Chapter 8

Section 8-1: Reflection and Transmission at Normal Incidence

Problem 8.1 A plane wave in air with an electric field amplitude of 10 V/m is incident normally upon the surface of a lossless, nonmagnetic medium with $\varepsilon_r = 25$. Determine:

(a) the reflection and transmission coefficients,

(b) the standing-wave ratio in the air medium, and

(c) the average power densities of the incident, reflected, and transmitted waves.

Solution:

(a)

$$\eta_1 = \eta_0 = 120\pi \quad (\Omega), \qquad \eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{120\pi}{5} = 24\pi \quad (\Omega).$$

From Eqs. (8.8a) and (8.9),

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{24\pi - 120\pi}{24\pi + 120\pi} = \frac{-96}{144} = -0.67,$$

$$\tau = 1 + \Gamma = 1 - 0.67 = 0.33.$$

(b)

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.67}{1 - 0.67} = 5.$$

(c) According to Eqs. (8.19) and (8.20),

$$S_{\text{av}}^{\text{i}} = \frac{|E_0^{\text{i}}|^2}{2\eta_0} = \frac{100}{2 \times 120\pi} = 0.13 \text{ W/m}^2,$$

$$S_{\text{av}}^{\text{r}} = |\Gamma|^2 S_{\text{av}}^{\text{i}} = (0.67)^2 \times 0.13 = 0.06 \text{ W/m}^2,$$

$$S_{\text{av}}^{\text{t}} = |\tau|^2 \frac{|E_0^{\text{i}}|^2}{2\eta_2} = |\tau|^2 \frac{\eta_1}{\eta_2} S_{\text{av}}^{\text{i}} = (0.33)^2 \times \frac{120\pi}{24\pi} \times 0.13 = 0.07 \text{ W/m}^2.$$

Problem 8.2 A plane wave traveling in medium 1 with $\varepsilon_{r1} = 2.25$ is normally incident upon medium 2 with $\varepsilon_{r2} = 4$. Both media are made of nonmagnetic, non-conducting materials. If the electric field of the incident wave is given by

$$\mathbf{E}^{\text{i}} = \hat{\mathbf{y}} 4 \cos(6\pi \times 10^9 t - 30\pi x) \quad (\text{V/m}),$$

(a) obtain time-domain expressions for the electric and magnetic fields in each of the two media, and
(b) determine the average power densities of the incident, reflected and transmitted waves.

Solution:
(a)

$$\mathbf{E}^{\mathrm{i}} = \hat{\mathbf{y}}\, 4\cos(6\pi\times 10^9 t - 30\pi x) \quad \text{(V/m)},$$

$$\eta_1 = \frac{\eta_0}{\sqrt{\varepsilon_{r_1}}} = \frac{\eta_0}{\sqrt{2.25}} = \frac{\eta_0}{1.5} = \frac{377}{1.5} = 251.33\ \Omega,$$

$$\eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_{r_2}}} = \frac{\eta_0}{\sqrt{4}} = \frac{377}{2} = 188.5\ \Omega,$$

$$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{1/2-1/1.5}{1/2+1/1.5} = -0.143,$$

$$\tau = 1+\Gamma = 1-0.143 = 0.857,$$

$$\mathbf{E}^{\mathrm{r}} = \Gamma\mathbf{E}^{\mathrm{i}} = -0.57\,\hat{\mathbf{y}}\cos(6\pi\times 10^9 t + 30\pi x) \quad \text{(V/m)}.$$

Note that the coefficient of x is positive, denoting the fact that $\mathbf{E}^{\mathrm{r}}$ belongs to a wave traveling in $-x$-direction.

$$\mathbf{E}_1 = \mathbf{E}^{\mathrm{i}}+\mathbf{E}^{\mathrm{r}} = \hat{\mathbf{y}}[4\cos(6\pi\times 10^9 t - 30\pi x) - 0.57\cos(6\pi\times 10^9 t + 30\pi x)] \quad \text{(A/m)},$$

$$\mathbf{H}^{\mathrm{i}} = \hat{\mathbf{z}}\frac{4}{\eta_1}\cos(6\pi\times 10^9 t - 30\pi x) = \hat{\mathbf{z}}\,15.91\cos(6\pi\times 10^9 t - 30\pi x) \quad \text{(mA/m)},$$

$$\mathbf{H}^{\mathrm{r}} = \hat{\mathbf{z}}\frac{0.57}{\eta_1}\cos(6\pi\times 10^9 t + 30\pi x) = \hat{\mathbf{z}}\,2.27\cos(6\pi\times 10^9 t + 30\pi x) \quad \text{(mA/m)},$$

$$\begin{aligned}\mathbf{H}_1 &= \mathbf{H}^{\mathrm{i}}+\mathbf{H}^{\mathrm{r}} \\ &= \hat{\mathbf{z}}[15.91\cos(6\pi\times 10^9 t - 30\pi x) + 2.27\cos(6\pi\times 10^9 t + 30\pi x)] \quad \text{(mA/m)}.\end{aligned}$$

Since $k_1 = \omega\sqrt{\mu\varepsilon_1}$ and $k_2 = \omega\sqrt{\mu\varepsilon_2}$,

$$k_2 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\,k_1 = \sqrt{\frac{4}{2.25}}\,30\pi = 40\pi \quad \text{(rad/m)},$$

$$\mathbf{E}_2 = \mathbf{E}^{\mathrm{t}} = \hat{\mathbf{y}}\,4\tau\cos(6\pi\times 10^9 t - 40\pi x) = \hat{\mathbf{y}}\,3.43\cos(6\pi\times 10^9 t - 40\pi x) \quad \text{(V/m)},$$

$$\mathbf{H}_2 = \mathbf{H}^{\mathrm{t}} = \hat{\mathbf{z}}\frac{4\tau}{\eta_2}\cos(6\pi\times 10^9 t - 40\pi x) = \hat{\mathbf{z}}\,18.19\cos(6\pi\times 10^9 t - 40\pi x) \quad \text{(mA/m)}.$$

(b)

$$\mathbf{S}^{\mathrm{i}}_{\mathrm{av}} = \hat{\mathbf{x}}\frac{4^2}{2\eta_1} = \frac{16}{2\times 251.33} = \hat{\mathbf{x}}\,31.8 \quad (\mathrm{mW/m^2}),$$

$$\mathbf{S}^{\mathrm{r}}_{\mathrm{av}} = -|\Gamma|^2 \mathbf{S}^{\mathrm{i}}_{\mathrm{av}} = -\hat{\mathbf{x}}(0.143)^2 \times 0.032 = -\hat{\mathbf{x}}0.65 \quad (\mathrm{mW/m^2}),$$

$$\mathbf{S}^{\mathrm{t}}_{\mathrm{av}} = \frac{|E^{\mathrm{t}}_0|^2}{2\eta_2} = \hat{\mathbf{x}}\tau^2 \frac{(4)^2}{2\eta_2} = \hat{\mathbf{x}}\frac{(0.86)^2 16}{2\times 188.5} = \hat{\mathbf{x}}31.17 \quad (\mathrm{mW/m^2}).$$

Within calculation error, $\mathbf{S}^{\mathrm{i}}_{\mathrm{av}} + \mathbf{S}^{\mathrm{r}}_{\mathrm{av}} = \mathbf{S}^{\mathrm{t}}_{\mathrm{av}}$.

Problem 8.3 A plane wave traveling in a medium with $\varepsilon_{r_1} = 9$ is normally incident upon a second medium with $\varepsilon_{r_2} = 4$. Both media are made of nonmagnetic, non-conducting materials. If the magnetic field of the incident plane wave is given by

$$\mathbf{H}^{\mathrm{i}} = \hat{\mathbf{z}}2\cos(2\pi\times 10^9 t - ky) \quad (\mathrm{A/m}),$$

(a) obtain time domain expressions for the electric and magnetic fields in each of the two media, and

(b) determine the average power densities of the incident, reflected and transmitted waves.

Solution:

(a) In medium 1,

$$u_{\mathrm{p}} = \frac{c}{\sqrt{\varepsilon_{r_1}}} = \frac{3\times 10^8}{\sqrt{9}} = 1\times 10^8 \quad (\mathrm{m/s}),$$

$$k_1 = \frac{\omega}{u_{\mathrm{p}}} = \frac{2\pi\times 10^9}{1\times 10^8} = 20\pi \quad (\mathrm{rad/m}),$$

$$\mathbf{H}^{\mathrm{i}} = \hat{\mathbf{z}}2\cos(2\pi\times 10^9 t - 20\pi y) \quad (\mathrm{A/m}),$$

$$\eta_1 = \frac{\eta_0}{\sqrt{\varepsilon_{r_1}}} = \frac{377}{3} = 125.67\ \Omega,$$

$$\eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_{r_2}}} = \frac{377}{2} = 188.5\ \Omega,$$

$$\mathbf{E}^{\mathrm{i}} = -\hat{\mathbf{x}}2\eta_1\cos(2\pi\times 10^9 t - 20\pi y) = -\hat{\mathbf{x}}251.34\cos(2\pi\times 10^9 t - 20\pi y) \quad (\mathrm{V/m}),$$

$$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{188.5-125.67}{188.5+125.67} = 0.2,$$

$$\tau = 1+\Gamma = 1.2,$$

$$\mathbf{E}^{\mathrm{r}} = -\hat{\mathbf{x}}251.34\times 0.2\cos(2\pi\times 10^9 t + 20\pi y) = -\hat{\mathbf{x}}50.27\cos(2\pi\times 10^9 t + 20\pi y) \quad (\mathrm{V/m}),$$

$$\mathbf{H}^r = -\hat{\mathbf{z}}\frac{50.27}{\eta_1}\cos(2\pi\times 10^9 t + 20\pi y)$$
$$= -\hat{\mathbf{z}}\,0.4\cos(2\pi\times 10^9 t + 20\pi y) \quad \text{(A/m)},$$

$$\mathbf{E}_1 = \mathbf{E}^i + \mathbf{E}^r$$
$$= -\hat{\mathbf{x}}[25.134\cos(2\pi\times 10^9 t - 20\pi y) + 50.27\cos(2\pi\times 10^9 t + 20\pi y)] \quad \text{(V/m)},$$
$$\mathbf{H}_1 = \mathbf{H}^i + \mathbf{H}^r = \hat{\mathbf{z}}[2\cos(2\pi\times 10^9 t - 20\pi y) - 0.4\cos(2\pi\times 10^9 t + 20\pi y)] \quad \text{(A/m)}.$$

In medium 2,

$$k_2 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\,k_1 = \sqrt{\frac{4}{9}}\times 20\pi = \frac{40\pi}{3} \quad \text{(rad/m)},$$
$$\mathbf{E}_2 = \mathbf{E}^t = -\hat{\mathbf{x}}\,251.34\,\tau\cos\left(2\pi\times 10^9 t - \frac{40\pi y}{3}\right)$$
$$= -\hat{\mathbf{x}}\,301.61\cos\left(2\pi\times 10^9 t - \frac{40\pi y}{3}\right) \quad \text{(V/m)},$$
$$\mathbf{H}_2 = \mathbf{H}^t = \hat{\mathbf{z}}\frac{301.61}{\eta_2}\cos\left(2\pi\times 10^9 t - \frac{40\pi y}{3}\right)$$
$$= \hat{\mathbf{z}}\,1.6\cos\left(2\pi\times 10^9 t - \frac{40\pi y}{3}\right) \quad \text{(A/m)}.$$

(b)

$$\mathbf{S}^i_{av} = \hat{\mathbf{y}}\frac{|E_0|^2}{2\eta_1} = \hat{\mathbf{y}}\frac{(251.34)^2}{2\times 125.67} = \hat{\mathbf{y}}\,251.34 \quad \text{(W/m}^2\text{)},$$
$$\mathbf{S}^r_{av} = -\hat{\mathbf{y}}\,|\Gamma|^2(251.34) = \hat{\mathbf{y}}\,10.05 \quad \text{(W/m}^2\text{)},$$
$$\mathbf{S}^t_{av} = \hat{\mathbf{y}}(251.34 - 10.05) = \hat{\mathbf{y}}\,241.29 \quad \text{(W/m}^2\text{)}.$$

Problem 8.4 A 200-MHz left-hand circularly polarized plane wave with an electric field modulus of 10 V/m is normally incident in air upon a dielectric medium with $\varepsilon_r = 4$ and occupying the region defined by $z \geq 0$.

(a) Write an expression for the electric field phasor of the incident wave, given that the field is a positive maximum at $z = 0$ and $t = 0$.

(b) Calculate the reflection and transmission coefficients.

(c) Write expressions for the electric field phasors of the reflected wave, the transmitted wave, and the total field in the region $z \leq 0$.

(d) Determine the percentages of the incident average power reflected by the boundary and transmitted into the second medium.

Solution:

(a)

$$k_1 = \frac{\omega}{c} = \frac{2\pi \times 2 \times 10^8}{3 \times 10^8} = \frac{4\pi}{3} \text{ rad/m},$$
$$k_2 = \frac{\omega}{u_{p_2}} = \frac{\omega}{c}\sqrt{\varepsilon_{r_2}} = \frac{4\pi}{3}\sqrt{4} = \frac{8\pi}{3} \text{ rad/m}.$$

LHC wave:

$$\widetilde{\mathbf{E}}^i = a_0(\hat{\mathbf{x}} + \hat{\mathbf{y}}e^{j\pi/2})e^{-jkz} = a_0(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-jkz},$$
$$\mathbf{E}^i(z,t) = \hat{\mathbf{x}}a_0\cos(\omega t - kz) - \hat{\mathbf{y}}a_0\sin(\omega t - kz),$$
$$|\mathbf{E}^i| = [a_0^2\cos^2(\omega t - kz) + a_0^2\sin^2(\omega t - kz)]^{1/2} = a_0 = 10 \quad \text{(V/m)}.$$

Hence,

$$\widetilde{\mathbf{E}}^i = 10(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-j4\pi z/3} \quad \text{(V/m)}.$$

(b)

$$\eta_1 = \eta_0 = 120\pi \quad (\Omega), \qquad \eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{\eta_0}{2} = 60\pi \quad (\Omega).$$

Equations (8.8a) and (8.9) give

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{60\pi - 120\pi}{60\pi + 120\pi} = \frac{-60}{180} = -\frac{1}{3}, \qquad \tau = 1 + \Gamma = \frac{2}{3}.$$

(c)

$$\widetilde{\mathbf{E}}^r = 10\Gamma(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{jk_1z} = -\frac{10}{3}(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{j4\pi z/3} \quad \text{(V/m)},$$
$$\widetilde{\mathbf{E}}^t = 10\tau(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-jk_2z} = \frac{20}{3}(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-j8\pi z/3} \quad \text{(V/m)},$$
$$\widetilde{\mathbf{E}}_1 = \widetilde{\mathbf{E}}^i + \widetilde{\mathbf{E}}^r = 10(\hat{\mathbf{x}} + j\hat{\mathbf{y}})\left[e^{-j4\pi z/3} - \frac{1}{3}e^{j4\pi z/3}\right] \quad \text{(V/m)}.$$

(d)

$$\% \text{ of reflected power} = 100 \times |\Gamma|^2 = \frac{100}{9} = 11.11\%,$$
$$\% \text{ of transmitted power} = 100 \times |\tau|^2\frac{\eta_1}{\eta_2} = 100 \times \left(\frac{2}{3}\right)^2 \times \frac{120\pi}{60\pi} = 88.89\%.$$

Problem 8.5 Repeat Problem 8.4 after replacing the dielectric medium with a poor conductor characterized by $\varepsilon_r = 2.25$, $\mu_r = 1$, and $\sigma = 10^{-4}$ S/m.

Solution:
(a) Medium 1:

$$\eta_1 = \eta_0 = 120\pi \quad (\Omega), \qquad k_1 = \frac{\omega}{c} = \frac{2\pi \times 2 \times 10^8}{3 \times 10^8} = \frac{4\pi}{3} \quad \text{(rad/m)}.$$

Medium 2:

$$\frac{\sigma_2}{\omega\varepsilon_2} = \frac{10^{-4} \times 36\pi}{2\pi \times 2 \times 10^8 \times 2.25 \times 10^{-9}} = 4 \times 10^{-3}.$$

Hence, medium 2 is a low-loss dielectric. From Table 7-1,

$$\begin{aligned}
\alpha_2 &= \frac{\sigma_2}{2}\sqrt{\frac{\mu_2}{\varepsilon_2}} \\
&= \frac{\sigma_2}{2}\frac{120\pi}{\sqrt{\varepsilon_{r_2}}} = \frac{\sigma_2}{2} \times \frac{120\pi}{\sqrt{2.25}} = \frac{10^{-4}}{2} \times \frac{120\pi}{1.5} = 1.26 \times 10^{-2} \quad \text{(NP/m)}, \\
\beta_2 &= \omega\sqrt{\mu_2\varepsilon_2} = \frac{\omega\sqrt{\varepsilon_{r_2}}}{c} = 2\pi \quad \text{(rad/m)}, \\
\eta_2 &= \sqrt{\frac{\mu_2}{\varepsilon_2}}\left(1 + \frac{j\sigma_2}{2\omega\varepsilon_2}\right) = \frac{120\pi}{\sqrt{\varepsilon_{r_2}}}\left(1 + j2 \times 10^{-3}\right) \simeq \frac{120\pi}{1.5} = 80\pi \quad (\Omega).
\end{aligned}$$

LHC wave:

$$\begin{aligned}
\widetilde{\mathbf{E}}^{\mathrm{i}} &= a_0(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-jk_1z}, \\
|\widetilde{\mathbf{E}}^{\mathrm{i}}| &= a_0 = 10 \quad \text{(V/m)}, \\
\widetilde{\mathbf{E}}^{\mathrm{i}} &= 10(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-j4\pi z/3} \quad \text{(V/m)}.
\end{aligned}$$

(b) According to Eqs. (8.8a) and (8.9),

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{80\pi - 120\pi}{80\pi + 120\pi} = -0.2, \qquad \tau = 1 + \Gamma = 1 - 0.2 = 0.8.$$

(c)

$$\begin{aligned}
\widetilde{\mathbf{E}}^{\mathrm{r}} &= 10\Gamma(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{jk_1z} = -2(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{j4\pi z/3} \quad \text{(V/m)}, \\
\widetilde{\mathbf{E}}^{\mathrm{t}} &= 10\tau(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-\alpha_2 z}e^{-j\beta_2 z} = 8(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-1.26\times10^{-2}z}e^{-j2\pi z} \quad \text{(V/m)}, \\
\widetilde{\mathbf{E}}_1 &= \widetilde{\mathbf{E}}^{\mathrm{i}} + \widetilde{\mathbf{E}}^{\mathrm{r}} = 10(\hat{\mathbf{x}} + j\hat{\mathbf{y}})[e^{-j4\pi z/3} - 0.2e^{j4\pi z/3}] \quad \text{(V/m)}.
\end{aligned}$$

(d)

$$\% \text{ of reflected power} = 100|\Gamma|^2 = 100(0.2)^2 = 4\%,$$
$$\% \text{ of transmitted power} = 100|\tau|^2\frac{\eta_1}{\eta_2} = 100(0.8)^2 \times \frac{120\pi}{80\pi} = 96\%.$$

Problem 8.6 A 50-MHz plane wave with electric field amplitude of 30 V/m is normally incident in air onto a semi-infinite, perfect dielectric medium with $\varepsilon_r = 36$. Determine (a) Γ, (b) the average power densities of the incident and reflected waves, and (c) the distance in the air medium from the boundary to the nearest minimum of the electric field intensity, $|\mathbf{E}|$.

Solution:
(a)

$$\eta_1 = \eta_0 = 120\pi \quad (\Omega), \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \frac{120\pi}{\sqrt{\varepsilon_{r_2}}} = \frac{120\pi}{6} = 20\pi \quad (\Omega),$$
$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{20\pi - 120\pi}{20\pi + 120\pi} = -0.71.$$

Hence, $|\Gamma| = 0.71$ and $\theta_\eta = 180°$.
(b)

$$S^{\mathrm{i}}_{\mathrm{av}} = \frac{|E^{\mathrm{i}}_0|^2}{2\eta_1} = \frac{(30)^2}{2 \times 120\pi} = 1.19 \quad (\mathrm{W/m^2}),$$
$$S^{\mathrm{r}}_{\mathrm{av}} = |\Gamma|^2 S^{\mathrm{i}}_{\mathrm{av}} = (0.71)^2 \times 1.19 = 0.60 \quad (\mathrm{W/m^2}).$$

(c) In medium 1 (air),

$$\lambda_1 = \frac{c}{f} = \frac{3 \times 10^8}{5 \times 10^7} = 6 \text{ m}.$$

From Eqs. (8.16) and (8.17),

$$l_{\max} = \frac{\theta_r \lambda_1}{4\pi} = \frac{\pi \times 6}{4\pi} = 1.5 \text{ m},$$
$$l_{\min} = l_{\max} - \frac{\lambda_1}{4} = 1.5 - 1.5 = 0 \text{ m (at the boundary)}.$$

Problem 8.7 What is the maximum amplitude of the total electric field in the air medium of Problem 8.6, and at what nearest distance from the boundary does it occur?

Solution: From Problem 8.6, $\Gamma = -0.71$ and $\lambda = 6$ m.

$$|\widetilde{\mathbf{E}}_1|_{\max} = (1+|\Gamma|)E_0^i = (1+0.71)\times 30 = 51.3 \text{ V/m},$$
$$l_{\max} = \frac{\theta_r\lambda_1}{4\pi} = \frac{\pi\times 6}{4\pi} = 1.5 \text{ m}.$$

Problem 8.8 Repeat Problem 8.6 after replacing the dielectric medium with a conductor with $\varepsilon_r = 1$, $\mu_r = 1$, and $\sigma = 2.78\times 10^{-3}$ S/m.

Solution:

(a) Medium 1:

$$\eta_1 = \eta_0 = 120\pi = 377 \quad (\Omega), \qquad \lambda_1 = \frac{c}{f} = \frac{3\times 10^8}{5\times 10^7} = 6 \text{ m},$$

Medium 2:

$$\frac{\sigma_2}{\omega\varepsilon_2} = \frac{2.78\times 10^{-3}\times 36\pi}{2\pi\times 5\times 10^7\times 10^{-9}} = 1.$$

Hence, Medium 2 is a quasi-conductor. From Eq. (7.70),

$$\begin{aligned}\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}\left(1-j\frac{\varepsilon_2''}{\varepsilon_2'}\right)^{-1/2} &= 120\pi\left(1-j\frac{\sigma_2}{\omega\varepsilon_2}\right)^{-1/2}\\ &= 120\pi(1-j1)^{-1/2}\\ &= 120\pi(\sqrt{2})^{-1/2}e^{j22.5^\circ} = (292.88+j121.31) \quad (\Omega).\end{aligned}$$

$$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{(292.88+j121.31)-377}{(292.88+j121.31)+377} = -0.09+j0.12 = 0.22\angle 114.5^\circ.$$

(b)

$$S_{\text{av}}^{\text{i}} = \frac{|E_0^{\text{i}}|^2}{2\eta_1} = \frac{30^2}{2\times 120\pi} = 1.19 \quad (\text{W/m}^2),$$
$$|S_{\text{av}}^{\text{r}}| = |\Gamma|^2 S_{\text{av}}^{\text{i}} = (0.22)^2(1.19) = 0.06 \quad (\text{W/m}^2).$$

(c) In medium 1 (air),

$$\lambda_1 = \frac{c}{f} = \frac{3\times 10^8}{5\times 10^7} = 6 \text{ m}.$$

For $\theta_r = 114.5° = 2$ rad, Eqs. (8.16) and (8.17) give

$$l_{\max} = \frac{\theta_r\lambda_1}{4\pi} + \frac{(0)\lambda_1}{2} = \frac{2(6)}{4} + 0 = 3 \text{ m},$$
$$l_{\min} = l_{\max} - \frac{\lambda_1}{4} = 3 - \frac{6}{4} = 3 - 1.5 = 1.5 \text{ m}.$$

Problem 8.9 The three regions shown in Fig. 8-32 (P8.9) contain perfect dielectrics. For a wave in medium 1 incident normally upon the boundary at $z = -d$, what combination of ε_{r_2} and d produce no reflection? Express your answers in terms of ε_{r_1}, ε_{r_3} and the oscillation frequency of the wave, f.

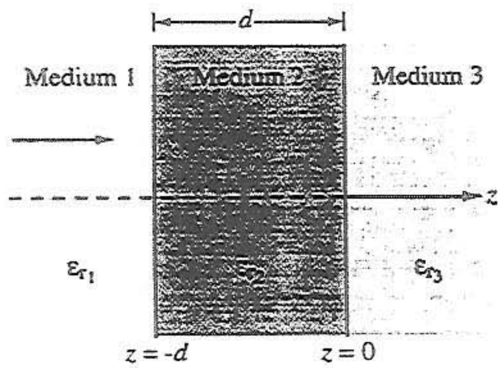


Figure P8.9: Three dielectric regions.

Solution: By analogy with the transmission-line case, there will be no reflection at $z = -d$ if medium 2 acts as a quarter-wave transformer, which requires that

$$d = \frac{\lambda_2}{4}$$

and

$$\eta_2 = \sqrt{\eta_1\eta_3}.$$

The second condition may be rewritten as

$$\frac{\eta_0}{\sqrt{\varepsilon_{r_2}}} = \left[\frac{\eta_0}{\sqrt{\varepsilon_{r_1}}}\frac{\eta_0}{\sqrt{\varepsilon_{r_3}}}\right]^{1/2}, \qquad \text{or} \quad \varepsilon_{r_2} = \sqrt{\varepsilon_{r_1}\varepsilon_{r_3}},$$

$$\lambda_2 = \frac{\lambda_0}{\sqrt{\varepsilon_{r_2}}} = \frac{c}{f\sqrt{\varepsilon_{r_2}}} = \frac{c}{f(\varepsilon_{r_1}\varepsilon_{r_3})^{1/4}},$$

and

$$d = \frac{c}{4f(\varepsilon_{r_1}\varepsilon_{r_3})^{1/4}}.$$

Problem 8.10 For the configuration shown in Fig. 8-32 (P8.9), use transmission-line equations (or the Smith chart) to calculate the input impedance at $z = -d$ for $\varepsilon_{r_1} = 1$, $\varepsilon_{r_2} = 9$, $\varepsilon_{r_3} = 4$, $d = 1.2$ m, and $f = 50$ MHz. Also determine the fraction of the incident average power density reflected by the structure. Assume all media are lossless and nonmagnetic.

Solution: In medium 2,

$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_{r_2}}} = \frac{c}{f\sqrt{\varepsilon_{r_2}}} = \frac{3\times 10^8}{5\times 10^7 \times 3} = 2 \text{ m}.$$

Hence,

$$\beta_2 = \frac{2\pi}{\lambda_2} = \pi \text{ rad/m}, \qquad \beta_2 d = 1.2\pi \text{ rad}.$$

At $z = -d$, the input impedance of a transmission line with load impedance Z_L is given by Eq. (2.63) as

$$Z_{in}(-d) = Z_0\left(\frac{Z_L + jZ_0 \tan\beta_2 d}{Z_0 + jZ_L \tan\beta_2 d}\right).$$

In the present case, $Z_0 = \eta_2 = \eta_0/\sqrt{\varepsilon_{r_2}} = \eta_0/3$ and $Z_L = \eta_3 = \eta_0/\sqrt{\varepsilon_{r_3}} = \eta_0/2$, where $\eta_0 = 120\pi$ (Ω). Hence,

$$Z_{in}(-d) = \eta_2\left(\frac{\eta_3 + j\eta_2 \tan\beta_2 d}{\eta_2 + j\eta_3 \tan\beta_2 d}\right) = \frac{\eta_0}{3}\left(\frac{\frac{1}{2} + j\left(\frac{1}{3}\right)\tan 1.2\pi}{\frac{1}{3} + j\left(\frac{1}{2}\right)\tan 1.2\pi}\right) = \eta_0(0.28 + j0.11).$$

At $z = -d$,

$$\Gamma = \frac{Z_{in} - Z_1}{Z_{in} + Z_1} = \frac{\eta_0(0.28 + j0.11) - \eta_0}{\eta_0(0.28 + j0.11) + \eta_0} = 0.57e^{-j21.62}.$$

Fraction of incident power reflected by the structure is $|\Gamma|^2 = |0.57|^2 = 0.33$.

Problem 8.11 Repeat Problem 8.10 after interchanging ε_{r_1} and ε_{r_3}.

Solution: In medium 2,

$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_{r_2}}} = \frac{c}{f\sqrt{\varepsilon_{r_2}}} = \frac{3\times 10^8}{5\times 10^7 \times 3} = 2 \text{ m}.$$

Hence,

$$\beta_2 = \frac{2\pi}{\lambda_2} = \pi \text{ rad/m}, \qquad \beta_2 d = 1.2\pi \text{ rad}.$$

At $z = -d$, the input impedance of a transmission line with impedance Z_L is given as Eq. (2.63),

$$Z_{in}(-d) = Z_0\left(\frac{Z_L + jZ_0 \tan\beta d}{Z_0 + jZ_L \tan\beta 2d}\right).$$

In the present case, $Z_0 = \eta_2 = \eta_0/\sqrt{\varepsilon_{r_2}} = \eta_0/3$, $Z_L = \eta_3 = \eta_0/\sqrt{\varepsilon_{r_1}} = \eta_0$, where $\eta_0 = 120\pi\ (\Omega)$. Hence,

$$\begin{aligned} Z_{in}(-d) &= \eta_2\left(\frac{\eta_3 + j\eta_2 \tan 1.2\pi}{\eta_2 + j\eta_3 \tan 1.2\pi}\right) \\ &= \frac{\eta_0}{3}\left(\frac{1+(j/3)\tan 1.2\pi}{(1/3)+j\tan 1.2\pi}\right) \\ &= \eta_0\left(\frac{1+(j/3)\tan 1.2\pi}{1+j3\tan 1.2\pi}\right) = (0.266 - j0.337)\eta_0 = 0.43\eta_0\angle{-51.7^\circ}. \end{aligned}$$

At $z = -d$,

$$\Gamma = \frac{Z_{in} - Z_1}{Z_{in} + Z_1} = \frac{0.43\angle{-51.7^\circ} - \frac{1}{2}}{0.43\angle{-51.7^\circ} + \frac{1}{2}} = 0.49\angle{-101.1^\circ}.$$

Fraction of incident power reflected by structure is $|\Gamma|^2 = 0.24$.

Problem 8.12 Orange light of wavelength 0.61 μm in air enters a block of glass with $\varepsilon_r = 2.25$. What color would it appear to a sensor embedded in the glass? The wavelength ranges of colors are violet (0.39 to 0.45 μm), blue (0.45 to 0.49 μm), green (0.49 to 0.58 μm), yellow (0.58 to 0.60 μm), orange (0.60 to 0.62 μm), and red (0.62 to 0.78 μm).

Solution: In the glass,

$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_r}} = \frac{0.61}{\sqrt{2.25}} = 0.407\ \mu\text{m}.$$

The light would appear violet.

Problem 8.13 A plane wave of unknown frequency is normally incident in air upon the surface of a perfect conductor. Using an electric-field meter, it was determined that the total electric field in the air medium is always zero when measured at a distance of 2.5 m from the conductor surface. Moreover, no such nulls were observed at distances closer to the conductor. What is the frequency of the incident wave?

Solution: The electric field of the standing wave is zero at the conductor surface, and the standing wave pattern repeats itself every $\lambda/2$. Hence,

$$\frac{\lambda}{2} = 2.5 \text{ m}, \qquad \text{or } \lambda = 5 \text{ m},$$

in which case

$$f = \frac{c}{\lambda} = \frac{3\times10^8}{5} = 6\times10^7 = 60 \text{ MHz}.$$

Problem 8.14 Consider a thin film of soap in air under illumination by yellow light with $\lambda = 0.6$ μm in vacuum. If the film is treated as a planar dielectric slab with $\varepsilon_r = 1.72$, surrounded on both sides by air, what film thickness would produce strong reflection of the yellow light at normal incidence?

Solution: The transmission line analogue of the soap-bubble wave problem is shown in Fig. P8.14(b) where the load Z_L is equal to η_0, the impedance of the air medium on the other side of the bubble. That is,

$$\eta_0 = 377\ \Omega, \qquad \eta_1 = \frac{377}{\sqrt{1.72}} = 287.5\ \Omega.$$

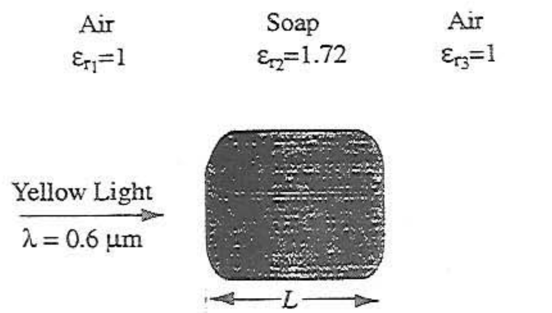
The normalized load impedance is

$$z_L = \frac{\eta_0}{\eta_1} = 1.31.$$

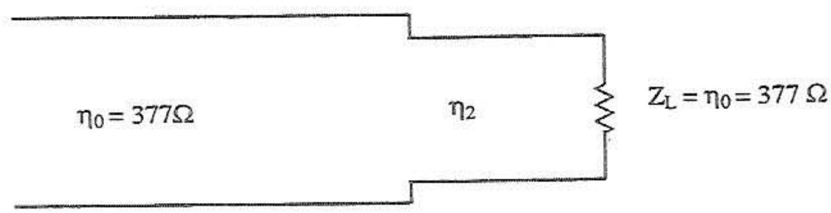
For the reflection by the soap bubble to be the largest, Z_{in} needs to be the most different from η_0. This happens when z_L is transformed through a length $\lambda/4$. Hence,

$$L = \frac{\lambda}{4} = \frac{\lambda_0}{4\sqrt{\varepsilon_r}} = \frac{0.6\ \mu\text{m}}{4\sqrt{1.72}} = 0.115\ \mu\text{m},$$

where λ is the wavelength of the soap bubble material. Strong reflections will also occur if the thickness is greater than L by integer multiples of $n\lambda/2 = (0.23n)$ μm.



(a) Yellow light incident on soap bubble.



(b) Transmission-line equivalent circuit

Figure P8.14: Diagrams for Problem 8.14.

Hence, in general

$$L = (0.115 + 0.23\,n)\ \mu\text{m}, \qquad n = 0, 1, 2, \ldots .$$

According to Section 2-7.5, transforming a load $Z_L = 377\ \Omega$ through a $\lambda/4$ section of $Z_0 = 287.5\ \Omega$ ends up presenting an input impedance of

$$Z_{in} = \frac{Z_0^2}{Z_L} = \frac{(287.5)^2}{377} = 219.25\ \Omega.$$

This Z_{in} is at the input side of the soap bubble. The reflection coefficient at that interface is

$$\Gamma = \frac{Z_{in} - \eta_0}{Z_{in} + \eta_0} = \frac{219.25 - 377}{219.25 + 377} = -0.27.$$

Any other thickness would produce a reflection coeffficient with a smaller magnitude.

Problem 8.15 A 5-MHz plane wave with electric field amplitude of 20 (V/m) is normally incident in air onto the plane surface of a semi-infinite conducting material with $\varepsilon_r = 4$, $\mu_r = 1$, and $\sigma = 100$ (S/m). Determine the average power dissipated (lost) per unit cross-sectional area in a 2-mm penetration of the conducting medium.

Solution: For convenience, let us choose $\mathbf{E}^i$ to be along $\hat{\mathbf{x}}$ and the incident direction to be $+\hat{\mathbf{z}}$. With

$$k_1 = \frac{\omega}{c} = \frac{2\pi \times 5 \times 10^6}{3 \times 10^8} = \frac{\pi}{30} \quad \text{(rad/m)},$$

we have

$$\mathbf{E}^i = \hat{\mathbf{x}}\, 20 \cos\left(\pi \times 10^7 t - \frac{\pi}{30} z\right) \quad \text{(V/m)},$$
$$\eta_1 = \eta_0 = 377\ \Omega.$$

From Table 7-1,

$$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon_r\varepsilon_0} = \frac{100 \times 36\pi}{\pi \times 10^7 \times 4 \times 10^{-9}} = 9 \times 10^4,$$

which makes the material a good conductor, for which

$$\alpha_2 = \sqrt{\pi f \mu \sigma} = \sqrt{\pi \times 5 \times 10^6 \times 4\pi \times 10^{-7} \times 100} = 44.43 \quad \text{(Np/m)},$$
$$\beta_2 = 44.43 \quad \text{(rad/m)},$$
$$\eta_{c_2} = (1 + j)\frac{\alpha_2}{\sigma} = (1 + j)\frac{44.43}{100} = 0.44(1 + j)\ \Omega.$$

According to the expression for $\mathbf{S}_{\text{av}_2}$ given in the answer to Exercise 8.3,

$$\mathbf{S}_{\text{av}_2} = \hat{\mathbf{z}}\, |\tau|^2 \frac{|E_0^i|^2}{2} e^{-2\alpha_2 z}\, \Re e\left(\frac{1}{\eta_{c_2}^*}\right).$$

The power lost is equal to the difference between $\mathbf{S}_{\text{av}_2}$ at $z = 0$ and $\mathbf{S}_{\text{av}_2}$ at $z = 2$ mm. Thus,

$$\begin{aligned} P' &= \text{power lost per unit cross-sectional area} \\ &= S_{\text{av}_2}(0) - S_{\text{av}_2}(z = 2\ \text{mm}) \\ &= |\tau|^2 \frac{|E_0^i|^2}{2} \Re e\left(\frac{1}{\eta_{c_2}^*}\right)[1 - e^{-2\alpha_2 z_1}] \end{aligned}$$

where $z_1 = 2$ mm.

$$\begin{aligned} \tau &= 1 + \Gamma \\ &= 1 + \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = 1 + \frac{0.44(1 + j) - 377}{0.44(1 + j) + 377} \approx 0.0023(1 + j) = 3.3 \times 10^{-3} e^{j45^\circ}. \end{aligned}$$

$$\Re e\left(\frac{1}{\eta^*_{c_2}}\right) = \Re e\left(\frac{1}{0.44(1+j)^*}\right)$$

$$= \Re e\left(\frac{1}{0.44(1-j)}\right) = \Re e\left(\frac{1+j}{0.44\times 2}\right) = \frac{1}{0.88} = 1.14,$$

$$P' = (3.3\times 10^{-3})^2\frac{20^2}{2}\times 1.14[1-e^{-2\times 44.43\times 2\times 10^{-3}}] = 4.04\times 10^{-4} \quad (\text{W/m}^2).$$

Problem 8.16 A 0.5-MHz antenna carried by an airplane flying over the ocean surface generates a wave that approaches the water surface in the form of a normally incident plane wave with an electric-field amplitude of 3,000 (V/m). Sea water is characterized by $\varepsilon_r = 72$, $\mu_r = 1$, and $\sigma = 4$ (S/m). The plane is trying to communicate a message to a submarine submerged at a depth d below the water surface. If the submarine's receiver requires a minimum signal amplitude of 0.1 (μV/m), what is the maximum depth d to which successful communication is still possible?

Solution: For sea water at 0.5 MHz,

$$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon} = \frac{4\times 36\pi}{2\pi\times 0.5\times 10^6\times 72\times 10^{-9}} = 2000.$$

Hence, sea water is a good conductor, in which case we use the following expressions from Table 7-1:

$$\alpha_2 = \sqrt{\pi f\mu\sigma} = \sqrt{\pi\times 0.5\times 10^6\times 4\pi\times 10^{-7}\times 4} = 2.81 \quad (\text{Np/m}),$$

$$\beta_2 = 2.81 \quad (\text{rad/m}),$$

$$\eta_{c_2} = (1+j)\frac{\alpha_2}{\sigma} = (1+j)\frac{2.81}{4} = 0.7(1+j)\ \Omega,$$

$$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{0.7(1+j)-377}{0.7(1+j)+377} = (-0.9963+j3.7\times 10^{-3}),$$

$$\tau = 1+\Gamma = 5.24\times 10^{-3}e^{j44.89^\circ},$$

$$|E^{\mathrm{t}}| = |\tau E^{\mathrm{i}}_0 e^{-\alpha_2 d}|.$$

We need to find the depth z at which $|E^{\mathrm{t}}| = 0.1\ \mu\text{V/m} = 10^{-7}$ V/m.

$$10^{-7} = 5.24\times 10^{-3}\times 3\times 10^3 e^{-2.81d},$$

$$e^{-2.81d} = 6.36\times 10^{-9},$$

$$-2.81d = \ln(6.36\times 10^{-9}) = -18.87,$$

or

$$d = 6.7 \quad (\text{m}).$$

Sections 8-2 and 8-3: Snell's Laws and Fiber Optics

Problem 8.17 A light ray is incident on a prism at an angle θ as shown in Fig. 8-33 (P8.17). The ray is refracted at the first surface and again at the second surface. In terms of the apex angle ϕ of the prism and its index of refraction n, determine the smallest value of θ for which the ray will emerge from the other side. Find this minimum θ for $n = 1.5$ and $\phi = 60°$.

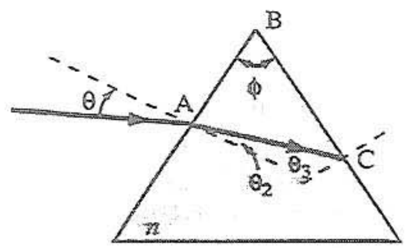


Figure P8.17: Prism of Problem 8.17.

Solution: For the beam to emerge at the second boundary, it is necessary that

$$\theta_3 < \theta_c,$$

where $\sin\theta_c = 1/n$. From the geometry of triangle ABC,

$$180° = \phi + (90° - \theta_2) + (90° - \theta_3),$$

or $\theta_2 = \phi - \theta_3$. At the first boundary, $\sin\theta = n\sin\theta_2$. Hence,

$$\sin\theta_{\rm min} = n\sin(\phi - \theta_3) = n\sin\left(\phi - \sin^{-1}\left(\frac{1}{n}\right)\right),$$

or

$$\theta_{\rm min} = \sin^{-1}\left[n\sin\left(\phi - \sin^{-1}\left(\frac{1}{n}\right)\right)\right].$$

For $n = 1.5$ and $\phi = 60°$,

$$\theta_{\rm min} = \sin^{-1}\left[1.5\sin(60° - \sin^{-1}\left(\frac{1}{1.5}\right)\right] = 27.92°.$$

Problem 8.18 For some types of glass, the index of refraction varies with wavelength. A prism made of a material with

$$n = 1.71 - \frac{4}{30}\lambda_0, \quad (\lambda_0 \text{ in } \mu\text{m}),$$

where λ_0 is the wavelength in vacuum, was used to disperse white light as shown in Fig. 8-34 (P8.18). The white light is incident at an angle of 50°, the wavelength λ_0 of red light is 0.7 μm and that of violet light is 0.4 μm. Determine the angular dispersion in degrees.

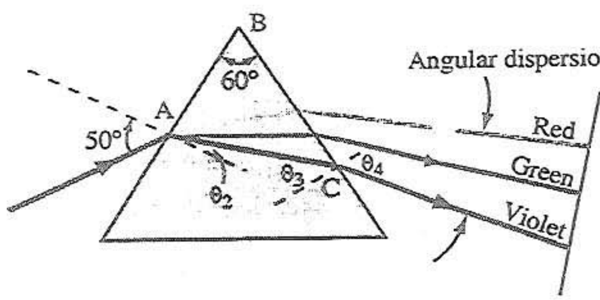


Figure P8.18: Prism of Problem 8.18.

Solution: For violet,

$$n_v = 1.71 - \frac{4}{30} \times 0.4 = 1.66, \qquad \sin\theta_2 = \frac{\sin\theta}{n_v} = \frac{\sin 50°}{1.66},$$

or

$$\theta_2 = 27.48°.$$

From the geometry of triangle ABC,

$$180° = 60° + (90° - \theta_2) + (90° - \theta_3),$$

or

$$\theta_3 = 60° - \theta_2 = 60 - 27.48° = 32.52°,$$

and

$$\sin\theta_4 = n_v \sin\theta_3 = 1.66 \sin 32.52° = 0.89,$$

or

$$\theta_4 = 63.18°.$$

For red,

$$n_r = 1.71 - \frac{4}{30} \times 0.7 = 1.62,$$
$$\theta_2 = \sin^{-1}\left[\frac{\sin 50°}{1.62}\right] = 28.22°,$$
$$\theta_3 = 60° - 28.22° = 31.78°,$$
$$\theta_4 = \sin^{-1}[1.62 \sin 31.78°] = 58.56°.$$

Hence, angular dispersion $= 63.18° - 58.56° = 4.62°$.

Problem 8.19 The two prisms in Fig. 8-35 (P8.19) are made of glass with $n = 1.52$. What fraction of the power density carried by the ray incident upon the top prism emerges from bottom prism? Neglect multiple internal reflections.

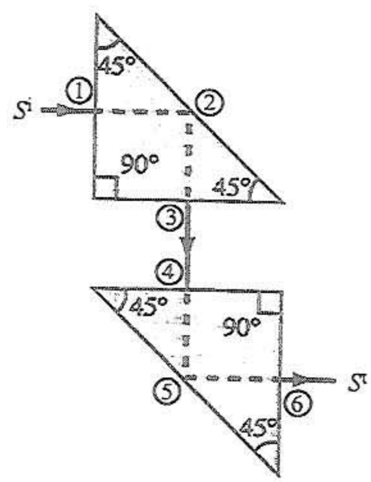


Figure P8.19: Periscope problem.

Solution: Using $\eta = \eta_0/n$, at interfaces 1 and 4,

$$\Gamma_a = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1 - 1.52}{1 + 1.52} = -0.21.$$

At interfaces 3 and 6,

$$\Gamma_b = -\Gamma_a = 0.21.$$

At interfaces 2 and 5,

$$\theta_c = \sin^{-1}\left(\frac{1}{n}\right) = \sin^{-1}\left(\frac{1}{1.52}\right) = 41.14°.$$

Hence, total internal reflection takes place at those interfaces. At interfaces 1, 3, 4 and 6, the ratio of power density transmitted to that incident is $(1-\Gamma^2)$. Hence,

$$\frac{S^t}{S^i} = (1-\Gamma^2)^4 = (1-(0.21)^2)^4 = 0.835.$$

Problem 8.20 A light ray incident at 45° passes through two dielectric materials with the indices of refraction and thicknesses given in Fig. 8-36 (P8.20). If the ray strikes the surface of the first dielectric at a height of 2 cm, at what height will it strike the screen?

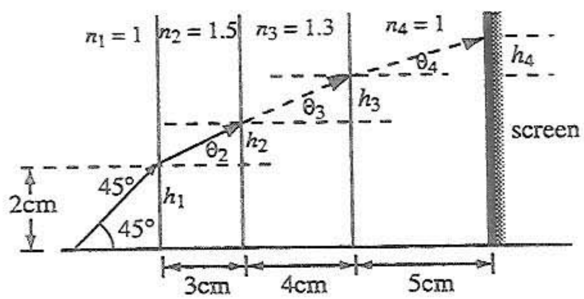


Figure P8.20: Light incident on a screen through a multi-layered dielectric (Problem 8.20).

Solution:

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1 = \frac{1}{1.5}\sin 45° = 0.47.$$

Hence,

$$\theta_2 = 28.13°,$$
$$h_2 = 3 \text{ cm} \times \tan\theta_2 = 3 \text{ cm} \times 0.53 = 1.6 \text{ cm},$$
$$\sin\theta_3 = \frac{n_2}{n_3}\sin\theta_2 = \frac{1.5}{1.3}\sin 28.13° = 0.54.$$

Hence,

$$\theta_3 = 32.96^\circ,$$
$$h_3 = 4 \text{ cm} \times \tan 32.96^\circ = 2.6 \text{ cm},$$
$$\sin\theta_4 = \frac{n_3}{n_4}\sin\theta_3 = 0.707.$$

Hence,

$$\theta_4 = 45^\circ,$$
$$h_4 = 5 \text{ cm} \times \tan 45^\circ = 5 \text{ cm}.$$

Total height $= h_1 + h_2 + h_3 + h_4 = (2 + 1.6 + 2.6 + 5) = 11.2$ cm.

Problem 8.21 Figure P8.21 depicts a beaker containing a block of glass on the bottom and water over it. The glass block contains a small air bubble at an unknown depth below the water surface. When viewed from above at an angle of 60°, the air bubble appears at a depth of 6.81 cm. What is the true depth of the air bubble?

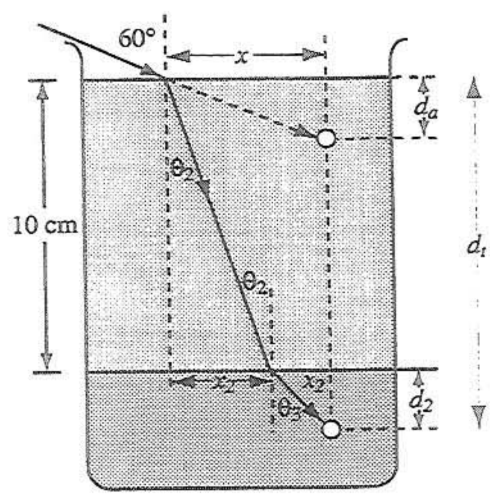


Figure P8.21: Apparent position of the air bubble in Problem 8.21.

Solution: Let

$$d_a = 6.81 \text{ cm} = \text{apparent depth},$$
$$d_t = \text{true depth}.$$

$$\theta_2 = \sin^{-1}\left[\frac{n_1}{n_2}\sin\theta_i\right] = \sin^{-1}\left[\frac{1}{1.33}\sin 60^\circ\right] = 40.6^\circ,$$
$$\theta_3 = \sin^{-1}\left[\frac{n_1}{n_3}\sin\theta_i\right] = \sin^{-1}\left[\frac{1}{1.6}\sin 60^\circ\right] = 32.77^\circ,$$
$$x_1 = (10 \text{ cm}) \times \tan 40.6^\circ = 8.58 \text{ cm},$$
$$x = d_a \cot 30^\circ = 6.81 \cot 30^\circ = 11.8 \text{ cm}.$$

Hence,

$$x_2 = x - x_1 = 11.8 - 8.58 = 3.22 \text{ cm},$$

and

$$d_2 = x_2 \cot 32.77^\circ = (3.22 \text{ cm}) \times \cot 32.77^\circ = 5 \text{ cm}.$$

Hence, $d_t = (10+5) = 15$ cm.

Problem 8.22 A glass semicylinder with $n = 1.5$ is positioned such that its flat face is horizontal, as shown in Fig. 8-38 (P8.22). Its horizontal surface supports a drop of oil, as shown. When light is directed radially toward the oil, total internal reflection occurs if θ exceeds 60°. What is the index of refraction of the oil?

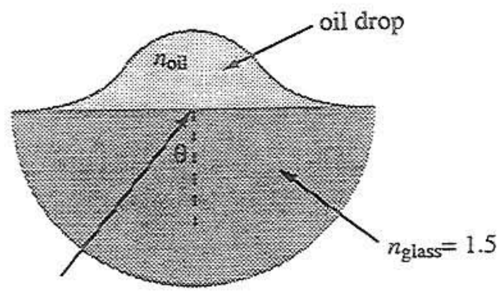


Figure P8.22: Oil drop on the flat surface of a glass semicylinder (Problem 8.22).

Solution:

$$\sin\theta_c = \frac{n_2}{n_1} = \frac{n_{oil}}{1.5},$$
$$n_{oil} = 1.5 \sin 60^\circ = 1.3.$$

Problem 8.23 A penny lies at the bottom of a water fountain at a depth of 30 cm. Determine the diameter of a piece of paper which, if placed to float on the surface of the water directly above the penny, would totally obscure the penny from view. Treat the penny as a point and assume that $n = 1.33$ for water.

Solution:

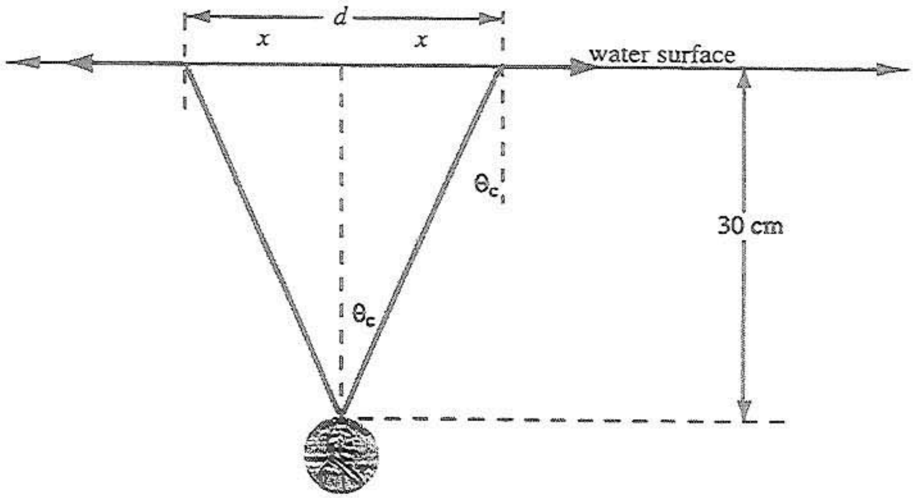


Figure P8.23: Light cone bounded by total internal reflection.

$$\theta_c = \sin^{-1}\left[\frac{1}{1.33}\right] = 48.75°,$$
$$d = 2x = 2[(30 \text{ cm})\tan\theta_c] = (60 \text{ cm}) \times \tan 48.75° = 68.42 \text{ cm}.$$

Problem 8.24 Suppose the optical fiber of Example 8-5 is submerged in water (with $n = 1.33$) instead of air. Determine θ_a and f_p in that case.

Solution: With $n_0 = 1.33$, $n_f = 1.52$ and $n_c = 1.49$, Eq. (8.40) gives

$$\sin\theta_a = \frac{1}{n_0}(n_f^2 - n_c^2)^{1/2} = \frac{1}{1.33}\left[(1.52)^2 - (1.49)^2\right]^{1/2} = 0.23,$$

or

$$\theta_a = 13.1°.$$

The data rate f_p given by Eq. (8.45) is not a function of n_0, and therefore it remains unchanged at 4.9 (Mb/s).

Problem 8.25 Equation (8.45) was derived for the case where the light incident upon the sending end of the optical fiber extends over the entire acceptance cone shown in Fig. 8-12(b). Suppose the incident light is constrained to a narrower range extending between normal incidence and θ', where $\theta' < \theta_a$.

(a) Obtain an expression for the maximum data rate f_p in terms of θ'.

(b) Evaluate f_p for the fiber of Example 8-5 when $\theta' = 3°$.

Solution:

(a) For $\theta_i = \theta'$,

$$\sin\theta_2 = \frac{1}{n_f}\sin\theta',$$

$$l_{max} = \frac{l}{\cos\theta_2} = \frac{l}{\sqrt{1-\sin^2\theta_2}} = \frac{l}{\sqrt{1-\left(\frac{\sin\theta'}{n_f}\right)^2}} = \frac{l n_f}{\sqrt{n_f^2-(\sin\theta')^2}},$$

$$t_{max} = \frac{l_{max}}{u_p} = \frac{l_{max} n_f}{c} = \frac{l n_f^2}{c\sqrt{n_f^2-(\sin\theta')^2}},$$

$$t_{min} = \frac{l}{u_p} = l\,\frac{n_f}{c},$$

$$\tau = \Delta t = t_{max} - t_{min} = l\,\frac{n_f}{c}\left[\frac{n_f}{\sqrt{n_f^2-(\sin\theta')^2}} - 1\right],$$

$$f_p = \frac{1}{2\tau} = \frac{c}{2 l n_f}\left[\frac{n_f}{\sqrt{n_f^2-(\sin\theta')^2}} - 1\right]^{-1} \quad \text{(bits/s)}.$$

(b) For:

$$n_f = 1.52,$$
$$\theta' = 3°,$$
$$l = 1 \text{ km},$$
$$c = 3\times10^8 \text{ m/s},$$
$$f_p = 166.33 \quad \text{(Mb/s)}.$$

Sections 8-4 and 8-5: Reflection and Transmission at Oblique Incidence

Problem 8.26 A plane wave in air with

$$\widetilde{\mathbf{E}}^{\mathrm{i}} = \hat{\mathbf{y}}\, 10 e^{-j(3x+4z)} \quad \text{(V/m)},$$

is incident upon the planar surface of a dielectric material, with $\varepsilon_r = 4$, occupying the half space $z \geq 0$. Determine:

(a) the polarization of the incident wave,
(b) the angle of incidence,
(c) the time-domain expressions for the reflected electric and magnetic fields,
(d) the time-domain expressions for the transmitted electric and magnetic fields, and
(e) the average power density carried by the wave in the dielectric medium.

Solution:

(a) $\widetilde{\mathbf{E}}^{\mathrm{i}} = \hat{\mathbf{y}}\, 10 e^{-j(3x+4z)}$ V/m.

Since $\mathbf{E}^{\mathrm{i}}$ is along $\hat{\mathbf{y}}$, which is perpendicular to the plane of incidence, the wave is perpendicularly polarized.

(b) From Eq. (8.48a), the argument of the exponential is

$$-jk_1(x\sin\theta_{\mathrm{i}} + z\cos\theta_{\mathrm{i}}) = -j(3x+4z).$$

Hence,

$$k_1 \sin\theta_{\mathrm{i}} = 3, \qquad k_1\cos\theta_{\mathrm{i}} = 4,$$

from which we determine that

$$\tan\theta_{\mathrm{i}} = \frac{3}{4} \qquad \text{or} \qquad \theta_{\mathrm{i}} = 36.87°,$$

and

$$k_1 = \sqrt{3^2+4^2} = 5 \quad \text{(rad/m)}.$$

Also,

$$\omega = u_{\mathrm{p}} k = ck = 3\times 10^8 \times 5 = 1.5\times 10^9 \quad \text{(rad/s)}.$$

(c)

$$\eta_1 = \eta_0 = 377\ \Omega,$$
$$\eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_{r_2}}} = \frac{\eta_0}{2} = 188.5\ \Omega,$$
$$\theta_{\mathrm{t}} = \sin^{-1}\left[\frac{\sin\theta_{\mathrm{i}}}{\sqrt{\varepsilon_{r_2}}}\right] = \sin^{-1}\left[\frac{\sin 36.87°}{\sqrt{4}}\right] = 17.46°,$$

$$\Gamma_\perp = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} = -0.41,$$
$$\tau_\perp = 1 + \Gamma_\perp = 0.59.$$

In accordance with Eq. (8.49a), and using the relation $E_0^r = \Gamma_\perp E_0^i$,

$$\widetilde{\mathbf{E}}^r = -\hat{\mathbf{y}}\,4.1\, e^{-j(3x-4z)},$$
$$\widetilde{\mathbf{H}}^r = (\hat{\mathbf{x}}\cos\theta_i + \hat{\mathbf{z}}\sin\theta_i)\,\frac{4.1}{\eta_0}\, e^{-j(3x-4z)},$$

where we used the fact that $\theta_i = \theta_r$ and the z-direction has been reversed.

$$\mathbf{E}^r = \Re e[\widetilde{\mathbf{E}}^r e^{j\omega t}] = -\hat{\mathbf{y}}\,4.1\cos(1.5\times 10^9 t - 3x + 4z) \quad \text{(V/m)},$$
$$\mathbf{H}^r = (\hat{\mathbf{x}}\,8.70 + \hat{\mathbf{z}}\,6.53)\cos(1.5\times 10^9 t - 3x + 4z) \quad \text{(mA/m)}.$$

(d) In medium 2,

$$k_2 = k_1\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = 5\sqrt{4} = 10 \quad \text{(rad/m)},$$

and

$$\theta_t = \sin^{-1}\left[\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_i\right] = \sin^{-1}\left[\frac{1}{2}\sin 36.87^\circ\right] = 17.46^\circ$$

and the exponent of $\mathbf{E}^t$ and $\mathbf{H}^t$ is

$$-jk_2(x\sin\theta_t + z\cos\theta_t) = -j10(x\sin 17.46^\circ + z\cos 17.46^\circ) = -j(3x + 9.54z).$$

Hence,

$$\widetilde{\mathbf{E}}^t = \hat{\mathbf{y}}\,10\times 0.59\, e^{-j(3x+9.54z)},$$
$$\widetilde{\mathbf{H}}^t = (-\hat{\mathbf{x}}\cos\theta_t + \hat{\mathbf{z}}\sin\theta_t)\,\frac{10\times 0.59}{\eta_2}\, e^{-j(3x+9.54z)}.$$
$$\mathbf{E}^t = \Re e[\widetilde{\mathbf{E}}^t e^{j\omega t}] = \hat{\mathbf{y}}\,5.90\cos(1.5\times 10^9 t - 3x - 9.54z) \quad \text{(V/m)},$$
$$\mathbf{H}^t = (-\hat{\mathbf{x}}\cos 17.46^\circ + \hat{\mathbf{z}}\sin 17.46^\circ)\,\frac{5.90}{188.5}\cos(1.5\times 10^9 t - 3x - 9.54z)$$
$$= (-\hat{\mathbf{x}}\,29.86 + \hat{\mathbf{z}}\,9.39)\cos(1.5\times 10^9 t - 3x - 9.54z) \quad \text{(mA/m)}.$$

(e)

$$S_{av}^t = \frac{|E_0^t|^2}{2\eta_2} = \frac{(5.90)^2}{2\times 188.5} = 0.09 \quad \text{(W/m}^2\text{)}.$$

Problem 8.27 Repeat Problem 8.26 for a wave in air with

$$\widetilde{\mathbf{H}}^{\mathrm{i}} = \hat{\mathbf{y}}\, 2 \times 10^{-2} e^{-j(8x+6z)} \quad \text{(A/m)},$$

incident upon the planar boundary of a dielectric medium ($z \geq 0$) with $\varepsilon_r = 9$.

Solution:

(a) $\widetilde{\mathbf{H}}^{\mathrm{i}} = \hat{\mathbf{y}}\, 2 \times 10^{-2} e^{-j(8x+6z)}$.

Since $\mathbf{H}^{\mathrm{i}}$ is along $\hat{\mathbf{y}}$, which is perpendicular to the plane of incidence, the wave is TM polarized, or equivalently, its electric field vector is parallel polarized (parallel to the plane of incidence).

(b) From Eq. (8.65b), the argument of the exponential is

$$-jk_1(x\sin\theta_{\mathrm{i}} + z\cos\theta_{\mathrm{i}}) = -j(8x + 6z).$$

Hence,

$$k_1 \sin\theta_{\mathrm{i}} = 8, \qquad k_1 \cos\theta_{\mathrm{i}} = 6,$$

from which we determine

$$\theta_{\mathrm{i}} = \tan^{-1}\left(\frac{8}{6}\right) = 53.13^\circ,$$

$$k_1 = \sqrt{6^2 + 8^2} = 10 \quad \text{(rad/m)}.$$

Also,

$$\omega = u_{\mathrm{p}} k = ck = 3 \times 10^8 \times 10 = 3 \times 10^9 \quad \text{(rad/s)}.$$

(c)

$$\eta_1 = \eta_0 = 377\ \Omega,$$

$$\eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_{r_2}}} = \frac{\eta_0}{3} = 125.67\ \Omega,$$

$$\theta_{\mathrm{t}} = \sin^{-1}\left[\frac{\sin\theta_{\mathrm{i}}}{\sqrt{\varepsilon_{r_2}}}\right] = \sin^{-1}\left[\frac{\sin 53.13^\circ}{\sqrt{9}}\right] = 15.47^\circ,$$

$$\Gamma_{\parallel} = \frac{\eta_2\cos\theta_{\mathrm{t}} - \eta_1\cos\theta_{\mathrm{i}}}{\eta_2\cos\theta_{\mathrm{t}} + \eta_1\cos\theta_{\mathrm{i}}} = -0.30,$$

$$\tau_{\parallel} = (1 + \Gamma_{\parallel})\frac{\cos\theta_{\mathrm{i}}}{\cos\theta_{\mathrm{t}}} = 0.44.$$

In accordance with Eqs. (8.65a) to (8.65d), $E_0^{\mathrm{i}} = 2 \times 10^{-2}\eta_1$ and

$$\widetilde{\mathbf{E}}^{\mathrm{i}} = (\hat{\mathbf{x}}\cos\theta_{\mathrm{i}} - \hat{\mathbf{z}}\sin\theta_{\mathrm{i}})\, 2 \times 10^{-2}\eta_1\, e^{-j(8x+6z)} = (\hat{\mathbf{x}} 4.52 - \hat{\mathbf{z}} 6.03)\, e^{-j(8x+6z)}.$$

$\widetilde{\mathbf{E}}^{\mathrm{r}}$ is similar to $\widetilde{\mathbf{E}}^{\mathrm{i}}$ except for reversal of z-components and multiplication of amplitude by $\Gamma_{\|}$. Hence, with $\Gamma_{\|} = -0.30$,

$$\mathbf{E}^{\mathrm{r}} = \mathfrak{Re}[\widetilde{\mathbf{E}}^{\mathrm{r}} e^{j\omega t}] = -(\hat{\mathbf{x}}1.36 + \hat{\mathbf{z}}1.81)\cos(3\times10^9 t - 8x + 6z) \text{ V/m},$$
$$\mathbf{H}^{\mathrm{r}} = \hat{\mathbf{y}}\,2\times10^{-2}\Gamma_{\|}\cos(3\times10^9 t - 8x + 6z)$$
$$= -\hat{\mathbf{y}}\,0.6\times10^{-2}\cos(3\times10^9 t - 8x + 6z) \text{ A/m}.$$

(d) In medium 2,

$$k_2 = k_1\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = 10\sqrt{9} = 30 \text{ rad/m},$$
$$\theta_{\mathrm{t}} = \sin^{-1}\left[\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\sin\theta_{\mathrm{i}}\right] = \sin^{-1}\left[\frac{1}{3}\sin 53.13^\circ\right] = 15.47^\circ,$$

and the exponent of $\mathbf{E}^{\mathrm{t}}$ and $\mathbf{H}^{\mathrm{t}}$ is

$$-jk_2(x\sin\theta_{\mathrm{t}} + z\cos\theta_{\mathrm{t}}) = -j30(x\sin 15.47^\circ + z\cos 15.47^\circ) = -j(8x + 28.91\,z).$$

Hence,

$$\widetilde{\mathbf{E}}^{\mathrm{t}} = (\hat{\mathbf{x}}\cos\theta_{\mathrm{t}} - \hat{\mathbf{z}}\sin\theta_{\mathrm{t}})E_0^{\mathrm{i}}\tau_{\|}e^{-j(8x+28.91\,z)}$$
$$= (\hat{\mathbf{x}}0.96 - \hat{\mathbf{z}}0.27)\,2\times10^{-2}\times377\times0.44\,e^{-j(8x+28.91\,z)}$$
$$= (\hat{\mathbf{x}}3.18 - \hat{\mathbf{z}}0.90)\,e^{-j(8x+28.91\,z)},$$
$$\widetilde{\mathbf{H}}^{\mathrm{t}} = \hat{\mathbf{y}}\frac{E_0^{\mathrm{i}}\tau_{\|}}{\eta_2}\,e^{-j(8x+28.91\,z)}$$
$$= \hat{\mathbf{y}}\,2.64\times10^{-2}e^{-j(8x+28.91\,z)},$$
$$\mathbf{E}^{\mathrm{t}} = \mathfrak{Re}\{\widetilde{\mathbf{E}}^{\mathrm{t}}e^{j\omega t}\}$$
$$= (\hat{\mathbf{x}}3.18 - \hat{\mathbf{z}}0.90)\cos(3\times10^9 t - 8x - 28.91\,z) \text{ V/m},$$
$$\mathbf{H}^{\mathrm{t}} = \hat{\mathbf{y}}\,2.64\times10^{-2}\cos(3\times10^9 t - 8x - 28.91\,z) \text{ A/m}.$$

(e)

$$S_{\mathrm{av}}^{\mathrm{t}} = \frac{|E_0^{\mathrm{t}}|^2}{2\eta_2} = \frac{|H_0^{\mathrm{t}}|^2}{2}\eta_2 = \frac{(2.64\times10^{-2})^2}{2}\times125.67 = 44 \text{ mW/m}^2.$$

Problem 8.28 Natural light is randomly polarized, which means that, on average, half the light energy is polarized along any given direction (in the plane orthogonal to the direction of propagation) and the other half of the energy is polarized along the direction orthogonal to the first polarization direction. Hence, when treating natural light incident upon a planar boundary, we can consider half of its energy to be in the form of parallel-polarized waves and the other half as perpendicularly polarized waves. Determine the fraction of the incident power reflected by the planar surface of a piece of glass with $n = 1.5$ when illuminated by natural light at 70°.

Solution: Assume the incident power is 1 W. Hence:
Incident power with parallel polarization = 0.5 W,
Incident power with perpendicular polarization = 0.5 W.
$\varepsilon_2/\varepsilon_1 = (n_2/n_1)^2 = n^2 = 1.5^2 = 2.25$. Equations (8.60) and (8.68) give

$$\Gamma_\perp = \frac{\cos 70^\circ - \sqrt{2.25 - \sin^2 70^\circ}}{\cos 70^\circ + \sqrt{2.25 - \sin^2 70^\circ}} = -0.55,$$

$$\Gamma_\parallel = \frac{-2.25\cos 70^\circ + \sqrt{2.25 - \sin^2 70^\circ}}{2.25\cos 70^\circ + \sqrt{2.25 - \sin^2 70^\circ}} = 0.21.$$

Reflected power with parallel polarization $= 0.5(\Gamma_\parallel)^2 = 0.5(0.21)^2 = 22$ mW,
Reflected power with perpendicular polarization $= 0.5(\Gamma_\perp)^2 = 0.5(0.55)^2 = 151.3$ mW.
Total reflected power $= 22 + 151.3 = 173.3$ mW, or 17.33%..

Problem 8.29 A parallel polarized plane wave is incident from air onto a dielectric medium with $\varepsilon_r = 9$ at the Brewster angle. What is the refraction angle?

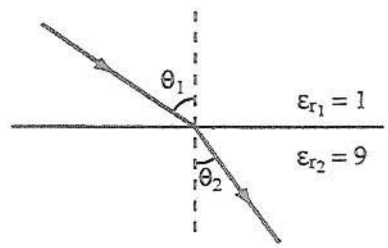


Figure P8.29: Geometry of Problem 8.29.

Solution: For nonmagnetic materials, Eq. (8.72) gives

$$\theta_1 = \theta_B = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon 1}} = \tan^{-1} 3 = 71.57^\circ.$$

But

$$\sin\theta_2 = \frac{\sin\theta_1}{\sqrt{\varepsilon_{r_2}}} = \frac{\sin\theta_1}{3} = \frac{\sin 71.57^\circ}{3} = 0.32,$$

or $\theta_2 = 18.44^\circ$.

Problem 8.30 A perpendicularly polarized wave in air is obliquely incident upon a planar glass-air interface at an incidence angle of 30°. The wave frequency is 600 THz (1 THz = 10^{12} Hz), which corresponds to green light, and the index of refraction of the glass is 1.6. If the electric field amplitude of the incident wave is 50 V/m, determine

(a) the reflection and transmission coefficients, and

(b) the instantaneous expressions for **E** and **H** in the glass medium.

Solution:

(a) For nonmagnetic materials, $(\varepsilon_2/\varepsilon_1) = (n_2/n_1)^2$. Using this relation in Eq. (8.60) gives

$$\Gamma_\perp = \frac{\cos\theta_i - \sqrt{(n_2/n_1)^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{(n_2/n_1)^2 - \sin^2\theta_i}} = \frac{\cos 30^\circ - \sqrt{(1.6)^2 - \sin^2 30^\circ}}{\cos 30^\circ + \sqrt{(1.6)^2 - \sin^2 30^\circ}} = -0.27,$$

$$\tau_\perp = 1 + \Gamma_\perp = 1 - 0.27 = 0.73.$$

(b) In the glass medium,

$$\sin\theta_t = \frac{\sin\theta_i}{n_2} = \frac{\sin 30^\circ}{1.6} = 0.31,$$

or $\theta_t = 18.21^\circ$.

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \frac{\eta_0}{n_2} = \frac{120\pi}{1.6} = 75\pi = 235.62 \quad (\Omega),$$

$$k_2 = \frac{\omega}{u_p} = \frac{2\pi f}{c/n} = \frac{2\pi f n}{c} = \frac{2\pi \times 600 \times 10^{12} \times 1.6}{3 \times 10^8} = 6.4\pi \times 10^6 \text{ rad/m},$$

$$E_0^t = \tau_\perp E_0^i = 0.73 \times 50 = 36.5 \text{ V/m}.$$

From Eqs. (8.49c) and (8.49d),

$$\widetilde{\mathbf{E}}_\perp^t = \hat{\mathbf{y}} E_0^t e^{-jk_2(x\sin\theta_t + z\cos\theta_t)},$$

$$\widetilde{\mathbf{H}}_\perp^t = (-\hat{\mathbf{x}}\cos\theta_t + \hat{\mathbf{z}}\sin\theta_t)\frac{E_0^t}{\eta_2} e^{-jk_2(x\sin\theta_t + z\cos\theta_t)},$$

and the corresponding instantaneous expressions are:

$$\mathbf{E}^{t}_{\perp}(x,z,t) = \hat{\mathbf{y}}36.5\cos(\omega t - k_2 x\sin\theta_t - k_2 z\cos\theta_t) \quad \text{(V/m)},$$
$$\mathbf{H}^{t}_{\perp}(x,z,t) = (-\hat{\mathbf{x}}\cos\theta_t - \hat{\mathbf{z}}\cos\theta_t)0.16\cos(\omega t - k_2 x\sin\theta_t - k_2 z\cos\theta_t) \quad \text{(A/m)},$$

with $\omega = 2\pi \times 10^{15}$ rad/s and $k_2 = 6.4\pi \times 10^6$ rad/m.

Problem 8.31 Show that the reflection coefficient $\Gamma_\perp$ can be written in the form

$$\Gamma_\perp = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}.$$

Solution: From Eq. (8.58a),

$$\Gamma_\perp = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{(\eta_2/\eta_1)\cos\theta_i - \cos\theta_t}{(\eta_2/\eta_1)\cos\theta_i + \cos\theta_t}.$$

Using Snell's law for refraction given by Eq. (8.31), we have

$$\frac{\eta_2}{\eta_1} = \frac{\sin\theta_t}{\sin\theta_i},$$

we have

$$\Gamma_\perp = \frac{\sin\theta_t\cos\theta_i - \cos\theta_t\sin\theta_i}{\sin\theta_t\cos\theta_i + \cos\theta_t\sin\theta_i} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}.$$

Problem 8.32 Show that for nonmagnetic media, the reflection coefficient $\Gamma_\parallel$ can be written in the form

$$\Gamma_\parallel = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)}.$$

Solution: From Eq. (8.66a), $\Gamma_\parallel$ is given by

$$\Gamma_\parallel = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \frac{(\eta_2/\eta_1)\cos\theta_t - \cos\theta_i}{(\eta_2/\eta_1)\cos\theta_t + \cos\theta_i}.$$

For nonmagnetic media, $\mu_1 = \mu_2 = \mu_0$ and

$$\frac{\eta_2}{\eta_1} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} = \frac{n_1}{n_2}.$$

Snell's law of refraction is

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2}.$$

Hence,

$$\Gamma_{\parallel} = \frac{\dfrac{\sin\theta_t}{\sin\theta_i}\cos\theta_t - \cos\theta_i}{\dfrac{\sin\theta_t}{\sin\theta_i}\cos\theta_t + \cos\theta_i} = \frac{\sin\theta_t\cos\theta_t - \sin\theta_i\cos\theta_i}{\sin\theta_t\cos\theta_t + \sin\theta_i\cos\theta_i}.$$

To show that the expression for $\Gamma_{\parallel}$ is the same as

$$\Gamma_{\parallel} = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)},$$

we shall proceed with the latter and show that it is equal to the former.

$$\frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)} = \frac{\sin(\theta_t - \theta_i)\cos(\theta_t + \theta_i)}{\cos(\theta_t - \theta_i)\sin(\theta_t + \theta_i)}.$$

Using the identities (from Appendix C):

$$2\sin x\cos y = \sin(x+y) + \sin(x-y),$$

and if we let $x = \theta_t - \theta_i$ and $y = \theta_t + \theta_i$ in the numerator, while letting $x = \theta_t + \theta_i$ and $y = \theta_t - \theta_i$ in the denominator, then

$$\frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)} = \frac{\sin(2\theta_t) + \sin(-2\theta_i)}{\sin(2\theta_t) + \sin(2\theta_i)}.$$

But $\sin 2\theta = 2\sin\theta\cos\theta$, and $\sin(-\theta) = -\sin\theta$, hence,

$$\frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)} = \frac{\sin\theta_t\cos\theta_t - \sin\theta_i\cos\theta_i}{\sin\theta_t\cos\theta_t + \sin\theta_i\cos\theta_i},$$

which is the intended result.

Problem 8.33 A parallel polarized beam of light with an electric field amplitude of 20 (V/m) is incident in air on polystyrene with $\mu_r = 1$ and $\varepsilon_r = 2.6$. If the incidence angle at the air–polystyrene planar boundary is 50°, determine

(a) the reflectivity and transmissivity, and

(b) the power carried by the incident, reflected, and transmitted beams if the spot on the boundary illuminated by the incident beam is 1 m^2 in area.

Solution:
(a) From Eq. (8.68),

$$\Gamma_{\parallel} = \frac{-(\varepsilon_2/\varepsilon_1)\cos\theta_i + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_i}}{(\varepsilon_2/\varepsilon_1)\cos\theta_i + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_i}}$$
$$= \frac{-2.6\cos 50^\circ + \sqrt{2.6 - \sin^2 50^\circ}}{2.6\cos 50^\circ + \sqrt{2.6 - \sin^2 50^\circ}} = -0.08,$$
$$R_{\parallel} = |\Gamma_{\parallel}|^2 = (0.08)^2 = 6.4 \times 10^{-3},$$
$$T_{\parallel} = 1 - R_{\parallel} = 0.9936.$$

(b)

$$P^i_{\parallel} = \frac{|E^i_{\parallel 0}|^2}{2\eta_1} A\cos\theta_i = \frac{(20)^2}{2 \times 120\pi} \times \cos 50^\circ = 0.34 \text{ W},$$
$$P^r_{\parallel} = R_{\parallel} P^i_{\parallel} = (6.4 \times 10^{-3}) \times 0.34 = 2.2 \times 10^{-3} \text{ W},$$
$$P^t_{\parallel} = T_{\parallel} P^i_{\parallel} = 0.9936 \times 0.34 = 0.338 \text{ W}.$$

Sections 8-6 to 8-8: Geometric Optics

Problem 8.34 A man is 2 m tall. How long should a mirror be, when placed in front of him at a distance d, in order for him to have a full-length view of his reflected image? Does the answer depend on d?

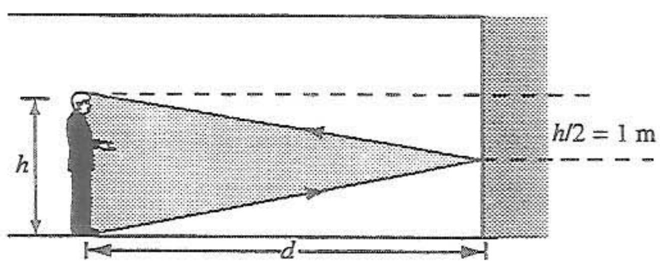


Figure P8.34: For the man to be able to see his feet, the mirror must be at least half the man's height.

Solution: The mirror needs to be only half his height and positioned as illustrated in the figure. The distance d is irrelevant.

Problem 8.35 A gas flame is 2 m from a screen, as shown in Fig. 8-39 (P8.35). A concave mirror is used to produce on the screen a real image of the flame, magnified four times. Determine s and the focal length of the mirror.

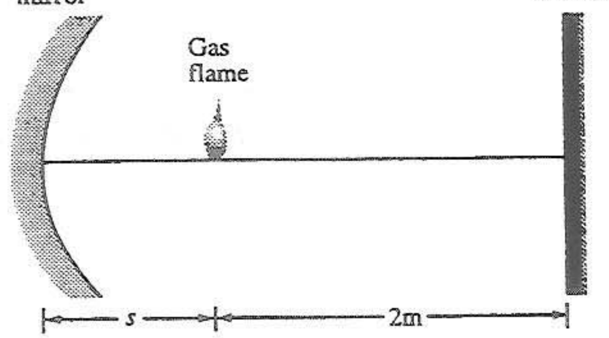


Figure P8.35: Imaging configuration of Problem 8.35.

Solution: From Table 8-4, if the image is real and magnified, then the object has to be between f and $2f$ from the mirror. Also, the image is inverted and at $s' = s + 2$. Hence,

$$M = -4 = \frac{h'}{h} = -\frac{s'}{s} = -\frac{s+2}{s} = -1 - \frac{2}{5}.$$

Consequently, $s = \frac{2}{3} = 0.67$ m.

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'} = \frac{1}{0.5} + \frac{1}{2.67},$$

which gives $f = 0.53$ m.

Problem 8.36 A dentist uses a concave mirror with a radius of curvature of 6 cm to view a filling in a tooth. If the mirror is placed at 2 cm from the tooth, by how many times will the image of the filling be magnified?

Solution:

$$R = 2f = 6 \text{ cm},$$
$$s = 2 \text{ cm},$$

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f},$$
$$\frac{1}{2} + \frac{1}{s'} = \frac{1}{3},$$

which yields $s' = -6$ cm. Hence,

$$M = -\frac{s'}{s} = -\frac{-6}{2} = 3.$$

Problem 8.37 A flower is 20 cm from a concave spherical mirror. If the image of the flower is formed at a distance of 80 cm from the mirror, determine (a) the mirror's radius of curvature, and (b) the lateral magnification.

Solution:

(a) $s = 20$ cm and $s' = 80$ cm. Hence, from Eq. (8.86),

$$\frac{1}{20} + \frac{1}{80} = \frac{2}{R},$$

or $R = 32$ cm.

(b) From Eq. (8.84),

$$M = \frac{-s'}{s} = \frac{-80}{20} = -4.$$

Problem 8.38 A candle in air is placed at a distance of 25 cm to the left of a glass medium with a concave spherical boundary characterized by a radius of curvature of 50 cm. If the index of refraction of glass is 1.5, determine the location of the candle's image.

Solution: With $n_1 = 1$, $n_2 = 1.5$, $s = 25$ cm, and $R = -50$ cm (the radius of curvature of a concave boundary is negative), Eq. (8.94) is

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}, \qquad \text{or} \qquad \frac{1}{25} + \frac{1.5}{s'} = \frac{1.5 - 1}{-50},$$

which gives $s' = -30$ cm.

Problem 8.39 A thin bi-convex lens with index of refraction of 1.6 has radii of curvature of 20 cm and 30 cm. Determine the location, magnification, and orientation of an object placed 20 cm from the lens.

Solution: From Table 8-5, a biconvex lens has a positive focal length. Hence, in Eq. (8.102), we set $R_1 = 20$ cm and $R_2 = 30$ cm:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = (1.6-1)\left(\frac{1}{20} - \frac{1}{30}\right),$$

which gives $f = 100$ cm. Application of Eq. (8.103) gives

$$\frac{1}{20} + \frac{1}{s'} = \frac{1}{100}, \qquad \text{or} \qquad s' = -25 \text{ cm}.$$

The image is virtual and erect, and $M = -s'/s = 1.25$.

Problem 8.40 Repeat Problem 8.39 for an object placed 50 cm from a planar-concave lens with a radius of curvature of 15 cm.

Solution: For a planar-concave lens (which has a negative focal length), we use $R_1 = \infty$ and $R_2 = 15$ cm in Eq. (8.102) to get

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = (1.6-1)\left(0 - \frac{1}{15}\right),$$

which gives $f = -25$ cm. Eq. (8.103) gives

$$\frac{1}{50} + \frac{1}{s'} = \frac{1}{-25}, \qquad \text{or} \qquad s' = -16.67 \text{ cm}.$$

The image is virtual and erect, and $M = -s'/s = 0.33$.

Problem 8.41 The curved face of a planar convex lens has a radius of curvature of 3 cm and the index of refraction of the lens material is 1.5. Determine the focal length of the lens when:

(a) the planar surface of the lens faces the light, and

(b) the convex surface of the lens faces the light.

Solution:

(a)

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = (1.5-1)\left(\frac{1}{\infty} - \frac{1}{-3}\right) = \frac{0.5}{3},$$

$$f = 6 \text{ cm}.$$

(b)

$$\frac{1}{f} = (1.5-1)\left(\frac{1}{3} - \frac{1}{\infty}\right) = \frac{0.5}{3},$$

$$f = 6 \text{ cm}.$$

Thus, f of a planar convex lens is positive regardless which side faces the incident light.

Problem 8.42 The image formed by a convex lens was observed to move by a distance of 1 cm when the object was moved from far away (essentially at infinity) to only 42 cm from the lens. What is the focal length of the lens?

Solution: When object is at infinity, image is at

$$s' = f.$$

When object is at $s = 42$ cm, image is at s'_2:

$$\frac{1}{42 \text{ cm}} + \frac{1}{s'_2} = \frac{1}{f}.$$

But $s'_2 = s' + 1 \text{ cm} = f + 1 \text{ cm}$. Hence,

$$\frac{1}{42 \text{ cm}} + \frac{1}{f + 1 \text{ cm}} = \frac{1}{f},$$

$$\frac{1}{42 \text{ cm}} = \frac{1}{f} - \frac{1}{f+1} = \frac{(f+1) - f}{f(f+1)} = \frac{1}{f(f+1)},$$

which yields the solution $f = 6$ cm.

Problem 8.43 Equation (8.102) defines the focal length of a thin lens in air.

(a) Derive an expression for f for the general case of a thin lens with index of refraction n_l immersed in a medium with index of refraction n_m.

(b) Determine the focal length of a bi-concave lens with radii of curvature of 10 cm and 15 cm and index of refraction of 1.5 when placed in air and when placed in water (with index of refraction of 1.33).

Solution:

(a) The thin-lens equation is given by Eq. (8.100) as

$$\frac{n_m}{s_1} + \frac{n_m}{s'_2} = (n_l - n_m)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

The focal length is defined as $f = s'_2$ when $s_1 = \infty$. Hence,

$$\frac{n_m}{f} = (n_l - n_m)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

or

$$\frac{1}{f} = \left(\frac{n_l - n_m}{n_m}\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

(b) A bi-concave lens has a negative focal length (when $n_l > n_m$). Hence, in above equations, we choose $R_1 = 15$ cm and $R_2 = 10$ cm:

$$\frac{1}{f} = (1.5-1)\left(\frac{1}{15} - \frac{1}{10}\right), \qquad \text{or} \quad f = -60 \text{ cm (in air)},$$

$$\frac{1}{f} = \left(\frac{1.5-1.33}{1.33}\right)\left(\frac{1}{15} - \frac{1}{10}\right), \qquad \text{or} \quad f = -235 \text{ cm (in water)}.$$

Problem 8.44 A positive lens is used to image an object placed 50 cm in front of it on a screen located 1 m behind the lens. What is the focal length of the lens?

Solution:

$$\frac{1}{50} + \frac{1}{100} = \frac{1}{f}, \qquad \text{or} \quad f = 33.33 \text{ cm}.$$

Problem 8.45 An imaging system consists of two thin positive lenses with focal lengths of 15 cm for the first lens and 5 cm for the second. If the two lenses are separated by a distance of 50 cm, locate the image of an object placed 25 cm in front of the first lens, relative to the location of the second lens.

Solution: Image due to first lens:

$$\frac{1}{s_1} + \frac{1}{s'_1} = \frac{1}{f}, \qquad \text{or} \qquad \frac{1}{25} + \frac{1}{s'_1} = \frac{1}{15},$$

which gives $s'_1 = 37.5$ cm. With respect to the second lens, $s_2 = 50 \text{ cm} - s'_1 = 12.5$ cm.

$$\frac{1}{12.5} + \frac{1}{s'_2} = \frac{1}{5}, \qquad \text{or} \qquad s'_2 = 8.34 \text{ cm}.$$

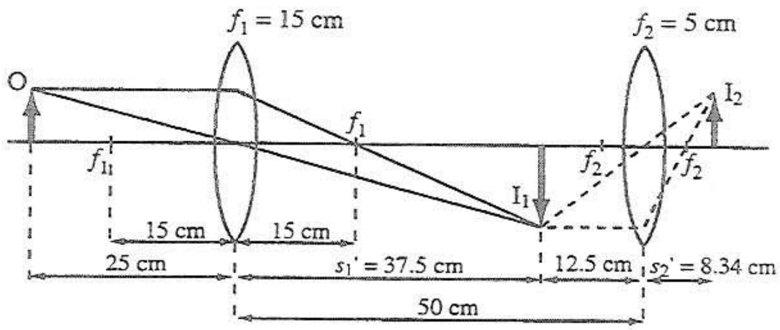


Figure P8.45: Two-lens arrangement of Problem 8.45.

Problem 8.46 Two thin lenses with focal lengths f_1 and f_2 are placed in contact with each other, as shown in Fig. 8-40 (P8.46). Show that F, the focal length of the combination, is given by

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}.$$

f_1 f_2

Figure P8.46: Two lenses in contact. The lenses are so thin that they may be considered to be at the same location (Problem 8.46).

Solution: The lens formula for the first lens is

$$\frac{1}{s_1} + \frac{1}{s_1'} = \frac{1}{f_1}. \tag{15}$$

For the second lens,

$$\frac{1}{s_2}+\frac{1}{s'_2}=\frac{1}{f_2}. \tag{16}$$

The object for the second lens is virtual because it is the image formed by the first lens, and since the two lenses are thin and in contact, we can consider them to be at the same location. Hence,

$$s_2=-s'_1, \tag{17}$$

and

$$\frac{1}{-s'_1}+\frac{1}{s'_2}=\frac{1}{f_2}. \tag{18}$$

Upon combining Eq. (15) and Eq. (18) and eliminating s'_1, we have

$$\frac{1}{s_1}+\left(\frac{1}{s'_2}-\frac{1}{f_2}\right)=\frac{1}{f_1},$$
$$\frac{1}{s_1}+\frac{1}{s'_2}=\frac{1}{f_1}+\frac{1}{f_2}\triangleq\frac{1}{F},$$
$$\frac{1}{s_1}+\frac{1}{s'_2}=\frac{1}{F},$$

where s_1 is the object distance and s'_2 is the image due to the combination of both lenses.

Problem 8.47 Two thin lenses of focal lengths $f_1 = 5$ cm and $f_2 = -10$ cm are separated by a distance of 5 cm, as shown in Fig. 8-41 (P8.47). Determine the location of the object.

Solution:

$$\frac{1}{7.5}+\frac{1}{s'_1}=\frac{1}{5},$$
$$\frac{1}{s'_1}=\frac{1}{5}-\frac{1}{7.5}=\frac{2.5}{37.5}=\frac{1}{15},$$
$$s'_1=15 \text{ cm},$$
$$\frac{1}{s_2}+\frac{1}{s'_2}=-\frac{1}{10}.$$

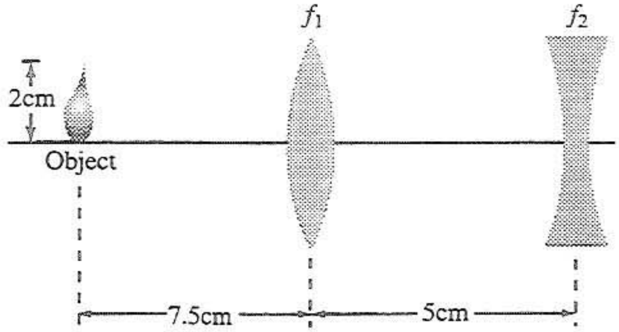


Figure P8.47: Imaging configuration of Problem 8.47.

From the figure, $|s_2| = 15 - 5 = 10$ cm, and since it is a virtual object, $s_2 = -10$ cm.

$$\frac{1}{-10} + \frac{1}{s_2'} = -\frac{1}{10},$$
$$s_2' = \infty.$$