Chapter 5

Sections 5-1: Forces and Torques

Problem 5.1 An electron with a speed of 4×10^6 m/s is projected along the positive x-direction into a medium containing a uniform magnetic flux density $\mathrm{B} = (\hat{\mathrm{x}}2 - \hat{\mathrm{z}}3)$ T. Given that $e = 1.6 \times 10^{-19}$ C and the mass of an electron is $m_e = 9.1 \times 10^{-31}$ kg, determine the initial acceleration vector of the electron (at the moment it is projected into the medium).

Solution: The acceleration vector of a free particle is the net force vector divided by the particle mass. Neglecting gravity, and using Eq. (5.3), we have

$$\mathrm{a} = \frac{\mathrm{F_m}}{m_e} = \frac{-e}{m_e}\mathrm{u} \times \mathrm{B} = \frac{-1.6 \times 10^{-19}}{9.1 \times 10^{-31}}(\hat{\mathrm{x}}4 \times 10^6) \times (\hat{\mathrm{x}}2 - \hat{\mathrm{z}}3)$$
$$= -\hat{\mathrm{y}}2.1 \times 10^{18} \quad (\mathrm{m/s^2}).$$

Problem 5.2 When a particle with charge q and mass m is introduced into a medium with a uniform field B such that the initial velocity of the particle u is perpendicular to B, as shown in Fig. 5-31 (P5.2), the magnetic force exerted on the particle causes it to move in a circle of radius a. By equating $\mathrm{F_m}$ to the centripetal force on the particle, determine a in terms of q, m, u, and B.

Solution: The centripetal force acting on the particle is given by $F_c = mu^2/a$.

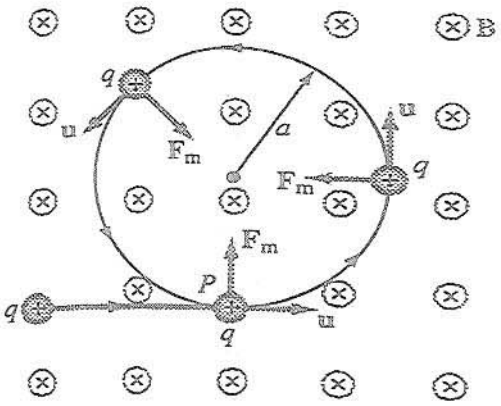

Figure P5.2: Particle of charge q projected with velocity u into a medium with a uniform field B perpendicular to u (Problem 5.2).

Equating F_c to F_m given by Eq. (5.4), we have $mu^2/a = quB\sin\theta$. Since the magnetic field is perpendicular to the particle velocity, $\sin\theta = 1$. Hence, $a = mu/qB$.

Problem 5.3 The circuit shown in Fig. 5-32 (P5.3) uses two identical springs to support a 10-cm-long horizontal wire with a mass of 5 g. In the absence of a magnetic field, the weight of the wire causes the springs to stretch a distance of 0.2 cm each. When a uniform magnetic field is turned on in the region containing the horizontal wire, the springs are observed to stretch an additional 0.5 cm. What is the intensity of the magnetic flux density $\mathbb{B}$?

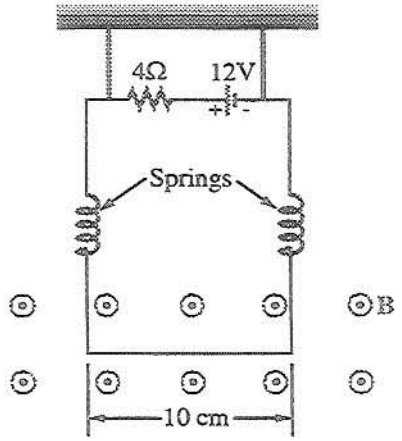

Figure P5.3: Configuration of Problem 5.3.

Solution: Springs are characterized by a spring constant k where $F = kd$ is the force exerted on the spring and d is the amount the spring is stretched from its rest configuration. In this instance, each spring sees half the weight of the wire:

$$F = \tfrac{1}{2}mg = kd, \qquad k = \frac{mg}{2d} = \frac{5\times10^{-3}\times9.8}{2\times2\times10^{-3}} = 12.25 \quad \text{(N/m)}.$$

Therefore, when the springs are further stretched by an additional 0.5 cm, this amounts to an additional force of $F = 12.25\,\text{N/m} \times (5\times10^{-3}\,\text{m}) = 61.25\,\text{mN}$ per spring, or a total additional force of $F = 0.123\,\text{N}$. This force is equal to the force exerted on the wire by the interaction of the magnetic field and the current as described by Eq. (5.12): $\mathbb{F}_m = I\ell \times \mathbb{B}$, where ℓ and $\mathbb{B}$ are at right angles. Moreover

$\ell \times \mathbb{B}$ is in the downward direction, and $I = V/R = 12\ \text{V}/4\ \Omega = 3\ \text{A}$. Therefore,

$$|\mathbb{F}_m| = |I||\ell||\mathbb{B}|, \qquad |\mathbb{B}| = \frac{|\mathbb{F}_m|}{|I||\ell|} = \frac{0.123}{3 \times 0.1} = 410 \quad (\text{mT}).$$

Problem 5.4 The rectangular loop shown in Fig. 5-33 (P5.4) consists of 20 closely wrapped turns and is hinged along the z-axis. The plane of the loop makes an angle of 30° with the y-axis, and the current in the windings is 0.5 A. What is the magnitude of the torque exerted on the loop in the presence of a uniform field $\mathbb{B} = \hat{y}1.2$ T? When viewed from above, is the expected direction of rotation clockwise or counterclockwise?

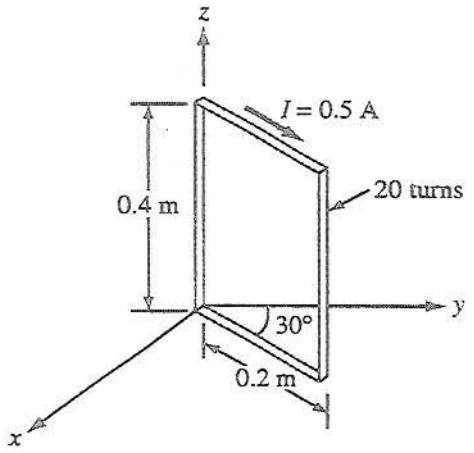

Figure P5.4: Hinged rectangular loop of Problem 5.4.

Solution: The magnetic torque on a loop is given by $\mathbb{T} = \mathbb{m} \times \mathbb{B}$ (Eq. (5.20)), where $\mathbb{m} = \hat{n}NIA$ (Eq. (5.19)). For this problem, it is given that $I = 0.5$ A, $N = 20$ turns, and $A = 0.2\ \text{m} \times 0.4\ \text{m} = 0.08\ \text{m}^2$. From the figure, $\hat{n} = -\hat{x}\cos 30° + \hat{y}\sin 30°$. Therefore, $\mathbb{m} = \hat{n}0.8\ (\text{A} \cdot \text{m}^2)$ and $\mathbb{T} = \hat{n}0.8\ (\text{A} \cdot \text{m}^2) \times \hat{y}1.2\ \text{T} = -\hat{z}0.83\ (\text{N} \cdot \text{m})$.. As the torque is negative, the direction of rotation is clockwise, looking from above.

Problem 5.5 In a cylindrical coordinate system, a 2-m-long straight wire carrying a current of 5 A in the positive z-direction is located at $r = 4$ cm, $\phi = \pi/2$, and $-1\ \text{m} \leq z \leq 1\ \text{m}$.

(a) If $\mathbb{B} = \hat{r}0.2\cos\phi$ (T), what is the magnetic force acting on the wire?

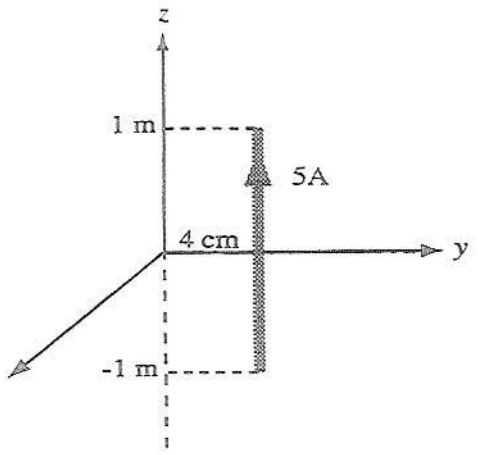

Figure P5.5: Problem 5.5.

(b) How much work is required to rotate the wire once about the z-axis in the negative ϕ-direction (while maintaining $r = 4$ cm)?

(c) At what angle ϕ is the force a maximum?

Solution:

(a)

$$\begin{aligned}\mathbb{F} &= I\ell \times \mathbb{B} \\ &= 5\hat{\mathbf{z}}2 \times [\hat{\mathbf{r}}0.2\cos\phi] \\ &= \hat{\boldsymbol{\phi}}2\cos\phi.\end{aligned}$$

At $\phi = \pi/2$, $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}$. Hence,

$$\mathbb{F} = -\hat{\mathbf{x}}2\cos(\pi/2) = 0.$$

(b)

$$\begin{aligned}W = \int_{\phi=0}^{2\pi} \mathbb{F}\cdot d\mathbb{l} &= \int_0^{2\pi} \hat{\boldsymbol{\phi}}[2\cos\phi]\cdot(-\hat{\boldsymbol{\phi}})r\,d\phi\Big|_{r=4\text{ cm}} \\ &= -2r\int_0^{2\pi}\cos\phi\,d\phi\Big|_{r=4\text{ cm}} = -8\times 10^{-2}[\sin\phi]_0^{2\pi} = 0.\end{aligned}$$

The force is in the $+\hat{\boldsymbol{\phi}}$-direction, which means that rotating it in the $-\hat{\boldsymbol{\phi}}$-direction would require work. However, the force varies as $\cos\phi$, which means it is positive

when $-\pi/2 \leq \phi \leq \pi/2$ and negative over the second half of the circle. Thus, work is provided by the force between $\phi = \pi/2$ and $\phi = -\pi/2$ (when rotated in the $-\hat{\phi}$-direction), and work is supplied for the second half of the rotation, resulting in a net work of zero.

(c) The force $\mathbf{F}$ is maximum when $\cos\phi = 1$, or $\phi = 0$.

Problem 5.6 A 20-turn rectangular coil with side $l = 15$ cm and $w = 5$ cm is placed in the y–z plane as shown in Fig. 5-34 (P5.6).

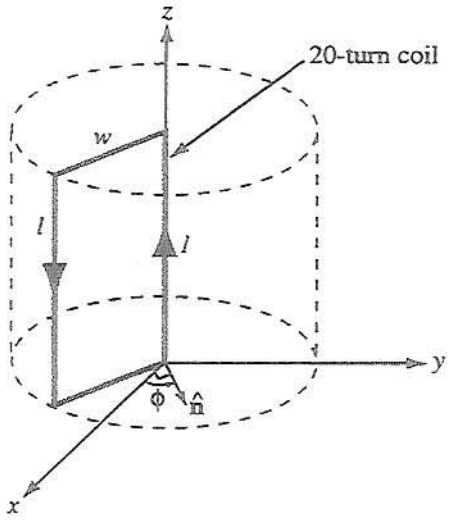

Figure P5.6: Rectangular loop of Problem 5.6.

(a) If the coil, which carries a current $I = 10$ A, is in the presence of a magnetic flux density

$$\mathbb{B} = 2 \times 10^{-2}(\hat{\mathbf{x}} + \hat{\mathbf{y}}2) \quad \text{(T)},$$

determine the torque acting on the coil.

(b) At what angle ϕ is the torque zero?

(c) At what angle ϕ is the torque maximum? Determine its value.

Solution:

(a) The magnetic field is in direction $(\hat{\mathbf{x}} + \hat{\mathbf{y}}2)$, which makes an angle $\phi_0 = \tan^{-1}\frac{2}{1} = 63.43°$.

The magnetic moment of the loop is

$$\mathbb{m} = \hat{\mathbf{n}}NIA = \hat{\mathbf{n}}20 \times 10 \times (15 \times 5) \times 10^{-4} = \hat{\mathbf{n}}1.5 \quad (\text{A}\cdot\text{m}^2),$$

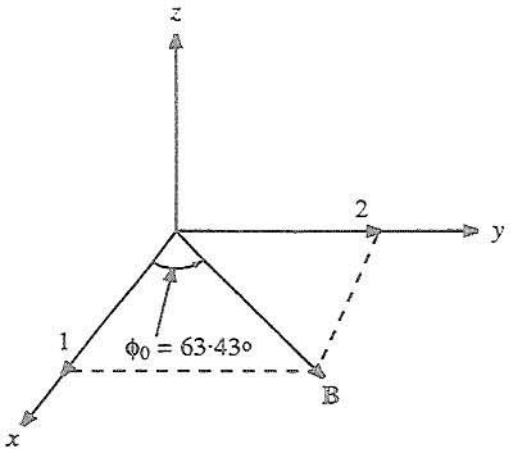

Figure P5.6: (a) Direction of $\mathbb{B}$.

where $\hat{\mathbf{n}}$ is the surface normal in accordance with the right-hand rule. When the loop is in the negative-y of the y–z plane, $\hat{\mathbf{n}}$ is equal to $\hat{\mathbf{x}}$, but when the plane of the loop is moved to an angle ϕ, $\hat{\mathbf{n}}$ becomes

$$\begin{aligned}\hat{\mathbf{n}} &= \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi,\\ \mathbb{T} &= \mathbb{m}\times\mathbb{B} = \hat{\mathbf{n}}\,1.5\times 2\times 10^{-2}(\hat{\mathbf{x}}+\hat{\mathbf{y}}2)\\ &= (\hat{\mathbf{x}}\cos\phi+\hat{\mathbf{y}}\sin\phi)\,1.5\times 2\times 10^{-2}(\hat{\mathbf{x}}+\hat{\mathbf{y}}2)\\ &= \hat{\mathbf{z}}3\times 10^{-2}[2\cos\phi-\sin\phi] \quad (\text{N}\cdot\text{m}).\end{aligned}$$

(b) The torque is zero when

$$2\cos\phi - \sin\phi = 0,$$

or

$$\tan\phi = 2, \qquad \phi = 63.43° \text{ or } -116.57°.$$

Thus, when $\hat{\mathbf{n}}$ is parallel to $\mathbb{B}$, $\mathbb{T} = 0$.

(c) The torque is a maximum when $\hat{\mathbf{n}}$ is perpendicular to $\mathbb{B}$, which occurs at

$$\phi = 63.43 \pm 90° = -26.57° \text{ or } +153.43°.$$

Mathematically, we can obtain the same result by taking the derivative of $\mathbb{T}$ and equating it to zero to find the values of ϕ at which $|\mathbb{T}|$ is a maximum. Thus,

$$\frac{\partial T}{\partial\phi} = \frac{\partial}{\partial\phi}(3\times 10^{-2}(2\cos\phi-\sin\phi)) = 0$$

or

$$-2\sin\phi + \cos\phi = 0,$$

which gives $\tan\phi = -\frac{1}{2}$, or

$$\phi = -26.57° \text{ or } 153.43°.$$

Section 5-2: Biot–Savart Law

Problem 5.7 An 8 cm × 12 cm rectangular loop of wire is situated in the x–y plane with the center of the loop at the origin and its long sides parallel to the x-axis. The loop has a current of 25 A flowing with clockwise direction (when viewed from above). Determine the magnetic field at the center of the loop.

Solution: The total magnetic field is the vector sum of the individual fields of each of the four wire segments: $\mathbb{B} = \mathbb{B}_1 + \mathbb{B}_2 + \mathbb{B}_3 + \mathbb{B}_4$. An expression for the magnetic field from a wire segment is given by Eq. (5.29).

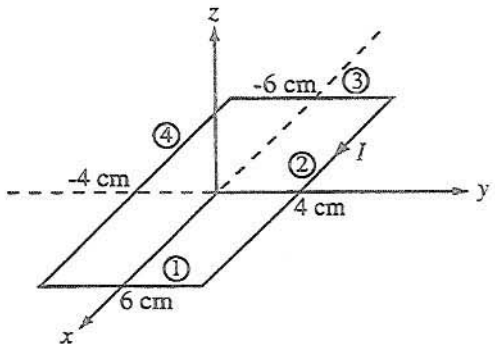

Figure P5.7: Problem 5.7.

For all segments shown in Fig. P5.7, the combination of the direction of the current and the right-hand rule gives the direction of the magnetic field as $-z$ direction at the origin. With $r = 6$ cm and $l = 8$ cm,

$$\mathbb{B}_1 = -\hat{z}\frac{\mu I l}{2\pi r\sqrt{4r^2 + l^2}}$$

$$= -\hat{z}\frac{4\pi \times 10^{-7} \times 25 \times 0.08}{2\pi \times 0.06 \times \sqrt{4 \times 0.06^2 + 0.08^2}} = -\hat{z}4.62 \times 10^{-5} \quad \text{(T)}.$$

For segment 2, $r = 4$ cm and $l = 12$ cm,

$$\mathbb{B}_2 = -\hat{z}\frac{\mu I l}{2\pi r\sqrt{4r^2+l^2}}$$
$$= -\hat{z}\frac{4\pi\times 10^{-7}\times 25\times 0.12}{2\pi\times 0.04\times\sqrt{4\times 0.04^2+0.12^2}} = -\hat{z}\,10.40\times 10^{-5} \quad \text{(T)}.$$

Similarly,

$$\mathbb{B}_3 = -\hat{z}\,4.62\times 10^{-5} \quad \text{(T)}, \qquad \mathbb{B}_4 = -\hat{z}\,10.40\times 10^{-5} \quad \text{(T)}.$$

The total field is then $\mathbb{B} = \mathbb{B}_1+\mathbb{B}_2+\mathbb{B}_3+\mathbb{B}_4 = -\hat{z}\,0.30$ (mT).

Problem 5.8 Use the approach outlined in Example 5-2 to develop an expression for the magnetic field $\mathbb{H}$ at an arbitrary point P due to the linear conductor defined by the geometry shown in Fig. 5-35 (P5.8). If the conductor extends between $z_1 = 3$ m and $z_2 = 7$ m and carries a current $I = 5$ A, find $\mathbb{H}$ at $P(2,\phi,0)$.

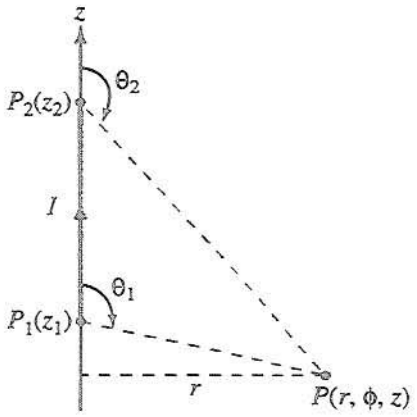

Figure P5.8: Current-carrying linear conductor of Problem 5.8.

Solution: The solution follows Example 5-2 up through Eq. (5.27), but the expressions for the cosines of the angles should be generalized to read as

$$\cos\theta_1 = \frac{z-z_1}{\sqrt{r^2+(z-z_1)^2}}, \qquad \cos\theta_2 = \frac{z-z_2}{\sqrt{r^2+(z-z_2)^2}}$$

instead of the expressions in Eq. (5.28), which are specialized to a wire centered at the origin. Plugging these expressions back into Eq. (5.27), the magnetic field is given as

$$\mathbf{H} = \hat{\phi}\frac{I}{4\pi r}\left(\frac{z-z_1}{\sqrt{r^2+(z-z_1)^2}} - \frac{z-z_2}{\sqrt{r^2+(z-z_2)^2}}\right).$$

For the specific geometry of Fig. P5.8,

$$\mathbf{H} = \hat{\phi}\frac{5}{4\pi\times 2}\left[\frac{0-3}{\sqrt{3^2+2^2}} - \frac{0-7}{\sqrt{7^2+2^2}}\right] = \hat{\phi}25.8\times 10^{-3}\ (\text{A/m}) = \hat{\phi}25.8 \quad (\text{mA/m}).$$

Problem 5.9 The loop shown in Fig. 5-36 (P5.9) consists of radial lines and segments of circles whose centers are at point P. Determine the magnetic field $\mathbf{H}$ at P in terms of a, b, θ, and I.

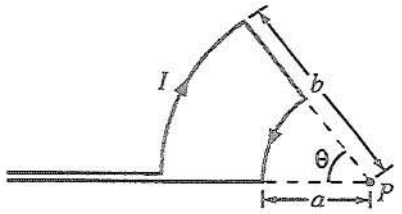

Figure P5.9: Configuration of Problem 5.9.

Solution: From the solution to Example 5-3, if we denote the z-axis as passing out of the page through point P, the magnetic field pointing out of the page at P due to the current flowing in the outer arc is $\mathbf{H}_{\text{outer}} = -\hat{z}I\theta/4\pi b$ and the field pointing out of the page at P due to the current flowing in the inner arc is $\mathbf{H}_{\text{inner}} = \hat{z}I\theta/4\pi a$. The other wire segments do not contribute to the magnetic field at P. Therefore, the total field flowing directly out of the page at P is

$$\mathbf{H} = \mathbf{H}_{\text{outer}} + \mathbf{H}_{\text{inner}} = \hat{z}\frac{I\theta}{4\pi}\left(\frac{1}{a}-\frac{1}{b}\right) = \hat{z}\frac{I\theta(b-a)}{4\pi ab}.$$

Problem 5.10 An infinitely long, thin conducting sheet defined over the space $0 \le x \le w$ and $-\infty \le y \le \infty$ is carrying a current with a uniform surface current

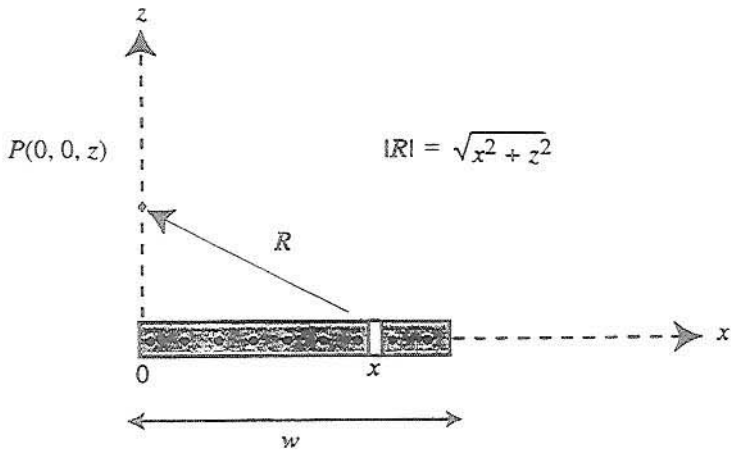

Figure P5.10: Conducting sheet of width w in x–y plane.

density $\mathbb{J}_s = \hat{y}5$ (A/m). Obtain an expression for the magnetic field at point $P(0,0,z)$ in Cartesian coordinates.

Solution: The sheet can be considered to be a large number of infinitely long but narrow wires each dx wide lying next to each other, with each carrying a current $I_x = J_s\,dx$. The wire at a distance x from the origin is at a distance vector $\mathbb{R}$ from point P, with

$$\mathbb{R} = -\hat{x}x + \hat{z}z.$$

Equation (5.30) provides an expression for the magnetic field due to an infinitely long wire carrying a current I as

$$\mathbb{H} = \frac{\mathbb{B}}{\mu_0} = \frac{\hat{\phi}I}{2\pi r}.$$

We now need to adapt this expression to the present situation by replacing I with $I_x = J_s\,dx$, replacing r with $R = (x^2+z^2)^{1/2}$, as shown in Fig. P5.10, and by assigning the proper direction for the magnetic field. From the Biot–Savart law, the direction of $\mathbb{H}$ is governed by $\mathbb{l} \times \mathbb{R}$, where $\mathbb{l}$ is the direction of current flow. In the present case, $\mathbb{l}$ is in the $\hat{y}$ direction. Hence, the direction of the field is

$$\frac{\mathbb{l}\times\mathbb{R}}{|\mathbb{l}\times\mathbb{R}|} = \frac{\hat{y}\times(-\hat{x}x+\hat{z}z)}{|\hat{y}\times(-\hat{x}x+\hat{z}z)|} = \frac{\hat{x}z+\hat{z}x}{(x^2+z^2)^{1/2}}.$$

Therefore, the field $d\mathbf{H}$ due to the current I_x is

$$d\mathbf{H} = \frac{\hat{\mathbf{x}}z + \hat{\mathbf{z}}x}{(x^2+z^2)^{1/2}} \frac{I_x}{2\pi R} = \frac{(\hat{\mathbf{x}}z + \hat{\mathbf{z}}x)J_s\,dx}{2\pi(x^2+z^2)},$$

and the total field is

$$\begin{aligned}
\mathbf{H}(0,0,z) &= \int_{x=0}^{w} (\hat{\mathbf{x}}z + \hat{\mathbf{z}}x)\frac{J_s\,dx}{2\pi(x^2+z^2)} \\
&= \frac{J_s}{2\pi}\int_{x=0}^{w} (\hat{\mathbf{x}}z + \hat{\mathbf{z}}x)\frac{dx}{x^2+z^2} \\
&= \frac{J_s}{2\pi}\left(\hat{\mathbf{x}}z\int_{x=0}^{w}\frac{dx}{x^2+z^2} + \hat{\mathbf{z}}\int_{x=0}^{w}\frac{x\,dx}{x^2+z^2}\right) \\
&= \frac{J_s}{2\pi}\left(\hat{\mathbf{x}}z\left(\frac{1}{z}\tan^{-1}\left(\frac{x}{z}\right)\right)\Big|_{x=0}^{w} + \hat{\mathbf{z}}\left(\tfrac{1}{2}\ln(x^2+z^2)\right)\Big|_{x=0}^{w}\right) \\
&= \hat{\mathbf{x}}\frac{J_s}{2\pi}\tan^{-1}\left(\frac{w}{z}\right) + \hat{\mathbf{z}}\tfrac{1}{2}(\ln(w^2+z^2) - \ln(0+z^2)) \quad \text{for } z \neq 0, \\
&= \hat{\mathbf{x}}\frac{J_s}{2\pi}\tan^{-1}\left(\frac{w}{z}\right) + \hat{\mathbf{z}}\tfrac{1}{2}\ln\left(\frac{w^2+z^2}{z^2}\right) \quad \text{(A/m)} \quad \text{for } z \neq 0.
\end{aligned}$$

An alternative approach is to employ Eq. (5.24a) directly.

Problem 5.11 An infinitely long wire carrying a 50-A current in the positive x-direction is placed along the x-axis in the vicinity of a 10-turn circular loop located in the x–y plane as shown in Fig. 5-37 (P5.11(a)). If the magnetic field at the center of the loop is zero, what is the direction and magnitude of the current flowing in the loop?

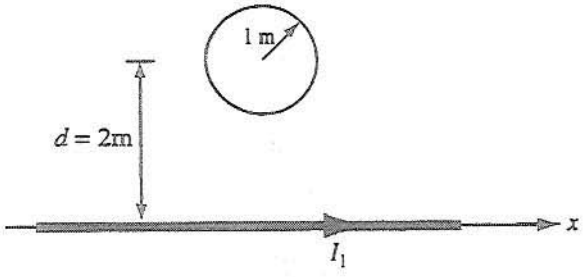

Figure P5.11: (a) Circular loop next to a linear current (Problem 5.11).

Solution: From Eq. (5.30), the magnetic flux density at the center of the loop due to

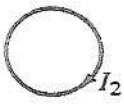

Figure P5.11: (b) Direction of I_2.

the wire is

$$\mathbb{B}_1 = \hat{z}\frac{\mu_0}{2\pi d}I_1$$

where $\hat{z}$ is out of the page. Since the net field is zero at the center of the loop, I_2 must be clockwise, as seen from above, in order to oppose I_1. The field due to I_2 is, from Eq. (5.35),

$$\mathbb{B} = \mu_0\mathbb{H} = -\hat{z}\frac{\mu_0 N I_2}{2a}.$$

Equating the magnitudes of the two fields, we obtain the result

$$\frac{NI_2}{2a} = \frac{I_1}{2\pi d},$$

or

$$I_2 = \frac{2aI_1}{2\pi Nd} = \frac{1\times 50}{\pi\times 2\times 10} = 0.8 \text{ A}.$$

Problem 5.12 Two infinitely long, parallel wires carry 6-A currents in opposite directions. Determine the magnetic flux density at point P in Fig. 5-38 (P5.12).

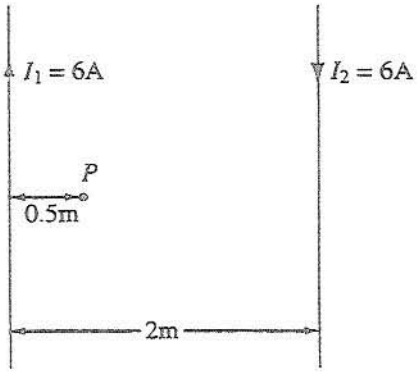

Figure P5.12: Arrangement for Problem 5.12.

Solution:

$$\mathbb{B} = \hat{\phi}\frac{\mu_0 I_1}{2\pi(0.5)} + \hat{\phi}\frac{\mu_0 I_2}{2\pi(1.5)} = \hat{\phi}\frac{\mu_0}{\pi}(6+2) = \hat{\phi}\frac{8\mu_0}{\pi} \quad \text{(T)}.$$

Problem 5.13 A long, East-West oriented power cable carrying an unknown current I is at a height of 8 m above the Earth's surface. If the magnetic flux density recorded by a magnetic-field meter placed at the surface is 12 μT when the current is flowing through the cable and 20 μT when the current is zero, what is the magnitude of I?

Solution: The power cable is producing a magnetic flux density that opposes Earth's, own magnetic field. An East–West cable would produce a field whose direction at the surface is along North–South. The flux density due to the cable is

$$B = (20 - 12)\,\mu\text{T} = 8\mu\text{T}.$$

As a magnet, the Earth's field lines are directed from the South Pole to the North Pole inside the Earth and the opposite on the surface. Thus the lines at the surface are from North to South, which means that the field created by the cable is from South to North. Hence, by the right-hand rule, the current direction is toward the East. Its magnitude is obtained from

$$8\,\mu\text{T} = 8 \times 10^{-6} = \frac{\mu_0 I}{2\pi d} = \frac{4\pi \times 10^{-7} I}{2\pi \times 8},$$

which gives $I = 320$ A.

Problem 5.14 Two parallel, circular loops carrying a current of 20 A each are arranged as shown in Fig. 5-39 (P5.14). The first loop is situated in the x–y plane with its center at the origin and the second loop's center is at $z = 2$ m. If the two loops have the same radius $a = 3$ m, determine the magnetic field at:

(a) $z = 0$,

(b) $z = 1$ m,

(c) $z = 2$ m.

Solution: The magnetic field due to a circular loop is given by (5.34) for a loop in the x–y plane carrying a current I in the $+\hat{\phi}$-direction. Considering that the bottom loop in Fig. P5.14 is in the x–y plane, but the current direction is along $-\hat{\phi}$,

$$\mathbf{H}_1 = -\hat{z}\frac{Ia^2}{2(a^2 + z^2)^{3/2}},$$

where z is the observation point along the z-axis. For the second loop, which is at a height of 2 m, we can use the same expression but z should be replaced with $(z - 2)$. Hence,

$$\mathbf{H}_2 = -\hat{z}\frac{Ia^2}{2[a^2 + (z-2)^2]^{3/2}}.$$

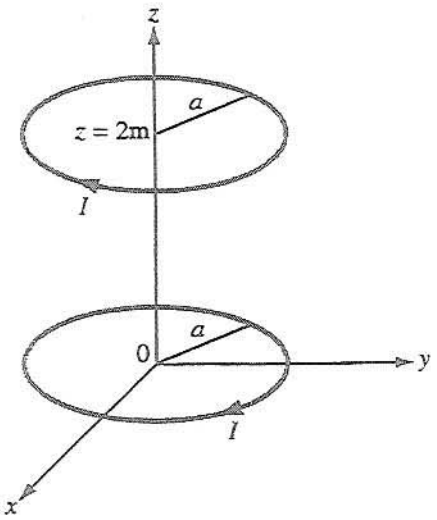

Figure P5.14: Parallel circular loops of Problem 5.14.

The total field is

$$\mathbb{H} = \mathbb{H}_1 + \mathbb{H}_2 = -\hat{z}\frac{Ia^2}{2}\left[\frac{1}{(a^2+z^2)^{3/2}} + \frac{1}{[a^2+(z-2)^2]^{3/2}}\right] \text{ A/m.}$$

(a) At $z = 0$, and with $a = 3$ m and $I = 20$ A,

$$\mathbb{H} = -\hat{z}\frac{20\times 9}{2}\left[\frac{1}{3^3} + \frac{1}{(9+4)^{3/2}}\right] = -\hat{z}5.25 \text{ A/m.}$$

(b) At $z = 1$ m (midway between the loops):

$$\mathbb{H} = -\hat{z}\frac{20\times 9}{2}\left[\frac{1}{(9+1)^{3/2}} + \frac{1}{(9+1)^{3/2}}\right] = -\hat{z}5.69 \text{ A/m.}$$

(c) At $z = 2$ m, $\mathbb{H}$ should be the same as at $z = 0$. Thus,

$$\mathbb{H} = -\hat{z}5.25 \text{ A/m.}$$

Section 5-3: Forces between Currents

Problem 5.15 The long, straight conductor shown in Fig. 5-40 (P5.15) lies in the plane of the rectangular loop at a distance $d = 0.1$ m. The loop has dimensions

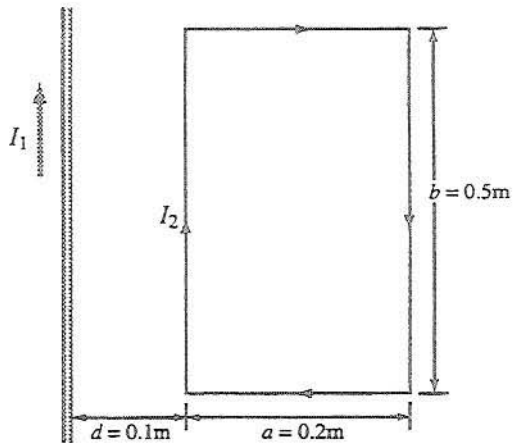

Figure P5.15: Current loop next to a conducting wire (Problem 5.15).

$a = 0.2$ m and $b = 0.5$ m, and the currents are $I_1 = 10$ A and $I_2 = 15$ A. Determine the net magnetic force acting on the loop.

Solution: The net magnetic force on the loop is due to the magnetic field surrounding the wire carrying current I_1. The magnetic forces on the loop as a whole due to the current in the loop itself are canceled out by symmetry. Consider the wire carrying I_1 to coincide with the z-axis, and the loop to lie in the $+x$ side of the x-z plane. Assuming the wire and the loop are surrounded by free space or other nonmagnetic material, Eq. (5.30) gives

$$\mathbb{B} = \hat{\phi}\frac{\mu_0 I_1}{2\pi r}.$$

In the plane of the loop, this magnetic field is

$$\mathbb{B} = \hat{y}\frac{\mu_0 I_1}{2\pi x}.$$

Then, from Eq. (5.12), the force on the side of the loop nearest the wire is

$$\mathbb{F}_{m1} = I_2 \ell \times \mathbb{B} = I_2(\hat{z}b) \times \left(\hat{y}\frac{\mu_0 I_1}{2\pi x}\right)\Big|_{x=d} = -\hat{x}\frac{\mu_0 I_1 I_2 b}{2\pi d}.$$

The force on the side of the loop farthest from the wire is

$$\mathbb{F}_{m2} = I_2 \ell \times \mathbb{B} = I_2(-\hat{z}b) \times \left(\hat{y}\frac{\mu_0 I_1}{2\pi x}\right)\Big|_{x=a+d} = \hat{x}\frac{\mu_0 I_1 I_2 b}{2\pi(a+d)}.$$

The other two sides do not contribute any net forces to the loop because they are equal in magnitude and opposite in direction. Therefore, the total force on the loop is

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_{m1} + \mathbf{F}_{m2} \\ &= -\hat{\mathbf{x}}\frac{\mu_0 I_1 I_2 b}{2\pi d} + \hat{\mathbf{x}}\frac{\mu_0 I_1 I_2 b}{2\pi(a+d)} \\ &= -\hat{\mathbf{x}}\frac{\mu_0 I_1 I_2 ab}{2\pi d(a+d)} \\ &= -\hat{\mathbf{x}}\frac{4\pi \times 10^{-7} \times 10 \times 15 \times 0.2 \times 0.5}{2\pi \times 0.1 \times 0.3} = -\hat{\mathbf{x}}10^{-4}\ (\text{N}) = -\hat{\mathbf{x}}0.1 \quad (\text{mN}).\end{aligned}$$

The force is pulling the loop toward the wire.

Problem 5.16 In the arrangement shown in Fig. 5-41 (P5.16), each of the two long, parallel conductors carries a current I, is supported by 8-cm-long strings, and has a mass per unit length of 0.3 g/cm. Due to the repulsive force acting on the conductors, the angle θ between the supporting strings is 10°. Determine the magnitude of I and the relative directions of the currents in the two conductors.

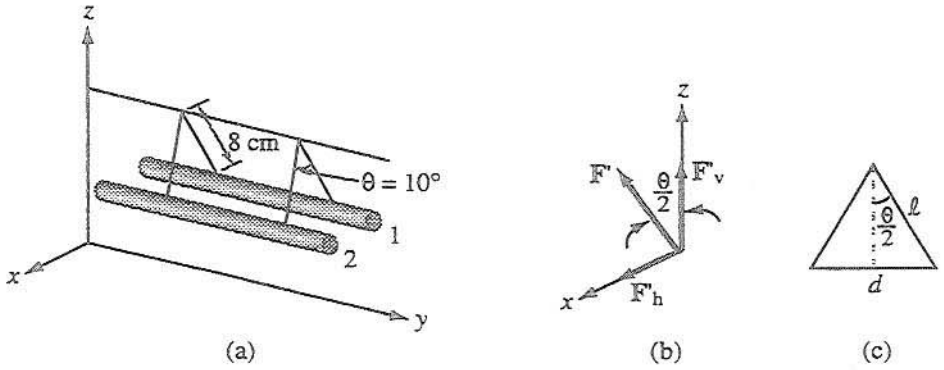

Figure P5.16: Parallel conductors supported by strings (Problem 5.16).

Solution: While the vertical component of the tension in the strings is counteracting the force of gravity on the wires, the horizontal component of the tension in the strings is counteracting the magnetic force, which is pushing the wires apart. According to Section 5-3, the magnetic force is repulsive when the currents are in opposite directions.

Figure P5.16(b) shows forces on wire 1 of part (a). The quantity F' is the tension force per unit length of wire due to the mass per unit length $m' = 0.3$ g/cm. The

vertical component of $\mathbf{F}'$ balances out the gravitational force,

$$F'_{\rm v} = m'g, \tag{9}$$

where $g = 9.8$ (m/s^2). But

$$F'_{\rm v} = F'\cos(\theta/2). \tag{10}$$

Hence,

$$F' = \frac{m'g}{\cos(\theta/2)}. \tag{11}$$

The horizontal component of $\mathbf{F}'$ must be equal to the repulsion magnitude force given by Eq. (5.42):

$$F'_{\rm h} = \frac{\mu_0 I^2}{2\pi d} = \frac{\mu_0 I^2}{2\pi[2\ell\sin(\theta/2)]}, \tag{12}$$

where d is the spacing between the wires and ℓ is the length of the string, as shown in Fig. P5.16(c). From Fig. 5.16(b),

$$F'_{\rm h} = F'\sin(\theta/2) = \frac{m'g}{\cos(\theta/2)}\sin(\theta/2) = m'g\tan(\theta/2). \tag{13}$$

Equating Eqs. (12) and (13) and then solving for I, we have

$$I = \sin(\theta/2)\sqrt{\frac{4\pi\ell m'g}{\mu_0\cos(\theta/2)}} = \sin 5^\circ\sqrt{\frac{4\pi\times 0.08\times 0.03\times 9.8}{4\pi\times 10^{-7}\cos 5^\circ}} = 42.4 \quad \text{(A)}.$$

Problem 5.17 An infinitely long, thin conducting sheet of width w along the x-direction lies in the x–y plane and carries a current I in the $-y$-direction. Determine (a) the magnetic field at a point P midway between the edges of the sheet and at a height h above it (Fig. 5-42 (P5.17)), and then (b) determine the force per unit length exerted on an infinitely long wire passing through point P and parallel to the sheet if the current through the wire is equal in magnitude but opposite in direction to that carried by the sheet.

Solution:

(a) The sheet can be considered to consist of a large number of infinitely long but narrow wires each dx wide lying next to each other, with each carrying a current

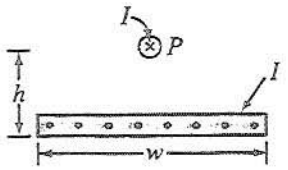

Figure P5.17: A linear current source above a current sheet (Problem 5.17).

$I_x = I\,dx/w$. If we choose the coordinate system shown in Fig. P5.17, the wire at a distance x from the origin is at a distance vector $\mathbb{R}$ from point P, with

$$\mathbb{R} = -\hat{\mathbb{x}}x + \hat{\mathbb{z}}h.$$

Equation (5.30) provides an expression for the magnetic field due to an infinitely long wire carrying a current I as

$$\mathbb{H} = \frac{\mathbb{B}}{\mu_0} = \hat{\phi}\frac{I}{2\pi r}.$$

We now need to adapt this expression to the present situation by replacing I with $I_x = I\,dx/w$, replacing r with $R = (x^2+h^2)^{1/2}$, and by assigning the proper direction for the magnetic field. From the Biot–Savart law, the direction of $\mathbb{H}$ is governed by $\mathbb{l}\times\mathbb{R}$, where $\mathbb{l}$ is the direction of current flow. In the present case, $\mathbb{l}$ is out of the page, which is the $-\hat{\mathbb{y}}$ direction. Hence, the direction of the field is

$$\frac{\mathbb{l}\times\mathbb{R}}{|\mathbb{l}\times\mathbb{R}|} = \frac{-\hat{\mathbb{y}}\times(-\hat{\mathbb{x}}x+\hat{\mathbb{z}}h)}{|-\hat{\mathbb{y}}\times(-\hat{\mathbb{x}}x+\hat{\mathbb{z}}h)|} = \frac{-(\hat{\mathbb{x}}h+\hat{\mathbb{z}}x)}{(x^2+h^2)^{1/2}}.$$

Therefore, the field $d\mathbb{H}$ due to current I_x is

$$d\mathbb{H} = \frac{-(\hat{\mathbb{x}}h+\hat{\mathbb{z}}x)}{(x^2+h^2)^{1/2}}\frac{I_x}{2\pi R} = \frac{-(\hat{\mathbb{x}}h+\hat{\mathbb{z}}x)I\,dx}{2\pi w(x^2+h^2)},$$

and the total field is

$$\begin{aligned}\mathbb{H}(0,0,h) &= \int_{x=-w/2}^{w/2} -(\hat{\mathbb{x}}h+\hat{\mathbb{z}}x)\frac{I\,dx}{2\pi w(x^2+h^2)} \\ &= \frac{-I}{2\pi w}\int_{x=-w/2}^{w/2}(\hat{\mathbb{x}}h+\hat{\mathbb{z}}x)\frac{dx}{x^2+h^2} \\ &= \frac{-I}{2\pi w}\left(\hat{\mathbb{x}}h\int_{x=-w/2}^{w/2}\frac{dx}{x^2+h^2} + \hat{\mathbb{z}}\int_{x=-w/2}^{w/2}\frac{x\,dx}{x^2+h^2}\right)\end{aligned}$$

$$= \frac{-I}{2\pi w}\left(\hat{\mathrm{x}}h\left(\frac{1}{h}\tan^{-1}\left(\frac{x}{h}\right)\right)\Big|_{x=-w/2}^{w/2} + \hat{\mathrm{z}}\left(\tfrac{1}{2}\ln(x^2+h^2)\right)\Big|_{x=-w/2}^{w/2}\right)$$

$$= -\hat{\mathrm{x}}\frac{I}{\pi w}\tan^{-1}\left(\frac{w}{2h}\right) \quad \text{(A/m)}.$$

At P in Fig. P5.17, the field is pointing to the left. The z-component could have been assumed zero with a symmetry argument. An alternative solution is to employ Eq. (5.24a) directly.

(b) From Eq. (5.9), a differential force is of the form $d\mathbf{F}_{\mathrm{m}} = I\,d\mathbf{l} \times \mathbf{B}$ or, assuming $d\mathbf{l} = \hat{\mathrm{a}}_\ell\, d\ell$, the force per unit length is given by

$$\mathbf{F}'_{\mathrm{m}} = \frac{\partial \mathbf{F}_{\mathrm{m}}}{\partial \ell} = I\hat{\mathrm{a}}_\ell \times \mathbf{B} = I\hat{\mathrm{y}} \times \left(-\hat{\mathrm{x}}\frac{\mu_0 I}{\pi w}\tan^{-1}\left(\frac{w}{2h}\right)\right) = \hat{\mathrm{z}}\frac{\mu_0 I^2}{\pi w}\tan^{-1}\left(\frac{w}{2h}\right) \quad \text{(N)}.$$

The force is repulsive; the wire is experiencing a force pushing it up.

Problem 5.18 Three long, parallel wires are arranged as shown in Fig. 5-43 (P5.18(a)). Determine the force per unit length acting on the wire carrying I_3.

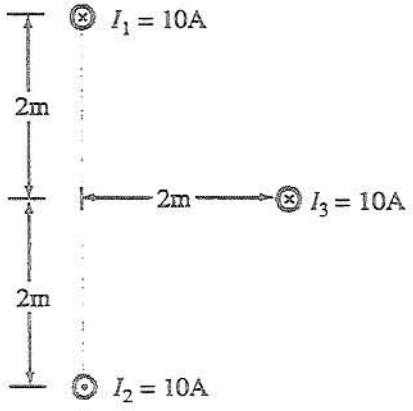

Figure P5.18: (a) Three parallel wires of Problem 5.18.

Solution: Since I_1 and I_2 are equal in magnitude and opposite in direction, and equidistant from I_3, our intuitive answer might be that the net force on I_3 is zero. As

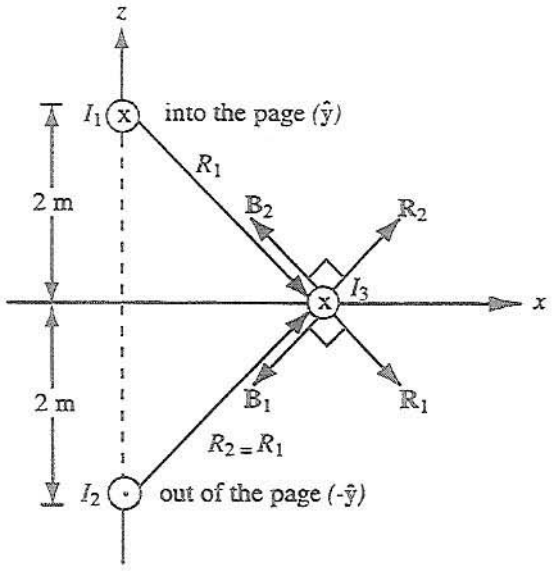

Figure P5.18: (b) $\mathbb{B}$ fields due to I_1 and I_2 at location of I_3.

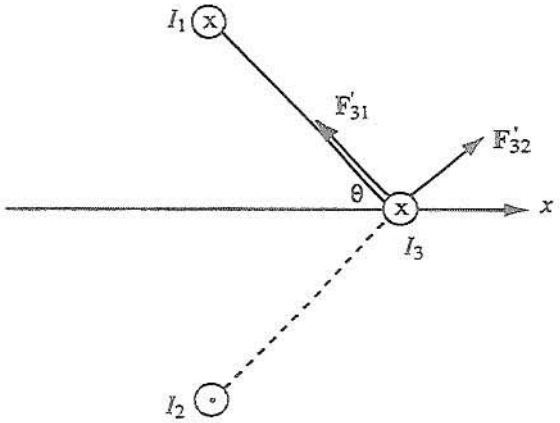

Figure P5.18: (c) Forces acting on I_3.

we will see, that's not the correct answer. The field due to I_1 (which is along $\hat{y}$) at location of I_3 is

$$\mathbb{B}_1 = \hat{b}_1 \frac{\mu_0 I_1}{2\pi R_1}$$

where $\hat{b}_1$ is the unit vector in the direction of $\mathbb{B}_1$ shown in the figure, which is

perpendicular to $\hat{\mathbf{R}}_1$. The force per unit length exerted on I_3 is

$$\mathbf{F}'_{31} = \frac{\mu_0 I_1 I_3}{2\pi R_1}(\hat{\mathbf{y}} \times \hat{\mathbf{b}}_1) = -\hat{\mathbf{R}}_1 \frac{\mu_0 I_1 I_3}{2\pi R_1}.$$

Similarly, the force per unit length excited on I_3 by the field due to I_2 (which is along $-\hat{\mathbf{y}}$) is

$$\mathbf{F}'_{32} = \hat{\mathbf{R}}_2 \frac{\mu_0 I_2 I_3}{2\pi R_2}.$$

The two forces have opposite components along $\hat{\mathbf{x}}$ and equal components along $\hat{\mathbf{z}}$. Hence, with $R_1 = R_2 = \sqrt{8}$ m and $\theta = \sin^{-1}(2/\sqrt{8}) = \sin^{-1}(1/\sqrt{2}) = 45°$,

$$\begin{aligned}\mathbf{F}'_3 = \mathbf{F}'_{31} + \mathbf{F}'_{32} &= \hat{\mathbf{z}}\left(\frac{\mu_0 I_1 I_3}{2\pi R_1} + \frac{\mu_0 I_2 I_3}{2\pi R_2}\right)\sin\theta \\ &= \hat{\mathbf{z}}2\left(\frac{4\pi \times 10^{-7} \times 10 \times 20}{2\pi \times \sqrt{8}}\right) \times \frac{1}{\sqrt{2}} = \hat{\mathbf{z}}2 \times 10^{-5} \text{ N/m}.\end{aligned}$$

Problem 5.19 A square loop placed as shown in Fig. 5-44 (P5.19) has 2-m sides and carries a current $I_1 = 5$ A. If a straight, long conductor carrying a current $I_2 = 10$ A is introduced and placed just above the midpoints of two of the loop's sides, determine the net force acting on the loop.

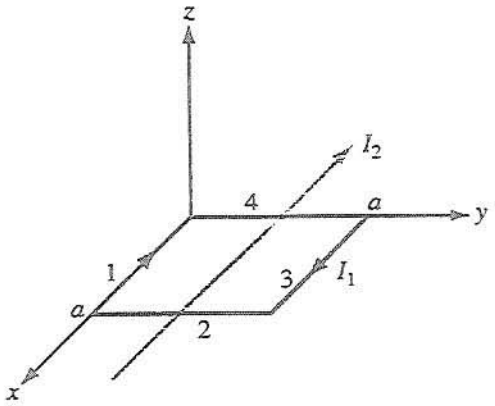

Figure P5.19: Long wire carrying current I_2, just above a square loop carrying I_1 (Problem 5.19).

Solution: Since I_2 is just barely above the loop, we can treat it as if it's in the same plane as the loop. For side 1, I_1 and I_2 are in the same direction, hence the force on

side 1 is attractive. That is,

$$\mathbf{F}_1 = \hat{\mathbf{y}} \frac{\mu_0 I_1 I_2 a}{2\pi(a/2)} = \hat{\mathbf{y}} \frac{4\pi \times 10^{-7} \times 5 \times 10 \times 2}{2\pi \times 1} = \hat{\mathbf{y}} 2 \times 10^{-5} \text{ N}.$$

I_1 and I_2 are in opposite directions for side 3. Hence, the force on side 3 is repulsive, which means it is also along $\hat{\mathbf{y}}$. That is, $\mathbf{F}_3 = \mathbf{F}_1$.

The net forces on sides 2 and 4 are zero. Total net force on the loop is

$$\mathbf{F} = 2\mathbf{F}_1 = \hat{\mathbf{y}} 4 \times 10^{-5} \text{ N}.$$

Section 5-4: Gauss's Law for Magnetism and Ampère's Law

Problem 5.20 Current I flows along the positive z-direction in the inner conductor of a long coaxial cable and returns through the outer conductor. The inner conductor has radius a, and the inner and outer radii of the outer conductor are b and c, respectively.

(a) Determine the magnetic field in each of the following regions: $0 \leq r \leq a$, $a \leq r \leq b$, $b \leq r \leq c$, and $r \geq c$.

(b) Plot the magnitude of $\mathbf{H}$ as a function of r over the range from $r = 0$ to $r = 10$ cm, given that $I = 10$ A, $a = 2$ cm, $b = 4$ cm, and $c = 5$ cm.

Solution:

(a) Following the solution to Example 5-5, the magnetic field in the region $r < a$,

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{rI}{2\pi a^2},$$

and in the region $a < r < b$,

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{I}{2\pi r}.$$

The total area of the outer conductor is $A = \pi(c^2 - b^2)$ and the fraction of the area of the outer conductor enclosed by a circular contour centered at $r = 0$ in the region $b < r < c$ is

$$\frac{\pi(r^2 - b^2)}{\pi(c^2 - b^2)} = \frac{r^2 - b^2}{c^2 - b^2}.$$

The total current enclosed by a contour of radius r is therefore

$$I_{\text{enclosed}} = I\left(1 - \frac{r^2 - b^2}{c^2 - b^2}\right) = I\frac{c^2 - r^2}{c^2 - b^2},$$

and the resulting magnetic field is

$$\mathbb{H} = \hat{\phi}\frac{I_{\text{enclosed}}}{2\pi r} = \hat{\phi}\frac{I}{2\pi r}\left(\frac{c^2 - r^2}{c^2 - b^2}\right).$$

For $r > c$, the total enclosed current is zero: the total current flowing on the inner conductor is equal to the total current flowing on the outer conductor, but they are flowing in opposite directions. Therefore, $\mathbb{H} = 0$.

(b) See Fig. P5.20.

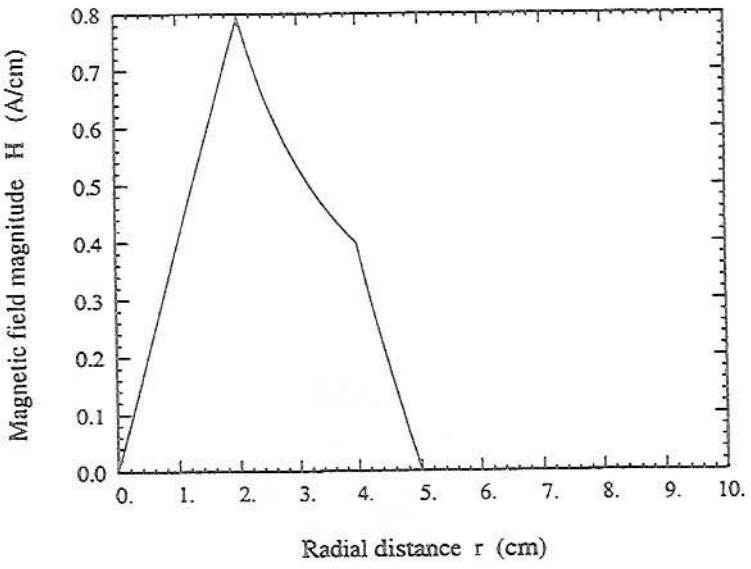

Figure P5.20: Problem 5.20(b).

Problem 5.21 A long cylindrical conductor whose axis is coincident with the z-axis has a radius a and carries a current characterized by a current density $\mathbb{J} = \hat{z}J_0/r$, where J_0 is a constant and r is the radial distance from the cylinder's axis. Obtain an expression for the magnetic field $\mathbb{H}$ for (a) $0 \leq r \leq a$ and (b) $r > a$.

Solution: This problem is very similar to Example 5-5.

(a) For $0 \leq r_1 \leq a$, the total current flowing within the contour C_1 is

$$I_1 = \iint \mathbb{J} \cdot d\mathbb{s} = \int_{\phi=0}^{2\pi}\int_{r=0}^{r_1}\left(\frac{\hat{z}J_0}{r}\right)\cdot(\hat{z}\,r\,dr\,d\phi) = 2\pi\int_{r=0}^{r_1} J_0\,dr = 2\pi r_1 J_0.$$

Therefore, since $I_1 = 2\pi r_1 H_1$, $H_1 = J_0$ within the wire and $\mathbf{H}_1 = \hat{\phi} J_0$.

(b) For $r \geq a$, the total current flowing within the contour is the total current flowing within the wire:

$$I = \iint \mathbf{J} \cdot ds = \int_{\phi=0}^{2\pi} \int_{r=0}^{a} \left(\frac{\hat{z} J_0}{r} \right) \cdot (\hat{z} r \, dr \, d\phi) = 2\pi \int_{r=0}^{a} J_0 \, dr = 2\pi a J_0.$$

Therefore, since $I = 2\pi r H_2$, $H_2 = J_0 a / r$ within the wire and $\mathbf{H}_2 = \hat{\phi} J_0 (a/r)$.

Problem 5.22 Repeat Problem 5.21 for a current density $\mathbf{J} = \hat{z} J_0 e^{-r}$.

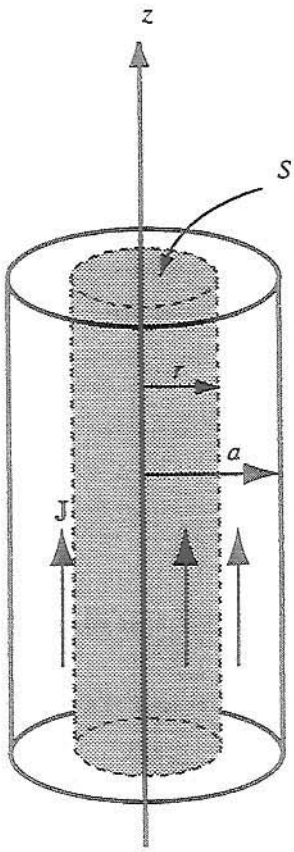

Figure P5.22: Cylindrical current.

Solution:

(a) For $r \le a$, Ampère's law is

$$\oint_c \mathbf{H}\cdot d\mathbf{l} = I = \int_S \mathbf{J}\cdot d\mathbf{s},$$

$$\hat{\phi}H\cdot\hat{\phi}2\pi r = \int_0^r \mathbf{J}\cdot d\mathbf{s} = \int_0^r \hat{z}J_0e^{-r}\cdot\hat{z}2\pi r\,dr,$$

$$2\pi rH = 2\pi J_0\int_0^r re^{-r}\,dr$$

$$= 2\pi J_0[-e^{-r}(r+1)]_0^r = 2\pi J_0[1-e^{-r}(r+1)].$$

Hence,

$$\mathbb{H} = \hat{\phi}H = \hat{\phi}\frac{J_0}{r}[1-e^{-r}(r+1)], \qquad \text{for } r \le a.$$

(b) For $r \ge a$,

$$2\pi rH = 2\pi J_0[-e^{-r}(r+1)]_0^a = 2\pi J_0[1-e^{-a}(a+1)],$$

$$\mathbb{H} = \hat{\phi}H = \hat{\phi}\frac{J_0}{r}\left[1-e^{-a}(a+1)\right], \qquad r \ge a.$$

Problem 5.23 In a certain conducting region, the magnetic field is given in cylindrical coordinates by

$$\mathbb{H} = \hat{\phi}\frac{4}{r}[1-(1+2r)e^{-2r}].$$

Find the current density $\mathbf{J}$.

Solution:

$$\mathbf{J} = \nabla\times\mathbf{H} = \hat{z}\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{4}{r}[1-(1+2r)e^{-2r}]\right)$$

$$= \hat{z}\frac{1}{r}[8e^{-2r}(1+2r)-8e^{-2r}] = \hat{z}16e^{-2r}\ \text{A/m}^2.$$

Section 5-5: Magnetic Potential

Problem 5.24 With reference to Fig. 5-10, (a) derive an expression for the vector magnetic potential A at a point P located at a distance r from the wire in the x–y plane, and then (b) derive B from A. Show that your result is identical with the expression given by Eq. (5.29), which was derived by applying the Biot–Savart law.

Solution:

(a) From the text immediately following Eq. (5.65), that equation may take the form

$$\begin{aligned}\mathbf{A} = \frac{\mu}{4\pi}\int_{\ell'}\frac{I}{R'}d\boldsymbol{\ell}' &= \frac{\mu_0}{4\pi}\int_{z'=-\ell/2}^{\ell/2}\frac{I}{\sqrt{z'^2+r^2}}\hat{\mathbf{z}}\,dz' \\ &= \frac{\mu_0}{4\pi}\left(\hat{\mathbf{z}}I\ln(z'+\sqrt{z'^2+r^2})\right)\Big|_{z'=-\ell/2}^{\ell/2} \\ &= \hat{\mathbf{z}}\frac{\mu_0 I}{4\pi}\ln\left(\frac{\ell/2+\sqrt{(\ell/2)^2+r^2}}{-\ell/2+\sqrt{(-\ell/2)^2+r^2}}\right) \\ &= \hat{\mathbf{z}}\frac{\mu_0 I}{4\pi}\ln\left(\frac{\ell+\sqrt{\ell^2+4r^2}}{-\ell+\sqrt{\ell^2+4r^2}}\right).\end{aligned}$$

(b) From Eq. (5.53),

$$\begin{aligned}\mathbf{B} &= \nabla\times\mathbf{A} \\ &= \nabla\times\left(\hat{\mathbf{z}}\frac{\mu_0 I}{4\pi}\ln\left(\frac{\ell+\sqrt{\ell^2+4r^2}}{-\ell+\sqrt{\ell^2+4r^2}}\right)\right) \\ &= -\hat{\boldsymbol{\phi}}\frac{\mu_0 I}{4\pi}\frac{\partial}{\partial r}\ln\left(\frac{\ell+\sqrt{\ell^2+4r^2}}{-\ell+\sqrt{\ell^2+4r^2}}\right) \\ &= -\hat{\boldsymbol{\phi}}\frac{\mu_0 I}{4\pi}\left(\frac{-\ell+\sqrt{\ell^2+4r^2}}{\ell+\sqrt{\ell^2+4r^2}}\right)\frac{\partial}{\partial r}\left(\frac{\ell+\sqrt{\ell^2+4r^2}}{-\ell+\sqrt{\ell^2+4r^2}}\right) \\ &= -\hat{\boldsymbol{\phi}}\frac{\mu_0 I}{4\pi}\left(\frac{-\ell+\sqrt{\ell^2+4r^2}}{\ell+\sqrt{\ell^2+4r^2}}\right) \\ &\times\left(\frac{(-\ell+\sqrt{\ell^2+4r^2})\frac{\partial}{\partial r}(\ell+\sqrt{\ell^2+4r^2})-(\ell+\sqrt{\ell^2+4r^2})\frac{\partial}{\partial r}(-\ell+\sqrt{\ell^2+4r^2})}{(-\ell+\sqrt{\ell^2+4r^2})^2}\right) \\ &= -\hat{\boldsymbol{\phi}}\frac{\mu_0 I}{4\pi}\left(\frac{(-\ell+\sqrt{\ell^2+4r^2})-(\ell+\sqrt{\ell^2+4r^2})}{(-\ell+\sqrt{\ell^2+4r^2})(\ell+\sqrt{\ell^2+4r^2})}\right)\frac{4r}{\sqrt{\ell^2+4r^2}} \\ &= -\hat{\boldsymbol{\phi}}\frac{\mu_0 I}{4\pi}\left(\frac{-2\ell}{4r^2}\right)\frac{4r}{\sqrt{\ell^2+4r^2}} = \hat{\boldsymbol{\phi}}\frac{\mu_0 I\ell}{2\pi r\sqrt{\ell^2+4r^2}} \quad (\mathrm{T}).\end{aligned}$$

which is the same as Eq. (5.29).

Problem 5.25 In a given region of space, the vector magnetic potential is given by $\mathbf{A} = \hat{\mathbf{x}}5\cos\pi y + \hat{\mathbf{z}}(2+\sin\pi x)$ (Wb/m).

(a) Determine B.

(b) Use Eq. (5.66) to calculate the magnetic flux passing through a square loop with 0.25-m-long edges if the loop is in the x–y plane, its center is at the origin, and its edges are parallel to the x- and y-axes.

(c) Calculate Φ again using Eq. (5.67).

Solution:

(a) From Eq. (5.53), $\mathbb{B} = \nabla \times \mathbb{A} = \hat{z}5\pi \sin \pi y - \hat{y}\pi \cos \pi x$.

(b) From Eq. (5.66),

$$\begin{aligned}\Phi = \iint \mathbb{B} \cdot ds &= \int_{y=-0.125\text{ m}}^{0.125\text{ m}} \int_{x=-0.125\text{ m}}^{0.125\text{ m}} (\hat{z}5\pi \sin \pi y - \hat{y}\pi \cos \pi x) \cdot (\hat{z}\, dx\, dy) \\ &= \left(\left(-5\pi x \frac{\cos \pi y}{\pi} \right) \Big|_{x=-0.125}^{0.125} \right) \Big|_{y=-0.125}^{0.125} \\ &= \frac{-5}{4} \left(\cos\left(\frac{\pi}{8}\right) - \cos\left(\frac{-\pi}{8}\right) \right) = 0.\end{aligned}$$

(c) From Eq. (5.67), $\Phi = \oint_C \mathbb{A} \cdot d\ell$, where C is the square loop in the x-y plane with sides of length 0.25 m centered at the origin. Thus, the integral can be written as

$$\Phi = \oint_C \mathbb{A} \cdot d\ell = S_{\text{front}} + S_{\text{back}} + S_{\text{left}} + S_{\text{right}},$$

where S_{front}, S_{back}, S_{left}, and S_{right} are the sides of the loop.

$$\begin{aligned}S_{\text{front}} &= \int_{x=-0.125}^{0.125} (\hat{x}5 \cos \pi y + \hat{z}(2 + \sin \pi x))|_{y=-0.125} \cdot (\hat{x}\, dx) \\ &= \int_{x=-0.125}^{0.125} 5 \cos \pi y|_{y=-0.125}\, dx \\ &= \left((5x \cos \pi y)|_{y=-0.125} \right) \Big|_{x=-0.125}^{0.125} = \frac{5}{4} \cos\left(\frac{-\pi}{8}\right) = \frac{5}{4} \cos\left(\frac{\pi}{8}\right), \\ S_{\text{back}} &= \int_{x=-0.125}^{0.125} (\hat{x}5 \cos \pi y + \hat{z}(2 + \sin \pi x))|_{y=0.125} \cdot (-\hat{x}\, dx) \\ &= -\int_{x=-0.125}^{0.125} 5 \cos \pi y|_{y=0.125}\, dx \\ &= \left((-5x \cos \pi y)|_{y=0.125} \right) \Big|_{x=-0.125}^{0.125} = -\frac{5}{4} \cos\left(\frac{\pi}{8}\right), \\ S_{\text{left}} &= \int_{y=-0.125}^{0.125} (\hat{x}5 \cos \pi y + \hat{z}(2 + \sin \pi x))|_{x=-0.125} \cdot (-\hat{y}\, dy)\end{aligned}$$

$$= -\int_{y=-0.125}^{0.125} 0|_{x=-0.125}\, dy = 0,$$

$$S_{\text{right}} = \int_{y=-0.125}^{0.125} (\hat{\mathbf{x}}5\cos\pi y + \hat{\mathbf{z}}(2+\sin\pi x))|_{x=0.125} \cdot (\hat{\mathbf{y}}\, dy)$$

$$= \int_{y=-0.125}^{0.125} 0|_{x=0.125}\, dy = 0.$$

Thus,

$$\Phi = \oint_c \mathbf{A}\cdot d\ell = S_{\text{front}} + S_{\text{back}} + S_{\text{left}} + S_{\text{right}} = \frac{5}{4}\cos\left(\frac{\pi}{8}\right) - \frac{5}{4}\cos\left(\frac{\pi}{8}\right) + 0 + 0 = 0.$$

Problem 5.26 A uniform current density given by

$$\mathbf{J} = \hat{\mathbf{z}}J_0 \quad (\text{A/m}^2),$$

gives rise to a vector magnetic potential

$$\mathbf{A} = -\hat{\mathbf{z}}\frac{\mu_0 J_0}{4}(x^2+y^2) \quad (\text{Wb/m}).$$

(a) Apply the vector Poisson's equation to confirm the above statement.

(b) Use the expression for $\mathbf{A}$ to find $\mathbf{H}$.

(c) Use the expression for $\mathbf{J}$ in conjunction with Ampère's law to find $\mathbf{H}$. Compare your result with that obtained in part (b).

Solution:

(a)

$$\nabla^2\mathbf{A} = \hat{\mathbf{x}}\nabla^2 A_x + \hat{\mathbf{y}}\nabla^2 A_y + \hat{\mathbf{z}}\nabla^2 A_z = \hat{\mathbf{z}}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left[-\mu_0\frac{J_0}{4}(x^2+y^2)\right]$$

$$= -\hat{\mathbf{z}}\mu_0\frac{J_0}{4}(2+2) = -\hat{\mathbf{z}}\mu_0 J_0.$$

Hence, $\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$ is verified.

(b)

$$\mathbf{H} = \frac{1}{\mu_0}\nabla\times\mathbf{A} = \frac{1}{\mu_0}\left[\hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right]$$

$$= \frac{1}{\mu_0}\left(\hat{\mathbf{x}}\frac{\partial A_z}{\partial y} - \hat{\mathbf{y}}\frac{\partial A_z}{\partial x}\right)$$

$$= \frac{1}{\mu_0}\left[\hat{\mathbf{x}}\frac{\partial}{\partial y}\left(-\mu_0\frac{J_0}{4}(x^2+y^2)\right) - \hat{\mathbf{y}}\frac{\partial}{\partial x}\left(-\mu_0\frac{J_0}{4}(x^2+y^2)\right)\right]$$

$$= -\hat{\mathbf{x}}\frac{J_0 y}{2} + \hat{\mathbf{y}}\frac{J_0 x}{2} \quad (\text{A/m}).$$

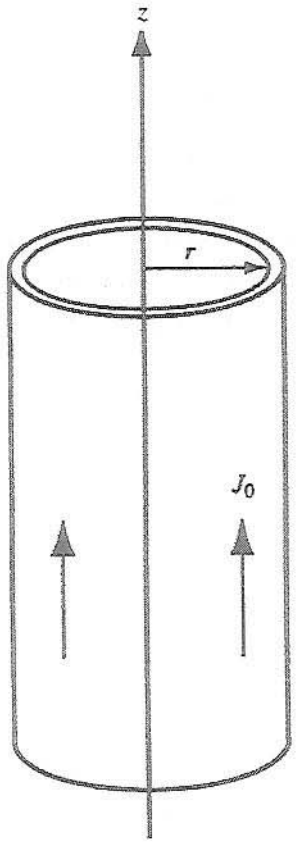

Figure P5.26: Current cylinder of Problem 5.26.

(c)

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I = \int_S \mathbf{J} \cdot d\mathbf{s},$$
$$\hat{\phi} H_\phi \cdot \hat{\phi} 2\pi r = J_0 \cdot \pi r^2,$$
$$\mathbf{H} = \hat{\phi} H_\phi = \hat{\phi} J_0 \frac{r}{2}.$$

We need to convert the expression from cylindrical to Cartesian coordinates. From Table 3-2,

$$\hat{\phi} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi = -\hat{\mathbf{x}} \frac{y}{\sqrt{x^2+y^2}} + \hat{\mathbf{y}} \frac{x}{\sqrt{x^2+y^2}},$$
$$r = \sqrt{x^2+y^2}.$$

Hence

$$\mathbf{H} = \left(-\hat{\mathbf{x}} \frac{y}{\sqrt{x^2+y^2}} + \hat{\mathbf{y}} \frac{x}{\sqrt{x^2+y^2}} \right) \cdot \frac{J_0}{2} \sqrt{x^2+y^2} = -\hat{\mathbf{x}} \frac{yJ_0}{2} + \hat{\mathbf{y}} \frac{xJ_0}{2},$$

which is identical with the result of part (b).

Problem 5.27 A thin current element extending between $z = -L/2$ and $z = L/2$ carries a current I along $+\hat{z}$ through a circular cross section of radius a.

(a) Find A at a point P located very far from the origin (assume R is so much larger than L that point P may be considered to be at approximately the same distance from every point along the current element).

(b) Determine the corresponding H.

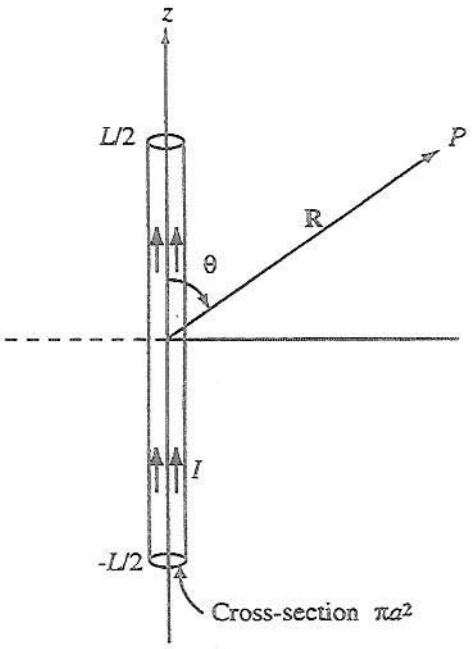

Figure P5.27: Current element of length L observed at distance $R \gg L$.

Solution:

(a) Since $R \gg L$, we can assume that P is approximately equidistant from all segments of the current element. Hence, with R treated as constant, (5.65) gives

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{v'} \frac{\mathbf{J}}{R'} \, dv' = \frac{\mu_0}{4\pi R} \int_{v'} \hat{z} \frac{I}{(\pi a^2)} \pi a^2 \, dz = \hat{z} \frac{\mu_0 I}{4\pi R} \int_{-L/2}^{L/2} dz = \hat{z} \frac{\mu_0 I L}{4\pi R}.$$

(b)

$$\begin{aligned}
\mathrm{H} &= \frac{1}{\mu_0}\nabla\times\mathrm{A} \\
&= \frac{1}{\mu_0}\left[\hat{\mathrm{x}}\frac{\partial A_z}{\partial y} - \hat{\mathrm{y}}\frac{\partial A_z}{\partial x}\right] \\
&= \frac{1}{\mu_0}\left\{\hat{\mathrm{x}}\frac{\partial}{\partial y}\left[\frac{\mu_0 IL}{4\pi}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)\right] - \hat{\mathrm{y}}\frac{\partial}{\partial x}\left[\frac{\mu_0 IL}{4\pi}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)\right]\right\} \\
&= \frac{IL}{4\pi}\left[\frac{-\hat{\mathrm{x}}y+\hat{\mathrm{y}}x}{(x^2+y^2+z^2)^{3/2}}\right].
\end{aligned}$$

Section 5-6: Magnetic Properties of Materials

Problem 5.28 In the model of the hydrogen atom proposed by Bohr in 1913, the electron moves around the nucleus at a speed of 2×10^6 m/s in a circular orbit of radius 5×10^{-11} m. What is the magnitude of the magnetic moment generated by the electron's motion?

Solution: From Eq. (5.69), the magnitude of the orbital magnetic moment of an electron is

$$|m_0| = \left|-\tfrac{1}{2}eur\right| = \tfrac{1}{2}\times1.6\times10^{-19}\times2\times10^6\times5\times10^{-11} = 8\times10^{-24} \quad (\mathrm{A\cdot m^2}).$$

Problem 5.29 Iron contains 8.5×10^{28} atoms/m^3. At saturation, the alignment of the electrons' spin magnetic moments in iron can contribute 1.5 T to the total magnetic flux density B. If the spin magnetic moment of a single electron is 9.27×10^{-24} (A·m^2), how many electrons per atom contribute to the saturated field?

Solution: From the first paragraph of Section 5-6.2, the magnetic flux density of a magnetized material is $\mathrm{B_m} = \mu_0\mathrm{M}$, where M is the vector sum of the microscopic magnetic dipoles within the material: $\mathrm{M} = N_\mathrm{e}\mathrm{m_s}$, where $\mathrm{m_s}$ is the magnitude of the spin magnetic moment of an electron in the direction of the mean magnetization, and N_e is net number of electrons per unit volume contributing to the bulk magnetization. If the number of electrons per atom contributing to the bulk magnetization is n_e, then $N_\mathrm{e} = n_\mathrm{e}N_\mathrm{atoms}$ where $N_\mathrm{atoms} = 8.5\times10^{28}$ atoms/m^3 is the number density of atoms for iron. Therefore,

$$n_\mathrm{e} = \frac{N_\mathrm{e}}{N_\mathrm{atoms}} = \frac{M}{m_\mathrm{s}N_\mathrm{atoms}} = \frac{B}{\mu_0 m_\mathrm{s}N_\mathrm{atoms}} = \frac{1.5}{4\pi\times10^{-7}\times9.27\times10^{-24}\times8.5\times10^{28}}$$
$$= 1.5 \quad \text{(electrons/atom)}.$$

Section 5-7: Magnetic Boundary Conditions

Problem 5.30 The x–y plane separates two magnetic media with magnetic permeabilities μ_1 and μ_2, as shown in Fig. 5-45 (P5.30). If there is no surface current at the interface and the magnetic field in medium 1 is

$$\mathbf{H}_1 = \hat{\mathbf{x}}H_{1x} + \hat{\mathbf{y}}H_{1y} + \hat{\mathbf{z}}H_{1z},$$

find:

(a) $\mathbf{H}_2$,

(b) θ_1 and θ_2, and

(c) evaluate $\mathbf{H}_2$, θ_1, and θ_2 for $H_{1x} = 3$ (A/m), $H_{1y} = 0$, $H_{1z} = 4$ (A/m), $\mu_1 = \mu_0$, and $\mu_2 = 4\mu_0$.

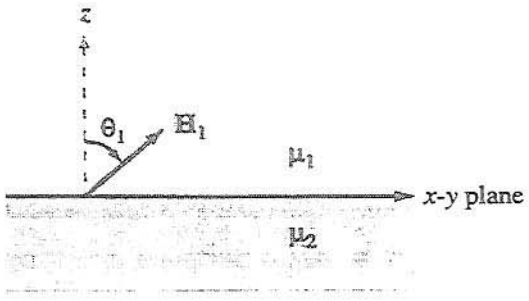

Figure P5.30: Adjacent magnetic media (Problem 5.30).

Solution:

(a) From (5.80),

$$\mu_1 H_{1n} = \mu_2 H_{2n},$$

and in the absence of surface currents at the interface, (5.85) states

$$H_{1t} = H_{2t}.$$

In this case, $H_{1z} = H_{1n}$, and H_{1x} and H_{1y} are tangential fields. Hence,

$$\mu_1 H_{1z} = \mu_2 H_{2z},$$
$$H_{1x} = H_{2x},$$
$$H_{1y} = H_{2y},$$

and

$$\mathbf{H}_2 = \hat{\mathbf{x}} H_{1x} + \hat{\mathbf{y}} H_{1y} + \hat{\mathbf{z}} \frac{\mu_1}{\mu_2} H_{1z}.$$

(b)

$$H_{1t} = \sqrt{H_{1x}^2 + H_{1y}^2},$$

$$\tan\theta_1 = \frac{H_{1t}}{H_{1z}} = \frac{\sqrt{H_{1x}^2 + H_{1y}^2}}{H_{1z}},$$

$$\tan\theta_2 = \frac{H_{2t}}{H_{2z}} = \frac{\sqrt{H_{1x}^2 + H_{1y}^2}}{\frac{\mu_1}{\mu_2} H_{1z}} = \frac{\mu_2}{\mu_1} \tan\theta_1.$$

(c)

$$\mathbf{H}_2 = \hat{\mathbf{x}} 3 + \hat{\mathbf{z}} \frac{1}{4} \cdot 4 = \hat{\mathbf{x}} 3 + \hat{\mathbf{z}} \quad \text{(A/m)},$$

$$\theta_1 = \tan^{-1}\left(\frac{3}{4}\right) = 36.86°,$$

$$\theta_2 = \tan^{-1}\left(\frac{3}{1}\right) = 71.57°.$$

Problem 5.31 Given that a current sheet with surface current density $\mathbf{J}_s = \hat{\mathbf{x}} 4$ (A/m) exists at $y = 0$, the interface between two magnetic media, and $\mathbf{H}_1 = \hat{\mathbf{z}} 8$ (A/m) in medium 1 ($y > 0$), determine $\mathbf{H}_2$ in medium 2 ($y < 0$).

Solution:

$$\mathbf{J}_s = \hat{\mathbf{x}} 4 \text{ A/m},$$
$$\mathbf{H}_1 = \hat{\mathbf{z}} 8 \text{ A/m}.$$

$\mathbf{H}_1$ is tangential to the boundary, and therefore $\mathbf{H}_2$ is also. With $\hat{\mathbf{n}}_2 = \hat{\mathbf{y}}$, from Eq. (5.84), we have

$$\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s,$$
$$\hat{\mathbf{y}} \times (\hat{\mathbf{z}} 8 - \mathbf{H}_2) = \hat{\mathbf{x}} 4,$$
$$\hat{\mathbf{x}} 8 - \hat{\mathbf{y}} \times \mathbf{H}_2 = \hat{\mathbf{x}} 4,$$

or

$$\hat{\mathbf{y}} \times \mathbf{H}_2 = \hat{\mathbf{x}} 4,$$

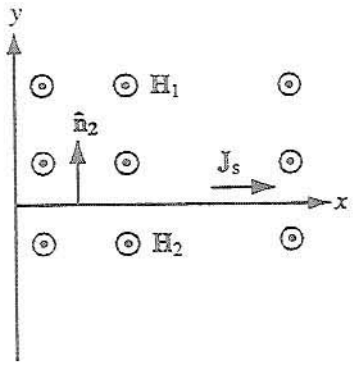

Figure P5.31: Adjacent magnetic media with $\mathbf{J}_s$ on boundary.

which implies that $\mathbf{H}_2$ does not have an x-component. Also, since $\mu_1 H_{1y} = \mu_2 H_{2y}$ and $\mathbf{H}_1$ does not have a y-component, it follows that $\mathbf{H}_2$ does not have a y-component either. Consequently, we conclude that

$$\mathbf{H}_2 = \hat{\mathbf{z}}4.$$

Problem 5.32 In Fig. 5-46 (P5.32), the plane defined by $x - y = 1$ separates medium 1 of permeability μ_1 from medium 2 of permeability μ_2. If no surface current exists on the boundary and

$$\mathbf{B}_1 = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 \quad (\text{T}),$$

find $\mathbf{B}_2$ and then evaluate your result for $\mu_1 = 5\mu_2$. Hint: Start out by deriving the equation for the unit vector normal to the given plane.

Solution: We need to find $\hat{\mathbf{n}}_2$. To do so, we start by finding any two vectors in the plane $x - y = 1$, and to do that, we need three non-collinear points in that plane. We choose $(0,-1,0)$, $(1,0,0)$, and $(1,0,1)$.

Vector $\mathbf{A}_1$ is from $(0,-1,0)$ to $(1,0,0)$:

$$\mathbf{A}_1 = \hat{\mathbf{x}}1 + \hat{\mathbf{y}}1.$$

Vector $\mathbf{A}_2$ is from $(1,0,0)$ to $(1,0,1)$:

$$\mathbf{A}_2 = \hat{\mathbf{z}}1.$$

Hence, if we take the cross product $\mathbf{A}_2 \times \mathbf{A}_1$, we end up in a direction normal to the given plane, from medium 2 to medium 1,

$$\hat{\mathbf{n}}_2 = \frac{\mathbf{A}_2 \times \mathbf{A}_1}{|\mathbf{A}_2 \times \mathbf{A}_1|} = \frac{\hat{\mathbf{z}}1 \times (\hat{\mathbf{x}}1 + \hat{\mathbf{y}}1)}{|\mathbf{A}_2 \times \mathbf{A}_1|} = \frac{\hat{\mathbf{y}}1 - \hat{\mathbf{x}}1}{\sqrt{1+1}} = \frac{\hat{\mathbf{y}}}{\sqrt{2}} - \frac{\hat{\mathbf{x}}}{\sqrt{2}}.$$

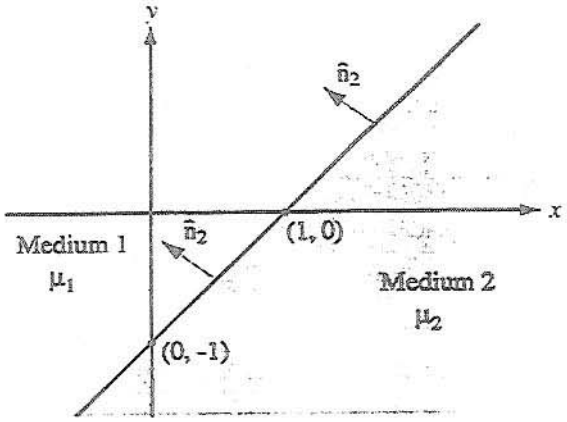

Figure P5.32: Magnetic media separated by the plane $x - y = 1$ (Problem 5.32).

In medium 1, normal component is

$$B_{1n} = \hat{n}_2 \cdot \mathbb{B}_1 = \left(\frac{\hat{y}}{\sqrt{2}} - \frac{\hat{x}}{\sqrt{2}}\right) \cdot (\hat{x}2 + \hat{y}3) = \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{2}} = \frac{1}{\sqrt{2}},$$

$$\mathbb{B}_{1n} = \hat{n}_2 B_{1n} = \left(\frac{\hat{y}}{\sqrt{2}} - \frac{\hat{x}}{\sqrt{2}}\right) \cdot \frac{1}{\sqrt{2}} = \frac{\hat{y}}{2} - \frac{\hat{x}}{2}.$$

Tangential component is

$$\mathbb{B}_{1t} = \mathbb{B}_1 - \mathbb{B}_{1n} = (\hat{x}2 + \hat{y}3) - \left(\frac{\hat{y}}{2} - \frac{\hat{x}}{2}\right) = \hat{x}2.5 + \hat{y}2.5.$$

Boundary conditions:

$$B_{1n} = B_{2n}, \quad \text{or} \quad \mathbb{B}_{2n} = \frac{\hat{y}}{2} - \frac{\hat{x}}{2},$$

$$H_{1t} = H_{2t}, \quad \text{or} \quad \frac{B_{2t}}{\mu_2} = \frac{B_{1t}}{\mu_1}.$$

Hence,

$$\mathbb{B}_{2t} = \frac{\mu_2}{\mu_1}\mathbb{B}_{1t} = \frac{\mu_2}{\mu_1}(\hat{x}2.5 + \hat{y}2.5).$$

Finally,

$$\mathbb{B}_2 = \mathbb{B}_{2n} + \mathbb{B}_{2t} = \left(\frac{\hat{y}}{2} - \frac{\hat{x}}{2}\right) + \frac{\mu_2}{\mu_1}(\hat{x}2.5 + \hat{y}2.5).$$

For $\mu_1 = 5\mu_2$,

$$\mathbb{B}_2 = \hat{y} \quad \text{(T)}.$$

Problem 5.33 The plane boundary defined by $z = 0$ separates air from a block of iron. If $\mathbb{B}_1 = \hat{\mathbf{x}}4 - \hat{\mathbf{y}}6 + \hat{\mathbf{z}}8$ in air ($z \geq 0$), find $\mathbb{B}_2$ in iron ($z \leq 0$), given that $\mu = 5000\mu_0$ for iron.

Solution: From Eq. (5.2),

$$\mathbb{H}_1 = \frac{\mathbb{B}_1}{\mu_1} = \frac{1}{\mu_1}(\hat{\mathbf{x}}4 - \hat{\mathbf{y}}6 + \hat{\mathbf{z}}8).$$

The z component is the normal component to the boundary at $z = 0$. Therefore, from Eq. (5.79), $B_{2z} = B_{1z} = 8$ while, from Eq. (5.85),

$$H_{2x} = H_{1x} = \frac{1}{\mu_1}4, \qquad H_{2y} = H_{1y} = -\frac{1}{\mu_1}6,$$

or

$$B_{2x} = \mu_2 H_{2x} = \frac{\mu_2}{\mu_1}4, \qquad B_{2y} = \mu_2 H_{2y} = -\frac{\mu_2}{\mu_1}6,$$

where $\mu_2/\mu_1 = \mu_r = 5000$. Therefore,

$$\mathbb{B}_2 = \hat{\mathbf{x}}20000 - \hat{\mathbf{y}}30000 + \hat{\mathbf{z}}8.$$

Problem 5.34 Show that if no surface current densities exist at the parallel interfaces shown in Fig. 5-47 (P5.34), the relationship between θ_4 and θ_1 is independent of μ_2.

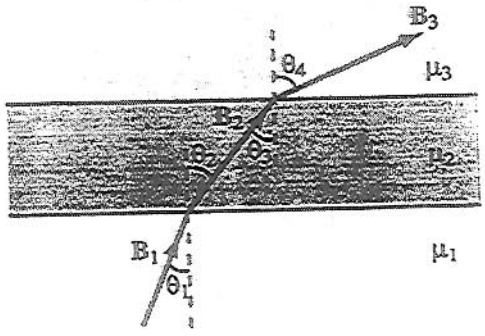

Figure P5.34: Three magnetic media with parallel interfaces (Problem 5.34).

Solution:

$$\tan\theta_1 = \frac{B_{1t}}{B_{1n}},$$

and

$$\tan\theta_2 = \frac{B_{2t}}{B_{2n}}.$$

But $B_{2n} = B_{1n}$ and $\dfrac{B_{2t}}{\mu_2} = \dfrac{B_{1t}}{\mu_1}$. Hence,

$$\tan\theta_2 = \frac{B_{1t}}{B_{1n}}\frac{\mu_2}{\mu_1} = \frac{\mu_2}{\mu_1}\tan\theta_1.$$

We note that $\theta_2 = \theta_3$ and

$$\tan\theta_4 = \frac{\mu_3}{\mu_2}\tan\theta_3 = \frac{\mu_3}{\mu_2}\tan\theta_2 = \frac{\mu_3}{\mu_2}\frac{\mu_2}{\mu_1}\tan\theta_1 = \frac{\mu_3}{\mu_1}\tan\theta_1,$$

which is independent of μ_2.

Sections 5-8 and 5-9: Inductance and Magnetic Energy

Problem 5.35 Obtain an expression for the self-inductance per unit length for the parallel wire transmission line of Fig. 5-27(a) in terms of a, d, and μ, where a is the radius of the wires, d is the axis-to-axis distance between the wires, and μ is the permeability of the medium in which they reside.

Solution: Let us place the two wires in the x–z plane and orient the current in one of them to be along the $+z$-direction and the current in the other one to be along the $-z$-direction, as shown in Fig. P5.35. From Eq. (5.30), the magnetic field at point $P(x,0,z)$ due to wire 1 is

$$\mathbb{B}_1 = \hat{\phi}\frac{\mu I}{2\pi r} = \hat{y}\frac{\mu I}{2\pi x},$$

where the permeability has been generalized from free space to any substance with permeability μ, and it has been recognized that in the x-z plane, $\hat{\phi} = \hat{y}$ and $r = x$ as long as $x > 0$.

Given that the current in wire 2 is opposite that in wire 1, the magnetic field created by wire 2 at point $P(x,0,z)$ is in the same direction as that created by wire 1, and it is given by

$$\mathbb{B}_2 = \hat{y}\frac{\mu I}{2\pi(d-x)}.$$

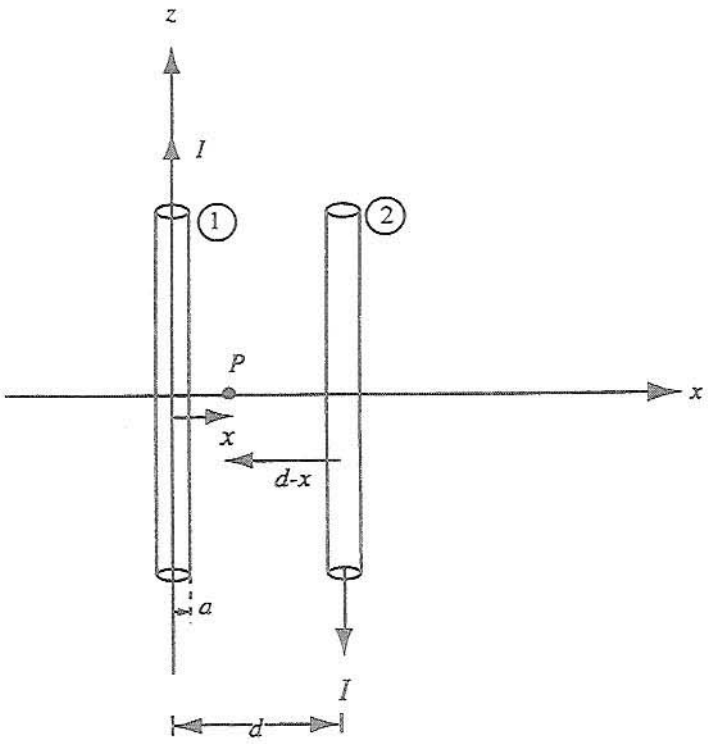

Figure P5.35: Parallel wire transmission line.

Therefore, the total magnetic field in the region between the wires is

$$\mathbb{B} = \mathbb{B}_1 + \mathbb{B}_2 = \hat{\mathrm{y}}\frac{\mu I}{2\pi}\left(\frac{1}{x} + \frac{1}{d-x}\right) = \hat{\mathrm{y}}\frac{\mu I d}{2\pi x(d-x)}.$$

From Eq. (5.91), the flux crossing the surface area between the wires over a length l of the wire structure is

$$\begin{aligned}\Phi = \iint_S \mathbb{B}\cdot d\mathrm{s} &= \int_{z=z_0}^{z_0+l}\int_{x=a}^{d-a}\left(\hat{\mathrm{y}}\frac{\mu I d}{2\pi x(d-x)}\right)\cdot(\hat{\mathrm{y}}\,dx\,dz)\\ &= \frac{\mu I l d}{2\pi}\left(\frac{1}{d}\ln\left(\frac{x}{d-x}\right)\right)\Bigg|_{x=a}^{d-a}\\ &= \frac{\mu I l}{2\pi}\left(\ln\left(\frac{d-a}{a}\right) - \ln\left(\frac{a}{d-a}\right)\right)\\ &= \frac{\mu I l}{2\pi}\times 2\ln\left(\frac{d-a}{a}\right) = \frac{\mu I l}{\pi}\ln\left(\frac{d-a}{a}\right).\end{aligned}$$

Since the number of 'turns' in this structure is 1, Eq. (5.93) states that the flux linkage is the same as magnetic flux: $\Lambda = \Phi$. Then Eq. (5.94) gives a total inductance over the length l as

$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{\mu l}{\pi} \ln\left(\frac{d-a}{a}\right) \quad \text{(H)}.$$

Therefore, the inductance per unit length is

$$L' = \frac{L}{l} = \frac{\mu}{\pi} \ln\left(\frac{d-a}{a}\right) \approx \frac{\mu}{\pi} \ln\left(\frac{d}{a}\right) \quad \text{(H/m)},$$

where the last approximation recognizes that the wires are thin compared to the separation distance (i.e., that $d \gg a$). This has been an implied condition from the beginning of this analysis, where the flux passing through the wires themselves have been ignored. This is the thin-wire limit in Table 2-1 for the two wire line.

Problem 5.36 A solenoid with a length of 20 cm and a radius of 5 cm consists of 400 turns and carries a current of 12 A. If $z = 0$ represents the midpoint of the solenoid, generate a plot for $|\mathbb{H}(z)|$ as a function of z along the axis of the solenoid for the range $-20 \text{ cm} \leq z \leq 20 \text{ cm}$ in 1-cm steps.

Solution: Let the length of the solenoid be $l = 20$ cm. From Eq. (5.88a) and Eq. (5.88b), $z = a\tan\theta$ and $a^2 + t^2 = a^2\sec^2\theta$, which implies that $z/\sqrt{z^2+a^2} = \sin\theta$. Generalizing this to an arbitrary observation point z' on the axis of the solenoid, $(z-z')/\sqrt{(z-z')^2+a^2} = \sin\theta$. Using this in Eq. (5.89),

$$\begin{aligned}
\mathbb{H}(0,0,z') = \frac{\mathbb{B}}{\mu} &= \hat{\mathbf{z}}\frac{nI}{2}(\sin\theta_2 - \sin\theta_1) \\
&= \hat{\mathbf{z}}\frac{nI}{2}\left(\frac{l/2-z'}{\sqrt{(l/2-z')^2+a^2}} - \frac{-l/2-z'}{\sqrt{(-l/2-z')^2+a^2}}\right) \\
&= \hat{\mathbf{z}}\frac{nI}{2}\left(\frac{l/2-z'}{\sqrt{(l/2-z')^2+a^2}} + \frac{l/2+z'}{\sqrt{(l/2+z')^2+a^2}}\right) \quad \text{(A/m)}.
\end{aligned}$$

A plot of the magnitude of this function of z' with $a = 5$ cm, $n = 400$ turns/20 cm = 20,000 turns/m, and $I = 12$ A appears in Fig. P5.36.

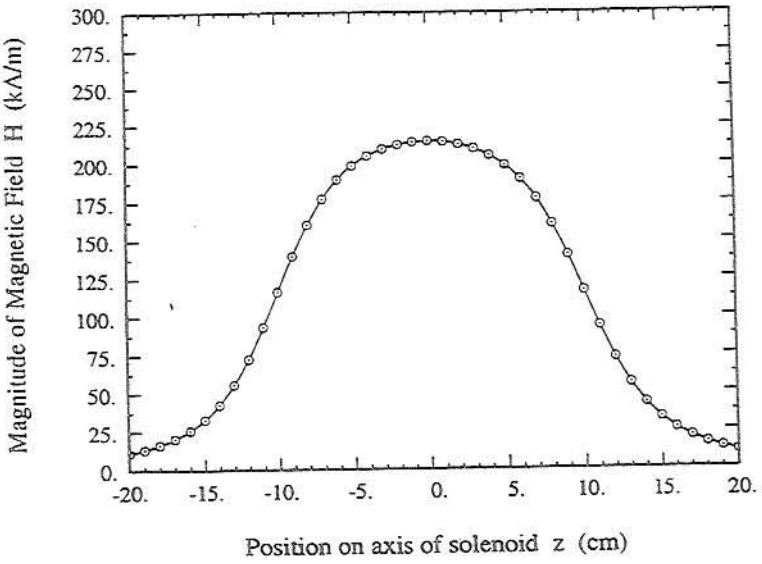

Figure P5.36: Problem 5.36.

Problem 5.37 In terms of the d-c current I, how much magnetic energy is stored in the insulating medium of a 2-m-long, air-filled section of a coaxial transmission line, given that the radius of the inner conductor is 5 cm and the inner radius of the outer conductor is 10 cm?

Solution: From Eq. (5.99), the inductance per unit length of an air-filled coaxial cable is given by

$$L' = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) \quad \text{(H/m)}.$$

Over a length of 2 m, the inductance is

$$L = 2L' = \frac{2 \times 4\pi \times 10^{-7}}{2\pi} \ln\left(\frac{10}{5}\right) = 277 \times 10^{-9} \quad \text{(H)}.$$

From Eq. (5.104), $W_m = LI^2/2 = 139I^2$ (nJ), where W_m is in nanojoules when I is in amperes. Alternatively, we can use Eq. (5.106) to compute W_m:

$$W_m = \frac{1}{2} \int_v \mu_0 H^2 \, dv.$$

From Eq. (5.97), $H = B/\mu_0 = I/2\pi r$, and

$$W_m = \frac{1}{2}\int_{z=0}^{2m}\int_{\phi=0}^{2\pi}\int_{r=a}^{b}\mu_0\left(\frac{I}{2\pi r}\right)^2 r\,dr\,d\phi\,dz = 139I^2 \quad \text{(nJ)}.$$

Problem 5.38 The rectangular loop shown in Fig. 5-48 (P5.38) is coplanar with the long, straight wire carrying the current $I = 20$ A. Determine the magnetic flux through the loop.

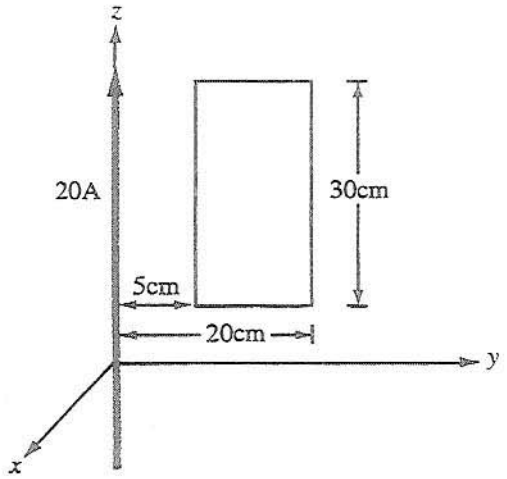

Figure P5.38: Loop and wire arrangement for Problem 5.38.

Solution: The field due to the long wire is, from Eq. (5.30),

$$\mathbb{B} = \hat{\phi}\frac{\mu_0 I}{2\pi r} = -\hat{\mathrm{x}}\frac{\mu_0 I}{2\pi r} = -\hat{\mathrm{x}}\frac{\mu_0 I}{2\pi y},$$

where in the plane of the loop, $\hat{\phi}$ becomes $-\hat{\mathrm{x}}$ and r becomes y.

The flux through the loop is along $-\hat{\mathrm{x}}$, and the magnitude of the flux is

$$\begin{aligned}\Phi = \int_S \mathbb{B}\cdot d\mathbb{s} &= \frac{\mu_0 I}{2\pi}\int_{5\text{ cm}}^{20\text{ cm}} -\frac{\hat{\mathrm{x}}}{y}\cdot -\hat{\mathrm{x}}(30\text{ cm}\times dy)\\ &= \frac{\mu_0 I}{2\pi}\times 0.3\int_{0.05}^{0.2}\frac{dy}{y}\\ &= \frac{0.3\mu_0}{2\pi}\times 20\times \ln\left(\frac{0.2}{0.05}\right) = 1.66\times 10^{-6} \quad \text{(Wb)}.\end{aligned}$$