(60 + j30) - 100}{(60 + j30) + 100} = 0.307e^{j132.5°}.$$

From Eq. (2.59),

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.307}{1 - 0.307} = 1.89.$$

From Eq. (2.63)

$$Z_{\text{in}} = Z_0 \left(\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l} \right)$$
$$= 100 \left(\frac{(60 + j30) + j100 \tan\left(\frac{2\pi \text{ rad}}{\lambda} 0.35\lambda\right)}{100 + j(60 + j30) \tan\left(\frac{2\pi \text{ rad}}{\lambda} 0.35\lambda\right)} \right) = (64.8 - j38.3)\ \Omega.$$

Problem 2.19 Show that the input impedance of a quarter-wavelength long lossless line terminated in a short circuit appears as an open circuit.

Solution:

$$Z_{\text{in}} = Z_0 \left(\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l} \right).$$

For $l = \frac{\lambda}{4}$, $\beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \frac{\pi}{2}$. With $Z_L = 0$, we have

$$Z_{\text{in}} = Z_0 \left(\frac{jZ_0 \tan\pi/2}{Z_0} \right) = j\infty \quad \text{(open circuit)}.$$

Problem 2.20 Show that at the position where the magnitude of the voltage on the line is a maximum the input impedance is purely real.

Solution: From Eq. (2.56), $l_{\max} = (\theta_r + 2n\pi)/2\beta$, so from Eq. (2.61), using polar representation for Γ,

$$Z_{\text{in}}(-l_{\max}) = Z_0 \left(\frac{1 + |\Gamma| e^{j\theta_r} e^{-j2\beta l_{\max}}}{1 - |\Gamma| e^{j\theta_r} e^{-j2\beta l_{\max}}} \right)$$
$$= Z_0 \left(\frac{1 + |\Gamma| e^{j\theta_r} e^{-j(\theta_r + 2n\pi)}}{1 - |\Gamma| e^{j\theta_r} e^{-j(\theta_r + 2n\pi)}} \right) = Z_0 \left(\frac{1 + |\Gamma|}{1 - |\Gamma|} \right),$$

which is real, provided Z_0 is real.

Problem 2.21 A voltage generator with $v_g(t) = 5\cos(2\pi \times 10^9 t)$ V and internal impedance $Z_g = 50\ \Omega$ is connected to a 50-Ω lossless air-spaced transmission line. The line length is 5 cm and it is terminated in a load with impedance $Z_L = (100 - j100)\ \Omega$. Find

(a) Γ at the load.

(b) Z_{in} at the input to the transmission line.

(c) the input voltage $\widetilde{V}_i$ and input current $\widetilde{I}_i$.

Solution:

(a) From Eq. (2.49a),

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(100 - j100) - 50}{(100 - j100) + 50} = 0.62e^{-j29.7^\circ}.$$

(b) All formulae for Z_{in} require knowledge of $\beta = \omega/u_p$. Since the line is an air line, $u_p = c$, and from the expression for $v_g(t)$ we conclude $\omega = 2\pi \times 10^9$ rad/s. Therefore

$$\beta = \frac{2\pi \times 10^9 \text{ rad/s}}{3 \times 10^8 \text{ m/s}} = \frac{20\pi}{3} \text{ rad/m}.$$

Then, using Eq. (2.63),

$$\begin{aligned} Z_{in} &= Z_0 \left(\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \right) \\ &= 50 \left(\frac{(100 - j100) + j50 \tan\left(\frac{20\pi}{3} \text{ rad/m} \times 5 \text{ cm}\right)}{50 + j(100 - j100) \tan\left(\frac{20\pi}{3} \text{ rad/m} \times 5 \text{ cm}\right)} \right) \\ &= 50 \left(\frac{(100 - j100) + j50 \tan\left(\frac{\pi}{3} \text{ rad}\right)}{50 + j(100 - j100) \tan\left(\frac{\pi}{3} \text{ rad}\right)} \right) = (12.5 - j12.7)\ \Omega. \end{aligned}$$

An alternative solution to this part involves the solution to part (a) and Eq. (2.61).

(c) In phasor domain, $\widetilde{V}_g = 5 \text{ V } e^{j0^\circ}$. From Eq. (2.64),

$$\widetilde{V}_i = \frac{\widetilde{V}_g Z_{in}}{Z_g + Z_{in}} = \frac{5 \times (12.5 - j12.7)}{50 + (12.5 - j12.7)} = 1.40e^{-j34.0^\circ} \quad \text{(V)},$$

and also from Eq. (2.64),

$$\widetilde{I}_i = \frac{\widetilde{V}_i}{Z_{in}} = \frac{1.4e^{-j34.0^\circ}}{(12.5 - j12.7)} = 78.4e^{j11.5^\circ} \quad \text{(mA)}.$$

Problem 2.22 A 6-m section of 150-Ω lossless line is driven by a source with

$$v_g(t) = 5\cos(8\pi \times 10^7 t - 30^\circ) \quad \text{(V)}$$

and $Z_g = 150\ \Omega$. If the line, which has a relative permittivity $\varepsilon_r = 2.25$, is terminated in a load $Z_L = (150 - j50)\ \Omega$, find

(a) λ on the line,

(b) the reflection coefficient at the load,

(c) the input impedance,

(d) the input voltage $\widetilde{V}_i$,
(e) the time-domain input voltage $v_i(t)$.

Solution:

$$v_g(t) = 5\cos(8\pi \times 10^7 t - 30°) \text{ V},$$
$$\widetilde{V}_g = 5e^{-j30°} \text{ V}.$$

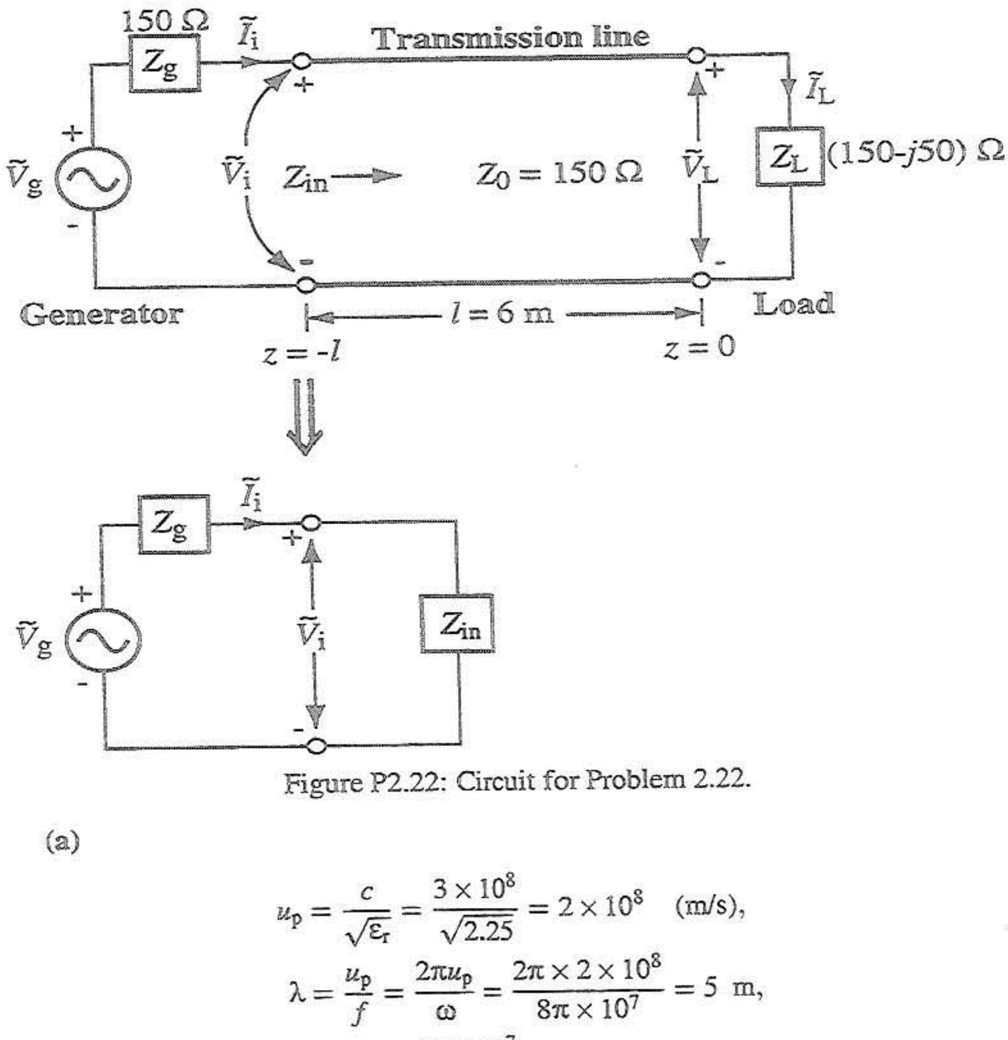

Figure P2.22: Circuit for Problem 2.22.

(a)

$$u_p = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3 \times 10^8}{\sqrt{2.25}} = 2 \times 10^8 \quad \text{(m/s)},$$

$$\lambda = \frac{u_p}{f} = \frac{2\pi u_p}{\omega} = \frac{2\pi \times 2 \times 10^8}{8\pi \times 10^7} = 5 \text{ m},$$

$$\beta = \frac{\omega}{u_p} = \frac{8\pi \times 10^7}{2 \times 10^8} = 0.4\pi \quad \text{(rad/m)},$$

$$\beta l = 0.4\pi \times 6 = 2.4\pi \quad \text{(rad)}.$$

$$\beta l = 0.4\pi \times 6 = 2.4\pi \quad \text{(rad)}.$$

Since this exceeds 2π (rad), we can subtract 2π, which leaves a remainder $\beta l = 0.4\pi$ (rad).

(b) $\Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \dfrac{150 - j50 - 150}{150 - j50 + 150} = \dfrac{-j50}{300 - j50} = 0.16\,e^{-j80.54^\circ}$.

(c)

$$\begin{aligned} Z_{in} &= Z_0 \left[\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}\right] \\ &= 150 \left[\frac{(150 - j50) + j150\tan(0.4\pi)}{150 + j(150 - j50)\tan(0.4\pi)}\right] = (115.70 + j27.42)\ \Omega. \end{aligned}$$

(d)

$$\begin{aligned} \widetilde{V}_i = \frac{\widetilde{V}_g Z_{in}}{Z_g + Z_{in}} &= \frac{5e^{-j30^\circ}(115.7 + j27.42)}{150 + 115.7 + j27.42} \\ &= 5e^{-j30^\circ}\left(\frac{115.7 + j27.42}{265.7 + j27.42}\right) \\ &= 5e^{-j30^\circ} \times 0.44\,e^{j7.44^\circ} = 2.2\,e^{-j22.56^\circ} \quad \text{(V)}. \end{aligned}$$

(e)

$$v_i(t) = \Re\mathfrak{e}[\widetilde{V}_i e^{j\omega t}] = \Re\mathfrak{e}[2.2\,e^{-j22.56^\circ} e^{j\omega t}] = 2.2\cos(8\pi \times 10^7 t - 22.56^\circ)\ \text{V}.$$

Problem 2.23 Two half-wave dipole antennas, each with impedance of 75 Ω, are connected in parallel through a pair of transmission lines, and the combination is connected to a feed transmission line, as shown in Fig. 2.39 (P2.23(a)). All lines are 50 Ω and lossless.

(a) Calculate Z_{in_1}, the input impedance of the antenna-terminated line, at the parallel juncture.

(b) Combine Z_{in_1} and Z_{in_2} in parallel to obtain Z'_L, the effective load impedance of the feedline.

(c) Calculate Z_{in} of the feedline.

Solution:

(a)

$$\begin{aligned} Z_{in_1} &= Z_0 \left[\frac{Z_{L_1} + jZ_0 \tan\beta l_1}{Z_0 + jZ_{L_1} \tan\beta l_1}\right] \\ &= 50 \left\{\frac{75 + j50\tan[(2\pi/\lambda)(0.2\lambda)]}{50 + j75\tan[(2\pi/\lambda)(0.2\lambda)]}\right\} = (35.20 - j8.62)\ \Omega. \end{aligned}$$

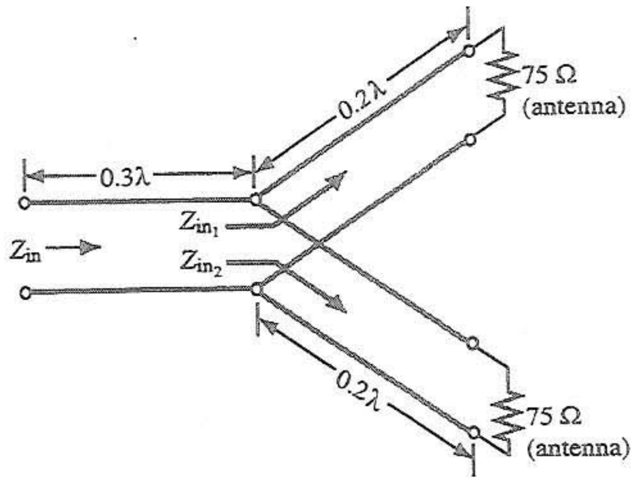

Figure P2.23: (a) Circuit for Problem 2.23.

(b)

$$Z'_L = \frac{Z_{in_1} Z_{in_2}}{Z_{in_1} + Z_{in_2}} = \frac{(35.20 - j8.62)^2}{2(35.20 - j8.62)} = (17.60 - j4.31)\ \Omega.$$

(c)

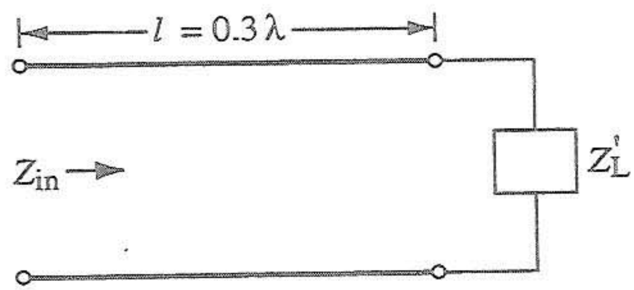

Figure P2.23: (b) Equivalent circuit.

$$Z_{in} = 50 \left\{ \frac{(17.60 - j4.31) + j50 \tan[(2\pi/\lambda)(0.3\lambda)]}{50 + j(17.60 - j4.31) \tan[(2\pi/\lambda)(0.3\lambda)]} \right\} = (107.57 - j56.7)\ \Omega.$$

Section 2-7: Special Cases

Problem 2.24 At an operating frequency of 200 MHz, it is desired to use a section of a lossless 50-Ω transmission line terminated in a short circuit to construct an equivalent load with reactance $X = 25\ \Omega$. If the phase velocity of the line is $0.75c$, what is the shortest possible line length that would exhibit the desired reactance at its input?

Solution:

$$\beta = \omega/u_p = \frac{(2\pi \text{ rad/cycle}) \times (200 \times 10^6 \text{ cycle/s})}{0.75 \times (3 \times 10^8 \text{ m/s})} = 5.59 \text{ rad/m}.$$

On a lossless short-circuited transmission line, the input impedance is always purely imaginary; i.e., $Z_{in}^{sc} = jX_{in}^{sc}$. Solving Eq. (2.68) for the line length,

$$l = \frac{1}{\beta}\tan^{-1}\left(\frac{X_{in}^{sc}}{Z_0}\right) = \frac{1}{5.59 \text{ rad/m}}\tan^{-1}\left(\frac{25\ \Omega}{50\ \Omega}\right) = \frac{(0.464 + n\pi) \text{ rad}}{5.59 \text{ rad/m}},$$

for which the smallest positive solution is 8.3 cm (with $n = 0$).

Problem 2.25 A lossless transmission line is terminated in a short circuit. How long (in wavelengths) should the line be in order for it to appear as an open circuit at its input terminals?

Solution: From Eq. (2.68), $Z_{in}^{sc} = jZ_0 \tan \beta l$. If $\beta l = (\pi/2 + n\pi)$, then $Z_{in}^{sc} = j\infty\ (\Omega)$. Hence,

$$l = \frac{\lambda}{2\pi}\left(\frac{\pi}{2} + n\pi\right) = \frac{\lambda}{4} + \frac{n\lambda}{2}.$$

This is evident from Figure 2.15(d).

Problem 2.26 The input impedance of a 31-cm-long lossless transmission line of unknown characteristic impedance was measured at 1 MHz. With the line terminated in a short circuit, the measurement yielded an input impedance equivalent to an inductor with inductance of 0.128 μH, and when the line was open circuited, the measurement yielded an input impedance equivalent to a capacitor with capacitance of 20 pF. Find Z_0 of the line, the phase velocity, and the relative permittivity of the insulating material.

Solution: Now $\omega = 2\pi f = 6.28 \times 10^6$ rad/s, so

$$Z_{in}^{sc} = j\omega L = j2\pi \times 10^6 \times 0.128 \times 10^{-6} = j0.804\ \Omega$$

and $Z_{\text{in}}^{\text{oc}} = 1/j\omega C = 1/(j2\pi \times 10^6 \times 20 \times 10^{-12}) = -j8000\ \Omega$.

From Eq. (2.74), $Z_0 = \sqrt{Z_{\text{in}}^{\text{sc}} Z_{\text{in}}^{\text{oc}}} = \sqrt{(j0.804\ \Omega)(-j8000\ \Omega)} = 80\ \Omega$. Using Eq. (2.75),

$$u_p = \frac{\omega}{\beta} = \frac{\omega l}{\tan^{-1}\sqrt{-Z_{\text{in}}^{\text{sc}}/Z_{\text{in}}^{\text{oc}}}}$$
$$= \frac{6.28 \times 10^6 \times 0.31}{\tan^{-1}\left(\pm\sqrt{-j0.804/(-j8000)}\right)} = \frac{1.95 \times 10^6}{(\pm 0.01 + n\pi)}\ \text{m/s},$$

where $n \geq 0$ for the plus sign and $n \geq 1$ for the minus sign. For $n = 0$, $u_p = 1.94 \times 10^8$ m/s $= 0.65c$ and $\varepsilon_r = (c/u_p)^2 = 1/0.65^2 = 2.4$. For other values of n, u_p is very slow and ε_r is unreasonably high.

Problem 2.27 A 60-Ω resistive load is preceded by a $\lambda/4$ section of a 50-Ω lossless line, which itself is preceded by another $\lambda/4$ section of a 100-Ω line. What is the input impedance?

Solution: The input impedance of the $\lambda/4$ section of line closest to the load is found from Eq. (2.77):

$$Z_{\text{in}} = \frac{Z_0^2}{Z_L} = \frac{50^2}{60} = 41.7\ \Omega.$$

The input impedance of the line section closest to the load can be considered as the load impedance of the next section of the line. By reapplying Eq. (2.77), the next section of $\lambda/4$ line is taken into account:

$$Z_{\text{in}} = \frac{Z_0^2}{Z_L} = \frac{100^2}{41.7} = 240\ \Omega.$$

Problem 2.28 A 100-MHz FM broadcast station uses a 300-Ω transmission line between the transmitter and a tower-mounted half-wave dipole antenna. The antenna impedance is 73 Ω. You are asked to design a quarter-wave transformer to match the antenna to the line.

(a) Determine the electrical length and characteristic impedance of the quarter-wave section.

(b) If the quarter-wave section is a two-wire line with $d = 2.5$ cm, and the spacing between the wires is made of polystyrene with $\varepsilon_r = 2.6$, determine the physical length of the quarter-wave section and the radius of the two wire conductors.

Solution:

(a) For a match condition, the input impedance of a load must match that of the transmission line attached to the generator. A line of electrical length $\lambda/4$ can be used. From Eq. (2.77), the impedance of such a line should be

$$Z_0 = \sqrt{Z_{\text{in}} Z_{\text{L}}} = \sqrt{300 \times 73} = 148\ \Omega.$$

(b)

$$\frac{\lambda}{4} = \frac{u_{\text{p}}}{4f} = \frac{c}{4\sqrt{\varepsilon_{\text{r}}}f} = \frac{3 \times 10^8}{4\sqrt{2.6} \times 100 \times 10^6} = 0.465\ \text{m},$$

and, from Table 2-2,

$$Z_0 = \frac{120}{\sqrt{\varepsilon}} \ln\left[\left(\frac{d}{2a}\right) + \sqrt{\left(\frac{d}{2a}\right)^2 - 1}\,\right]\ \Omega.$$

Hence,

$$\ln\left[\left(\frac{d}{2a}\right) + \sqrt{\left(\frac{d}{2a}\right)^2 - 1}\,\right] = \frac{148\sqrt{2.6}}{120} = 1.99,$$

which leads to

$$\left(\frac{d}{2a}\right) + \sqrt{\left(\frac{d}{2a}\right)^2 - 1} = 7.31,$$

and whose solution is $a = d/7.44 = 25\ \text{cm}/7.44 = 3.36$ mm.

Problem 2.29 A 50-MHz generator with $Z_{\text{g}} = 50\ \Omega$ is connected to a load $Z_{\text{L}} = (50 - j25)\ \Omega$. The time-average power transferred from the generator into the load is maximum when $Z_{\text{g}} = Z_{\text{L}}^*$, where Z_{L}^* is the complex conjugate of Z_{L}. To achieve this condition without changing Z_{g}, the effective load impedance can be modified by adding an open-circuited line in series with Z_{L}, as shown in Fig. 2-40 (P2.29). If the line's $Z_0 = 100\ \Omega$, determine the shortest length of line (in wavelengths) necessary for satisfying the maximum-power-transfer condition.

Solution: Since the real part of Z_{L} is equal to Z_{g}, our task is to find l such that the input impedance of the line is $Z_{\text{in}} = +j25\ \Omega$, thereby cancelling the imaginary part of Z_{L} (once Z_{L} and the input impedance the line are added in series). Hence, using Eq. (2.73),

$$-j100\cot\beta l = j25,$$

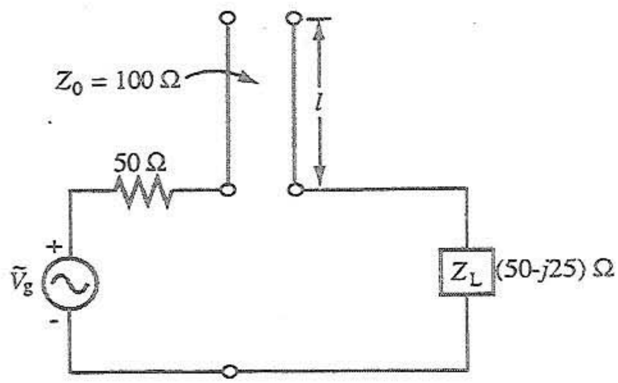

Figure P2.29: Transmission-line arrangement for Problem 2.29.

or

$$\cot\beta l = -\frac{25}{100} = -0.25,$$

which leads to

$$\beta l = -1.326 \text{ or } 1.816.$$

Since l cannot be negative, the first solution is discarded. The second solution leads to

$$l = \frac{1.816}{\beta} = \frac{1.816}{(2\pi/\lambda)} = 0.29\lambda.$$

Problem 2.30 A 50-Ω lossless line of length $l = 0.375\lambda$ connects a 200-MHz generator with $\widetilde{V}_g = 150$ V and $Z_g = 50\ \Omega$ to a load Z_L. Determine the time-domain current through the load for:

(a) $Z_L = (50 - j50)\ \Omega$,

(b) $Z_L = 50\ \Omega$,

(c) $Z_L = 0$ (short circuit).

Solution:

(a) $Z_L = (50 - j50)\ \Omega$, $\beta l = \frac{2\pi}{\lambda} \times 0.375\lambda = 2.36$ (rad) $= 135°$.

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 - j50 - 50}{50 - j50 + 50} = \frac{-j50}{100 - j50} = 0.45\,e^{-j63.43°}.$$

Application of Eq. (2.63) gives:

$$Z_{in} = Z_0\left[\frac{Z_L + jZ_0\tan\beta l}{Z_0 + jZ_L\tan\beta l}\right] = 50\left[\frac{(50 - j50) + j50\tan 135°}{50 + j(50 - j50)\tan 135°}\right] = (100 + j50)\ \Omega.$$

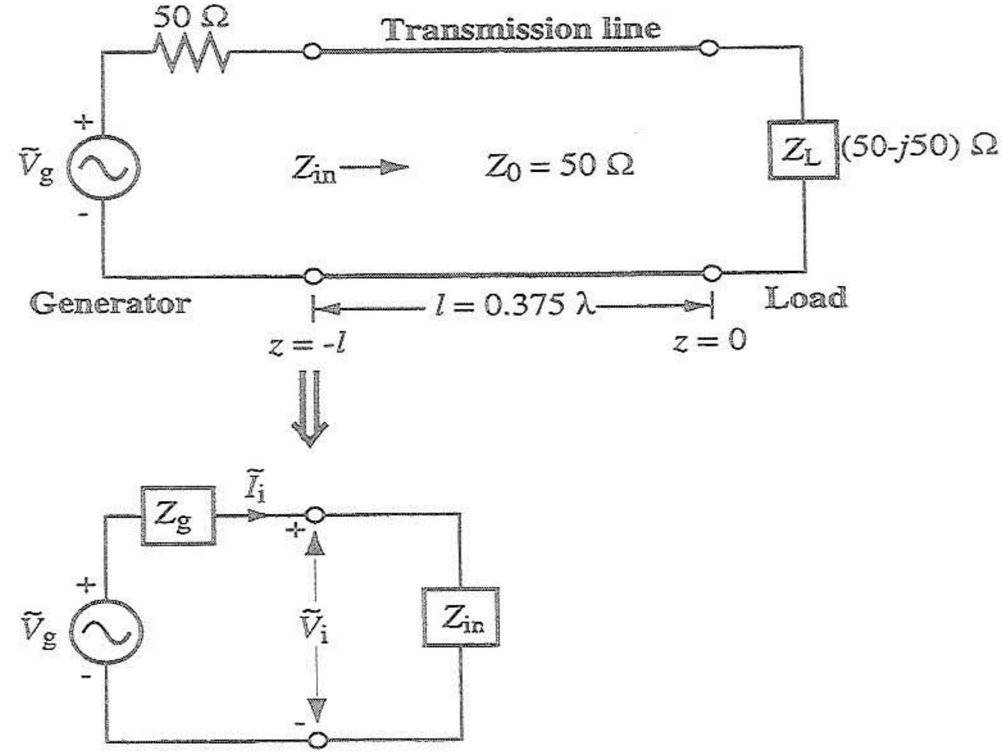

Figure P2.30: Circuit for Problem 2.30(a).

Using Eq. (2.66) gives

$$
\begin{aligned}
V_0^+ &= \left(\frac{\widetilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right) \\
&= \frac{150(100+j50)}{50+(100+j50)}\left(\frac{1}{e^{j135^\circ} + 0.45\,e^{-j63.43^\circ}e^{-j135^\circ}}\right) \\
&= 75\,e^{-j135^\circ} \quad \text{(V)}, \\
\widetilde{I}_L &= \frac{V_0^+}{Z_0}(1-\Gamma) = \frac{75\,e^{-j135^\circ}}{50}(1 - 0.45\,e^{-j63.43^\circ}) = 1.34\,e^{-j108.44^\circ} \quad \text{(A)}, \\
i_L(t) &= \Re\mathrm{e}[\widetilde{I}_L e^{j\omega t}] \\
&= \Re\mathrm{e}[1.34\,e^{-j108.44^\circ} e^{j4\pi\times 10^8 t}] \\
&= 1.34\cos(4\pi\times 10^8 t - 108.44^\circ) \quad \text{(A)}.
\end{aligned}
$$

(b)

$$Z_L = 50\ \Omega,$$
$$\Gamma = 0,$$
$$Z_{in} = \qquad\qquad Z_0 = 50\ \Omega,$$
$$V_0^+ = \frac{150 \times 50}{50+50}\left(\frac{1}{e^{j135^\circ}+0}\right) = 75e^{-j135^\circ} \quad \text{(V)},$$
$$\widetilde{I}_L = \frac{V_0^+}{Z_0} = \frac{75}{50}e^{-j135^\circ} = 1.5e^{-j135^\circ} \quad \text{(A)},$$
$$i_L(t) = \mathfrak{Re}[1.5e^{-j135^\circ}e^{j4\pi\times 10^8 t}] = 1.5\cos(4\pi \times 10^8 t - 135^\circ) \quad \text{(A)}.$$

(c)

$$Z_L = 0,$$
$$\Gamma = -1,$$
$$Z_{in} = Z_0\left(\frac{0 + jZ_0 \tan 135^\circ}{Z_0 + 0}\right) = jZ_0 \tan 135^\circ = -j50 \quad (\Omega),$$
$$V_0^+ = \frac{150(-j50)}{50-j50}\left(\frac{1}{e^{j135^\circ}-e^{-j135^\circ}}\right) = 75e^{-j135^\circ} \quad \text{(V)},$$
$$\widetilde{I}_L = \frac{V_0^+}{Z_0}[1-\Gamma] = \frac{75e^{-j135^\circ}}{50}[1+1] = 3e^{-j135^\circ} \quad \text{(A)},$$
$$i_L(t) = 3\cos(4\pi \times 10^8 t - 135^\circ) \quad \text{(A)}.$$

Section 2-8: Power Flow on Lossless Line

Problem 2.31 A generator with $\widetilde{V}_g = 100$ V and $Z_g = 50\ \Omega$ is connected to a load $Z_L = 75\ \Omega$ through a 50-Ω lossless line of length $l = 0.15\lambda$.

(a) Compute Z_{in}, the input impedance of the line at the generator end.

(b) Compute $\widetilde{I}_i$ and $\widetilde{V}_i$.

(c) Compute the time-average power delivered to the line, $P_{in} = \frac{1}{2}\mathfrak{Re}[\widetilde{V}_i\widetilde{I}_i^*]$.

(d) Compute $\widetilde{V}_L$, $\widetilde{I}_L$, and the time-average power delivered to the load, $P_L = \frac{1}{2}\mathfrak{Re}[\widetilde{V}_L\widetilde{I}_L^*]$. How does P_{in} compare to P_L? Explain.

(e) Compute the time average power delivered by the generator, P_g, and the time average power dissipated in Z_g. Is conservation of power satisfied?

Solution:

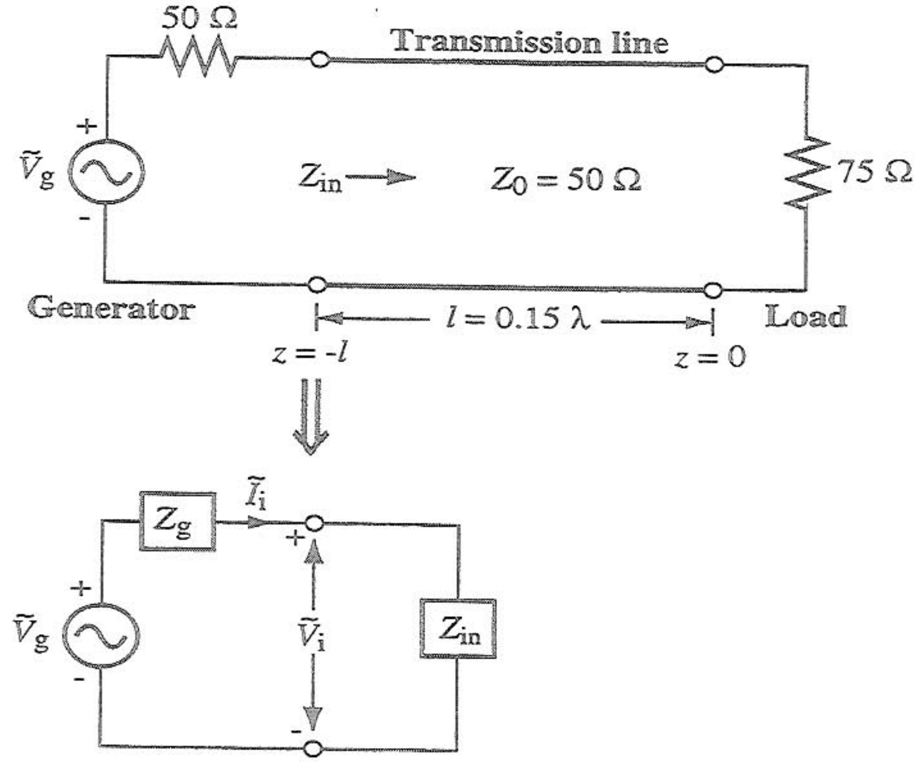

Figure P2.31: Circuit for Problem 2.31.

(a)

$$\beta l = \frac{2\pi}{\lambda} \times 0.15\lambda = 54°,$$
$$Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}\right] = 50 \left[\frac{75 + j50 \tan 54°}{50 + j75 \tan 54°}\right] = (41.25 - j16.35)\ \Omega.$$

(b)

$$\widetilde{I}_i = \frac{\widetilde{V}_g}{Z_g + Z_{in}} = \frac{100}{50 + (41.25 - j16.35)} = 1.08\, e^{j10.16°} \quad \text{(A)},$$
$$\widetilde{V}_i = \widetilde{I}_i Z_{in} = 1.08\, e^{j10.16°}(41.25 - j16.35) = 47.86\, e^{-j11.46°} \quad \text{(V)}.$$

(c)

$$P_{in} = \frac{1}{2}\Re e[\widetilde{V}_i \widetilde{I}_i^*] = \frac{1}{2}\Re e[47.86\, e^{-j11.46^\circ} \times 1.08\, e^{-j10.16^\circ}]$$
$$= \frac{47.86 \times 1.08}{2}\cos(21.62^\circ) = 24 \quad \text{(W)}.$$

(d)

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{75-50}{75+50} = 0.2,$$
$$V_0^+ = \widetilde{V}_i \left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right) = \frac{47.86\, e^{-j11.46^\circ}}{e^{j54^\circ} + 0.2\, e^{-j54^\circ}} = 50e^{-j54^\circ} \quad \text{(V)},$$
$$\widetilde{V}_L = V_0^+(1+\Gamma) = 50e^{-j54^\circ}(1+0.2) = 60e^{-j54^\circ} \quad \text{(V)},$$
$$\widetilde{I}_L = \frac{V_0^+}{Z_0}(1-\Gamma) = \frac{50e^{-j54^\circ}}{50}(1-0.2) = 0.8\, e^{-j54^\circ} \quad \text{(A)},$$
$$P_L = \frac{1}{2}\Re e[\widetilde{V}_L \widetilde{I}_L^*] = \frac{1}{2}\Re e[60e^{-j54^\circ} \times 0.8\, e^{j54^\circ}] = 24 \quad \text{(W)}.$$

$P_L = P_{in}$, which is as expected because the line is lossless; power input to the line ends up in the load.

(e)

Power delivered by generator:

$$P_g = \frac{1}{2}\Re e[\widetilde{V}_g \widetilde{I}_i] = \frac{1}{2}\Re e[100 \times 1.08\, e^{j10.16^\circ}] = 54\cos(10.16^\circ) = 53.15 \quad \text{(W)}.$$

Power dissipated in Z_g:

$$P_{Zg} = \frac{1}{2}\Re e[\widetilde{I}_i \widetilde{V}_{Zg}] = \frac{1}{2}\Re e[\widetilde{I}_i \widetilde{I}_i^* Z_g] = \frac{1}{2}|\widetilde{I}_i|^2 Z_g = \frac{1}{2}(1.08)^2 \times 50 = 29.15 \quad \text{(W)}.$$

Note 1: $P_g = P_{Z_g} + P_{in} = 53.15$ W.

Problem 2.32 If the two-antenna configuration shown in Fig. 2-41 (P2.32) is connected to a generator with $\widetilde{V}_g = 250$ V and $Z_g = 50\ \Omega$, how much average power is delivered to each antenna?

Solution: Since line 2 is $\lambda/2$ in length, the input impedance is the same as $Z_{L_1} = 75\ \Omega$. The same is true for line 3. At junction C–D, we now have two 75-Ω impedances in parallel, whose combination is $75/2 = 37.5\ \Omega$. Line 1 is $\lambda/2$ long. Hence at A–C, input impedance of line 1 is 37.5 Ω, and

$$\widetilde{I}_i = \frac{\widetilde{V}_g}{Z_g + Z_{in}} = \frac{250}{50+37.5} = 2.86 \quad \text{(A)},$$

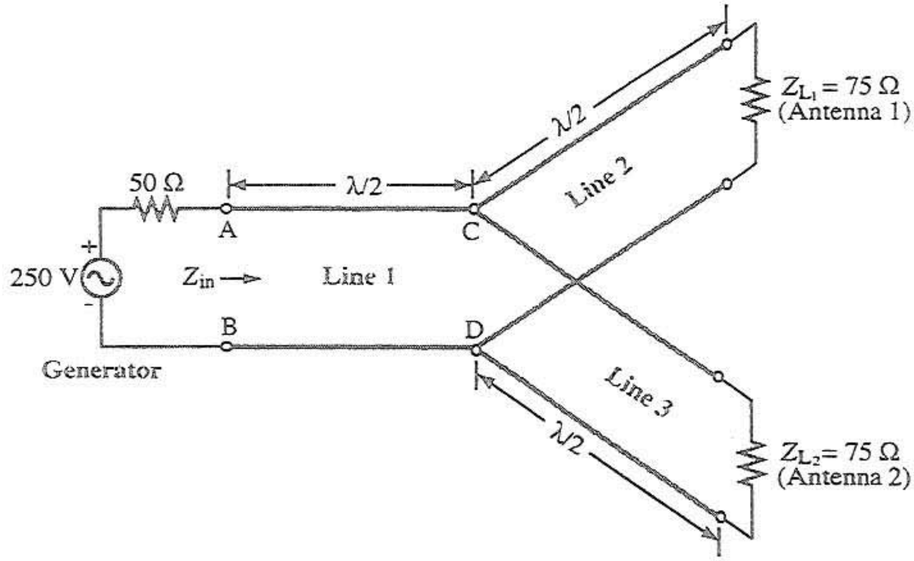

Figure P2.32: Antenna configuration for Problem 2.32.

$$P_{in} = \frac{1}{2}\Re e[\widetilde{I}_i \widetilde{V}_i^*] = \frac{1}{2}\Re e[\widetilde{I}_i \widetilde{I}_i^* \widetilde{Z}_{in}^*] = \frac{(2.86)^2 \times 37.5}{2} = 153.37 \quad \text{(W)}.$$

This is divided equally between the two antennas. Hence, each antenna receives $\frac{153.37}{2} = 76.68$ (W).

Problem 2.33 For the circuit shown in Fig. 2-42 (P2.33), calculate the average incident power, the average reflected power, and the average power transmitted into the infinite 100-Ω line. The $\lambda/2$ line is lossless and the infinitely long line is slightly lossy. (Hint: The input impedance of an infinitely long line is equal to its characteristic impedance so long as $\alpha \neq 0$.)

Solution: Considering the semi-infinite transmission line as equivalent to a load (since all power sent down the line is lost to the rest of the circuit), $Z_L = Z_1 = 100\ \Omega$. Since the feed line is $\lambda/2$ in length, Eq. (2.76) gives $Z_{in} = Z_L = 100\ \Omega$ and $\beta l = (2\pi/\lambda)(\lambda/2) = \pi$, so $e^{\pm j\beta l} = -1$. From Eq. (2.49a),

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100 - 50}{100 + 50} = \frac{1}{3}.$$

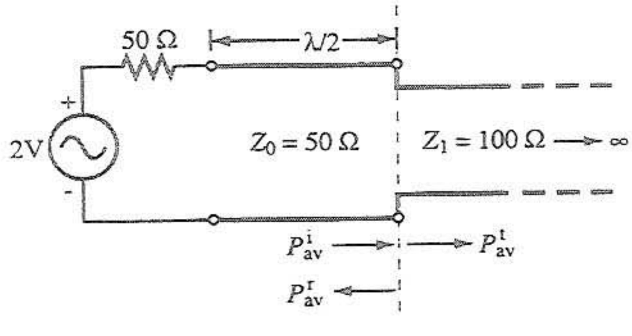

Figure P2.33: Line terminated in an infinite line.

Also, converting the generator to a phasor gives $\widetilde{V}_g = 2e^{j0^\circ}$ (V). Plugging all these results into Eq. (2.66),

$$V_0^+ = \left(\frac{\widetilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right) = \left(\frac{2 \times 100}{50 + 100}\right)\left(\frac{1}{(-1) + \frac{1}{3}(-1)}\right)$$
$$= 1e^{j180^\circ} = -1 \quad \text{(V)}.$$

From Eqs. (2.84), (2.85), and (2.86),

$$P^i_{av} = \frac{|V_0^+|^2}{2Z_0} = \frac{|1e^{j180^\circ}|^2}{2 \times 50} = 10.0 \text{ mW},$$
$$P^r_{av} = -|\Gamma|^2 P^i_{av} = -\left|\frac{1}{3}\right|^2 \times 10 \text{ mW} = -1.1 \text{ mW},$$
$$P^t_{av} = P_{av} = P^i_{av} + P^r_{av} = 10.0 \text{ mW} - 1.1 \text{ mW} = 8.9 \text{ mW}.$$

Problem 2.34 An antenna with a load impedance $Z_L = (75 + j25)\ \Omega$ is connected to a transmitter through a 50-Ω lossless transmission line. If under matched conditions (50-Ω load), the transmitter can deliver 10 W to the load, how much power does it deliver to the antenna? Assume $Z_g = Z_0$.

Solution: From Eqs. (2.66) and (2.61),

$$V_0^+ = \left(\frac{\widetilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right)$$
$$= \left(\frac{\widetilde{V}_g Z_0\left[(1+\Gamma e^{-j2\beta l})/(1-\Gamma e^{-j2\beta l})\right]}{Z_0 + Z_0\left[(1+\Gamma e^{-j2\beta l})/(1-\Gamma e^{-j2\beta l})\right]}\right)\left(\frac{e^{-j\beta l}}{1+\Gamma e^{-j2\beta l}}\right)$$
$$= \frac{\widetilde{V}_g e^{-j\beta l}}{(1-\Gamma e^{-j2\beta l}) + (1+\Gamma e^{-j2\beta l})}$$
$$= \frac{\widetilde{V}_g e^{-j\beta l}}{(1-\Gamma e^{-j2\beta l}) + (1+\Gamma e^{-j2\beta l})} = \tfrac{1}{2}\widetilde{V}_g e^{-j\beta l}.$$

Thus, in Eq. (2.86),

$$P_{av} = \frac{|V_0^+|^2}{2Z_0}(1-|\Gamma|^2) = \frac{|\frac{1}{2}\widetilde{V}_g e^{-j\beta l}|^2}{2Z_0}(1-|\Gamma|^2) = \frac{|\widetilde{V}_g|^2}{8Z_0}(1-|\Gamma|^2).$$

Under the matched condition, $|\Gamma| = 0$ and $P_L = 10$ W, so $|\widetilde{V}_g|^2/8Z_0 = 10$ W.

When $Z_L = (75 + j25)\ \Omega$, from Eq. (2.49a),

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(75+j25)\ \Omega - 50\ \Omega}{(75+j25)\ \Omega + 50\ \Omega} = 0.277e^{j33.6^\circ},$$

so $P_{av} = 10\text{ W}\,(1-|\Gamma|^2) = 10\text{ W}\,(1-0.277^2) = 9.23$ W.

Section 2-9: Smith Chart

Problem 2.35 Use the Smith chart to find the reflection coefficient corresponding to a load impedance:

(a) $Z_L = 3Z_0$,
(b) $Z_L = (2-2j)Z_0$,
(c) $Z_L = -2jZ_0$,
(d) $Z_L = 0$ (short circuit).

Solution: Refer to Fig. P2.35.

(a) Point A is $z_L = 3 + j0$. $\Gamma = 0.5e^{0^\circ}$
(b) Point B is $z_L = 2 - j2$. $\Gamma = 0.62e^{-29.7^\circ}$
(c) Point C is $z_L = 0 - j2$. $\Gamma = 1.0e^{-53.1^\circ}$
(d) Point D is $z_L = 0 + j0$. $\Gamma = 1.0e^{180.0^\circ}$

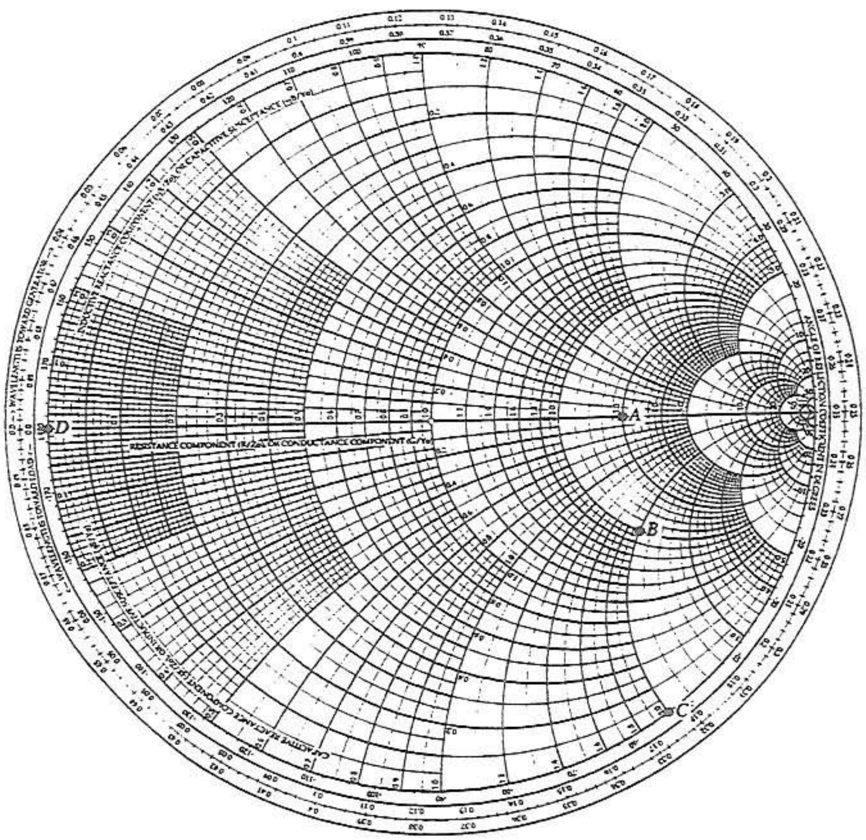

Figure P2.35: Solution of Problem 2.35.

Problem 2.36 Use the Smith chart to find the normalized load impedance corresponding to a reflection coefficient:

(a) $\Gamma = 0.5$,
(b) $\Gamma = 0.5\angle 60^\circ$,
(c) $\Gamma = -1$,
(d) $\Gamma = 0.3\angle -30^\circ$,
(e) $\Gamma = 0$,
(f) $\Gamma = j$.

Solution: Refer to Fig. P2.36.

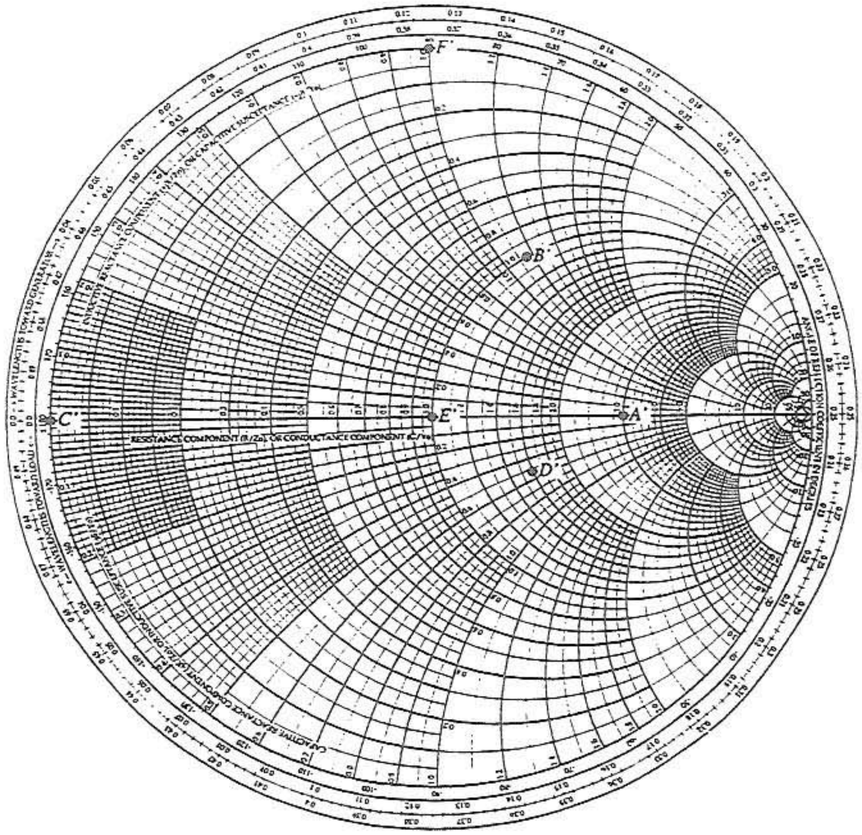

Figure P2.36: Solution of Problem 2.36.

(a) Point A' is $\Gamma = 0.5$ at $z_L = 3 + j0$.
(b) Point B' is $\Gamma = 0.5e^{j60^\circ}$ at $z_L = 1 + j1.15$.
(c) Point C' is $\Gamma = -1$ at $z_L = 0 + j0$.
(d) Point D' is $\Gamma = 0.3e^{-j30^\circ}$ at $z_L = 1.60 - j0.53$.
(e) Point E' is $\Gamma = 0$ at $z_L = 1 + j0$.
(f) Point F' is $\Gamma = j$ at $z_L = 0 + j1$.

Problem 2.37 On a lossless transmission line terminated in a load $Z_L = 100\ \Omega$, the standing-wave ratio was measured to be 2.5. Use the Smith chart to find the two possible values of Z_0.

Solution: Refer to Fig. P2.37. $S = 2.5$ is at point $L1$ and the constant SWR circle is shown. z_L is real at only two places on the SWR circle, at $L1$, where $z_L = S = 2.5$, and $L2$, where $z_L = 1/S = 0.4$. so $Z_{01} = Z_L / z_{L1} = 100\ \Omega/2.5 = 40\ \Omega$ and $Z_{02} = Z_L / z_{L2} = 100\ \Omega/0.4 = 250\ \Omega$.

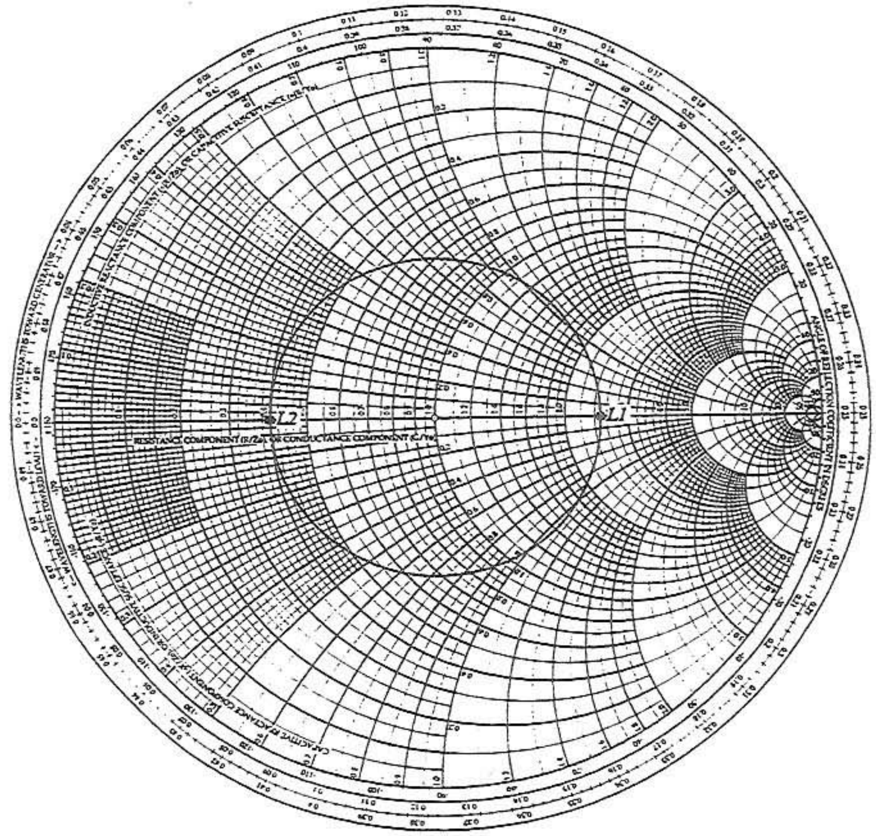

Figure P2.37: Solution of Problem 2.37.

Problem 2.38 A lossless 50-Ω transmission line is terminated in a load with $Z_L = (50 + j25)\ \Omega$. Use the Smith chart to find

(a) the reflection coefficient Γ,

(b) the standing-wave ratio,

(c) the input impedance at 0.35λ from the load,

(d) the input admittance at 0.35λ from the load,
(e) the shortest line length for which the input impedance is purely resistive,
(f) the position of the first voltage maximum from the load.

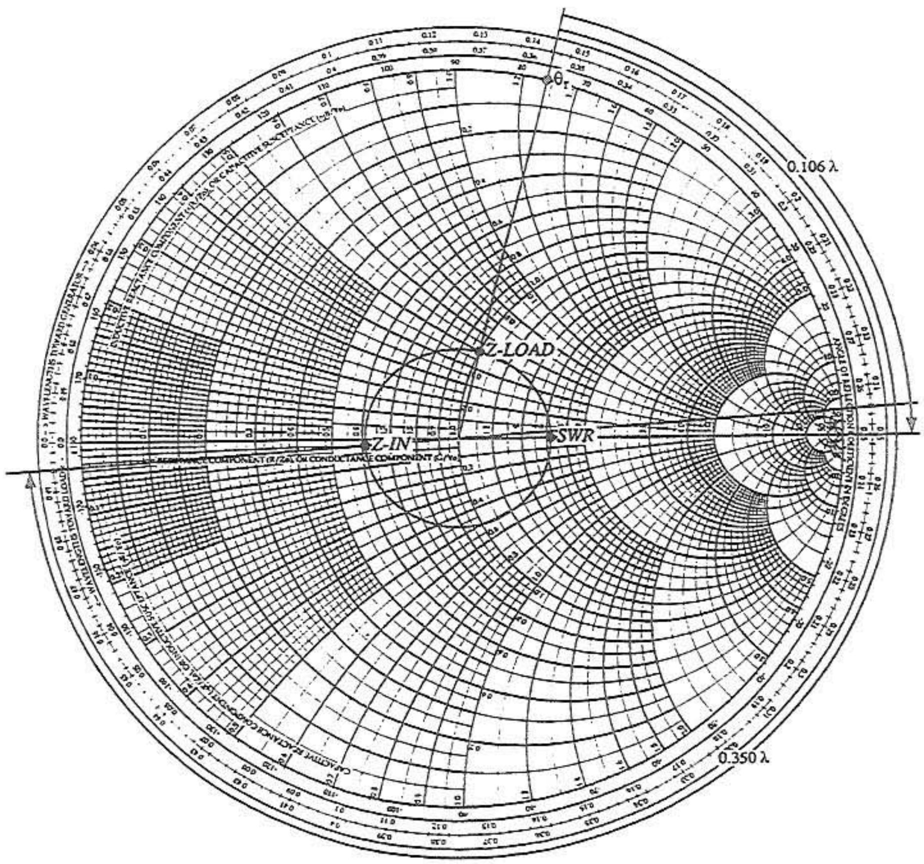

Figure P2.38: Solution of Problem 2.38.

Solution: Refer to Fig. P2.38. The normalized impedance

$$z_L = \frac{(50 + j25)\ \Omega}{50\ \Omega} = 1 + j0.5$$

is at point *Z-LOAD*.

(a) $\Gamma = 0.24e^{j76.0^\circ}$ The angle of the reflection coefficient is read of that scale at the point θ_r.