Chapter 6

Sections 6-1 to 6-6: Faraday's Law and its Applications

Problem 6.1 The switch in the bottom loop of Fig. 6-17 (P6.1) is closed at $t = 0$ and then opened at a later time t_1. What is the direction of the current I in the top loop (clockwise or counterclockwise) at each of these two times?

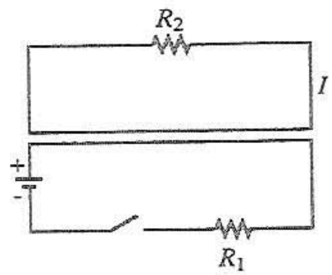

Figure P6.1: Loops of Problem 6.1.

Solution: The magnetic coupling will be strongest at the point where the wires of the two loops come closest. When the switch is closed the current in the bottom loop will start to flow clockwise, which is from left to right in the top portion of the bottom loop. To oppose this change, a current will momentarily flow in the bottom of the top loop from right to left. Thus the current in the top loop is momentarily clockwise when the switch is closed. Similarly, when the switch is opened, the current in the top loop is momentarily counterclockwise.

Problem 6.2 The loop in Fig. 6-18 (P6.2) is in the x–y plane and $\mathbf{B} = \hat{\mathbf{z}}B_0 \sin \omega t$ with B_0 positive. What is the direction of I ($\hat{\phi}$ or $-\hat{\phi}$) at (a) $t = 0$, (b) $\omega t = \pi/4$, and (c) $\omega t = \pi/2$?

Solution: $I = V_{\text{emf}}/R$. Since the single-turn loop is not moving or changing shape with time, $V^{\text{m}}_{\text{emf}} = 0$ V and $V_{\text{emf}} = V^{\text{tr}}_{\text{emf}}$. Therefore, from Eq. (6.8),

$$I = V^{\text{tr}}_{\text{emf}}/R = \frac{-1}{R}\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}.$$

If we take the surface normal to be $+\hat{\mathbf{z}}$, then the right hand rule gives positive flowing current to be in the $+\hat{\phi}$ direction.

$$I = \frac{-A}{R}\frac{\partial}{\partial t}B_0 \sin \omega t = \frac{-AB_0\omega}{R}\cos \omega t \quad \text{(A)},$$

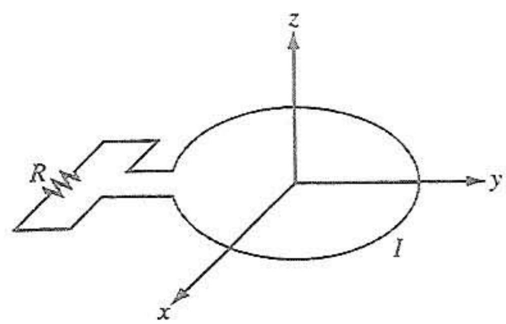

Figure P6.2: Loop of Problem 6.2.

where A is the area of the loop.

(a) A, ω and R are positive quantities. At $t = 0$, $\cos\omega t = 1$ so $I < 0$ and the current is flowing in the $-\hat{\phi}$ direction (so as to produce an induced magnetic field that opposes $\mathbf{B}$).

(b) At $\omega t = \pi/4$, $\cos\omega t = \sqrt{2}/2$ so $I < 0$ and the current is still flowing in the $-\hat{\phi}$ direction.

(c) At $\omega t = \pi/2$, $\cos\omega t = 0$ so $I = 0$. There is no current flowing in either direction.

Problem 6.3 A coil consists of 100 turns of wire wrapped around a square frame of sides 0.25 m. The coil is centered at the origin with each of its sides parallel to the x- or y-axis. Find the induced emf across the open-circuited ends of the coil if the magnetic field is given by

(a) $\mathbf{B} = \hat{\mathbf{z}}\,10e^{-2t}$ (T),

(b) $\mathbf{B} = \hat{\mathbf{z}}\,10\cos x\,\cos 10^3 t$ (T),

(c) $\mathbf{B} = \hat{\mathbf{z}}\,10\cos x\,\sin 2y\,\cos 10^3 t$ (T).

Solution: Since the coil is not moving or changing shape, $V_{\text{emf}}^{\text{m}} = 0$ V and $V_{\text{emf}} = V_{\text{emf}}^{\text{tr}}$. From Eq. (6.6),

$$V_{\text{emf}} = -N\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = -N\frac{d}{dt}\int_{-0.125}^{0.125}\int_{-0.125}^{0.125} \mathbf{B}\cdot(\hat{\mathbf{z}}\,dx\,dy),$$

where $N = 100$ and the surface normal was chosen to be in the $+\hat{\mathbf{z}}$ direction.

(a) For $\mathbf{B} = \hat{\mathbf{z}}10e^{-2t}$ (T),

$$V_{\text{emf}} = -100\frac{d}{dt}(10e^{-2t}(0.25)^2) = 125e^{-2t} \quad \text{(V)}.$$

(b) For $\mathbf{B} = \hat{\mathbf{z}}10\cos x \cos 10^3 t$ (T),

$$V_{\text{emf}} = -100\frac{d}{dt}\left(10\cos 10^3 t \int_{x=-0.125}^{0.125}\int_{y=-0.125}^{0.125} \cos x \, dx\, dy\right) = 62.3 \sin 10^3 t \quad \text{(kV)}.$$

(c) For $\mathbf{B} = \hat{\mathbf{z}}10\cos x \sin 2y \cos 10^3 t$ (T),

$$V_{\text{emf}} = -100\frac{d}{dt}\left(10\cos 10^3 t \int_{x=-0.125}^{0.125}\int_{y=-0.125}^{0.125} \cos x \sin 2y \, dx\, dy\right) = 0.$$

Problem 6.4 A stationary conducting loop with internal resistance of 0.5 Ω is placed in a time-varying magnetic field. When the loop is closed, a current of 2.5 A flows through it. What will the current be if the loop is opened to create a small gap and a 2-Ω resistor is connected across its open ends?

Solution: V_{emf} is independent of the resistance which is in the loop. Therefore, when the loop is intact and the internal resistance is only 0.5 Ω,

$$V_{\text{emf}} = 2.5 \text{ A} \times 0.5 \ \Omega = 1.25 \text{ V}.$$

When the small gap is created, the total resistance in the loop is infinite and the current flow is zero. With a 2-Ω resistor in the gap,

$$I = V_{\text{emf}}/(2\ \Omega + 0.5\ \Omega) = 1.25 \text{ V}/2.5\ \Omega = 0.5 \quad \text{(A)}.$$

Problem 6.5 A circular-loop TV antenna with 0.01 m^2 area is in the presence of a uniform-amplitude 300-MHz signal. When oriented for maximum response, the loop develops an emf with a peak value of 20 (mV). What is the peak magnitude of $\mathbf{B}$ of the incident wave?

Solution: TV loop antennas have one turn. At maximum orientation, Eq. (6.5) evaluates to $\Phi = \int \mathbf{B} \cdot d\mathbf{s} = \pm BA$ for a loop of area A and a uniform magnetic field with magnitude $B = |\mathbf{B}|$. Since we know the frequency of the field is $f = 300$ MHz, we can express B as $B = B_0 \cos(\omega t + \alpha_0)$ with $\omega = 2\pi \times 300 \times 10^6$ rad/s and α_0 an arbitrary reference phase. From Eq. (6.6),

$$V_{\text{emf}} = -N\frac{d\Phi}{dt} = -A\frac{d}{dt}[B_0 \cos(\omega t + \alpha_0)] = AB_0 \omega \sin(\omega t + \alpha_0).$$

V_{emf} is maximum when $\sin(\omega t + \alpha_0) = 1$. Hence,

$$20 \times 10^{-3} = AB_0\omega = 10^{-2} \times B_0 \times 6\pi \times 10^8,$$

which yields $B_0 = 1.06$ (nA/m).

Problem 6.6 The square loop shown in Fig. 6-19 (P6.6) is coplanar with a long, straight wire carrying a current

$$i(t) = 2.5\cos 2\pi \times 10^4 t \quad \text{(A)}.$$

(a) Determine the emf induced across a small gap created in the loop.

(b) Determine the direction and magnitude of the current that would flow through a 4-Ω resistor connected across the gap. The loop has an internal resistance of 1 Ω.

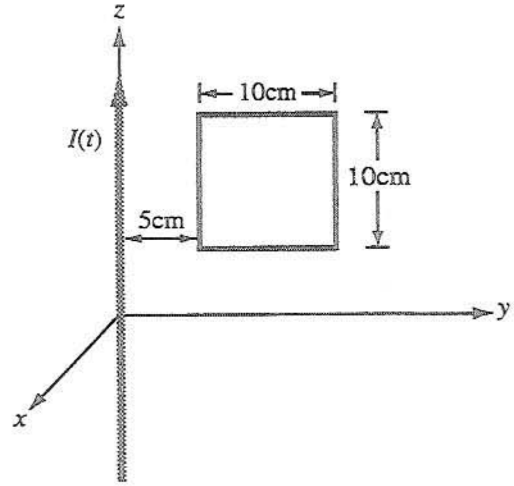

Figure P6.6: Loop coplanar with long wire (Problem 6.6).

Solution:

(a) The magnetic field due to the wire is

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r} = -\hat{\mathbf{x}} \frac{\mu_0 I}{2\pi y},$$

where in the plane of the loop, $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}$ and $r = y$. The flux passing through the loop is

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_{5\text{ cm}}^{15\text{ cm}} \left(-\hat{\mathbf{x}}\frac{\mu_0 I}{2\pi y}\right) \cdot [-\hat{\mathbf{x}}\, 10\text{ (cm)}]\, dy$$
$$= \frac{\mu_0 I \times 10^{-1}}{2\pi} \ln\frac{15}{5}$$
$$= \frac{4\pi \times 10^{-7} \times 2.5\cos(2\pi \times 10^4 t) \times 10^{-1}}{2\pi} \times 1.1$$
$$= 0.55 \times 10^{-7} \cos(2\pi \times 10^4 t) \quad \text{(Wb)}.$$

$$V_{\text{emf}} = -\frac{d\Phi}{dt} = 0.55 \times 2\pi \times 10^4 \sin(2\pi \times 10^4 t) \times 10^{-7}$$
$$= 3.45 \times 10^{-3} \sin(2\pi \times 10^4 t) \quad \text{(V)}.$$

(b)

$$I_{\text{ind}} = \frac{V_{\text{emf}}}{4+1} = \frac{3.45 \times 10^{-3}}{5} \sin(2\pi \times 10^4 t) = 0.69 \sin(2\pi \times 10^4 t) \quad \text{(mA)}.$$

At $t = 0$, $\mathbf{B}$ is a maximum, it points in $-\hat{\mathbf{x}}$-direction, and since it varies as $\cos(2\pi \times 10^4 t)$, it is decreasing. Hence, the induced current has to be CCW when looking down on the loop, as shown in the figure.

Problem 6.7 The rectangular conducting loop shown in Fig. 6-20 (P6.7) rotates at 6,000 revolutions per minute in a uniform magnetic flux density given by

$$\mathbf{B} = \hat{\mathbf{y}}\, 50 \quad \text{(mT)}.$$

Determine the current induced in the loop if its internal resistance is 0.5 Ω.

Solution:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \hat{\mathbf{y}}\, 50 \times 10^{-3} \cdot \hat{\mathbf{y}}(2 \times 3 \times 10^{-4}) \cos\phi(t) = 3 \times 10^{-5} \cos\phi(t),$$
$$\phi(t) = \omega t = \frac{2\pi \times 6 \times 10^3}{60} t = 200\pi t \quad \text{(rad/s)},$$
$$\Phi = 3 \times 10^{-5} \cos(200\pi t) \quad \text{(Wb)},$$
$$V_{\text{emf}} = -\frac{d\Phi}{dt} = 3 \times 10^{-5} \times 200\pi \sin(200\pi t) = 18.85 \times 10^{-3} \sin(200\pi t) \quad \text{(V)},$$
$$I_{\text{ind}} = \frac{V_{\text{emf}}}{0.5} = 37.7 \sin(200\pi t) \quad \text{(mA)}.$$

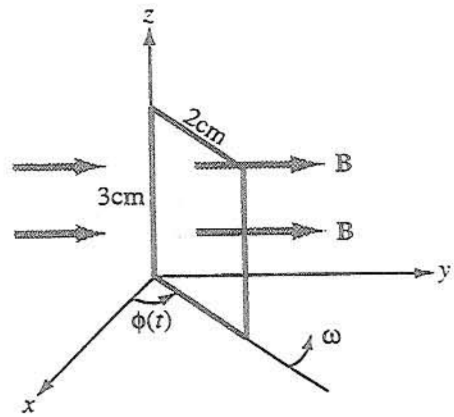

Figure P6.7: Rotating loop in a magnetic field (Problem 6.7).

The direction of the current is CW (if looking at it along $-\hat{\mathbf{x}}$-direction) when the loop is in the first quadrant ($0 \leq \phi \leq \pi/2$). The current reverses direction in the second quadrant, and reverses again every quadrant.

Problem 6.8 A rectangular conducting loop 5 cm $\times$10 cm with a small air gap in one of its sides is spinning at 7200 revolutions per minute. If the field $\mathbf{B}$ is normal to the loop axis and its magnitude is 5×10^{-6} T, what is the peak voltage induced across the air gap?

Solution:

$$\omega = \frac{2\pi \text{ rad/cycle} \times 7200 \text{ cycles/min}}{60 \text{ s/min}} = 240\pi \text{ rad/s},$$
$$A = 5 \text{ cm} \times 10 \text{ cm}/(100 \text{ cm/m})^2 = 5.0 \times 10^{-3} \text{ m}^2.$$

From Eqs. (6.36) or (6.38), $V_{\text{emf}} = A\omega B_0 \sin\omega t$; it can be seen that the peak voltage is

$$V_{\text{emf}}^{\text{peak}} = A\omega B_0 = 5.0 \times 10^{-3} \times 240\pi \times 5 \times 10^{-6} = 18.85 \quad (\mu\text{V}).$$

Problem 6.9 A 50-cm-long metal rod rotates about the z-axis at 180 revolutions per minute, with end 1 fixed at the origin as shown in Fig. 6-21 (P6.9). Determine the induced emf V_{12} if $\mathbf{B} = \hat{\mathbf{z}}3 \times 10^{-4}$ T.

Solution: Since $\mathbf{B}$ is constant, $V_{\text{emf}} = V_{\text{emf}}^{\text{m}}$. The velocity $\mathbf{u}$ for any point on the bar is given by $\mathbf{u} = \hat{\boldsymbol{\phi}} r\omega$, where

$$\omega = \frac{2\pi \text{ rad/cycle} \times (180 \text{ cycles/min})}{(60 \text{ s/min})} = 6\pi \text{ rad/s}.$$

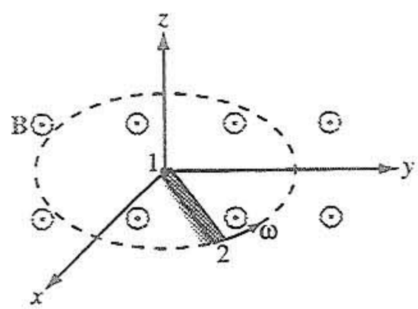

Figure P6.9: Rotating rod of Problem 6.9.

From Eq. (6.24),

$$V_{12} = V_{\text{emf}}^{\text{m}} = \int_2^1 (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = \int_{r=0.5}^{0} (\hat{\boldsymbol{\phi}}6\pi r \times \hat{\mathbf{z}}3 \times 10^{-4}) \cdot \hat{\mathbf{r}}\, dr$$

$$= 18\pi \times 10^{-4} \int_{r=0.5}^{0} r\, dr$$

$$= 9\pi \times 10^{-4} r^2 \Big|_{0.5}^{0}$$

$$= -9\pi \times 10^{-4} \times 0.25 = -707 \quad (\mu\text{V}).$$

Problem 6.10 The loop shown in Fig. 6-22 (P6.10) moves away from a wire carrying a current $I_1 = 10$ (A) at a constant velocity $\mathbf{u} = \hat{\mathbf{y}}5$ (m/s). If $R = 10\ \Omega$ and the direction of I_2 is as defined in the figure, find I_2 as a function of y_0, the distance between the wire and the loop. Ignore the internal resistance of the loop.

Solution: Assume that the wire carrying current I_1 is in the same plane as the loop. The two identical resistors are in series, so $I_2 = V_{\text{emf}}/2R$, where the induced voltage is due to motion of the loop and is given by Eq. (6.26):

$$V_{\text{emf}} = V_{\text{emf}}^{\text{m}} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}.$$

The magnetic field $\mathbf{B}$ is created by the wire carrying I_1. Choosing $\hat{\mathbf{z}}$ to coincide with the direction of I_1, Eq. (5.30) gives the external magnetic field of a long wire to be

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I_1}{2\pi r}.$$

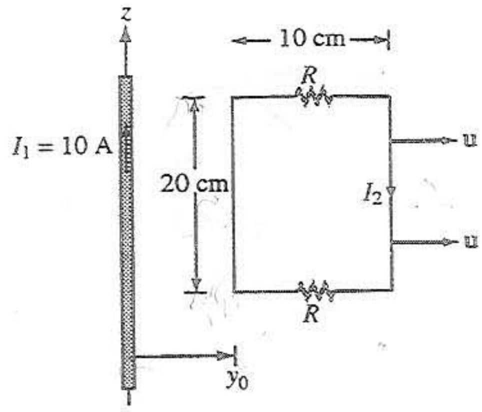

Figure P6.10: Moving loop of Problem 6.10.

For positive values of y_0 in the y-z plane, $\hat{\mathbf{y}} = \hat{\mathbf{r}}$, so

$$\mathbf{u} \times \mathbf{B} = \hat{\mathbf{y}}|\mathbf{u}| \times \mathbf{B} = \hat{\mathbf{r}}|\mathbf{u}| \times \hat{\boldsymbol{\phi}} \frac{\mu_0 I_1}{2\pi r} = \hat{\mathbf{z}} \frac{\mu_0 I_1 u}{2\pi r}.$$

Integrating around the four sides of the loop with $d\mathbf{l} = \hat{\mathbf{z}}\, dz$ and the limits of integration chosen in accordance with the assumed direction of I_2, and recognizing that only the two sides without the resistors contribute to $V_{\text{emf}}^{\text{m}}$, we have

$$\begin{aligned} V_{\text{emf}}^{\text{m}} &= \int_0^{0.2} \left(\hat{\mathbf{z}} \frac{\mu_0 I_1 u}{2\pi r} \right) \bigg|_{r=y_0} \cdot (\hat{\mathbf{z}}\, dz) + \int_{0.2}^{0} \left(\hat{\mathbf{z}} \frac{\mu_0 I_1 u}{2\pi r} \right) \bigg|_{r=y_0+0.1} \cdot (\hat{\mathbf{z}}\, dz) \\ &= \frac{4\pi \times 10^{-7} \times 10 \times 5 \times 0.2}{2\pi} \left(\frac{1}{y_0} - \frac{1}{y_0 + 0.1} \right) \\ &= 2 \times 10^{-6} \left(\frac{1}{y_0} - \frac{1}{y_0 + 0.1} \right) \quad \text{(V)}, \end{aligned}$$

and therefore

$$I_2 = \frac{V_{\text{emf}}^{\text{m}}}{2R} = 100 \left(\frac{1}{y_0} - \frac{1}{y_0 + 0.1} \right) \quad \text{(nA)}.$$

Problem 6.11 The conducting cylinder shown in Fig. 6-23 (P6.11) rotates about its axis at 1,200 revolutions per minute in a radial field given by

$$\mathbf{B} = \hat{\mathbf{r}}\, 6 \quad \text{(T)}.$$

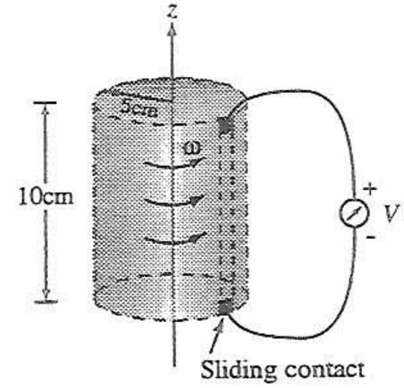

Figure P6.11: Rotating cylinder in a magnetic field (Problem 6.11).

The cylinder, whose radius is 5 cm and height 10 cm, has sliding contacts at its top and bottom connected to a voltmeter. Determine the induced voltage.

Solution: The surface of the cylinder has velocity $\mathbf{u}$ given by

$$\mathbf{u} = \hat{\phi}\,\omega r = \hat{\phi}\,2\pi \times \frac{1,200}{60} \times 5 \times 10^{-2} = \hat{\phi}\,2\pi \quad \text{(m/s)},$$

$$V_{12} = \int_0^L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = \int_0^{0.1} (\hat{\phi}\,2\pi \times \hat{\mathbf{r}}6) \cdot \hat{\mathbf{z}}\,dz = -3.77 \quad \text{(V)}.$$

Problem 6.12 The electromagnetic generator shown in Fig. 6-12 is connected to an electric bulb with a resistance of 100 Ω. If the loop area is 0.1 m^2 and it rotates at 3,600 revolutions per minute in a uniform magnetic flux density $B_0 = 0.2$ T, determine the amplitude of the current generated in the light bulb.

Solution: From Eq. (6.38), the sinusoidal voltage generated by the a-c generator is $V_{\text{emf}} = A\omega B_0 \sin(\omega t + C_0) = V_0 \sin(\omega t + C_0)$. Hence,

$$V_0 = A\omega B_0 = 0.1 \times \frac{2\pi \times 3,600}{60} \times 0.2 = 7.54 \quad \text{(V)},$$

$$I = \frac{V_0}{R} = \frac{7.54}{100} = 75.4 \quad \text{(mA)}.$$

Problem 6.13 The circular disk shown in Fig. 6-24 (P6.13) lies in the x–y plane and rotates with uniform angular velocity ω about the z-axis. The disk is of radius a and is present in a uniform magnetic flux density $\mathbf{B} = \hat{\mathbf{z}}B_0$. Obtain an expression for the emf induced at the rim relative to the center of the disk.

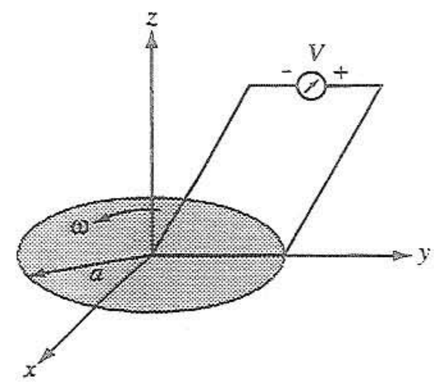

Figure P6.13: Rotating circular disk in a magnetic field (Problem 6.13).

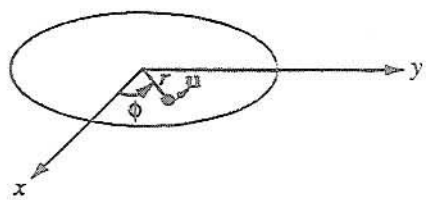

Figure P6.13: (a) Velocity vector $\mathbf{u}$.

Solution: At a radial distance r, the velocity is

$$\mathbf{u} = \hat{\boldsymbol{\phi}}\, \omega r$$

where ϕ is the angle in the x–y plane shown in the figure. The induced voltage is

$$V = \int_0^a (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = \int_0^a [(\hat{\boldsymbol{\phi}}\, \omega r) \times \hat{\mathbf{z}} B_0] \cdot \hat{\mathbf{r}}\, dr.$$

$\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}}$ is along $\hat{\mathbf{r}}$. Hence,

$$V = \omega B_0 \int_0^a r\, dr = \frac{\omega B_0 a^2}{2}.$$

Section 6-7: Displacement Current

Problem 6.14 The plates of a parallel-plate capacitor have areas 10 cm^2 each and are separated by 1 cm. The capacitor is filled with a dielectric material with

$\varepsilon = 4\varepsilon_0$, and the voltage across it is given by $V(t) = 20\cos 2\pi \times 10^6 t$ (V). Find the displacement current.

Solution: Since the voltage is of the form given by Eq. (6.46) with $V_0 = 20$ V and $\omega = 2\pi \times 10^6$ rad/s, the displacement current is given by Eq. (6.49):

$$\begin{aligned} I_d &= -\frac{\varepsilon A}{d} V_0 \omega \sin \omega t \\ &= -\frac{4 \times 8.854 \times 10^{-12} \times 10 \times 10^{-4}}{1 \times 10^{-2}} \times 20 \times 2\pi \times 10^6 \sin(2\pi \times 10^6 t) \\ &= -445 \sin(2\pi \times 10^6 t) \quad (\mu\text{A}). \end{aligned}$$

Problem 6.15 A coaxial capacitor of length $l = 6$ cm uses an insulating dielectric material with $\varepsilon_r = 9$. The radii of the cylindrical conductors are 0.5 cm and 1 cm. If the voltage applied across the capacitor is

$$V(t) = 100 \sin(120\pi t) \quad \text{(V)},$$

what is the displacement current?

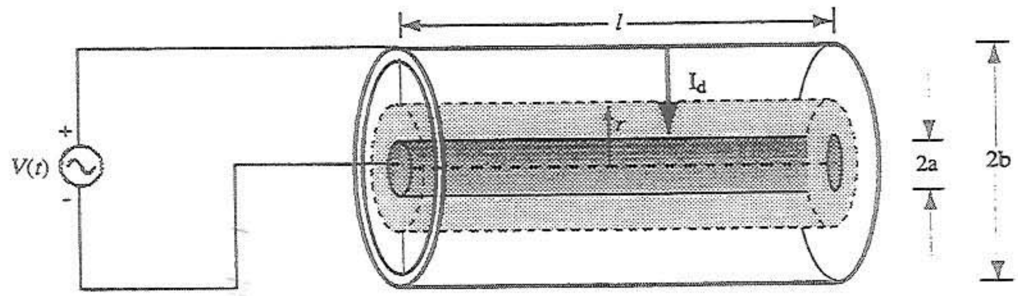

Figure P6.15:

Solution: To find the displacement current, we need to know **E** in the dielectric space between the cylindrical conductors. From Eqs. (4.114) and (4.115),

$$\mathbf{E} = -\hat{\mathbf{r}} \frac{Q}{2\pi\varepsilon r l},$$
$$V = \frac{Q}{2\pi\varepsilon l} \ln\left(\frac{b}{a}\right).$$

Hence,

$$\mathbf{E} = -\hat{\mathbf{r}} \frac{V}{r \ln\left(\frac{b}{a}\right)} = -\hat{\mathbf{r}} \frac{100 \sin(120\pi t)}{r \ln 2} = -\hat{\mathbf{r}} \frac{144.3}{r} \sin(120\pi t) \quad \text{(V/m)},$$

$$\begin{aligned}\mathbf{D} &= \varepsilon\mathbf{E}\\ &= \varepsilon_r\varepsilon_0\mathbf{E}\\ &= -\hat{\mathbf{r}}\,9\times 8.85\times 10^{-12}\times\frac{144.3}{r}\sin(120\pi t)\\ &= -\hat{\mathbf{r}}\,\frac{1.15\times 10^{-8}}{r}\sin(120\pi t)\quad (\text{C/m}^2).\end{aligned}$$

The displacement current flows between the conductors through an imaginary cylindrical surface of length l and radius r. The current flowing from the outer conductor to the inner conductor along $-\hat{\mathbf{r}}$ crosses surface $\mathbf{S}$ where

$$\mathbf{S} = -\hat{\mathbf{r}}\,2\pi rl.$$

Hence,

$$\begin{aligned}I_\text{d} = \frac{\partial\mathbf{D}}{\partial t}\cdot\mathbf{S} &= -\hat{\mathbf{r}}\frac{\partial}{\partial t}\left(\frac{1.15\times 10^{-8}}{r}\sin(120\pi t)\right)\cdot(-\hat{\mathbf{r}}\,2\pi rl)\\ &= 1.15\times 10^{-8}\times 120\pi\times 2\pi l\cos(120\pi t)\\ &= 1.63\cos(120\pi t)\quad (\mu\text{A}).\end{aligned}$$

Alternatively, since the coaxial capacitor is lossless, its displacement current has to be equal to the conduction current flowing through the wires connected to the voltage sources. The capacitance of a coaxial capacitor is given by (4.116) as

$$C = \frac{2\pi\varepsilon l}{\ln\left(\frac{b}{a}\right)}.$$

The current is

$$I = C\frac{dV}{dt} = \frac{2\pi\varepsilon l}{\ln\left(\frac{b}{a}\right)}[120\pi\times 100\cos(120\pi t)] = 1.63\cos(120\pi t)\quad (\mu\text{A}),$$

which is the same answer we obtained before.

Problem 6.16 The parallel-plate capacitor shown in Fig. 6-25 (P6.16) is filled with a lossy dielectric material of relative permittivity ε_r and conductivity σ. The separation between the plates is d and each plate is of area A. The capacitor is connected to a time-varying voltage source $V(t)$.

(a) Obtain an expression for I_c, the conduction current flowing between the plates inside the capacitor, in terms of the given quantities.

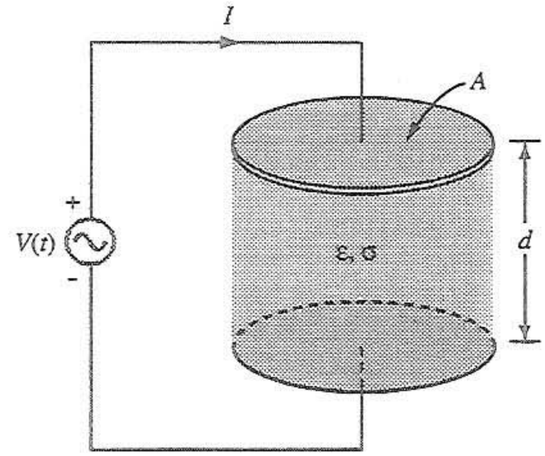

Figure P6.16: Parallel-plate capacitor containing a lossy dielectric material (Problem 6.16).

(b) Obtain an expression for I_d, the displacement current flowing inside the capacitor.

(c) Based on your expression for parts (a) and (b), give an equivalent-circuit representation for the capacitor.

(d) Evaluate the values of the circuit elements for $A = 2$ cm^2, $d = 0.5$ cm, $\varepsilon_r = 4$, $\sigma = 2.5$ (S/m), and $V(t) = 10\cos(3\pi \times 10^3 t)$ (V).

Solution:

(a)

$$R = \frac{d}{\sigma A}, \qquad I_c = \frac{V}{R} = \frac{V\sigma A}{d}.$$

(b)

$$E = \frac{V}{d}, \qquad I_d = \frac{\partial D}{\partial t} \cdot A = \varepsilon A \frac{\partial E}{\partial t} = \frac{\varepsilon A}{d} \frac{\partial V}{\partial t}.$$

(c) The conduction current is directly proportional to V, as characteristic of a resistor, whereas the displacement current varies as $\partial V/\partial t$, which is characteristic of a capacitor. Hence,

$$R = \frac{d}{\sigma A} \qquad \text{and} \qquad C = \frac{\varepsilon A}{d}.$$

(d)

$$R = \frac{0.5 \times 10^{-2}}{2.5 \times 2 \times 10^{-4}} = 10\ \Omega,$$

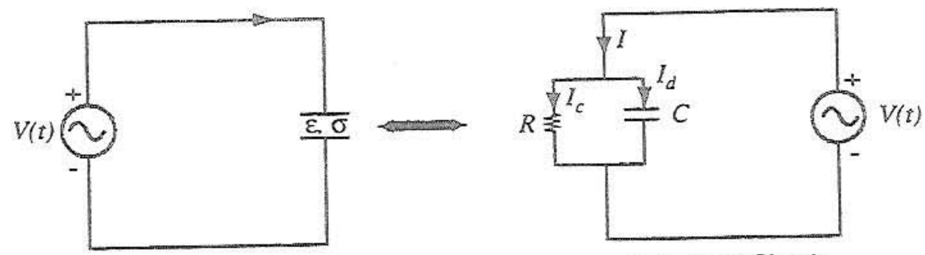

Figure P6.16: (a) Equivalent circuit.

$$C = \frac{4 \times 8.85 \times 10^{-12} \times 2 \times 10^{-4}}{0.5 \times 10^{-2}} = 1.42 \times 10^{-12} \text{ F.}$$

Problem 6.17 An electromagnetic wave propagating in seawater has an electric field with a time variation given by $\mathbf{E} = \hat{\mathbf{z}}E_0 \cos \omega t$. If the permittivity of water is $81\varepsilon_0$ and its conductivity is 4 (S/m), find the ratio of the magnitudes of the conduction current density to displacement current density at each of the following frequencies: (a) 1 kHz, (b) 1 MHz, (c) 1 GHz, (d) 100 GHz.

Solution: From Eq. (6.44), the displacement current density is given by

$$\mathbf{J}_d = \frac{\partial}{\partial t}\mathbf{D} = \varepsilon \frac{\partial}{\partial t}\mathbf{E}$$

and, from Eq. (4.67), the conduction current is $\mathbf{J} = \sigma \mathbf{E}$. Converting to phasors and taking the ratio of the magnitudes,

$$\left| \frac{\widetilde{\mathbf{J}}}{\widetilde{\mathbf{J}}_d} \right| = \left| \frac{\sigma \widetilde{\mathbf{E}}}{j\omega\varepsilon_r\varepsilon_0 \widetilde{\mathbf{E}}} \right| = \frac{\sigma}{\omega\varepsilon_r\varepsilon_0}.$$

(a) At $f = 1$ kHz, $\omega = 2\pi \times 10^3$ rad/s, and

$$\left| \frac{\widetilde{\mathbf{J}}}{\widetilde{\mathbf{J}}_d} \right| = \frac{4}{2\pi \times 10^3 \times 81 \times 8.854 \times 10^{-12}} = 888 \times 10^3.$$

The displacement current is negligible.

(b) At $f = 1$ MHz, $\omega = 2\pi \times 10^6$ rad/s, and

$$\left| \frac{\widetilde{\mathbf{J}}}{\widetilde{\mathbf{J}}_d} \right| = \frac{4}{2\pi \times 10^6 \times 81 \times 8.854 \times 10^{-12}} = 888.$$

The displacement current is practically negligible.

(c) At $f = 1$ GHz, $\omega = 2\pi \times 10^9$ rad/s, and

$$\left|\frac{\widetilde{\mathbf{J}}}{\widetilde{\mathbf{J}}_{\mathrm{d}}}\right| = \frac{4}{2\pi \times 10^9 \times 81 \times 8.854 \times 10^{-12}} = 0.888.$$

Neither the displacement current nor the conduction current are negligible.

(d) At $f = 100$ GHz, $\omega = 2\pi \times 10^{11}$ rad/s, and

$$\left|\frac{\widetilde{\mathbf{J}}}{\widetilde{\mathbf{J}}_{\mathrm{d}}}\right| = \frac{4}{2\pi \times 10^{11} \times 81 \times 8.854 \times 10^{-12}} = 8.88 \times 10^{-3}.$$

The conduction current is practically negligible.

Sections 6-9 and 6-10: Continuity Equation and Charge Dissipation

Problem 6.18 At $t = 0$, charge density ρ_{v0} was introduced into the interior of a material with a relative permittivity $\varepsilon_r = 4\varepsilon_0$. If at $t = 1$ μs the charge density has dissipated down to $10^{-3}\rho_{v0}$, what is the conductivity of the material?

Solution: We start by using Eq. (6.61) to find τ_r:

$$\rho_v(t) = \rho_{v0}e^{-t/\tau_r},$$

or

$$10^{-3}\rho_{v0} = \rho_{v0}e^{-10^{-6}/\tau_r},$$

which gives

$$\ln 10^{-3} = -\frac{10^{-6}}{\tau_r},$$

or

$$\tau_r = -\frac{10^{-6}}{\ln 10^{-3}} = 1.45 \times 10^{-7} \quad \text{(s)}.$$

But $\tau_r = \varepsilon/\sigma = 4\varepsilon_0/\sigma$. Hence

$$\sigma = \frac{4\varepsilon_0}{\tau_r} = \frac{4 \times 8.854 \times 10^{-12}}{1.45 \times 10^{-7}} = 2.44 \times 10^{-4} \quad \text{(S/m)}.$$

Problem 6.19 If the current density in a conducting medium is given by

$$\mathbf{J}(x,y,z;t) = (\hat{\mathbf{x}}z - \hat{\mathbf{y}}3y^2 + \hat{\mathbf{z}}2x)\cos\omega t,$$

determine the corresponding charge distribution $\rho_v(x,y,z;t)$.

Solution: Eq. (6.58) is given by

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho_v}{\partial t}. \tag{14}$$

The divergence of $\mathbf{J}$ is

$$\begin{aligned}\nabla\cdot\mathbf{J} &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\cdot(\hat{\mathbf{x}}z - \hat{\mathbf{y}}3y^2 + \hat{\mathbf{z}}2x)\cos\omega t \\ &= -3\frac{\partial}{\partial y}(y^2\cos\omega t) = -6y\cos\omega t.\end{aligned}$$

Using this result in Eq. (14) and then integrating both sides with respect to t gives

$$\rho_v = -\int(\nabla\cdot\mathbf{J})\,dt = -\int -6y\cos\omega t\,dt = \frac{6y}{\omega}\sin\omega t + C_0,$$

where C_0 is a constant of integration.

Problem 6.20 In a certain medium, the direction of current density $\mathbf{J}$ points in the radial direction in cylindrical coordinates and its magnitude is independent of both ϕ and z. Determine $\mathbf{J}$, given that the charge density in the medium is

$$\rho_v = \rho_0 r\cos\omega t \quad (\text{C/m}^3).$$

Solution: Based on the given information,

$$\mathbf{J} = \hat{\mathbf{r}}J_r(r).$$

With $J_\phi = J_z = 0$, in cylindrical coordinates the divergence is given by

$$\nabla\cdot\mathbf{J} = \frac{1}{r}\frac{\partial}{\partial r}(rJ_r).$$

From Eq. (6.54),

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho_v}{\partial t} = -\frac{\partial}{\partial t}(\rho_0 r\cos\omega t) = \rho_0 r\omega\sin\omega t.$$

Hence

$$\frac{1}{r}\frac{\partial}{\partial r}(rJ_r) = \rho_0 r\omega \sin \omega t,$$
$$\frac{\partial}{\partial r}(rJ_r) = \rho_0 r^2 \omega \sin \omega t,$$
$$\int_0^r \frac{\partial}{\partial r}(rJ_r)\,dr = \rho_0 \omega \sin \omega t \int_0^r r^2\,dr,$$
$$rJ_r\big|_0^r = (\rho_0 \omega \sin \omega t)\left.\frac{r^3}{3}\right|_0^r,$$
$$J_r = \frac{\rho_0 \omega r^2}{3}\sin \omega t,$$

and

$$\mathbf{J} = \hat{\mathbf{r}} J_r = \hat{\mathbf{r}}\frac{\rho_0 \omega r^2}{3}\sin \omega t \quad (\text{A/m}^2).$$

Problem 6.21 If we were to characterize how good a material is as an insulator by its resistance to dissipating charge, which of the following two materials is the better insulator?

Dry Soil: $\varepsilon_r = 2.5$, $\sigma = 10^{-4}$ (S/m)
Fresh Water: $\varepsilon_r = 80$, $\sigma = 10^{-3}$ (S/m)

Solution: Relaxation time constant $\tau_r = \frac{\varepsilon}{\sigma}$.

For dry soil, $$\tau_r = \frac{2.5}{10^{-4}} = 2.5 \times 10^4 \text{ s}.$$
For fresh water, $$\tau_r = \frac{80}{10^{-3}} = 8 \times 10^4 \text{ s}.$$

Since it takes longer for charge to dissipate in fresh water, it is a better insulator than dry soil.

Sections 6-11: Electromagnetic Potentials

Problem 6.22 The electric field of an electromagnetic wave propagating in air is given by

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}4\cos(6\times 10^8 t - 2z) + \hat{\mathbf{y}}3\sin(6\times 10^8 t - 2z) \quad (\text{V/m}).$$

Find the associated magnetic field $\mathbf{H}(z,t)$.

Solution: Converting to phasor form, the electric field is given by

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}4e^{-j2z} - j\hat{\mathbf{y}}3e^{-j2z} \quad \text{(V/m)},$$

which can be used with Eq. (6.87) to find the magnetic field:

$$\begin{aligned}
\widetilde{\mathbf{H}}(z) &= \frac{1}{-j\omega\mu}\nabla\times\widetilde{\mathbf{E}} \\
&= \frac{1}{-j\omega\mu}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4e^{-j2z} & -j3e^{-j2z} & 0 \end{vmatrix} \\
&= \frac{1}{-j\omega\mu}(\hat{\mathbf{x}}6e^{-j2z} - \hat{\mathbf{y}}\,j8e^{-j2z}) \\
&= \frac{j}{6\times10^{8}\times4\pi\times10^{-7}}(\hat{\mathbf{x}}6 - \hat{\mathbf{y}}\,j8)e^{-j2z} = j\hat{\mathbf{x}}8.0e^{-j2z} + \hat{\mathbf{y}}10.6e^{-j2z} \quad \text{(mA/m)}.
\end{aligned}$$

Converting back to instantaneous values, this is

$$\mathbf{H}(t,z) = -\hat{\mathbf{x}}8.0\sin(6\times10^{8}t - 2z) + \hat{\mathbf{y}}10.6\cos(6\times10^{8}t - 2z) \quad \text{(mA/m)}.$$

Problem 6.23 The magnetic field in a dielectric material with $\varepsilon = 4\varepsilon_0$, $\mu = \mu_0$, and $\sigma = 0$ is given by

$$\mathbf{H}(y,t) = \hat{\mathbf{x}}5\cos(2\pi\times10^{7}t + ky) \quad \text{(A/m)}.$$

Find k and the associated electric field $\mathbf{E}$.

Solution: In phasor form, the magnetic field is given by $\widetilde{\mathbf{H}} = \hat{\mathbf{x}}5e^{jky}$ (A/m). From Eq. (6.86),

$$\widetilde{\mathbf{E}} = \frac{1}{j\omega\varepsilon}\nabla\times\widetilde{\mathbf{H}} = \frac{-jk}{j\omega\varepsilon}\hat{\mathbf{z}}5e^{jky}$$

and, from Eq. (6.87),

$$\widetilde{\mathbf{H}} = \frac{1}{-j\omega\mu}\nabla\times\widetilde{\mathbf{E}} = \frac{-jk^2}{-j\omega^2\varepsilon\mu}\hat{\mathbf{x}}5e^{jky},$$

which, together with the original phasor expression for $\widetilde{\mathbf{H}}$, implies that

$$k = \omega\sqrt{\varepsilon\mu} = \frac{\omega\sqrt{\varepsilon_r}}{c} = \frac{2\pi\times10^{7}\sqrt{4}}{3\times10^{8}} = \frac{4\pi}{30} \quad \text{(rad/m)}.$$

Inserting this value in the expression for $\widetilde{\mathbf{E}}$ above,

$$\widetilde{\mathbf{E}} = -\hat{\mathbf{z}}\frac{4\pi/30}{2\pi\times10^7\times4\times8.854\times10^{-12}}5e^{j4\pi y/30} = -\hat{\mathbf{z}}941e^{j4\pi y/30} \quad \text{(V/m)}.$$

Problem 6.24 Given an electric field

$$\mathbf{E} = \hat{\mathbf{x}}E_0\sin ay\cos(\omega t - kz),$$

where E_0, a, ω, and k are constants, find $\mathbf{H}$.

Solution:

$$\begin{aligned}
\mathbf{E} &= \hat{\mathbf{x}}E_0\sin ay\cos(\omega t - kz),\\
\widetilde{\mathbf{E}} &= \hat{\mathbf{x}}E_0\sin ay\, e^{-jkz},\\
\widetilde{\mathbf{H}} &= -\frac{1}{j\omega\mu}\nabla\times\widetilde{\mathbf{E}}\\
&= -\frac{1}{j\omega\mu}\left[\hat{\mathbf{y}}\frac{\partial}{\partial z}(E_0\sin ay\, e^{-jkz}) - \hat{\mathbf{z}}\frac{\partial}{\partial y}(E_0\sin ay\, e^{-jkz})\right]\\
&= \frac{E_0}{\omega\mu}[\hat{\mathbf{y}}k\sin ay - \hat{\mathbf{z}}\, ja\cos ay]e^{-jkz},\\
\mathbf{H} &= \mathfrak{Re}[\widetilde{\mathbf{H}}e^{j\omega t}]\\
&= \mathfrak{Re}\left\{\frac{E_0}{\omega\mu}[\hat{\mathbf{y}}k\sin ay + \hat{\mathbf{z}}a\cos ay\, e^{-j\pi/2}]e^{-jkz}e^{j\omega t}\right\}\\
&= \frac{E_0}{\omega\mu}\left[\hat{\mathbf{y}}k\sin ay\cos(\omega t - kz) + \hat{\mathbf{z}}a\cos ay\cos\left(\omega t - kz - \frac{\pi}{2}\right)\right]\\
&= \frac{E_0}{\omega\mu}[\hat{\mathbf{y}}k\sin ay\cos(\omega t - kz) + \hat{\mathbf{z}}a\cos ay\sin(\omega t - kz)].
\end{aligned}$$

Problem 6.25 The electric field radiated by a short dipole antenna is given in spherical coordinates by

$$\mathbf{E}(R,\theta;t) = \hat{\boldsymbol{\theta}}\frac{2\times10^{-2}}{R}\sin\theta\,\cos(6\pi\times10^8t - 2\pi R) \quad \text{(V/m)}.$$

Find $\mathbf{H}(R,\theta;t)$.

Solution: Converting to phasor form, the electric field is given by

$$\widetilde{\mathbf{E}}(R,\theta) = \hat{\boldsymbol{\theta}}E_\theta = \hat{\boldsymbol{\theta}}\frac{2\times10^{-2}}{R}\sin\theta e^{-j2\pi R} \quad \text{(V/m)},$$

which can be used with Eq. (6.87) to find the magnetic field:

$$\widetilde{\mathbf{H}}(R,\theta) = \frac{1}{-j\omega\mu}\nabla\times\widetilde{\mathbf{E}} = \frac{1}{-j\omega\mu}\left[\hat{\mathbf{R}}\frac{1}{R\sin\theta}\frac{\partial E_\theta}{\partial\phi} + \hat{\boldsymbol{\phi}}\frac{1}{R}\frac{\partial}{\partial R}(RE_\theta)\right]$$
$$= \frac{1}{-j\omega\mu}\hat{\boldsymbol{\phi}}\frac{2\times10^{-2}}{R}\sin\theta\frac{\partial}{\partial R}(e^{-j2\pi R})$$
$$= \hat{\boldsymbol{\phi}}\frac{2\pi}{6\pi\times10^8\times4\pi\times10^{-7}}\frac{2\times10^{-2}}{R}\sin\theta e^{-j2\pi R}$$
$$= \hat{\boldsymbol{\phi}}\frac{53}{R}\sin\theta\, e^{-j2\pi R}\quad(\mu\text{A/m}).$$

Converting back to instantaneous value, this is

$$\mathbf{H}(R,\theta;t) = \hat{\boldsymbol{\phi}}\frac{53}{R}\sin\theta\cos(6\pi\times10^8 t - 2\pi R)\quad(\mu\text{A/m}).$$

Problem 6.26 A Hertzian dipole is a short conducting wire carrying an approximately constant current over its length l. If such a dipole is placed along the z-axis with its midpoint at the origin and if the current flowing through it is $i(t) = I_0\cos\omega t$, find

(a) the retarded vector potential $\widetilde{\mathbf{A}}(R,\theta,\phi)$ at an observation point $Q(R,\theta,\phi)$ in a spherical coordinate system, and

(b) the magnetic field phasor $\widetilde{\mathbf{H}}(R,\theta,\phi)$.

Assume l to be sufficiently small so that the observation point is approximately equidistant to all points on the dipole; that is, assume that $R' \simeq R$.

Solution:

(a) In phasor form, the current is given by $\widetilde{I} = I_0$. Explicitly writing the volume integral in Eq. (6.84) as a double integral over the wire cross section and a single integral over its length,

$$\widetilde{\mathbf{A}} = \frac{\mu}{4\pi}\int_{-l/2}^{l/2}\iint_s \frac{\widetilde{\mathbf{J}}(\mathbf{R}_i)\,e^{-jkR'}}{R'}\,ds\,dz,$$

where s is the wire cross section. The wire is infinitesimally thin, so that R' is not a function of x or y and the integration over the cross section of the wire applies only to the current density. Recognizing that $\widetilde{J} = \hat{\mathbf{z}}I_0/s$, and employing the relation $R' \approx R$,

$$\widetilde{\mathbf{A}} = \hat{\mathbf{z}}\frac{\mu I_0}{4\pi}\int_{-l/2}^{l/2}\frac{e^{-jkR'}}{R'}\,dz \approx \hat{\mathbf{z}}\frac{\mu I_0}{4\pi}\int_{-l/2}^{l/2}\frac{e^{-jkR}}{R}\,dz = \hat{\mathbf{z}}\frac{\mu I_0 l}{4\pi R}e^{-jkR}.$$

In spherical coordinates, $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$, and therefore

$$\widetilde{\mathbf{A}} = (\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)\frac{\mu I_0 l}{4\pi R}e^{-jkR}.$$

(b) From Eq. (6.85),

$$\begin{aligned}\widetilde{\mathbf{H}} &= \frac{1}{\mu}\nabla\times\widetilde{\mathbf{A}} = \frac{I_0 l}{4\pi}\nabla\times\left[(\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)\frac{e^{-jkR}}{R}\right] \\ &= \frac{I_0 l}{4\pi}\hat{\boldsymbol{\phi}}\frac{1}{R}\left(\frac{\partial}{\partial R}\left(-\sin\theta e^{-jkR}\right) - \frac{\partial}{\partial\theta}\left(\cos\theta\frac{e^{-jkR}}{R}\right)\right) \\ &= \hat{\boldsymbol{\phi}}\frac{I_0 l \sin\theta e^{-jkR}}{4\pi R}\left(jk + \frac{1}{R}\right).\end{aligned}$$

Problem 6.27 The magnetic field in a given dielectric medium is given by

$$\mathbf{H} = \hat{\mathbf{y}}\,6\cos 2z\sin(2\times10^7 t - 0.1x) \quad \text{(A/m)},$$

where x and z are in meters. Determine:

(a) $\mathbf{E}$,

(b) the displacement current density $\mathbf{J}_d$, and

(c) the charge density ρ_v.

Solution:

(a)

$$\begin{aligned}\mathbf{H} &= \hat{\mathbf{y}}\,6\cos 2z\sin(2\times10^7 t - 0.1x) = \hat{\mathbf{y}}\,6\cos 2z\cos(2\times10^7 t - 0.1x - \pi/2), \\ \widetilde{\mathbf{H}} &= \hat{\mathbf{y}}\,6\cos 2z\,e^{-j0.1x}e^{-j\pi/2} = -\hat{\mathbf{y}}\,j6\cos 2z\,e^{-j0.1x}, \\ \widetilde{\mathbf{E}} &= \frac{1}{j\omega\varepsilon}\nabla\times\widetilde{\mathbf{H}} \\ &= \frac{1}{j\omega\varepsilon}\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & -j6\cos 2z\,e^{-j0.1x} & 0\end{vmatrix} \\ &= \frac{1}{j\omega\varepsilon}\left\{\hat{\mathbf{x}}\left[-\frac{\partial}{\partial z}(-j6\cos 2z\,e^{-j0.1x})\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}(-j6\cos 2z\,e^{-j0.1x})\right]\right\} \\ &= \hat{\mathbf{x}}\left(-\frac{12}{\omega\varepsilon}\sin 2z\,e^{-j0.1x}\right) + \hat{\mathbf{z}}\left(\frac{j0.6}{\omega\varepsilon}\cos 2z\,e^{-j0.1x}\right).\end{aligned}$$

From the given expression for $\mathbf{H}$,

$$\omega = 2\times10^7 \quad \text{(rad/s)},$$

$$\beta = 0.1 \quad \text{(rad/m)}.$$

Hence,

$$u_p = \frac{\omega}{\beta} = 2 \times 10^8 \quad \text{(m/s)},$$

and

$$\varepsilon_r = \left(\frac{c}{u_p}\right)^2 = \left(\frac{3 \times 10^8}{2 \times 10^8}\right)^2 = 2.25.$$

Using the values for ω and ε, we have

$$\widetilde{\mathbf{E}} = (-\hat{\mathbf{x}}30\sin 2z + \hat{\mathbf{z}}\, j1.5\cos 2z) \times 10^3 e^{-j0.1x} \quad \text{(V/m)},$$

$$\mathbf{E} = \left[-\hat{\mathbf{x}}30\sin 2z\cos(2\times 10^7 t - 0.1x) - \hat{\mathbf{z}}\,1.5\cos 2z\sin(2\times 10^7 t - 0.1x)\right] \quad \text{(kV/m)}.$$

(b)

$$\widetilde{\mathbf{D}} = \varepsilon\widetilde{\mathbf{E}} = \varepsilon_r\varepsilon_0\widetilde{\mathbf{E}} = (-\hat{\mathbf{x}}0.6\sin 2z + \hat{\mathbf{z}}\, j0.03\cos 2z) \times 10^{-6} e^{-j0.1x} \quad \text{(C/m}^2\text{)},$$

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t},$$

or

$$\widetilde{\mathbf{J}}_d = j\omega\widetilde{\mathbf{D}} = (-\hat{\mathbf{x}}\, j12\sin 2z - \hat{\mathbf{z}}0.6\cos 2z)e^{-j0.1x},$$

$$\begin{aligned}\mathbf{J}_d &= \mathfrak{Re}[\widetilde{\mathbf{J}}_d e^{j\omega t}] \\ &= \left[\hat{\mathbf{x}}12\sin 2z\sin(2\times 10^7 t - 0.1x) - \hat{\mathbf{z}}0.6\cos 2z\cos(2\times 10^7 t - 0.1x)\right] \quad \text{(A/m}^2\text{)}.\end{aligned}$$

(c) We can find ρ_v from

$$\nabla\cdot\mathbf{D} = \rho_v$$

or from

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho_v}{\partial t}.$$

Applying Maxwell's equation,

$$\rho_v = \nabla\cdot\mathbf{D} = \varepsilon\nabla\cdot\mathbf{E} = \varepsilon_r\varepsilon_0\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z}\right)$$

yields

$$\begin{aligned}\rho_v &= \varepsilon_r\varepsilon_0\left\{\frac{\partial}{\partial x}\left[-30\sin 2z\cos(2\times 10^7 t - 0.1x)\right]\right. \\ &\qquad \left. + \frac{\partial}{\partial z}\left[-1.5\cos 2z\sin(2\times 10^7 t - 0.1x)\right]\right\} \\ &= \varepsilon_r\varepsilon_0\left[-3\sin 2z\sin(2\times 10^7 t - 0.1x) + 3\sin 2z\sin(2\times 10^7 t - 0.1x)\right] = 0.\end{aligned}$$