Chapter 4

Sections 4-2: Charge and Current Distributions

Problem 4.1 A cube 2 m on a side is located in the first octant in a Cartesian coordinate system, with one of its corners at the origin. Find the total charge contained in the cube if the charge density is given by $\rho_v = xy^2 e^{-2z}$ (mC/m^3).

Solution: For the cube shown in Fig. P4.1, application of Eq. (4.5) gives

$$Q = \int_{\mathcal{V}} \rho_v \, d\mathcal{V} = \int_{x=0}^{2} \int_{y=0}^{2} \int_{z=0}^{2} xy^2 e^{-2z} \, dx \, dy \, dz$$

$$= \left(\frac{-1}{12} x^2 y^3 e^{-2z} \right) \Bigg|_{x=0}^{2} \Bigg|_{y=0}^{2} \Bigg|_{z=0}^{2} = \frac{8}{3}(1 - e^{-4}) = 2.62 \text{ mC}.$$

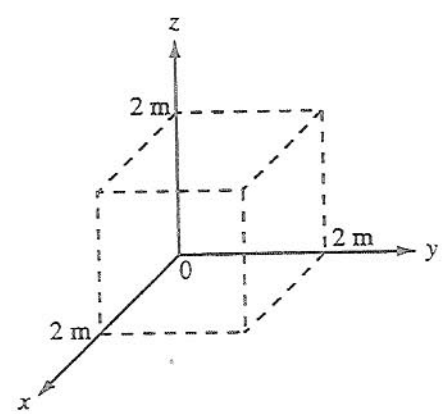


Figure P4.1: Cube of Problem 4.1.

Problem 4.2 Find the total charge contained in a cylindrical volume defined by $r \le 2$ m and $0 \le z \le 3$ m if $\rho_v = 10rz$ (mC/m^3).

Solution: For the cylinder shown in Fig. P4.2, application of Eq. (4.5) gives

$$Q = \int_{z=0}^{3} \int_{\phi=0}^{2\pi} \int_{r=0}^{2} 10rz \, r \, dr \, d\phi \, dz$$

$$= \left(\frac{5}{3} r^3 \phi z^2 \right) \Bigg|_{r=0}^{2} \Bigg|_{\phi=0}^{2\pi} \Bigg|_{z=0}^{3} = 240\pi \text{ (mC)} = 0.754 \text{ C}.$$

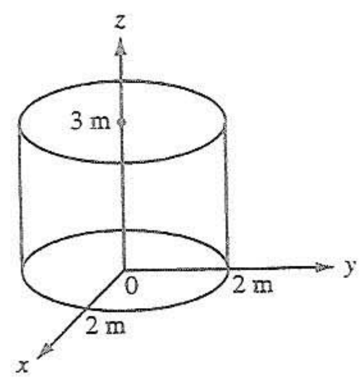


Figure P4.2: Cylinder of Problem 4.2.

Problem 4.3 Find the total charge contained in a cone defined by $R \leq 2$ m and $0 \leq \theta \leq \pi/4$, given that $\rho_v = 20R^2 \cos^2\theta$ (mC/m^3).

Solution: For the cone of Fig. P4.3, application of Eq. (4.5) gives

$$\begin{aligned} Q &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/4} \int_{R=0}^{2} 20R^2 \cos^2\theta \, R^2 \sin\theta \, dR \, d\theta \, d\phi \\ &= \left(\frac{-4}{3} R^5 \phi \cos^3\theta \right) \Bigg|_{R=0}^{2} \Bigg|_{\theta=0}^{\pi/4} \Bigg|_{\phi=0}^{2\pi} \\ &= \frac{256\pi}{3} \left(1 - \left(\frac{\sqrt{2}}{2} \right)^3 \right) = 173.3 \text{ (mC)} = 0.173 \text{ C}. \end{aligned}$$

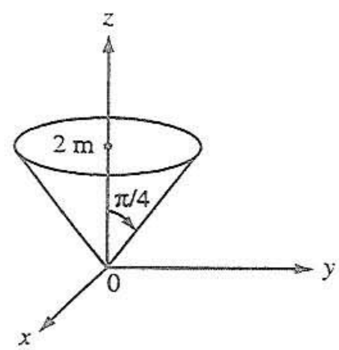


Figure P4.3: Cone of Problem 4.3.

Problem 4.4 If the line charge density is given by $\rho_l = 12y^2$ (mC/m), find the total charge distributed on the y-axis from $y = -5$ to $y = 5$.

Solution:

$$Q = \int_{-5}^{5} \rho_l \, dy = \int_{-5}^{5} 12y^2 \, dy = \left. \frac{12y^3}{3} \right|_{-5}^{5} = 1000 \text{ mC} = 1 \text{ C}.$$

Problem 4.5 Find the total charge on a circular disk defined by $r \leq a$ and $z = 0$ if:
(a) $\rho_s = \rho_{s0} \sin\phi$ (C/m^2),
(b) $\rho_s = \rho_{s0} \sin^2\phi$ (C/m^2),
(c) $\rho_s = \rho_{s0} e^{-r}$ (C/m^2),
(d) $\rho_s = \rho_{s0} e^{-r} \sin^2\phi$ (C/m^2),
where ρ_{s0} is a constant.

Solution:
(a)

$$Q = \int \rho_s \, ds = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} \sin\phi \; r \, dr \, d\phi = -\rho_{s0} \left. \frac{r^2}{2} \right|_0^a \cos\phi \Big|_0^{2\pi} = 0.$$

(b)

$$Q = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} \sin^2\phi \; r \, dr \, d\phi = \rho_{s0} \left. \frac{r^2}{2} \right|_0^a \int_0^{2\pi} \left(\frac{1 - \cos 2\phi}{2} \right) d\phi$$

$$= \frac{\rho_{s0} a^2}{4} \left(\phi - \frac{\sin 2\phi}{2} \right) \Bigg|_0^{2\pi} = \frac{\pi a^2}{2} \rho_{s0}.$$

(c)

$$Q = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} e^{-r} r\,dr\,d\phi = 2\pi\rho_{s0} \int_{0}^{a} re^{-r}\,dr$$
$$= 2\pi\rho_{s0} \left[-re^{-r} - e^{-r}\right]_{0}^{a}$$
$$= 2\pi\rho_{s0}[1 - e^{-a}(1+a)].$$

(d)

$$Q = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} e^{-r} \sin^2\phi \; r\,dr\,d\phi$$
$$= \rho_{s0} \int_{r=0}^{a} re^{-r}\,dr \int_{\phi=0}^{2\pi} \sin^2\phi\,d\phi$$
$$= \rho_{s0}[1 - e^{-a}(1+a)] \cdot \pi = \pi\rho_{s0}[1 - e^{-a}(1+a)].$$

Problem 4.6 If $\mathbf{J} = \hat{\mathbf{y}}2xz$ (A/m^2), find the current I flowing through a square with corners at $(0,0,0)$, $(2,0,0)$, $(2,0,2)$, and $(0,0,2)$.

Solution: Using Eq. (4.12), the net current flowing through the square shown in Fig. P4.6 is

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} = \int_{x=0}^{2} \int_{z=0}^{2} (\hat{\mathbf{y}}2xz)\bigg|_{y=0} \cdot (\hat{\mathbf{y}}\,dx\,dz) = \left(\frac{x^2z^2}{2}\right)\bigg|_{x=0}^{2}\bigg|_{z=0}^{2} = 8 \text{ A}.$$

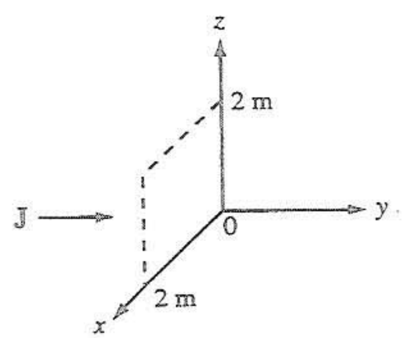


Figure P4.6: Square surface.

Problem 4.7 If $\mathbf{J} = \hat{\mathbf{R}}25/R$ (A/m^2), find I through the surface $R = 5$ m.

Solution: Using Eq. (4.12), we have

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left(\hat{\mathbf{R}} \frac{25}{R} \right) \cdot (\hat{\mathbf{R}} R^2 \sin\theta \, d\theta \, d\phi)$$

$$= -25R\phi \cos\theta \Bigg|_{R=5} \Bigg|_{\theta=0}^{\pi} \Bigg|_{\phi=0}^{2\pi} = 500\pi = 1,570.8 \quad \text{(A)}.$$

Problem 4.8 An electron beam shaped like a circular cylinder of radius r_0 carries a charge density given by

$$\rho_v = \left(\frac{-\rho_0}{1+r^2} \right) \quad \text{(C/m}^3\text{)},$$

where ρ_0 is a positive constant and the beam's axis is coincident with the z-axis.

(a) Determine the total charge contained in length L of the beam.

(b) If the electrons are moving in the $+z$-direction with uniform speed u, determine the magnitude and direction of the current crossing the z-plane.

Solution:

(a)

$$Q = \int_{r=0}^{r_0} \int_{z=0}^{L} \rho_v \, d\mathcal{V} = \int_{r=0}^{r_0} \int_{z=0}^{L} \left(\frac{-\rho_0}{1+r^2} \right) 2\pi r \, dr \, dz$$

$$= -2\pi\rho_0 L \int_0^{r_0} \frac{r}{1+r^2} \, dr = -\pi\rho_0 L \ln(1+r_0^2).$$

(b)

$$\mathbf{J} = \rho_v \mathbf{u} = -\hat{\mathbf{z}} \frac{u\rho_0}{1+r^2} \quad \text{(A/m}^2\text{)},$$

$$I = \int \mathbf{J} \cdot d\mathbf{s}$$

$$= \int_{r=0}^{r_0} \int_{\phi=0}^{2\pi} \left(-\hat{\mathbf{z}} \frac{u\rho_0}{1+r^2} \right) \cdot \hat{\mathbf{z}} r \, dr \, d\phi$$

$$= -2\pi u\rho_0 \int_0^{r_0} \frac{r}{1+r^2} \, dr = -\pi u\rho_0 \ln(1+r_0^2) \quad \text{(A)}.$$

Current direction is along $-\hat{\mathbf{z}}$.

Section 4-3: Coulomb's Law

Problem 4.9 A square with sides 2 m each has a charge of 20 μC at each of its four corners. Determine the electric field at a point 5 m above the center of the square.

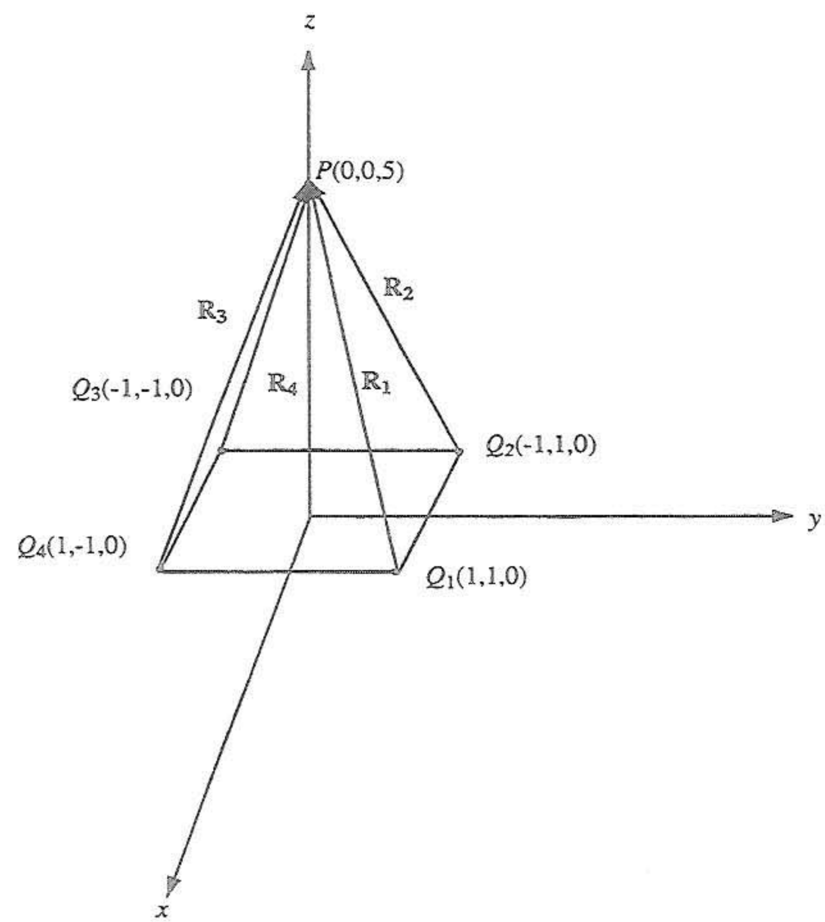


Figure P4.9: Square with charges at the corners.

Solution: The distance $|R|$ between any of the charges and point P is

$|R| = \sqrt{1^2+1^2+5^2} = \sqrt{27}$.

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{\mathbf{R}_1}{|\mathbf{R}|^3}+\frac{\mathbf{R}_2}{|\mathbf{R}|^3}+\frac{\mathbf{R}_3}{|\mathbf{R}|^3}+\frac{\mathbf{R}_4}{|\mathbf{R}|^3}\right]$$
$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{-\hat{\mathbf{x}}-\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}+\frac{\hat{\mathbf{x}}-\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}+\frac{-\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}+\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}\right]$$
$$= \hat{\mathbf{z}}\frac{5Q}{(27)^{3/2}\pi\varepsilon_0} = \hat{\mathbf{z}}\frac{5\times 20\ \mu\text{C}}{(27)^{3/2}\pi\varepsilon_0} = \frac{0.71}{\pi\varepsilon_0}\times 10^{-6}\ \text{(V/m)} = \hat{\mathbf{z}}25.61\ \text{(kV/m)}.$$

Problem 4.10 Three point charges, each with $q = 3$ nC, are located at the corners of a triangle in the x–y plane, with one corner at the origin, another at $(2\text{ cm}, 0, 0)$, and the third at $(0, 2\text{ cm}, 0)$. Find the force acting on the charge located at the origin.

Solution: Use Eq. (4.19) to determine the electric field at the origin due to the other two point charges [Fig. P4.10]:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\left[\frac{3\text{nC}\,(-\hat{\mathbf{x}}0.02)}{(0.02)^3}\right]+\frac{3\text{nC}\,(-\hat{\mathbf{y}}0.02)}{(0.02)^3} = -67.4(\hat{\mathbf{x}}+\hat{\mathbf{y}})\ \text{(kV/m) at } \mathbf{R}=0.$$

Employ Eq. (4.14) to find the force $\mathbf{F} = q\mathbf{E} = -202.2(\hat{\mathbf{x}}+\hat{\mathbf{y}})$ (μN).

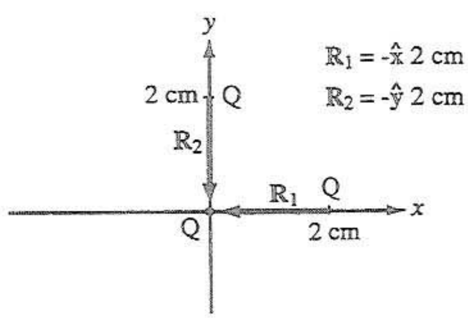


Figure P4.10: Locations of charges in Problem 4.10.

Problem 4.11 Charge $q_1 = 4\ \mu$C is located at $(1\text{ cm}, 1\text{ cm}, 0)$ and charge q_2 is located at $(0, 0, 4\text{ cm})$. What should q_2 be so that $\mathbf{E}$ at $(0, 2\text{ cm}, 0)$ has no y-component?

Solution: For the configuration of Fig. P4.11, use of Eq. (4.19) gives

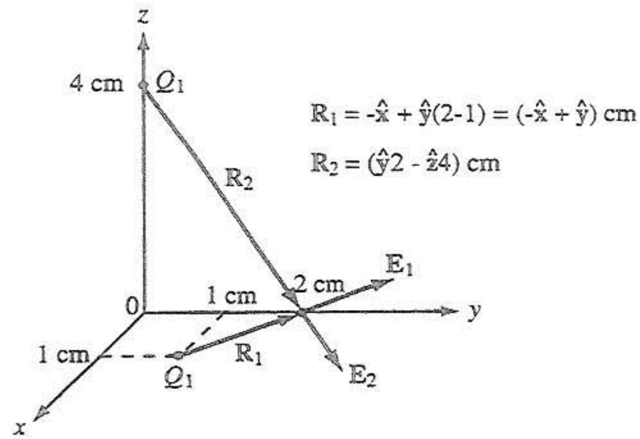


Figure P4.11: Locations of charges in Problem 4.11.

$$\begin{aligned}\mathbf{E}(\mathbf{R}=\hat{\mathbf{y}}2\text{cm}) &= \frac{1}{4\pi\varepsilon}\left[\frac{4\mu\text{C}(-\hat{\mathbf{x}}+\hat{\mathbf{y}})\times 10^{-2}}{(2\times 10^{-2})^{3/2}}+\frac{q_2(\hat{\mathbf{y}}2-\hat{\mathbf{z}}4)\times 10^{-2}}{(20\times 10^{-2})^{3/2}}\right]\\ &= \frac{1}{4\pi\varepsilon}[-\hat{\mathbf{x}}14.14\times 10^{-6}+\hat{\mathbf{y}}(14.14\times 10^{-6}+0.224q_2)\\ &\quad -\hat{\mathbf{z}}0.447q_2] \quad \text{(V/m)}.\end{aligned}$$

If $E_y = 0$, then $q_2 = -14.14\times 10^{-6}/0.224 \approx -63.13$ (μC).

Problem 4.12 A line of charge with uniform density $\rho_l = 4$ (μC/m) exists in air along the z-axis between $z = 0$ and $z = 5$ cm. Find $\mathbf{E}$ at (0,10 cm,0).

Solution: Use of Eq. (4.21c) for the line of charge shown in Fig. P4.12 gives

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\varepsilon_0}\int_{l'}\hat{\mathbf{R}}'\frac{\rho_l\,dl'}{R'^2},\\ R' &= \hat{\mathbf{y}}0.1-\hat{\mathbf{z}}z\\ &= \frac{1}{4\pi\varepsilon_0}\int_{z=0}^{0.05}(4\times 10^{-6})\frac{(\hat{\mathbf{y}}0.1-\hat{\mathbf{z}}z)}{[(0.1)^2+z^2]^{3/2}}\,dz\\ &= \frac{4\times 10^{-6}}{4\pi\varepsilon_0}\left[\frac{\hat{\mathbf{y}}10z+\hat{\mathbf{z}}}{\sqrt{(0.1)^2+z^2}}\right]\Bigg|_{z=0}^{0.05}\\ &= 35.93\times 10^3[\hat{\mathbf{y}}4.47-\hat{\mathbf{z}}1.06] = \hat{\mathbf{y}}160.7\times 10^3-\hat{\mathbf{z}}38.1\times 10^3 \quad \text{(V/m)}.\end{aligned}$$

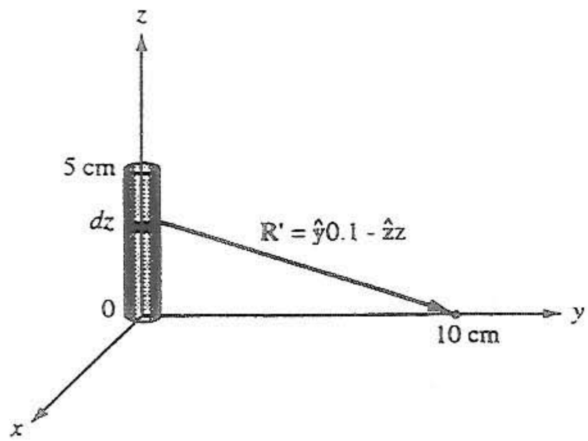


Figure P4.12: Line charge.

Problem 4.13 Electric charge is distributed along an arc located in the x–y plane and defined by $r = 2$ cm and $0 \le \phi \le \pi/4$. If $\rho_l = 5$ (μC/m), find $\mathbf{E}$ at $(0,0,z)$ and then evaluate it at (a) the origin, (b) $z = 5$ cm, and (c) $z = -5$ cm.

Solution: For the arc of charge shown in Fig. P4.13, $dl = r\,d\phi = 0.02\,d\phi$, and $\mathbf{R}' = -\hat{\mathbf{r}}0.02 + \hat{\mathbf{z}}z$. Use of Eq. (4.21c) gives

$$\mathbf{E} = \int \frac{\hat{\mathbf{R}}\,dq}{4\pi\varepsilon R^2} = \int \mathbf{R}\,\frac{\rho_l r\,dl}{4\pi\varepsilon R^3} = \int_0^{\pi/4} \frac{(-\hat{\mathbf{r}}r + \hat{\mathbf{z}}z)\rho_l r}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}}\,d\phi$$

Since $\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$

$$\begin{aligned}
\mathbf{E} &= \left\{ \int_0^{\pi/4} -\hat{\mathbf{x}}r\cos\phi - \hat{\mathbf{y}}r\sin\phi\,d\phi + \hat{\mathbf{z}}\frac{\pi}{4}z \right\} \frac{\rho_l r}{4\pi\varepsilon_0(r^2+z^2)^{3/2}} \\
&= \left\{ (-\sin\phi)r\hat{\mathbf{x}}\Big|_0^{\pi/4} + (\cos\phi)r\hat{\mathbf{y}}\Big|_0^{\pi/4} + \frac{\pi}{4}z\hat{\mathbf{z}} \right\} \frac{\rho_l r}{4\pi\varepsilon_0(r^2+z^2)^{3/2}} \\
&= \left\{ \left(-\frac{\sqrt{2}}{2}\right)(0.02)\hat{\mathbf{x}} + \left(\frac{\sqrt{2}}{2} - 1\right)(0.02)\hat{\mathbf{y}} + \left(\frac{\pi}{4}\right)z\hat{\mathbf{z}} \right\} \frac{899}{(r^2+z^2)^{3/2}}
\end{aligned}$$

(a) At the origin, $\mathbf{E} = -\hat{\mathbf{x}}1.6 - \hat{\mathbf{y}}0.66$ MV/m
(b) At $z = 5$ cm, $\mathbf{E} = -\hat{\mathbf{x}}81.4 - \hat{\mathbf{y}}33.7 + \hat{\mathbf{z}}226$ kV/m
(c) At $z = -5$ cm, $\mathbf{E} = -\hat{\mathbf{x}}81.4 - \hat{\mathbf{y}}33.7 - \hat{\mathbf{z}}226$ kV/m

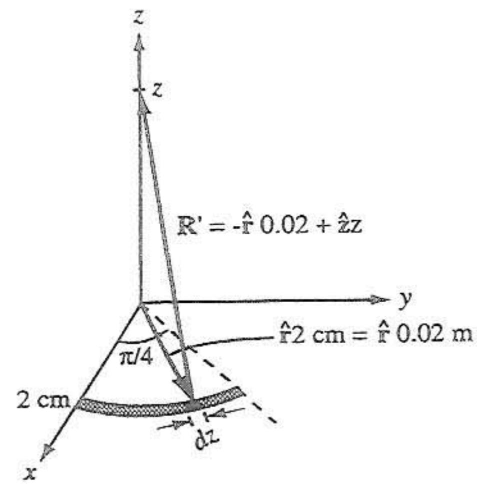


Figure P4.13: Line charge along an arc.

Problem 4.14 A line of charge with uniform density ρ_l extends between $z = -L/2$ and $z = L/2$ along the z-axis. Apply Coulomb's law to obtain an expression for the electric field at any point $P(r, \phi, 0)$ on the x–y plane. Show that your result reduces to the expression given by Eq. (4.33) as the length L is extended to infinity.

Solution: Consider an element of charge of height dz at height z. Call it element 1. The electric field at P due to this element is $d\mathbf{E}_1$. Similarly, an element at $-z$ produces $d\mathbf{E}_2$. These two electric fields have equal z-components, but in opposite directions, and hence they will cancel. Their components along $\hat{\mathbf{r}}$ will add. Thus, the net field due to both elements is

$$d\mathbf{E} = d\mathbf{E}_1 + d\mathbf{E}_2 = \hat{\mathbf{r}}\frac{2\rho_l \cos\theta \, dz}{4\pi\varepsilon_0 R^2} = \frac{\hat{\mathbf{r}}\rho_l \cos\theta \, dz}{2\pi\varepsilon_0 R^2}.$$

where the $\cos\theta$ factor provides the components of $d\mathbf{E}_1$ and $d\mathbf{E}_2$ along $\hat{\mathbf{r}}$.

Our integration variable is z, but it will be easier to integrate over the variable θ from $\theta = 0$ to

$$\theta_0 = \sin^{-1}\frac{L/2}{\sqrt{r^2 + (L/2)^2}}.$$

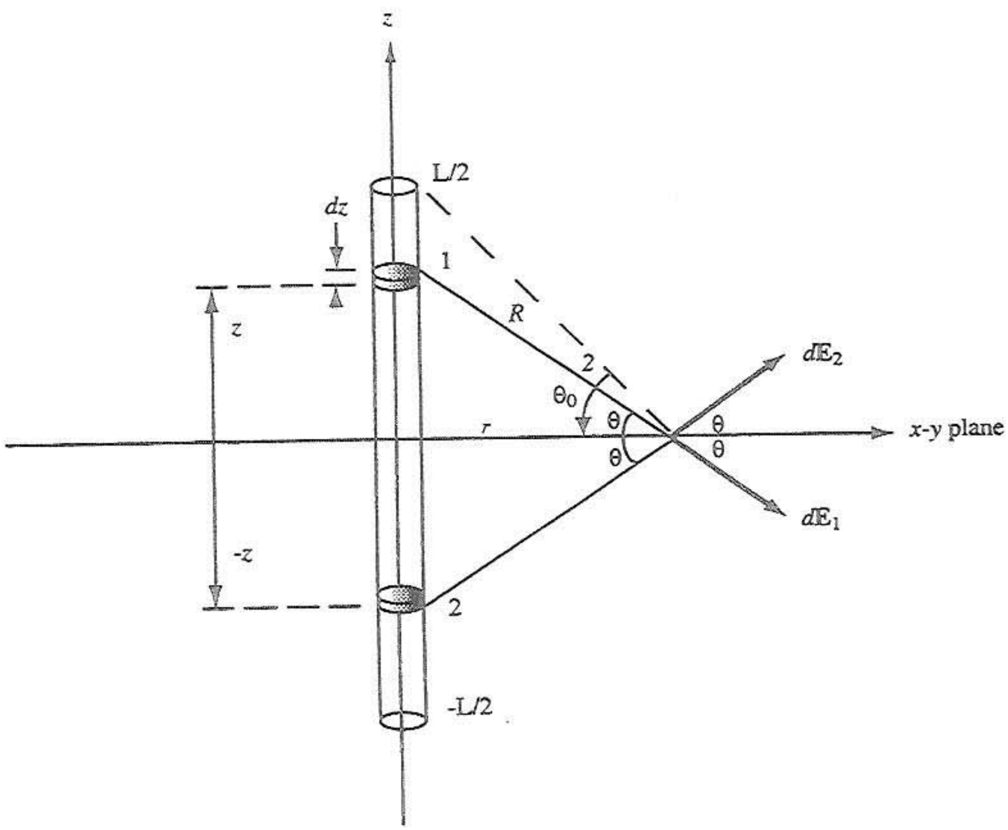


Figure P4.14: Line charge of length L.

Hence, with $R = r/\cos\theta$, and $z = r\tan\theta$ and $dz = r\sec^2\theta\, d\theta$, we have

$$\mathbf{E} = \int_{z=0}^{L/2} d\mathbf{E} = \int_{\theta=0}^{\theta_0} d\mathbf{E} = \int_0^{\theta_0} \hat{\mathbf{r}}\, \frac{\rho_l}{2\pi\varepsilon_0} \frac{\cos^3\theta}{r^2}\, r\sec^2\theta\, d\theta$$

$$= \hat{\mathbf{r}}\, \frac{\rho_l}{2\pi\varepsilon_0 r} \int_0^{\theta_0} \cos\theta\, d\theta$$

$$= \hat{\mathbf{r}}\, \frac{\rho_l}{2\pi\varepsilon_0 r} \sin\theta_0 = \hat{\mathbf{r}}\, \frac{\rho_l}{2\pi\varepsilon_0 r} \frac{L/2}{\sqrt{r^2 + (L/2)^2}}.$$

For $L \gg r$,

$$\frac{L/2}{\sqrt{r^2 + (L/2)^2}} \approx 1,$$

and

$$\mathbf{E} = \hat{\mathbf{r}}\,\frac{\rho_l}{2\pi\varepsilon_0 r} \quad \text{(infinite line of charge).}$$

Problem 4.15 Repeat Example 4-5 for the circular disk of charge of radius a, but in the present case assume the surface charge density to vary with r as

$$\rho_s = \rho_{s0} r^2 \quad (\text{C/m}^2),$$

where ρ_{s0} is a constant.

Solution: We start with the expression for $d\mathbf{E}$ given in Example 4-5 but we replace ρ_s with $\rho_{s0}r^2$:

$$d\mathbf{E} = \hat{\mathbf{z}}\,\frac{h}{4\pi\varepsilon_0(r^2+h^2)^{3/2}}\,(2\pi\rho_{s0}r^3\,dr),$$

$$\mathbf{E} = \hat{\mathbf{z}}\,\frac{\rho_{s0}h}{2\varepsilon_0}\int_0^a \frac{r^3\,dr}{(r^2+h^2)^{3/2}}\,.$$

To perform the integration, we use

$$R^2 = r^2 + h^2,$$

$$2R\,dR = 2r\,dr,$$

$$\begin{aligned}
\mathbf{E} &= \hat{\mathbf{z}}\,\frac{\rho_{s0}h}{2\varepsilon_0}\int_h^{(a^2+h^2)^{1/2}} \frac{(R^2-h^2)\,dR}{R^2} \\
&= \hat{\mathbf{z}}\,\frac{\rho_{s0}h}{2\varepsilon_0}\left[\int_h^{(a^2+h^2)^{1/2}} dR - \int_h^{(a^2+h^2)^{1/2}} \frac{h^2}{R^2}\,dR\right] \\
&= \hat{\mathbf{z}}\,\frac{\rho_{s0}h}{2\varepsilon_0}\left[\sqrt{a^2+h^2} + \frac{h^2}{\sqrt{a^2+h^2}} - 2h\right].
\end{aligned}$$

Problem 4.16 Multiple charges at different locations are said to be in equilibrium if the force acting on any one of them is identical in magnitude and direction to the force acting on any of the others. Suppose we have two negative charges, one located at the origin and carrying charge $-9e$, and the other located on the positive x-axis at a distance d from the first one and carrying charge $-36e$. Determine the location, polarity and magnitude of a third charge whose placement would bring the entire system into equilibrium.

Solution: If

$$\mathbf{F}_1 = \text{force on } Q_1,$$

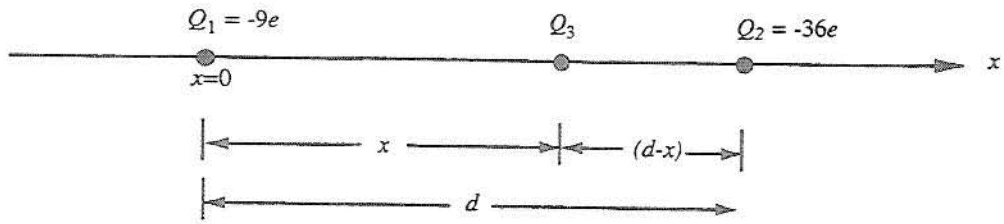


Figure P4.16: Three collinear charges.

$$\mathbf{F}_2 = \text{force on } Q_2,$$
$$\mathbf{F}_3 = \text{force on } Q_3,$$

then equilibrium means that

$$\mathbf{F}_1 = \mathbf{F}_2 = \mathbf{F}_3.$$

The two original charges are both negative, which mean they would repel each other. The third charge has to be positive and has to lie somewhere between them in order to counteract their repulsion force. The forces acting on charges Q_1, Q_2, and Q_3 are respectively

$$\mathbf{F}_1 = \frac{\hat{\mathbf{R}}_{21}Q_1Q_2}{4\pi\varepsilon_0 R_{21}^2} + \frac{\hat{\mathbf{R}}_{31}Q_1Q_3}{4\pi\varepsilon_0 R_{31}^2} = -\hat{\mathbf{x}}\frac{324e^2}{4\pi\varepsilon_0 d^2} + \hat{\mathbf{x}}\frac{9eQ_3}{4\pi\varepsilon_0 x^2},$$
$$\mathbf{F}_2 = \frac{\hat{\mathbf{R}}_{12}Q_1Q_2}{4\pi\varepsilon_0 R_{12}^2} + \frac{\hat{\mathbf{R}}_{32}Q_3Q_2}{4\pi\varepsilon_0 R_{32}^2} = \hat{\mathbf{x}}\frac{324e^2}{4\pi\varepsilon_0 d^2} - \hat{\mathbf{x}}\frac{36eQ_3}{4\pi\varepsilon_0 (d-x)^2},$$
$$\mathbf{F}_3 = \frac{\hat{\mathbf{R}}_{13}Q_1Q_3}{4\pi\varepsilon_0 R_{13}^2} + \frac{\hat{\mathbf{R}}_{23}Q_2Q_3}{4\pi\varepsilon_0 R_{23}^2} = -\hat{\mathbf{x}}\frac{9eQ_3}{4\pi\varepsilon_0 x^2} + \hat{\mathbf{x}}\frac{36eQ_3}{4\pi\varepsilon_0 (d-x)^2}.$$

Hence, equilibrium requires that

$$-\frac{324e}{d^2} + \frac{9Q_3}{x^2} = \frac{324e}{d^2} - \frac{36Q_3}{(d-x)^2} = -\frac{9Q_3}{x^2} + \frac{36Q_3}{(d-x)^2}.$$

Solution of the above equations yields

$$Q_3 = 4e, \qquad x = \frac{d}{3}.$$

Section 4-4: Gauss's Law

Problem 4.17 Three infinite lines of charge, all parallel to the z-axis, are located at the three corners of the kite-shaped arrangement shown in Fig. 4-29 (P4.17). If the two right triangles are symmetrical and of equal corresponding sides, show that the electric field is zero at the origin.

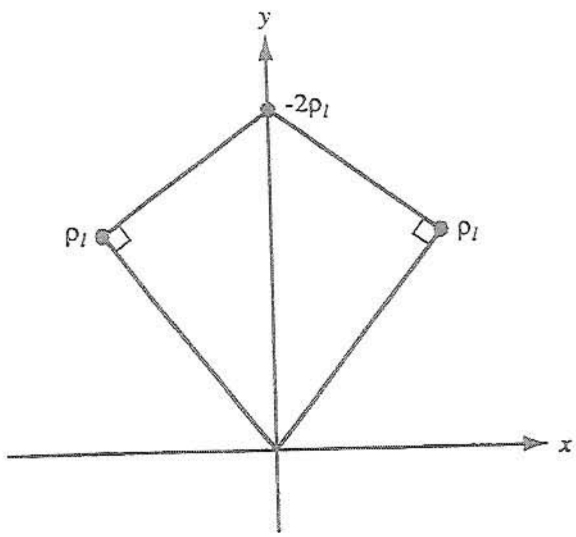


Figure P4.17: Kite-shaped arrangment of line charges for Problem 4.17.

Solution: The field due to an infinite line of charge is given by Eq. (4.33). In the present case, the total $\mathbf{E}$ at the origin is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3.$$

The components of $\mathbf{E}_1$ and $\mathbf{E}_2$ along $\hat{\mathbf{x}}$ cancel and their components along $-\hat{\mathbf{y}}$ add. Also, $\mathbf{E}_3$ is along $\hat{\mathbf{y}}$ because the line charge on the y-axis is negative. Hence,

$$\mathbf{E} = -\hat{\mathbf{y}}\,\frac{2\rho_l \cos\theta}{2\pi\varepsilon_0 R_1} + \hat{\mathbf{y}}\,\frac{2\rho_l}{2\pi\varepsilon_0 R_2}\,.$$

But $\cos\theta = R_1/R_2$. Hence,

$$\mathbf{E} = -\hat{\mathbf{y}}\,\frac{\rho_l}{\pi\varepsilon_0 R_1}\,\frac{R_1}{R_2} + \hat{\mathbf{y}}\,\frac{\rho_l}{\pi\varepsilon_0 R_2} = 0.$$

Problem 4.18 Three infinite lines of charge, $\rho_{l_1} = 5$ (nC/m), $\rho_{l_2} = -5$ (nC/m), and $\rho_{l_3} = 5$ (nC/m), are all parallel to the z-axis. If they pass through the respective points

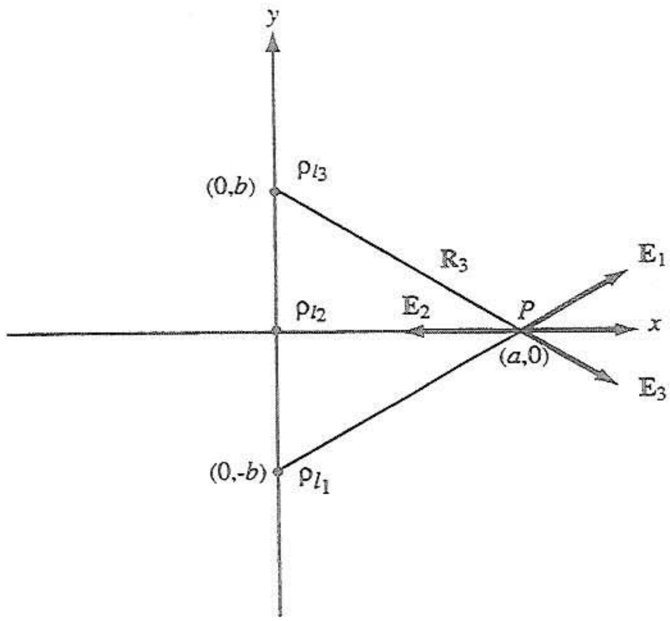


Figure P4.18: Three parallel line charges.

$(0,-b)$, $(0,0)$, and $(0,b)$ in the x–y plane, find the electric field at $(a,0,0)$. Evaluate your result for $a = 2$ cm and $b = 1$ cm.

Solution:

$$\rho_{l_1} = 5 \quad \text{(nC/m)},$$
$$\rho_{l_2} = -5 \quad \text{(nC/m)},$$
$$\rho_{l_3} = \rho_{l_1},$$
$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3.$$

Components of line charges 1 and 3 along y cancel and components along x add. Hence, using Eq. (4.33),

$$\mathbf{E} = \hat{\mathbf{x}}\,\frac{2\rho_{l_1}}{2\pi\varepsilon_0 R_1}\cos\theta + \hat{\mathbf{x}}\,\frac{\rho_{l_2}}{2\pi\varepsilon_0 a}.$$

with $\cos\theta = \dfrac{a}{\sqrt{a^2+b^2}}$ and $R_1 = \sqrt{a^2+b^2}$,

$$\mathbf{E} = \frac{\hat{\mathbf{x}}5}{2\pi\varepsilon_0}\left[\frac{2a}{a^2+b^2} - \frac{1}{a}\right] \times 10^{-9} \quad \text{(V/m)}.$$

For $a = 2$ cm and $b = 1$ cm,

$$\mathbf{E} = \hat{\mathbf{x}}2.70 \quad \text{(kV/m)}.$$

Problem 4.19 A horizontal strip lying in the x–y plane is of width d in the y-direction and infinitely long in the x-direction. If the strip is in air and has a uniform charge distribution ρ_s, use Coulomb's law to obtain an explicit expression for the electric field at a point P located at a distance h above the centerline of the strip. Extend your result to the special case where d is infinite and compare it with Eq. (4.25).

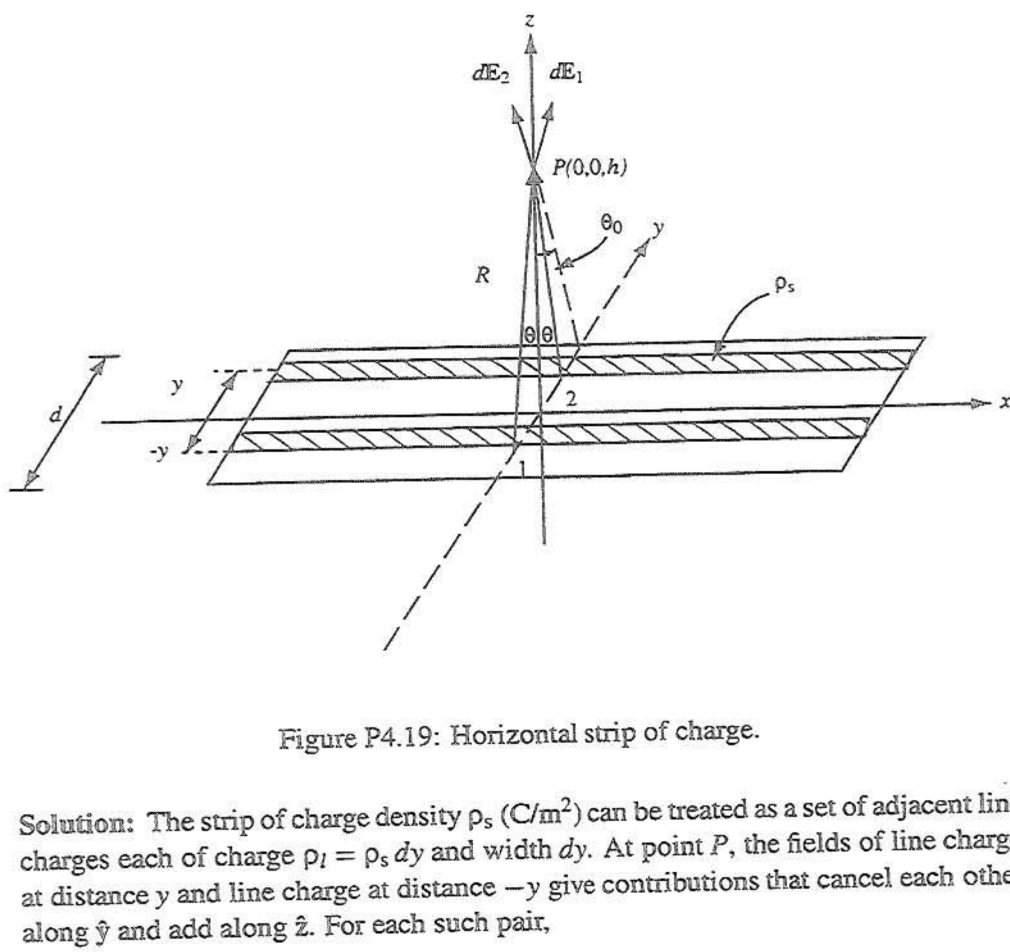


Figure P4.19: Horizontal strip of charge.

Solution: The strip of charge density ρ_s (C/m^2) can be treated as a set of adjacent line charges each of charge $\rho_l = \rho_s\,dy$ and width dy. At point P, the fields of line charge at distance y and line charge at distance $-y$ give contributions that cancel each other along $\hat{\mathbf{y}}$ and add along $\hat{\mathbf{z}}$. For each such pair,

$$d\mathbf{E} = \hat{\mathbf{z}}\,\frac{2\rho_s\,dy\cos\theta}{2\pi\varepsilon_0 R}\,.$$

With $R = h/\cos\theta$, we integrate from $y = 0$ to $d/2$, which corresponds to $\theta = 0$ to $\theta_0 = \sin^{-1}[(d/2)/(h^2+(d/2)^2)^{1/2}]$. Thus,

$$\mathbf{E} = \int_0^{d/2} d\mathbf{E} = \hat{\mathbf{z}}\frac{\rho_s}{\pi\varepsilon_0}\int_0^{d/2}\frac{\cos\theta}{R}\,dy = \hat{\mathbf{z}}\frac{\rho_s}{\pi\varepsilon_0}\int_0^{\theta_0}\frac{\cos^2\theta}{h}\cdot\frac{h}{\cos^2\theta}\,d\theta$$
$$= \hat{\mathbf{z}}\frac{\rho_s}{\pi\varepsilon_0}\theta_0.$$

For an infinitely wide sheet, $\theta_0 = \pi/2$ and $\mathbf{E} = \hat{\mathbf{z}}\dfrac{\rho_s}{2\varepsilon_0}$, which is identical with Eq. (4.25).

Problem 4.20 Given the electric flux density

$$\mathbf{D} = \hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y) \quad (\text{C/m}^2),$$

determine

(a) ρ_v by applying Eq. (4.26),

(b) the total charge Q enclosed in a cube 2 m on a side, located in the first octant with three of its sides coincident with the x-, y-, and z-axes and one of its corners at the origin, and

(c) the total charge Q in the cube, obtained by applying Eq. (4.29).

Solution:

(a) By applying Eq. (4.26)

$$\rho_v = \nabla\cdot\mathbf{D} = \frac{\partial}{\partial x}(2x+2y) + \frac{\partial}{\partial y}(3x-2y) = 0.$$

(b) Integrate the charge density over the volume as in Eq. (4.27):

$$Q = \int_{\mathcal{V}}\nabla\cdot\mathbf{D}\,d\mathcal{V} = \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} 0\,dx\,dy\,dz = 0.$$

(c) Apply Gauss' law to calculate the total charge from Eq. (4.29)

$$Q = \oint\mathbf{D}\cdot d\mathbf{s} = F_{\text{front}} + F_{\text{back}} + F_{\text{right}} + F_{\text{left}} + F_{\text{top}} + F_{\text{bottom}},$$

$$F_{\text{front}} = \int_{y=0}^{2}\int_{z=0}^{2}(\hat{\mathbf{x}}2(x+y)+\hat{\mathbf{y}}(3x-2y))\Big|_{x=2}\cdot(\hat{\mathbf{x}}\,dz\,dy)$$
$$= \int_{y=0}^{2}\int_{z=0}^{2} 2(x+y)\Big|_{x=2}\,dz\,dy = \left(2z\left(2y+\frac{1}{2}y^2\right)\Big|_{z=0}^{2}\right)\Big|_{y=0}^{2} = 24,$$

$$F_{\text{back}} = \int_{y=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y))\bigg|_{x=0} \cdot (-\hat{\mathbf{x}}\,dz\,dy)$$

$$= -\int_{y=0}^{2}\int_{z=0}^{2} 2(x+y)\bigg|_{x=0} dz\,dy = -\left(zy^2\Big|_{z=0}^{2}\right)\Bigg|_{y=0}^{2} = -8,$$

$$F_{\text{right}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y))\bigg|_{y=2} \cdot (\hat{\mathbf{y}}\,dz\,dx)$$

$$= \int_{x=0}^{2}\int_{z=0}^{2} (3x-2y)\bigg|_{y=2} dz\,dx = \left(z\left(\frac{3}{2}x^2 - 4x\right)\Bigg|_{z=0}^{2}\right)\Bigg|_{x=0}^{2} = -4,$$

$$F_{\text{left}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y))\bigg|_{y=0} \cdot (-\hat{\mathbf{y}}\,dz\,dx)$$

$$= -\int_{x=0}^{2}\int_{z=0}^{2} (3x-2y)\bigg|_{y=0} dz\,dx = -\left(z\left(\frac{3}{2}x^2\right)\Bigg|_{z=0}^{2}\right)\Bigg|_{x=0}^{2} = -12,$$

$$F_{\text{top}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y))\bigg|_{z=2} \cdot (\hat{\mathbf{z}}\,dy\,dx)$$

$$= \int_{x=0}^{2}\int_{z=0}^{2} 0\bigg|_{z=2} dy\,dx = 0,$$

$$F_{\text{bottom}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y))\bigg|_{z=0} \cdot (\hat{\mathbf{z}}\,dy\,dx)$$

$$= \int_{x=0}^{2}\int_{z=0}^{2} 0\bigg|_{z=0} dy\,dx = 0.$$

Thus $Q = \oint \mathbf{D} \cdot d\mathbf{s} = 24 - 8 - 4 - 12 + 0 + 0 = 0.$

Problem 4.21 Repeat Problem 4.20 for $\mathbf{D} = \hat{\mathbf{x}}xy^2z^3$ (C/m^2).

Solution:

(a) From Eq. (4.26), $\rho_v = \nabla \cdot \mathbf{D} = \dfrac{\partial}{\partial x}(xy^2z^3) = y^2z^3$.

(b) Total charge Q is given by Eq. (4.27):

$$Q = \int_{\mathcal{V}} \nabla \cdot \mathbf{D}\, d\mathcal{V} = \int_{z=0}^{2}\int_{y=0}^{2}\int_{x=0}^{2} y^2z^3\,dx\,dy\,dz = \frac{xy^3z^4}{12}\Bigg|_{x=0}^{2}\Bigg|_{y=0}^{2}\Bigg|_{z=0}^{2} = \frac{64}{3}\text{ C}.$$

(c) Using Gauss' law we have

$$\oint_S \mathbf{D}\cdot d\mathbf{s} = F_{\text{front}} + F_{\text{back}} + F_{\text{right}} + F_{\text{left}} + F_{\text{top}} + F_{\text{bottom}}.$$

Note that $\mathbf{D} = \hat{\mathbf{x}}D_x$, so only F_{front} and F_{back} (integration over $\hat{\mathbf{z}}$ surfaces) will contribute to the integral.

$$F_{\text{front}} = \int_{z=0}^{2}\int_{y=0}^{2} (\hat{\mathbf{x}}xy^2z^3)\Big|_{x=2} \cdot (\hat{\mathbf{x}}\,dy\,dz)$$

$$= \int_{z=0}^{2}\int_{y=0}^{2} xy^2z^3\Big|_{x=2} dy\,dz = \left(2\left(\frac{y^3z^4}{12}\right)\Big|_{y=0}^{2}\right)\Big|_{z=0}^{2} = \frac{64}{3},$$

$$F_{\text{back}} = \int_{z=0}^{2}\int_{y=0}^{2} (\hat{\mathbf{x}}xy^2z^3)\Big|_{x=0} \cdot (-\hat{\mathbf{x}}\,dy\,dz) = -\int_{z=0}^{2}\int_{y=0}^{2} xy^2z^3\Big|_{x=0} dy\,dz = 0.$$

Thus $Q = \oint \mathbf{D}\cdot d\mathbf{s} = \frac{64}{3} + 0 + 0 + 0 + 0 + 0 = \frac{64}{3}$ C.

Problem 4.22 Charge Q_1 is uniformly distributed over a thin spherical shell of radius a, and charge Q_2 is uniformly distributed over a second spherical shell of radius b, with $b > a$. Apply Gauss's law to find $\mathbf{E}$ in the regions $R < a$, $a < R < b$, and $R > b$.

Solution: Using symmetry considerations, we know $\mathbf{D} = \hat{\mathbf{R}}D_R$. From Table 3.1, $d\mathbf{s} = \hat{\mathbf{R}}R^2 \sin\theta\, d\theta\, d\phi$ for an element of a spherical surface. Using Gauss's law in integral form (Eq. (4.29)),

$$\oint_S \mathbf{D}\cdot d\mathbf{s} = Q_{\text{tot}},$$

where Q_{tot} is the total charge enclosed in S. For a spherical surface of radius R,

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} (\hat{\mathbf{R}}D_R)\cdot(\hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi) = Q_{\text{tot}},$$

$$D_R R^2(2\pi)[-\cos\theta]_0^{\pi} = Q_{\text{tot}},$$

$$D_R = \frac{Q_{\text{tot}}}{4\pi R^2}.$$

From Eq. (4.15), we know a linear, isotropic material has the constitutive relationship $\mathbf{D} = \varepsilon\mathbf{E}$. Thus, we find $\mathbf{E}$ from $\mathbf{D}$.

(a) In the region $R < a$,

$$Q_{\rm tot} = 0, \qquad \mathbf{E} = \hat{\mathbf{R}}E_{\rm R} = \frac{\hat{\mathbf{R}}Q_{\rm tot}}{4\pi R^2\varepsilon} = 0 \quad \text{(V/m)}.$$

(b) In the region $a < R < b$,

$$Q_{\rm tot} = Q_1, \qquad \mathbf{E} = \hat{\mathbf{R}}E_{\rm R} = \frac{\hat{\mathbf{R}}Q_1}{4\pi R^2\varepsilon} \quad \text{(V/m)}.$$

(c) In the region $R > b$,

$$Q_{\rm tot} = Q_1 + Q_2, \qquad \mathbf{E} = \hat{\mathbf{R}}E_{\rm R} = \frac{\hat{\mathbf{R}}(Q_1 + Q_2)}{4\pi R^2\varepsilon} \quad \text{(V/m)}.$$

Problem 4.23 The electric flux density inside a dielectric sphere of radius a centered at the origin is given by

$$\mathbf{D} = \hat{\mathbf{R}}\rho_0 R \quad \text{(C/m}^2\text{)},$$

where ρ_0 is a constant. Find the total charge inside the sphere.

Solution:

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{s} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \hat{\mathbf{R}}\rho_0 R \cdot \hat{\mathbf{R}}R^2 \sin\theta \, d\theta \, d\phi \bigg|_{R=a}$$
$$= 2\pi\rho_0 a^3 \int_0^{\pi} \sin\theta \, d\theta = -2\pi\rho_0 a^3 \cos\theta \big|_0^{\pi} = 4\pi\rho_0 a^3 \quad \text{(C)}.$$

Problem 4.24 In a certain region of space, the charge density is given in cylindrical coordinates by the function:

$$\rho_{\rm v} = 20re^{-r} \quad \text{(C/m}^3\text{)}.$$

Apply Gauss's law to find $\mathbf{D}$.

Solution:

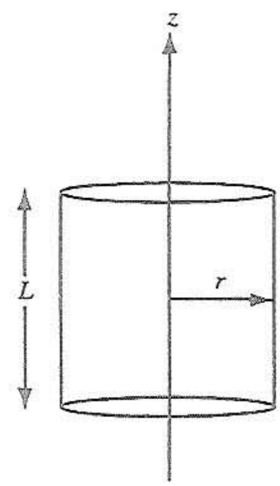


Figure P4.24: Gaussian surface.

Method 1: Integral Form of Gauss's Law

Since ρ_v varies as a function of r only, so will $\mathbb{D}$. Hence, we construct a cylinder of radius r and length L, coincident with the z-axis. Symmetry suggests that $\mathbb{D}$ has the functional form $\mathbb{D} = \hat{\mathbf{r}} D$. Hence,

$$\oint_S \mathbb{D} \cdot d\mathbf{s} = Q,$$

$$\int \hat{\mathbf{r}} D \cdot d\mathbf{s} = D(2\pi r L),$$

$$Q = 2\pi L \int_0^r 20 r e^{-r} \cdot r\, dr$$

$$= 40\pi L[-r^2 e^{-r} + 2(1 - e^{-r}(1+r))],$$

$$\mathbb{D} = \hat{\mathbf{r}} D = \hat{\mathbf{r}} 20 \left[\frac{2}{r}(1 - e^{-r}(1+r)) - r e^{-r} \right].$$

Method 2: Differential Method

$$\nabla \cdot \mathbb{D} = \rho_v, \qquad \mathbb{D} = \hat{\mathbf{r}} D_r,$$

with D_r being a function of r.

$$\frac{1}{r} \frac{\partial}{\partial r}(r D_r) = 20 r e^{-r},$$

$$\frac{\partial}{\partial r}(rD_r) = 20r^2e^{-r},$$
$$\int_0^r \frac{\partial}{\partial r}(rD_r)\,dr = \int_0^r 20r^2e^{-r}\,dr,$$
$$rD_r = 20[2(1-e^{-r}(1+r)) - r^2e^{-r}],$$
$$\mathbf{D} = \hat{\mathbf{r}}\,rD_r = \hat{\mathbf{r}}\,20\left[\frac{2}{r}(1-e^{-r}(1+r)) - re^{-r}\right].$$

Problem 4.25 An infinitely long cylindrical shell extending between $r = 1$ m and $r = 3$ m contains a uniform charge density ρ_{v0}. Apply Gauss's law to find $\mathbf{D}$ in all regions.

Solution: For $r < 1$ m, $\mathbf{D} = 0$.

For $1 \leq r \leq 3$ m,

$$\oint_S \hat{\mathbf{r}}D_r \cdot d\mathbf{s} = Q,$$
$$D_r \cdot 2\pi rL = \rho_{v0} \cdot \pi L(r^2 - 1^2),$$
$$\mathbf{D} = \hat{\mathbf{r}}D_r = \hat{\mathbf{r}}\,\frac{\rho_{v0}\pi L(r^2-1)}{2\pi rL} = \hat{\mathbf{r}}\,\frac{\rho_{v0}(r^2-1)}{2r}, \qquad 1 \leq r \leq 3 \text{ m}.$$

For $r \geq 3$ m,

$$D_r \cdot 2\pi rL = \rho_{v0}\pi L(3^2 - 1^2) = 8\rho_{v0}\pi L,$$
$$\mathbf{D} = \hat{\mathbf{r}}D_r = \hat{\mathbf{r}}\,\frac{4\rho_{v0}}{r}, \qquad r \geq 3 \text{ m}.$$

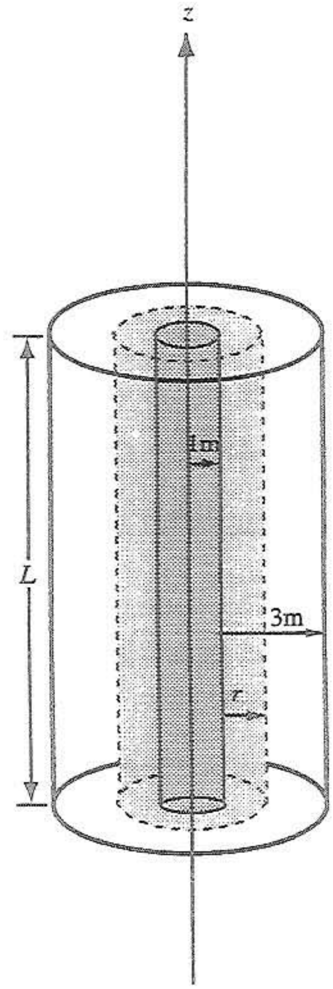


Figure P4.25: Cylindrical shell.

Problem 4.26 If the charge density increases linearly with distance from the origin such that $\rho_v = 0$ at the origin and $\rho_v = 10$ C/m^3 at $R = 2$ m, find the corresponding variation of $\mathbf{D}$.

Solution:

$$\rho_v(R) = a + bR,$$
$$\rho_v(0) = a = 0,$$

$$\rho_v(2) = 2b = 10.$$

Hence, $b = 5$.

$$\rho_v(R) = 5R \quad (\text{C/m}^3).$$

Applying Gauss's law to a spherical surface of radius R,

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_V \rho_v \, d\mathcal{V},$$

$$D_R \cdot 4\pi R^2 = \int_0^R 5R \cdot 4\pi R^2 \, dR = 20\pi \frac{R^4}{4},$$

$$D_R = \frac{5}{4} R^2 \quad (\text{C/m}^2),$$

$$\mathbf{D} = \hat{\mathbf{R}} D_R = \hat{\mathbf{R}} \frac{5}{4} R^2 \quad (\text{C/m}^2).$$

Section 4-5: Electric Potential

Problem 4.27 A square in the x–y plane in free space has a point charge of $+Q$ at corner $(a/2, a/2)$ and the same at corner $(a/2, -a/2)$ and a point charge of $-Q$ at each of the other two corners.

(a) Find the electric potential at any point P along the x-axis.

(b) Evaluate V at $x = a/2$.

Solution: $R_1 = R_2$ and $R_3 = R_4$.

$$V = \frac{Q}{4\pi\varepsilon_0 R_1} + \frac{Q}{4\pi\varepsilon_0 R_2} + \frac{-Q}{4\pi\varepsilon_0 R_3} + \frac{-Q}{4\pi\varepsilon_0 R_4} = \frac{Q}{2\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_3}\right)$$

with

$$R_1 = \sqrt{\left(x - \frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2},$$

$$R_3 = \sqrt{\left(x + \frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2}.$$

At $x = a/2$,

$$R_1 = \frac{a}{2},$$

$$R_3 = \frac{a\sqrt{5}}{2},$$

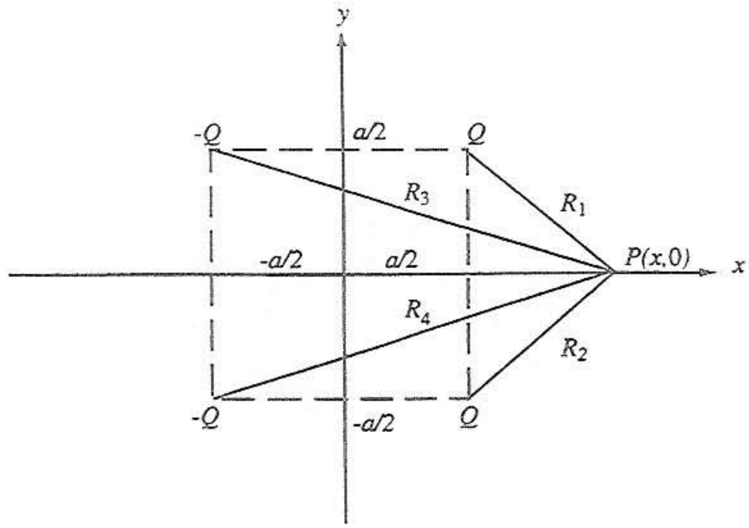


Figure P4.27: Potential due to four point charges.

$$V = \frac{Q}{2\pi\varepsilon_0}\left(\frac{2}{a} - \frac{2}{\sqrt{5}a}\right) = \frac{0.55Q}{\pi\varepsilon_0 a}.$$

Problem 4.28 The circular disk of radius a shown in Fig. 4-7 (P4.28) has uniform charge density ρ_s across its surface.

(a) Obtain an expression for the electric potential V at a point $P(0,0,z)$ on the z-axis.

(b) Use your result to find $\mathbf{E}$ and then evaluate it for $z = h$. Compare your final expression with Eq. (4.24), which was obtained on the basis of Coulomb's law.

Solution:

(a) Consider a ring of charge at a radial distance r. The charge contained in width dr is

$$dq = \rho_s(2\pi r\,dr) = 2\pi\rho_s r\,dr.$$

The potential at P is

$$dV = \frac{dq}{4\pi\varepsilon_0 R} = \frac{2\pi\rho_s r\,dr}{4\pi\varepsilon_0(r^2+z^2)^{1/2}}.$$

The potential due to the entire disk is

$$V = \int_0^a dV = \frac{\rho_s}{2\varepsilon_0}\int_0^a \frac{r\,dr}{(r^2+z^2)^{1/2}} = \frac{\rho_s}{2\varepsilon_0}(r^2+z^2)^{1/2}\Big|_0^a = \frac{\rho_s}{2\varepsilon_0}\left[(a^2+z^2)^{1/2} - z\right].$$

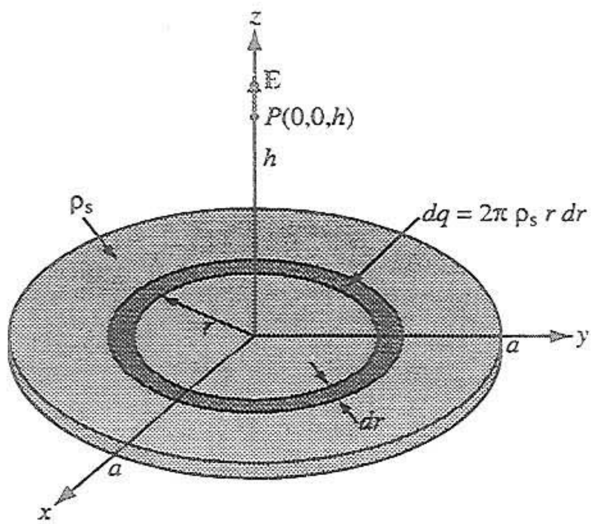


Figure P4.28: Circular disk of charge.

(b)

$$\mathbf{E} = -\nabla V = -\hat{\mathbf{x}}\,\frac{\partial V}{\partial x} - \hat{\mathbf{y}}\,\frac{\partial V}{\partial y} - \hat{\mathbf{z}}\,\frac{\partial V}{\partial z} = \hat{\mathbf{z}}\,\frac{\rho_s}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{a^2+z^2}}\right].$$

The expression for $\mathbf{E}$ reduces to Eq. (4.24) when $z = h$.

Problem 4.29 A circular ring of charge of radius a lies in the x–y plane and is centered at the origin. If the ring is in air and carries a uniform density ρ_l, (a) show that the electrical potential at $(0,0,z)$ is given by $V = \rho_l a/[2\varepsilon_0(a^2+z^2)^{1/2}]$, and (b) find the corresponding electric field $\mathbf{E}$.

Solution:

(a) For the ring of charge shown in Fig. P4.29, using Eq. (3.67) in Eq. (4.48c) gives

$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon_0}\int_{l'}\frac{\rho_l}{R'}\,dl' = \frac{1}{4\pi\varepsilon_0}\int_{\phi'=0}^{2\pi}\frac{\rho_l}{\sqrt{a^2+r^2-2ar\cos(\phi'-\phi)+z^2}}\,a\,d\phi'.$$

Point $(0,0,z)$ in Cartesian coordinates corresponds to $(r,\phi,z) = (0,\phi,z)$ in cylindrical coordinates. Hence, for $r = 0$,

$$V(0,0,z) = \frac{1}{4\pi\varepsilon_0}\int_{\phi'=0}^{2\pi}\frac{\rho_l}{\sqrt{a^2+z^2}}\,a\,d\phi' = \frac{\rho_l a}{2\varepsilon_0\sqrt{a^2+z^2}}.$$

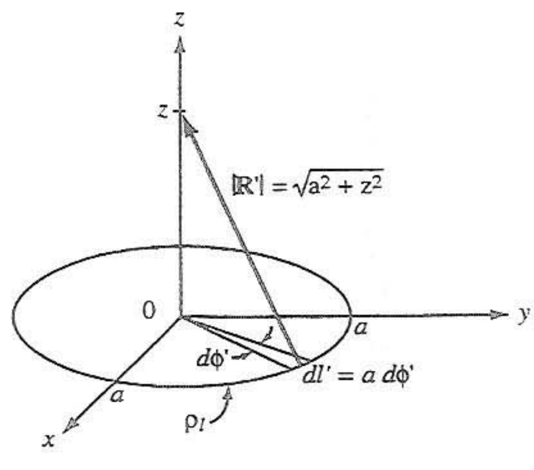


Figure P4.29: Ring of charge.

(b) From Eq. (4.51),

$$\mathbf{E} = -\nabla V = -\hat{\mathbf{z}}\frac{\rho_l a}{2\varepsilon_0}\frac{\partial}{\partial z}(a^2+z^2)^{-1/2} = \hat{\mathbf{z}}\frac{\rho_l a}{2\varepsilon_0}\frac{z}{(a^2+z^2)^{3/2}} \quad \text{(V/m)}.$$

Problem 4.30 Show that the electric potential difference V_{12} between two points in air at radial distances r_1 and r_2 from an infinite line of charge with density ρ_l along the z-axis is $V_{12} = (\rho_l/2\pi\varepsilon_0)\ln(r_2/r_1)$.

Solution: From Eq. (4.33), the electric field due to an infinite line of charge is

$$\mathbf{E} = \hat{\mathbf{r}}E_r = \hat{\mathbf{r}}\,\frac{\rho_l}{2\pi\varepsilon_0 r}.$$

Hence, the potential difference is

$$V_{12} = -\int_{r_2}^{r_1}\mathbf{E}\cdot d\mathbf{l} = -\int_{r_2}^{r_1}\frac{\hat{\mathbf{r}}\rho_l}{2\pi\varepsilon_0 r}\cdot\hat{\mathbf{r}}\,dr = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{r_2}{r_1}\right).$$

Problem 4.31 Find the electric potential V at a location a distance b from the origin in the x–y plane due to a line charge with charge density ρ_l and of length l. The line charge is coincident with the z-axis and extends from $z = -l/2$ to $z = l/2$.

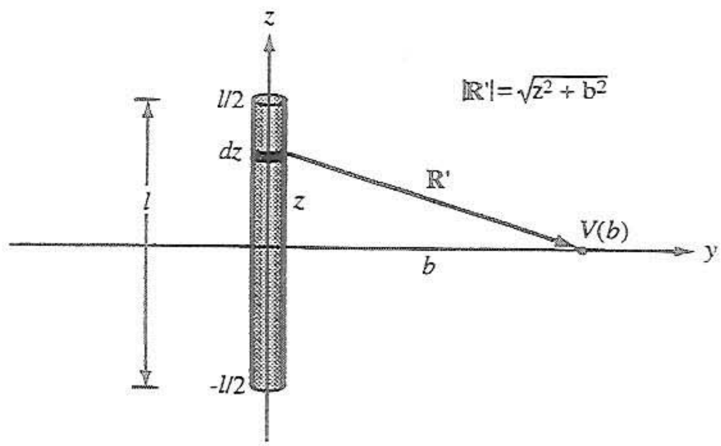


Figure P4.31: Line of charge of length ℓ.

Solution: From Eq. (4.48c), we can find the voltage at a distance b away from a line of charge [Fig. P4.31]:

$$V(b) = \frac{1}{4\pi\varepsilon}\int_{l'}\frac{\rho_l}{R'}\,dl' = \frac{\rho_l}{4\pi\varepsilon}\int_{-l/2}^{l/2}\frac{dz}{\sqrt{z^2+b^2}} = \frac{\rho_l}{4\pi\varepsilon}\ln\left(\frac{l+\sqrt{l^2+4b^2}}{-l+\sqrt{l^2+4b^2}}\right).$$

Problem 4.32 For the electric dipole shown in Fig. 4-13, $d = 1$ cm and $|\mathbf{E}| = 2$ (mV/m) at $R = 1$ m and $\theta = 0°$. Find $\mathbf{E}$ at $R = 2$ m and $\theta = 90°$.

Solution: For $R = 1$ m and $\theta = 0°$, $|\mathbf{E}| = 2$ mV/m, we can solve for q using Eq. (4.56):

$$\mathbf{E} = \frac{qd}{4\pi\varepsilon_0 R^3}(\hat{\mathbf{R}}2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta).$$

Hence,

$$|\mathbf{E}| = \left(\frac{qd}{4\pi\varepsilon_0}\right)2 = 2 \text{ mV/m} \quad \text{at } \theta = 0°,$$

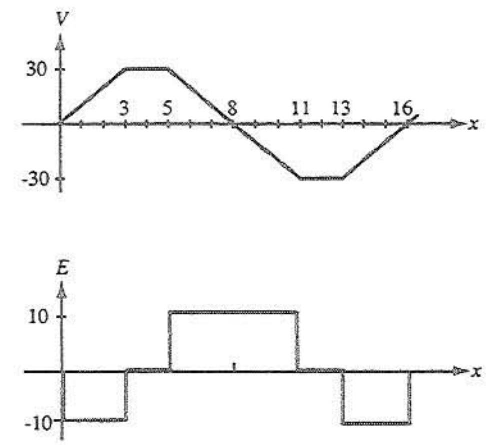
$$q = \frac{10^{-3}\times 4\pi\varepsilon_0}{d} = \frac{10^{-3}\times 4\pi\varepsilon_0}{10^{-2}} = 0.4\pi\varepsilon_0 \quad \text{(C)}.$$

Again using Eq. (4.56) to find $\mathbf{E}$ at $R = 2$ m and $\theta = 90°$, we have

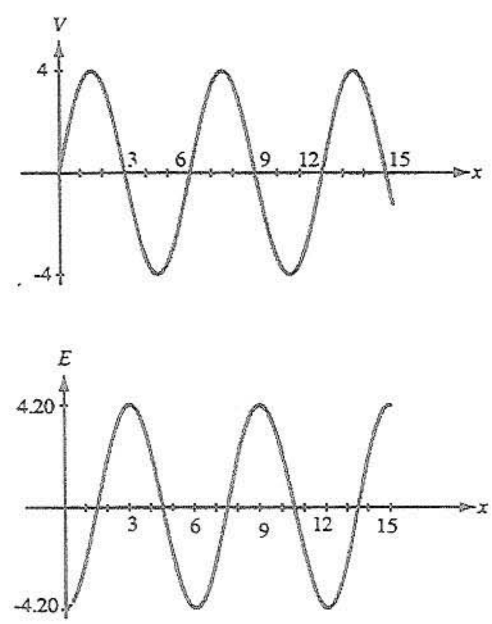
$$\mathbf{E} = \frac{0.4\pi\varepsilon_0\times 10^{-2}}{4\pi\varepsilon_0\times 2^3}(\hat{\mathbf{R}}(0) + \hat{\boldsymbol{\theta}}) = \hat{\boldsymbol{\theta}}\frac{1}{8} \quad \text{(mV/m)}.$$

Problem 4.33 For each of the following distributions of the electric potential V, sketch the corresponding distribution of $\mathbf{E}$ (in all cases, the vertical axis is in volts and the horizontal axis is in meters):

Solution:



(a)



(b)

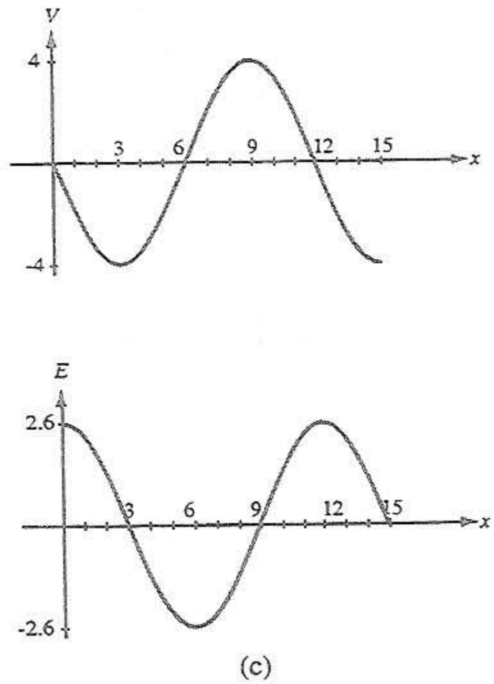


Figure P4.33: Electric potential distributions of Problem 4.33.

Problem 4.34 Given the electric field

$$\mathbf{E} = \hat{\mathbf{R}}\,\frac{12}{R^2} \quad \text{(V/m)},$$

find the electric potential of point A with respect to point B where A is at $+2$ m and B at -4 m, both on the z-axis.

Solution:

$$V_{AB} = V_A - V_B = -\int_B^A \mathbf{E} \cdot d\mathbf{l}.$$

Along z-direction, $\hat{\mathbf{R}} = \hat{\mathbf{z}}$ and $\mathbf{E} = \hat{\mathbf{z}}\,\frac{12}{z^2}$ for $z \geq 0$, and $\hat{\mathbf{R}} = -\hat{\mathbf{z}}$ and $\mathbf{E} = -\hat{\mathbf{z}}\,\frac{12}{z^2}$ for $z \leq 0$. Hence,

$$V_{AB} = -\int_{-4}^{2} \hat{\mathbf{R}}\,\frac{12}{z^2} \cdot \hat{\mathbf{z}}\,dz = -\left[\int_{-4}^{0} -\hat{\mathbf{z}}\,\frac{12}{z^2} \cdot \hat{\mathbf{z}}\,dz + \int_{0}^{2} \hat{\mathbf{z}}\,\frac{12}{z^2} \cdot \hat{\mathbf{z}}\,dz\right] = 3 \text{ V}.$$

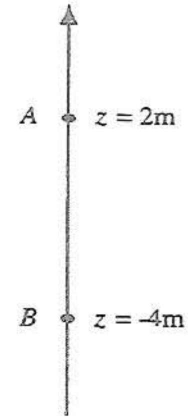


Figure P4.34: Potential between B and A.

Problem 4.35 An infinitely long line of charge with uniform density $\rho_l = 6$ (nC/m) lies in the x–y plane parallel to the y-axis at $x = 2$ m. Find the potential V_{AB} at point $A(3\text{ m}, 0, 4\text{ m})$ in Cartesian coordinates with respect to point $B(0,0,0)$ by applying the result of Problem 4.30.

Solution: According to Problem 4.30,

$$V = \frac{\rho_l}{2\pi\varepsilon_0} \ln\left(\frac{r_2}{r_1}\right)$$

where r_1 and r_2 are the distances of A and B. In this case,

$$r_1 = \sqrt{(3-2)^2 + 4^2} = \sqrt{17}\text{ m},$$
$$r_2 = 2\text{ m}.$$

Hence,

$$V_{AB} = \frac{6 \times 10^{-9}}{2\pi \times 8.85 \times 10^{-12}} \ln\left(\frac{2}{\sqrt{17}}\right) = -78.06\text{ V}.$$

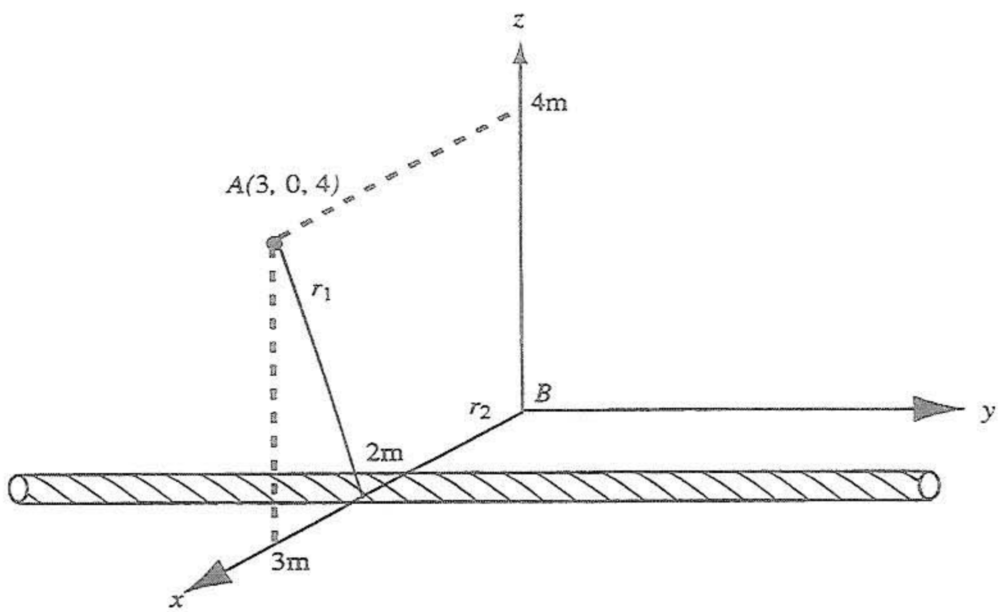


Figure P4.35: Line of charge parallel to y-axis.

Problem 4.36 The x–y plane contains a uniform sheet of charge with $\rho_{s_1} = 0.2$ (nC/m^2) and a second sheet with $\rho_{s_2} = -0.2$ (nC/m^2) occupies the plane $z = 6$ m. Find V_{AB}, V_{BC}, and V_{AC} for $A(0,0,6\text{ m})$, $B(0,0,0)$, and $C(0,-2\text{ m},2\text{ m})$.

Solution: We start by finding the $\mathbf{E}$ field in the region between the plates. For any point above the x–y plane, $\mathbf{E}_1$ due to the charge on x–y plane is, from Eq. (4.25),

$$\mathbf{E}_1 = \hat{\mathbf{z}}\,\frac{\rho_{s_1}}{2\varepsilon_0}.$$

In the region below the top plate, $\mathbf{E}$ would point downwards for positive ρ_{s_2} on the top plate. In this case, $\rho_{s_2} = -\rho_{s_1}$. Hence,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \hat{\mathbf{z}}\,\frac{\rho_{s_1}}{2\varepsilon_0} - \hat{\mathbf{z}}\,\frac{\rho_{s_2}}{2\varepsilon_0} = \hat{\mathbf{z}}\,\frac{2\rho_{s_1}}{2\varepsilon_0} = \hat{\mathbf{z}}\,\frac{\rho_{s_1}}{\varepsilon_0}.$$

Since $\mathbf{E}$ is along $\hat{\mathbf{z}}$, only change in position along z can result in change in voltage.

$$V_{AB} = -\int_0^6 \hat{\mathbf{z}}\,\frac{\rho_{s_1}}{\varepsilon_0}\cdot\hat{\mathbf{z}}\,dz = -\frac{\rho_{s_1}}{\varepsilon_0}\,z\Big|_0^6 = -\frac{6\rho_{s_1}}{\varepsilon_0} = -\frac{6\times 0.2\times 10^{-9}}{8.85\times 10^{-12}} = -135.59\text{ V}.$$

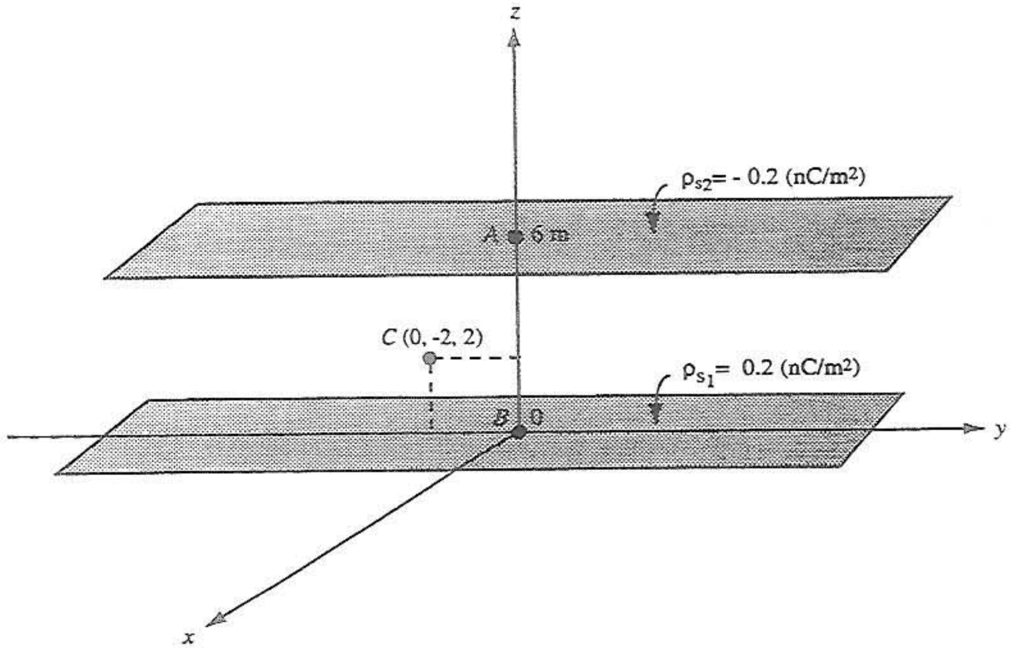


Figure P4.36: Two parallel planes of charge.

The voltage at C depends only on the z-coordinate of C. Hence, with point A being at the lowest potential and B at the highest potential,

$$V_{BC} = \frac{-2}{6} V_{AB} = -\frac{(-135.59)}{3} = 45.20 \text{ V},$$
$$V_{AC} = V_{AB} + V_{BC} = -135.59 + 45.20 = -90.39 \text{ V}.$$

Section 4-7: Conductors

Problem 4.37 A cylindrical bar of silicon has a radius of 2 mm and a length of 5 cm. If a voltage of 5 V is applied between the ends of the bar and $\mu_e = 0.13$ (m^2/V·s), $\mu_h = 0.05$ (m^2/V·s), $N_e = 1.5 \times 10^{16}$ electrons/m^3, and $N_h = N_e$, find

(a) the conductivity of silicon,
(b) the current I flowing in the bar,
(c) the drift velocities $\mathbf{u}_e$ and $\mathbf{u}_h$,
(d) the resistance of the bar, and
(e) the power dissipated in the bar.

Solution:

(a) Conductivity is given in Eq. (4.65),

$$\sigma = (N_e\mu_e + N_h\mu_h)e$$
$$= (1.5\times10^{16})(0.13+0.05)(1.6\times10^{-19}) = 4.32\times10^{-4} \quad \text{(S/m)}.$$

(b) Similarly to Example 4.8, parts b and c,

$$I = JA = \sigma EA = (4.32\times10^{-4})\left(\frac{5\text{V}}{0.05}\right)(\pi(2\times10^{-3})^2) = 542.9 \quad \text{(nA)}.$$

(c) From Eqs. (4.62a) and (4.62b),

$$\mathbf{u}_e = -\mu_e\mathbf{E} = -(0.13)(100)\frac{\mathbf{E}}{|\mathbf{E}|} = -13\frac{\mathbf{E}}{|\mathbf{E}|} \quad \text{(m/s)},$$
$$\mathbf{u}_h = \mu_h\mathbf{E} = +(0.05)(100)\frac{\mathbf{E}}{|\mathbf{E}|} = 5\frac{\mathbf{E}}{|\mathbf{E}|} \quad \text{(m/s)}.$$

(d) To find the resistance, we use what we calculated above,

$$R = \frac{V}{I} = \frac{5\text{V}}{542.9\text{ nA}} = 9.21 \quad (\text{M}\Omega).$$

(e) Power dissipated in the bar is $P = IV = (5\text{V})(542.9\text{ nA}) = 2.7\ (\mu\text{W})$.

Problem 4.38 Repeat Problem 4.37 for a bar of germanium with $\mu_e = 0.4$ (m²/V·s), $\mu_h = 0.2$ (m²/V·s), and $N_e = N_h = 2.4\times10^{19}$ electrons or holes/m³.

Solution:

(a) Conductivity is given in Eq. (4.65),

$$\sigma = (N_e\mu_e + N_u\mu_u)e = (2.4\times10^{19})(0.4+0.2)(1.6\times10^{-19}) = 2.3 \quad \text{(S/m)}.$$

(b) Similarly to Example 4.8, parts b and c,

$$I = JA = \sigma EA = (2.3)\left(\frac{5\text{V}}{0.05}\right)(\pi(2\times10^{-3})^2) = 2.89 \quad \text{(mA)}.$$

(c) From Eqs. (4.62a) and (4.62b),

$$\mathbf{u}_e = -\mu_e\mathbf{E} = -(0.4)(100)\frac{\mathbf{E}}{|\mathbf{E}|} = -40\frac{\mathbf{E}}{|\mathbf{E}|} \quad \text{(m/s)},$$
$$\mathbf{u}_h = \mu_h\mathbf{E} = (0.2)(100)\frac{\mathbf{E}}{|\mathbf{E}|} = 20\frac{\mathbf{E}}{|\mathbf{E}|} \quad \text{(m/s)}.$$

(d) To find the resistance, we use what we calculated above,

$$R = \frac{V}{I} = \frac{5\text{V}}{2.89 \text{ mA}} = 1.73 \quad (\text{k}\Omega).$$

(e) Power dissipated in the bar is $P = IV = (5\text{V})(2.89 \text{ mA}) = 14.5$ (mW).

Problem 4.39 A 100-m-long conductor of uniform cross section has a voltage drop of 2 V between its ends. If the density of the current flowing through it is 7×10^5 (A/m^2), identify the material of the conductor.

Solution: We know that conductivity characterizes a material:

$$\mathbf{J} = \sigma\mathbf{E}, \qquad 7 \times 10^5 \text{ (A/m}^2) = \sigma\left(\frac{2 \text{ (V)}}{100 \text{ (m)}}\right), \qquad \sigma = 3.5 \times 10^7 \quad \text{(S/m)}.$$

From Table B-2, we find that aluminum has $\sigma = 3.5 \times 10^7$ (S/m).

Problem 4.40 A coaxial resistor of length l consists of two concentric cylinders. The inner cylinder has radius a and is made of a material with conductivity σ_1, and the outer cylinder, extending between $r = a$ and $r = b$, is made of a material with conductivity σ_2. If the two ends of the resistor are capped with conducting plates, show that the resistance between the two ends is $R = l/[\pi(\sigma_1 a^2 + \sigma_2(b^2 - a^2))]$.

Solution: Due to the conducting plates, the ends of the coaxial resistor are each uniform at the same potential. Hence, the electric field everywhere in the resistor will be parallel to the axis of the resistor, in which case the two cylinders can be considered to be two separate resistors in parallel. Then, from Eq. (4.70),

$$\frac{1}{R} = \frac{1}{R_{\text{inner}}} + \frac{1}{R_{\text{outer}}} = \frac{\sigma_1 A_1}{l_1} + \frac{\sigma_2 A_2}{l_2} = \frac{\sigma_1 \pi a^2}{l} + \frac{\sigma_2 \pi(b^2 - a^2)}{l},$$

or

$$R = \frac{l}{\pi(\sigma_1 a^2 + \sigma_2(b^2 - a^2))} \quad (\Omega).$$

Problem 4.41 Apply the result of Problem 4.40 to find the resistance of a 10-cm-long hollow cylinder (Fig. P4.41) made of carbon with $\sigma = 3 \times 10^4$ (S/m).

Solution: From Problem 4.40, we know that for two concentric cylinders,

$$R = \frac{l}{\pi(\sigma_1 a^2 + \sigma_2(b^2 - a^2))} \quad (\Omega).$$

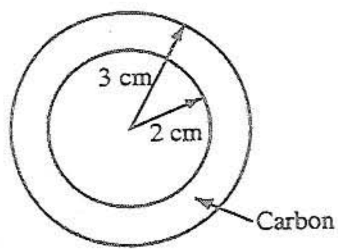


Figure P4.41: Cross section of hollow cylinder of Problem 4.41.

For air $\sigma_1 = 0$ (S/m), $\sigma_2 = 3 \times 10^4$ (S/m); hence,

$$R = \frac{0.1}{3\pi \times 10^4((0.03)^2 - (0.02)^2)} = 2.1 \quad (\text{m}\Omega).$$

Problem 4.42 A 2×10^{-3}-mm-thick square sheet of aluminum has 10 cm × 10 cm faces. Find:

(a) the resistance between opposite edges on a square face, and

(b) the resistance between the two square faces. (See Appendix B for the electrical constants of materials).

Solution:

(a)

$$R = \frac{l}{\sigma A}.$$

For aluminum, $\sigma = 3.5 \times 10^7$ (S/m) [Appendix B].

$$l = 10 \text{ cm}, \qquad A = 10 \text{ cm} \times 2 \times 10^{-3} \text{ mm} = 20 \times 10^{-2} \times 10^{-6} = 2 \times 10^{-7} \text{ m}^2,$$

$$R = \frac{10 \times 10^{-2}}{3.5 \times 10^7 \times 2 \times 10^{-7}} = 14 \quad (\text{m}\Omega).$$

(b) Now, $l = 2 \times 10^{-3}$ mm and $A = 10 \text{ cm} \times 10 \text{ cm} = 10^{-2} \text{ m}^2$.

$$R = \frac{2 \times 10^{-6}}{3.5 \times 10^7 \times 10^{-2}} = 5.71 \text{ p}\Omega.$$

Section 4-9: Boundary Conditions

Problem 4.43 With reference to Fig. 4-19, find $\mathbf{E}_1$ if $\mathbf{E}_2 = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}4$ (V/m), $\varepsilon_1 = 2\varepsilon_0$, $\varepsilon_2 = 18\varepsilon_0$, and the boundary has a surface charge density $\rho_s = 7.08 \times 10^{-11}$ (C/m^2). What angle does $\mathbf{E}_2$ make with the z-axis?

Solution: We know that $\mathbf{E}_{1t} = \mathbf{E}_{2t}$ for any 2 media. Hence, $\mathbf{E}_{1t} = \mathbf{E}_{2t} = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2$. Also, $(\mathbf{D}_1 - \mathbf{D}_2) \cdot \hat{\mathbf{n}} = \rho_s$ (from Table 4.3). Hence, $\varepsilon_1(\mathbf{E}_1 \cdot \hat{\mathbf{n}}) - \varepsilon_2(\mathbf{E}_2 \cdot \hat{\mathbf{n}}) = \rho_s$, which gives

$$E_{1z} = \frac{\rho_s + \varepsilon_2 E_{2z}}{\varepsilon_1} = \frac{7.08 \times 10^{-11}}{2\varepsilon_0} + \frac{18(4)}{2} = \frac{7.08 \times 10^{-11}}{2 \times 8.85 \times 10^{-12}} + 36 = 40 \quad \text{(V/m)}.$$

Hence, $\mathbf{E}_1 = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}40$ (V/m). Finding the angle $\mathbf{E}_2$ makes with the z-axis:

$$\mathbf{E}_2 \cdot \hat{\mathbf{z}} = |\mathbf{E}_2| \cos\theta, \qquad 4 = \sqrt{9+4+16}\cos\theta, \qquad \theta = \cos^{-1}\left(\frac{4}{\sqrt{29}}\right) = 42°.$$

Problem 4.44 An infinitely long dielectric cylinder with $\varepsilon_{1r} = 4$ and described by $r \leq 10$ cm is surrounded by a material with $\varepsilon_{2r} = 8$. If $\mathbf{E}_1 = \hat{\mathbf{r}}r^2 \sin\phi - \hat{\boldsymbol{\phi}}3r^2 \cos\phi + \hat{\mathbf{z}}3$ (V/m) in the cylinder region, find $\mathbf{E}_2$ and $\mathbf{D}_2$ in the surrounding region. Assume that no free charges exist along the cylinder's boundary.

Solution: Using Table 4-3, $E_{1t} = E_{2t}$, and $\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$.

$$\mathbf{E}_{2t} = \mathbf{E}_{1t} = -\hat{\boldsymbol{\phi}}3r^2 \cos\phi + \hat{\mathbf{z}}3 \quad \text{(V/m)},$$
$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n} \quad (\text{since } \rho_s = 0),$$
$$\hat{\mathbf{r}} \cdot \mathbf{E}_2 = E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n} = \frac{1}{2} r^2 \sin\phi.$$

Thus,

$$\mathbf{E}_2 = \hat{\mathbf{r}}\frac{1}{2}r^2 \sin\phi - \hat{\boldsymbol{\phi}}3r^2 \cos\phi + \hat{\mathbf{z}}3,$$
$$\mathbf{D}_2 = \varepsilon_2 \mathbf{E}_2 = 8\varepsilon_0 \left(\hat{\mathbf{r}}\frac{1}{2}r^2 \sin\phi - \hat{\boldsymbol{\phi}}3r^2 \cos\phi + \hat{\mathbf{z}}3\right) \quad \text{(C/m}^2\text{)}.$$

Problem 4.45 A 2-cm dielectric sphere with $\varepsilon_{1r} = 3$ is embedded in a medium with $\varepsilon_{2r} = 9$. If $\mathbf{E}_2 = \hat{\mathbf{R}}3\cos\theta - \hat{\boldsymbol{\theta}}3\sin\theta$ (V/m) in the surrounding region, find $\mathbf{E}_1$ and $\mathbf{D}_1$ in the sphere.

Solution: Using Table 4.3, $E_{1t} = E_{2t}$, and $\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$,

$$\rho_s = 0, \qquad D_{R_1} = D_{R_2},$$

$$E_{R_1} = \frac{\varepsilon_2}{\varepsilon_1} E_{R_2} = \left(\frac{9}{3}\right) 3\cos\theta = 9\cos\theta,$$
$$E_{1t} = E_{2t} = -3\sin\theta.$$

So

$$\mathbf{E}_1 = \hat{\mathbf{R}}9\cos\theta - \hat{\boldsymbol{\theta}}3\sin\theta \quad \text{(V/m)},$$
$$\mathbf{D}_1 = \varepsilon_1\mathbf{E}_1 = 3\varepsilon_0(\hat{\mathbf{R}}9\cos\theta - \hat{\boldsymbol{\theta}}3\sin\theta) = \varepsilon_0(\hat{\mathbf{R}}27\cos\theta - \hat{\boldsymbol{\theta}}9\sin\theta) \quad \text{(C/m}^2\text{)}.$$

Problem 4.46 If $\mathbf{E} = \hat{\mathbf{R}}50$ (V/m) at the surface of a 5-cm conducting sphere centered at the origin, what is the total charge Q on the sphere's surface?

Solution: From Table 4-3, $\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$. $\mathbf{E}_2$ inside the sphere is zero, since we assume it is a perfect conductor. Hence, for a sphere with surface area $S = 4\pi a^2$,

$$D_{1\text{R}} = \rho_s, \qquad E_{1\text{R}} = \frac{\rho_s}{\varepsilon_0} = \frac{Q}{S\varepsilon_0},$$
$$Q = E_{\text{R}} S\varepsilon_0 = (50)4\pi(0.05)^2\varepsilon_0 = \frac{\pi\varepsilon_0}{2} \quad \text{(C)}.$$

Problem 4.47 Figure 4-34(a) (P4.47) shows three planar dielectric slabs of equal thickness but with different dielectric constants. If $\mathbf{E}_0$ in air makes an angle of 45° with respect to the z-axis, find the angle of $\mathbf{E}$ in each of the other layers.

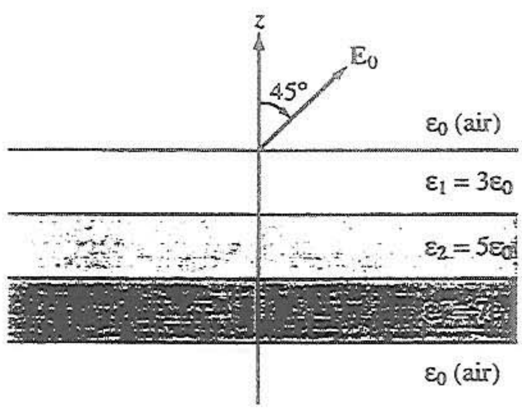


Figure P4.47: Dielectric slabs in Problem 4.47.

Solution: Labeling the upper air region as region 0 and using Eq. (4.99),

$$\theta_1 = \tan^{-1}\left(\frac{\varepsilon_1}{\varepsilon_0}\tan\theta_0\right) = \tan^{-1}(3\tan 45°) = 71.6°,$$
$$\theta_2 = \tan^{-1}\left(\frac{\varepsilon_2}{\varepsilon_1}\tan\theta_1\right) = \tan^{-1}\left(\frac{5}{3}\tan 71.6°\right) = 78.7°,$$
$$\theta_3 = \tan^{-1}\left(\frac{\varepsilon_3}{\varepsilon_2}\tan\theta_2\right) = \tan^{-1}\left(\frac{7}{5}\tan 78.7°\right) = 81.9°.$$

In the lower air region, the angle is again 45°.

Sections 4-10 and 4-11: Capacitance and Electrical Energy

Problem 4.48 Determine the force of attraction in a parallel-plate capacitor with $A = 10$ cm^2, $d = 1$ cm, and $\varepsilon_r = 4$ if the voltage across it is 50 V.

Solution: From Eq. (4.131),

$$\mathbf{F} = -\hat{\mathbf{z}}\frac{\varepsilon A|\mathbf{E}|^2}{2} = -\hat{\mathbf{z}}2\varepsilon_0(10\times10^{-4})\left(\frac{50}{0.01}\right)^2 = -\hat{\mathbf{z}}442.7\times10^{-9} \quad \text{(N)}.$$

Problem 4.49 Dielectric breakdown occurs in a material whenever the magnitude of the field $\mathbf{E}$ exceeds the dielectric strength anywhere in that material. In the coaxial capacitor of Example 4-12,

(a) At what value of r is $|E|$ maximum?

(b) What is the breakdown voltage if $a = 1$ cm, $b = 2$ cm, and the dielectric material is mica with $\varepsilon_r = 6$?

Solution:

(a) From Eq. (4.114), $\mathbf{E} = -\hat{\mathbf{r}}\rho_l/2\pi\varepsilon r$ for $a < r < b$. Thus, it is evident that $|\mathbf{E}|$ is maximum at $r = a$.

(b) The dielectric breaks down when $|\mathbf{E}| > 200$ (MV/m) (see Table 4-2), or

$$|\mathbf{E}| = \frac{\rho_l}{2\pi\varepsilon r} = \frac{\rho_l}{2\pi(6\varepsilon_0)(10^{-2})} = 200 \quad \text{(MV/m)},$$

which gives $\rho_l = (200 \text{ MV/m})(2\pi)6(8.854\times10^{-12})(0.01) = 667.6$ (μC/m).

From Eq. (4.115), we can find the voltage corresponding to that charge density,

$$V = \frac{\rho_l}{2\pi\varepsilon}\ln\left(\frac{b}{a}\right) = \frac{(667.6\,\mu\text{C/m})}{12\pi(8.854\times10^{-12}\text{ F/m})}\ln(2) = 1.39 \quad \text{(MV)}.$$

Thus, $V = 1.39$ (MV) is the breakdown voltage for this capacitor.

Problem 4.50 An electron with charge $Q_e = -1.6 \times 10^{-19}$ C and mass $m_e = 9.1 \times 10^{-31}$ kg is injected at a point adjacent to the negatively charged plate in the region between the plates of an air-filled parallel-plate capacitor with separation of 1 cm and rectangular plates each 10 cm^2 in area Fig. 4-33 (P4.50). If the voltage across the capacitor is 10 V, find

(a) the force acting on the electron,

(b) the acceleration of the electron, and

(c) the time it takes the electron to reach the positively charged plate, assuming that it starts from rest.

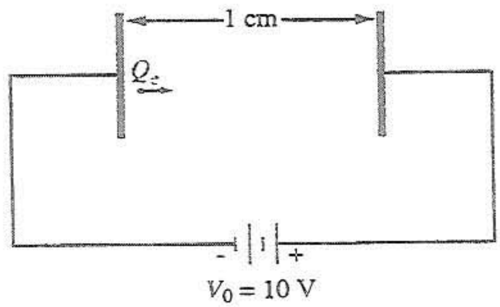


Figure P4.50: Electron between charged plates of Problem 4.50.

Solution:

(a) The electric force acting on a charge Q_e is given by Eq. (4.14) and the electric field in a capacitor is given by Eq. (4.112). Combining these two relations, we have

$$F = Q_e E = Q_e \frac{V}{d} = -1.6 \times 10^{-19} \frac{10}{0.01} = -1.6 \times 10^{-16} \quad \text{(N)}.$$

The force is directed from the negatively charged plate towards the positively charged plate.

(b)

$$a = \frac{F}{m} = \frac{1.6 \times 10^{-16}}{9.1 \times 10^{-31}} = 1.76 \times 10^{14} \quad \text{(m/s}^2\text{)}.$$

(c) The electron does not get fast enough at the end of its short trip for relativity to manifest itself; classical mechanics is adequate to find the transit time. From classical mechanics, $d = d_0 + u_0 t + \frac{1}{2}at^2$, where in the present case the start position is $d_0 = 0$, the total distance traveled is $d = 1$ cm, the initial velocity $u_0 = 0$, and the acceleration is given by part (b). Solving for the time t,

$$t = \sqrt{\frac{2d}{a}} = \sqrt{\frac{2 \times 0.01}{1.76 \times 10^{14}}} = 10.7 \times 10^{-9}\ \text{s} = 10.7 \quad \text{(ns)}.$$

Problem 4.51 In a dielectric medium with $\varepsilon_r = 4$, the electric field is given by

$$\mathbf{E} = \hat{\mathbf{x}}(x^2 + 2z) + \hat{\mathbf{y}}x^2 - \hat{\mathbf{z}}(y + z) \quad \text{(V/m)}.$$

Calculate the electrostatic energy stored in the region $-1\ \text{m} \leq x \leq 1\ \text{m},\ 0 \leq y \leq 2\ \text{m}$, and $0 \leq z \leq 3$ m.

Solution: Electrostatic potential energy is given by Eq. (4.124),

$$\begin{aligned} W_e &= \frac{1}{2}\int_{\mathcal{V}} \varepsilon |\mathbf{E}|^2\, d\mathcal{V} = \frac{\varepsilon}{2}\int_{z=0}^{3}\int_{y=0}^{2}\int_{x=-1}^{1} [(x^2 + 2z)^2 + x^4 + (y+z)^2]\, dx\, dy\, dz \\ &= \frac{4\varepsilon_0}{2}\left(\left(\left(\frac{2}{5}x^5 yz + \frac{2}{3}z^2 x^3 y + \frac{4}{3}z^3 xy + \frac{1}{12}(y+z)^4 x\right)\Bigg|_{x=-1}^{1}\right)\Bigg|_{y=0}^{2}\right)\Bigg|_{z=0}^{3} \\ &= \frac{4\varepsilon_0}{2}\left(\frac{1304}{5}\right) = 4.62 \times 10^{-9} \quad \text{(J)}. \end{aligned}$$

Problem 4.52 Figure 4-34a (P4.52(a)) depicts a capacitor consisting of two parallel, conducting plates separated by a distance d. The space between the plates contains two adjacent dielectrics, one with permittivity ε_1 and surface area A_1 and another with ε_2 and A_2. The objective of this problem is to show that the capacitance C of the configuration shown in Fig. 4-34a (P4.52(a)) is equivalent to two capacitances in parallel, as illustrated in Fig. 4-34b (P4.52(b)), with

$$C = C_1 + C_2, \tag{4.132}$$

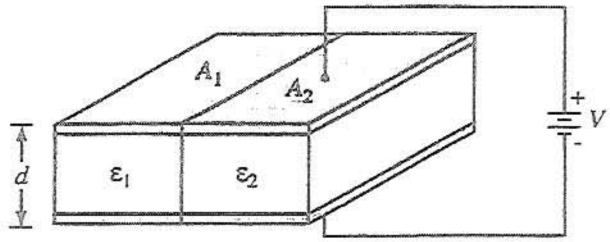
where

$$C_1 = \frac{\varepsilon_1 A_1}{d}, \tag{4.133}$$

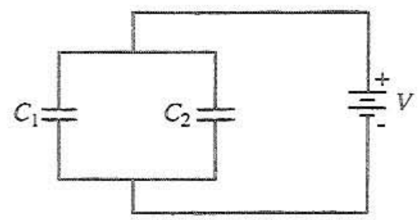
$$C_2 = \frac{\varepsilon_2 A_2}{d}. \tag{4.134}$$

To this end, you are asked to proceed as follows:

(a) Find the electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ in the two dielectric layers.



(a)



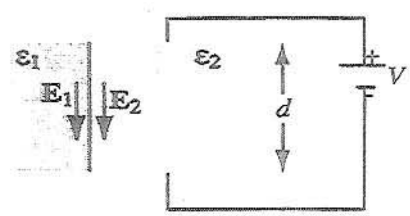
(b)

Figure P4.52: (a) Capacitor with parallel dielectric section, and (b) equivalent circuit.

(b) Calculate the energy stored in each section and use the result to calculate C_1 and C_2.

(c) Use the total energy stored in the capacitor to obtain an expression for C. Show that Eq. (4.132) is indeed a valid result.

Solution:



(c)

Figure P4.52: (c) Electric field inside of capacitor.

(a) Within each dielectric section, **E** will point from the plate with positive voltage to the plate with negative voltage, as shown in Fig. P4-52(c). From $V = Ed$,

$$E_1 = E_2 = \frac{V}{d}.$$

(b)

$$W_{e_1} = \frac{1}{2}\varepsilon_1 E_1^2 \cdot v = \frac{1}{2}\varepsilon_1 \frac{V^2}{d^2} \cdot A_1 d = \frac{1}{2}\varepsilon_1 V^2 \frac{A_1}{d}.$$

But, from Eq. (4.121),

$$W_{e_1} = \frac{1}{2} C_1 V^2.$$

Hence $C_1 = \varepsilon_1 \frac{A_1}{d}$. Similarly, $C_2 = \varepsilon_2 \frac{A_2}{d}$.

(c) Total energy is

$$W_e = W_{e_1} + W_{e_2} = \frac{1}{2}\frac{V^2}{d}(\varepsilon_1 A_1 + \varepsilon_2 A_2) = \frac{1}{2} C V^2.$$

Hence,

$$C = \frac{\varepsilon_1 A_1}{d} + \frac{\varepsilon_2 A_2}{d} = C_1 + C_2.$$

Problem 4.53 Use the result of Problem 4.52 to determine the capacitance for each of the following configurations:

(a) conducting plates are on top and bottom faces of rectangular structure in Fig. 4-35(a) (P4.53(a)),

(b) conducting plates are on front and back faces of structure in Fig. 4-35(a) (P4.53(a)),

(c) conducting plates are on top and bottom faces of the cylindrical structure in Fig. 4-35(b) (P4.53(b)).

Solution:

(a) The two capacitors share the same voltage; hence they are in parallel.

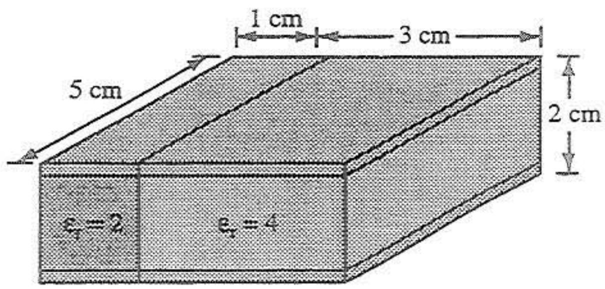
$$C_1 = \varepsilon_1 \frac{A_1}{d} = 2\varepsilon_0 \frac{(5 \times 1) \times 10^{-4}}{2 \times 10^{-2}} = 5\varepsilon_0 \times 10^{-2},$$

$$C_2 = \varepsilon_2 \frac{A_2}{d} = 4\varepsilon_0 \frac{(5 \times 3) \times 10^{-4}}{2 \times 10^{-2}} = 30\varepsilon_0 \times 10^{-2},$$

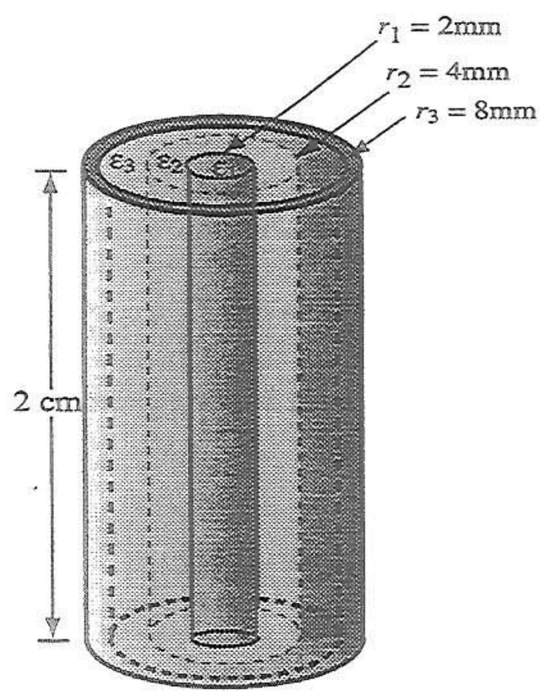
$$C = C_1 + C_2 = (5\varepsilon_0 + 30\varepsilon_0) \times 10^{-2} = 0.35\varepsilon_0 = 3.1 \times 10^{-12} \text{ F}.$$

(b)

$$C_1 = \varepsilon_1 \frac{A_1}{d} = 2\varepsilon_0 \frac{(2 \times 1) \times 10^{-4}}{5 \times 10^{-2}} = 0.8\,\varepsilon_0 \times 10^{-2},$$



(a)



$\varepsilon_1 = 8\varepsilon_0$; $\varepsilon_2 = 4\varepsilon_0$; $\varepsilon_3 = 2\varepsilon_0$

(b)

Figure P4.53: Dielectric sections for Problems 4.53 and 4.55.

$$C_2 = \varepsilon_2 \frac{A_2}{d} = 4\varepsilon_0 \frac{(3\times 2)\times 10^{-4}}{5\times 10^{-2}} = \frac{24}{5}\varepsilon_0 \times 10^{-2},$$
$$C = C_1 + C_2 = 0.5\times 10^{-12}\text{ F}.$$

(c)

$$C_1 = \varepsilon_1 \frac{A_1}{d} = 8\varepsilon_0 \frac{(\pi r_1^2)}{2\times 10^{-2}} = \frac{4\pi\varepsilon_0}{10^{-2}}(2\times 10^{-3})^2 = 0.04\times 10^{-12}\text{ F},$$
$$C_2 = \varepsilon_2 \frac{A_2}{d} = 4\varepsilon_0 \frac{(\pi r_2^2)}{2\times 10^{-2}} = \frac{2\pi\varepsilon_0}{10^{-2}}(4\times 10^{-3})^2 = 0.09\times 10^{-12}\text{ F},$$
$$C_3 = \varepsilon_3 \frac{A_3}{d} = 2\varepsilon_0 \frac{(\pi r_3^2)}{2\times 10^{-2}} = \frac{\pi\varepsilon_0}{10^{-2}}(8\times 10^{-3})^2 = 0.18\times 10^{-12}\text{ F},$$
$$C = C_1 + C_2 + C_3 = 0.31\times 10^{-12}\text{ F}.$$

Problem 4.54 The capacitor shown in Fig. 4-36 (P4.54) consists of two parallel dielectric layers. We wish to use energy considerations to show that the equivalent capacitance of the overall capacitor, C, is equal to the series combination of the capacitances of the individual layers, C_1 and C_2, namely

$$C = \frac{C_1 C_2}{C_1 + C_2}, \tag{4.136}$$

where

$$C_1 = \varepsilon_1 \frac{A}{d_1}, \qquad C_2 = \varepsilon_2 \frac{A}{d_2}.$$

(a) Let V_1 and V_2 be the electric potentials across the upper and lower dielectrics, respectively. What are the corresponding electric fields E_1 and E_2? By applying the appropriate boundary condition at the interface between the two dielectrics, obtain explicit expressions for E_1 and E_2 in terms of ε_1, ε_2, V, and the indicated dimensions of the capacitor.

(b) Calculate the energy stored in each of the dielectric layers and then use the sum to obtain an expression for C.

(c) Show that C is given by Eq. (4.136).

Solution:

(a) If V_1 is the voltage across the top layer and V_2 across the bottom layer, then

$$V = V_1 + V_2,$$

and

$$E_1 = \frac{V_1}{d_1}, \qquad E_2 = \frac{V_2}{d_2}.$$

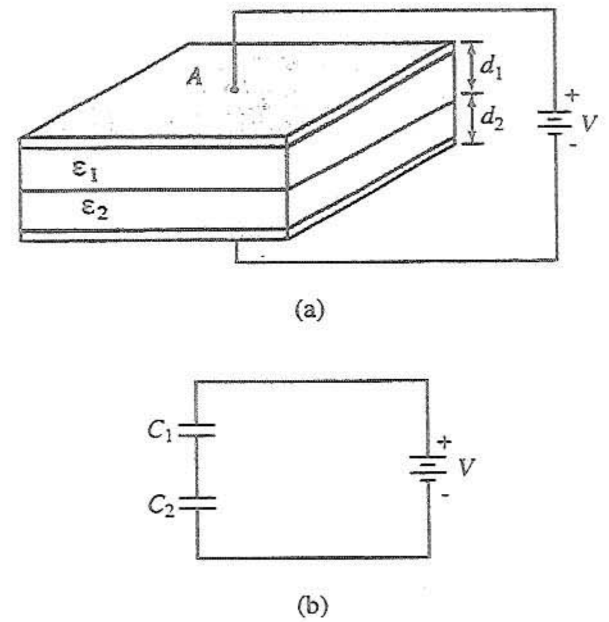


Figure P4.54: (a) Capacitor with parallel dielectric layers, and (b) equivalent circuit (Problem 4.54).

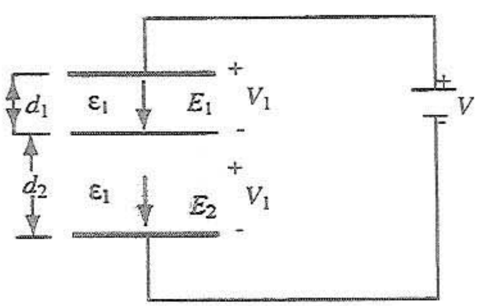


Figure P4.54: (c) Electric fields inside of capacitor.

According to boundary conditions, the normal component of $\mathbf{D}$ is continuous across the boundary (in the absence of surface charge). This means that at the interface between the two dielectric layers,

$$D_{1n} = D_{2n}$$

or

$$\varepsilon_1 E_1 = \varepsilon_2 E_2.$$

Hence,

$$V = E_1 d_1 + E_2 d_2 = E_1 d_1 + \frac{\varepsilon_1 E_1}{\varepsilon_2} d_2,$$

which can be solved for E_1:

$$E_1 = \frac{V}{d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2}.$$

Similarly,

$$E_2 = \frac{V}{d_2 + \frac{\varepsilon_2}{\varepsilon_1} d_1}.$$

(b)

$$W_{e_1} = \frac{1}{2}\varepsilon_1 E_1^2 \cdot \mathcal{V}_1 = \frac{1}{2}\varepsilon_1 \left(\frac{V}{d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2} \right)^2 \cdot A d_1 = \frac{1}{2} V^2 \left[\frac{\varepsilon_1 \varepsilon_2^2 A d_1}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} \right],$$

$$W_{e_2} = \frac{1}{2}\varepsilon_2 E_2^2 \cdot \mathcal{V}_2 = \frac{1}{2}\varepsilon_2 \left(\frac{V}{d_2 + \frac{\varepsilon_2}{\varepsilon_1} d_1} \right)^2 \cdot A d_2 = \frac{1}{2} V^2 \left[\frac{\varepsilon_1^2 \varepsilon_2 A d_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2} \right],$$

$$W_e = W_{e_1} + W_{e_2} = \frac{1}{2} V^2 \left[\frac{\varepsilon_1 \varepsilon_2^2 A d_1 + \varepsilon_1^2 \varepsilon_2 A d_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2} \right].$$

But $W_e = \frac{1}{2} C V^2$, hence,

$$C = \frac{\varepsilon_1 \varepsilon_2^2 A d_1 + \varepsilon_1^2 \varepsilon_2 A d_2}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} = \varepsilon_1 \varepsilon_2 A \frac{(\varepsilon_2 d_1 + \varepsilon_1 d_2)}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} = \frac{\varepsilon_1 \varepsilon_2 A}{\varepsilon_2 d_1 + \varepsilon_1 d_2}.$$

(c) Multiplying numerator and denominator of the expression for C by $A/d_1 d_2$, we have

$$C = \frac{\frac{\varepsilon_1 A}{d_1} \cdot \frac{\varepsilon_2 A}{d_2}}{\frac{\varepsilon_1 A}{d_1} + \frac{\varepsilon_2 A}{d_2}} = \frac{C_1 C_2}{C_1 + C_2},$$

where

$$C_1 = \frac{\varepsilon_1 A}{d_1}, \qquad C_2 = \frac{\varepsilon_2 A}{d_2}.$$

Problem 4.55 Use the expressions given in Problem 4.54 to determine the capacitance for the configurations in Fig. 4.35(a) (P4.55) when the conducting plates are placed on the right and left faces of the structure.

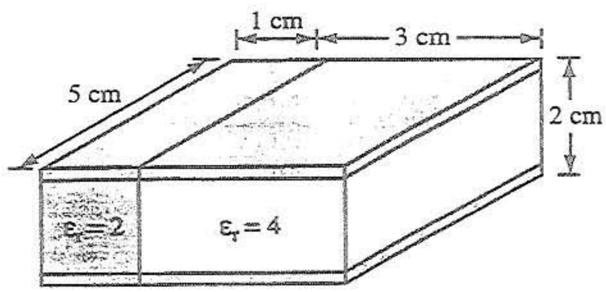


Figure P4.55: Dielectric section for Problem 4.55.

Solution:

$$C_1 = \varepsilon_1 \frac{A}{d_1} = 2\varepsilon_0 \frac{(2\times 5)\times 10^{-4}}{1\times 10^{-2}} = 20\varepsilon_0 \times 10^{-2} = 1.77\times 10^{-12} \text{ F},$$

$$C_2 = \varepsilon_2 \frac{A}{d_2} = 4\varepsilon_0 \frac{(2\times 5)\times 10^{-4}}{3\times 10^{-2}} = 1.18\times 10^{-12} \text{ F},$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{1.77\times 1.18}{1.77+1.18} \times 10^{-12} = 0.71\times 10^{-12} \text{ F}.$$

Section 4-12: Image Method

Problem 4.56 With reference to Fig. 4-37 (P4.56), charge Q is located at a distance d above a grounded half-plane located in the x–y plane and at a distance d from another grounded half-plane in the x–z plane. Use the image method to

(a) establish the magnitudes, polarities, and locations of the images of charge Q with respect to each of the two ground planes (as if each is infinite in extent), and

(b) then find the electric potential and electric field at an arbitrary point $P(0, y, z)$.

Solution:

(a) The original charge has magnitude and polarity $+Q$ at location $(0, d, d)$. Since the negative y-axis is shielded from the region of interest, there might as well be a conducting half-plane extending in the $-y$ direction as well as the $+y$ direction. This

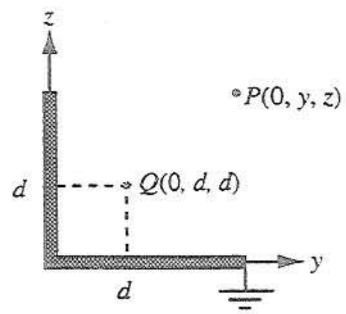


Figure P4.56: Charge Q next to two perpendicular, grounded, conducting half planes.

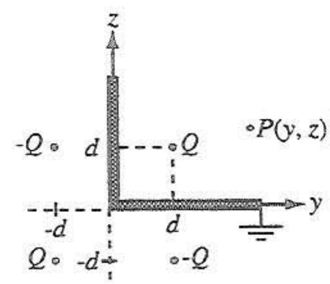


Figure P4.56: (a) Image charges.

ground plane gives rise to an image charge of magnitude and polarity $-Q$ at location $(0,d,-d)$. In addition, since charges exist on the conducting half plane in the $+z$ direction, an image of this conducting half plane also appears in the $-z$ direction. This ground plane in the x-z plane gives rise to the image charges of $-Q$ at $(0,-d,d)$ and $+Q$ at $(0,-d,-d)$.

(b) Using Eq. (4.47) with $N = 4$,

$$V(x,y,z) = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z-d)|} - \frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z-d)|} + \frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z+d)|} - \frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z+d)|}\right)$$

$$= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{\sqrt{x^2+(y-d)^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+(y+d)^2+(z-d)^2}} + \frac{1}{\sqrt{x^2+(y+d)^2+(z+d)^2}} - \frac{1}{\sqrt{x^2+(y-d)^2+(z+d)^2}}\right)$$

$$= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{\sqrt{x^2+y^2-2yd+z^2-2zd+2d^2}} - \frac{1}{\sqrt{x^2+y^2+2yd+z^2-2zd+2d^2}} + \frac{1}{\sqrt{x^2+y^2+2yd+z^2+2zd+2d^2}} - \frac{1}{\sqrt{x^2+y^2-2yd+z^2+2zd+2d^2}}\right) \quad \text{(V)}.$$

From Eq. (4.51),

$$\mathbf{E} = -\nabla V$$

$$= \frac{Q}{4\pi\varepsilon}\left(\nabla\frac{1}{\sqrt{x^2+(y-d)^2+(z-d)^2}} - \nabla\frac{1}{\sqrt{x^2+(y+d)^2+(z-d)^2}} + \nabla\frac{1}{\sqrt{x^2+(y+d)^2+(z+d)^2}} - \nabla\frac{1}{\sqrt{x^2+(y-d)^2+(z+d)^2}}\right)$$

$$= \frac{Q}{4\pi\varepsilon}\left(\frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z-d)}{(x^2+(y-d)^2+(z-d)^2)^{3/2}} - \frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z-d)}{(x^2+(y+d)^2+(z-d)^2)^{3/2}} + \frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z+d)}{(x^2+(y+d)^2+(z+d)^2)^{3/2}} - \frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z+d)}{(x^2+(y-d)^2+(z+d)^2)^{3/2}}\right) \quad \text{(V/m)}.$$

Problem 4.57 Conducting wires above a conducting plane carry currents I_1 and I_2 in the directions shown in Fig. 4-38 (P4.57). Keeping in mind that the direction of a current is defined in terms of the movement of positive charges, what are the directions of the image currents corresponding to I_1 and I_2?

Solution:

(a) In the image current, movement of negative charges downward = movement of positive charges upward. Hence, image of I_1 is same as I_1.

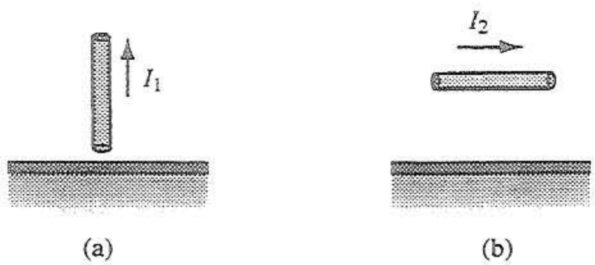


Figure P4.57: Currents above a conducting plane (Problem 4.57).

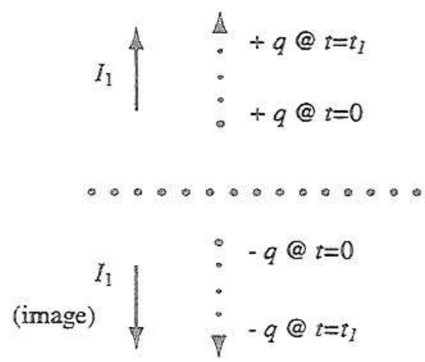


Figure P4.57: (a) Solution for part (a).

(b) In the image current, movement of negative charges to right = movement of positive charges to left.

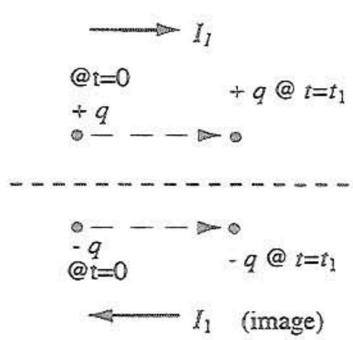


Figure P4.57: (b) Solution for part (b).

Problem 4.58 Use the image method to find the capacitance per unit length of an infinitely long conducting cylinder of radius a situated at a distance d from a parallel conducting plane, as shown in Fig. 4-39 (P4.58).

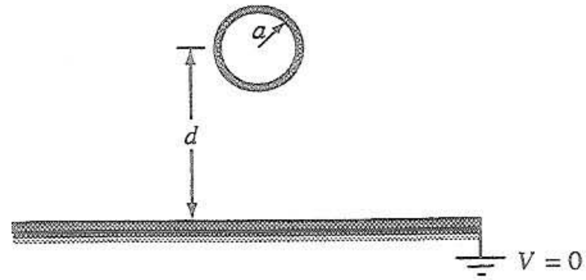


Figure P4.58: Conducting cylinder above a conducting plane (Problem 4.58).

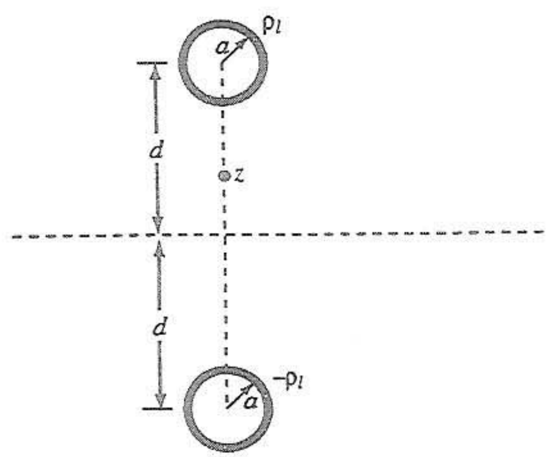


Figure P4.58: (a) Cylinder and its image.

Solution: Let us distribute charge ρ_l (C/m) on the conducting cylinder. Its image cylinder at $z = -d$ will have charge density $-\rho_l$.

For the line at $z = d$, the electric field at any point z (at a distance of $d - z$ from the center of the cylinder) is, from Eq. (4.33),

$$\mathbf{E}_1 = -\hat{\mathbf{z}}\frac{\rho_l}{2\pi\varepsilon_0(d-z)}$$

where $-\hat{\mathbf{z}}$ is the direction away from the cylinder. Similarly for the image cylinder at distance $(d+z)$ and carrying charge $-\rho_l$,

$$\mathbf{E}_2 = \hat{\mathbf{z}}\frac{(-\rho_l)}{2\pi\varepsilon_0(d+z)} = -\hat{\mathbf{z}}\frac{\rho_l}{2\pi\varepsilon_0(d+z)}.$$

The potential difference between the cylinders is obtained by integrating the total electric field from $z = -(d-a)$ to $z = (d-a)$:

$$\begin{aligned}
V &= -\int_2^1 (\mathbf{E}_1 + \mathbf{E}_2)\cdot\hat{\mathbf{z}}\,dz \\
&= -\int_{-(d-a)}^{d-a} -\hat{\mathbf{z}}\frac{\rho_l}{2\pi\varepsilon_0}\left(\frac{1}{d-z}+\frac{1}{d+z}\right)\cdot\hat{\mathbf{z}}\,dz \\
&= \frac{\rho_l}{2\pi\varepsilon_0}\int_{-(d-a)}^{d-a}\left(\frac{1}{d-z}+\frac{1}{d+z}\right)dz \\
&= \frac{\rho_l}{2\pi\varepsilon_0}\left[-\ln(d-z)+\ln(d+z)\right]_{-(d-a)}^{d-a} \\
&= \frac{\rho_l}{2\pi\varepsilon_0}\left[-\ln(a)+\ln(2d-a)+\ln(2d-a)-\ln(a)\right] \\
&= \frac{\rho_l}{\pi\varepsilon_0}\ln\left(\frac{2d-a}{a}\right).
\end{aligned}$$

For a length L, $Q = \rho_l L$ and

$$C = \frac{Q}{V} = \frac{\rho_l L}{(\rho_l/\pi\varepsilon_0)\ln[(2d-a)/a]},$$

and the capacitance per unit length is

$$C' = \frac{C}{L} = \frac{\pi\varepsilon_0}{\ln[(2d/a)-1]} \quad \text{(C/m)}.$$