1.1

Background:

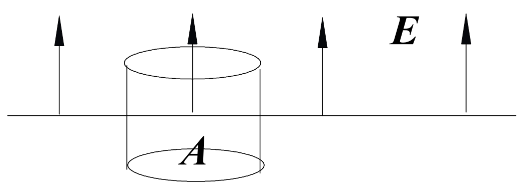
$$\text{Gauss's Law: } \int \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0}$$

Also, a conductor is defined as a material in which the conducting electrons move freely if an external electric field is applied. Thus in static equilibrium, there is no electric field present within a conductor; similarly an electric field parallel to the surface of the conductor would cause charges to move on the surface, and so this electric field cannot exist in static equilibrium. We conclude only electric fields perpendicular to the suface of the conductor can exist.

a) From the above arguments, the excess charge lies completely on the surface.

b) Consider a closed hollow conductor. Now bring up a collection of charges on the outside (bring them slowly so that static equillibrium alway obtains.) From our above arguments, the electric field lines from the charges never penetrate the conductor, so the hollow region within is shielded. On the other hand, if charges are placed within the hollow part of the conductor, electric fields exist throughout the interior because Gauss's law shows the electric field is non-zero for any surface within the interior which encloses the charges brought in.

c) Consider the Gaussian pillbox



Gauss's Law gives

$$\int \vec{E} \cdot d\vec{a} = AE = \frac{A\sigma}{\varepsilon_0} \to E = \frac{\sigma}{\varepsilon_0}$$

1.3

In general we take the charge density to be of the form $\rho = f(\vec{r})\delta,$ where $f(\vec{r})$ is determined by physical constraints, such as $\int \rho d^3x = Q.$

a) variables: $r, \theta, \phi.$ $d^3x = d\phi d\cos\theta r^2 dr$

$$\rho = f(\vec{r})\delta(r-R) = f(r)\delta(r-R) = f(R)\delta(r-R)$$

$$\int \rho d^3x = f(R)\int r^2 dr d\Omega d(r-R) = 4\pi f(R)R^2 = Q \to f(R) = \frac{Q}{4\pi R^2}$$

$$\rho(\vec{r}) = \frac{Q}{4\pi R^2}\delta(r-R)$$

b) variables: $r, \phi, z.$ $d^3x = d\phi dz r dr$

$$\rho(\vec{r}) = f(\vec{r})\delta(r-b) = f(b)\delta(r-b)$$

$$\int \rho d^3x = f(b)\int dz d\phi r dr \delta(r-b) = 2\pi f(b) bL = \lambda L \to f(b) = \frac{\lambda}{2\pi b}$$

$$\rho(\vec{r}) = \frac{\lambda}{2\pi b}\delta(r-b)$$

c) variables: $r, \phi, z.$ $d^3x = d\phi dz r dr$ Choose the center of the disk at the origin, and the z-axis perpendicular to the plane of the disk

$$\rho(\vec{r}) = f(\vec{r})\delta(z)\theta(R-r) = f\delta(z)\theta(R-r)$$

where $\theta(R-r)$ is a step function.

$$\int \rho d^3x = f\int \delta(z)\theta(R-r) d\phi dz r dr = 2\pi\frac{R^2}{2}f = Q \to f = \frac{Q}{\pi R^2}$$

$$\rho(\vec{r}) = \frac{Q}{\pi R^2}\delta(z)\theta(R-r)$$

d) variables: $r, \theta, \phi.$ $d^3x = d\phi d\cos\theta r^2 dr$

$$\rho(\vec{r}) = f(\vec{r})\delta(\cos\theta)\theta(R-r) = f(r)\delta(\cos\theta)\theta(R-r)$$

$$\int \rho d^3x = \int f(r)\delta(\cos\theta)\theta(R-r) d\phi d\cos\theta r^2 dr = \int_0^R [f(r)r] r dr d\phi = 2\pi N\int_0^R r dr = \pi R^2 N = Q$$

where I've used the fact that $r dr d\phi$ is an element of area and that the charge density is uniformly distributed over area.

$$\rho(\vec{r}) = \frac{Q}{\pi R^2 r}\delta(\cos\theta)\theta(R-r)$$

1.4

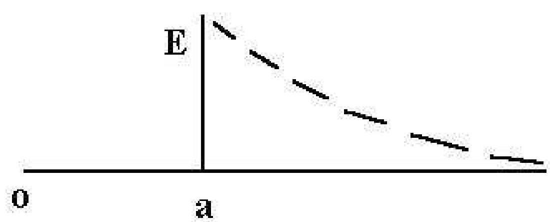
Gauss's Law:

$$\int \vec{E} \bullet d\vec{a} = \frac{Q_{enclosed}}{\varepsilon_0}$$

a) Conducting sphere: all of the charge is on the surface $\sigma = \frac{Q}{4\pi a^2}$

$$E 4\pi r^2 = 0,\ r < a \qquad E 4\pi r^2 = \frac{Q}{\varepsilon_0},\ r \sum a$$

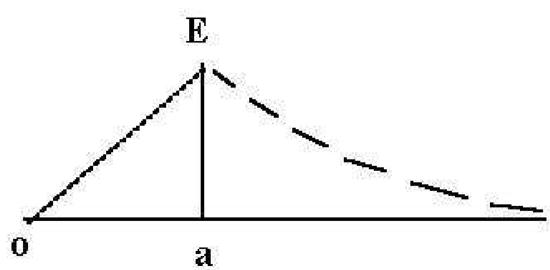
$$E = 0,\ r < a \qquad \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r},\quad r \sum a$$



b) Uniform charge density: $\rho = \frac{Q}{\frac{4}{3}\pi a^3},\ r < a,\ \rho = 0,\ r \sum a.$

$$E 4\pi r^2 = \frac{Q r^3}{\varepsilon_0 a^3} \to E = \frac{Q r}{4\pi\varepsilon_0 a^3},\quad r < a$$

$$E 4\pi r^2 = \frac{Q}{\varepsilon_0} \to E = \frac{Q}{4\pi\varepsilon_0 r^2},\quad r \sum a$$



c) $\rho = A\, r^n$

$$Q = 4\pi \int r^2 dr A r^n = 4\pi A a^{n+3}/(n+3) \to \rho = \frac{(n+3)Q}{4\pi a^{n+3}} r^n,\ r < a$$

$$\rho = 0,\quad r \sum a$$

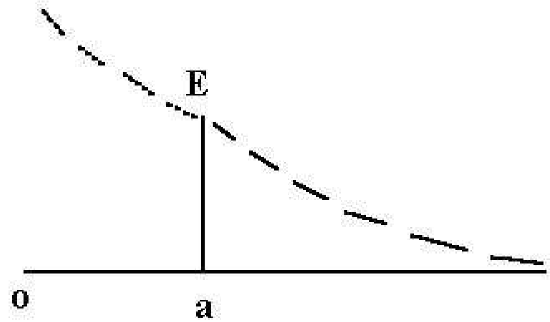
$$E 4\pi r^2 = \frac{(n+3)Q}{4\pi\varepsilon_0 a^{n+3}} 4\pi r^{n+3}/(n+3) \to E = \frac{Q}{4\pi\varepsilon_0 r^2}\left(\frac{r^{n+3}}{a^{n+3}}\right),\quad r < a$$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2},\quad r \sum a$$

1. $n = -2$.

$$E = \frac{Q}{4\pi\varepsilon_0 r a},\quad r < a$$

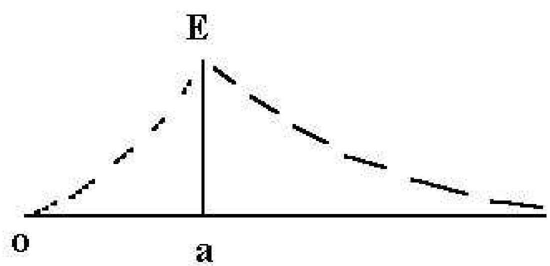
$$E = \frac{Q}{4\pi\varepsilon_0 r^2}, \quad r \sum a$$



2. $n = 2$.

$$E = \frac{Qr^3}{4\pi\varepsilon_0 a^5}, \quad r < a$$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}, \quad r \sum a$$



1.5

$$\phi(\vec{r}) = \frac{qe^{-\alpha r}\left(1 + \frac{\alpha r}{2}\right)}{4\pi\varepsilon_0 r}$$

$$\nabla^2\phi(\vec{r}) = -\frac{\rho}{\varepsilon_0}, \qquad \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

$$\nabla^2\phi = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(\frac{e^{-\alpha r}}{r} + \frac{\alpha}{2}e^{-\alpha r}\right)\right]$$

Using $\nabla^2\frac{1}{r} = -4\pi\delta(\vec{r})$

$$\nabla^2\phi = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left[-\alpha r e^{-\alpha r} + e^{-\alpha r}r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right) - \frac{\alpha^2}{2}r^2e^{-\alpha r}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[-\frac{\alpha}{r^2}e^{-\alpha r} + \frac{\alpha^2e^{-\alpha r}}{r} + \frac{\alpha e^{-\alpha r}}{r^2} - 4\pi\delta(\vec{r}) - \frac{\alpha^2e^{-\alpha r}}{r} + \frac{\alpha^3e^{-\alpha r}}{2}\right]$$

$$= -\frac{1}{\varepsilon_0}\left[q\delta(\vec{r}) - \frac{q\alpha^3}{8\pi}e^{-\alpha r}\right]$$

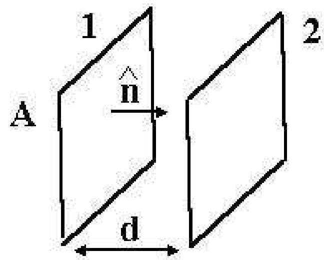
$$\rho(\vec{r}) = q\delta(\vec{r}) - \frac{q\alpha^3}{8\pi}e^{-\alpha r}$$

That is, the charge distribution consists of a positive point charge at the origin, plus an exponentially decreasing negatively charged cloud.

1.8

We will be using Gauss's law $\int \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$

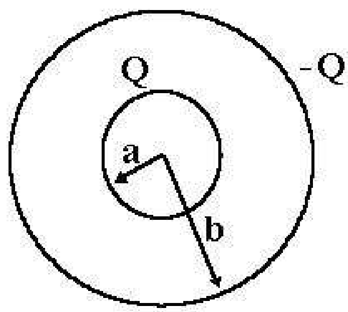
a) 1) Parallel plate capacitor



From Gauss's law $E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0} = \frac{\phi_{12}}{d} \to Q = \frac{A\varepsilon_0\phi_{12}}{d}$

$$W = \frac{\varepsilon_0}{2}\int E^2 d^3x = \frac{\varepsilon_0 E^2 Ad}{2} = \frac{\varepsilon_0\left(\frac{Q}{A\varepsilon_0}\right)^2 Ad}{2} = \frac{1}{2\varepsilon_0}\frac{Q^2}{A}d = \frac{1}{2\varepsilon_0}\frac{\left(\frac{A\varepsilon_0\phi_{12}}{d}\right)^2}{A}d = \frac{1}{2}\varepsilon_0 A\frac{\phi_{12}^2}{d}$$

2) Spherical capacitor



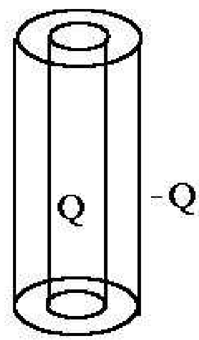
From Gauss's law, $E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2},\ a < r < b.$

$$\phi_{12} = \int_a^b E dr = \frac{Q}{4\pi\varepsilon_0}\int_a^b r^{-2} dr = \frac{1}{4}\frac{Q}{\pi\varepsilon_0}\frac{(b-a)}{ba} \to Q = \frac{4\pi\varepsilon_0 ba\phi_{12}}{(b-a)}$$

$$W = \frac{\varepsilon_0}{2}\int E^2 d^3x = \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0}\right)^2\int_a^b 4\pi\frac{r^2 dr}{r^4} = \frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0}\right)^2\left(-4\pi\frac{-b+a}{ba}\right) = \frac{1}{8\varepsilon_0}\frac{Q^2}{\pi}\frac{(b-a)}{ba}$$

$$W = \frac{1}{8\varepsilon_0}\frac{\left(\frac{4\pi\varepsilon_0 ba\phi_{12}}{(b-a)}\right)^2}{\pi}\frac{(b-a)}{ba} = 2\pi\varepsilon_0 ba\frac{\phi_{12}^2}{(b-a)}$$

3) Cylindrical conductor



From Gauss's law, $E 2\pi r L = \frac{\lambda L}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$

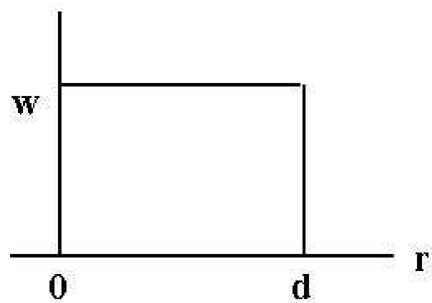
$$\phi_{12} = \int_a^b E dr = \frac{Q}{2\pi\varepsilon_0 L}\int_a^b \frac{dr}{r} = \frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right)$$

$$W = \frac{\varepsilon_0}{2}\int E^2 d^3x = \frac{\varepsilon_0}{2}\left(\frac{Q}{2\pi\varepsilon_0 L}\right)^2 2\pi L\int_a^b \frac{r dr}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{L}\ln\left(\frac{b}{a}\right)$$

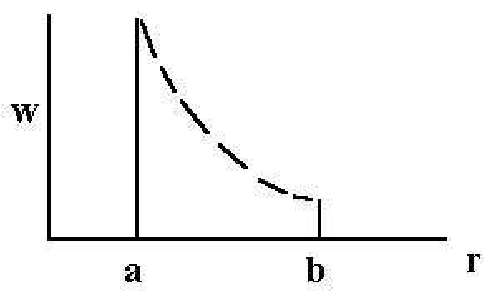
$$W = \frac{1}{4\pi\varepsilon_0}\frac{\left(\frac{2\pi\varepsilon_0 L\phi_{12}}{\ln\left(\frac{b}{a}\right)}\right)^2}{L}\ln\left(\frac{b}{a}\right) = \pi\varepsilon_0 L\frac{\phi_{12}^2}{\ln\frac{b}{a}}$$

b) $w = \frac{\varepsilon_0}{2}E^2$

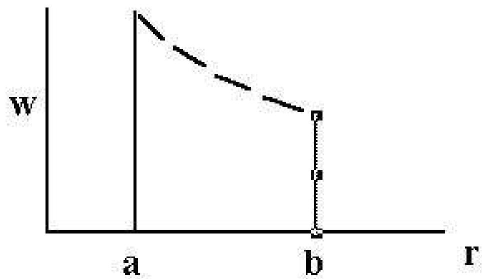
1) $w(r) = \frac{\varepsilon_0}{2}\left(\frac{Q}{A\varepsilon_0}\right)^2 = \frac{1}{2\varepsilon_0}\frac{Q^2}{A^2}$ $0 < r < d, = 0$ otherwise.



2) $w(r) = \frac{\varepsilon_0}{2}\left(\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\right)^2 = \frac{1}{32\varepsilon_0\pi^2}\frac{Q^2}{r^4},\ a < r < b, = 0,$ otherwise

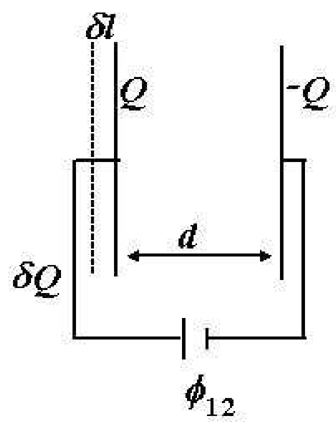


3) $w(r) = \frac{\varepsilon_0}{2}\left(\frac{Q}{2\pi\varepsilon_0 L r}\right)^2 = \frac{1}{8\varepsilon_0}\frac{Q^2}{\pi^2 L^2 r^2},\ = 0$ otherwise.


w
a
b
r

1.9

I will be using the principle of virtual work. In the figure below, $F\delta l$ is the work done by an external force. If F is along δl (ie. is positive), then the force between the plates is attractive. This work goes into increasing the electrostatic energy carried by the electric field and into forcing charge into the battery holding the plates at constant potential ϕ_{12}.



Conservation of energy gives

$$F\delta l = \delta W + \delta Q\phi_{12}$$

or

$$F = \frac{\partial W}{\partial l} + \left|\frac{\partial Q}{\partial l}\right|\phi_{12}$$

From problem 1.8,

a) Charge fixed.

1) Parallel plate capacitor

$$W = \frac{1}{2}\varepsilon_0 A\frac{\phi_{12}^2}{d},\ \phi_{12} = \frac{dQ}{A\varepsilon_0} \rightarrow W = \frac{1}{2}\varepsilon_0 A\frac{\left(\frac{dQ}{A\varepsilon_0}\right)^2}{d} = \frac{d}{2\varepsilon_0 A}Q^2$$

$$\frac{\partial Q}{\partial l} = 0,\ F = \frac{\partial W}{\partial l} = \frac{Q^2}{2\varepsilon_0 A}\ \text{(attractive)}$$

2) Parallel cylinder capacitor

$$\phi_{12} = \frac{\lambda}{\varepsilon_0}\ln(\frac{d}{a}),\ a = \sqrt{a_1 a_2}$$

$$W = \frac{1}{2}Q\phi_{12} \rightarrow F = \frac{\partial W}{\partial l} = \frac{1}{2}Q\frac{\lambda}{\varepsilon_0}\frac{\partial}{\partial d}\ln(\frac{d}{a}) = \frac{1}{2}\frac{\lambda Q}{\varepsilon_0 d}\ \text{(attractive)}$$

b) Potential fixed

1) Parallel plate capacitor

Using Gauss's law, $Q = \frac{\phi_{12}A\varepsilon_0}{d},\ \left|\frac{\partial}{\partial l}Q\right| = \frac{\phi_{12}A\varepsilon_0}{d^2}$

$$\rightarrow F = -\frac{1}{2}\varepsilon_0 A\frac{\phi_{12}^2}{d^2} + \frac{\phi_{12}^2 A\varepsilon_0}{d^2} = \frac{1}{2}\varepsilon_0 A\frac{\phi_{12}^2}{d^2} = \frac{1}{2}\varepsilon_0 A\frac{\left(\frac{Qd}{\varepsilon_0 A}\right)^2}{d^2} = \frac{1}{2\varepsilon_0 A}Q^2$$

2) Parallel cylinder capacitor

$$W = \frac{1}{2}Q\phi_{12},\ \text{and}\ Q = \frac{\varepsilon_0 L\phi_{12}}{\ln(\frac{d}{a})}$$

so

$$W = \frac{1}{2}\frac{\varepsilon_0 L\phi_{12}^2}{\ln(\frac{d}{a})},\ \frac{\partial W}{\partial l} = -\frac{1}{2}\varepsilon_0 L\frac{\phi_{12}^2}{\left(\ln^2\frac{d}{a}\right)d}$$

$$\left|\frac{\partial Q}{\partial l}\right| = \varepsilon_0 L\frac{\phi_{12}}{\left(\ln^2\frac{d}{a}\right)d}$$

$$F = -\frac{1}{2}\varepsilon_0 L\frac{\phi_{12}^2}{\left(\ln^2\frac{d}{a}\right)d} + \varepsilon_0 L\frac{\phi_{12}^2}{\left(\ln^2\frac{d}{a}\right)d} = \frac{1}{2}\varepsilon_0 L\frac{\phi_{12}^2}{\left(\ln^2\frac{d}{a}\right)d}$$

1.10 I will base the solution on the application of Green's theorem, which results in eq. 1.36 from the textbook:

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{x}')}{R} d^3x' + \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\frac{\partial\phi}{\partial n'} - \phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right] da'$$

Since the volume includes no charge, the first term on the rhs vanishes. For the second term

$$\frac{\partial\phi}{\partial n'} = \vec{\nabla}\phi \cdot \hat{n}' = -\vec{E}\cdot\hat{n}'$$

Note

$$\oint_S \vec{E}\cdot\hat{n}' da' = \int_V \vec{\nabla}' \cdot \vec{E} d^3x' \text{ by the divergence theorem}$$

Using the fact that

$$\vec{\nabla}' \cdot \vec{E} = \rho(\vec{x}')/\varepsilon_0$$

then the second term of the first equation also vanishes, since the volume integrated over contains no charge. Since $\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) = -\frac{1}{R^2}$, where R is the radius of the sphere, and I'm taking the origin at the center of the sphere,

$$\phi(\vec{x}) = \frac{1}{4\pi R^2}\oint_S \phi(\vec{x}')da' = \text{ mean value of the potential over the sphere.}$$

More Problems for Chapter 1

Problem 1.5
From Poisson's equation $\nabla^2\Phi = -\rho/\epsilon_0$, we have the charge density $\rho = -\epsilon_0\nabla^2\Phi$. Let $f \equiv \frac{1}{r}$ and $g \equiv e^{-\alpha r}(1+\frac{\alpha r}{2})$, then

$$\rho = -\epsilon_0\nabla^2\Phi = -\frac{q}{4\pi}(g\nabla^2 f + 2\nabla f\cdot\nabla g + f\nabla^2 g)$$

Note that

$$\nabla f = -\frac{\vec{r}}{r^3}, \quad \nabla^2 f = -4\pi\delta^3(\vec{r})$$

$$\nabla g = -\frac{1}{2}\alpha e^{-\alpha r}(1+\alpha r)\frac{\vec{r}}{r}, \quad \nabla^2 g = \frac{\alpha}{2r}e^{-\alpha r}(-2-2\alpha r+\alpha^2 r^2)$$

Plug them into the charge density

$$\rho = -\frac{q}{4\pi}(-4\pi\delta^3(\vec{r}) + \frac{1}{2}\alpha^3 e^{-\alpha r}) = q\delta^3(\vec{r}) - \frac{q}{8\pi}\alpha^3 e^{-\alpha r}$$

The first term represents a point charge q at the origin and the second term is due to a continuous distributed volume charge. Note that the total charge is zero:

$$\int \rho d^3x = 0$$

Problem 1.6
(a) Applying Gauss's law to the plate with a Gaussian pillbox, one gets the electric field due to a flat surface charge distribution to be $\sigma/2\epsilon_0$, where $\sigma = Q/A$ is the surface charge density. The contributions from the two plates add up in between the two plates and cancel outside. The total electric field in between the two plates is

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}.$$

The potential difference between the two plates

$$V \equiv \Phi_+ - \Phi_- = -\int_-^+ \vec{E}\cdot d\vec{\ell} = \frac{Qd}{\epsilon_0 A}$$

The capacitance

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d}.$$

(b) Assuming the inner sphere has charge Q and the outer has $-Q$, applying Gauss's law with a Gaussian sphere of radius r $(a < r < b)$:

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$$

The potential difference

$$V \equiv \Phi_+ - \Phi_- = -\int_-^+ \vec{E}\cdot d\vec{\ell} = \frac{Q}{4\pi\epsilon_0}\frac{b-a}{ab}$$

The capacitance is therefore

$$C = \frac{Q}{V} = 4\pi\epsilon_0 \frac{ab}{b-a}.$$

(c) Again assuming the inner cylinder has charge Q and the outer has $-Q$, applying Gauss's law with a cylindrical surface as the Gaussian surface:

$$\vec{E} = \frac{Q}{2\pi\epsilon_0}\frac{\vec{r}}{r^2}$$

The potential difference

$$V = -\int_{-}^{+} \vec{E}\cdot d\vec{\ell} = \frac{Q}{2\pi\epsilon_0}\ln\frac{b}{a}$$

The capacitance per unit length

$$C = \frac{Q}{V} = \frac{2\pi\epsilon_0}{\ln\frac{b}{a}}$$

Problem 1.7

From Gauss's law, the field due to one conductor is

$$\vec{E} = \frac{Q}{2\pi\epsilon_0}\frac{1}{r}$$

where Q is the charge per unit length and r is the perpendicular distance from the point of interest to the conductor. Along the perpendicular line joining the two conductors, the fields due to the two conductors are in the same direction. Therefore, the total field along the line is:

$$E = \frac{Q}{2\pi\epsilon_0}\frac{1}{r_+} + \frac{Q}{2\pi\epsilon_0}\frac{1}{r_-}$$

where r_+ and r_- are the perpendicular distances to the positively and negatively charged conductors respectively. $\vec{E}$ points from $+Q$ to $-Q$. The potential difference

$$V = \Phi_+ - \Phi_- = -\int_{-}^{+} \vec{E}\cdot d\vec{\ell} = \frac{Q}{2\pi\epsilon_0}\{\int_{a_1}^{d-a_2}\frac{1}{r_+}dr_+ + \int_{a_2}^{d-a_1}\frac{1}{r_-}dr_-\} = \frac{Q}{2\pi\epsilon_0}\ln\frac{(d-a_1)(d-a_2)}{a_1a_2}$$

The capacitance per unit length

$$C = \frac{Q}{V} = 2\pi\epsilon_0/\ln\frac{(d-a_1)(d-a_2)}{a_1a_2} \approx 2\pi\epsilon_0/\ln\frac{d^2}{(\sqrt{a_1a_2})^2} = \pi\epsilon_0/\ln\frac{d}{a}$$

where a is the geometrical mean of a_1 and a_2: $a = \sqrt{a_1a_2}$.

Problem 1.9

(a) *Parallel plate capacitor*

The negatively charged plate experiences a field of

$$E = \frac{\sigma}{2\epsilon_0} = \frac{Q}{2\epsilon_0 A}$$

due to the positively charged plate, where Q is the total charge on the plate. Therefore, the attractive force between the two plates is

$$F = QE = \frac{Q^2}{2\epsilon_0 A}$$

Parallel cylinder capacitor
Again, one conductor experiences an electric field of

$$E \approx \frac{Q}{2\pi\epsilon_0}\frac{1}{d}$$

from the other conductor. Here Q is the charge per unit length. Therefore, the attractive force per unit length between them is

$$F = QE = \frac{Q^2}{2\pi\epsilon_0}\frac{1}{d}$$

(b) The force should be the same except that Q should be replaced by CV.
Parallel plate capacitor

$$F = \frac{\epsilon_0 A V^2}{2d^2}$$

Parallel cylinder capacitor

$$F = \frac{\pi\epsilon_0 V^2}{2d\ln(b/a)^2}$$

2.1

We will work in cylidrical coordinate, (ρ, z, ϕ), with the charge q located at the point $\vec{d} = d\hat{z}$, and the conducting plane is in the $z = 0$ plane.

Then we know from class the potential is given by

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\left|\vec{x}-\vec{d}\right|} - \frac{q}{\left|\vec{x}+\vec{d}\right|}\right]$$

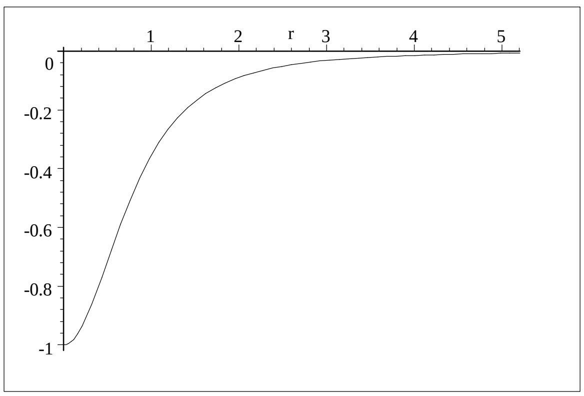
$$E_z = -\frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial z}\left[\frac{1}{\left((z-d)^2+\rho^2\right)^{1/2}} - \frac{1}{\left((z+d)^2+\rho^2\right)^{1/2}}\right]$$

$$E_z = \frac{q}{4\pi\varepsilon_0}\left[\frac{z-d}{\left((z-d)^2+\rho^2\right)^{3/2}} - \frac{z+d}{\left((z+d)^2+\rho^2\right)^{3/2}}\right]$$

a) $\sigma = \varepsilon_0 E_z(z=0)$

$$\sigma = \varepsilon_0\frac{q}{4\pi\varepsilon_0}\left[\frac{-d}{\left((-d)^2+\rho^2\right)^{3/2}} - \frac{+d}{\left((+d)^2+\rho^2\right)^{3/2}}\right] = -\frac{q}{2\pi d^2}\frac{1}{\left(1^2+\left(\frac{\rho}{d}\right)^2\right)^{\frac{3}{2}}}$$

Plotting $\frac{-1}{\left(1^2+r^2\right)^{\frac{3}{2}}}$ gives



b) Force of charge on plane

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q(-q)}{(2d)^2}(-\hat{z}) = \frac{1}{4\times 4\pi\varepsilon_0}\frac{q^2}{d^2}\hat{z}$$

c)

$$\frac{F}{A} = w = \frac{\varepsilon_0}{2}E^2 = \frac{\sigma^2}{2\varepsilon_0} = \frac{1}{2\varepsilon_0}\left(-\frac{q}{2\pi d^2}\frac{1}{\left(1^2+\left(\frac{\rho}{d}\right)^2\right)^{\frac{3}{2}}}\right)^2 = \frac{1}{8\varepsilon_0}\frac{q^2}{\pi^2 d^4\left(1+\frac{\rho^2}{d^2}\right)^3}$$

$$F = 2\pi\frac{q^2}{8\varepsilon_0\pi^2 d^4}\int_0^\beta \frac{\rho}{\left(1+\frac{\rho^2}{d^2}\right)^3}d\rho = 2\pi\frac{q^2}{8\varepsilon_0\pi^2 d^4}\left(\frac{1}{4}d^2\right) = \frac{1}{4\times 4\pi\varepsilon_0}\frac{q^2}{d^2}$$

d)

$$W = \int_d^\beta F dz = \frac{q^2}{4 \times 4\pi\varepsilon_0} \int_d^\beta \frac{dz}{z^2} = \frac{q^2}{4 \times 4\pi\varepsilon_0 d}$$

e)

$$W = \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \sum_{i,j,i\neq j} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} = -\frac{q^2}{2 \times 4\pi\varepsilon_0 d}$$

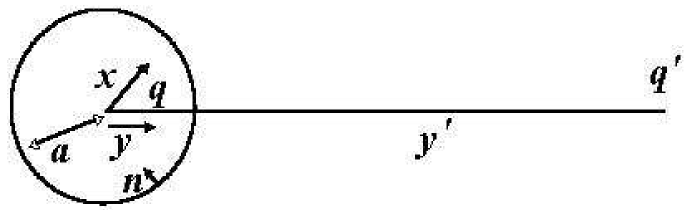
Notice parts d) and e) are not equal in magnitude, because in d) the image moves when q moves.

f) 1 Angstrom = 10^{-10}m, $q = e = 1.6 \times 10^{-19}$C.

$$W = \frac{q^2}{4 \times 4\pi\varepsilon_0 d} = e\frac{e}{4 \times 4\pi\varepsilon_0 d} = e\frac{1.6 \times 10^{-19}}{4 \times 10^{-10}} 9 \times 10^9 \text{ V} = 3.6 \text{ eV}$$

2.2

The system is described by



a) Using the method of images

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\vec{x}-\vec{y}|} + \frac{q'}{|\vec{x}-\vec{y}'|}\right]$$

with $y' = \frac{a^2}{y}$, and $q' = -q\frac{a}{y}$

b) $\sigma = -\varepsilon_0 \frac{\partial}{\partial n}\phi|_{x=a} = +\varepsilon_0 \frac{\partial}{\partial x}\phi|_{x=a}$

$$\sigma = \varepsilon_0 \frac{1}{4\pi\varepsilon_0}\frac{\partial}{\partial x}\left[\frac{q}{(x^2+y^2-2xy\cos\gamma)^{1/2}} + \frac{q'}{(x^2+y'^2-2xy'\cos\gamma)^{1/2}}\right]$$

$$\sigma = -q\frac{1}{4\pi}\frac{a\left(1-\frac{y^2}{a^2}\right)}{(y^2+a^2-2ay\cos\gamma)^{3/2}}$$

Note

$$q_{induced} = a^2\int \sigma d\Omega = -q\frac{1}{4\pi}a^2 2\pi a\left(1-\frac{y^2}{a^2}\right)\int_{-1}^{1}\frac{dx}{(y^2+a^2-2ayx)^{3/2}}, \text{ where } x = \cos\gamma$$

$$q_{induced} = -\frac{q}{2}a(a^2-y^2)\frac{2}{a(a^2-y^2)} = -q$$

c)

$$|F| = \left|\frac{qq'}{4\pi\varepsilon_0(y'-y)^2}\right| = \frac{1}{4\pi\varepsilon_0}\frac{q^2ay}{(a^2-y^2)}, \text{ the force is attractive, to the right.}$$

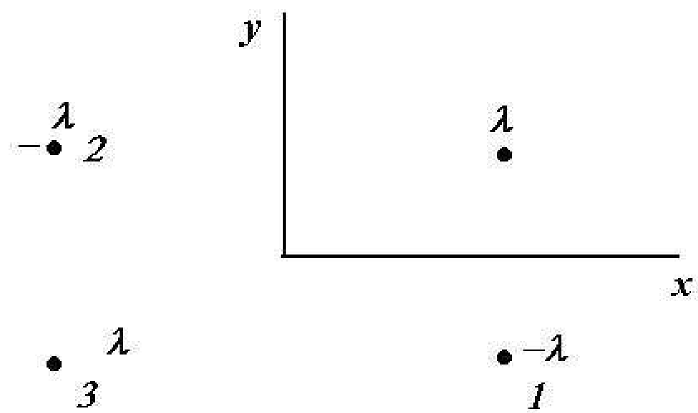
d) If the conductor were fixed at a different potential, or equivalently if extra charge were put on the conductor, then the potential would be

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\vec{x}-\vec{y}|} + \frac{q'}{|\vec{x}-\vec{y}'|}\right] + V$$

and obviously the electric field in the sphere and induced charge on the inside of the sphere would remain unchanged.

2.3

The system is described by



a) Given the potenial for a line charge in the problem, we write down the solution from the figure,

$$\phi_T = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\frac{R^2}{(\vec{x}-\vec{x}_o)^2} - \ln\frac{R^2}{(\vec{x}-\vec{x}_{o1})^2} - \ln\frac{R^2}{(\vec{x}-\vec{x}_{o2})^2} + \ln\frac{R^2}{(\vec{x}-\vec{x}_{o3})^2}\right]$$

Looking at the figure when $y = 0,\ (\vec{x}-\vec{x}_o)^2 = (\vec{x}-\vec{x}_{o1})^2,\ (\vec{x}-\vec{x}_{o2})^2 = (\vec{x}-\vec{x}_{o3})^2$, so $\phi_T|_{y=0} = 0$
Similarly, when $x = 0,\ (\vec{x}-\vec{x}_o)^2 = (\vec{x}-\vec{x}_{o2})^2,\ (\vec{x}-\vec{x}_{o1})^2 = (\vec{x}-\vec{x}_{o3})^2,$ so $\phi_T|_{x=0} = 0$
On the surface $\phi_T = 0,$ so $\delta\phi_T = 0,$ however,

$$\delta\phi_T = \frac{\partial\phi_T}{\partial x_t}\delta x_t = 0 \to \frac{\partial\phi_T}{\partial x_t} = 0 \to E_t = 0$$

b) We remember

$$\sigma = -\varepsilon_0\frac{\partial\phi_T}{\partial y} = \frac{-\lambda}{\pi}\left[\frac{y_0}{(x-x_0)^2+y_0^2} - \frac{y_0}{(x+x_0)^2+y_0^2}\right]$$

where I've applied the symmetries derived in a). Let

$$\sigma/\lambda = \frac{-1}{\pi}\left[\frac{y_0}{(x-x_0)^2+y_0^2} - \frac{y_0}{(x+x_0)^2+y_0^2}\right]$$

This is an easy function to plot for various combinations of the position of the original line charge (x_0, y_0).

c) If we integrate over a strip of width $\Delta z,$ we find, where we use the integral

$$\int_0^\infty \frac{1}{(x\mp x_0)^2+y_0^2}dx = \frac{1}{2}\frac{\pi \pm 2\arctan\frac{x_0}{y_0}}{y_0}$$

$$\Delta Q = \int_0^\infty \sigma dx\Delta z \to \frac{\Delta Q}{\Delta z} = \int_0^\infty \sigma dx = \frac{-2\lambda}{\pi}\tan^{-1}\left(\frac{x_0}{y_0}\right)$$

and the total charge induced on the plane is $-\infty,$ as expected.
d)
Expanding

$$\ln\left(\frac{R^2}{(x-x_0)^2+(y-y_0)^2}\right)-\ln\left(\frac{R^2}{(x-x_0)^2+(y+y_0)^2}\right)-\ln\left(\frac{R^2}{(x+x_0)^2+(y-y_0)^2}\right)+\ln\left(\frac{R^2}{(x+x_0)^2+(y+\ldots}\right.$$

to lowest non-vanishing order in $x_0,\ y_0$ gives

$$16\frac{xy}{(x^2+y^2)^2}y_0x_0$$

so

$$\phi \to \phi_{asym} = \frac{4\lambda}{\pi\varepsilon_0}\frac{xy}{(x^2+y^2)^2}y_0x_0$$

This is the quadrupole contribution.

2.5

a)

$$W = \int_r^\infty |F| dy = \frac{q^2 a}{4\pi\varepsilon_0} \int_r^\infty \frac{dy}{y^3 \left(1 - \frac{a^2}{y^2}\right)^2} = \frac{q^2 a}{8\pi\varepsilon_0 (r^2 - a^2)}$$

Let us compare this to disassemble the charges

$$-W' = -\frac{1}{8\pi\varepsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} = \frac{1}{4\pi\varepsilon_0} \left[\frac{aq^2}{r} \frac{1}{r\left(1 - \frac{a^2}{r^2}\right)} \right] = \frac{q^2 a}{4\pi\varepsilon_0 (r^2 - a^2)} > W$$

The reason for this difference is that in the first expression W, the image charge is moving and changing size, whereas in the second, whereas in the second, they don't.

b) In this case

$$W = \int_r^\infty |F| dy = \frac{q}{4\pi\varepsilon_0} \left[\int_r^\infty \frac{Q dy}{y^2} - qa^3 \int_r^\infty \frac{(2y^2 - a^2)}{y(y^2 - a^2)^2} dy \right]$$

Using standard integrals, this gives

$$W = \frac{1}{4\pi\varepsilon_0} \left[\frac{q^2 a}{2(r^2 - a^2)} - \frac{q^2 a}{2r^2} - \frac{qQ}{r} \right]$$

On the other hand

$$-W' = \frac{1}{4\pi\varepsilon_0} \left[\frac{aq^2}{(r^2 - a^2)} - \frac{q(Q + \frac{a}{r} q)}{r} \right] = \frac{1}{4\pi\varepsilon_0} \left[\frac{aq^2}{(r^2 - a^2)} - \frac{q^2 a}{r^2} - \frac{qQ}{r} \right]$$

The first two terms are larger than those found in W for the same reason as found in a), whereas the last term is the same, because Q is fixed on the sphere.

2.6

We are considering two conducting spheres of radii r_a and r_b respectively. The charges on the spheres are Q_a and Q_b.

a) The process is that you start with $q_a(1)$ and $q_b(1)$ at the centers of the spheres, and sphere *a* then is an equipotential from charge $q_a(1)$ but not from $q_b(1)$ and vice versa. To correct this we use the method of images for spheres as discussed in class. This gives the iterative equations given in the text.

b) $q_a(1)$ and $q_b(1)$ are determined from the two requirements

$$\sum_{j=1}^{\beta} q_a(j) = Q_a \text{ and } \sum_{j=1}^{\beta} q_b(j) = Q_b$$

As a program equation, we use a do-loop of the form

$$\sum_{j=2}^{n} \left[q_a(j) = \frac{-r_a q_b(j-1)}{d_b(j-1)}\right]$$

and similar equations for $q_b(j),\ x_a(j),\ x_b(j),\ d_a(j), d_b(j)$. The potential outside the spheres is given by

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left(\sum_{j=1}^{n} \frac{q_a(j)}{\left|\vec{x} - x_a(j)\hat{k}\right|} + \sum_{j=1}^{n} \frac{q_b(j)}{\left|\vec{x} - d_b(j)\hat{k}\right|}\right)$$

This potential is constant on the surface of the spheres by construction.

And the force between the spheres is

$$F = \frac{1}{4\pi\varepsilon_0}\sum_{j,k} \frac{q_a(j)q_b(k)}{[d - x_a(j) - x_b(k)]^2}$$

c) Now we take the special case $Q_a = Q_b,\ r_a = r_b = R,\ d = 2R.$ Then we find, using the iteration equations

$$x_a(j) = x_b(j) = x(j)$$

$$x(1) = 0,\ x(2) = R/2,\ x(3) = 2R/3, \text{ or } x(j) = \frac{(j-1)}{j}R$$

$$q_a(j) = q_b(j) = q(j)$$

$$q(j) = q,\ \ q(2) = -q/2,\ q(3) = q/3, \text{ or } q(j) = \frac{(-1)^{j+1}}{j}q$$

So, as $n \to \beta$

$$\sum_{j=1}^{\beta} q(j) = q\sum_{j=1}^{\beta} \frac{(-1)^{j+1}}{j} = q\ln 2 = Q \to q = \frac{Q}{\ln 2}$$

The force between the spheres is

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{R^2}\sum_{j,k} \frac{(-1)^{j+k}}{jk\left[2 - \frac{(j-1)}{j} - \frac{(k-1)}{k}\right]^2} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{R^2}\sum_{j,k} \frac{(-1)^{j+k}jk}{(j+k)^2}$$

Evaluating the sum numerically

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{R^2}(0.0739) = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{R^2}\frac{1}{(\ln 2)^2}(0.0739)$$

Comparing this to the force between the charges located at the centers of the spheres

$$F_p = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{R^2 4}$$

Comparing the two results, we see

$$F = 4\frac{1}{(\ln 2)^2}(0.0739)F_p = 0.615F_p$$

On the surface of the sphere

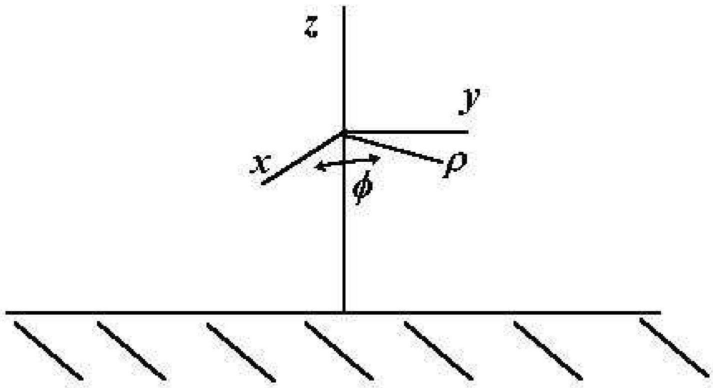
$$\phi = \frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{\beta}\frac{q(j)}{R - x(j)} = \frac{q}{4\pi\varepsilon_0 R}\sum_{j=1}^{\beta}(-1)^{j+1}$$

$$\text{Notice } \frac{1}{1+1} = \sum_{j=1}^{\beta}(-1)^{j+1}$$

$$\text{So } \phi = \frac{1}{4\pi\varepsilon_0}\frac{q}{2R} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2\ln 2R} = \frac{Q}{C} \rightarrow \frac{C}{4\pi\varepsilon_0 R} = 2\ln 2 = 1.386$$

2.7

The system is described by



a) The Green's function, which vanishes on the surface is obviously

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{1}{|\vec{x}-\vec{x}_I'|}$$

where

$$\vec{x}' = x'\hat{\imath} + y'\hat{\jmath} + z'\hat{k},\ \vec{x}_I' = x'\hat{\imath} + y'\hat{\jmath} - z'\hat{k}$$

b) There is no free charge distribution, so the potential everywhere is determined by the potential on the surface. From Eq. (1.44)

$$\phi(\vec{x}) = -\frac{1}{4\pi}\int_S \phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}da'$$

Note that $\hat{n}'$ is in the $-z$ direction, so

$$\frac{\partial}{\partial n'}G(\vec{x},\vec{x}')|_{z'=0} = -\frac{\partial}{\partial z}G(\vec{x},\vec{x}')|_{z'=0} = -\frac{2z}{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}}$$

So

$$\phi(\vec{x}) = \frac{z}{2\pi}V\int_0^a\int_0^{2\pi}\frac{\rho' d\rho' d\phi'}{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}}$$

where $x'=\rho'\cos\phi', y' = \rho'\sin\phi'$.

c) If $\rho = 0,$ or equivalently $x = y = 0,$

$$\phi(z) = \frac{z}{2\pi}V\int_0^a\int_0^{2\pi}\frac{\rho' d\rho' d\phi'}{[\rho'^2+z^2]^{3/2}} = zV\int_0^a\frac{\rho d\rho}{[\rho^2+z^2]^{3/2}}$$

$$\phi(z) = zV\left(-\frac{z-\sqrt{(a^2+z^2)}}{z\sqrt{(a^2+z^2)}}\right) = V\left(1-\frac{z}{\sqrt{(a^2+z^2)}}\right)$$

d)

$$\phi(\vec{x}) = \frac{z}{2\pi}V\int_0^a\int_0^{2\pi}\frac{\rho' d\rho' d\phi'}{\left[(\vec{\rho}-\vec{\rho}')^2+z^2\right]^{3/2}}$$

In the integration choose the x –axis parallel to $\vec{\rho}$, then $\vec{\rho}\cdot\vec{\rho}' = \rho\rho'\cos\phi'$

$$\phi(\vec{x}) = \frac{z}{2\pi}V\int_0^a\int_0^{2\pi}\frac{\rho' d\rho' d\phi'}{[\rho^2+\rho'^2-2\vec{\rho}\cdot\vec{\rho}'+z^2]^{3/2}}$$

Let $r^2 = \rho^2 + z^2$, so

$$\phi(\vec{x}) = \frac{z}{2\pi}\frac{V}{r^3}\int_0^a\int_0^{2\pi}\frac{\rho' d\rho' d\phi'}{\left[1+\frac{\rho'^2-2\vec{\rho}\cdot\vec{\rho}'}{r^2}\right]^{3/2}}$$

We expand the denominator up to factors of $O(1/r^4)$, (and change notation $\phi' \to \theta, \rho' \to \alpha, r^2 \to \frac{1}{\beta^2}$)

$$\phi(\vec{x}) = \frac{z}{2\pi}\frac{V}{r^3}\int_0^a\int_0^{2\pi}\frac{\alpha d\alpha d\theta}{\left[1+\frac{\alpha^2-2\vec{\rho}\cdot\vec{\alpha}}{r^2}\right]^{3/2}}$$

where the denominator in this notation is written

$$\frac{1}{[1+\beta^2(\alpha^2-2\rho\alpha\cos\theta)]^{3/2}}$$

or, after expanding,

$$\phi(\vec{x}) = \frac{z}{2\pi}\frac{V}{r^3}\int_0^a\alpha d\alpha\int_0^{2\pi}\left(1-\frac{3}{2}\beta^2\alpha^2+3\beta^2\rho\alpha\cos\theta+\frac{15}{8}\beta^4\alpha^4-\frac{15}{2}\beta^4\alpha^3\rho\cos\theta+\frac{15}{2}\beta^4\rho^2\alpha^2\cos^2\theta\right)d\theta$$

Integrating over θ gives

$$\phi(\vec{x}) = \frac{z}{2\pi}\frac{V}{r^3}\int_0^a\alpha\left(2\pi+\frac{15}{4}\beta^4\alpha^4\pi-3\beta^2\alpha^2\pi+\frac{15}{2}\beta^4\rho^2\alpha^2\pi\right)d\alpha$$

Integrating over α yields

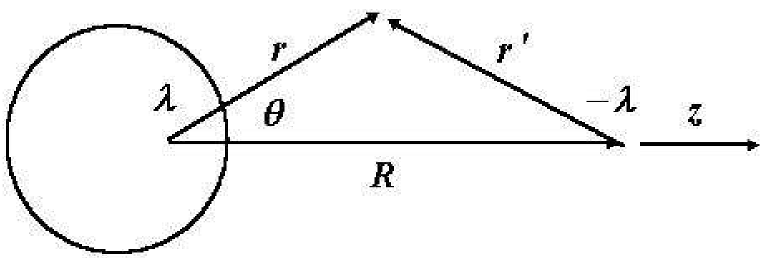
$$\phi(\vec{x}) = \frac{z}{2\pi}\frac{V}{r^3}\left(\frac{5}{8}\beta^4\pi a^6-\frac{3}{4}a^4\beta^2\pi+\frac{15}{8}a^4\beta^4\rho^2\pi+\pi a^2\right)$$

or

$$\phi(\vec{x}) = \frac{Va^2}{2(\rho^2+z^2)^{3/2}}\left[1-\frac{3}{4}\frac{a^2}{(\rho^2+z^2)}+\frac{5}{8}\left(\frac{a^4+3\rho^2a^2}{(\rho^2+z^2)^2}\right)\right]$$

2.8

The system is pictured below



a) Using the known potential for a line charge, the two line charges above give the potential

$$\phi(\vec{r}) = \frac{1}{2\pi\varepsilon_0}\lambda \ln\frac{r'}{r} = V, \text{ a constant. Let us define } V' = 4\pi\varepsilon_0 V$$

Then the above equation can be written

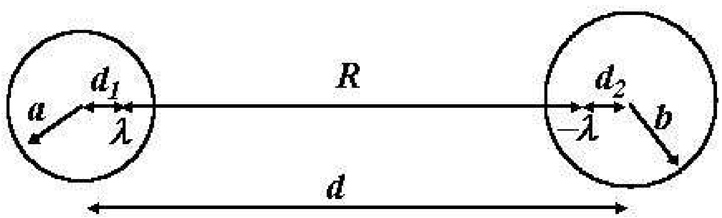
$$\left(\frac{r'}{r}\right)^2 = e^{\frac{V'}{\lambda}} \text{ or } r'^2 = r^2 e^{\frac{V'}{\lambda}}$$

Writing $r'^2 = \left(\vec{r} - \vec{R}\right)^2$, the above can be written

$$\left(\vec{r} + \hat{z}\frac{R}{(e^{\frac{V'}{\lambda}} - 1)}\right)^2 = \frac{R^2 e^{\frac{V'}{\lambda}}}{(e^{\frac{V'}{\lambda}} - 1)^2}$$

The equation is that of a circle whose center is at $-\hat{z}\frac{R}{e^{\frac{V'}{\lambda}}-1}$, and whose radius is $a = \frac{Re^{\frac{V'}{2\lambda}}}{(e^{\frac{V'}{\lambda}}-1)}$

b) The geometry of the system is shown in the fugure.



Note that

$$d = R + d_1 + d_2$$

with

$$d_1 = \frac{R}{e^{\frac{V'_a}{\lambda}}-1}, \quad d_2 = \frac{R}{e^{\frac{-V'_b}{\lambda}}-1}$$

and

$$a = \frac{Re^{\frac{V'_a}{2\lambda}}}{(e^{\frac{V'_a}{\lambda}} - 1)}, \; b = \frac{Re^{\frac{-V'_b}{2\lambda}}}{(e^{\frac{-V'_b}{\lambda}} - 1)}$$

Forming

$$d^2 - a^2 - b^2 = \left(R + \frac{R}{e^{\frac{V_a'}{\lambda}-1}} + \frac{R}{e^{\frac{-V_b'}{\lambda}-1}} \right)^2 - \left(\frac{Re^{\frac{V_a'}{2\lambda}}}{(e^{\frac{V_a'}{\lambda}} - 1)} \right)^2 - \left(\frac{Re^{\frac{-V_b'}{2\lambda}}}{(e^{\frac{-V_b'}{\lambda}} - 1)} \right)^2$$

or

$$d^2 - a^2 - b^2 = \frac{R^2\left(e^{\frac{V_a'-V_b'}{\lambda}} + 1\right)}{(e^{\frac{V_a'}{\lambda}} - 1)(e^{\frac{-V_b'}{\lambda}} - 1)}$$

Thus we can write

$$\frac{d^2 - a^2 - b^2}{2ab} = \frac{\left(e^{\frac{V_a'-V_b'}{\lambda}} + 1\right)}{2e^{\frac{V_a'}{2\lambda}}e^{\frac{-V_b'}{2\lambda}}} = \frac{e^{\frac{V_a'-V_b'}{2\lambda}} + e^{\frac{-(V_a'-V_b')}{2\lambda}}}{2} = \cosh\left(\frac{V_a' - V_b'}{2\lambda}\right)$$

or

$$\frac{V_a - V_b}{\lambda} = \frac{1}{2\pi\varepsilon_0}\cosh^{-1}\left(\frac{d^2 - a^2 - b^2}{2ab}\right)$$

$$\text{Capacitance/unit length} = \frac{C}{L} = \frac{Q/L}{V_a - V_b} = \frac{\lambda}{V_a - V_b} = \frac{2\pi\varepsilon_0}{\cosh^{-1}\left(\frac{d^2-a^2-b^2}{2ab}\right)}$$

c) Suppose $a^2 << d^2$, and $b^2 << d^2$, and $a' = \sqrt{ab}$, then

$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\cosh^{-1}\left(\frac{d^2-a^2-b^2}{2a'^2}\right)} = \frac{2\pi\varepsilon_0}{\cosh^{-1}\left(\frac{d^2(1-(a^2+b^2)/d^2)}{2a'^2}\right)}$$

$$\cosh^{-1}\left(\frac{d^2(1-(a^2+b^2)/d^2)}{2a'^2}\right) = \frac{2\pi\varepsilon_0 L}{C}$$

$$\left(\frac{d^2(1-(a^2+b^2)/d^2)}{2a'^2}\right) = \frac{e^{\frac{2\pi\varepsilon_0 L}{C}}}{2} + \text{ negligible terms if } \frac{2\pi\varepsilon_0 L}{C} \sum\sum 1$$

or

$$\ln\left(\frac{d^2(1-(a^2+b^2)/d^2)}{a'^2}\right) = \frac{2\pi\varepsilon_0 L}{C}$$

or

$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{d^2(1-(a^2+b^2)/d^2)}{a'^2}\right)}$$

Let us defind $\alpha^2 = (a^2 + b^2)/d^2$, then

$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{d^2(1-\alpha^2)}{a'^2}\right)} = 2\pi\frac{\varepsilon_0}{\ln\frac{d^2}{a'^2}} + 2\pi\frac{\varepsilon_0}{\ln^2\frac{d^2}{a'^2}}\alpha^2 + O(\alpha^4)$$

The first term of this result agree with problem 1.7, and the second term gives the appropriate correction asked for.

d) In this case, we must take the opposite sign for $d^2 - a^2 - b^2$, since $a^2 + b^2 \sum d^2$. Thus

$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\cosh^{-1}\left(\frac{a^2+b^2-d^2}{2a'^2}\right)}$$

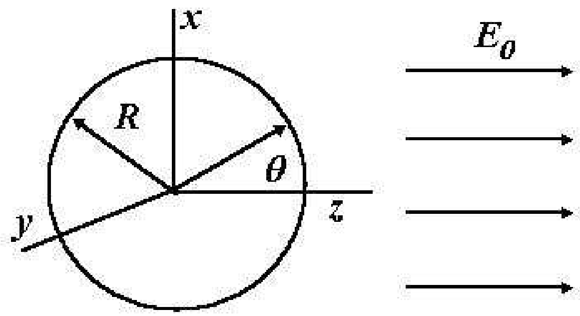
If we use the identiy, $\ln(x + \sqrt{x^2 - 1}) = \cosh^{-1}(x)$, G.&R., p. 50., then for $d = 0$

$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{a^2+b^2}{2ab} + \frac{a^2-b^2}{2ab}\right)} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{a}{b}\right)}$$

in agreement with problem 1.6.

2.9

The system is pictured below



a) We have treated this probem in class. We found the charge density induced was

$$\sigma = 3\varepsilon_0 E_0 \cos\theta$$

We also note the radial force/unit area outward from the surface is $\sigma^2/2\varepsilon_0$. Thus the force on the right hand hemisphere is, using $x = \cos\theta$

$$F_z = \frac{1}{2\varepsilon_0}\int \sigma^2 \hat{z} \cdot d\vec{a} = \frac{1}{2\varepsilon_0}(3\varepsilon_0 E_0)^2 2\pi R^2 \int_0^1 x^3 dx = \frac{1}{2\varepsilon_0}(3\varepsilon_0 E_0)^2 2\pi R^2/4 = \frac{9}{4}\pi\varepsilon_0 E_0^2 R^2$$

An equal force acting in the opposite direction would be required to keep the hemispheres from sparating.

b) Now the charge density is

$$\sigma = 3\varepsilon_0 E_0 \cos\theta + \frac{Q}{4\pi R^2} = 3\varepsilon_0 E_0 x + \frac{Q}{4\pi R^2} = 3\varepsilon_0 E_0\left(x + \frac{Q}{12\pi\varepsilon_0 E_0 R^2}\right)$$

$$F_z = \frac{1}{2\varepsilon_0}\int \sigma^2 \hat{z} \cdot d\vec{a} = \frac{1}{2\varepsilon_0}(3\varepsilon_0 E_0)^2 2\pi R^2 \int_0^1 x\left(x + \frac{Q}{12\pi\varepsilon_0 E_0 R^2}\right)^2 dx$$

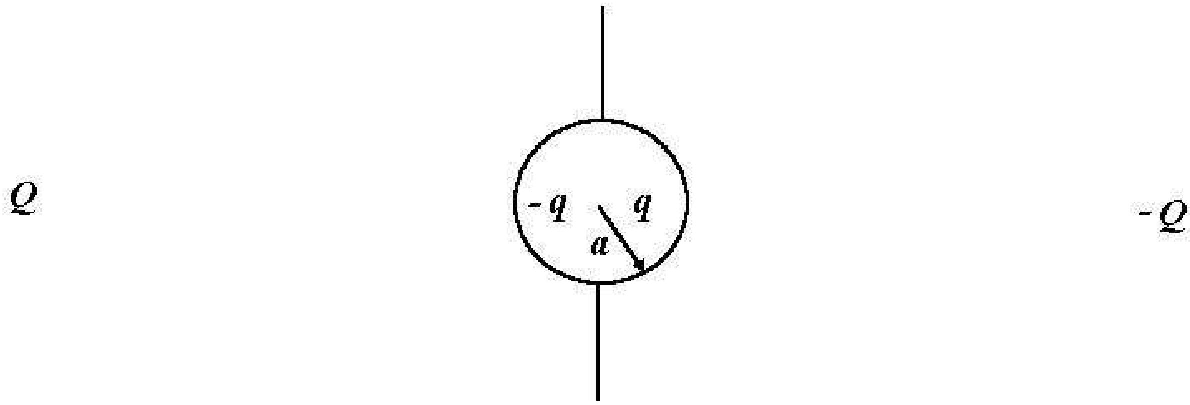
Thus

$$F_z = \frac{9}{4}\pi\varepsilon_0 E_0^2 R^2 + \frac{1}{2}QE_0 + \frac{1}{32\varepsilon_0\pi R^2}Q^2$$

An equal force acting in the opposite direction would be required to keep the himispheres from separating.

2.10

As done in class we simulate the electric field E_0 by two charges at ∞



$$\sigma = 3\varepsilon_0 E_0 \cos\theta$$

a) This charge distribution simulates the given system for $\cos\theta \sum 0$. We have treated this probem in class. The potential is given by

$$\phi(\vec{x}) = -E_0\left(1 - \frac{a^3}{r^3}\right) r\cos\theta$$

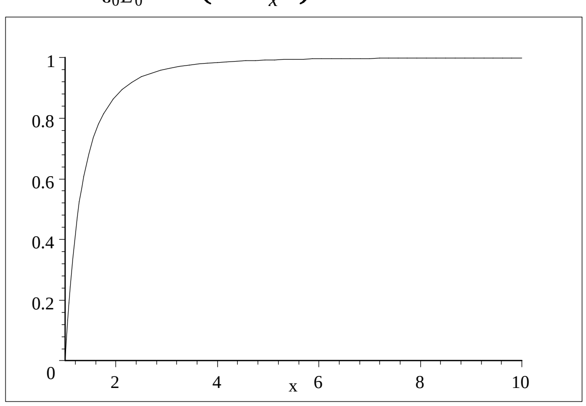
Using

$$\sigma = -\varepsilon_0 \frac{\partial}{\partial n}\phi|_{surface}$$

We have the charge density on the plate to be

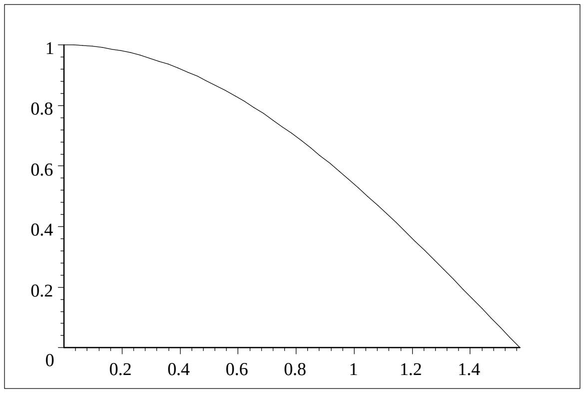
$$\sigma_{plate} = -\varepsilon_0 \frac{\partial}{\partial z}\phi|_{z=0} = \varepsilon_0 E_0\left(1 - \frac{a^3}{\rho^3}\right)$$

For purposes of plotting, consider $\frac{\sigma_{plate}}{\varepsilon_0 E_0} = \left(1 - \frac{1}{x^3}\right)$



$$\sigma_{boss} = 3\varepsilon_0 E_0 \cos\theta =$$

For plotting, we use $\frac{\sigma_{boss}}{3\varepsilon_0 E_0} = \cos\theta$



b)

$$q = 3\varepsilon_0 E_0 2\pi a^2 \int_0^1 x dx = 3\pi\varepsilon_0 E_0 a^2$$

c) Now we have

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\left|\vec{r}-\vec{d}\right|} + \frac{q'}{\left|\vec{r}-\vec{d}'\right|} - \frac{q}{\left|\vec{r}+\vec{d}\right|} - \frac{q'}{\left|\vec{r}+\vec{d}'\right|} \right]$$

where $q' = -q\frac{a}{d}$, $d' = \frac{a^2}{d}$.

$$\sigma = -\varepsilon_0 \frac{\partial}{\partial r}\phi|_{r=a} = \frac{-q}{4\pi}\left[\frac{(d^2-a^2)}{a\left|\vec{a}-\vec{d}\right|^3} - \frac{(d^2-a^2)}{a\left|\vec{a}+\vec{d}\right|^3} \right]$$

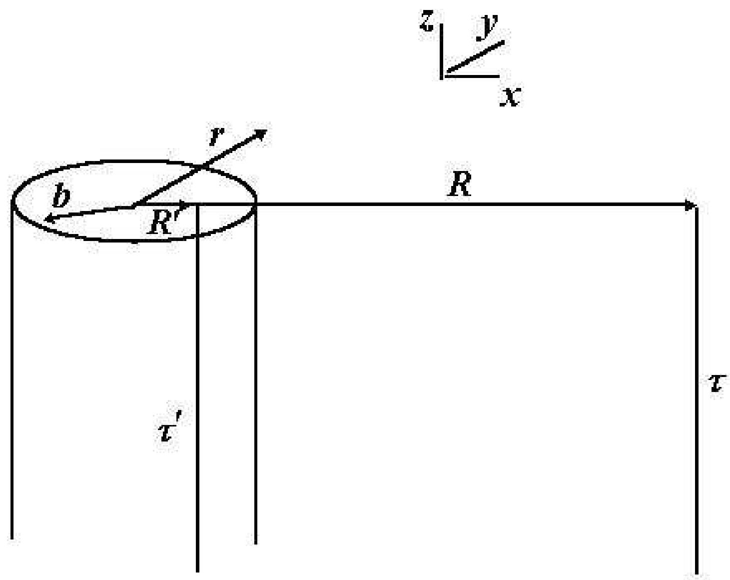
$$q_{ind} = 2\pi a^2\left(\frac{-q}{4\pi}\right)\int_0^1\left[\frac{(d^2-a^2)}{a\left|\vec{a}-\vec{d}\right|^3} - \frac{(d^2-a^2)}{a\left|\vec{a}+\vec{d}\right|^3} \right]dx$$

$$q_{ind} = \frac{-qa^2(d^2-a^2)}{2a}\left[\frac{1}{da}\left(\frac{1}{d-a} - \frac{1}{\sqrt{a^2+d^2}} + \frac{1}{d+a} - \frac{1}{\sqrt{a^2+d^2}} \right) \right]$$

$$q_{ind} = \frac{-1}{2}q\frac{d^2-a^2}{d}\left[\frac{2d}{d^2-a^2} - \frac{2}{\sqrt{a^2+d^2}} \right] = -q\left[1 - \frac{(d^2-a^2)}{d\sqrt{a^2+d^2}} \right]$$

2.11

The system is pictured in the following figure:



a) The potential for a line charge is (see problem 2.3)

$$\phi(\vec{r}) = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r_\infty}{r}\right)$$

Thus for this system

$$\phi(\vec{r}) = \frac{\tau}{2\pi\varepsilon_0} \ln\left(\frac{r_\infty}{|\vec{r}-\vec{R}|}\right) + \frac{\tau'}{2\pi\varepsilon_0} \ln\left(\frac{r_\infty}{|\vec{r}-\vec{R}'|}\right)$$

To determine τ' and R', we need two conditions:

I) As $r \to \infty,$ we want $\phi \to 0,$ so $\tau' = -\tau.$

II) $\phi(\vec{r} = b \quad) = \phi(\vec{r} = -b \quad)$ or

$$\ln\left(\frac{b-R'}{R-b}\right) = \ln\left(\frac{b+R'}{b+R}\right)$$

or

$$\left(\frac{b-R'}{R-b}\right) = \left(\frac{b+R'}{b+R}\right)$$

This is an equation for R' with the solution

$$R' = \frac{b^2}{R}$$

The same condition we found for a sphere.

b)

$$\phi(\vec{r}) = \frac{\tau}{4\pi\varepsilon_0}\ln\left[\frac{r^2+\frac{b^4}{R^2}-2r\frac{b^2}{R}\cos\phi}{r^2+R^2-2rR\cos\phi}\right]$$

as $r \to \infty$

$$\phi(\vec{r}) = \frac{\tau}{4\pi\varepsilon_0}\ln\left[\frac{1-2b^2\cos\phi/rR}{1-2R\cos\phi/r}\right] = \frac{\tau}{4\pi\varepsilon_0}\ln\left[1-\frac{2}{rR}(b^2-R^2)\cos\phi\right]$$

Using

$$\ln(1+x) = x-\frac{1}{2}x^2+\frac{1}{3}x^3+O(x^4)$$

$$\phi(\vec{r}) = -\frac{\tau}{2\pi\varepsilon_0}\frac{1}{rR}(b^2-R^2)\cos\phi$$

c)

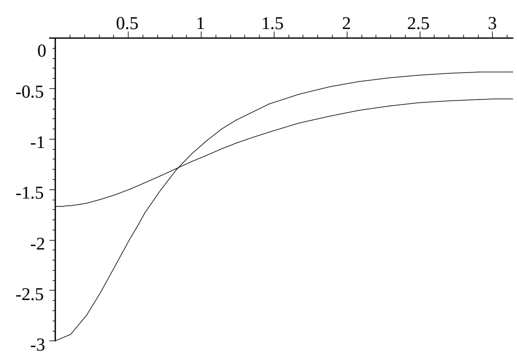
$$\sigma = -\varepsilon_0\frac{\partial}{\partial r}\phi|_{r=b} = -\frac{\tau}{4\pi}\frac{\partial}{\partial r}\ln\left[\frac{r^2+\frac{b^4}{R^2}-2r\frac{b^2}{R}\cos\phi}{r^2+R^2-2rR\cos\phi}\right]_{r=b}$$

$$\sigma = \frac{\tau}{2\pi b}\left[\frac{1-y^2}{y^2+1-2y\cos\phi}\right]$$

where $y = R/b$. Plotting $\sigma/\left(\frac{\tau}{2\pi b}\right) = \left[\frac{1-y^2}{y^2+1-2y\cos\phi}\right]$, for $y = 2,4,$ gives

$$g(y) = \frac{1-y^2}{y^2+1-2y\cos\phi}$$

$g(2), g(4) = -\frac{3}{5-4\cos\phi}, -\frac{15}{17-8\cos\phi}$



d) If the line charges are a distance d apart, then the electric field at τ from τ' is, using Gauss's law

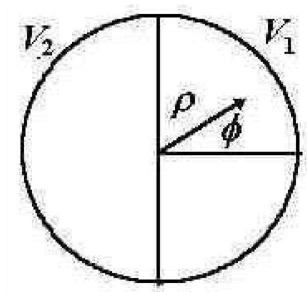
$$E = \frac{\tau'}{2\pi\varepsilon_0 d}$$

The force on τ is τLE, ie,

$$F = \frac{\tau\tau' L}{2\pi\varepsilon_0 d} = -\frac{\tau^2 L}{2\pi\varepsilon_0 d},$$ and the force is attractive.

2.13

The system is pictured in the following figure:



a) Notice from the figure, $\Phi(\rho,-\phi) = \Phi(\rho,\phi);$ thus from Eq. (2.71) in the text,

$$\Phi(\rho,\phi) = a_0 + \sum_{n=1}^{\beta} a_n \rho^n \cos(n\phi)$$

$$\int_{-\pi/2}^{3\pi/2} \Phi(b,\phi) = 2\pi a_0 = \pi V_1 + \pi V_2 \rightarrow a_0 = \frac{V_1 + V_2}{2}$$

Using

$$\int_{-\pi/2}^{3\pi/2} \cos m\phi \cos n\phi d\phi = \delta_{nm}\pi$$

Applying this to $\Phi,$ only odd terms m contribute in the sum and

$$a_m = \frac{2(V_1 - V_2)}{\pi m b^m}(-1)^{\frac{m-1}{2}}$$

Thus

$$\Phi(\rho,\phi) = \frac{V_1 + V_2}{2} + \frac{2(V_1 - V_2)}{\pi} \operatorname{Im} \sum_{m \text{ odd}} \frac{i^m \rho^m e^{im\phi}}{m b^m}$$

Using

$$2\sum_{m \text{ odd}} \frac{x^m}{m} = \ln\left[\frac{(1+x)}{(1-x)}\right]$$

and

$$\operatorname{Im} \ln(A + iB) = \tan^{-1}(B/A)$$

we get

$$\Phi(\rho,\phi) = \frac{V_1 + V_2}{2} + \frac{(V_1 - V_2)}{\pi} \tan^{-1}\left(\frac{2\frac{\rho}{b}\cos\phi}{\left(1 - \frac{\rho^2}{b^2}\right)}\right)$$

as desired.
b)

$$\sigma = -\varepsilon_0 \frac{\partial}{\partial \rho} \Phi(\rho,\phi)|_{\rho=b}$$

$$\sigma = -\varepsilon_0 \frac{(V_1 - V_2)}{\pi} \frac{\partial}{\partial \rho} \tan^{-1}\left(\frac{2\frac{\rho}{b}\cos\phi}{\left(1 - \frac{\rho^2}{b^2}\right)} \right)_{\rho=b}$$

$$\sigma = -2\varepsilon_0 \frac{V_1 - V_2}{\pi} b(\cos\phi) \frac{b^2 + \rho^2}{b^4 - 2b^2\rho^2 + \rho^4 + 4\rho^2 b^2 \cos^2\phi} |_{\rho=b} = -\varepsilon_0 \frac{V_1 - V_2}{\pi b \cos\phi}$$

:

2.22

a) Using the fact that for the interior problem, the normal derivative is outward, rather than inward, we have the potenial given by the negative of Eq. (2.21), which takes the form, when $\theta = 0$ and $\gamma = \theta'$

$$\Phi(z) = -\frac{Va\left(z^2-a^2\right)2\pi}{4\pi}\int_0^1\left(\frac{1}{(a^2+z^2-2azx)^{3/2}}-\frac{1}{(a^2+z^2+2azx)^{3/2}}\right)dx$$

where I've replaced $\cos\theta'$ by x in the integral. The integral yields

$$\Phi(z) = \frac{Va}{z}\left(1-\frac{(a^2-z^2)}{a\sqrt{a^2+z^2}}\right)$$

$$\Phi(z) = \frac{Va}{z}\left[\frac{3}{2}\frac{z^2}{a^2}+\left(-\frac{7}{8}\right)\frac{z^4}{a^4}+O\left(z^6\right)\right]$$

which agrees with Eq. (2.27) if $\cos\theta = 1$.

b) For $z > a$, we have, using Eq. (2.22)

$$E_z = -\frac{\partial}{\partial z}V\left(1-\frac{(z^2-a^2)}{z\sqrt{a^2+z^2}}\right) = E_z(z) = \frac{Va^2}{(a^2+z^2)^{\frac{3}{2}}}\left(3+\frac{a^2}{z^2}\right)$$

For $|z| < a$,

$$E_z(z) = -\frac{\partial}{\partial z}\frac{Va}{z}\left(1-\frac{(a^2-z^2)}{a\sqrt{a^2+z^2}}\right) = E_z(z) = -\frac{V}{a}\left(-\frac{a^2}{z^2}+\frac{3a^3+a^5/z^2}{(a^2+z^2)^{\frac{3}{2}}}\right)$$

in agreement with the book. Expanding the second form in a Taylor series exapansion about $z = 0$ gives

$$E_z = -\frac{3}{2a}V + \frac{21}{8}\frac{V}{a^3}z^2 - \frac{55}{16}\frac{V}{a^5}z^4 + O\left(z^6\right)$$

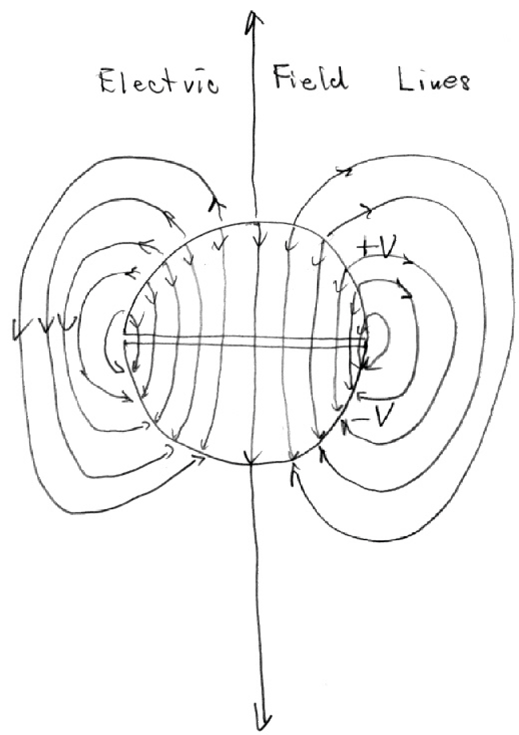
which shows $E_z(0) = -\frac{3}{2a}V$, as required. Also, from the second form

$$E_z(a) = -\frac{V}{a}\left(-1+\sqrt{2}\right)$$

From the first form, on the outside, we get

$$E_z(a) = \frac{\sqrt{2}V}{a}$$

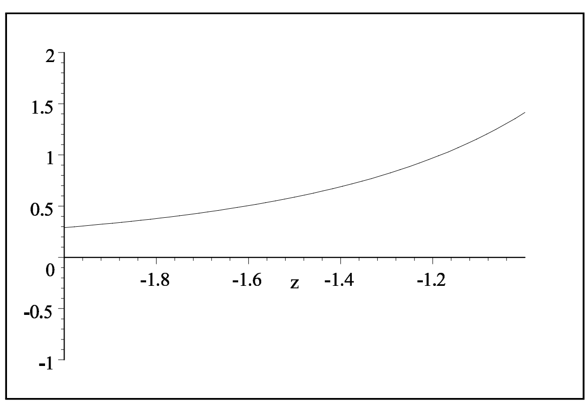
c) First look at a plot of the field lines:



Next, look at $E(z)$ in the region (-$2a$, $2a$). I will make the plot in units of a.

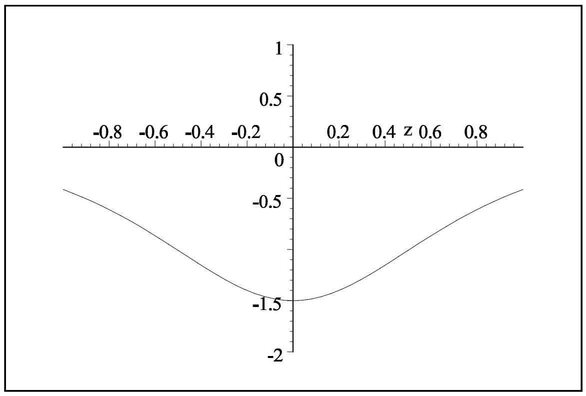
$$E(z) = \frac{1}{(1+z^2)^{\frac{3}{2}}}\left(3+\frac{1}{z^2}\right)$$

$E(z)$



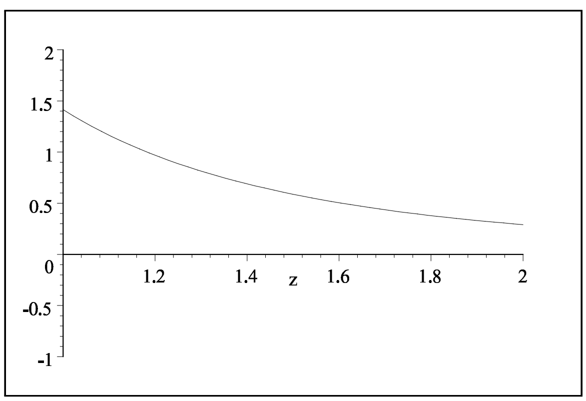
$$E(z) = -\left(-\frac{1}{z^2} + \frac{3 + 1/z^2}{(1+z^2)^{\frac{3}{2}}}\right)$$

$E(z)$



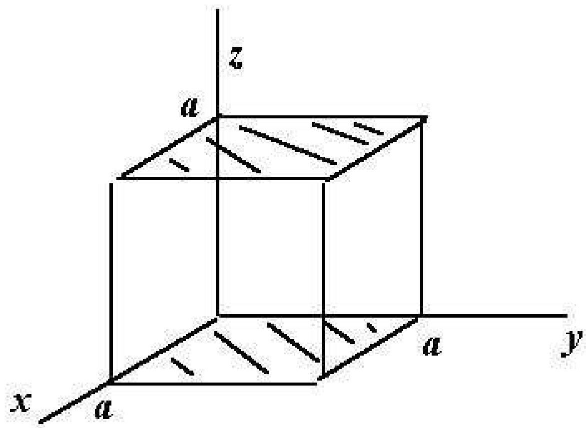
$$E(z) = \frac{1}{(1+z^2)^{\frac{3}{2}}}\left(3 + \frac{1}{z^2}\right)$$

$E(z)$



2.23

The system is pictured in the following figure:



a) As suggested in the text and in class, we will superpose solutions of the form (2.56) for the two sides with $V(x,y,z) = V$.

1) First consider the side $V(x,y,a) = z$:

$$\Phi_1(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z)$$

with $\alpha_n = \frac{n\pi}{a},\ \beta_m = \frac{m\pi}{a},\ \gamma_{nm} = \frac{\pi}{a}\sqrt{n^2+m^2}$. Projecting out A_{nm} using the orthogonality of the sine functions,

$$A_{nm} = \frac{16V}{\sinh(\gamma_{nm} a) nm\pi^2}$$

where both n, and m are odd. (Later we will use $n = 2p+1,\ m = 2q+1$)

2) In order to express $\Phi_2(x,y,z)$ in a form like the above, we make the coordinate transformation

$$x' = y,\ y' = x,\ z' = -z+a$$

So

$$\Phi_2(x,y,z) = \Phi_1(x',y',z') = \Phi_1(y,x,= z+a)$$

$$\Phi(x,y,z) = \Phi_1(x,y,z) + \Phi_2(x,y,z)$$

b)

$$\Phi(\frac{a}{2},\frac{a}{2},\frac{a}{2}) = \frac{16 \cdot V}{\pi^2} \sum_{p,q=0}^{\infty} \frac{(-1)^{p+q}}{(2p+1)(2q+1)\cosh\left(\gamma_{nm}\frac{a}{2}\right)}$$

where I have used the identity

$$\sinh(\gamma_{nm} a) = 2\sinh(\frac{\gamma_{nm} a}{2})\cosh(\frac{\gamma_{nm} a}{2})$$

Let $f(p,q) \equiv \sum_{p,q=0}^{\infty} \frac{(-1)^{p+q}}{(2p+1)(2q+1)\cosh\left(\sqrt{(2p+1)^2+(2q+1)^2}\,\frac{\pi}{2}\right)}$

(p,q)	$f(p,q)$	Error	Sum
0,0	0.213484	4.4%	.214384
1,0	–0.004641	2.13%	0.20974
0,1	–0.004641	0.013%	0.20510
1,1	0.0002835	0.015%	0.20539

The first three terms give an accuracy of 3 significant figures.

$$\Phi\left(\frac{a}{2},\frac{a}{2},\frac{a}{2}\right) = \frac{16\cdot 0.20539}{\pi^2}V = 0.33296V$$

$$\Phi_{av}\left(\frac{a}{2},\frac{a}{2},\frac{a}{2}\right) = \frac{2}{6}V = 0.333\ldots V$$

c)

$$\sigma(x,y,a) = -\varepsilon_0\frac{\partial}{\partial z}\Phi|_{z=a}$$

$$\sigma(x,y,a) = -\frac{16\varepsilon_0}{\pi^2}V$$

$$= -\frac{16\varepsilon_0}{\pi^2}V\sum_{n,m\text{ odd}}^{\infty}\sin(\alpha_n x)\sin(\beta_m y)\left[\frac{(\cosh(\gamma_{nm}a)-1}{\sinh(\gamma_{nm}a)}\right]$$

$$\sigma(x,y,a) = -\frac{16\varepsilon_0}{\pi^2}V\sum_{n,m\text{ odd}}^{\infty}\sin(\alpha_n x)\sin(\beta_m y)\tanh\left(\frac{\gamma_{nm}a}{2}\right)$$

More Problems for Chapter 2

Problem 2.2

(a) Let the point charge q at $\vec{r}_o$ and the image charge q_i at $\vec{r}_i$. The potential for a point $\vec{r}$ is

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}_o|} + \frac{1}{4\pi\epsilon_0}\frac{q_i}{|\vec{r}-\vec{r}_i|}$$

On the surface of the sphere, the potential is zero every where, *i.e.*, $\Phi(r=a)=0$. This is only possible if

$$q_i = -q\frac{a}{r_o} \qquad \text{and} \qquad r_i = \frac{a^2}{r_o}.$$

The potential inside the sphere is therefore

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}_o|} - \frac{1}{4\pi\epsilon_0}\frac{a}{r_o}\frac{q}{|\vec{r}-a^2\vec{r}_o/r_o^2|}$$

or

$$\Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\{\frac{1}{\sqrt{r^2+r_o^2-2rr_o\cos\beta}} - \frac{a}{\sqrt{r^2r_o^2+a^4-2a^2rr_o\cos\beta}}\}$$

where β is the angle between $\vec{r}$ and $\vec{r}_o$.

(b) The induced inner surface charge density

$$\sigma = \epsilon_0\frac{\partial\Phi}{\partial r}|_{r=a} = \frac{q}{4\pi a^2}\frac{a}{r_o}\frac{1-\frac{a^2}{r_o^2}}{(1+\frac{a^2}{r_o^2}-2\frac{a}{r_o}\cos\beta)^{3/2}} = -\frac{q}{4\pi a^2}\frac{a(a^2-r_o^2)}{(a^2+r_o^2-2ar_o\cos\beta)^{3/2}}$$

Note the total charge on the inner surface is

$$\oint \sigma da = -q \qquad (a > r_o)$$

There is no charge on the outer surface.

(c) The force on charge q

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q\cdot q_i}{(r_o-r_i)^2}\frac{\vec{r}_o-\vec{r}_i}{|\vec{r}_o-\vec{r}_i|} = \frac{q^2}{4\pi\epsilon_0}\frac{ar_o}{(a^2-r_o^2)^2}\frac{\vec{r}_o}{r_o}$$

(d) If the sphere is kept at a fixed potential V:

(a) The potential inside is raised by a constant V

(b) No change to the inner surface charge density, but there will be uniformly distributed charge on the outer surface.

(c) No change to the force.

If the sphere has a total charge Q:

(a) The potential inside is raised by a constant equal to the potential of the sphere.

(b) No change to the inner surface charge density, but there will be $Q+q$ uniformly distributed on the outer surface.

(c) No change to the force.

Problem 2.4
(a) The force on the point charge q is

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}[1 - \frac{R^3(2d^2 - R^2)}{d(d^2 - R^2)^2}]\frac{\vec{d}}{d}$$

At the point where $F = 0$, the force changes from repulsive to attractive. Solving

$$F = 0 = \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}[1 - \frac{R^3(2d^2 - R^2)}{d(d^2 - R^2)^2}],$$

one gets (with the help of the `Mathematica`)

$$d = \frac{1}{2}(1 + \sqrt{5})R$$

Therefore at a distance smaller than $d - R = \frac{1}{2}(\sqrt{5} - 1)R \approx 0.618R$ from the surface, the charge q is attracted to the sphere.
(b) When the charge is close to the surface $d = R + a \sim R$. The force

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}[1 - \frac{R^3(2d^2 - R^2)}{d(d^2 - R^2)^2}] \approx \frac{1}{4\pi\epsilon_0}\frac{q^2}{R^2}[1 - \frac{R^5}{R((R+a)^2 - R^2)^2}] \approx -\frac{1}{16\pi\epsilon_0}\frac{q^2}{a^2}$$

(c) The limiting force on the point charge q is determined by itself and its image charge and therefore is independent of the charge on the conductor. However, the location where the force changes between repulsive and attractive is a function of the charge on the sphere.
For $Q = 2q$,

$$F = 0 = \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}[2 - \frac{R^3(2d^2 - R^2)}{d(d^2 - R^2)^2}]$$

Using for example the `Mathematica` program to solve the above equation, one gets $d = 1.4276R$, *i.e.*, the switch point is a distance $0.4276R$ away from the surface.
For $Q = \frac{1}{2}q$,

$$F = 0 = \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}[1/2 - \frac{R^3(2d^2 - R^2)}{d(d^2 - R^2)^2}]$$

Again, with the help of programs like `Mathematica`, one gets $d = 1.8823R$. The switch over is a distance $0.8823R$ away from the surface.

Problem 2.5
(a) The electric force on the charge q is

$$\vec{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}\frac{(a/r)^3}{(1 - a^2/r^2)^2}\frac{\vec{r}}{r}$$

The work done to the charge

$$W = -\int_r^\infty \vec{F}\cdot d\vec{\ell} = \int_r^\infty |F(r')|dr' = \frac{q^2 a}{4\pi\epsilon_0}\int_r^\infty \frac{dr'}{r'^3(1 - a^2/r'^2)^2} = \frac{q^2 a}{8\pi\epsilon_0(r^2 - a^2)}$$

Alternatively, we could consider calculating W from the energy conservation:

$$U_q + U_{q'} + U_{qq'} + W = U_q \quad \Rightarrow \quad W = -U_{q'} - U_{qq'}$$

where U_q and $U_{q'}$ are self energies of the point charge q and the induced charge on the sphere and $U_{qq'}$ is the interaction energy between q and the induced charge at r. The problem with this approach is that $U_{q'}$ is difficult to calculate.

(b) The work done to the charge

$$W = -\int_r^\infty \vec{F}\cdot d\vec{\ell} = -\frac{q}{4\pi\epsilon_0}[\int_r^\infty \frac{Q}{r'^2}dr' - qa^3\int_r^\infty \frac{2r'^2-a^2}{r'(r'^2-a^2)^2}dr'] = \frac{q^2a}{8\pi\epsilon_0(r^2-a^2)} - \frac{1}{4\pi\epsilon_0}[\frac{q^2a}{2r^2}+\frac{qQ}{r}]$$

On the approach of energy conservation, see the discussion above.

Problem 2.9

(a) The surface charge density on the sphere is (Eq. 2.15)

$$\sigma = 3\epsilon_0 E_0 \cos\theta$$

The electrostatic pressure

$$\vec{P} = \frac{\sigma^2}{2\epsilon_0}\vec{n}$$

The total electrostatic force on one of the hemisphere

$$F = \int P_z da = a^2\int_0^{\pi/2}\sin\theta d\theta\int_0^{2\pi}d\phi\frac{\sigma^2}{2\epsilon_0}\cos\theta = \frac{9}{4}\pi a^2\epsilon_0 E_0^2$$

(b) The surface charge density on the sphere is

$$\sigma = 3\epsilon_0 E_0\cos\theta + \frac{Q}{4\pi a^2}$$

The total force

$$F = \int P_z da = a^2\int_0^{\pi/2}\sin\theta d\theta\int_0^{2\pi}d\phi\frac{\sigma^2}{2\epsilon_0}\cos\theta = \frac{9}{4}\pi a^2\epsilon_0 E_0^2 + \frac{1}{2}E_0 Q + \frac{1}{32\pi\epsilon_0}\frac{Q^2}{a^2}$$

More Problems for Chapter 2

Problem 2.10
The problem is very similar to a grounded conducting sphere in a uniform electric field E_0, the case discussed in Section 2.5. The region between the two plates is identical to one of the half spaces of the above mentioned problem, where the half spaces are defined by a plane perpendicular to the field and cut the sphere into two hemispheres.
(a) In a spherical coordinate system centered at the origin of the boss with $+z$ pointing to the other conductor, the potential between the two plates is given by Eq. (2.14):

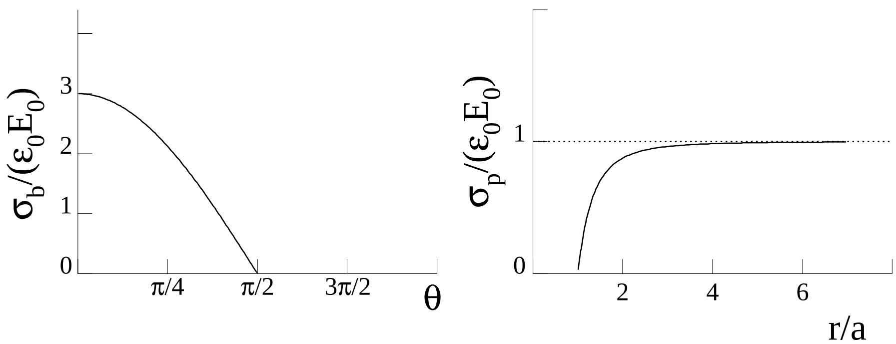
$$\Phi(r,\theta) = -E_0(r - \frac{a^3}{r^2})\cos\theta$$

The surface charge density on the boss:

$$\sigma_b(a,\theta) = \epsilon_0 E_r|_{r=a} = -\epsilon_0 \frac{\partial\Phi}{\partial r}|_{r=a} = 3\epsilon_0 E_0 \cos\theta \qquad 0 < \theta < \frac{\pi}{2}$$

The density on the plane:

$$\sigma_p(r,\theta=\frac{\pi}{2}) = \epsilon_0 E_\perp|_{\theta=\pi/2} = -\epsilon_0 E_\theta|_{\theta=\pi/2} = \epsilon_0 \frac{1}{r}\frac{\partial\Phi}{\partial\theta}|_{\theta=\pi/2} = \epsilon_0 E_0(1 - \frac{a^3}{r^3})$$



(b) The total charge on the boss

$$Q_b = \int \sigma_b da = 3\epsilon_0 E_0 a^2 \int_0^{\pi/2} \cos\theta \sin\theta d\theta \int_0^{2\pi} d\phi = 3\pi\epsilon_0 E_0 a^2$$

(c) The boundary condition $\Phi = 0$ on the plane and the boss can be meet by placing three image charges, one $-qa/d$ at $z = a^2/d$, one $+qa/d$ at $z = -a^2/d$ and the third one $-q$ at $z = -d$. The induced charge distribution on the boss is the sum of those induced by charge pairs $(q, -qa/d)$ and $(-q, qa/d)$:

$$\sigma(\theta) = \sigma_1(\theta) + \sigma_2(\theta)$$

where σ_1 and σ_2 are given by Eq. (2.5):

$$\sigma_1(\theta) = -\frac{q}{4\pi a^2}\frac{a(d^2-a^2)}{(a^2+d^2-2ad\cos\theta)^{3/2}} \quad \text{where} \quad 0 < \theta < \frac{\pi}{2} \quad \text{due to charges} \quad (q, -qa/d)$$

$$\sigma_2(\theta) = +\frac{q}{4\pi a^2}\frac{a(d^2-a^2)}{(a^2+d^2-2ad\cos\theta)^{3/2}} \quad \text{where} \quad \frac{\pi}{2} < \theta < \pi \quad \text{due to charges} \quad (-q, qa/d)$$

The total induced charge on the boss

$$q' = \int(\sigma_1+\sigma_2)da = -\frac{q}{4\pi a^2}a(d^2-a^2)2\pi a^2\{\int_0^{\pi/2}d\theta - \int_{\pi/2}^{\pi}d\theta\}\frac{\sin\theta}{(a^2+d^2-2ad\cos\theta)^{3/2}} = -q(1-\frac{d^2-a^2}{d\sqrt{d^2+a^2}})$$

Problem 2.11

The electric field and potential due to a line charge are

$$\vec{E}(\rho,\phi) = \frac{\tau}{2\pi\epsilon_0}\frac{1}{\rho}\frac{\vec{\rho}}{\rho}, \quad \Phi(\rho,\phi) = C - \frac{\tau}{2\pi\epsilon_0}\ln\rho$$

where C, τ and ρ are, respectively, a constant, the line charge density, and the distance to the line charge. Note that the potential does not depend on z (along the axis).

(a) The image charge has to be in the plane formed by the cylinder axis and the line charge. Choose a cylindrical coordinate system with the origin at the axis of the cylinder and the direction from the origin to the line charge as the x axis. In this case, the line charge is at $x = R$. Let τ' and R' be the image line charge density and the polar distance to the axis, the potential at a point (ρ,ϕ) is given by

$$\Phi(\rho,\phi) = \Phi_0 - \frac{\tau}{2\pi\epsilon_0}\ln r - \frac{\tau'}{2\pi\epsilon_0}\ln r'$$

Φ_0 is another constant. r and r' are distances from the point to the line charge τ and the image charge τ':

$$r = \sqrt{\rho^2+R^2-2\rho R\cos\phi}; \qquad r' = \sqrt{\rho^2+R'^2-2\rho R'\cos\phi}$$

Therefore

$$\Phi(\rho,\phi) = \Phi_0 - \frac{1}{4\pi\epsilon_0}\{\tau\ln(\rho^2+R^2-2R\rho\cos\phi) + \tau'\ln(\rho^2+R'^2-2R'\rho\cos\phi)\}$$

At the surface of the cylinder ($\rho = b$):

$$\Phi(b,\phi) = \Phi_0 - \frac{1}{4\pi\epsilon_0}\{\tau\ln(b^2+R^2-2Rb\cos\phi) + \tau'\ln(b^2+R'^2-2R'b\cos\phi)\} = \text{Constant}$$

independent of ϕ. This is only possible if

$$\begin{aligned} \tau' &= -\tau \\ b^2+R^2 &= A(b^2+R'^2) \\ bR &= A(bR') \end{aligned}$$

A is another constant. It is easy to determine R' from the above equation to be $R' = b^2/R$. Therefore, the line image charge is at $x = +b^2/R$ with a line charge density $-\tau$.

(b) Potential at any point (ρ,ϕ):

$$\Phi(\rho,\phi) = \Phi_0 - \frac{\tau}{4\pi\epsilon_0}\ln\{\frac{\rho^2+R^2-2R\rho\cos\phi}{\rho^2+(b^2/R)^2-2(b^2/R)\rho\cos\phi}\}$$

As $\rho\to\infty$, $\Phi(\rho,\phi)\to\Phi_0$. Since the potential vanishes at infinity, one gets Φ_0. The potential at a point (ρ,ϕ):

$$\Phi(\rho,\phi) = -\frac{\tau}{4\pi\epsilon_0}\ln\{\frac{\rho^2+R^2-2R\rho\cos\phi}{\rho^2+(b^2/R)^2-2(b^2/R)\rho\cos\phi}\}$$

Note that the potential on the cylinder surface is

$$\Phi(b,\phi) = -\frac{\tau}{2\pi\epsilon_0}\ln\{\frac{R}{b}\}$$

For a point far away from the cylinder,

$$\frac{\rho^2 + R^2 - 2R\rho\cos\phi}{\rho^2 + (b^2/R)^2 - 2(b^2/R)\rho\cos\phi} \approx \frac{1 - 2(R/\rho)\cos\phi}{1 - 2(b^2/R\rho)\cos\phi} \approx 1 + 2\frac{b^2 - R^2}{R\rho}\cos\phi$$

Therefore the asymptotic form of the potential far from the cylinder:

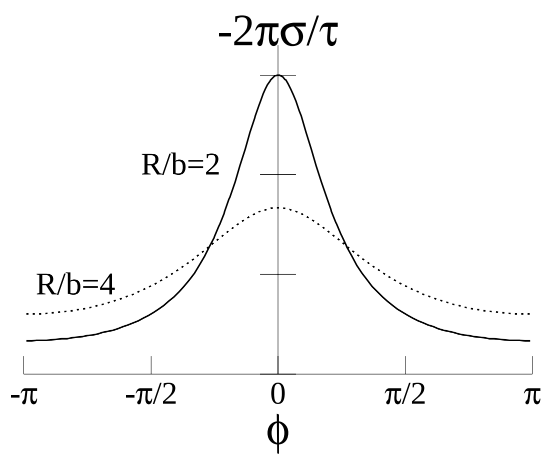
$$\Phi(\rho,\phi) \approx -\frac{\tau}{4\pi\epsilon_0}\ln\{1 + 2\frac{b^2 - R^2}{R\rho}\cos\phi\} \approx -\frac{\tau}{2\pi\epsilon_0}\frac{b^2 - R^2}{R\rho}\cos\phi$$

(c) Since the cylinder has a constant potential, the potential inside the cylinder must be the same constant (see the next problem for proof). Therefore, the surface charge density on the cylinder

$$\sigma(\phi) = \epsilon_0 E_\rho|_{\rho=b} = -\epsilon_0\frac{\partial\Phi}{\partial\rho}|_{\rho=b} = -\frac{\tau}{2\pi}\frac{R^2 - b^2}{b(b^2 + R^2 - 2bR\cos\phi)}$$

Note that the total surface charge per unit length

$$\tau'' = \int_0^{2\pi}\sigma(\phi)bd\phi = -\tau$$



(d) The force on the charge per unit length:

$$\vec{F} = \tau\vec{E} = -\frac{\tau^2}{2\pi\epsilon_0}\frac{1}{R - b^2/R}\frac{\vec{\rho}}{\rho} = -\frac{\tau^2}{2\pi\epsilon_0}\frac{R}{R^2 - b^2}\frac{\vec{\rho}}{\rho}$$

where $\vec{E}$ is the electric field at the line charge due to the image charge. The force is attractive.

The Green function for the Dirichlet condition

It is interesting to derive the Green function for the Dirichlet condition for the case of cylinder. The Dirichlet condition $\Phi(b,\phi) = 0$ leads to $\Phi_0 = \frac{\tau}{4\pi\epsilon_0}\ln(R^2/b^2)$. The potential at (ρ,ϕ) due to the line charge τ at (ρ',ϕ') with the Dirichlet boundary condition is:

$$\Phi(\rho,\phi) = \frac{\tau}{4\pi\epsilon_0}\ln\{\frac{\rho'^2[\rho^2 + (b^2/\rho')^2 - 2(b^2/\rho')\rho\cos(\phi' - \phi)]}{b^2[\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi' - \phi)]}\}$$

The Green function is the potential at (ρ,ϕ) due to a line charge with density $\tau = 4\pi\epsilon_0$ at (ρ',ϕ'):

$$G_D(\rho,\phi,\rho',\phi') = \ln\{\frac{\rho^2\rho'^2 + b^4 - 2b^2\rho\rho'\cos(\phi' - \phi)}{b^2[\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi' - \phi)]}\}$$

Problem 2.13
Integrals and identities useful for the problem:

$$\int_0^{2\pi} \frac{d\phi}{a^2+b^2-2ab\cos\phi} = \frac{2\pi}{|a-b|(a+b)}$$

$$\int \frac{\cos\phi d\phi}{a^2-b^2\cos^2\phi} = \frac{\tan^{-1}(b\sin\phi/\sqrt{a^2-b^2})}{b\sqrt{a^2-b^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

(a) The cylinder can be viewed as a superposition of two cylinders with the following potentials: We can regard this problem as a superposition of two problems:

$$\Phi_1(b,\phi) = (V_1+V_2)/2$$

$$\Phi_2(b,-\frac{\pi}{2}<\phi<\frac{\pi}{2}) = \frac{V_1-V_2}{2} \qquad \Phi_2(b,\frac{\pi}{2}<\phi<\frac{3\pi}{2}) = \frac{V_2-V_1}{2}$$

The potential inside the first cylinder (see problem 2.12):

$$\Phi_1(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi}\frac{V_1+V_2}{2}\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi' = \frac{V_1+V_2}{2}$$

The potential inside the second cylinder:

$$\Phi_2(\rho,\phi) = \Phi_{2a}(\rho,\phi)+\Phi_{2b}(\rho,\phi)$$

where

$$\Phi_{2a}(\rho,\phi) = \frac{V_1-V_2}{4\pi}\int_{-\pi/2}^{\pi/2}\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi'$$

$$\Phi_{2b}(\rho,\phi) = -\frac{V_1-V_2}{4\pi}\int_{\pi/2}^{3\pi/2}\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi'$$

Let $\phi''=\phi'-\pi$, then

$$\Phi_{2b}(\rho,\phi) = -\frac{V_1-V_2}{4\pi}\int_{-\pi/2}^{\pi/2}\frac{b^2-\rho^2}{b^2+\rho^2+2b\rho\cos(\phi''-\phi)}d\phi''$$

Combining Φ_{2a} and Φ_{2b}:

$$\Phi_2(\rho,\phi) = \frac{V_1-V_2}{4\pi}(b^2-\rho^2)\int_{-\pi/2}^{\pi/2}\frac{4b\rho\cos(\phi'-\phi)}{(b^2+\rho^2)^2-4b^2\rho^2\cos^2(\phi'-\phi)}d\phi'$$

$$= \frac{(V_1-V_2)}{\pi}b\rho(b^2-\rho^2)\frac{1}{2b\rho(b^2-\rho^2)}\tan^{-1}\{\frac{2b\rho}{b^2-\rho^2}\sin(\phi'-\phi)\}|_{\phi'=-\pi/2}^{\phi'=\pi/2}$$

$$= \frac{V_1-V_2}{\pi}\tan^{-1}(\frac{2b\rho}{b^2-\rho^2}\cos\phi)$$

Thus the potential inside the original cylinder

$$\Phi(\rho,\phi) = \Phi_1(\rho,\phi) + \Phi_2(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\{\frac{2b\rho}{b^2-\rho^2}\cos\phi\}$$

(b) Let σ be the surface charge density, the electric fields just inside and outside the surface are given by Gauss's law:

$$\vec{E}_{\text{in}} = -\frac{\sigma}{2\epsilon_0}\frac{\vec{\rho}}{\rho}, \qquad \vec{E}_{\text{out}} = \frac{\sigma}{2\epsilon_0}\frac{\vec{\rho}}{\rho}$$

Therefore,

$$\sigma = -2\epsilon_0\vec{E}_{\text{in}}\cdot\frac{\vec{\rho}}{\rho}|_{\rho=b} = -2\epsilon_0 E_\rho|_{\rho=b} = 2\epsilon_0\frac{\partial\Phi}{\partial\rho}|_{\rho=b} = \frac{2\epsilon_0(V_1-V_2)}{\pi b\cos\phi}$$

Problem 2.21

First of all, the Poisson integral solution is the solution of Problem 2.12, *i.e.*, the potential inside a cylinder:

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi'$$

Cauchy's theorem is expressed using complex variable z where $z=\rho e^{i\phi}$. Let z be inside the curve C and $F(z)=\Phi(z)$, we then have

$$\Phi(z) = \frac{1}{2\pi i}\oint\frac{\Phi(z')}{z'-z}dz'$$

Exploiting the hint, the image point of z is $(b^2/|z|)e^{i\phi} = b^2/z^*$ (here z^* is the complex conjugate of z) which lies outside the curve C. Therefore

$$\frac{1}{2\pi i}\oint\frac{\Phi(z')}{z'-b^2/z^*}dz' = 0$$

Subtract this zero integral from the $\Phi(z)$:

$$\Phi(z) = \frac{1}{2\pi i}\oint\Phi(z')\{\frac{1}{z'-z} - \frac{1}{z'-b^2/z^*}\}dz'$$

For the Poisson integral problem, the curve C is circle of radius b, z' is on the curve, *i.e.*, $z'z'^* = b^2$. Using the identity

$$z' - \frac{b^2}{z^*} = z' - z'z'^*/z^* = \frac{z'}{z^*}(z^*-z'^*)$$

one gets

$$\frac{1}{z'-z} - \frac{1}{z'-b^2/z^*} = \frac{1}{z'-z} + \frac{z^*/z'}{z'^*-z^*} = \frac{z'^*-z^*+z^*-zz^*/z'}{|z'-z|^2} = \frac{1}{z'}\frac{z'z^*-zz^*}{|z'-z|^2}$$

Plugging in to the potential $\Phi(z)$:

$$\Phi(z) = \frac{1}{2\pi i}\oint\Phi(z')z'\frac{z'z'^*-zz^*}{|z'-z|^2}\frac{dz'}{z'}$$

In polar coordinate system, $z=\rho e^{i\phi}$, $z'=be^{i\phi'}$, therefore

$$zz^* = \rho^2, \qquad z'z'^* = b^2, \qquad \frac{dz'}{z'} = id\phi'$$

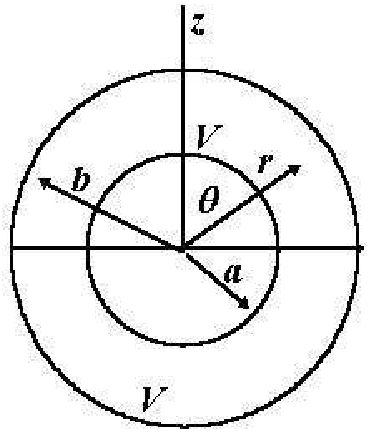
$$|z'-z|^2 = \rho^2+b^2-2b\rho\cos(\phi'-\phi)$$

The potential inside the curve in polar coordinates

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi'$$

3.1

The system is pictured in the following figure:



$\int_0^1 P_l(x)dx$

The problem is symmetric around the z axis so

$$\phi(r,\theta) = \sum_l (A_l r^l + B_l r^{-l-1}) P_l(\cos\theta)$$

The A_l and B_l are determined by the conditions

1)

$$\int_{-1}^{1} \phi(a,x) P_l(x) dx = \frac{2}{2l+1}(A_l a^l + B_l a^{-l-1})$$

2)

$$\int_{-1}^{1} \phi(b,x) P_l(x) dx = \frac{2}{2l+1}(A_l b^l + B_l b^{-l-1})$$

Solving these two equations gives

$$A_l = \frac{2l+1}{2(a^{2l+1} - b^{2l+1})}\left[a^{l+1} \int_{-1}^{1} \phi(a,x) P_l(x) dx - b^{l+1} \int_{-1}^{1} \phi(b,x) P_l(x) dx \right]$$

$$B_l = a^{l+1} \frac{2l+1}{2} \int_{-1}^{1} \phi(a,x) P_l(x) dx - A_l a^{2l+1}$$

Using

$$\int_{-1}^{1} \phi(a,x) P_l(x) dx = V \int_0^1 P_l(x) dx$$

$$\int_{-1}^{1} \phi(b,x) P_l(x) dx = V \int_{-1}^{0} P_l(x) dx = V(-1)^l \int_0^1 P_l(x) dx$$

So

$$A_l = \frac{2l+1}{2(a^{2l+1} - b^{2l+1})} V[a^{l+1} - b^{l+1}(-1)^l] \int_0^1 P_l(x) dx$$

$$B_l = a^{l+1} \frac{2l+1}{2} V \int_0^1 P_l(x) dx - A_l a^{2l+1}$$

Note that

$$\int_0^1 P_l(x)dx = \frac{1}{2}\int_{-1}^1 P_l(x)dx$$

for l even. For even $l \sum 0$, $\int_0^1 P_l(x)dx = 0$. Thus we have

$$\int_0^1 P_0(x)dx = 1;\ \int_0^1 P_1(x)dx = \frac{1}{2},\ \int_0^1 P_3(x)dx = -\frac{1}{8}$$

and

$$A_0 = \frac{V}{2},\ A_1 = \frac{3}{4(a^3-b^3)}V(a^2+b^2),\ A_3 = -\frac{7}{16(a^7-b^7)}V(a^4+b^4)$$

$$B_0 = \frac{1}{2}Va - \frac{1}{2}Va = 0$$

$$B_1 = \frac{3}{4}a^2V - \frac{3}{4(a^3-b^3)}V(a^2+b^2)a^3 = \frac{3}{4}Va^2b^2\frac{b+a}{-a^3+b^3}$$

$$B_3 = -\frac{7}{16}a^4V - a^7\left(-\frac{7}{16(a^7-b^7)}V(a^4+b^4)\right) = -\frac{7}{16}Va^4b^4\frac{b^3+a^3}{-a^7+b^7}$$

As $b \to \infty$, only the B_l terms (and A_0) survive. Thus using the general expression for $\int_0^1 P_l(x)dx$ given by (3.26)

$$\phi(r,\theta) = \frac{V}{2}\left[P_0(x) + \frac{3}{2}\frac{a^3}{r^2}P_1(x) - \frac{7}{8}\frac{a^4}{r^4}P_3(x) + \ldots\ldots\right]$$

Let's now solve the problem neglecting the outer sphere (since $b \to \infty$) using the Green's function result

(2.19) this integral give, for $\cos\theta = 1$

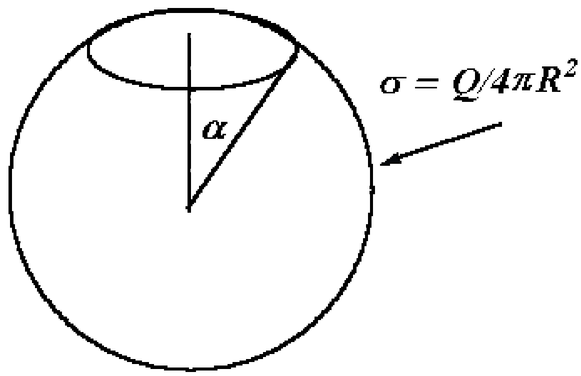
$$\phi(r,\theta) = \frac{V}{2}(1-\rho^2)\left[\frac{1}{(1-\rho)} - \frac{1}{\sqrt{(1+\rho^2)}}\right]$$

with $\rho = a/r$. Expanding the above,

$$\phi(r,\theta) = \frac{V}{2}\left[\frac{a}{r} + \frac{3}{2}\frac{a^2}{r^2} - \frac{7}{8}\frac{a^4}{r^4} + \ldots\ldots\right]$$

Comparing with our previous solution with $x = 1$, we see the Green's function solution differs by having a B_0 term and by not having an A_0 term. All the other higher power terms agree in the series. This difference is due to having a potential at ∞ in the original problem.

3.2 The charge distribution is shown by



a) We see the charge distribution is given by

$$\rho\,(\vec{r}) = N\theta(\cos\alpha - \cos\theta)\delta(r - a)$$

where N is determined by the requirement $\int d^3r\rho\,(\vec{r}) = Q$, or

$$\rho\,(\vec{r}) = \frac{Q}{4\pi R^2}\theta(\cos\alpha - \cos\theta)\delta(r - R)$$

Expanding $\theta(\cos\alpha - \cos\theta)$ in terms of Legendre polynomials,

$$\theta(\cos\alpha - \cos\theta) = \sum_l A_l P_l(\cos\theta)$$

or

$$A_l = \frac{2l+1}{2}\int_{-1}^{\cos\alpha} P_l(x)dx$$

Using Mathematica 4, I get

$$\int_{-1}^{\cos\alpha} P_l(x)dx = \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1}$$

Notice for $l = 0$ in the above, $P_{-1}(\cos\alpha) \equiv -1$.

$$A_l = \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2}$$

This problem has azimuthal symmetry, so we can write in general (when $r < R$)

$$\phi(\vec{r}) = \sum_l B_l r^l P_l(\cos\theta)$$

where now θ is the polar angle of the vector $\vec{r}$. Choosing $\vec{r} \,||\, \hat{z}$,

$$\phi(\vec{r} = r\hat{z}) = \sum_l B_l r^l P_l(1)$$

On the other hand we can write

$$\phi(\vec{r} = r\hat{z}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{4\pi R^2}\sum_l \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2} \int \frac{d\phi d\cos\theta r'^2 dr' P_l(\cos\theta)\delta(r' - R)}{|\vec{r} - \vec{r}'|}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{4\pi R^2} 2\pi \sum_{ll'} \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2} \frac{r^l R^2}{R^{l+1}} \int_{-1}^{1} dx P_l(x) P_{l'}(x)$$

Using $\int_{-1}^{1} dx P_l(x) P_{l'}(x) = \frac{2}{2l+1}\delta_{ll''}$

$$\phi(\vec{r} = r\hat{z}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2}\sum_l \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1}\frac{r^l}{R^{l+1}}$$

Then for general directions of $\vec{r}$,

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2}\sum_l \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1}\frac{r^l}{R^{l+1}} P_l(\cos\theta)$$

If $\vec{r}$ is on the outside, we know that R and r are interchanged in the expansion

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2}\sum_l \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1}\frac{R^l}{r^{l+1}} P_l(\cos\theta)$$

b) By symmetry, at the origin the electric field is along $\hat{z}$.

$$\vec{E}(0) = -\frac{\partial}{\partial z}\phi(0)\hat{z} =$$

$$= -\hat{z}\frac{\partial}{\partial z}\frac{1}{4\pi\varepsilon_0}\frac{Q}{2}\frac{(P_2(\cos\alpha) - P_0(\cos\alpha))}{3R^2} z + \text{ terms that vanish}$$

$$\vec{E}(0) = \frac{Q}{12\pi\varepsilon_0 R^2}\left(1 - P_2(\cos\alpha)\right)$$

c) Consider the case where α is very small. Using our general expression for $\phi(\vec{r})$, we see we need to expand $P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)$. (I will keep the leading terms.)

$$P_l(\cos\alpha) = \sum_n \frac{1}{n!}\frac{d^n}{dx^n} P_l(x)|_{x=1}(x-1)^n \approx P_l(1) + \frac{d}{dx}P_l(x)|_{x=1}(x-1)$$

$$((x-1) = \sqrt{1-\varepsilon^2} - 1) = -\frac{1}{2}\varepsilon^2 + O\left(\varepsilon^4\right)$$

where $\varepsilon = \sin\alpha$.

$$P_l(\cos\alpha) = 1 - \frac{1}{2}\frac{d}{dx}P_l(1)\varepsilon^2$$

$$P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha) = \frac{\varepsilon^2}{2}\left(\frac{d}{dx}P_{l-1}(1) - \frac{d}{dx}P_{l+1}(1)\right) = -\frac{\varepsilon^2\,(2l+1)\,P_l(1)}{2}$$

where I have used Eq. (3.28) and these formulas apply for $l > 0$.
So

$$\phi(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 R} - \frac{1}{4\pi\varepsilon_0}\frac{Q\varepsilon^2}{4}\sum_l \frac{r^l}{R^{l+1}}P_l(\cos\theta) = \frac{Q}{4\pi\varepsilon_0 R} - \frac{1}{4\pi\varepsilon_0}\frac{Q\varepsilon^2}{4\,|R\hat{z} - \vec{r}|}$$

That is, the potential is just that of a uniformly charged sphere plus a point charge $= -\,Q\frac{\text{(solid angle subtended by empty cap)}}{4\pi}$, located at the point $R\hat{z}$.
The electric field for this point charge is obviously given by

$$\vec{E}(\vec{r}) = \frac{-1}{4\pi\varepsilon_0}\frac{Q\varepsilon^2\,(\vec{r} - R\hat{z})}{4\,|R\hat{z} - \vec{r}|^3}$$

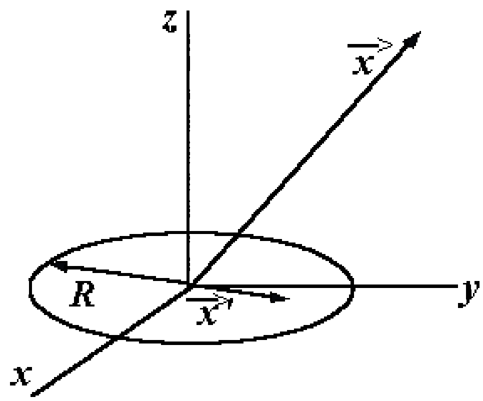
If the charge were located on a small cap at the bottom of the sphere, ie, if $\alpha \to \pi - \beta$, then clearly in analogy with what we have already done, we can see that it would act like a point charge $= Q\frac{\text{(solid angle subtended by cap)}}{4\pi}$
and located at the point $-R\hat{z}$. Then the potential is

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q\varepsilon^2}{4\,|R\hat{z} + \vec{r}|}$$

where now $\varepsilon = \sin\beta$.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q\varepsilon^2\,(\vec{r} + R\hat{z})}{4\,|R\hat{z} + \vec{r}|^3}$$

3.3 The system is described by



a) We will calculate the potential when the field point is along the $z-$ axis, then generalize to any $\vec{x}$.

$$\phi(\vec{x}=z\hat{z}) = \frac{1}{4\pi\varepsilon_0}2\pi\int_0^R \frac{\sigma\rho' d\rho'}{\sqrt{z^2+\rho'^2}} = \frac{C}{2\varepsilon_0}\int_0^R \frac{\rho' d\rho'}{\sqrt{R^2-\rho'^2}\sqrt{z^2+\rho'^2}}$$

$$= \frac{C}{2\varepsilon_0}\tan^{-1}(\frac{R}{z})$$

where I actually mean the absolute value of z here, and where $\sigma = \frac{C}{\sqrt{R^2-\rho'^2}}$. If $z=0$, then $\phi = V$, so $\left(\frac{1}{4}\frac{C}{\varepsilon_0}\pi\right) = V$, or $C = \frac{4\varepsilon_0 V}{\pi}$. Now if $z > R$

$$\phi = \frac{2V}{\pi}\tan^{-1}(\frac{R}{z})$$

Now

$$\tan^{-1}(\frac{R}{z}) = \frac{1}{z}R + \left(-\frac{1}{3z^3}\right)R^3 + \frac{1}{5z^5}R^5 + O\left(R^7\right)$$

which we generalize

$$\phi = \frac{2V}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{R}{z}\right)^{2n+1}$$

But in general

$$\phi = \sum_l B_l\left(\frac{1}{z}\right)^{l+1}P_l(1)$$

Thus $l = 2n$, and in general

$$\phi = \frac{2V}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{R}{r}\right)^{2n+1} P_{2n}(\cos\theta)$$

b) If $r < R$,

$$\phi(\vec{x}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

At $r = R$, the two forms should be equal, so

$$A_l R^l = \frac{2V}{\pi} \frac{(-1)^n}{2n+1}$$

with $l = 2n$, as before.

$$\phi(\vec{x}) = \frac{2V}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{r}{R}\right)^{2n} P_{2n}(\cos\theta)$$

c)

$$C = Q/V = \frac{4\varepsilon_0 V}{\pi V} 2\pi \int_0^R \frac{\rho d\rho}{\sqrt{R^2 - \rho^2}} = 8\varepsilon_0 R$$

3.4

Slice the sphere equally by n planes slicing through the z axis, subtending angle $\Delta\phi$ about this axis with the surface of each slice of the pie alternating as $\pm V$.

$$\phi(r,\theta,\phi) = \sum_{l,m} A_{lm} r^l Y_l^m(\theta,\phi)$$

so

$$A_{lm} = \frac{1}{a^l}\int d\Omega (Y_l^m(\theta,\phi))^* \phi(a,\theta,\phi)$$

Symmetries:

$$A_{l-m} = (-1)^m (A_{lm})^*$$

$$\phi(r,\theta,\phi + 2\Delta\phi) = \phi(r,\theta,\phi)$$

where

$$\Delta\phi = \frac{2\pi}{2n}$$

Thus

$$m = \pm n, \text{ and integral multiples thereof}$$

$$\phi(-\vec{r}) = -\phi(\vec{r}),\ n = 1$$

$$\phi(-\vec{r}) = \phi(\vec{r}),\ n > 1$$

Since

$$PY_l^m(\theta,\phi) = (-1)^l Y_l^m(\theta,\phi)$$

Then

$$l \text{ is odd for } n = 1;\ \ l \text{ is even for } n > 1$$

Thus we only have contributions of $l \geq n$. Using

$$A_{lm} = \frac{1}{a^l}\int d\Omega (Y_l^m(\theta,\phi))^* \phi(a,\theta,\phi)$$

The integral over ϕ can be done trivially, since the integrand is just $e^{-im\phi}$ leaving the desired answer in terms of an integral over $\cos\theta$.

$n = 1$ case: I am going to keep only the lowest novanishing terms, involving A_{11} and A_{1-1}.

$$\phi = r(A_{11}Y_1^1 + A_{1-1}Y_1^{-1}) = r(A_{11}Y_1^1 + (A_{11}Y_1^1)^*) = 2r\,\mathrm{Re}(A_{11}Y_1^1)$$

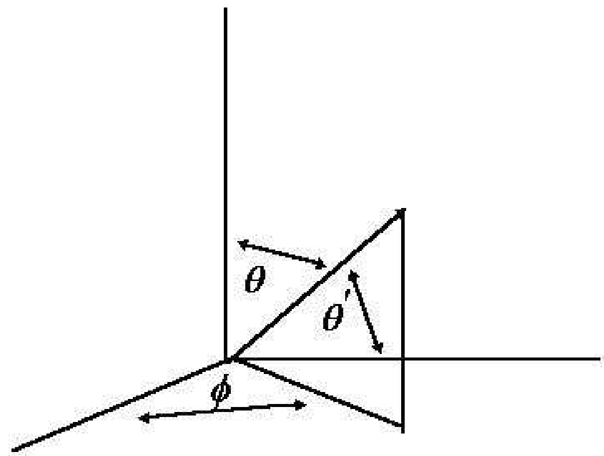
$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}(1-x^2)^{1/2}e^{i\phi}$$

$$A_{11} = -\frac{1}{a}\sqrt{\frac{3}{8\pi}}V\left[\int_{-1}^{1}(1-x^2)^{1/2}dx\right]\left[\int_0^\pi e^{-i\phi}d\phi - \int_\pi^{2\pi} e^{-i\phi}d\phi\right]$$

$$A_{11} = \frac{2i\pi}{a}\sqrt{\frac{3}{8\pi}}\,V$$

$$\phi = 2r\,\mathrm{Re}\left[\left(\frac{2i\pi}{a}\sqrt{\frac{3}{8\pi}}\,V\right)\left(-\sqrt{\frac{3}{8\pi}}\,\sin\theta e^{i\phi}\right)\right] = \frac{3r}{2a}V\sin\theta\sin\phi$$

From the figure



we see

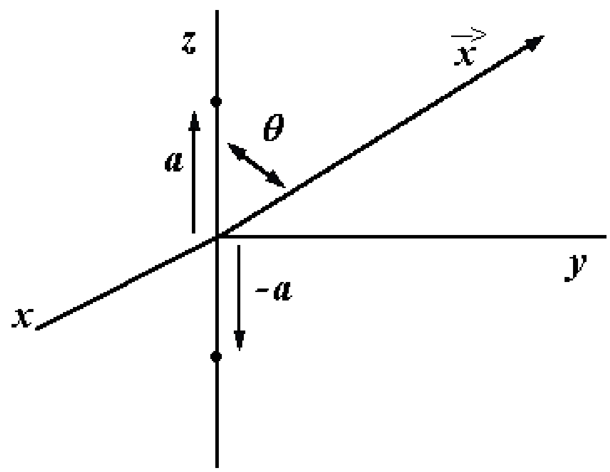
$$\sin\theta\sin\phi = \cos\theta'$$

So

$$\phi = \frac{3r}{2a}V\cos\theta' = V\left[\frac{3}{2}\frac{r}{a}P_1(\cos\theta') + \ldots\ldots\right]$$

The other terms, for $l = 2, 3,$ can be obtained in the same way in agreement with the result of (3.36)

3.6 The system is described by



a) From the figure, we can write

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{x}-\vec{a}|} - \frac{1}{|\vec{x}+\vec{a}|}\right]$$

And using the familiar expansion of $\frac{1}{|\vec{x}-\vec{a}|}$, this expression can be written

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\sum_l \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l [P_l(\theta) - P_l(\pi-\theta)]$$

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\sum_l \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l \left(1+(-1)^{l+1}\right) P_l(\theta)$$

This can be written in terms of spherical harmonics using $P_l(\theta) = \sqrt{\frac{4\pi}{2l+1}}Y_l^0(\theta,\phi)$.
b) We are given $r > a$, so

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\sum_l \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l \left(1+(-1)^{l+1}\right) P_l(\theta)$$

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\sum_l \frac{1}{r}\left(\frac{a}{r}\right)^l \left(1+(-1)^{l+1}\right) P_l(\theta); a \to 0 = \frac{q2a}{4\pi\varepsilon_0 r^2}P_1(\theta)$$

$$\phi(\vec{x}) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

c) The electric dipole is the particular solution, and ϕ_0 is the homogeneous solution which is a solution to Laplace's equation: by superposition,

$$\phi(\vec{x}) = \phi_p + \phi_0$$

$$\phi(\vec{x}) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} + \sum_l A_l r^l P_l(\theta)$$

The boundary condition we must satisfy is that $\phi(|\vec{x}| = b) = 0$, so

$$\frac{p\cos\theta}{4\pi\varepsilon_0 b^2} + \sum_l A_l b^l P_l(\theta) = 0$$

$$\rightarrow A_l = \begin{matrix} 0, \; l \neq 1 \\ -\frac{p}{4\pi\varepsilon_0 b^3}, \; l = 1 \end{matrix}$$

$$\phi(\vec{x}) = \frac{p\cos\theta}{4\pi\varepsilon_0}\left(\frac{1}{r^2} - \frac{r}{b^3}\right)$$

3.7

a) We first work the problem in the absence of the sphere, using the superposition principle,

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{x}-a\hat{z}|} - \frac{2}{|\vec{x}|} + \frac{1}{|\vec{x}+a\hat{z}|}\right]$$

We know

$$\frac{1}{|\vec{x}-a\,(\pm\hat{z})|} = \sum_l \frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l P_l(\pm\cos\theta)$$

where $r_>, r_<$ are the larger, smaller of a, r respectively and $\hat{r}\cdot(\pm\hat{z}) = \pm\cos\theta$. Since $P_l(-\cos\theta) = (-1)^l P_l(\cos\theta)$

$$\phi(\vec{x}) = \frac{2q}{4\pi\varepsilon_0}\sum_{l\text{ even}}\left(\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l - \frac{\delta_{l0}}{r}\right)P_l(\cos\theta)$$

As $a \to 0$,

$$\phi(\vec{x}) = \frac{2q}{4\pi\varepsilon_0}\sum_{l\text{ even}}\left(\frac{1}{r}\left(\frac{a}{r}\right)^l - \frac{\delta_{l0}}{r}\right)P_l(\cos\theta) \to \frac{2qa^2}{4\pi\varepsilon_0 r^3}P_2 \to \frac{Q}{2\pi\varepsilon_0 r^3}P_2(\cos\theta)$$

b) We can write the general solution as the sum of a particular and homogeneous part $\phi = \phi_p + \phi_0$, where $\nabla^2\phi_p = -\rho/\varepsilon_0$ and $\nabla^2\phi_0 = 0$. Clearly, we can take as ϕ_p the solution of part a) and choose ϕ_0 to satisfy the BC's. The non-trivial solution is in the region $r < b$, where $\phi_0 = \sum_l A_l r^l P_l(\cos\theta)$. At $r = b$, we must have

$$(\phi_p + \phi_0)\,|_{r=b} = 0 = \frac{2q}{4\pi\varepsilon_0}\sum_{l\text{ even}}\left(\frac{1}{b}\left(\frac{a}{b}\right)^l - \frac{\delta_{l0}}{b}\right)P_l(\cos\theta) + \sum_l A_l b^l P_l(\cos\theta)$$

thus

$$A_l = \left.\begin{array}{c} 0 \\ -\frac{a^l}{b^{2l+1}}\frac{q}{2\pi\varepsilon_0} \end{array}\right\} \begin{array}{c} l\text{ odd} \\ l\text{ even}, > 0 \end{array}$$

1. $r > a$

$$\phi(\vec{x}) = \frac{q}{2\pi\varepsilon_0}\left[\sum_{l\text{ even}}\left(\frac{1}{r}\left(\frac{a}{r}\right)^l - \frac{\delta_{l0}}{r}\right) - \sum_{l\text{ even},>0}\frac{a^l}{b^{2l+1}}r^l\right]P_l(\cos\theta)$$

2. $r < a$

$$\phi(\vec{x}) = \frac{q}{2\pi\varepsilon_0}\left[\sum_{l\text{ even}}\left(\frac{1}{a}\left(\frac{r}{a}\right)^l - \frac{\delta_{l0}}{r}\right) - \sum_{l\text{ even},>0}\frac{a^l}{b^{2l+1}}r^l\right]P_l(\cos\theta)$$

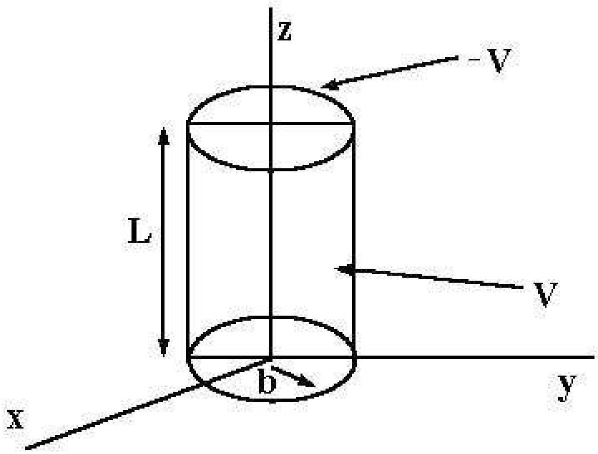
As $a \to 0$, the potential is dominated by the lowest non-vanishing term of expression 1.:

$$\phi(\vec{x}) = \frac{q}{2\pi\varepsilon_0}\left(\frac{a^2}{r^3} - \frac{a^2}{b^5}r^2\right) P_2(\cos\theta)$$

$$\phi(\vec{x}) = \frac{Q}{2\pi\varepsilon_0 r^3}\left(1 - \frac{r^5}{b^5}\right) P_2(\cos\theta)$$

3.10

This problem is described by



a) From the class notes

$$\Phi(\rho, z, \phi) = \sum_{n\nu} (A_{n\nu} \sin\nu\phi + B_{n\nu} \cos\nu\phi) I_\nu\left(\frac{n\pi\rho}{L}\right) \sin\left(\frac{n\pi z}{L}\right)$$

where

$$A_{n\nu} = \frac{2}{\pi L} \frac{1}{I_\nu\left(\frac{n\pi b}{L}\right)} \int_0^L \int_0^{2\pi} V(\phi, z) \sin\left(\frac{n\pi a}{L}\right) \sin(\nu\phi) d\phi dz, \qquad \nu \sum 0$$

$$B_{n\nu} = \frac{2}{\pi L} \frac{1}{I_\nu\left(\frac{n\pi b}{L}\right)} \int_0^L \int_0^{2\pi} V(\phi, z) \sin\left(\frac{n\pi a}{L}\right) \cos(\nu\phi) d\phi dz, \qquad \nu \ \Phi\ 0$$

$$B_{n\nu} = \frac{1}{\pi L} \frac{1}{I_\nu\left(\frac{n\pi b}{L}\right)} \int_0^L \int_0^{2\pi} V(\phi, z) \sin\left(\frac{n\pi a}{L}\right) d\phi dz, \qquad \nu = 0$$

Noting

$$\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\nu\phi d\phi - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin\nu\phi d\phi \right] = 0$$

we conclude $A_{n\nu} = 0$. Similarly, noting

$$\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\nu\phi d\phi - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\nu\phi d\phi \right] = \frac{4(-1)^m}{2m+1}, \qquad m = 0, 1, 2, \ldots.$$

where I've recognized that ν must be odd, ie, $\nu = 2m + 1$. Also

$$\int_0^L \sin\left(\frac{n\pi z}{L}\right) dz = \frac{2}{(2l+1)\pi}, \qquad l = 0, 1, 2, \ldots..$$

where again I've recognized that n must be odd, ie, $n = 2l + 1$. Thus

$$B_{n\nu} = \frac{16(-1)^m V}{\pi^2 I_{2m+1}\left(\frac{n\pi b}{L}\right)(2l+1)(2m+1)}$$

b) Now $z = L/2, \quad L \sum\sum b,\ L \sum\sum \rho$. Then from the class notes

$$I_{2m+1}\left(\frac{(2l+1)\pi\rho}{L}\right) \sim \frac{1}{\Gamma(2m+2)}\left[\frac{(2l+1)\pi\rho}{2L}\right]^{m+1}$$

Also

$$\sin\left[\frac{(2l+1)\pi}{L}\right] = (-1)^l$$

so

$$\Phi(\rho,z,\phi) = \sum_{l,m} \frac{16(-1)^{l+m}V}{\pi^2(2l+1)(2m+1)}\left(\frac{\rho}{b}\right)^{2m+1}\cos[(2m+1)\phi]$$

Using

$$\tan^{-1(x)} = \sum_{l=0}^{\infty} \frac{x^{2l+1}}{1l+1}(-1)^l$$

$$\frac{\pi}{4} = \tan^{-1}(1) = \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1}$$

so

$$\Phi(\rho,z,\phi) = \frac{4V}{\pi}\sum_m \frac{(-1)^m}{2m+1}\left(\frac{\rho}{b}\right)^{2m+1}\cos[(2m+1)\phi]$$

Remembering from problem 2.13 that

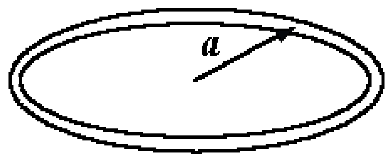
$$\sum_m \frac{(-1)^m}{2m+1}\left(\frac{\rho}{b}\right)^{2m+1}\cos[(2m+1)\phi] = \frac{1}{2}\tan^{-1}\left[\frac{2\left(\frac{\rho}{b}\right)\cos\phi}{\left(1-\frac{\rho^2}{b^2}\right)}\right]$$

we find

$$\Phi(\rho,z,\phi) = \frac{2V}{\pi}\tan^{-1}\left[\frac{2\left(\frac{\rho}{b}\right)\cos\phi}{\left(1-\frac{\rho^2}{b^2}\right)}\right]$$

which is the answer for problem 2.13.

3.12 The system is described by



a) From Eq. (3.106)

$$\Phi(\rho,\phi,z)=\sum_{m=0}^{\infty}\int_0^\infty dke^{-kz}J_m(k\rho)\left[A_m(k)\sin m\phi+B_m(k)\cos m\phi\right]$$

where from Eq. (3.109),

$$\left.\begin{matrix}A_m(k)\\B_m(k)\end{matrix}\right\}=\frac{k}{\pi}\int_0^\infty d\rho\rho\int_0^{2\pi}d\phi V\left(\rho,\phi\right)J_m(k\rho)\left\{\begin{matrix}\sin m\phi\\\cos m\phi\end{matrix}\right.$$

where we use $\frac{1}{2}B_0$ for $m=0$.

b) Using cylidrical coordinates, with the origin at the center of the disc, then we have $\rho=0$, and can use the small argument expansion for $J_m(k\rho)$

$$J_m(k\rho)|_{\rho=0}=\frac{\delta_{m0}}{\Gamma(1)}+O((k\rho)^2)=\delta_{m0}$$

$$\Phi(0,\phi,z)=\frac{1}{2}\int_0^\infty dke^{-kz}B_0(k)$$

And, using Mathematica 4,

$$B_0(k)=2kV\int_0^a d\rho\rho J_0(k\rho)=2kV\frac{a}{k}J_1(ka)=2VaJ_1(ka)$$

Thus, again using Mathematica 4,

$$\Phi(0,\phi,z)=Va\int_0^\infty dke^{-kz}J_1(ka)=V\frac{\sqrt{z^2+a^2}-z}{\sqrt{z^2+a^2}}=V\left(1-\frac{z}{\sqrt{z^2+a^2}}\right)$$

c) We notice that for this $V(\rho,\phi)$, which is independent of ϕ, that all $A_m(k)$ vanish, and that only B_0 is nonzero. Again

$$B_0(k)=2kV\int_0^a d\rho\rho J_0(k\rho)=2kV\frac{a}{k}J_1(ka)=2VaJ_1(ka)$$

$$\Phi(a,\phi,z)=Va\int_0^\infty dke^{-kz}J_0(ka)J_1(ka)$$

Using Mathematic 4,

$$\int_0^\infty dk e^{-kz} J_0(ka) J_1(ka) = \frac{1}{2a}\left(1 - \frac{zk}{\pi a} K(k)\right)$$

where $k = \frac{2a}{\sqrt{z^2+a^2}}$, and the complete elliptic integral of the first kind is defined by

$$K(k) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}$$

Thus

$$\Phi(a,\phi,z) = \frac{V}{2}\left(1 - \frac{zk}{\pi a} K(k)\right)$$

Jackson 3.15

Solution - Part A

The Green's function can be expanded as:

$$G(x,x') = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} A_{n,m} e^{im\phi} \sin\left(\frac{n\pi z}{L}\right) R(\rho)$$

Inserting this into the differential equation for the Green's function:

$$\nabla^2 G(x,x') = -4\pi\delta(x-x')$$

multiplying each side by $e^{-im\phi}$ and $\sin\left(\frac{n\pi z}{L}\right)$ and integrating over z and ϕ gives:

$$A_{n,m} 2\pi \frac{L}{2}\left(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{dR}{d\rho} - \left(\frac{m^2}{\rho^2} + \frac{n^2\pi^2}{L^2}\right) R\right) = -4\pi e^{-im\phi'} \sin\left(\frac{n\pi z'}{L}\right)\frac{\delta(\rho-\rho')}{\rho}$$

If

$$A_{n,m} = \frac{4}{L} e^{-im\phi'} \sin\left(\frac{n\pi z'}{L}\right)$$

then

$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{dR}{d\rho} - \left(\frac{m^2}{\rho^2} + \frac{n^2\pi^2}{L^2}\right) R = -\frac{\delta(\rho-\rho')}{\rho}$$

For $\rho \neq \rho'$, this is the equation for the modified Bessel functions $I_m\left(\frac{n\pi}{L}\rho\right)$ and $K_m\left(\frac{n\pi}{L}\rho\right)$. $K_m \to 0$ as $\rho \to \infty$ and I_m stays finite as $\rho \to 0$. Thus if $\rho < \rho'$, the $R \propto I_m\left(\frac{n\pi}{L}\rho\right)$ and if $\rho > \rho'$, the $R \propto K_m\left(\frac{n\pi}{L}\rho\right)$. To maintain the symmetry between ρ and ρ',

$$R \propto I_m\left(\frac{n\pi}{L}\rho_<\right) K_m\left(\frac{n\pi}{L}\rho_>\right)$$

The proportionality constant is determined by the condition on the break in the slope

$$\left.\frac{dR}{d\rho_>}\right|_{\rho_<,\rho_>=\rho} - \left.\frac{dR}{d\rho_<}\right|_{\rho_<,\rho_>=\rho} = -\frac{1}{\rho}.$$

Using the identities

$$\begin{aligned}
2\frac{d}{dz}I_m(z) &= I_{m+1}(z) + I_{m-1}(z) \\
-2\frac{d}{dz}K_m(z) &= K_{m+1}(z) + K_{m-1}(z) \\
\frac{1}{z} &= I_{m-1}(z)K_m(z) + I_m(z)K_{m-1}(z)
\end{aligned}$$

gives

$$R = I_m\left(\frac{n\pi}{L}\rho_<\right) K_m\left(\frac{n\pi}{L}\rho_>\right)$$

Combining these all together gives the result in Jackson.

More Problems for Chapter 3

Problem 3.1
The problem is azimuth symmetric and therefore the general solution is

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty}(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}})P_\ell(\cos\theta)$$

The coefficients A_ℓ and B_ℓ are to be determined by the following boundary conditions:

$$\Phi(r=a,\theta) = Vh(\frac{\pi}{2}-\theta); \qquad \Phi(r=b,\theta) = Vh(\theta-\frac{\pi}{2})$$

where $h(x)$ is a step function, *i.e.*, $h(x)=1$ if $x>0$ and $h(x)=0$ if $x<0$. Applying boundary condition at $r=a$:

$$\sum_\ell (A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}})P_\ell(\cos\theta) = Vh(\frac{\pi}{2}-\theta)$$

Multiplying both sides with $P_{\ell'}(\cos\theta)\sin\theta$ and integrating over θ:

$$\sum_\ell (A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}})\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta d\theta = V\int_0^{\pi/2} P_{\ell'}(\cos\theta)\sin\theta d\theta$$

Note that

$$\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta d\theta = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

$$\int_0^{\pi/2} P_{\ell'}(\cos\theta)\sin\theta d\theta = (-\frac{1}{2})^{(\ell'-1)/2}\frac{(\ell'-2)!!}{2(\frac{\ell'+1}{2})!} \qquad (\ell' = \text{odd})$$

$$= \int_0^1 P_{\ell'}(x)dx = \frac{1}{2}\int_{-1}^{+1} P_{\ell'}(x)P_0(x)dx = \delta_{\ell'0} \qquad (\ell' = \text{even})$$

Therefore

$$A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}} = Vc_\ell$$

$$c_\ell = \begin{cases} \frac{1}{2} & \text{for } \ell = 0 \\ 0 & \text{for other even } \ell \\ (-\frac{1}{2})^{\frac{\ell-1}{2}}\frac{2\ell+1}{4}\frac{(\ell-2)!!}{(\frac{\ell+1}{2})!} & \text{for odd } \ell \end{cases}$$

Applying the boundary condition at $r=b$ leads to a similar equation:

$$A_\ell b^\ell + \frac{B_\ell}{b^{\ell+1}} = (-1)^\ell Vc_\ell$$

Solving for A_ℓ and B_ℓ:

$$A_\ell = Vc_\ell\frac{a^{\ell+1}-(-1)^\ell b^{\ell+1}}{a^{2\ell+1}-b^{2\ell+1}}, \qquad B_\ell = Vc_\ell(ab)^{\ell+1}\frac{(-1)^\ell a^\ell - b^\ell}{a^{2\ell+1}-b^{2\ell+1}}$$

Therefore

$$A_0 = Vc_0 = \frac{V}{2}; \qquad A_1 = \frac{a^2+b^2}{a^3-b^3}Vc_1 = -\frac{3}{4}V\frac{a^2+b^2}{b^3-a^3}$$

$$A_2 = 0; \qquad A_3 = \frac{a^4+b^4}{a^7-b^7}Vc_3 = \frac{7}{16}V\frac{a^4+b^4}{b^7-a^7}; \qquad A_4 = 0$$

$$B_0 = 0; \qquad B_1 = -(ab)^2\frac{a+b}{a^3-b^3}Vc_1 = \frac{3}{4}V(ab)^2\frac{a+b}{b^3-a^3}; \qquad B_2 = 0$$

$$B_3 = -(ab)^4\frac{a^3+b^3}{a^7-b^7}Vc_3 = -\frac{7}{16}V(ab)^4\frac{a^3+b^3}{b^7-a^7}; \qquad B_4 = 0$$

The potential

$$\Phi(r,\theta) = V\left\{\frac{1}{2} - \frac{3}{4}\{\frac{a^2+b^2}{b^3-a^3}r - (ab)^2\frac{a+b}{b^3-a^3}\frac{1}{r^2}\}P_1(\cos\theta) + \frac{7}{16}\{\frac{a^4+b^4}{b^7-a^7}r^3 - (ab)^4\frac{a^3+b^3}{b^7-a^7}\frac{1}{r^4}\}P_3(\cos\theta) + ...\right\}$$

The limiting case of $a \to 0$:

$$\Phi(r,\theta) \to V\left\{\frac{1}{2} - \frac{3}{4}\frac{r}{b}P_1(\cos\theta) + \frac{7}{16}(\frac{r}{b})^3P_3(\cos\theta) + ...\right\}$$

The limiting case of $b \to \infty$:

$$\Phi(r,\theta) \to V\left\{\frac{1}{2} + \frac{3}{4}(\frac{a}{r})^2P_1(\cos\theta) - \frac{7}{16}(\frac{a}{r})^4P_3(\cos\theta) + ...\right\}$$

Problem 3.4

Using integrals and identities:

$$\int_{-1}^{+1} P_1^1(x)dx = -\frac{\pi}{2}; \qquad \int_{-1}^{+1} P_1^{-1}(x)dx = \frac{\pi}{4}$$

$$\int_{-1}^{+1} P_3^1(x)dx = -\frac{3\pi}{16}; \qquad \int_{-1}^{+1} P_3^{-1}(x)dx = \frac{\pi}{64}$$

$$\int_{-1}^{+1} P_3^3(x)dx = -\frac{45\pi}{8}; \qquad \int_{-1}^{+1} P_3^{-3}(x)dx = \frac{\pi}{128}$$

$$\sin(3\alpha) = -4\sin^3\alpha + 3\sin\alpha$$

This problem is not ϕ symmetric. The general solution is therefore

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(A_{\ell m}r^\ell + \frac{B_{\ell m}}{r^{\ell+1}})Y_{\ell m}(\theta,\phi)$$

Since there is no point charge inside the sphere, the potential has to be finite. Therefore, $B_{\ell m} = 0$:

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}A_{\ell m}r^\ell Y_{\ell m}(\theta,\phi)$$

However, the potential is invariant under $\phi \to \phi + 2\pi/n$ transformation, *i.e.*:

$$e^{im\phi} = e^{im(\phi+2\pi/n)}, \quad \Rightarrow \quad \cos(2\frac{m}{n}\pi) = 1 \quad \Rightarrow m = kn, \quad (k = \pm 1, \pm 2, ...)$$

(a) The coefficients $A_{\ell m}$ are to be determined by the potential at the surface:

$$\Phi(r = a, \theta, \phi) = V(-1)^j \ \text{ for } \ \frac{\pi}{n} j \le \phi < \frac{\pi}{n}(j+1) \quad \text{where } j = 0, 1, 2, ..., 2n-1$$

$$A_{\ell m} a^\ell = \int \Phi(r = a, \theta, \phi) Y^*_{\ell m} d\Omega = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} \int_{-1}^{1} P_\ell^m(x) dx \int_0^{2\pi} \Phi(r = a, \theta, \phi) e^{-im\phi} d\phi$$

Note that

$$\int_0^{2\pi} \Phi(r = a, \theta, \phi) e^{-im\phi} d\phi = V \sum_{j=0}^{2n-1} \int_{j\pi/n}^{(j+1)\pi/n} (-1)^j e^{-im\phi} d\phi = \frac{iV}{m}\{e^{-im\pi/n} - 1\} \sum_{j=0}^{2n-1} (-e^{-im\pi/n})^j$$

In order for the above integral to be non-vanishing, m/n must be an odd number. In this case,

$$\sum_{j=0}^{2n-1} (-e^{-im\pi/n})^j = 2n; \quad \text{and} \quad \int_0^{2\pi} \Phi(r = a, \theta, \phi) e^{-im\phi} d\phi = -4i\frac{Vn}{m}$$

Furthermore, the associated Legendre function $P_\ell^m(x)$ is even if $\ell + m$ =even and odd if $\ell + m$ =odd. Consequently, $A_{\ell m}$'s are non-vanishing only if ℓ and m are either both odd or both even numbers. Therefore,

$$\ell = \text{even} \quad \text{if} \quad n = \text{even}; \quad \ell = \text{odd} \quad \text{if} \quad n = \text{odd}$$

Since m/n is odd, $m = (2k+1)n, \ k = 0, \pm 1, \pm 2, ...$ The potential inside the sphere is

$$\Phi(r, \theta, \phi) = \sum_{k=-\infty}^{\infty} \sum_{\ell \ge |(2k+1)|n}^{\infty} \frac{1 + (-1)^{\ell+n}}{2} A_{\ell,(2k+1)n} Y_{\ell,(2k+1)n}(\theta, \phi)$$

The first non-vanishing $A_{\ell m}$'s are $A_{n,n}$ and $A_{n,-n}$:

$$A_{n,n} = \frac{-4iV}{a^n}\sqrt{\frac{2n+1}{4\pi}\frac{1}{(2n)!}} \int_{-1}^{+1} P_n^n(x) dx; \quad A_{n,-n} = \frac{4iV}{a^n}\sqrt{\frac{2n+1}{4\pi}(2n)!} \int_{-1}^{+1} P_n^{-n}(x) dx$$

(b) For $n = 1$, the non-vanishing $A_{\ell m}$'s up to $\ell = 3$ are $A_{11}, A_{1,-1}, A_{31}, A_{3,-1}, A_{33}, A_{3,-3}$:

$$A_{11} = \frac{-4iV}{a}\sqrt{\frac{3}{8\pi}} \int_{-1}^{+1} P_1^1(x) dx = \frac{iV}{a}\sqrt{\frac{3\pi}{2}}; \qquad A_{1,-1} = \frac{4iV}{a}\sqrt{\frac{3}{2\pi}} \int_{-1}^{+1} P_1^{-1}(x) dx = \frac{iV}{a}\sqrt{\frac{3\pi}{2}} = A_{11}$$

$$A_{31} = \frac{-4iV}{a^3}\sqrt{\frac{7}{48\pi}} \int_{-1}^{+1} P_3^1(x) dx = \frac{iV}{16a^3}\sqrt{21\pi}; \qquad A_{3,-1} = \frac{4iV}{a^3}\sqrt{\frac{84}{4\pi}} \int_{-1}^{+1} P_3^{-1}(x) dx = \frac{iV}{16a^3}\sqrt{21\pi} = A_{31}$$

$$A_{33} = -\frac{4iV}{3a^3}\sqrt{\frac{7}{2880\pi}} \int_{-1}^{+1} P_3^3(x) dx = \frac{iV}{16a^3}\sqrt{35\pi}; \quad A_{3,-3} = \frac{4iV}{3a^3}\sqrt{\frac{1260}{\pi}} \int_{-1}^{+1} P_3^{-3}(x) dx = \frac{iV}{16a^3}\sqrt{35\pi} = A_{33}$$

The potential expansion (up to the term $\ell = 3$) becomes:

$$\Phi(r, \theta, \phi) = r(A_{11}Y_{11} + A_{1,-1}Y_{1,-1}) + r^3(A_{31}Y_{31} + A_{3-1}Y_{3,-1} + A_{33}Y_{33} + A_{3-3}Y_{3,-3} + ...$$

$$= rA_{11}(Y_{11} - Y^*_{11}) + r^3 A_{31}(Y_{31} - Y^*_{31}) + r^3 A_{33}(Y_{33} - Y^*_{33})$$

where

$$rA_{11}(Y_{11}-Y_{11}^*) = 2iV\sqrt{\frac{3\pi}{2}}\frac{r}{a}\sqrt{\frac{3}{8\pi}}\sin\theta(e^{-i\phi}-e^{i\phi}) = \frac{3V}{2}\frac{r}{a}\sin\theta\sin\phi$$

$$r^3A_{31}(Y_{31}-Y_{31}^*) = \frac{iV}{8}\sqrt{21\pi}\frac{r^3}{a^3}\frac{1}{4}\sqrt{\frac{21}{4\pi}}\sin\theta(5\cos^2\theta-1)(e^{-i\phi}-e^{i\phi}) = \frac{21}{64}V\frac{r^3}{a^3}\sin\theta(5\cos^2\theta-1)\sin\phi$$

$$r^3A_{33}(Y_{33}-Y^*33) = \frac{iV}{8}\sqrt{35\pi}\frac{r^3}{a^3}\frac{1}{4}\sqrt{\frac{35}{4\pi}}\sin^3\theta(e^{-i3\phi}-e^{i3\phi}) = \frac{35}{64}V\frac{r^3}{a^3}\sin^3\theta\sin(3\phi)$$

Combining the above three terms, we have

$$\Phi(r,\theta,\phi) = V\left\{\frac{3}{2}(\frac{r}{a})\sin\theta\sin\phi + \frac{7}{64}(\frac{r}{a})^3\{3\sin\theta(5\cos^2\theta-1)\sin\phi + 5\sin^3\theta\sin(3\phi)\}\right\} + ...$$

$$= V\left\{\frac{3}{2}(\frac{r}{a})\sin\theta\sin\phi + \frac{7}{64}(\frac{r}{a})^3\{3\sin\theta(5\cos^2\theta-1) - 20\sin^3\theta\sin^3\phi + 15\sin^3\theta\sin\phi\}\right\} + ...$$

Translating to Cartesian coordinates ($x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$):

$$\Phi(x,y,z) = V\left\{\frac{3}{2}\frac{y}{a} + \frac{7}{64}\{3\frac{y}{a}(5\frac{z^2}{a^2} - \frac{r^2}{a^2}) - 20\frac{y^3}{a^3} + 15\frac{y}{a}(\frac{r^2}{a^2} - \frac{z^2}{a^5})\}\right\}$$

Rotating coordinates ($x \to y$, $y \to z$, and $z \to x$):

$$\Phi(x,y,z) = V\left\{\frac{3}{2}\frac{z}{a} + \frac{7}{64}\{3\frac{z}{a}(5\frac{x^2}{a^2} - \frac{r^2}{a^2} - 20\frac{z^3}{a^3} + 15\frac{z}{a}(\frac{r^2}{a^2} - \frac{x^2}{a^2})\}\right\}$$

Translating back to Spherical coordinates in the rotated system ($x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$):

$$\Phi(r,\theta,\phi) = V\left\{\frac{3}{2}\frac{r}{a}\cos\theta + \frac{7}{64}\frac{r^3}{a^3}\{3\cos\theta(5\sin^2\theta\cos^2\phi - 1) - 20\cos^3\theta + 15\cos\theta(1-\sin^2\theta\sin^2\phi)\}\right\}$$

$$= V\left\{\frac{3}{2}\frac{r}{a}\cos\theta - \frac{7}{8}\frac{r^3}{a^3}\{\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\}\right\}$$

$$= V\left\{\frac{3}{2}\frac{r}{a}P_1(\cos\theta) - \frac{7}{8}\frac{r^3}{a^3}P_3(\cos\theta) + ...\right\}$$

agrees with Eq. (3.36).

Problem 3.7

(a) The problem is ϕ−symmetric. The potential for an arbitrary point (r,θ,ϕ) is

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{r_+} + \frac{1}{r_-} - \frac{2}{r}\right\}$$

where r_+ and r_- are distances from the point to the charge at $z = +a$ and $z = -a$ respectively:

$$r_+ = \sqrt{r^2+a^2-2ar\cos\theta}; \qquad r_- = \sqrt{r^2+a^2+2ar\cos\theta}$$

Therefore,

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}} + \frac{1}{\sqrt{r^2+a^2+2ar\cos\theta}} - \frac{2}{r}\right\}$$

To find the limiting form of the potential as $a \to 0$, we expand r_+ and r_- as:

$$\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}} = \frac{1}{r}\frac{1}{\sqrt{1-2\frac{a}{r}\cos\theta+(\frac{a}{r})^2}} = \sum_{\ell=0}^{\infty}\frac{a^\ell}{r^{\ell+1}}P_\ell(\cos\theta)$$

$$\frac{1}{\sqrt{r^2+a^2+2ar\cos\theta}} = \frac{1}{r}\frac{1}{\sqrt{1+2\frac{a}{r}\cos\theta+(\frac{a}{r})^2}} = \sum_{\ell=0}^{\infty}(-1)^\ell\frac{a^\ell}{r^{\ell+1}}P_\ell(\cos\theta)$$

Using these expansions, the potential can be written as

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\left\{\sum_{\ell=0}^{\infty}\{1+(-1)^\ell\}\frac{a^\ell}{r^{\ell+1}}P_\ell(\cos\theta) - \frac{2}{r}\right\} = \frac{q}{2\pi\epsilon_0}\sum_{n=1}^{\infty}\frac{a^{2n}}{r^{2n+1}}P_{2n}(\cos\theta)$$

In the limit of $a \to 0$ while keeping qa^2 constant:

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\left\{2\frac{a^2}{r^3}P_2(\cos\theta)+2\frac{a^4}{r^5}P_4(\cos\theta)+....\right\} \to \frac{Q}{2\pi\epsilon_0}\frac{1}{r^3}P_2(\cos\theta)$$

(b) With the grounded sphere, the potential at (r,θ,ϕ) are superpositions of those of the three charges and their imagine charges. Denoting Φ_+, Φ_- and Φ_0 the potentials of charges at $z=+a,-a,0$ and of their respective imagine charges, we have

$$\Phi_+(r,\theta) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{\sqrt{r^2+a^2-2ar\cos\theta}} + \frac{-qb/a}{\sqrt{r^2+(b^2/a)^2-2r(b^2/a)\cos\theta}}\right\}$$

$$\Phi_-(r,\theta) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{\sqrt{r^2+a^2+2ar\cos\theta}} + \frac{-qb/a}{\sqrt{r^2+(b^2/a)^2+2r(b^2/a)\cos\theta}}\right\}$$

$$\Phi_0(r,\theta) = \frac{1}{4\pi\epsilon_0}\lim_{r_0\to 0}\left\{\frac{-2q}{\sqrt{r^2+r_0^2-2rr_0\cos\theta}} + \frac{2qb/r_0}{\sqrt{r^2+(b^2/r_0)^2-2r(b^2/r_0)\cos\theta}}\right\}$$

$$= \frac{q}{2\pi\epsilon_0}\left\{-\frac{1}{r}+\lim_{r_0\to 0}\frac{1}{b\sqrt{1+(rr_0/b)^2-2(rr_0/b^2)\cos\theta}}\right\}$$

$$= \frac{q}{2\pi\epsilon_0}\left\{-\frac{1}{r}+\frac{1}{b}\lim_{r_0\to 0}\sum_{\ell=0}^{\infty}P_\ell(\cos\theta)(\frac{rr_0}{b^2})^\ell\right\} = \frac{q}{2\pi\epsilon_0}\left\{\frac{1}{b}-\frac{1}{r}\right\}$$

For $r < a$, Φ_+ and Φ_- can be expanded in terms of r/a or ar/b:

$$\Phi_+ = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{a}\frac{1}{\sqrt{1+(\frac{r}{a})^2-2(\frac{r}{a})\cos\theta}} - \frac{1}{b}\frac{1}{\sqrt{1+(\frac{ar}{b^2})^2-2(\frac{ar}{b^2})\cos\theta}}\right\} = \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\left\{\frac{r^\ell}{a^{\ell+1}}-\frac{a^\ell r^\ell}{b^{2\ell+1}}\right\}P_\ell(\cos\theta)$$

$$\Phi_- = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{a}\frac{1}{\sqrt{1+(\frac{r}{a}^2+2\frac{r}{a}\cos\theta}} - \frac{1}{b}\frac{1}{\sqrt{1+(\frac{ar}{b^2})^2+2(\frac{ar}{b^2})\cos\theta}}\right\} = \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}(-1)^\ell\left\{\frac{r^\ell}{a^{\ell+1}}-\frac{a^\ell r^\ell}{b^{2\ell+1}}\right\}P_\ell(\cos\theta)$$

Adding Φ_+, Φ_- and Φ_0 together:

$$\Phi(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\{(1+(-1)^\ell\}\left\{\frac{r^\ell}{a^{\ell+1}}-\frac{a^\ell r^\ell}{b^{2\ell+1}}\right\}P_\ell(\cos\theta) + \frac{q}{2\pi\epsilon_0}\left\{\frac{1}{b}-\frac{1}{r}\right\}$$

$$= \frac{q}{2\pi\epsilon_0}\left\{\frac{1}{b}-\frac{1}{r}+\sum_{n=0}^{\infty} r^{2n}\{\frac{1}{a^{2n+1}}-\frac{a^{2n}}{b^{4n+1}}\}P_{2n}(\cos\theta)\right\}$$

For $r > a$, Φ_+ and Φ_- have to be expanded in terms of a/r or ar/b:

$$\Phi_+ = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{r}\frac{1}{\sqrt{1+(\frac{a}{r})^2-2\frac{a}{r}\cos\theta}}-\frac{1}{b}\frac{1}{\sqrt{1+(\frac{ar}{b^2})^2-2\frac{ar}{b^2}\cos\theta}}\right\} = \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\left\{\frac{a^\ell}{r^{\ell+1}}-\frac{a^\ell r^\ell}{b^{2\ell+1}}\right\}P_\ell(\cos\theta)$$

$$\Phi_- = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{r}\frac{1}{\sqrt{1+(\frac{a}{r})^2+2\frac{a}{r}\cos\theta}}-\frac{1}{b}\frac{1}{\sqrt{1+(\frac{ar}{b^2})^2+2(\frac{ar}{b^2})\cos\theta}}\right\} = \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}(-1)^\ell\left\{\frac{a^\ell}{r^{\ell+1}}-\frac{a^\ell r^\ell}{b^{2\ell+1}}\right\}P_\ell(\cos\theta)$$

The total potential for $r > a$:

$$\Phi(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\{(1+(-1)^\ell\}\left\{\frac{a^\ell}{r^{\ell+1}}-\frac{a^\ell r^\ell}{b^{2\ell+1}}\right\}P_\ell(\cos\theta)+\frac{q}{2\pi\epsilon_0}\left\{\frac{1}{b}-\frac{1}{r}\right\}$$

$$= \frac{q}{2\pi\epsilon_0}\left\{\frac{1}{b}-\frac{1}{r}+\sum_{n=0}^{\infty}\{\frac{a^{2n}}{r^{2n+1}}-\frac{a^{2n}r^{2n}}{b^{4n+1}}\}P_{2n}(\cos\theta)\right\}$$

$$= \frac{q}{2\pi\epsilon_0}\sum_{n=1}^{\infty}a^{2n}\left\{\frac{1}{r^{2n+1}}-\frac{r^{2n}}{b^{4n+1}}\}P_{2n}(\cos\theta)\right\}$$

In the limit of $a \to 0$:

$$\Phi(r,\theta) \to \frac{1}{2\pi\epsilon_0}qa^2\left\{\frac{1}{r^3}-\frac{r^2}{b^5}\right\}P_2(\cos\theta) = \frac{1}{2\pi\epsilon_0}\frac{Q}{r^3}\left\{1-\frac{r^5}{b^5}\right\}P_2(\cos\theta)$$

Problem 3.9

Useful integrals:

$$\int_0^{2\pi}\sin(m\phi)\sin(n\phi)d\phi = \pi\delta_{mn}, \qquad \int_0^{2\pi}\cos(m\phi)\cos(n\phi)d\phi = \pi\delta_{mn}$$

$$\int_0^{L}\sin(m\pi z/L)\sin(n\pi z/L)dz = \frac{L}{2}\delta_{mn}, \qquad \int_0^{L}\cos(m\pi z/L)\cos(n\imath z/L)dz = \frac{L}{2}\delta_{mn}$$

This is a problem of solving Laplace's equation $\nabla^2\Phi = 0$ with the following boundary conditions:

$$\Phi(\rho,\phi,z=0) = 0, \quad \Phi(\rho,\phi,z=L) = 0, \quad \Phi(\rho=b,\phi,z) = V(\phi,z)$$

Assuming the solution has the form:

$$\Phi(\rho,\phi,z) \sim R(\rho)Q(\phi)Z(z)$$

and plugging into the Laplace's equation, one gets:

$$Q(\phi) \sim A\cos(\nu\phi)+B\sin(\nu\phi), \quad Z(z) \sim C\cos(kz)+D\sin(kz), \quad R(\rho) \sim EI_\nu(k\rho)+FK_\nu(k\rho)$$

where A, B, C, D, E, F are constants. Applying generation consideration and boundary conditions:

- $Q(\phi+2\pi) = Q(\phi) \quad \Rightarrow \quad \nu = m,\ m = 0,1,2,...$
- Finite potential at $\rho = 0 \quad \Rightarrow \quad F = 0$;

- $\Phi = 0$ at $z = 0 \quad \Rightarrow \quad C = 0$;
- $\Phi = 0$ at $z = L \quad \Rightarrow \quad kL = n\pi,\ n = 0, 1, 2, ...$

Therefore, the complete general solution of the potential is

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} I_m(\frac{n\pi}{L}\rho) \sin(\frac{n\pi}{L}z)\{A_{mn}\sin(m\phi) + B_{mn}\cos(m\phi)\}$$

The coefficients A_{mn} and B_{mn} are to be determined from the boundary condition at $\rho = b$:

$$V(\phi, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} I_m(\frac{n\pi}{L}b) \sin(\frac{n\pi}{L}z)\{A_{mn}\sin(m\phi) + B_{mn}\cos(m\phi)\}$$

This is a double Fourier transform of $V(\phi, z)$. Multiplying $\sin(m'\phi)$ and integrating over ϕ:

$$\int_0^{2\pi} d\phi V(\phi, z) \sin(m'\phi) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} I_m(\frac{n\pi}{L}b) \sin(\frac{n\pi}{L}z) A_{mn} \cdot (\pi\delta_{m'm}) = \pi \sum_{n=0}^{\infty} I_{m'}(\frac{n\pi}{L}b) \sin(\frac{n\pi}{L}z) A_{m'n}$$

Multiplying $\sin(\frac{n'\pi}{L}z)$ and integrating over z:

$$\int_0^L dz \int_0^{2\pi} d\phi V(\phi, z) \sin(m'\phi) \sin(\frac{n'\pi}{L}z) = \pi \sum_{n=0}^{\infty} I'_m(\frac{n\pi}{L}b) A_{m'n} \cdot (\frac{L}{2}\delta_{n'n}) = \pi \frac{L}{2} I_{m'}(\frac{n'\pi}{L}b) A_{m'n'}$$

Therefore

$$A_{mn} = \frac{2}{\pi L I_m(n\pi b/L)} = \int_0^L dz \int_0^{2\pi} d\phi V(\phi, z) \sin(m\phi) \sin(\frac{n\pi}{L}z)$$

Similarly

$$B_{mn} = \frac{2}{\pi L I_m(n\pi b/L)} = \int_0^L dz \int_0^{2\pi} d\phi V(\phi, z) \cos(m\phi) \sin(\frac{n\pi}{L}z)$$

As usual, B_{0n} is to be replaced by $B_{0n}/2$.

Problem 3.10
Useful identifies:

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} = \tan^{-1}(x) \qquad \sum_{n=1}^{\infty}\frac{x^n}{n} = \ln(1-x)$$

(a) From the result of Problem 3.9:

$$\Phi(\rho,\phi,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} I_m(\frac{n\pi\rho}{L})\sin(\frac{n\pi}{L}z)\{A_{mn}\sin(m\phi)+B_{mn}\cos(m\phi)\}$$

with

$$A_{mn} = \frac{2}{\pi L I_m(n\pi b/L)}\int_0^L dz\int_0^{2\pi} d\phi V(\phi,z)\sin(m\phi)\sin(\frac{n\pi}{L}z)$$

$$= \frac{2}{\pi L I_m(n\pi b/L)}\int_0^L \sin(\frac{n\pi}{L}z)dz\left\{\int_{-\pi/2}^{\pi/2}(+V)\sin(m\phi)d\phi + \int_{\pi/2}^{3\pi/2}(-V)\sin(m\phi)d\phi\right\}$$

$$= \frac{2}{\pi L I_m(n\pi b/L)}\frac{L}{n\pi}\{1-(-1)^n\}\cdot 0 = 0$$

$$B_{mn} = \frac{2}{\pi L I_m(n\pi b/L)}\int_0^L dz\int_0^{2\pi} d\phi V(\phi,z)\cos(m\phi)\sin(\frac{n\pi}{L}z)$$

$$= \frac{2}{\pi L I_m(n\pi b/L)}\int_0^L \sin(\frac{n\pi}{L}z)dz\left\{\int_{-\pi/2}^{\pi/2}(+V)\cos(m\phi)d\phi + \int_{\pi/2}^{3\pi/2}(-V)\cos(m\phi)d\phi\right\}$$

$$= \frac{2}{\pi L I_m(n\pi b/L)}\frac{L}{n\pi}\{(-1)^n-1\}\frac{V}{m}\{3\sin(\frac{m\pi}{2})-\sin(\frac{3m\pi}{2})\}$$

B_{mn}'s are non-vanishing if both m and n are odd numbers. Let $m=2k+1$ and $n=2\ell+1$:

$$B_{2k+1,2\ell+1} = \frac{16(-1)^{k+1}V}{(2k+1)(2\ell+1)\pi^2}\frac{1}{I_{2k+1}\{(2\ell+1)\pi b/L\}}$$

Therefore, the potential

$$\Phi(\rho,\phi,z) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{16(-1)^{k+1}V}{(2k+1)(2\ell+1)\pi^2}\frac{I_{2k+1}\{(2\ell+1)\pi\rho/L\}}{I_{2k+1}\{(2\ell+1)\pi b/L\}}\sin\{\frac{(2\ell+1)\pi}{L}z\}\cos\{(2k+1)\phi\}$$

(b) $L \gg b \Rightarrow \pi\rho/L \ll 1$ and $\pi b/L \ll 1$.

$$\frac{I_{2k+1}\{(2\ell+1)\pi\rho/L\}}{I_{2k+1}\{(2\ell+1)\pi b/L\}} \Rightarrow (\frac{\rho}{b})^{2k+1}$$

The potential at $z=L/2$:

$$\Phi(\rho,\phi,z) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{16(-1)^{k+1}V}{(2k+1)(2\ell+1)\pi^2}(\frac{\rho}{b})^{2k+1}(-1)^{\ell}\cos\{(2k+1)\phi\}$$

$$= \left\{ \sum_{k=0}^{\infty} \frac{16(-1)^{k+1}V}{(2k+1)\pi^2 L} (\frac{\rho}{b})^{2k+1} \cos\{(2k+1)\phi\} \right\} \left\{ \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{2\ell+1} \right\}$$

$$= \frac{4V}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1} (\frac{\rho}{b})^{2k+1} \cos\{(2k+1)\phi\}$$

$$= \frac{4V}{\pi} Re \left\{ \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1} (\frac{\rho}{b})^{2k+1} e^{i(2k+1)\phi} \right\}$$

Note the summation

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1} (\frac{\rho}{b} e^{i\phi})^{2k+1} = i \sum_{k=0}^{\infty} \frac{1}{2k+1} (i\frac{\rho}{b} e^{i\phi})^{2k+1} = \frac{1}{2} i \sum_{p=1}^{\infty} \frac{1-(-1)^p}{p} (i\frac{\rho}{b} e^{i\phi})^p$$

$$= \frac{1}{2} i \left\{ \sum_{p=1}^{\infty} \frac{1}{p} (i\frac{\rho}{b} e^{i\phi})^p - \sum_{p=1}^{\infty} \frac{1}{p} (-i\frac{\rho}{b} e^{i\phi})^p \right\}$$

$$= \frac{1}{2} i \ln \left\{ \frac{1 - i\rho e^{i\phi}/b}{1 + i\rho e^{i\phi}/b} \right\}$$

Furthermore

$$\ln \frac{1 - i\rho e^{i\phi}/b}{1 + i\rho e^{i\phi}/b} = \ln \frac{b + \rho \sin\phi - i\rho\cos\phi}{b - \rho\sin\phi + i\rho\cos\phi}$$

$$= \ln \frac{b^2 - \rho^2 - 2ib\rho\cos\phi}{b^2 + \rho^2 - 2b\rho\sin\phi}$$

$$= \ln(Ae^{i\alpha}) = \ln(A) + i\alpha$$

where

$$A = \sqrt{\frac{b^2 + \rho^2 + 2b\rho\sin\phi}{b^2 + \rho^2 - 2b\rho\sin\phi}}, \qquad \alpha = -\tan^{-1}\{\frac{2b\rho\cos\phi}{b^2 - \rho^2}\}$$

Therefore,

$$Re \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} (\frac{\rho}{b})^{2k+1} e^{i(2k+1)\phi} \right\} = Re \left\{ \frac{1}{2} i\{\ln(A) + i\alpha\} \right\} = -\frac{1}{2}\alpha = \frac{1}{2} \tan^{-1}\{\frac{2b\rho\cos\phi}{b^2 - \rho^2}\}$$

The potential at $z = L/2$ for $L \gg b$ is

$$\Phi(\rho, \phi) = \frac{2V}{\pi} \tan^{-1}\{\frac{2b\rho\cos\phi}{b^2 - \rho^2}\}$$

agrees with the result of Problem 2.13(a).

Problem 3.14

(a) The potential inside is given by

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{r'})G(\vec{r},\vec{r'})d\tau' - \frac{1}{4\pi}\oint \Phi(a,\theta',\phi')\frac{\partial G}{\partial n'}da'$$

Since the sphere is grounded, the potential on the surface $\Phi(a,\theta',\phi') = 0$. The Green function for inside the sphere

$$G(\vec{r},\vec{r'}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\{\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}}\}Y^*_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi)$$

As the result of azimuthal symmetry, $m = 0$, the Green function is simplified:

$$G(\vec{r},\vec{r'}) = \sum_{\ell=0}^{\infty}\{\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}}\}P_\ell(\cos\theta)P_\ell(\cos\theta')$$

To proceed further, one needs to figure out the charge density $\rho(r,\theta,\phi)$. Since the density is non-zero only along the $z-$axis and since it is invariant under $z \leftrightarrow -z$, the charge density must of the form:

$$\rho(\vec{r}) \propto \delta(\cos\theta - 1) + \delta(\cos\theta + 1)$$

Furthermore, since the charge density vanishes for $z^2 > d^2$,

$$\rho(\vec{r}) \propto \Theta(d - r)$$

where $\Theta(x)$ is a step function, *i.e.* $\Theta(x) = 1$ if $x > 0$ and $\Theta(x) = 0$ if $x < 0$. Therefore,

$$\rho(r,\theta,\phi) = f(r)\Theta(d-r)\{\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\}$$

Since the linear charge density varies as $d^2 - z^2$ along the z and the total charge is Q, one gets the linear charge density along the z as:

$$\rho_z(z) = C(d^2 - z^2), \quad \int_{-d}^{d}\rho_z(z)dz = Q, \quad \Rightarrow \quad C = \frac{3Q}{4d^3}, \quad \rho_z(z) = \frac{3Q}{4d^3}(d^2 - z^2)$$

The charge in a spherical shell of radius $r(< d)$ and thickness dr:

$$2\rho_z(z)dz|_{z=r} = \int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta d\theta\rho(r,\theta,\phi)(r^2dr)$$

The above equation leads to

$$f(r) = \frac{3Q}{8\pi d^3}\frac{d^2 - r^2}{r^2}$$

Therefore, the charge density inside the sphere

$$\rho(\vec{r}) = \frac{3Q}{8\pi d^3}\frac{d^2 - r^2}{r^2}\Theta(d-r)\{\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\}$$

The potential inside the sphere

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{r'})G(\vec{r},\vec{r'})d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\frac{3Q}{8\pi d^3}\sum_{\ell=0}^{\infty}P_\ell(\cos\theta)\cdot 2\pi\cdot\{P_\ell(1) + P_\ell(-1)\}\cdot\int_0^d \frac{d^2 - r'^2}{r'^2}\left\{\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}}\right\}r'^2dr'$$

$$= \frac{3Q}{16\pi\epsilon_0 d^3}\sum_{\ell=0}^{\infty}\{1+(-1)^\ell\}P_\ell(\cos\theta)\int_0^d (d^2-r'^2)\left\{\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}}\right\}dr'$$

For $r > d$:

$$\Phi(\vec{r}) = \frac{3Q}{16\pi\epsilon_0 d^3}\sum_{\ell=0}^{\infty}\{1+(-1)^\ell\}P_\ell(\cos\theta)\int_0^d (d^2-r'^2)\left\{\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}}\right\}dr'$$

$$= \frac{3Q}{16\pi\epsilon_0 d^3}\sum_{\ell=0}^{\infty}\{1+(-1)^\ell\}P_\ell(\cos\theta)\int_0^d (d^2-r'^2)\left\{\frac{r'^\ell}{r^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}}\right\}dr'$$

$$= \frac{3Q}{16\pi\epsilon_0 d^3}\sum_{\ell=0}^{\infty}\{1+(-1)^\ell\}P_\ell(\cos\theta)\frac{2d^{\ell+3}}{(\ell+1)(\ell+3)}\left\{\frac{1}{r^{\ell+1}} - \frac{r^\ell}{b^{2\ell+1}}\right\}$$

$$= \frac{3Q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}P_{2n}(\cos\theta)\frac{d^{2n}}{(2n+1)(2n+3)}\left\{\frac{1}{r^{2n+1}} - \frac{r^{2n}}{b^{4n+1}}\right\}$$

The integral for the case of $r < d$ is messier and no need to evaluate it.
(b) The surface charge density

$$\sigma = -\epsilon_0 E_\perp|_{r=b} = \epsilon_0\frac{\partial\Phi}{\partial r}|_{r=b} = -\frac{3Q}{4\pi}\sum_{n=0}^{\infty}\frac{4n+1}{(2n+1)(2n+3)}P_{2n}(\cos\theta)\frac{d^{2n}}{b^{2n+2}}$$

(c) In the limit of $d \to 0$:

$$\Phi(\vec{r}) \Rightarrow \frac{3Q}{4\pi\epsilon_0}P_0(\cos\theta)\frac{1}{3}(\frac{1}{r}-\frac{1}{b}) = \frac{Q}{4\pi\epsilon_0}(\frac{1}{r}-\frac{1}{b})$$

This is the field inside a grounded sphere due to a point charge Q at the origin. The surface charge density

$$\sigma \Rightarrow -\frac{3Q}{4\pi}\frac{1}{3}\frac{1}{b^2} = -\frac{Q}{4\pi b^2}$$

The surface charge is uniformly distributed in this case.

Problem 3.22
The Green function $G(\rho,\phi;\rho',\phi')$ is the solution of the following Poisson equation:

$$\nabla^2 G = -4\pi\delta^3(\vec{r}-\vec{r'}) = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')$$

From the results of Problem 2.24, for $\phi \neq \phi'$, the angular solution $Q(\phi)$ is of the form $Q_m(\phi) \sim \sin(m\pi\phi/\beta)$ and the functions $\sin(m\pi\phi/\beta)$ are complete:

$$\delta(\phi-\phi') = \frac{2}{\beta}\sum_{m=1}^{\infty}\sin(m\pi\phi/\beta)\sin(m\pi\phi'/\beta)$$

Therefore,

$$\nabla^2 G(\rho,\phi;\rho',\phi') = -\frac{8\pi}{\rho\beta}\delta(\rho-\rho')\sum_{m=1}^{\infty}\sin(m\pi\phi/\beta)\sin(m\pi\phi'/\beta)$$

Expanding the Green function in terms of $\sin(m\pi\phi/\beta)\sin(m\pi\phi'/\beta)$:

$$G(\rho,\phi;\rho',\phi') = \sum_{m=1}^{\infty}g_m(\rho,\rho')\sin(m\pi\phi/\beta)\sin(m\pi\phi'/\beta)$$

and plugging into the above Poisson equation:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial g_m}{\partial\rho}) - \frac{1}{\rho^2}(\frac{m\pi}{\beta})^2 = -\frac{8\pi}{\beta\rho}\delta(\rho-\rho')$$

For $\rho \neq \rho'$, the above equation reduces to the radial equation of the Poisson equation in Cylindrical coordinates without $z-$dependence, and the two independent solutions are $\rho^{m\pi/\beta}$ and $\rho^{-m\pi/\beta}$:

$$g_m(\rho,\rho') = A_m\rho^{m\pi/\beta} + \frac{B_m}{\rho^{m\pi/\beta}}$$

For $\rho < \rho'$, the boundary condition $g_m(\rho \to 0, \rho') = 0$ leads to

$$B_m = 0, \qquad g_m(\rho,\rho') = A_m(\rho')\rho^{m\pi/\beta}$$

For $\rho > \rho'$, the boundary condition $g_m(\rho = a, \rho') = 0$ leads to

$$A_m = -\frac{B_m}{a^{2m\pi/\beta}}, \qquad g_m(\rho,\rho') = B_m(\rho')\left\{\frac{1}{\rho^{m\pi/\beta}} - \frac{\rho^{m\pi/\beta}}{a^{2m\pi/\beta}}\right\}$$

Note that the radial function $g_m(\rho,\rho')$ is invariant under $\rho \leftrightarrow \rho'$, this is only possible if

$$g_m = C_m\rho_<^{m\pi/\beta}\left\{\frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}}\right\}$$

where $\rho_> = \max(\rho,\rho')$, $\rho_< = \min(\rho,\rho')$ and C_m's are constants independent of ρ and ρ'. Integrating the above radial equation:

$$\int_{\rho'-\epsilon}^{\rho'+\epsilon} d\rho\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial g_m}{\partial\rho}) - \frac{1}{\rho^2}(\frac{m\pi}{\beta})^2\right\} = -\int_{\rho'-\epsilon}^{\rho'+\epsilon}\frac{8\pi}{\beta\rho}\delta(\rho-\rho')d\rho$$

and letting $\epsilon \to 0$:

$$\frac{\partial g_m}{\partial\rho}|_{\rho'_+} - \frac{\partial g_m}{\partial\rho}|_{\rho'_-} = -\frac{8\pi}{\beta\rho'}$$

Evaluating the above equation:

$$-C_m\frac{1}{\rho'}\frac{m\pi}{\beta}\{1+(\frac{\rho'}{a})^{2m\pi/\beta}\} - C_m\frac{1}{\rho'}\frac{m\pi}{\beta}\{1-(\frac{\rho'}{a})^{2m\pi/\beta}\} = -\frac{8\pi}{\beta\rho'}$$

Therefore, $C_m = 4/m$ and the radial function:

$$g_m(\rho,\rho') = \frac{4}{m}\rho_<^{m\pi/\beta}\left\{\frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}}\right\}$$

Combining radial and angular solutions, we get the Green function:

$$G(\rho,\phi;\rho',\phi') = \sum_{m=1}^{\infty}\frac{4}{m}\rho_<^{m\pi/\beta}\left\{\frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}}\right\}\sin(\frac{m\pi\phi}{\beta})\sin(\frac{m\pi\phi'}{\beta})$$

Problem 3.24
Useful integral:

$$\int_0^b \rho' J_0(k\rho')d\rho' = \frac{b}{k}J_1(kb); \quad \int_a^b \rho' I_0(k\rho')d\rho' = \frac{b}{k}I_1(kb) - \frac{a}{k}I_1(ka); \quad \int_a^b \rho' K_0(k\rho')d\rho' = \frac{a}{k}K_1(ka) - \frac{b}{k}K_1(kb)$$

(a) Using the results of Problem 3.23, we get three forms of expansions of the Green function ($m = 0$ due to $\phi-$invariance):

$$G(\rho, z; \rho', z') = \frac{4}{a}\sum_{n=1}^{\infty} \frac{J_0(x_{0n}\rho/a)J_0(x_{0n}\rho'/a)}{x_{0n}J_1^2(x_{0n})\sinh(x_{0n}L/a)} \sinh(\frac{x_{0n}z_<}{a})\sinh\{\frac{x_{0n}}{a}(L - z_>)\}$$

$$G(\rho, z; \rho', z') = \frac{4}{L}\sum_{n=1}^{\infty} \sin(\frac{n\pi z}{L})\sin(\frac{n\pi z'}{L})\frac{I_0(n\pi\rho_</L)}{I_0(n\pi a/L)}\left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho_>}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho_>}{L})\right\}$$

$$G(\rho, z; \rho', z') = \frac{8}{La^2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sin(k\pi z/L)\sin(k\pi z'/L)J_0(x_{0n}\rho/a)J_0(x_{0n}\rho'/a)}{\{(x_{0n}/a)^2 + (k\pi/L)^2\}J_1^2(x_{0n})}$$

The potential inside the cylinder is given by

$$\Phi(\rho, z) = \frac{1}{4\pi\epsilon_0}\int \rho_c(\vec{r'})G(\vec{r}, \vec{r'})d\tau' - \frac{1}{4\pi}\int \Phi(\vec{r'})\frac{\partial G}{\partial n'}da' = -\frac{V}{2}\int_0^b \frac{\partial G}{\partial z'}|_{z'=L}\rho' d\rho'$$

(i)

$$\frac{\partial G}{\partial z'}|_{z'=L} = \frac{4}{a}\sum_{n=1}^{\infty} \frac{J_0(x_{0n}\rho/a)J_0(x_{0n}\rho'/a)}{x_{0n}J_1^2(x_{0n})\sinh(x_{0n}L/a)}\frac{\partial}{\partial z'}\left\{\sinh(\frac{x_{0n}}{a}z)\sinh(\frac{x_{0n}}{a}(L - z'))\right\}|_{z'=L}$$

$$= -\frac{4}{a^2}\sum_{n=1}^{\infty} \frac{J_0(x_{0n}\rho/a)J_0(x_{0n}\rho'/a)}{J_1^2(x_{0n})\sinh(x_{0n}L/a)}\sinh(\frac{x_{0n}}{a}z)$$

$$\Phi(\rho, z) = \frac{2V}{a^2}\sum_{n=1}^{\infty} \frac{J_0(x_{0n}\rho/a)\sinh(x_{0n}z/a)}{J_1^2(x_{0n})\sinh(x_{0n}L/a)}\int_0^b \rho' J_0(x_{0n}\rho'/a)d\rho'$$

$$= 2V\frac{b}{a}\sum_{n=1}^{\infty} \frac{J_0(x_{0n}\rho/a)J_1(x_{0n}b/a)\sinh(x_{0n}z/a)}{x_{0n}J_1^2(x_{0n})\sinh(x_{0n}L/a)}$$

(ii)

$$\frac{\partial G}{\partial z'}|_{z'=L} = \frac{4\pi}{L^2}\sum_{n=1}^{\infty} n(-1)^n \sin(\frac{n\pi z}{L})\frac{I_0(n\pi\rho_</L)}{I_0(n\pi a/L)}\left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho_>}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho_>}{L})\right\}$$

For $\rho > b$: $\rho_< = \rho'$ and $\rho_> = \rho$:

$$\Phi(\rho, z) = -\frac{2\pi V}{L^2}\sum_{n=1}^{\infty} n(-1)^n \sin(\frac{n\pi z}{L})\left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho}{L})\right\}\int_0^b \rho'\frac{I_0(n\pi\rho'/L)}{I_0(n\pi a/L)}d\rho'$$

$$= -2V\frac{b}{L}\sum_{n=1}^{\infty} (-1)^n \sin(\frac{n\pi z}{L})\left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho}{L})\right\}\frac{I_1(n\pi b/L)}{I_0(n\pi a/L)}$$

For $\rho < b$, the integration has to be break up from $0 \to \rho$ and $\rho \to b$:

$$\Phi(\rho, z) = -\frac{2\pi V}{L^2}\sum_{n=1}^{\infty} n(-1)^n \sin(\frac{n\pi z}{L})\left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho}{L})\right\}\int_0^\rho \rho' \frac{I_0(n\pi\rho'/L)}{I_0(n\pi a/L)}d\rho'$$

$$-\frac{2\pi V}{L^2}\sum_{n=1}^{\infty} n(-1)^n \sin(\frac{n\pi z}{L})\frac{I_0(n\pi\rho/L)}{I_0(n\pi a/L)}\int_\rho^b \left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho'}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho'}{L})\right\}\rho' d\rho'$$

$$= -\frac{2V}{L}\sum_{n=1}^{\infty}(-1)^n \frac{\sin(n\pi z/L)}{I_0(n\pi a/L)}\left\{I_1(\frac{n\pi\rho}{L})\{\rho I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho}{L}) - \rho K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho}{L})\}\right.$$

$$\left. + I_0(\frac{n\pi\rho}{L})\{I_0(\frac{n\pi a}{L})(\rho K_1(\frac{n\pi\rho}{L}) - bK_1(\frac{n\pi b}{L})) - K_0(\frac{n\pi a}{L})(bI_1(\frac{n\pi b}{L}) - \rho I_1(\frac{n\pi\rho}{L}))\right\}$$

$$= -\frac{2V}{L}\sum_{n=1}^{\infty}(-1)^n \frac{\sin(n\pi z/L)}{I_0(n\pi a/L)}$$

$$\left\{\rho I_0(\frac{n\pi a}{L})\{I_1(\frac{n\pi\rho}{L})K_0(\frac{n\pi\rho}{L}) - I_0(\frac{n\pi\rho}{L})K_1(\frac{n\pi\rho}{L})\} - bI_0(\frac{n\pi\rho}{L})\{I_0(\frac{n\pi a}{L})K_1(\frac{n\pi b}{L}) + K_0(\frac{n\pi a}{L})I_1(\frac{n\pi b}{L})\}\right\}$$

(iii)

$$\frac{\partial G}{\partial z'}|_{z'=L} = \frac{8\pi}{L^2 a^2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{k(-1)^k \sin(k\pi z/L)J_0(x_{0n}\rho/a)J_0(x_{0n}\rho'/a)}{\{(x_{0n}/a)^2 + (k\pi/L)^2\}J_1^2(x_{0n})}$$

$$\Phi(\rho, z) = -\frac{4\pi V}{L^2 a^2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{k(-1)^k \sin(k\pi z/L)J_0(x_{0n}\rho/a)}{\{(x_{0n}/a)^2 + (k\pi/L)^2\}J_1^2(x_{0n})}\int_0^b \rho' J_0(x_{0n}\rho'/a)d\rho'$$

$$= -4\pi V\frac{b}{L^2 a}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{k(-1)^k \sin(k\pi z/L)J_0(x_{0n}\rho/a)J_1(x_{0n}b/a)}{x_{0n}\{(x_{0n}/a)^2 + (k\pi/L)^2\}J_1^2(x_{0n})}$$

(b) For $L = 4b$ and $a = 2b$:
(i)

$$\Phi(\rho = 0, z = \frac{L}{2}) = V\sum_{n=1}^{\infty}\frac{J_1(x_{0n}/2)\sinh(x_{0n})}{x_{0n}J_1^2(x_{0n})\sinh(2x_{0n})}$$

(ii)

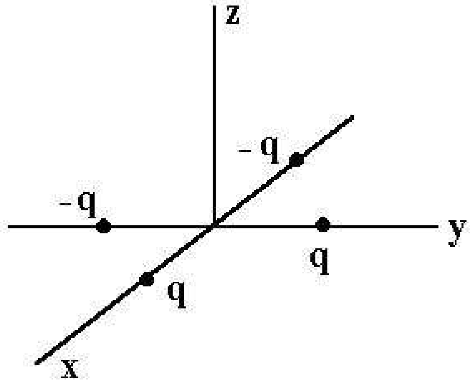
$$\Phi(\rho = 0, z = \frac{L}{2}) = \frac{1}{2}V\sum_{n=1}^{\infty}(-1)^n\frac{\sin(n\pi/2)}{I_0(n\pi/2)}\{I_0(n\pi/2)K_1(n\pi/4) + K_0(n\pi/2)I_1(n\pi/4)\}$$

(iii)

$$\Phi(\rho = 0, z = \frac{L}{2}) = -2\pi V\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{k(-1)^k \sin(k\pi/2)J_1(x_{0n}/2)}{x_{0n}\{4x_{0n}^2 + (k\pi)^2\}J_1^2(x_{0n})}$$

No need to work out numerical numbers.

4.1



$$q_{lm} = \int r^l Y_l^{m*}(\theta,\phi)\rho(\vec{x})d^3x = \sum_i q_i r_l^l Y_l^{m*}(\theta_i,\phi_i)$$

Using

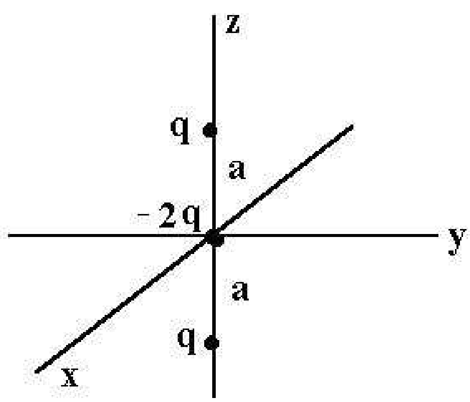
$$Y_l^{m*}(\theta,\phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(x)e^{--m\phi} = N_l^m P_l^m(x)e^{--m\phi}$$

From the figure we get

$$q_{lm} = a^l N_l^m P_l^m(0)q[(1-(-1)^m)(1-i^m)] = 0, \text{ for } m \text{ even, so } m = 2n+1,\ n = 0,1,2,\ldots$$

$$q_{lm} = 2qa^l N_l^m P_l^m(0)[(1-(-1)^n i)]$$

b) The figure for this system is



Since the sum of the charges equals zero, $l \geq 1$.

$$q_{lm} = qa^l[Y_l^{m*}(x=1,\phi) + Y_l^{m*}(x=-1,\phi)] = qa^l N_l^m[P_l^m(1) + P_l^m(-1)]$$

From the Rodrigues formula for $P_l^m(x)$, we see $P_l^m(\pm 1) = 0$, for $m \neq 0$. So

$$q_{lm} = qa^l N_l^0[1+(-1)^l]P_l(1)$$

Thus l is even, but $l \neq 0$

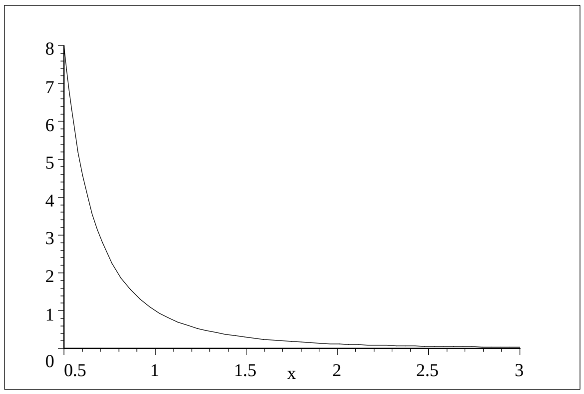
$$q_{lm} = 2qa^l N_l^0$$

c) Using the fact that $N_l^0 = \sqrt{\frac{2l+1}{4\pi}}$ and $Y_l^0 = \sqrt{\frac{2l+1}{4\pi}} P_l$

$$\Phi(\vec{x}) = \sum_{l=2}^{\beta} (2qa^l)\frac{P_l(x)}{r^{l+1}} \approx \frac{2qa^2}{r^3} P_2(x = 0 \text{ on x-y plane})$$

$$\Phi(\vec{x}) = -\frac{qa^2}{r^3}$$

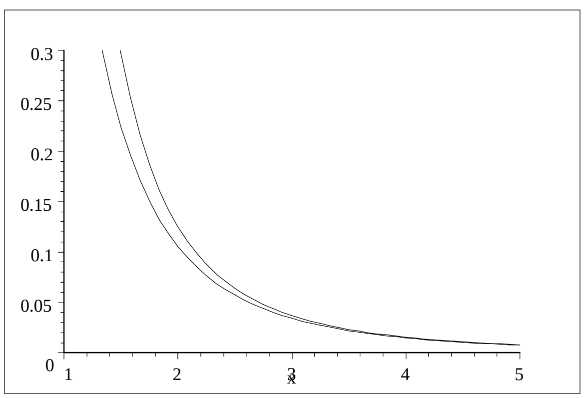
Let us plot $\Phi(\vec{x})/(-q/a)$, ie, $\frac{1}{(\frac{r}{a})^3} = \frac{1}{x^3}$



The exact answer on the x-y plane is

$$\Phi(\vec{x}) = \frac{-q}{a}\left[\frac{2}{x} - \frac{2}{x\sqrt{1+\frac{1}{x^2}}}\right] = \frac{-q}{a}\left(\left(\frac{1}{x}\right)^3 - \frac{3}{4}\left(\frac{1}{x}\right)^5 + \frac{5}{8}\left(\frac{1}{x}\right)^7 - \frac{35}{64}\left(\frac{1}{x}\right)^9 + \ldots\right)$$

So let's plot $\frac{1}{x^3}, \frac{2}{x} - \frac{2}{x\sqrt{1+\frac{1}{x^2}}}$



where the smaller is the exact answer.

4.2

We want to show that we can obtain the potential and potential energy of an elementary diplole:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{x}}{r^3}$$

$$W = -\vec{p}\cdot\vec{E}(0)$$

from the general formulas

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{x}')d^3x'}{|\vec{x}-\vec{x}'|}$$

$$W = \int\rho(\vec{x})\Phi(\vec{x})d^3x$$

using the effective charge density

$$\rho_{eff} = -\vec{p}\cdot\vec{\nabla}\delta(\vec{x})$$

where I've chosen the origin to be at $\vec{x}_0$.

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int\frac{-\vec{p}\cdot\vec{\nabla}'\delta(\vec{x}')d^3x'}{|\vec{x}-\vec{x}'|} = -\frac{1}{4\pi\varepsilon_0}\vec{p}\cdot\int\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\delta(\vec{x}')d^3x'$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{x}}{r^3}$$

Similarly,

$$W = \int\rho(\vec{x})\Phi(\vec{x})d^3x = -\int\vec{p}\cdot\vec{\nabla}\delta(\vec{x})\Phi(\vec{x})d^3x = \vec{p}\cdot\int\delta(\vec{x})\vec{\nabla}\Phi(\vec{x})d^3x = -\vec{p}\cdot\vec{E}(0)$$

4.6

a) We know that

$$W = -\frac{1}{6}\sum_i Q_{ii}\frac{\partial}{\partial x_i}E_i(0)$$

The problem is cylindrically symmetric, so $Q_{11} = Q_{22}$. Using the fact that the trace of the quadrupole tensor is zero, we see

$$Q_{11} = Q_{22} = -\frac{1}{2}Q_{33}$$

The book defines the quadrupole moment in nucleii to be $Q = \frac{1}{e}Q_{33}$. The electric field in our formula for W refers to the external electric field, so within the nucleus $\vec{\nabla}\cdot\vec{E} = 0$, or

$$\frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y = -\frac{\partial}{\partial z}E_z$$

Thus

$$W = -\frac{eQ}{6}\left(\frac{\partial}{\partial z}E_z - \frac{1}{2}\left(\frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y\right)\right)_0 = -\frac{eQ}{6}\left(\frac{\partial}{\partial z}E_z - \frac{1}{2}\left(-\frac{\partial}{\partial z}E_z\right)\right)_0$$

$$W = -\frac{eQ}{6}\left(\frac{\partial}{\partial z}E_z\right)_0\left(1+\frac{1}{2}\right) = -\frac{eQ}{4}\left(\frac{\partial}{\partial z}E_z\right)_0$$

b)

$$\left(\frac{\partial}{\partial z}E_z\right)_0 = -\frac{4W}{eQ} = -\frac{4W}{eQ\left(\frac{e}{4\pi\varepsilon_0 a_0^3}\right)}\left(\frac{e}{4\pi\varepsilon_0 a_0^3}\right)$$

Now from the particle data book,

$$\frac{e^2}{4\pi\varepsilon_0} = \alpha\hbar c = \frac{\alpha hc}{2\pi}, \text{ with } \alpha = 1/137$$

So

$$\frac{4W}{eQ\left(\frac{e}{4\pi\varepsilon_0 a_0^3}\right)} = \frac{4\,(W/h)\,2\pi a_0^3}{Q\alpha c} = \frac{4\cdot 10^7\text{sec}^{-1}2\pi\left(0.529\times 10^{-10}\right)^3\text{m}^3}{2\times 10^{-28}\text{m}^2\,(1/137)\times 3\times 10^8\text{m/sec}} = 0.085$$

$$\left(\frac{\partial}{\partial z}E_z\right)_0 = -0.085\left(\frac{e}{4\pi\varepsilon_0 a_0^3}\right)$$

c) Let us assume the spheroid is gotten by a rotation about the semimajor axis. The equation for a spheroid is given by

$$\frac{x^2+y^2}{b^2} + \frac{z^2}{a^2} = 1$$

The volume of the spheroid is

$$V = \int_0^{2\pi} d\phi \int_0^b \rho d\rho \int_{-a\sqrt{1-\rho^2/b^2}}^{a\sqrt{1-\rho^2/b^2}} dz = \frac{4\pi}{3} ab^2$$

where $\rho^2 = x^2 + y^2$.
Thus the charge density of the nucleus is

$$\rho_c = \frac{3Ze}{4\pi ab^2}$$

$$Q_{33} = \rho_c 2\pi \int_0^b \rho d\rho \int_{-a\sqrt{1-\rho^2/b^2}}^{a\sqrt{1-\rho^2/b^2}} \left(2z^2 - \rho^2\right) dz$$

$$Q_{33} = \rho_c 2\pi \int_0^b \rho \left(\frac{2}{3} a \sqrt{\left(\frac{b^2-\rho^2}{b^2}\right)} \frac{2a^2b^2 - 2a^2\rho^2 - 3\rho^2 b^2}{b^2} \right) d\rho = \rho_c 2\pi \frac{4ab^2\left(a^2-b^2\right)}{15}$$

$$Q_{33} = \left(\frac{3Ze}{4\pi ab^2}\right) 2\pi \frac{4ab^2\left(a^2-b^2\right)}{15} = \frac{2}{5} Ze \left(a^2 - b^2\right)$$

So

$$Q = \frac{2}{5} Z \left(a^2 - b^2\right) = \frac{4}{5} Z (a-b)(a+b)/2 = \frac{4}{5} ZR (a-b)$$

Or

$$\frac{(a-b)}{R} = \frac{5Q}{4ZR^2} = \frac{5 \cdot 2.5 \times 10^{-28} \text{m}^2}{4 \cdot 63 \cdot \left(7 \times 10^{-15}\right)^2 \text{m}^2} = 0.101$$

4.7

a) Since ρ does not depend on $\phi,$ we can write it in terms of spherical harmonics with $m = 0.$ First note

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\left(1-\sin^2\theta\right) - \frac{1}{2}\right)$$

or

$$\sin^2\theta = -\frac{2}{3}\sqrt{\frac{4\pi}{5}}\,Y_2^0 + \sqrt{4\pi}\,\frac{2}{3}Y_0^0$$

Thus only the $m = 0, l = 0, 2$ multipoles contribute.

$$q_{00} = \frac{2\sqrt{4\pi}}{3}\int_0^\beta r^2\left(\frac{1}{64\pi}r^2e^{-r}\right)dr = \frac{2\sqrt{4\pi}}{3}\,\frac{3}{8\pi} = \frac{1}{2\sqrt{\pi}}$$

$$q_{20} = -\frac{2}{3}\sqrt{\frac{4\pi}{5}}\int_0^\beta r^4\left(\frac{1}{64\pi}r^2e^{-r}\right)dr = -\frac{2}{3}\sqrt{\frac{4\pi}{5}}\,\frac{45}{4\pi} = -3\frac{\sqrt{5}}{\sqrt{\pi}}$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[4\pi q_{00}\frac{Y_0^0}{r} + 4\pi q_{20}\frac{Y_2^0}{5r^3}\right] = \frac{1}{4\pi\varepsilon_0}\left[\sqrt{4\pi}\,q_{00}\frac{P_0}{r} + \sqrt{\frac{4\pi}{5}}\,q_{20}\frac{P_2}{r^3}\right]$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{P_0}{r} - 6\frac{P_2}{r^3}\right]$$

b)

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{x}')d^3x'}{|\vec{x}-\vec{x}'|}$$

Using

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{lm}\frac{1}{(2l+1)r_>}\left(\frac{r_<}{r_>}\right)^l Y_l^{m*}(\theta',\phi')Y_l^m(\theta,\phi)$$

, we see only the $l = 0, 2$ and $m = 0$ terms of the expansion contribute in the potential. Next take $r' > r.$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}4\pi\sum_{lm}\frac{1}{(2l+1)}r^lY_l^m(\theta,\phi)\int Y_l^{m*}(\theta',\phi')r'^2d\Omega'\,\frac{\rho(\vec{x}')}{r'^{l+1}}dr'$$

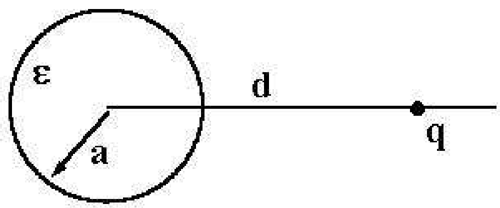
$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}4\pi\left[Y_0^0\sqrt{4\pi}\,\frac{2}{3}\int_0^\beta\left(\frac{1}{64\pi}r^2e^{-r}\right)rdr + \frac{Y_2^0}{5}r^2\left(-\frac{2}{3}\sqrt{\frac{4\pi}{5}}\int_0^\beta\left(\frac{1}{64\pi}r^2e^{-r}\right)\frac{1}{r}dr\right)\right]$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}4\pi\left[Y_0^0\sqrt{4\pi}\,\frac{2}{3}\,\frac{3}{32\pi} + \frac{Y_2^0}{5}r^2\left(-\frac{2}{3}\sqrt{\frac{4\pi}{5}}\right)\frac{1}{64\pi}\right]$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}4\pi\left[P_0\frac{2}{3}\,\frac{3}{32\pi} + \frac{P_2}{5}r^2\left(-\frac{2}{3}\right)\frac{1}{64\pi}\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{P_0}{4} - \frac{r^2P_2}{120}\right]$$

4.9

a) The system is described by



Since there is azimuthal symmetry, choosing the z-axis through q,

$$\Phi_{out} = \frac{1}{4\pi\varepsilon_0}\left(\sum_l B_l r^{-l-1}P_l + \frac{q}{|\vec{x}-\vec{x}'|}\right)$$

$$\Phi_{out} = \frac{1}{4\pi\varepsilon_0}\left(\sum_l B_l r^{-l-1}P_l + \frac{q}{r_>}\sum_l\left(\frac{r_<}{r_>}\right)^l P_l\right)$$

$$\Phi_{in} = \frac{1}{4\pi\varepsilon_0}\left(\sum_l A_l r^l P_l + \frac{q}{r_>}\sum_l\left(\frac{r_<}{r_>}\right)^l P_l\right)$$

Boundary conditions: At the surface, $r' = d = r_>,\ r = a = r_<$.

1) $\Phi_{out} = \Phi_{in}|_{r=a}$, or

$$B_l = A_l a^{2l+1}$$

2) $\varepsilon\frac{\partial}{\partial r}\Phi_{in} = \frac{\partial}{\partial r}\Phi_{out}|_{r=a}$, or letting $k = \frac{\varepsilon}{\varepsilon_0}$

$$k\left[\sum_l lA_l a^{l-1}P_l + \frac{q}{d}l\left(\frac{a^{l-1}}{d^l}\right)P_l\right] = \left[\sum_l -(l+1)B_l a^{-l-2}P_l + \frac{q}{d}l\left(\frac{a^{l-1}}{d^l}\right)P_l\right]$$

$$= \left[\sum_l -(l+1)A_l a^{l-1}P_l + \frac{q}{d}l\left(\frac{a^{l-1}}{d^l}\right)P_l\right]$$

or

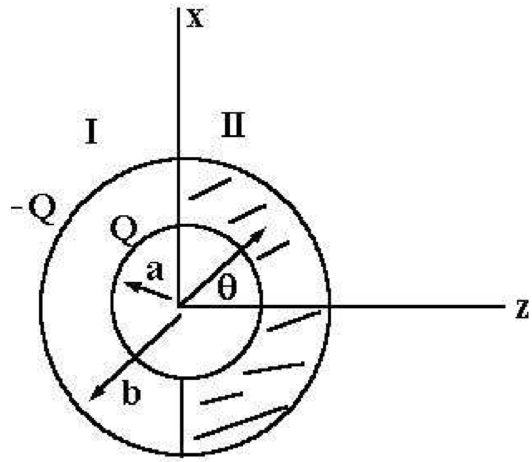
$$A_l = \frac{a(1-k)l}{[(1+k)l+1]d^{l+1}}$$

$$B_l = \frac{a(1-k)la^{2l+1}}{[(1+k)l+1]d^{l+1}}$$

Remember that $P_l = \sqrt{\frac{4\pi}{2+1}}\ Y_l^0$, and substitute the above coefficients into the expansion to get the answer requested by the problem.

4.10

The system is described by



a) Since there is azimutal symmetry,

$$\Phi(r,\theta) = \sum_l (A_l r^l + B_l r^{-l-1}) P_l(\cos\theta)$$

Also

$$\vec{D} = \varepsilon(\vec{x})\vec{E} = -\varepsilon(\vec{x})\vec{\nabla}\Phi(r,\theta)$$

$$D_r = -\varepsilon(\vec{x}) \sum_l (l A_l r^{l-1} - (l+1) B_l r^{-l-2}) P_l(\cos\theta)$$

between the spheres,

$$\int D_r d\Omega r^2 = Q, \text{ and is independent of } r.$$

Thus

$$A_l = 0,\ B_l = 0,\ l\ \Phi\ 0 \rightarrow D_r = \frac{\varepsilon(\vec{x})B_0}{r^2}$$

$$\int D_r d\Omega r^2 = 2\pi B_0 \left(\varepsilon_0 \int_{-1}^{0} d\cos\theta + \varepsilon \int_0^1 d\cos\theta \right) = 2\pi B_0(\varepsilon_0 + \varepsilon) = Q$$

$$B_0 = \frac{Q}{2\pi\varepsilon_0(1 + \frac{\varepsilon}{\varepsilon_0})}$$

$$\vec{E} = \frac{Q}{2\pi\varepsilon_0(1 + \frac{\varepsilon}{\varepsilon_0})r^2}\hat{r}$$

b)

$$\int D_r dA = D_r A = \sigma_f A \rightarrow \sigma_f = D_r = \varepsilon(\vec{x})E_r$$

$$\sigma_f = \frac{\varepsilon Q}{2\pi\varepsilon_0(1 + \frac{\varepsilon}{\varepsilon_0})r^2}, \qquad \cos\theta \geq 0$$

$$\sigma_f = \frac{Q}{2\pi(1+\frac{\varepsilon}{\varepsilon_0})r^2}, \qquad \cos\theta < 0$$

c)

$$\int \rho_{pol} dV = \sigma_{pol} A = \int -\vec{\nabla} \cdot \vec{P} dV = -PA \to \sigma_{pol} = -P = -\varepsilon_0 \chi_e E$$

$$\sigma_{pol} = -(\varepsilon(\vec{x})/\varepsilon_0 - 1)\frac{Q}{2\pi(1+\frac{\varepsilon}{\varepsilon_0})r^2}$$

Notice

$$\sigma_{pol} + \sigma_f = \sigma_{tot} = \frac{Q}{2\pi(1+\frac{\varepsilon}{\varepsilon_0})r^2} = \varepsilon_0 E, \quad \text{as expected.}$$

More Problems for Chapter 4

Problem 4.1
(a) The charge density

$$\rho(\vec{r}) = \frac{q}{r^2}\delta(r-a)\delta(\cos\theta)\{\delta(\phi)+\delta(\phi-\frac{\pi}{2})-\delta(\phi-\pi)-\delta(\phi-\frac{3\pi}{2})\}$$

The multipole moment

$$q_{\ell m} = \int Y^*_{\ell m}(\theta',\phi')r'^{\ell}\rho(\vec{r'})d\tau' = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{\ell+m)!}}qa^{\ell}P^m_{\ell}(0)\{1+e^{-im\pi/2}-e^{-im\pi}-e^{-i3m\pi/2}\}$$

$$= \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{\ell+m)!}}qa^{\ell}P^m_{\ell}(0)\left\{\{1-(-1)^m\}\{1+e^{-im\pi/2}\}\right\}$$

Since $P^m_{\ell}(x)$ is odd if $\ell+m$ =odd, $P^m_{\ell}(0)$ vanishes unless $\ell+m$ =even. Furthermore, $q_{\ell m}$ vanishes if m is even. Therefore, for non-vanishing $q_{\ell m}$, both ℓ and m must be odd. Let $\ell = 2j+1$ and $m = 2k+1$:

$$q_{\ell=2j+1,m=2k+1} = 2\{1+(-1)^{k+1}i\}qa^{\ell}\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P^m_{\ell}(0)$$

$$= -\frac{1-(-1)^k i}{2^{\ell}\ell!}qa^{\ell}\sqrt{\frac{2\ell+1}{\pi}\frac{(\ell-m)!}{(\ell+m)!}}\left\{\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^{\ell}\right\}|_{x=0}$$

The first two sets of non-vanishing moments are

$$q_{1,\pm1} = \mp(1\mp i)\sqrt{\frac{3}{2\pi}}qa$$

$$q_{3,\pm1} = \pm(1\mp i)\sqrt{\frac{21}{16\pi}}qa^3$$

$$q_{3,\pm3} = \mp(1\pm i)\sqrt{\frac{35}{16\pi}}qa^3$$

(b) The charge density:

$$\rho(\vec{r}) = \frac{q}{2\pi r^2}\{\delta(r-a)\delta(\cos\theta-1)+\delta(r-a)\delta(\cos\theta+1)-\delta(r)\}$$

The multipole moment

$$q_{\ell m} = \int Y^*_{\ell m}(\theta',\phi')r'^{\ell}\rho(\vec{r'})d\tau' = \frac{q}{2\pi}\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\int_0^{2\pi}e^{-im\phi'}d\phi'\{a^{\ell}P^m_{\ell}(1)+a^{\ell}P^m_{\ell}(-1)-2\delta_{\ell,0}\}$$

$$= q\sqrt{\frac{2\ell+1}{4\pi}}\{a^{\ell}P_{\ell}(1)+a^{\ell}P_{\ell}(-1)-2\delta_{\ell 0}\}\delta_{m0} = q\sqrt{\frac{2\ell+1}{4\pi}}\{a^{\ell}(1+(-1)^{\ell})-2\delta_{\ell 0}\}\delta_{m0}$$

The first two sets of non-vanishing moments are:

$$q_{2,0} = \sqrt{\frac{5}{\pi}}qa^2; \qquad q_{2,m\neq 0} = 0$$

$$q_{4,0} = \sqrt{\frac{9}{\pi}} q a^4; \qquad q_{4,m\neq 0} = 0$$

(c) The potential

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{\ell m}\frac{4\pi}{2\ell+1} q_{\ell m}\frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}} = \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{a^\ell\{1+(-1)^\ell\}-2\delta_{\ell,0}}{r^{\ell+1}}P_\ell(\cos\theta)$$

$$= \frac{q}{2\pi\epsilon_0}\sum_{k=1}^{\infty}\frac{a^{2k}}{r^{2k+1}}P_{2k}(\cos\theta) = \frac{q}{4\pi\epsilon_0}\frac{a^2}{r^3}(3\cos^2\theta - 1) + ...$$

In the $x-y$ plane, $\cos\theta = 0$:

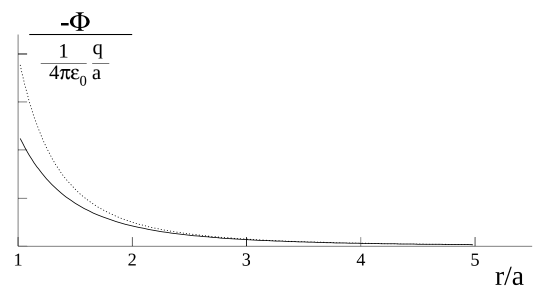
$$\Phi(r,\theta=\pi/2) = -\frac{q}{4\pi\epsilon_0}\frac{a^2}{r^3} + ...$$

(d) The exact potential in the $(x-y)$ plane

$$\Phi(r,\theta=\pi/2) = \frac{1}{4\pi\epsilon_0}\left\{\frac{2q}{\sqrt{r^2+a^2}} - \frac{2q}{r}\right\}$$

$$= \frac{q}{2\pi\epsilon_0}\frac{1}{r}\left\{\frac{1}{\sqrt{1+(a/r)^2}} - 1\right\} = -\frac{q}{4\pi\epsilon_0}\frac{a^2}{r^3} + ...$$

agrees with the result of (c).



The potential (in units of $(q/4\pi\epsilon_0 a)$) in $x-y$ plane as functions of r/a. The dotted line is the approximation from (c) and the solid line is the exact calculation of (d).

Problem 4.2
The potential at $\vec{r}$ due to a point dipole $\vec{p}$ at $\vec{r_0}$:

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\vec{p}\cdot\frac{\vec{r}-\vec{r_0}}{|\vec{r}-\vec{r_0}|^3} = \frac{1}{4\pi\epsilon_0}\int_V\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}\cdot\left\{\vec{p}\delta^3(\vec{r'}-\vec{r_0})\right\}d\tau' = \frac{1}{4\pi\epsilon_0}\int_V\nabla'\left\{\frac{1}{|\vec{r}-\vec{r'}|}\right\}\cdot\left\{\vec{p}\delta^3(\vec{r'}-\vec{r_0})\right\}d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int_V\nabla'\cdot\left\{\frac{\vec{p}\delta^3(\vec{r'}-\vec{r_0})}{|\vec{r}-\vec{r'}|}\right\}d\tau' - \frac{1}{4\pi\epsilon_0}\int_V\frac{1}{|\vec{r}-\vec{r'}|}\nabla'\cdot\left\{\vec{p}\delta^3(\vec{r'}-\vec{r_0})\right\}d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\oint\frac{\vec{p}\cdot\vec{n'}}{|\vec{r}-\vec{r'}|}\delta^3(\vec{r'}-\vec{r_0})da' + \frac{1}{4\pi\epsilon_0}\int_V\frac{1}{|\vec{r}-\vec{r'}|}\left\{-\vec{p}\cdot\nabla'\delta^3(\vec{r'}-\vec{r_0})\right\}d\tau'$$

where $\vec{r_0}$ is inside the volume V. Therefore, the surface integral vanishes:

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{1}{|\vec{r}-\vec{r'}|}\left\{-\vec{p}\cdot\nabla'\delta^3(\vec{r'}-\vec{r_0})\right\}d\tau'$$

which is the potential by an effective charge density

$$\rho_{\rm eff}(\vec{r}) = -\vec{p}\cdot\nabla\delta^3(\vec{r}-\vec{r_0})$$

The energy of the dipole in an electric field:

$$W = -\vec{p}\cdot\vec{E}(\vec{r_0}) = \int_V \delta^3(\vec{r'}-\vec{r_0})\left\{-\vec{p}\cdot\vec{E}(\vec{r'})\right\}d\tau' = \int_V \delta^3(\vec{r'}-\vec{r_0})\vec{p}\cdot\nabla'\Phi(\vec{r'})d\tau'$$

$$= \int_V \left\{\nabla'\cdot\{\vec{p}\Phi(\vec{r'})\delta^3(\vec{r'}-\vec{r_0})\} - \Phi(\vec{r'})\nabla'\cdot\{\vec{p}\delta^3(\vec{r'}-\vec{r_0})\}\right\}d\tau'$$

$$= \oint(\vec{p}\cdot\vec{n'})\Phi(\vec{r'})\delta^3(\vec{r'}-\vec{r_0})da' + \int_V\left\{-\vec{p}\cdot\nabla'\delta^3(\vec{r'}-\vec{r_0})\right\}\Phi(\vec{r'})d\tau'$$

Again the surface integral vanishes since $\vec{r_0}\in V$. Therefore,

$$W = \int_V\left\{-\vec{p}\cdot\nabla'\delta^3(\vec{r'}-\vec{r_0})\right\}\Phi(\vec{r'})d\tau'$$

which is the energy of a distribution of charge density

$$\rho_{\rm eff}(\vec{r}) = -\vec{p}\cdot\nabla\delta^3(\vec{r}-\vec{r_0})$$

Problem 4.10

(a) The electric fields in the two regions must be the same (otherwise, it will lead to different potential differences between the inner and the other spheres in the two regions). Applying Gauss's law in dielectrics on a Gaussian surface of radius r ($a<r<b$) and noting $\vec{D}$ is along the radial direction by symmetry

$$\oint\vec{D}\cdot\vec{n}da = Q; \quad\Rightarrow\quad (\epsilon E+\epsilon_0 E)2\pi r^2 = Q$$

Therefore, the electric field everywhere between the sphere is

$$\vec{E} = \frac{Q}{2\pi(\epsilon+\epsilon_0)}\frac{\vec{r}}{r^3}$$

(b) The free surface charge densities on the inner sphere are:

$$\sigma = \epsilon_0\vec{E}(r=a)_\perp = \frac{Q}{2\pi a^2}\frac{\epsilon_0}{\epsilon+\epsilon_0} \qquad \text{the region without the dielectric}$$

$$\sigma = \epsilon\vec{E}(r=a)_\perp = \frac{Q}{2\pi a^2}\frac{\epsilon}{\epsilon+\epsilon_0} \qquad \text{the region with the dielectric}$$

(c) The polarization in the region with the dielectric:

$$\vec{P} = (\epsilon-\epsilon_0)\vec{E} = \frac{Q}{2\pi}\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\frac{\vec{r}}{r^3}$$

Therefore, the polarization surface charge density

$$\sigma_b = \{\vec{P}\cdot\vec{n}\}_{r=a} = -P_r(r=a) = -\frac{Q}{2\pi a^2}\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}$$

In the region without the dielectric, the polarization surface charge density $\sigma_b = 0$.

Problem 4.13

At equilibrium, the electrostatic force balances the gravity. For a fixed potential difference V, the electrostatic force is given by

$$F_e = \frac{dW}{dh} = \frac{1}{2}V^2\frac{dC}{dh}$$

where C is the total capacitance of the section above the liquid surface:

$$C = C_h + C_{\ell-h}$$

Here C_h is the capacitance of the section with the liquid in between the two electrodes and $C_{\ell-h}$ is the capacitance of the section above, ℓ is the height above the liquid surface. Note for a cylindrical capacitor in vacuum, the capacitance per unit length is

$$C_0 = \frac{2\pi\epsilon_0}{\ln(b/a)}$$

Therefore,

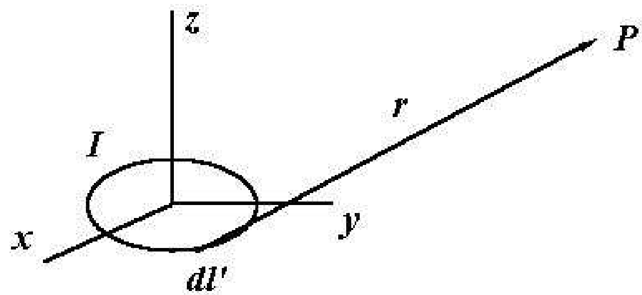
$$C = \frac{2\pi\epsilon_0(1+\chi_e)h}{\ln(b/a)} + \frac{2\pi\epsilon_0(\ell-h)}{\ln(b/a)} = \frac{2\pi\epsilon_0}{\ln(b/a)}\{\chi_e h + \ell\}$$

$$F_e = \frac{1}{2}V^2\frac{dC}{dh} = \frac{\pi\epsilon_0\chi_e V^2}{\ln(b/a)}, \qquad F_g = \rho\pi(b^2-a^2)hg$$

$$F_e = F_g \qquad \Rightarrow \qquad \chi_e = \frac{(b^2-a^2)\rho hg\ln(b/a)}{\epsilon_0 V^2}$$

5.1

The system is described by



We want to show

$$\phi_m = -\frac{\mu_0 I}{4\pi}\Omega$$

Suppose the observation point is moved by a displacement $\delta\vec{x}$, or equivlently that the loop is displaced by $-\delta\vec{x}$.

If we are to have $\vec{B} = -\vec{\nabla}\phi_m$, then

$$\delta\phi_m = -\delta\vec{x} \cdot \vec{B}$$

Using the law of Biot and Savart,

$$\delta\phi_m = -\frac{\mu_0 I}{4\pi}\oint \delta\vec{x} \cdot \frac{\left(d\vec{l}' \times \vec{r}\right)}{r^3} = -\frac{\mu_0 I}{4\pi}\oint \vec{r} \cdot \frac{\left(\delta\vec{x} \times d\vec{l}'\right)}{r^3} = -\frac{\mu_0 I}{4\pi}\oint \hat{r} \cdot \frac{\left(\delta\vec{x} \times d\vec{l}'\right)}{r^2}$$

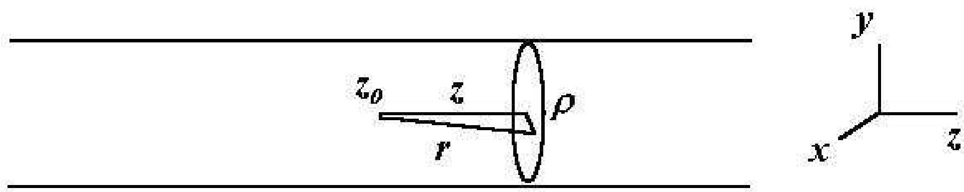
$$\delta\phi_m = -\frac{\mu_0 I}{4\pi}\oint \frac{\hat{r} \cdot \delta(dA)}{r^2} = -\frac{\mu_0 I}{4\pi}\delta\Omega$$

Or,

$$\phi_m = -\frac{\mu_0 I}{4\pi}\Omega$$

5.2

a) The system is described by



First consider a point at the axis of the solenoid at point z_0. Using the results of problem 5.1,

$$d\phi_m = \frac{\mu_0}{4\pi} NIdz\Omega$$

From the figure,

$$\Omega = \int \frac{\hat{r} \cdot d\vec{A}}{r^2} = \int \frac{dA\cos\theta}{r^2} = 2\pi z \int_0^R \frac{\rho d\rho}{(\rho^2 + z^2)^{3/2}} = 2\pi\left(-\frac{z}{\sqrt{(R^2 + z^2)}} + 1 \right)$$

$$\phi_m = \frac{\mu_0}{2} NI \int_{z_0}^{\beta} z\left(-\frac{1}{\sqrt{(R^2 + z^2)}} + \frac{1}{z} \right) dz = \frac{\mu_0}{2} NI\left(-z_0 + \sqrt{(R^2 + z_0^2)} \right)$$

$$B_r = -\frac{\mu_0}{2} NI \frac{\partial}{\partial z_0}\left(-z_0 + \sqrt{(R^2 + z_0^2)} \right) = \frac{\mu_0}{2} NI \frac{-z_0 + \sqrt{(R^2 + z_0^2)}}{\sqrt{(R^2 + z_0^2)}}$$

In the limit $z_0 \to 0$

$$B_r = \frac{\mu_0}{2} NI$$

By symmetry, thej loops to the left of z_0 give the same contribution, so

$$B = B_l + B_r = \mu_0 NI$$

$$H = NI$$

By symmetry, $\vec{B}$ is directed along the z axis, so

$$\delta\phi_m = -\delta\vec{\rho} \cdot \vec{B} = 0$$

if $\delta\vec{\rho}$ is directed $\perp$ to the z axis. Thus for a given z, ϕ_m is independent of ρ, and consequently

$$H = NI$$

everywhere within the solenoid.

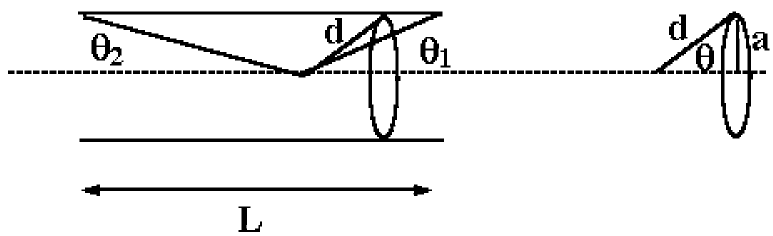
If you are on the outside of the solenoid at position z_0, by symmetry the magnetic field must be in the z direction. Thus using the above argument, ϕ_m must not depend on ρ. Set us take ρ far away from the axis of the solenoid, so that we can replace the loops by elementary dipoles $\vec{m}$ directed along the z axis. Thus for any point z_0 we will have a contributions

$$\phi_m \propto \left(\frac{\vec{m} \cdot \vec{r}_1}{r_1^3} + \frac{\vec{m} \cdot \vec{r}_2}{r_2^3} \right)$$

where $\vec{m} \cdot \vec{r}_1 = -\vec{m} \cdot \vec{r}_2$ and $r_1 = r_2$. Thus

$$H = 0$$

5.3 The system is described by



The law of Biot and Savart says

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l}\times\hat{r}}{r^2}$$

From the figure, for one loop

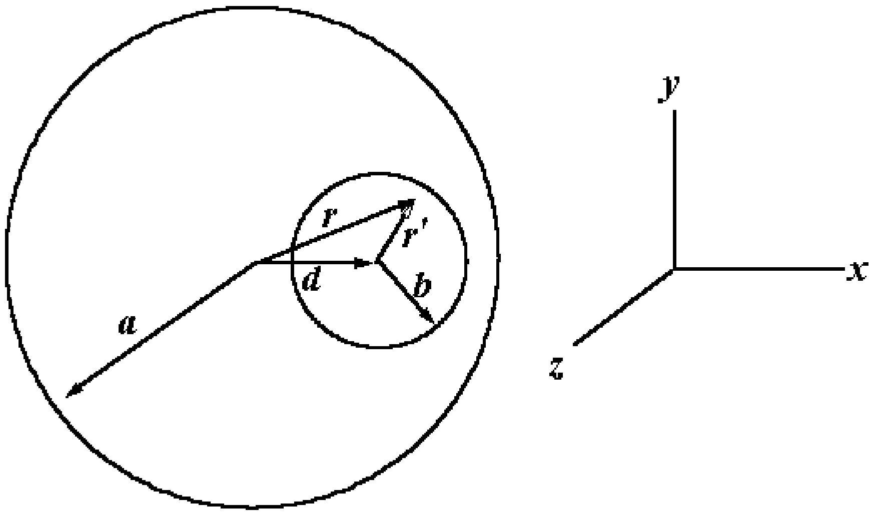
$$B_z = \frac{\mu_0}{4\pi}\frac{I2\pi a\sin\theta}{d^2} = \frac{\mu_0}{4\pi}\frac{I2\pi\sin^3\theta}{a}$$

As $NL \to \infty$, $dN = Ndz$, but $\frac{d\theta}{dz} = \frac{\sin\theta}{d}$, $d = \frac{a}{\sin\theta}$, so $dN = N\frac{ad\theta}{\sin^2\theta}$

$$B_{ztot} = \int B_z dN = \frac{\mu_0}{4\pi}I2\pi N\int_{\theta_2}^{\pi-\theta_1}\sin\theta d\theta = \frac{\mu_0 IN}{2}\left[\cos\theta_2 - \cos\left(\pi-\theta_1\right)\right]$$

$$B_{ztot} = \frac{\mu_0 IN}{2}\left[\cos\theta_2 + \cos\theta_1\right]$$

5.6 We may choose the coordinate system so the currents and hole are aligned as



Here, I'm taking the *z* axis as out of the paper. Then, applying the superposition principle, we can replace this system by one where a current $\vec{J}$ fills the whole wire and is in the z direction, while an opposite current $\vec{J'} = -\vec{J}$ flows in a wire the size of the hole and is located where the hole previously was.

From Ampere's law we can work out the magnitude of the magnetic flux density

$$\int \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{a} = \mu_0 J \pi r^2 = B 2\pi r \rightarrow B = \frac{\mu_0 J r}{2}$$

Similarly

$$B' = \frac{\mu_0 J r'}{2}$$

Putting in the directions

$$\vec{B} = \frac{\mu_0 J \hat{z} \times \vec{r}}{2}$$

and

$$\vec{B'} = \frac{\mu_0 J \left(-\hat{z}\right) \times \vec{r'}}{2}$$

$$\vec{B}_{tot} = \vec{B} + \vec{B'} = \frac{\mu_0 J \hat{z} \times \left(\vec{r} - \vec{r'}\right)}{2}$$

However, from the figure, $\vec{r} = \vec{d} + \vec{r'}$, so

$$\vec{B}_{tot} = \frac{\mu_0 J \hat{z} \times \vec{d}}{2} = \frac{\mu_0 J d}{2} \hat{y}$$

Thus we conclude the magnetic flux density in the hole is a constant, $B_{tot} = \frac{\mu_0 J d}{2}$, and it is directed in the y direction.

5.8

Using the same arguments that lead to Eq. (5.35), we can write

$$A_\phi = \frac{\mu_0}{4\pi}\int \frac{d^3x' \cos\phi' J_\phi(r',\theta')}{|\vec{x}-\vec{x}'|}$$

Choose $\vec{x}$ in the $x-z$ plane. Then we use the expansion

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l,m} \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_l^{m*}(\theta',\phi')Y_l^m(\theta,0)$$

The $\cos\phi'$ factor leads to only an $m = 1$ contribution in the expansion. Using

$$Y_l^m(\theta,0) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)$$

and $\frac{(l-1)!}{(l+1)!} = \frac{1}{l(l+1)}$, we have on the **inside**

$$A_\phi = \frac{\mu_0}{4\pi}\sum_l \frac{1}{l(l+1)} r^l P_l^1(\cos\theta)\int d^3x' \frac{P_l^1(\cos\theta')J_\phi(r',\theta')}{r'^{l+1}}$$

which can be written

$$A_\phi = -\frac{\mu_0}{4\pi}\sum_l m_l r^l P_l^1(\cos\theta)$$

with

$$m_l = -\frac{1}{l(l+1)}\int d^3x' \frac{P_l^1(\cos\theta')J_\phi(r',\theta')}{r'^{l+1}}$$

A similar expression can be written on the **outside** by redefining $r_<$ and $r_>$.

5.10

a) From Eq. (5.35)

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\frac{I}{a}\int \frac{r'^2 dr' d\Omega' \sin\theta' \cos\varphi' \delta(\cos\theta')\delta(r'-a)}{|\vec{x}-\vec{x}'|}$$

Using the expansion of $1/|\vec{x}-\vec{x}'|$ given by Eq. (3.149),

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{4}{\pi}\int_0^\beta dk \cos[k(z-z')]\left\{\frac{1}{2}I_0(k\rho_<)K_0(k\rho_>) + \sum_{m=1}^{\beta}\cos[m(\varphi-\varphi')]I_m(k\rho_<)K_m(k\rho_>)\right\}$$

We orient the coordinate system so $\varphi = 0,$ and because of the $\cos\varphi'$ factor, $m = 1.$ Thus,

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\frac{I}{a}\frac{4\pi}{\pi}\int_0^\beta dk \int r'^2 dr' d\cos\theta' \sin\theta' \delta(\cos\theta')\delta(r'-a)\cos(kz)I_1(k\rho_<)K_1(k\rho_>)$$

$$A_\phi(r,\theta) = \frac{\mu_0}{\pi}aI\int_0^\beta dk \cos(kz)I_1(k\rho_<)K_1(k\rho_>)$$

where $\rho_<(\rho_>)$ is the smaller (larger) of a and ρ.

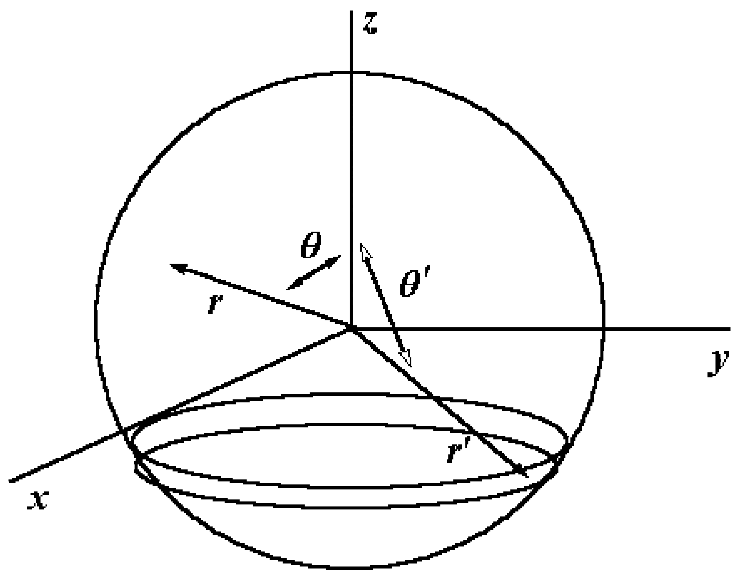
b) From problem 3.16 b),

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{m=-\beta}^{\beta}\int_0^\beta dk e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')e^{-k|z|}$$

Note $z' = 0,$ and $\phi = 0,$ so

$$A_\phi = \frac{\mu_0 Ia}{2}\int_0^\beta dk e^{-k|z|}J_1(k\rho)J_1(ka)$$

5.13 We may choose the coordinate system so the $\vec{r}$ lies in the $x-z$ plane:



The vector potential is given by

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} d^3 x'}{|\vec{r} - \vec{r}'|}$$

Noting $\vec{J} d^3 x' \to \Delta I d\vec{l}'$, where

$$\Delta I = \frac{\Delta Q}{\tau} = \frac{\sigma a^2 d\Omega'}{2\pi/\omega}$$

$$d\vec{l}' = a \left|\sin\theta'\right| d\phi' \hat{\phi}'$$

Since

$$\hat{\phi}' = \cos\phi' \hat{y} - \sin\phi' \hat{x}$$

By symmetry, the x-component of $\vec{A}$ vanishes, so

$$A_y = \frac{\mu_0}{4\pi} \sigma \omega a^3 \int \frac{\sin\theta' \cos\phi' d\Omega'}{|\vec{r} - \vec{r}'|}$$

where I've used

$$Y_1^1(\theta', \phi') = -\sqrt{\frac{3}{8\pi}} \sin\theta' e^{im\phi'}$$

$$A_y = \frac{\mu_0}{4\pi}\sigma\omega a^3\left(-\frac{1}{2}\sqrt{\frac{8\pi}{3}}\right)\int\frac{(Y_1^1(\theta',\phi')+Y_1^{*1}(\theta',\phi'))d\Omega'}{|\vec{r}-\vec{r}'|}$$

Using the expansion

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l,m}\frac{4\pi}{2l+1}Y_l^{m*}(\theta',\phi')Y_l^m(\theta,0)$$

and the fact that $Y_l^m(\theta,0)$ is real, we see only the $l = 1$, $m = 1$ terms contribute.

$$A_y = \frac{\mu_0}{4\pi}\sigma\omega a^3\left(-\sqrt{\frac{8\pi}{3}}\right)\frac{4\pi}{3}\frac{r_<}{r_>^2}Y_1^1(\theta,0) = \frac{\mu_0}{4\pi}\sigma\omega a^3\frac{4\pi}{3}\frac{r_<}{r_>^2}\sin\theta$$

If we take $\vec{r}$ to be in an arbitrary direction $A_y \to A_\phi$. Also, noting $Q = 4\pi a^2$,

$$A_\phi = \frac{\mu_0}{4\pi}\frac{Q\omega a}{3}\frac{r_<}{r_>^2}\sin\theta$$

Thus on the inside:

$$A_\phi = \frac{\mu_0}{4\pi}\frac{Q\omega}{3}\frac{r}{a}\sin\theta$$

outside:

$$A_\phi = \frac{\mu_0}{4\pi}\frac{Q\omega}{3}\frac{a^2}{r^2}\sin\theta$$

Remembering for this case

$$\vec{B} = \vec{\nabla}\times\vec{A} = \hat{r}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\right] + \hat{\theta}\left[-\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]$$

Thus on the

inside:

$$B_r = \frac{\mu_0}{4\pi}\frac{Q\omega}{3}\frac{2\cos\theta}{a}, \qquad B_\theta = -\frac{\mu_0}{4\pi}\frac{Q\omega}{3}\frac{2\sin\theta}{a}$$

outside:

$$B_r = \frac{\mu_0}{4\pi}\frac{Q\omega}{3}\frac{a^2 2\cos\theta}{r^3}, \qquad B_\theta = \frac{\mu_0}{4\pi}\frac{Q\omega}{3}\frac{a^2\sin\theta}{r^3}$$

5.14 This problem corresponds to $\vec{J} = 0$, so we have the equations

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \text{and } \vec{\nabla} \times \vec{H} = 0$$

from which it follow that

$$\vec{H} = -\vec{\nabla}\Phi$$

where $\vec{B} = \mu\vec{H}$. From the first two equations we have the boundary conditions at an interface:

$$B_{1\perp} = B_{2\perp}, \quad \text{and } H_{1||} = H_{2||}$$

From the discussion on p. 76 of the text, the potential is independent of z and can be expanded as

$$\Phi(\rho, \phi) = \sum_m \left[A_m\rho^m + B_m\rho^{-m}\right](C_m \sin m\phi + D_m \cos m\phi)$$

Because the system is odd under reflection through the y axis, which I take to be along $\vec{B}_0$, there are no cosine terms in the expansion. In the region III, outside the cylinder, as $\rho \to \infty$, $-\vec{\nabla}\Phi = \vec{H} = H_0\hat{y}$. Thus $\Phi_{III} \to -H_0 y = -H_0\rho\sin\phi$. Here $H_0 = B_0/\mu_0$. The boundary conditions can be satisfied if only the $m = 1$ terms are kept in the expansion, and we know that the solution which satisfies the boundary conditions is unique. Thus we have the expansions

Region I, $\rho < a$:

$$\Phi_I = A\rho\sin\phi$$

Region II, $a < \rho < b$.

$$\Phi_{II} = \left[C\rho + D\rho^{-1}\right]\sin\phi$$

Region III, $b < \rho$.

$$\Phi_{III} = -H_0\rho\sin\phi + E\rho^{-1}\sin\phi$$

Applying the boundary conditions, we have the four conditions

$$\Phi_I|_{\rho=a} = \Phi_{II}|_{\rho=a}$$

$$\mu_0\frac{\partial}{\partial\rho}\Phi_I|_{\rho=a} = \mu\frac{\partial}{\partial\rho}\Phi_{II}|_{\rho=a}$$

$$\Phi_{II}|_{\rho=b} = \Phi_{III}|_{\rho=b}$$

$$\mu\frac{\partial}{\partial\rho}\Phi_{II}|_{\rho=b} = \mu_0\frac{\partial}{\partial\rho}\Phi_{III}|_{\rho=b}$$

These four boundary conditions allow us to solve for A, C, D, and E, with the result that

$$A = \frac{4H_0 b^2 \mu_r}{d}$$

$$C = \frac{2H_0 b^2 (\mu_r + 1)}{d}$$

$$D = \frac{2H_0 (\mu_r - 1) a^2 b^2}{d}$$

$$E = \frac{H_0 b^2 \left[2(\mu_r + 1) b^2 + 2(\mu_r - 1) a^2 + d\right]}{d}$$

where

$$d = a^2(\mu_r - 1)^2 - b^2(\mu_r + 1)^2$$

and the relative permealbility is

$$\mu_r = \frac{\mu}{\mu_0}$$

With these expressions

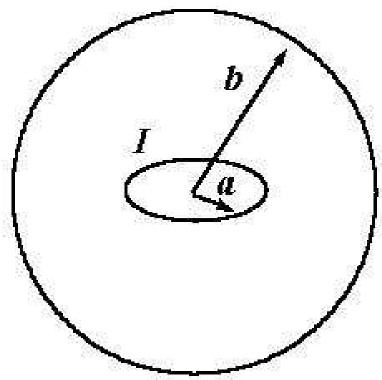
$$\bar{B}_I = -\mu_0 \vec{\nabla} \Phi_I$$

$$\bar{B}_{II} = -\mu \vec{\nabla} \Phi_{II}$$

$$\bar{B}_{III} = -\mu_0 \vec{\nabla} \Phi_{III}$$

5.16

a) The system is shown in the figure



I shall use the magnetic potential approach and will call inside the sphere region 1 and outside the sphere region 2.

$$\phi_1 = \phi_{loop} + \sum_l A_l r^l P_l$$

$$\phi_2 = \phi_{loop} + \sum_l B_l r^{-l-1} P_l$$

where $\vec{H} = -\vec{\nabla}\phi,$ and we have the boundary conditions,

$$H_{1\|} = H_{2\|} \rightarrow \phi_1(r = b) = \phi_2(r = b)$$

$$\mu_0 \frac{\partial}{\partial r}\phi_1(r = b) = \mu \frac{\partial}{\partial r}\phi_2(r = b)$$

We are given that $b >> a,$ so

$$\phi_{loop} = \frac{1}{4\pi}\frac{m\cos\theta}{r^2}$$

with $m = \pi a^2 I.$ (From the form of $\phi_{loop,}$ only the $l = 1$ term contributes.) The boundary conditions give

$$A_1 b_1 = B_1 b^{-1-1}$$

$$-\frac{2\mu_0 m}{4\pi b^3} + \mu_0 A_1 = -\frac{2\mu m}{4\pi b^3} - 2\mu B_1 b^{-3}$$

So

$$A_1 = -\frac{2}{4\pi}\frac{m}{b^3}\frac{(\mu - \mu_0)}{(2\mu + \mu_0)}$$

On the inside, at the center of the loop

$$\vec{H} = -\vec{\nabla}\phi_{loop} - \vec{\nabla}A_1 r\cos\theta$$

From Eq. (5.40), we are given $-\vec{\nabla}\phi_{loop}$ at the center of the loop, which is directed in the z direction.

$$H_z = \frac{1}{\mu_0}(-B_\theta) - A_1$$

If $\mu >> \mu_0$

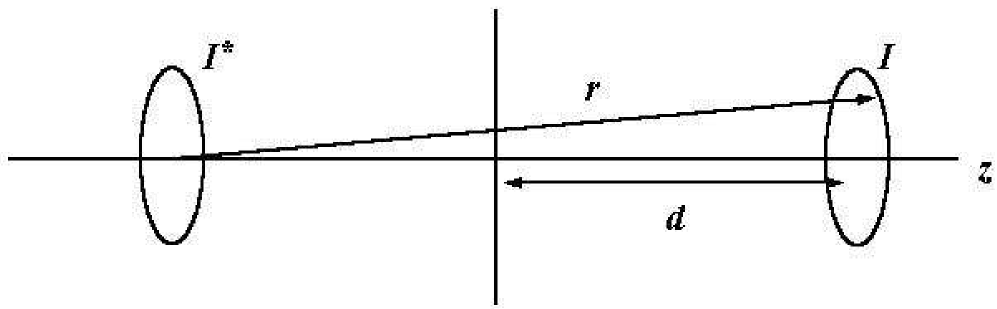
$$A_1 \rightarrow -\frac{1}{4\pi}\frac{m}{b^3}$$

and from (5.40), at $r = 0$

$$H_z = \frac{I}{2a} + \frac{1}{4\pi}\frac{m}{b^3} = \frac{I}{2a} + \frac{I}{4}\frac{a^2}{b^3} = \frac{I}{2a}\left(1 + \frac{a^3}{2b^3}\right)$$

5.18

a) From the results of Problem 5.17, we can replace the problem stated by the system



where I* is equidistant from the interface and is equal to $I^* = \frac{\mu_r - 1}{\mu_r + 1} I$. The radius of each current loop is a. Now from Eq. (5.7)

$$\vec{F}(on\ I) = I \int \vec{dl} \times B(\vec{r})$$

$$\vec{dl} \times \vec{B} = \vec{dl} \times \vec{B}_r + \vec{dl} \times \vec{B}_\theta = dlB_r\left(-\hat{\theta}\right) + dlB_\theta \hat{r}$$

By symmetry, only the z – component survives, so, from the figure

$$\left(\vec{dl} \times \vec{B}\right) \cdot \hat{z} = dlB_r\left(\frac{a}{\sqrt{4d^2 + a^2}}\right) + dlB_\theta\left(\frac{2d}{\sqrt{4d^2 + a^2}}\right)$$

So

$$F_z = \frac{2\pi a I}{\sqrt{4d^2 + a^2}}\left[aB_r + 2dB_\theta\right]$$

with B_r and B_θ given by Eqs. (5.48) and (5.49) and $\cos\theta = \frac{2d}{\sqrt{4d^2+a^2}}$, $r = \sqrt{4d^2 + a^2}$, and $I \to I^*$.

c) To determine the limiting term, simply let $r \to 2d$ and take the lowest non-vanishing term in the expansion of the magnetic flux density.

$$F_z = \frac{\pi a I}{d}\left[aB_r + 2dB_\theta\right]$$

$$F_z = \frac{\pi a I}{d}\left[a\left(\frac{\mu_0 I^* a}{4d}\frac{a}{(2d)^2}\right) + 2d\left(-\frac{\mu_0 I^* a^2}{4}\left(\frac{1}{(2d)^3}\right)\right)\left(-\frac{a}{2d}\right)\right]$$

$$F_z \to -\frac{3\pi\mu_0}{32}\frac{a^4 I \times I^*}{d^4}$$

The minus sign shows the force is attractive if I and I^* are in the same direction. This same result can be gotten more directly, using

$$F_z = \nabla_z (m B_z)$$

with $m = \pi a^2 I,$ and (from Eq. (5.64))

$$B_z = \frac{\mu_0}{4\pi} \left(\frac{2m^*}{z^3} \right)$$

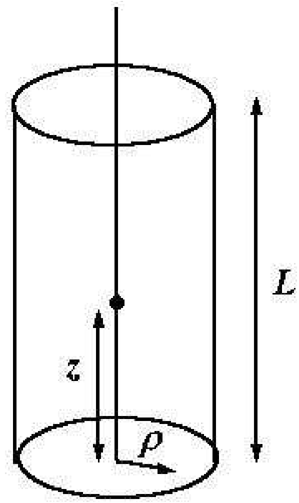
with $m^* = \pi a^2 I^*,$ and $z = 2d$

$$F_z = \frac{\mu_0}{4\pi} 2\pi a^2 I^* \pi a^2 I \left(-\frac{3}{(2d)^4} \right) = -\frac{3\pi\mu_0}{32} \frac{a^4 I \times I^*}{d^4}$$

with agrees with out previous result.

5,19

The system is described by



The effective volume magnetic charge density is zero, since $\vec{M}$ is constant within the cylinder. The effective surface charge density ($\hat{n} \cdot \vec{M}$ from Eq. (5.99)) is M_0, on the top surface and $-M_0$ on the bottom surface. From the bottom surface the potential is (for $z > 0$)

$$\Phi_b = \frac{1}{4\pi}(-M_0)2\pi \int_0^a \frac{\rho d\rho}{(\rho^2 + z^2)^{1/2}} = -\frac{M_0}{2}\left(\sqrt{(a^2 + z^2)} - z\right)$$

By symmetry, the potential from the top surface is (on the inside)

$$\Phi_t = \frac{M_0}{2}\left(\sqrt{\left(a^2 + (L-z)^2\right)} - (L-z)\right)$$

The total magnetic potential is

$$\Phi = \Phi_b + \Phi_t = -\frac{M_0}{2}\left(\sqrt{(a^2 + z^2)} - z\right) + \frac{M_0}{2}\left(\sqrt{\left(a^2 + (L-z)^2\right)} - (L-z)\right)$$

So, on the inside of the cylinder,

$$H_z = -\frac{\partial}{\partial z}\left(-\frac{M_0}{2}\left(\sqrt{(a^2 + z^2)} - z\right) + \frac{M_0}{2}\left(\sqrt{\left(a^2 + (L-z)^2\right)} - (L-z)\right)\right)$$

$$H_z = -\frac{M_0}{2}\left[2 - \frac{z}{\sqrt{(a^2 + z^2)}} - \frac{L-z}{\sqrt{\left(a^2 + (L-z)^2\right)}}\right]$$

while above the cylinder,

$$H_z = -\frac{M_0}{2}\left[-\frac{z}{\sqrt{(a^2+z^2)}} + \frac{z-L}{\sqrt{\left(a^2+(L-z)^2\right)}} \right]$$

with a similar expression below the cylinder.

$$\vec{B} = \mu_0\left(\vec{H}+\vec{M}\right)$$

Thus inside the cylinder,

$$B_z = \mu_0\left[-\frac{M_0}{2}\left[2 - \frac{z}{\sqrt{(a^2+z^2)}} - \frac{L-z}{\sqrt{\left(a^2+(L-z)^2\right)}} \right] + M_0 \right]$$

$$B_z = \frac{\mu_0 M_0}{2}\left(\frac{z}{\sqrt{(a^2+z^2)}} + \frac{L-z}{\sqrt{\left(a^2+(L-z)^2\right)}} \right)$$

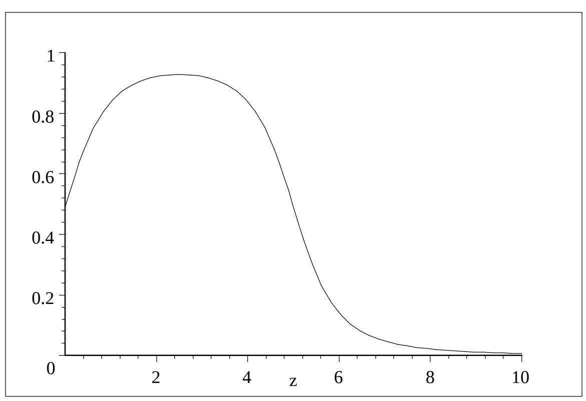
while above the cylinder,

$$B_z = \frac{\mu_0 M_0}{2}\left[\frac{z}{\sqrt{(a^2+z^2)}} - \frac{z-L}{\sqrt{\left(a^2+(L-z)^2\right)}} \right]$$

First we plot B_z in units of a for $L = 5a$

$$g(z) = \begin{cases} \frac{1}{2}\left(\frac{z}{\sqrt{(1+z^2)}} + \frac{5-z}{\sqrt{(1+(5-z)^2)}} \right) & \text{if} \quad z < 5 \\ \frac{1}{2}\left(\frac{z}{\sqrt{(1+z^2)}} - \frac{z-5}{\sqrt{(1+(5-z)^2)}} \right) & \text{if} \quad 5 < z \end{cases}$$

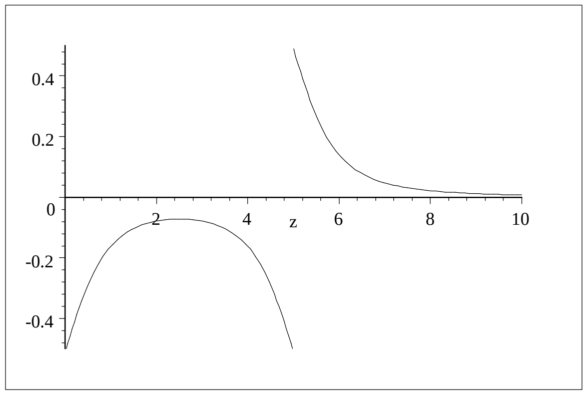
$g(z)$



And similarly, H_z in units of a for $L = 5a$.

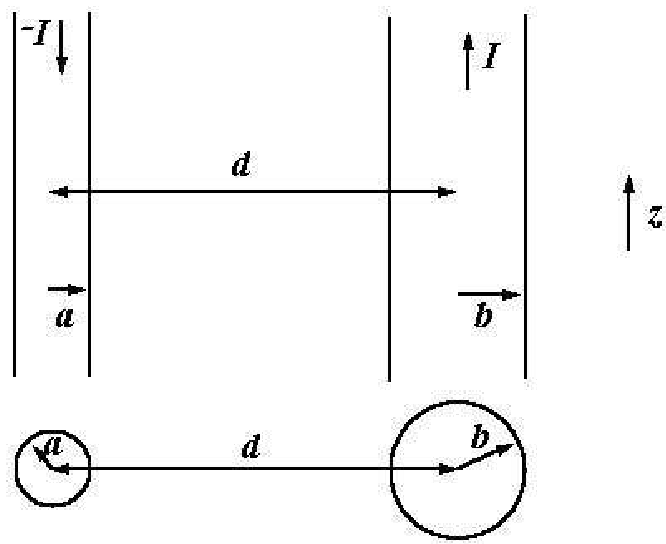
$$f(z) = \begin{cases} -\frac{1}{2}\left(2 - \frac{z}{\sqrt{(1+z^2)}} - \frac{5-z}{\sqrt{(1+(5-z)^2)}}\right) & \text{if} \quad z < 5 \\ -\frac{1}{2}\left(-\frac{z}{\sqrt{(1+z^2)}} + \frac{z-5}{\sqrt{(1+(5-z)^2)}}\right) & \text{if} \quad 5 < z \end{cases}$$

f(z)



5.26

The system is described by



Since the wires are nonpermeable, $\mu = \mu_0$. The system is made of parts with cylindrical symmetry, so we can determine B using Ampere's law.

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}, \text{ or } \quad \int \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{a}$$

On the **outside** of each wire,

$$\int \vec{B} \cdot d\vec{l} = B2\pi\rho = \mu_0 I \to B_{out} = \frac{\mu_0 I}{2\pi\rho}$$

On the **inside** of each wire

$$\int \vec{B} \cdot d\vec{l} = B2\pi\rho = \mu_0 I \frac{\rho^2}{R^2}, \;\; B_{in} = \frac{\mu_0 I}{2\pi} \frac{\rho}{R^2} \text{ with } R = a, b$$

From the right-hand rule, the B from each wire is in the $\hat{\phi}$ direction. From the above figure, using the general expression for the vector potential, we see $\vec{A}$ is in the $\pm\hat{z}$ direction. Since $\vec{\nabla} \times \vec{A} = \vec{B}$,

$$B_z = -\frac{\partial}{\partial\rho} A_z \to A_z = -\int B_z d\rho$$

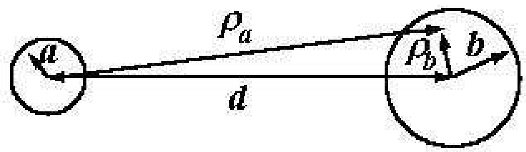
Thus

$$A_z = \begin{cases} -\frac{\mu_0 I}{2\pi}\left(\ln\frac{\rho}{R} + C\right) = -\frac{\mu_0 I}{4\pi}\left(\ln\frac{\rho^2}{R^2} + 1\right) & \text{on the outside} \\ -\frac{\mu_0 I}{4\pi}\frac{\rho^2}{R^2}, & \text{on the inside} \end{cases}$$

where I've determined $C = 1/2$, from the requirement that A_z be continuous at $\rho = R$. Let l be the length of the wire. Then we know the total potential energy is given by

$$W = \frac{1}{2}\int \vec{J}\cdot\vec{A}d^3x = \frac{l}{2}\int[J_a A da_a + J_b A da_b]$$

Consider the second term $\frac{l}{2}\int J_b A da_b$. The system is pictured as



From the figure

$$\vec{\rho}_a = \vec{d} + \vec{\rho}_b, \quad \rho_a^2 = d^2 + \rho_b^2 - 2d\rho_b\cos\phi$$

so, since $J_b = \frac{I}{\pi b^2}$

$$\frac{l}{2}\int J_b A da_b = \frac{l}{2}\frac{I}{\pi b^2}\int[A_{out}(\rho_a) + A_{in}(\rho_{b)})]\rho_b d\rho_b d\phi$$

$$= \frac{l}{2}\frac{I}{\pi b^2}\frac{\mu_0 I}{4\pi}\int\left[\ln\frac{\rho_a^2}{a^2} + 1 - \frac{\rho_b^2}{b^2}\right]\rho_b d\rho_b d\phi$$

$$\simeq \frac{l}{2}\frac{I}{\pi b^2}\frac{\mu_0 I}{4\pi}2\pi\int_0^b\left(\ln\frac{d^2}{a^2} + 1 - \frac{\rho_b^2}{b^2}\right)\rho_b d\rho_b$$

$$= \frac{l}{2}\frac{I}{\pi b^2}\frac{\mu_0 I}{4\pi}2\pi\frac{1}{4}b^2\left(1 + 2\ln\frac{d^2}{a^2}\right) = \frac{l}{2}\left(\frac{\mu_0}{4\pi}\right)\left(\frac{1}{2} + 2\ln\frac{d}{a}\right)I^2$$

The first term $\frac{l}{2}\int J_a A da_a$ is equal to

$$\frac{l}{2}\int J_a A da_a = \frac{l}{2}\left(\frac{\mu_0}{4\pi}\right)\left(\frac{1}{2} + 2\ln\frac{d}{b}\right)I^2$$

Thus

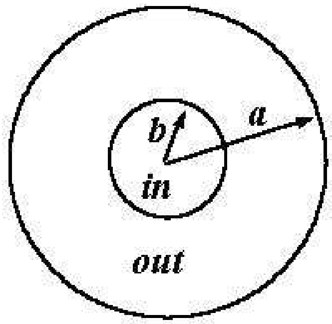
$$W = \frac{l}{2}\left(\frac{\mu_0}{4\pi}\right)\left(1 + 2\ln\frac{d^2}{ab}\right)I^2 = \frac{l}{2}\frac{L}{l}I^2$$

or

$$\frac{L}{l} = \frac{\mu_0}{4\pi}\left(1 + 2\ln\frac{d^2}{ab}\right)$$

5.27

The system is described by



Using Ampere’s law in integral form

$$\int \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed}$$

we get

$$B = \frac{\mu_0 I}{2\pi}\frac{\rho}{b^2},\ \rho < b$$

$$B = \frac{\mu_0 I}{2\pi}\frac{1}{\rho},\ b < \rho < a$$

$$B = 0,\ \rho > a$$

Now the energy in the magnetic field is given by (l is the length of the wires)

$$W = \frac{1}{2}\int \vec{B} \cdot \vec{H} d^3x = \frac{1}{2\mu_0}\int B^2 d^3x$$

$$= \frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi}\right)^2 l\left[2\pi\int_0^b \left(\frac{\rho}{b^2}\right)^2 \rho d\rho + 2\pi\int_b^a \left(\frac{1}{\rho}\right)^2 \rho d\rho\right]$$

$$= \frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi}\right)^2 l\pi\left(\frac{1}{2} + 2\ln\frac{a}{b}\right) = \frac{l}{2}\frac{L}{l}I^2$$

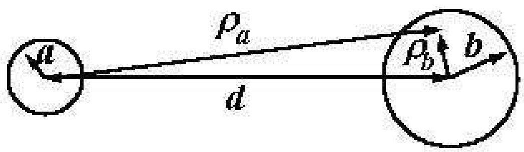
$$\rightarrow \frac{L}{l} = \frac{\mu_0}{4\pi}\left(\frac{1}{2} + 2\ln\frac{a}{b}\right)$$

If the inner wire is hollow, $B = 0,\ \rho < b,$ so

$$\frac{L}{l} = \frac{\mu_0}{2\pi}\ln\frac{a}{b}$$

5.29

The system is described by



This problem is very much like 5.26, except the wires are superconducting. We know from section 5.13 that the magnetic field within a superconductor is zero. We will be using

$$W = \frac{1}{2}\int \vec{J}\cdot\vec{A}d^3x = \frac{l}{2}\int [J_a A da_a + J_b A da_b]$$

Using the same arguments as applied in problem 5.26,

$$A_z = \begin{cases} -\frac{\mu I}{2\pi}\left(\ln\frac{\rho}{R} + C\right) = -\frac{\mu I}{4\pi}\left(\ln\frac{\rho^2}{R^2} + 0\right) & \text{on the outside} \\ 0, & \text{on the inside} \end{cases}$$

Thus if we consider the second term $\frac{l}{2}\int J_b A da_b$,

$$\frac{l}{2}\int J_b A da_b = \frac{l}{2}\frac{I}{\pi b^2}\int [A_{out}(\rho_a) + A_{in}(\rho_{b)})]\rho_b d\rho_b d\phi$$

$$\simeq \frac{l}{2}\frac{I}{\pi b^2}\frac{\mu I}{4\pi}2\pi\int_0^b \ln\frac{d^2}{a^2}\rho_b d\rho_b = \frac{l}{2}\left(\frac{\mu}{4\pi}\right)\left(2\ln\frac{d}{a}\right)I^2$$

The first term $\frac{l}{2}\int J_a A da_a$ is equal to

$$\frac{l}{2}\int J_a A da_a = \frac{l}{2}\left(\frac{\mu}{4\pi}\right)\left(2\ln\frac{d}{b}\right)I^2$$

Thus

$$W = \frac{l}{2}\left(\frac{\mu}{4\pi}\right)\left(2\ln\frac{d^2}{ab}\right)I^2 = \frac{l}{2}\frac{L}{l}I^2$$

so

$$\frac{L}{l} = \left(\frac{\mu}{4\pi}\right)\left(2\ln\frac{d^2}{ab}\right)$$

Now using the methods of problem 1.6, assuming the left wire has charge Q, and the right wire charge $-Q$, we find

$$\phi_{12} = \int_b^{d-a} E dr = \frac{\frac{Q}{l}}{2\pi\varepsilon}\int_b^{d-a}\left(\frac{1}{r} + \frac{1}{d-r}\right)dr \simeq \frac{\frac{Q}{l}}{2\pi\varepsilon}\ln\frac{d^2}{ab}$$

$$\frac{C}{l} = \frac{\frac{Q}{l}}{\phi_{12}} = \frac{2\pi\varepsilon}{\ln\frac{d^2}{ab}}$$

Thus

$$\frac{L}{l} \times \frac{C}{l} = \left(\frac{\mu}{4\pi}\right)\left(2\ln\frac{d^2}{ab}\right) \times \frac{2\pi\varepsilon}{\ln\frac{d^2}{ab}} = \mu\varepsilon$$

More Problems for Chapter 5

Problem 5.6

Using the principle of superposition, the magnetic field at $\vec{r}$ in the cavity is equal to that of a conductor without the hole minus that of a smaller conductor filling the hole with the same volume current densities, *i.e.*:

$$\vec{B}_{\text{cavity}}(\vec{r}) = \vec{B}_{\text{nohole}}(\vec{r}) - \vec{B}_{\text{hole}}(\vec{r})$$

In the polar coordinate system with the $z-$axis along the cylinder axis, $\vec{B}_{\text{nohole}}$ can be calculated from Ampere's law:

$$\vec{B}_{\text{nohole}}(\vec{r}) = \frac{\mu_0 I}{2\pi\rho}\hat{\phi} = \frac{\mu_0 \pi \rho^2 J}{2\pi\rho}\hat{\phi} = \frac{1}{2}\mu_0 J\rho\hat{\phi}$$

where $\hat{\phi}$ is the unit vector along the $\phi-$direction and J is the volume current density. Similarly, we have

$$\vec{B}_{\text{hole}}(\vec{r}) = \frac{1}{2}\mu_0 J\rho'\hat{\phi}'$$

where ρ' and ϕ' are measured with respect to the axis of the hole. Therefore,

$$\vec{B}_{\text{cavity}}(\vec{r}) = \frac{1}{2}\mu_0 J(\rho\hat{\phi} - \rho'\hat{\phi}')$$

Let $\vec{d}$ be the vector from the cylinder axis to the hole axis, we have:

$$\vec{\rho} - \vec{\rho'} = \vec{d}$$

Cross multiplying the above equation with $\hat{z}$ (the unit vector along the $z-$direction) and noting $\hat{z} \times \hat{\rho} = \hat{\phi}$, we get:

$$\rho\hat{\phi} - \rho'\hat{\phi}' = \hat{z} \times \vec{d}$$

Consequently, the field inside the cavity:

$$\vec{B}_{\text{cavity}}(\vec{r}) = \frac{1}{2}\mu_0 J\hat{z} \times \vec{d}$$

The field is uniform and in the direction perpendicular to the line joinning the axes of the cylinder and the hole. In terms of the current I on the cylinder:

$$J = \frac{I}{\pi(a^2 - b^2)}; \qquad \Rightarrow \qquad \vec{B}_{\text{cavity}}(\vec{r}) = \frac{\mu_0 I}{2\pi(a^2 - b^2)}\hat{z} \times \vec{d}$$

Problem 5.13

The rotating surface charges result surface currents. In the spherical coordinate system with the $z-$axis along the rotation axis, the surface current density

$$\vec{K}(\vec{r}) = \sigma\vec{v}(\vec{r}) = \sigma\omega a \sin\theta\hat{\phi}$$

where $\hat{\phi}$ is the unit vector along the $\phi-$direction. Therefore, the vector potential

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\oint \frac{\vec{K}(\vec{r'})}{|\vec{r} - \vec{r'}|}da' = \frac{1}{4\pi}\mu_0\sigma\omega a^3 \int \frac{\sin\theta'\hat{\phi}'}{|\vec{r} - \vec{r'}|}d\Omega'$$

To carry out the above integral, we project $\hat{\phi}'$ along fixed $x-$ and $y-$directions and expand $1/|\vec{r} - \vec{r'}|$ in spherical harmonics.

$$\hat{\phi}' = -\sin\phi'\hat{x} + \cos\phi'\hat{y}$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell m} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi)$$

where $r_< = \min(a, r)$ and $r_> = \max(a, r)$. The vector potential is therefore:

$$\vec{A}(\vec{r}) = \mu_0 \omega \sigma a^3 \sum_{\ell m} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}(\theta, \phi) \int Y^*_{\ell m}(\theta', \phi')(-\sin\theta' \sin\phi' \hat{x} + \sin\theta' \cos\phi' \hat{y}) d\Omega'$$

Note that $\sin\theta' \sin\phi'$ and $\sin\theta' \cos\phi'$ can be written in terms of $Y_{1,1}$:

$$Y_{1,1}(\theta, \phi) = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}; \quad \Rightarrow \quad \sin\theta' \sin\phi' = -\sqrt{\frac{8\pi}{3}} Im\{Y_{1,1}(\theta', \phi')\}; \quad \sin\theta' \cos\phi' = -\sqrt{\frac{8\pi}{3}} Re\{Y_{1,1}(\theta', \phi')\}$$

and the integrals can be carried out using the orthogonality properties of spherical harmonics:

$$Y_{\ell m}(\theta, \phi) \int Y^*_{\ell m}(\theta', \phi') \sin\theta' \sin\phi' d\Omega' = -\sqrt{\frac{8\pi}{3}} Im \left\{ Y_{\ell m}(\theta, \phi) \int Y^*_{\ell m}(\theta', \phi') Y_{1,1}(\theta', \phi') d\Omega' \right\} = \sin\theta \sin\phi \delta_{\ell,1} \delta_{m,1}$$

$$Y_{\ell m}(\theta, \phi) \int Y^*_{\ell m}(\theta', \phi') \sin\theta' \cos\phi' d\Omega' = -\sqrt{\frac{8\pi}{3}} Re \left\{ Y_{\ell m}(\theta, \phi) \int Y^*_{\ell m}(\theta', \phi') Y_{1,1}(\theta', \phi') d\Omega' \right\} = \sin\theta \cos\phi \delta_{\ell,1} \delta_{m,1}$$

As results of the above, the vector potential is

$$\vec{A}(\vec{r}) = \mu_0 \omega \sigma a^3 \sum_{\ell m} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} \delta_{\ell,1} \delta_{m,1} \sin\theta(-\sin\phi \hat{x} + \cos\phi \hat{y}) = \frac{1}{3} \mu_0 \omega \sigma a^3 \sin\theta \frac{r_<}{r_>^2} \hat{\phi}$$

Inside the sphere, $r_< = r$ and $r_> = a$:

$$\vec{A}(\vec{r}) = \frac{1}{3} \mu_0 \omega \sigma a(-y\hat{x} + x\hat{y}) = \frac{1}{3} \mu_0 \omega \sigma a r \sin\theta \hat{\phi} = \frac{1}{3} \mu_0 \sigma a \, \vec{\omega} \times \vec{r}$$

$$\vec{B}(\vec{r}) = \nabla \times \vec{A} = \frac{2}{3} \mu_0 \omega \sigma a \hat{z} = \frac{2}{3} \mu_0 \sigma a \, \vec{\omega}$$

The field inside the sphere is uniform and point to $z-$direction. Outside the sphere, $r_< = a$ and $r_> = r$:

$$\vec{A}(\vec{r}) = \frac{1}{3} \mu_0 \omega \sigma a^4 \frac{\sin\theta}{r^2} \hat{\phi} = \frac{1}{3} \mu_0 \sigma a^4 \frac{\vec{\omega} \times \vec{r}}{r^3}$$

$$\vec{B}(\vec{r}) = \nabla \times \vec{A} = \frac{1}{3} \mu_0 \omega \sigma a^4 \{ \frac{2\cos\theta}{r^3} \hat{r} + \frac{\sin\theta}{r^3} \hat{\theta} \} = \frac{\mu_0}{4\pi} \frac{3\hat{r}(\hat{r} \cdot \vec{m}) - \vec{m}}{r^3}$$

This is the field due to a magnetic dipole

$$\vec{m} = \frac{4}{3} \pi a^3 (\sigma a \, \vec{\omega})$$

Problem 5.18

(a) From the result of Prob. 5.17, the magnetic field at the current loop can be calculated by replacing the medium with an image current of magnitude

$$I' = \frac{\mu - \mu_0}{\mu + \mu_0} I$$

In a spherical coordinate system with its $x - y$ plane defined by the imagine current loop, its origin at the center of the loop and its $z-$axis pointing to the current loop I, the magnetic field due to the imagine current is given by Eq. (5.48) and (5.49). At the location of the current loop,

$$r_< = a, \quad r_> = r = \sqrt{a^2 + 4d^2}, \quad \cos\theta = \frac{2d}{\sqrt{a^2 + 4d^2}}$$

Therefore,

$$B_r = \frac{1}{2}\mu_0 I' a \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n n!} \frac{a^{2n+1}}{(a^2+4d^2)^{n+3/2}} P_{2n+1}(\cos\theta)$$

$$B_\theta = -\frac{1}{4}\mu_0 I' a \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n (n+1)!} \frac{a^{2n+1}}{(a^2+4d^2)^{n+3/2}} P^1_{2n+1}(\cos\theta)$$

The force on the current loop:

$$\vec{F} = \oint \vec{I} \times \vec{B} d\ell = \oint I\hat{\phi} \times (B_r \hat{r} + B_\theta \hat{\theta}) d\ell = \oint (IB_r\hat{\theta} - IB_\theta \hat{r}) d\ell$$

Note that both B_r and B_θ are constants of the integration and that

$$\oint \hat{\theta} d\ell = -\sin\theta \hat{z} \oint d\ell = -2\pi a \sin\theta \hat{z}, \qquad \oint \hat{r} d\ell = \cos\theta \hat{z} \oint d\ell = 2\pi a \cos\theta \hat{z}$$

The force acting on the loop:

$$\vec{F} = IB_r \oint \hat{\theta} d\ell - IB_\theta \oint \hat{r} d\ell = -2\pi a I (B_r \sin\theta + B_\theta \cos\theta)\hat{z}$$

$$= -\pi\mu_0 a^2 I I' \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n} \frac{a^{2n+1}}{(a^2+4d^2)^{n+2}} \left\{ \frac{a}{n!} P_{2n+1}(\cos\theta) - \frac{d}{(n+1)!} P^1_{2n+1}(\cos\theta) \right\} \hat{z}$$

(c) For $d \gg a$, the force is dominated by the $n = 0$ term:

$$\vec{F} = -\pi\mu_0 a^2 I I' \frac{a}{(a^2+4d^2)^2} \{a P_1(\cos\theta) - d P^1_1(\cos\theta)\} \hat{z}$$

$$= -\pi\mu_0 a^2 I I' \frac{a}{(a^2+4d^2)^2} \{a\cos\theta + d\sin\theta)\} \hat{z}$$

$$= -\pi\mu_0 a^2 I I' \frac{a}{(a^2+4d^2)^2} \left\{ a \frac{2d}{\sqrt{a^2+4d^2}} + d \frac{a}{\sqrt{a^2+4d^2}} \right\} \hat{z}$$

$$= -3\pi\mu_0 I I' \frac{a^4 d}{(a^2+4d^2)^{5/2}} \hat{z} \approx -\frac{3}{32}\pi\mu_0 I^2 \frac{\mu-\mu_0}{\mu+\mu_0} \frac{a^4}{d^4} \hat{z}$$

The force is attractive. Alternatively, for $d \gg a$, both current loops can be approximated as point dipoles with dipole moments

$$\vec{m} = I(\pi a^2)\hat{z}, \qquad \vec{m'} = I'(\pi a^2)\hat{z}$$

The magnetic field at $\vec{m}$ due to $\vec{m'}$:

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{3\hat{z}(\hat{z}\cdot\vec{m'}) - \vec{m'}}{z^3}$$

where $z = \sqrt{a^2+4d^2}$ is the separation between the two dipoles. Consequently, the potential energy:

$$U = -\vec{m}\cdot\vec{B} = -\frac{\mu_0}{4\pi} \frac{3(\hat{z}\cdot\vec{m})(\hat{z}\cdot\vec{m'}) - \vec{m}\cdot\vec{m'}}{z^3} = -\frac{\mu_0}{2\pi} \frac{mm'}{z^3}$$

and the force

$$\vec{F} = -\nabla U = -\frac{\partial U}{\partial z}|_{z=\sqrt{a^2+4d^2}}\ \hat{z} = -\frac{3\mu_0}{2\pi}\frac{mm'}{(a^2+4d^2)^2}\hat{z}$$

$$= -\frac{3}{2}\pi\mu_0 II'\frac{a^4}{(a^2+4d^2)^2}\hat{z} \approx -\frac{3}{32}\pi\mu_0 I^2\frac{\mu-\mu_0}{\mu+\mu_0}\frac{a^4}{d^4}\hat{z}$$

agrees with the result from (a).

Problem 5.19

There is no free current. Therefore, the scalar potential approach is applicable.

$$\Phi_M(\vec{r}) = \frac{1}{4\pi}\int\frac{-\nabla\cdot\vec{M}}{|\vec{r}-\vec{r'}|}d\tau' + \frac{1}{4\pi}\oint\frac{\vec{M}\cdot\vec{n'}}{|\vec{r}-\vec{r'}|}da'$$

The effective magnetic volume charge $\rho_M = -\nabla\cdot\vec{M} = 0$. In the cylindrical coordinate system with its origin at the center of the cylinder and its $z-$axis along the axis of the cylinder in the magnetization direction,

$$\Phi_M(\vec{r}) = \frac{1}{4\pi}\oint\frac{\vec{M}\cdot\vec{n'}}{|\vec{r}-\vec{r'}|}da' = \frac{1}{4\pi}\int_{\text{top}}\frac{M_0}{|\vec{r}-\vec{r'}|}da' - \frac{1}{4\pi}\int_{\text{bottom}}\frac{M_0}{|\vec{r}-\vec{r'}|}da'$$

Along the $z-$ axis,

$$\Phi_M(z) = \frac{M_0}{4\pi}(2\pi)\int_0^a d\rho'\left\{\frac{1}{\sqrt{\rho'^2+(z-L/2)^2}} - \frac{1}{\sqrt{\rho'^2+(z+L/2)^2}}\right\}$$

$$= \frac{M_0}{2}\left\{\sqrt{a^2+(z-\frac{L}{2})^2} - |z-\frac{L}{2}| - \sqrt{a^2+(z-\frac{L}{2})^2} + |z+\frac{L}{2}|\right\}$$

Therefore,

$$\Phi_M(z) = \frac{M_0}{2}\left\{\sqrt{a^2+(z-\frac{L}{2})^2} - \sqrt{a^2+(z-\frac{L}{2})^2} + L\right\} \qquad \text{for} \quad z > \frac{L}{2}$$

$$\Phi_M(z) = \frac{M_0}{2}\left\{\sqrt{a^2+(z-\frac{L}{2})^2} - \sqrt{a^2+(z-\frac{L}{2})^2} + 2z\right\} \qquad \text{for} \quad -\frac{L}{2} < z < \frac{L}{2}$$

$$\Phi_M(z) = \frac{M_0}{2}\left\{\sqrt{a^2+(z-\frac{L}{2})^2} - \sqrt{a^2+(z-\frac{L}{2})^2} - L\right\} \qquad \text{for} \quad z < -\frac{L}{2}$$

The auxiliary fields along the $z-$axis:

$$\vec{H}(z) = -\frac{\partial\Phi_M(z)}{\partial z}\hat{z} = -\frac{M_0}{2}\left\{\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}} + 2\right\}\hat{z} \quad \text{inside the cylinder}$$

$$\vec{H}(z) = -\frac{\partial\Phi_M(z)}{\partial z}\hat{z} = -\frac{M_0}{2}\left\{\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}}\right\}\hat{z} \quad \text{outside the cylinder}$$

The magnetic fields along the $z-$axis:

$$\vec{B}(z) = \mu_0(\vec{H}+\vec{M}) = -\frac{1}{2}\mu_0 M_0\left\{\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}}\right\}\hat{z} \quad \text{inside the cylinder}$$

$$\vec{B}(z) = \mu_0\vec{H} = -\frac{1}{2}\mu_0 M_0\left\{\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}}\right\}\hat{z} \quad \text{outside the cylinder}$$

More Problems for Chapter 5

Problem 5.10

Useful identities:

$$\frac{\partial J_1(k\rho)}{\partial\rho} = \frac{k}{2}\{J_0(k\rho) - J_2(k\rho)\}; \quad \frac{\partial I_1(k\rho)}{\partial\rho} = \frac{k}{2}\{I_0(k\rho) + I_2(k\rho)\}; \quad \frac{\partial K_1(k\rho)}{\partial\rho} = -\frac{k}{2}\{K_0(k\rho) + K_2(k\rho)\}$$

$$\int_0^\infty dk k\cos(kz)K_1(ka) = \frac{a\pi}{2(a^2+z^2)^{3/2}}; \quad \int_0^\infty dk k e^{-k|z|}J_1(ka) = \frac{a}{(a^2+z^2)^{3/2}}$$

The vector potential $\vec{A}$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint\frac{d\vec{\ell'}}{|\vec{r}-\vec{r'}|} = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{\hat{\phi}'}{|\vec{r}-\vec{r'}|}d\phi'$$

where $\hat{\phi}'$ is the unit vector along the ϕ' direction:

$$\hat{\phi}' = -\sin\phi'\hat{x} + \cos\phi'\hat{y}$$

Consequently

$$\vec{A}(\vec{r}) = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{-\sin\phi'\hat{x} + \cos\phi'\hat{y}}{|\vec{r}-\vec{r'}|}d\phi'$$

(a) Using the expansion of Eq. (3.148) with $z' = 0$:

$$\frac{1}{|\vec{r}-\vec{r'}|} = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')}\int_0^\infty dk\cos(kz)I_m(k\rho_<)K_m(k\rho_>)$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I a}{4\pi}\frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\cos(kz)I_m(k\rho_<)K_m(k\rho_>)\int_0^{2\pi}e^{im(\phi-\phi')}(-\sin\phi'\hat{x} + \cos\phi'\hat{y})d\phi'$$

where $\rho_< = \min(a,\rho)$ and $\rho_> = \max(a,\rho)$. The integrals

$$\int_0^{2\pi}e^{im(\phi-\phi')}\sin\phi' d\phi' = Im\left\{e^{im\phi}\int_0^{2\pi}e^{i(1-m)\phi'}d\phi'\right\} = Im\left\{e^{im\phi}2\pi\delta_{m,1}\right\} = 2\pi\sin\phi\delta_{m,1}$$

$$\int_0^{2\pi}e^{im(\phi-\phi')}\cos\phi' d\phi' = Re\left\{e^{im\phi}\int_0^{2\pi}e^{i(1-m)\phi'}d\phi'\right\} = Re\left\{e^{im\phi}2\pi\delta_{m,1}\right\} = 2\pi\cos\phi\delta_{m,1}$$

Noting $(-\sin\phi\hat{x} + \cos\phi\hat{y}) = \hat{\phi}$, the vector potential is

$$\vec{A}(\vec{r}) = \frac{\mu_0 I a}{4\pi}\frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\cos(kz)I_m(k\rho_<)K_m(k\rho_>)(2\pi\delta_{m,1})(-\sin\phi\hat{x} + \cos\phi\hat{y})$$

$$= \frac{\mu_0 I a}{\pi}\int_0^\infty dk\cos(kz)I_1(k\rho_<)K_1(k\rho_>)dk\hat{\phi}$$

i.e.

$$A_\phi(\rho,z) = \frac{\mu_0 I a}{\pi}\int_0^\infty dk\cos(kz)I_1(k\rho_<)K_1(k\rho_>)$$

(b) Using the expansion of Prob. 3.16(b) and noting $\rho' = a, z' = 0$:

$$\frac{1}{|\vec{r} - \vec{r'}|} = \sum_{m=-\infty}^{\infty} \int_0^\infty dk e^{im(\phi-\phi')} J_m(k\rho) J_m(ka) e^{-k|z|}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I a}{4\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty dk J_m(k\rho) J_m(ka) e^{-k|z|} \int_0^{2\pi} e^{im(\phi-\phi')}(-\sin\phi' \hat{x} + \cos\phi' \hat{y}) d\phi'$$

$$= \frac{\mu_0 I a}{4\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty dk J_m(k\rho) J_m(ka) e^{-k|z|} (2\pi \delta_{m,1})(-\sin\theta \hat{x} + \cos\theta \hat{y})$$

$$= \frac{\mu_0 I a}{2} \int_0^\infty dk J_1(k\rho) J_1(ka) e^{-k|z|} \hat{\phi}$$

i.e.,

$$A_\phi(\rho, z) = \frac{\mu_0 I a}{2} \int_0^\infty dk e^{-k|z|} J_1(ka) J_1(k\rho)$$

(c)

$$\vec{B} = \nabla \times \vec{A} = -\frac{\partial A_\phi}{\partial z}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi)\hat{z} = B_\rho \hat{\rho} + B_z \hat{z}$$

Expansion of (a):

$$B_\rho(\rho, z) = -\frac{\partial A_\phi}{\partial z} = \frac{\mu_0 I a}{\pi} \int_0^\infty dk k \sin(kz) I_1(k\rho_<) K_1(k\rho_>)$$

$$B_z(\rho, z) = \frac{\partial A_\phi}{\partial \rho} + \frac{1}{\rho} A_\phi = \frac{\mu_0 I a}{\pi} \int_0^\infty dk \cos(kz) \left\{ \frac{\partial}{\partial \rho} + \frac{1}{\rho} \right\} \{I_1(k\rho_<) K_1(k\rho_>)\}$$

On the $z-$axis ($\rho = 0$), $\rho_< = 0$ and $\rho_> = a$:

$$B_\rho(0, z) = \frac{\mu_0 I a}{\pi} \int_0^\infty dk k \sin(kz) I_1(0) K_1(ka) = 0$$

$$B_z(0, z) = \frac{\mu_0 I a}{\pi} \int_0^\infty dk \cos(kz) \lim_{\rho \to 0} \left\{ \frac{\partial}{\partial \rho} + \frac{1}{\rho} \right\} \{I_1(k\rho) K_1(ka)\}$$

$$= \frac{\mu_0 I a}{\pi} \int_0^\infty dk k \cos(kz) K_1(ka)$$

$$= \frac{\mu_0 I a}{\pi} \frac{a\pi}{2} \frac{1}{(a^2 + z^2)^{3/2}} = \frac{\mu_0 I a^2}{2} \frac{1}{(a^2 + z^2)^{3/2}}$$

Expansion of (b):

$$B_\rho(\rho, z) = -\frac{\partial A_\phi}{\partial \rho} = \frac{\mu_0 I a}{2} \int_0^\infty dk J_1(ka) J_1(k\rho) \frac{\partial}{\partial z} e^{-k|z|}$$

$$B_z(\rho, z) = \frac{\partial A_\phi}{\partial \rho} + \frac{1}{\rho} A_\phi = \frac{\mu_0 I a}{2} \int_0^\infty e^{-k|z|} J_1(ka) \left\{ \frac{\partial}{\partial \rho} + \frac{1}{\rho} \right\} J_1(k\rho)$$

On the $z-$axis, $\rho = 0$:

$$B_\rho(0,z) = \frac{\mu_0 I a}{2}\int_0^\infty dk J_1(ka) J_1(0)\frac{\partial}{\partial z} e^{-k|z|} = 0$$

$$B_z(0,z) = \frac{\mu_0 I a}{2}\int_0^\infty dk e^{-k|z|} J_1(ka) \lim_{\rho\to 0}\left\{\frac{\partial}{\partial \rho} + \frac{1}{\rho}\right\} J_1(k\rho)$$

$$= \frac{\mu_0 I a}{2}\int_0^\infty dk k e^{-k|z|} J_1(ka)$$

$$= \frac{\mu_0 I a^2}{2}\frac{1}{(a^2+z^2)^{3/2}}$$

Problem 5.20
Useful identity:

$$\int_V (\vec{C}\cdot\nabla)\vec{D} d\tau = -\int_V (\nabla\cdot\vec{C})\vec{D} d\tau + \oint (\vec{C}\cdot\vec{n})\vec{D} da$$

(a) The force on the bound volume and surface currents

$$\vec{F} = \int_V (\nabla\times\vec{M})\times\vec{B}_e d\tau + \oint (\vec{M}\times\vec{n})\times\vec{B}_e da$$

Using the product rule:

$$(\nabla\times\vec{M})\times\vec{B}_e = -\vec{B}_e\times(\nabla\times\vec{M}) = (\vec{M}\cdot\nabla)\vec{B}_e + (\vec{B}_e\cdot\nabla)\vec{M} + \vec{M}\times(\nabla\times\vec{B}_e) - \nabla(\vec{M}\cdot\vec{B}_e)$$

Since $\vec{B}_e$ is an external field, we have $\nabla\times\vec{B}_e = 0$ in the region of interest. So

$$(\nabla\times\vec{M})\times\vec{B}_e = (\vec{M}\cdot\nabla)\vec{B}_e + (\vec{B}_e\cdot\nabla)\vec{M} - \nabla(\vec{M}\cdot\vec{B}_e)$$

Also

$$(\vec{M}\times\vec{n})\times\vec{B}_e = -\vec{B}_e\times(\vec{M}\times\vec{n}) = -(\vec{B}_e\cdot\vec{n})\vec{M} + (\vec{B}_e\cdot\vec{M})\vec{n}$$

The force

$$\vec{F} = \int_V \left\{\vec{M}\cdot\nabla)\vec{B}_e + (\vec{B}_e\cdot\nabla)\vec{M} - \nabla(\vec{M}\cdot\vec{B}_e)\right\} d\tau + \oint \left\{(\vec{B}_e\cdot\vec{M})\vec{n} - (\vec{B}_e\cdot\vec{n})\vec{M}\right\} da$$

$$= \int_V \left\{\vec{M}\cdot\nabla)\vec{B}_e + (\vec{B}_e\cdot\nabla)\vec{M}\right\} d\tau - \oint \left\{(\vec{B}_e\cdot\vec{n})\vec{M}\right\} da$$

Using the identity

$$\int_V (\vec{C}\cdot\nabla)\vec{D} d\tau = -\int_V (\nabla\cdot\vec{C})\vec{D} d\tau + \oint (\vec{C}\cdot\vec{n})\vec{D} da$$

on the first two integrals:

$$\vec{F} = -\int_V (\nabla\cdot\vec{M})\vec{B}_e d\tau + \oint (\vec{M}\cdot\vec{n})\vec{B}_e da - \int_V (\nabla\cdot\vec{B}_e)\vec{M} d\tau + \oint (\vec{B}_e\cdot\vec{n})\vec{M} da - \oint (\vec{B}_e\cdot\vec{n})\vec{M} da$$

$$= -\int_V (\nabla\cdot\vec{M})\vec{B}_e d\tau + \oint (\vec{M}\cdot\vec{n})\vec{B}_e da$$

Problem 5.22

Since there is no free current, the magnetic scalar potential approach is applicable. Within the infinite permeable medium, the auxiliary field $\vec{H}$ vanishes. Since $\vec{n} \times \vec{H}$ is continuous at the surface, $\vec{H}$ outside the medium must be perpendicular to the surface. Consequently, the surface is at equipotential and we can set its potential to be zero. Therefore, this problem can be treated like the analogous electrostatic problems with a polarized bar against a perfect conductor at zero potential. We can use the method of image and replace the medium with an image bar magnet with magnetization $-\vec{M}$. From the result of Prob. 5.20, the force on the bar magnet is given by

$$\vec{F} = \int_V \rho_M \vec{B} d\tau + \oint \sigma_M \vec{B} da$$

where $\vec{B}$ is the field due to the image bar magnet. ρ_M, σ_M are the effective volume and surface magnetic charge densities respectively:

$$\rho_M = -\nabla \cdot \vec{M} = 0, \qquad \sigma_M = \vec{M} \cdot \vec{n}$$

In a coordinate system with its origin at the joint between the bar and the image and its $z-$axis along the $\vec{M}$ direction, the magnetic field due to the image bar along the $z-$axis is given by (Prob. 5.19, shifting $z-$axis by $L/2$):

$$\vec{B}(z) = -\frac{1}{2}\mu_0 M \left\{ \frac{z}{\sqrt{a^2+z^2}} - \frac{z+L}{\sqrt{a^2+(z+L)^2}} \right\} \hat{z}$$

where a is the effective radius of the bar ($\pi a^2 = A$): Assuming the bar is long and narrow, the force is then

$$\vec{F} = \oint (\vec{M} \cdot \vec{n}) \vec{B} da \approx AM\{B(L) - B(0)\}\hat{z}$$

$$= -\mu_0 A M^2 L \left\{ \frac{1}{\sqrt{a^2+L^2}} - \frac{1}{\sqrt{a^2+4L^2}} \right\} \hat{z} \approx -\frac{1}{2}\mu_0 A M^2 \hat{z}$$

The force points to $-\hat{z}$ direction, *i.e.*, the bar is attracted to the medium.

Problem 5.27

From the Ampere's law, we can calculate the magnetic fields in the three regions:

$$B(\rho) = \frac{\mu_c I}{\pi b^2}\rho \quad (0 < \rho < b); \quad B(\rho) = \frac{\mu_0 I}{2\pi\rho} \quad (b < \rho < a); \quad B(\rho) = 0 \quad (\rho > a)$$

where μ_c is the permeability of the conductor. The $\vec{B}$ is along the $\hat{\phi}$. The magnetic energy per unit length

$$W = \frac{1}{2}LI^2 = \frac{1}{2\mu_c}\int_0^b B^2 \rho d\rho d\phi + \frac{1}{2\mu_0}\int_b^a B^2 \rho d\rho d\phi$$

$$= \frac{1}{2\mu_c} \cdot 2\pi \int_0^b \left\{ \frac{\mu_c I}{2\pi b^2}\rho \right\}^2 \rho d\rho + \frac{1}{2\mu_0} \cdot 2\pi \int_b^a \left\{ \frac{\mu_0 I}{2\pi\rho} \right\}^2 \rho d\rho$$

$$= \frac{I^2}{4\pi}\left\{ \frac{\mu_c}{4} + \mu_0 \ln\frac{a}{b} \right\}$$

The self-inductance L per unit length

$$L = \frac{1}{4\pi}\left\{ \frac{\mu_c}{4} + \mu_0 \ln\frac{a}{b} \right\}$$

For the case of $\mu_c = \mu_0$, it simplifies to:

$$L = \frac{\mu_0}{4\pi}\left\{\frac{1}{4} + \ln\frac{a}{b}\right\}$$

If the inner conductor is a thin hollow tube, there will be no magnetic field inside the hollow tube. Consequently the self-inductance per unit length would be:

$$L = \frac{\mu_0}{4\pi}\ln\frac{a}{b}$$

Problem 5.29

Inside an ideal conductor, the electric and magnetic fields vanish. Furthermore, all charges and currents are on the conductor surfaces. In a cylindrical coordinate system with the $z-$axis along the direction of the two conductors, the surface charge and current densities are $z-$independent. Let $\sigma_1(\rho,\phi)$ and $\sigma_2(\rho,\phi)$ be the surface charge densities on the two conductors, the surface current densities are then $\sigma_1(\rho,\phi)v$ and $\sigma_2(\rho,\phi)v$ along the $z-$direction, where v is the speed of the charges. Consequently, the scalar and vector potentials are:

$$\Phi(\rho,\phi) = \frac{1}{4\pi\epsilon}\left\{\int\frac{\sigma_1(\rho',\phi')}{|\vec{r}-\vec{r'}|}da'_1 + \int\frac{\sigma_2(\rho',\phi')}{|\vec{r}-\vec{r'}|}da'_2\right\}$$

$$A_z(\rho,\phi) = \frac{\mu}{4\pi}\left\{\int\frac{\sigma_1(\rho',\phi')v}{|\vec{r}-\vec{r'}|}da'_1 + \int\frac{\sigma_2(\rho',\phi')v}{|\vec{r}-\vec{r'}|}da'_2\right\} = \mu\epsilon v\Phi(\rho,\phi)$$

where $\int da'_1$ and $\int da'_2$ integrate over the surfaces of the two conductors. Note that $A_\rho = A_\phi = 0$. Let $\pm q$ be the total charge per unit length on the two conductors, then we have $qv = I$. Therefore,

$$A_z(\rho,\phi) = \mu\epsilon\frac{I}{q}\Phi(\rho,\phi)$$

The electric and magnetic fields are

$$E^2 = \{-\nabla\Phi\}^2 = \left\{\frac{\partial\Phi}{\partial\rho}\right\}^2 + \left\{\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\right\}^2$$

$$B^2 = |\nabla\times\vec{A}|^2 = \left\{\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}\right\}^2 + \left\{\frac{\partial A_z}{\partial\rho}\right\}^2 = \left\{\mu\epsilon\frac{I}{q}\right\}^2 E^2$$

The energies in electric and magnetic fields per unit length are:

$$W_E = \frac{1}{2}\frac{q^2}{C} = \frac{\epsilon}{2}\int E^2 d\tau$$

$$W_B = \frac{1}{2}LI^2 = \frac{1}{2\mu}\int B^2 d\tau = \frac{1}{2}\mu\epsilon^2\frac{I^2}{q^2}\int E^2 d\tau$$

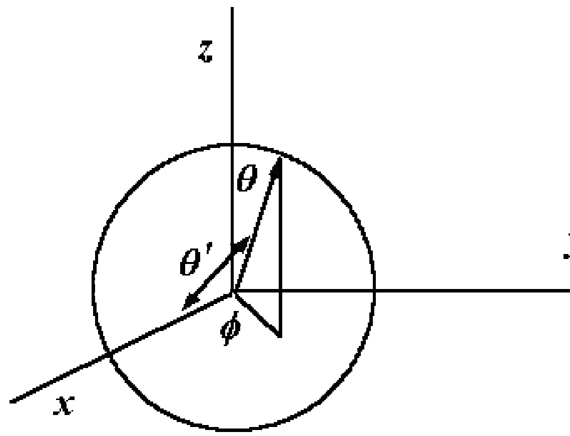
Taking the ratio of W_B and W_E leads to the final result:

$$LC = \mu\epsilon$$

PHY 5346
Homework Set 12 Solutions – Kimel

1. 6.8

The physical system is shown as



We know from Maxwell's equations that $-\vec{\nabla}\cdot\vec{M}$ plays the role of the effective magnetic charge density. Using the fact that

$$\vec{M} = \frac{1}{2}\left(\vec{x}\times\vec{J}\right)$$

and the fact that $\vec{J} = \rho_{pol}\vec{v}$, where $\rho_{pol} = \delta(r-a)\,\sigma_{pol}$, where σ_{pol} is given by equation (4.58) of the textbook:

$$\sigma_{pol} = 3\varepsilon_0\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)E_0\cos\theta' = 3\varepsilon_0\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)E_0\sin\theta\cos\phi$$

Using the figure

$$\vec{M} = \frac{1}{2}\left(\vec{x}\times\vec{J}\right) = \frac{1}{2}\sigma_{pol}va\sin\theta\delta(r-a)\left(-\hat{\theta}\right)$$

Thus

$$\rho_m = -\vec{\nabla}\cdot\vec{M} = \sigma_{pol}v\cos\theta\delta(r-a) = a\omega 3\varepsilon_0\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)E_0\sin\theta\cos\phi\cos\theta\delta(r-a)$$

This can be written

$$\rho_m = a\omega 3\varepsilon_0\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)E_0\left(-\sqrt{\frac{8\pi}{15}}\right)\left(\frac{Y_2^1 - Y_2^{-1}}{2}\right)\delta(r-a)$$

Using

$$q_{lm} = \int Y_l^{m*} r^l \rho d^3x$$

there are only two moments which survive for this distribution

$$q_{2\pm1} = \pm a^5 \omega 3\varepsilon_0 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) E_0 \left(-\sqrt{\frac{8\pi}{15}}\right) \frac{1}{2}$$

Using (on the outside of the sphere)

$$\phi_m = \sum_{lm} \frac{1}{2l+1} q_{lm} \frac{Y_l^m}{r^{l+1}}$$

$$\phi_m = a^5 \omega 3\varepsilon_0 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) E_0 \left(-\sqrt{\frac{8\pi}{15}}\right) \frac{1}{5} \left(\frac{Y_2^1 - Y_2^{-1}}{2}\right)$$

Or

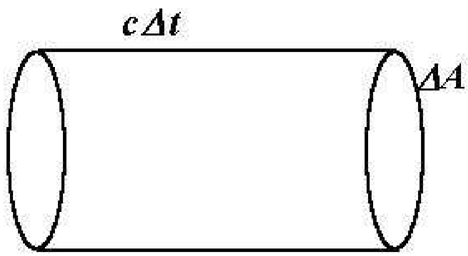
$$\phi_m = a^5 \omega 3\varepsilon_0 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) E_0 \left(-\sqrt{\frac{8\pi}{15}}\right) \frac{1}{5} \left(\frac{Y_2^1 - Y_2^{-1}}{2}\right) \frac{1}{r^3}$$

$$\phi_m = \frac{3}{5} \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) \omega \varepsilon_0 E_0 \left(\frac{a^5}{r^5}\right) xz$$

Repeat the same steps to get the potential on the inside of the sphere.

PHY 5346
Homework Set 12 Solutions – Kimel

1. 6.11
a) Consider the momentum contained in the volume



$$\Delta p = \Delta t c \Delta A g$$

$$F = \frac{\Delta p}{\Delta t} = c\Delta A g$$

$$P = \frac{F}{\Delta A} = cg$$

where I'm using the time averaged quantities. In class we found

$$cg = \frac{1}{c}S = \frac{1}{2}\varepsilon_0|E_0|^2 = u$$

Thus

$$P = u$$

b) We are given

$$S = 1.4 \times 10^3 \mathrm{W/m^2}$$

But we know $P = u = \frac{S}{c}$. From Newton's second law

$$a = \frac{F}{m} = \frac{F/A}{m/A} = \frac{S/c}{m/A} = \frac{1.4 \times 10^3 \mathrm{W/m^2}}{3 \times 10^8 \mathrm{m/s} \times 1 \times 10^{-3} \mathrm{kg/m^2}} = 4.66 \times 10^{-3} \mathrm{m/s^2}$$

In the solar wind, there are approximately 10×10^4 protons/$(m^2 - \mathrm{sec})$, with average velocity $v = 4 \times 10^5 \mathrm{m/s}$.

$$\frac{\Delta p}{\Delta t A} = P = 10 \times 10^4 \times 4 \times 10^5 \times 1.67 \times 10^{-27} = 6.68 \times 10^{-17} \mathrm{N/m^2}$$

:

$$a = \frac{F}{m} = \frac{F/A}{m/A} = \frac{P}{m/A} = \frac{6.68 \times 10^{-17} \mathrm{N/m^2}}{1 \times 10^{-3} \mathrm{kg/m^2}} = 6.68 \times 10^{-14} \mathrm{m/s^2}$$

6.21

a) I'm going to represent the dipole as a charge $-q$ at $\vec{r}_0$ and a charge q at $\vec{r}_0 + \vec{l}$. We take the limit

$$q\vec{l} \to \vec{p}$$

Thus

$$\rho = q\left[\delta(\vec{x} - \vec{r}_0 - \vec{l}) - \delta(\vec{x} - \vec{r}_0)\right]$$

Expanding around $\vec{l} = 0$ give

$$\rho(\vec{x}) = q\vec{\nabla}\delta(\vec{x} - \vec{r}_0) \cdot \left(-\vec{l}\right) = -\vec{p} \cdot \vec{\nabla}\delta(\vec{x} - \vec{r}_0)$$

As we've shown before for a collection of charges with charge density ρ and velocity $\vec{v}$

$$\vec{J} = \rho\vec{v} = -\vec{v}\left(\vec{p} \cdot \vec{\nabla}\right)\delta(\vec{x} - \vec{r}_0)$$

b) The magnetic dipole moment is given by

$$\vec{m} = \frac{1}{2}\int \vec{x} \times \vec{J} d^3x = -\frac{1}{2}\int \vec{x} \times \vec{v}\left(\vec{p} \cdot \vec{\nabla}\right)\delta(\vec{x} - \vec{r}_0)d^3x$$

Integrating by parts

$$\frac{1}{2}\vec{p} \cdot \int \vec{\nabla}\left(\vec{x} \times \vec{v}\right)\delta(\vec{x} - \vec{r}_0)d^3x$$

Look at the n^{th} component of the vector $\vec{p} \cdot \vec{\nabla}\left(\vec{x} \times \vec{v}\right)$

$$\left[\vec{p} \cdot \vec{\nabla}\left(\vec{x} \times \vec{v}\right)\right]_n = \sum_{ilm} p_i \partial_i \varepsilon_{lmn} x_l v_m = \sum_{lm} \varepsilon_{lmn} p_l v_m = [\vec{p} \times \vec{v}]_n$$

Thus

$$\vec{m} = \frac{1}{2}\int \vec{p} \times \vec{v}\left(\vec{x}\right)\delta(\vec{x} - \vec{r}_0)d^3x = \frac{1}{2}\vec{p} \times \vec{v}\left(\vec{r}_0\right)$$

Similarly

$$Q_{ij} = \int \left(3x_i x_j - r^2\delta_{ij}\right)\rho(\vec{x})d^3x = \int \left(3x_i x_j - r^2\delta_{ij}\right)\left[-\vec{p} \cdot \vec{\nabla}\delta(\vec{x} - \vec{r}_0)\right]d^3x$$

Integrating by parts

$$Q_{ij} = \sum_l \int p_l \partial_l \left(3x_i x_j - \sum_k x_k^2 \delta_{ij}\right)\delta(\vec{x} - \vec{r}_0)d^3x$$

$$Q_{ij} = \int \left(3p_i x_j + 3p_j x_i - 2\sum_l p_l x_l \delta_{ij}\right)\delta(\vec{x} - \vec{r}_0)d^3x$$

$$Q_{ij} = 3p_i x_{0j} + 3p_j x_{0i} - 2\vec{p} \cdot \vec{r}_0 \delta_{ij}$$

More Problems for Chapter 6

Problem 6.4
(a) Since the sphere is uniformly magnetized with magnetic moment $m = 4\pi R^3 M/3$, the magnetization is therefore M. The magnetic field inside the sphere is then given by Eq. (5.105):

$$\vec{B} = \frac{2}{3}\mu_0 \vec{M}$$

Given the Ohm's law in a moving conductor $\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B})$ and the fact that there is no current flowing inside the conductor, the electric field inside the conductor must be:

$$\vec{E} = -\vec{v} \times \vec{B} = -(\vec{\omega} \times \vec{r}) \times \vec{B} = \vec{B} \times (\vec{\omega} \times \vec{r}) = (\vec{B} \cdot \vec{r})\vec{\omega} - (\vec{B} \cdot \vec{\omega})\vec{r}$$

$$= \frac{2}{3}\mu_0 M(r\cos\theta\vec{\omega} - \omega\vec{r}) = \frac{2}{3}\mu_0 M\{z\omega\hat{z} - \omega(x\hat{x} + y\hat{y} + z\hat{z})\} = -\frac{2}{3}\mu_0 M\omega(x\hat{x} + y\hat{y})$$

Here we have chosen Spherical and Cartesian coordinates with their origins at the center of the sphere and their $z-$axis along the magnetization direction. Therefore, the volume charge density is then given by the Coulomb's law:

$$\rho = \epsilon_0 \nabla \cdot \vec{E} = -\frac{4}{3}\epsilon_0\mu_0 M\omega = -\frac{m\omega}{\pi c^2 R^3}$$

The total volume charge

$$Q_\rho = \int \rho d\tau = -\frac{4m\omega}{3c^2}$$

Note that rotating electric charges will result in an additional magnetic field. However, this field is suppressed by a factor of v/c compared with the field from magnetization and therefore ignored. (c) The surface charge distribution is $\phi-$symmetric and therefore can be written as

$$\sigma(\theta) = \sum_\ell \sigma_\ell P_\ell(\cos\theta)$$

Since the conductor is uncharged:

$$\oint \sigma(\theta)da + \int_V \rho d\tau = 0 \quad \Rightarrow \quad \sigma_0 = -\frac{1}{4\pi R^2}\rho\frac{4}{3}\pi R^3 = \frac{m\omega}{3\pi c^2 R^2} = -\frac{Q_\rho}{4\pi R^2}$$

From $\vec{E} = -\vec{v} \times \vec{B} = -(\vec{\omega} \times \vec{r}) \times \vec{B}$, we note that the electric field along the $z-$axis is zero and therefore the scalar potential is constant along the axis. For a point ($r < R$) inside the sphere, the potential due to the volume charge is (can be calculated in a variety of ways such as using Gauss's law to calculate the field first and then integrate):

$$\Phi_\rho(r) = \frac{\rho}{6\epsilon_0}(3R^2 - r^2)$$

On the $z-$axis:

$$\Phi_\rho(z) = \frac{\rho}{6\epsilon_0}(3R^2 - z^2)$$

The potential due to the surface charge for a point on the $z-$axis:

$$\Phi_\sigma(z) = \frac{1}{4\pi\epsilon_0}\oint \frac{\sigma(\theta')}{|\vec{r} - \vec{r'}|}da'$$

Note that

$$\frac{1}{|\vec{r} - \vec{r'}|} = \frac{1}{R}\frac{1}{\sqrt{1 + (z/R)^2 - 2(z/R)\cos\theta'}} = \sum_\ell \frac{z^\ell}{R^{\ell+1}}P_\ell(\cos\theta')$$

Consequently

$$\Phi_\sigma(z) = \frac{R^2}{2\epsilon_0}\sum_{\ell,\ell'}\frac{z^\ell\sigma_{\ell'}}{R^{\ell+1}}\int P_\ell(\cos\theta')P_{\ell'}(\cos\theta')d\cos\theta' = \frac{R^2}{\epsilon_0}\sum_\ell\frac{\sigma_\ell}{2\ell+1}\frac{z^\ell}{R^{\ell+1}}$$

The combined potential along the $z-$axis:

$$\Phi(z) = \Phi_\rho(z) + \Phi_\sigma(z) = \frac{\rho}{6\epsilon_0}(3R^2 - z^2) + \frac{R^2}{\epsilon_0}\sum_\ell\frac{\sigma_\ell}{2\ell+1}\frac{z^\ell}{R^{\ell+1}}$$

$\Phi(z)$ is independent of z only if

$$\sigma_2 = \frac{5}{6}\rho R = -\frac{5}{6}\frac{m\omega}{\pi c^2R^2}, \quad \text{and} \quad \sigma_\ell = 0 \ (\ell \neq 0, 2)$$

The surface charge density is then

$$\sigma(\theta) = \sigma_0 + \sigma_2 P_2(\cos\theta) = \frac{m\omega}{3\pi c^2R^2} - \frac{5m\omega}{6\pi c^2R^2}P_2(\cos\theta) = \frac{m\omega}{3\pi c^2R^2}(1 - \frac{5}{2}P_2(\cos\theta))$$

(b) Multipole moments $q_{\ell m}$ have contributions from both volume and surface charge distributions:

$$q_{\ell m} = \int Y^*_{\ell m}(\theta,\phi)r^\ell\rho d\tau + \oint Y^*_{\ell m}(\theta,\phi)R^\ell\sigma(\theta)da$$

$$= \sqrt{\frac{2\ell+1}{4\pi}}\left\{\rho\int_0^R r^{\ell+2}dr\int_{-1}^{+1}P_\ell^m(\cos\theta)d\cos\theta + R^\ell\int_{-1}^{+1}P_\ell^m(\cos\theta)\sigma(\theta)d\cos\theta\right\}\int_0^{2\pi}e^{im\phi}d\phi$$

$$= \sqrt{\frac{2\ell+1}{4\pi}}\left\{\rho\int_0^R r^{\ell+2}dr\int_{-1}^{+1}P_\ell(\cos\theta)d\cos\theta + R^\ell\int_{-1}^{+1}P_\ell(\cos\theta)\sigma(\theta)d\cos\theta\right\}2\pi\delta_{m,0}$$

Since $\sigma(\theta)$ is even in $\cos\theta$, the integral over $\cos\theta$ vanishes for odd ℓ values. Furthermore, the monopole moment also vanishes as a result of zero net charge on the sphere. Therefore, quadrupole moments are the lowest order non-vanishing moments. The quadrupole moment tensor has contributions from both the volume and the surface charge distributions:

$$Q_{ij} = \int(3x_ix_j - r^2\delta_{ij})\rho d\tau + \oint(3x_ix_j - R^2\delta_{ij})\sigma(\theta)da$$

Due to the symmetry in ϕ and in z, the only non-vanishing components are Q_{11}, Q_{22} and Q_{33}.

$$Q_{33} = \rho\int(3z^2 - r^2)d\tau + R^2\int(3z^2 - R^2)\sigma(\theta)d\Omega$$

$$= 2\pi\rho\int_0^R r^4dr\int_{-1}^{+1}(3\cos^2\theta - 1)d\cos\theta + 2\pi R^4\int_{-1}^{+1}(3\cos^2\theta - 1)\sigma(\theta)d\cos\theta$$

$$= 4\pi R^4\frac{m\omega}{3\pi c^2R^2}\int_{-1}^{+1}P_2(\cos\theta)\{1 - \frac{5}{2}P_2(\cos\theta)\}d\cos\theta = -\frac{4m\omega R^2}{3c^2}$$

The $\phi-$symmetry of charge distribution and the fact that the tensor is traceless lead to:

$$Q_{11} = Q_{22} = -\frac{1}{2}Q_{33} = \frac{2m\omega R^2}{3c^2}$$

(d) The electromotive force

$$\mathcal{E} = \int \vec{E}\cdot d\vec{\ell} = -\int \vec{v}\times\vec{B}\cdot d\vec{\ell} = \int_0^{\pi/2} \frac{2}{3}\mu_0 M(R\cos\theta\vec{\omega} - \omega\vec{r})\cdot(Rd\theta\hat{\theta})$$

$$= \frac{2}{3}\mu_0 M\omega R^2 \int_0^{\pi/2} \sin\theta\cos\theta d\theta = \frac{1}{3}\mu_0 M\omega R^2 = \frac{\mu_0 m\omega}{4\pi R}$$

Additional stuff for my record
The potential due to the surface charge

$$\Phi_\sigma(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint \frac{\sigma(\theta')}{|\vec{r}-\vec{r}'|}da' = \frac{R^2}{2\epsilon_0}\int \sigma(\theta')\sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\theta')P_\ell(\cos\theta)d\cos\theta'$$

$$= \frac{R^2\sigma_0}{2\epsilon_0}\sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\theta)\int P_\ell(\cos\theta')\left\{1-\frac{5}{2}P_2(\cos\theta')\right\}d\cos\theta'$$

$$= \frac{R^2\sigma_0}{2\epsilon_0}\left\{\frac{2}{r_>} - \frac{r_<^2}{r_>^3}P_2(\cos\theta)\right\} = \frac{\mu_0}{3\pi}m\omega\left\{\frac{1}{r_>} - \frac{r_<^2}{r_>^3}\frac{P_2(\cos\theta)}{2}\right\}$$

The total potential outside the sphere ($r_< = a$ and $r_> = r$):

$$\Phi(\vec{r}) = \Phi_\rho(\vec{r}) + \Phi_\sigma(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q_\rho}{r} + \frac{R^2\sigma_0}{\epsilon_0 r} - \frac{a^4\sigma_0}{2\epsilon_0 r^3}P_2(\cos\theta) = -\frac{m\omega R^2}{6\pi c^2\epsilon_0}\frac{P_2(\cos\theta)}{r^3} = -\frac{\mu_0}{6\pi}\frac{R^2}{r^3}P_2(\cos\theta)$$

The electric field outside the sphere:

$$\vec{E}(\vec{r}) = -\nabla\Phi(\vec{r}) = -\frac{\mu_0}{4\pi}\frac{m\omega R^2}{r^4}\left\{(3\cos^2\theta - 1)\hat{r} + 2\sin\theta\cos\theta\hat{\theta}\right\}$$

The total potential inside the sphere ($r_< = r$ and $r_> = a$):

$$\Phi(\vec{r}) = \Phi_\rho(\vec{r}) + \Phi_\sigma(\vec{r}) = \frac{\rho}{6\epsilon_0}(3R^2 - r^2) + \frac{a\sigma_0}{\epsilon_0} - \frac{\sigma_0 r^2}{2\epsilon_0 a}P_2(\cos\theta)$$

$$= -\frac{m\omega}{6\pi c^2\epsilon_0 R}\left\{1 - \frac{r^2}{R^2}(1 - P_2(\cos\theta))\right\} = -\frac{\mu_0}{6\pi}\frac{m\omega}{R}\left\{1 - \frac{r^2}{R^2}(1 - P_2(\cos\theta))\right\}$$

Problem 6.8
In an external uniform electric field $\vec{E}$, the sphere is uniformly polarized with the polarization given by Eq. (4.57):

$$\vec{P} = 3\epsilon_0\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\vec{E}_0$$

Therefore, the bound volume and surface charge densities are:

$$\rho_b = -\nabla\cdot\vec{P} = 0, \qquad \sigma_b = \vec{P}\cdot\vec{n}$$

where $\vec{n}$ is the normal vector on the sphere surface. Since the sphere is rotating, the bound surface charge results an effective surface current with density:

$$\vec{K}_M = \sigma_b\vec{v} = (\vec{P}\cdot\vec{n})(\vec{\omega}\times\vec{r})|_{r=a} = a(\vec{P}\cdot\vec{n})(\omega\times\vec{n})$$

Comparing with the effective surface current density $\vec{K}_M = \vec{M} \times \vec{n}$ due to magnetization $\vec{M}$, we identify $a(\vec{P} \cdot \vec{n})\vec{\omega}$ as an effective magnetization. Therefore, the effective magnetic surface charge density

$$\sigma_M(\theta, \phi) = \vec{M}_{\text{eff.}} \cdot \vec{n} = a(\vec{P} \cdot \vec{n})(\vec{\omega} \cdot \vec{n}) = 3\epsilon_0 a \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}(\vec{E}_0 \cdot \vec{n})(\omega \cos\theta) = 3\epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} a\omega E_0 \sin\theta \cos\theta \cos\phi$$

The magnetic scalar potential $\Phi_M(\vec{r})$ (Eq. (5.100)):

$$\Phi_M(\vec{r}) = \frac{1}{4\pi} \oint \frac{\sigma_M}{|\vec{r} - \vec{r'}|} da' = \frac{3}{4\pi} \epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} a^3 \omega E_0 \int \frac{\sin\theta' \cos\theta' \cos\phi'}{|\vec{r} - \vec{r'}|} d\Omega'$$

Using the identity:

$$\sin\theta' \cos\theta' \cos\phi' = -\sqrt{\frac{8\pi}{15}} Re\left\{Y_{21}(\theta', \phi')\right\}$$

and expanding $1/|\vec{r} - \vec{r'}|$ using spherical harmonics, the integral becomes:

$$\int \frac{\sin\theta' \cos\theta' \cos\phi'}{|\vec{r} - \vec{r'}|} d\Omega' = -\sqrt{\frac{8\pi}{15}} Re\left\{\sum_{\ell,m} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}(\theta, \phi) \int Y_{\ell m}^*(\theta', \phi') Y_{21}(\theta', \phi') d\Omega'\right\}$$

$$= -\sqrt{\frac{8\pi}{15}} Re\left\{\sum_{\ell,m} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}(\theta, \phi) \delta_{\ell,2} \delta_{m,1}\right\}$$

$$= \frac{4\pi}{5} \frac{r_<^2}{r_>^3} \left\{-\sqrt{\frac{8\pi}{15}} Re\{Y_{21}(\theta, \phi)\}\right\}$$

$$= \frac{4\pi}{5} \frac{r_<^2}{r_>^3} \sin\theta \cos\theta \cos\phi$$

where $r_< = \min(r, a)$ and $r_> = \max(r, a)$. Therefore, the scalar potential

$$\Phi_M(\vec{r}) = \frac{1}{4\pi} \oint \frac{\sigma_M}{|\vec{r} - \vec{r'}|} da' = \frac{3}{4\pi} \epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} a^3 \omega E_0 \left\{\frac{4\pi}{5} \frac{r_<^2}{r_>^3} \sin\theta \cos\theta \cos\phi\right\}$$

$$= \frac{3}{5} \epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \omega E_0 \frac{a^3 r_<^2}{r^2 r_>^3} (r \sin\theta \cos\phi)(r \cos\theta)$$

Note

$$\frac{a^3 r_<^2}{r^2 r_>^3} = \frac{a^3 r_<^2 r_>^2}{r^2 r_>^5} = \frac{a^3 r^2 a^2}{r^2 r_>^5} = \left\{\frac{a}{r_>}\right\}^5$$

The magnetic field $\vec{H}$ can be determined from $\Phi_M(\vec{r})$:

$$\Phi_M(\vec{r}) = \frac{3}{5} \epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \omega E_0 \left\{\frac{a}{r_>}\right\}^5 \cdot xz = \frac{1}{5} P\omega \left\{\frac{a}{r_>}\right\}^5 \cdot xz$$

What if the electric field is along the rotational axis?
The effective magnetic surface charge density:

$$\sigma_M(\theta, \phi) = a(\vec{P} \cdot \vec{n})(\vec{\omega} \cdot \vec{n}) = 3\epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} a\omega E_0 \cos^2\theta$$

$$\Phi_M(\vec{r}) = \frac{1}{4\pi}\oint \frac{\sigma_M}{|\vec{r}-\vec{r}'|}da' = \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}a^3\omega\epsilon_0 E_0\left\{\frac{1}{r_>}+\frac{2}{5}\frac{r_<^2}{r_>^3}P_2(\cos\theta)\right\}$$

Problem 6.11
(a) The momentum conservation equation

$$\frac{d}{dt}(\vec{P}_{\text{fields}}+\vec{P}_{\text{mech.}}) = \oint\sum_j T_{ij}n_j da = -\oint(-\vec{\vec{T}})\cdot\vec{n}da$$

implies that the projection of the momentum flow along the direction of $\vec{n}$ is given by $-\vec{\vec{T}}\cdot\vec{n}$ where $\vec{\vec{T}}$ is the Maxwell stress (momentum) tensor:

$$T_{ij} = \epsilon_0(E_iE_j - \frac{1}{2}E^2\delta_{ij}) + \mu_0(H_iH_j - \frac{1}{2}H^2\delta_{ij})$$

Physically $-T_{ij}$ is the rate at which the i^{th}-component of the momentum is crossing a unit area in the $j^{th}-$direction. In a Cartesian coordinate system with the $z-$axis along the wave propagation direction and $\vec{E}$ along the $x-$direction:

$$\vec{E} = E\hat{x}, \quad \vec{H} = H\hat{y}$$

The i^{th} component of the linear momentum flowing into the surface (in the direction $\vec{n} = \hat{z}$) per unit time per unit cross section is therefore

$$p_i = \sum_j(-T_{ij})n_j = -T_{i3} = -\epsilon_0(E_iE_3 - \frac{1}{2}E^2\delta_{i,3}) - \mu_0(H_iH_3 - \frac{1}{2}H^2\delta_{i,3}) = \frac{1}{2}(\epsilon_0E^2+\mu_0H^2)\delta_{i,3}$$

In the chosen coordinate system, the only non-vanishing component is p_z. The force exerted on the wave from the surface per unit area (according to Newton's second law):

$$F_z = \Delta p_z = (0 - p_z) = -\frac{1}{2}(\epsilon_0E^2+\mu_0H^2)$$

Therefore, the radiation pressure on the surface (Newton's third law):

$$P_z = -F_z = \frac{1}{2}(\epsilon_0E^2+\mu_0H^2)$$

which is the energy density in the electromagnetic wave.

Problem 6.13
(a) Note: only need to work out the first non-zero terms in electric/magnetic fields. To a good approximation, the conductors are at equipotential and have uniform surface charge distributions. Choose a Cartesian coordinate system with its origin at the center of the capacitor, the $x-$axis parallel to the edge a and pointing to the current feed, the $y-$ axis perpendicular to the two planes. Let $Q(t) = Q_0e^{-i\omega t}$ be the total charge on the bottom plate, the electric field in between the plates is therefore

$$\vec{E}(\vec{r},t) = \frac{\sigma(t)}{\epsilon_0}\hat{y} = \frac{1}{\epsilon_0}\frac{Q_0}{ab}e^{-i\omega t}\hat{y}$$

The charge on the $x' < x$ portion of the bottom plate is:

$$Q(\vec{r},t) = b(x+\frac{a}{2})\sigma(t) = \frac{Q_0}{ab}e^{-i\omega t}b(x+\frac{a}{2}) = (\frac{1}{2}+\frac{x}{a})Q_0e^{-i\omega t}$$

The surface current density

$$\vec{K}(\vec{r},t) = -\frac{1}{b}\frac{\partial Q(\vec{r},t)}{\partial t}\hat{x} = i\omega(\frac{1}{2}+\frac{x}{a})\frac{Q_0}{b}e^{-i\omega t}\hat{x}$$

Note that K is maximum at $x = a/2$ and zero at $x = -a/2$ as expected. In between, the conduction current looses its strength to the displacement current. (b) The electric energy density and energy

$$W_e = \frac{1}{4}\vec{E}\cdot\vec{D}^* = \frac{1}{4}\epsilon_0|E|^2 = \frac{1}{4\epsilon_0}\frac{Q_0^2}{(ab)^2} \quad \Rightarrow \quad \int W_e d\tau = \frac{Q_0^2}{4\epsilon_0}\frac{d}{ab}$$

The magnetic energy density and energy

$$W_m = \frac{1}{4}\vec{B}\cdot\vec{H}^* = \frac{1}{4\mu_0}|B|^2 = \frac{\mu_0}{4}\omega^2(\frac{1}{2}+\frac{x}{a})^2\frac{Q_0^2}{b^2} \quad \Rightarrow \quad \int W_m d\tau = \frac{\mu_0\omega^2 adQ_0^2}{12b}$$

The input current

$$I(t) = \frac{dQ(t)}{dt} = -i\omega Q_0 e^{-i\omega t}$$

The reactance

$$X = \frac{4\omega}{|I|^2}\int(W_m - W_e)d\tau = \frac{4\omega}{\omega^2 Q_0^2}\left\{\frac{\mu_0\omega^2 adQ_0^2}{12b} - \frac{Q_0^2 d}{4\epsilon_0 ab}\right\} = \frac{\mu_0\omega ad}{3b} - \frac{d}{\epsilon_0\omega ab} = \omega L - \frac{1}{\omega C}$$

where $L = \mu_0 ad/3b$ and $C = \epsilon_0 ab/d$. Therefore, X is equivalent to the reactance of a capacitor C connecting in series with an inductor L.

More Problems for Chapter 6

Problem 6.14

(a) The charge on the plate

$$Q(t) = \int_0^t I(t')dt' = \frac{I_0}{\omega}\sin\omega t$$

assuming there is no static charge. In a cylindrical coordinate system, the electric field is along the z and the magnetic field is along the ϕ based on the symmetry of the problem. Let (E_0, E_1) and (B_0, B_1) be the first two non-zero terms in the electric and magnetic field expansions:

$$\vec{E}(\vec{r}) = (E_0 + E_1)\hat{z}; \qquad \vec{B}(\vec{r}) = (B_0 + B_1)\hat{\phi}$$

The first order of the field is given by

$$E_0 = \frac{\sigma}{\epsilon_0} = \frac{Q(t)}{\pi a^2 \epsilon_0} = \frac{1}{\pi\epsilon_0}\frac{I_0}{\omega a^2}\sin(\omega t)$$

From Ampere-Maxwell's law, changing in electric field results in magnetic field:

$$\nabla \times \vec{B} = \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \Rightarrow \quad \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_0) = \mu_0\epsilon_0\frac{\partial E_0}{\partial t} = \frac{\mu_0 I_0}{\pi a^2}\cos(\omega t) \quad \Rightarrow \quad B_0 = \frac{\mu_0 I_0}{2\pi a}\frac{\rho}{a}\cos(\omega t)$$

Note that there is no static magnetic field. The oscillating magnetic field gives rise to additional electric field:

$$\nabla \times \vec{E}_1 = -\frac{\partial \vec{B}_0}{\partial t} \quad \Rightarrow \quad \frac{\partial E_1}{\partial \rho} = -\frac{\mu_0 I}{2\pi a}\frac{\rho}{a}\omega\sin(\omega t) \quad \Rightarrow \quad E_1 = -\frac{\mu_0 I_0}{4\pi}\frac{\rho^2}{a^2}\omega\sin(\omega t)$$

This additional electric field in turn contributes to the magnetic field according to Ampere-Maxwell's law:

$$\nabla \times \vec{B}_1 = \mu_0\epsilon_0\frac{\partial E_1}{\partial t} \quad \Rightarrow \quad \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_1) = -\frac{\mu_0 I_0}{4\pi c^2}\frac{\omega\rho^2}{a^2}\cos(\omega t) \quad \Rightarrow \quad B_1 = -\frac{\mu_0 I_0}{16\pi c^2}\frac{\rho^3}{a^2}\omega^2\cos(\omega t)$$

Combining two contributions together,

$$\vec{E}(\vec{r}) = (E_0 + E_1)\hat{z} = \left\{\frac{1}{\pi\epsilon_0}\frac{I_0}{\omega a^2}\sin(\omega t) - \frac{\mu_0 I_0}{4\pi}\frac{\rho^2}{a^2}\omega\sin(\omega t)\right\}\hat{z} = \frac{1}{\pi\epsilon_0}\frac{I_0}{\omega a^2}\sin(\omega t)\left\{1 - \frac{\rho^2}{4c^2}\omega^2\right\}\hat{z}$$

$$\vec{B}(\vec{r}) = (B_0 + B_1)\hat{\phi} = \left\{\frac{\mu_0 I_0}{2\pi a}\frac{\rho}{a}\cos(\omega t) - \frac{\mu_0 I_0}{16\pi c^2}\frac{\rho^3}{a^2}\omega^2\cos(\omega t)\right\}\hat{\phi} = \frac{\mu_0 I_0}{2\pi a}\frac{\rho}{a}\cos(\omega t)\left\{1 - \frac{\rho^2}{8c^2}\omega^2\right\}\hat{\phi}$$

(b) In complex notation, the average energy densities in electric and magnetic fields

$$w_e = \frac{1}{4}\vec{E}\cdot\vec{D}^* = \frac{1}{4}\epsilon_0|E|^2 \quad \Rightarrow \quad w_m = \frac{1}{4}\vec{B}\cdot\vec{H}^* = \frac{1}{4\mu_0}|B|^2$$

Converting the electric and magnetic fields obtained in (a) into complext notation:

$$\vec{E}(\vec{r}) = Re\left\{i\frac{1}{\pi\epsilon_0}\frac{I_0}{\omega a^2}\{1 - \frac{\rho^2}{4c^2}\omega^2\}e^{-i\omega t}\right\}\hat{z}$$

$$\vec{B}(\vec{r}) = Re\left\{\frac{\mu_0 I_0}{2\pi a}\frac{\rho}{a}\{1 - \frac{\rho^2}{8c^2}\omega^2\}e^{-i\omega t}\right\}\hat{\phi}$$

Therefore the total average energies in electric and magnetic fields are:

$$\int w_E d\tau = \frac{1}{4}\epsilon_0(2\pi d)\int_0^a |E|^2 \rho d\rho \approx \frac{1}{2}\pi\epsilon_0 d\{\frac{1}{\pi\epsilon_0}\frac{I_0}{\omega a^2}\}^2\int_0^a (1-\frac{\rho^2}{2c^2}\omega^2)\rho d\rho = \frac{1}{4\pi\epsilon_0}\frac{I_0^2 d}{\omega^2 a^2}\left\{1-\frac{a^2}{4c^2}\omega^2\right\}$$

$$\int w_m d\tau = \frac{1}{4\mu_0}(2\pi d)\int_0^a |B|^2\rho d\rho \approx \frac{\pi d}{2\mu_0}\left\{\frac{\mu_0 I_0}{2\pi a^2}\right\}^2\int_0^a (1-\frac{\rho^2}{8c^2}\omega^2)\rho^3 d\rho = \frac{\mu_0}{32\pi}I_0^2 d\left\{1-\frac{a^2}{6c^2}\omega^2\right\}$$

The total charge on the plate to the second order:

$$Q = \int \sigma da = \int \epsilon_0 E da = 2\pi\epsilon_0 \frac{i}{\pi\epsilon_0}\frac{I_0}{\omega a^2}e^{-i\omega t}\int_0^a (1-\frac{\rho^2}{4c^2}\omega^2)\rho d\rho = i\frac{I_0}{\omega}\left\{1-\frac{a^2}{8c^2}\omega^2\right\}e^{-i\omega t}$$

Therefore

$$|I_i|^2 = \omega^2|Q|^2 = I_0^2\left\{1-\frac{a^2}{8c^2}\omega^2\right\}^2 \approx I_0^2\left\{1-\frac{a^2}{4c^2}\omega^2\right\} \quad \Rightarrow \quad I_0^2 \approx |I_i|^2\left\{1+\frac{a^2}{4c^2}\omega^2\right\}$$

Plugging into $\int w_e d\tau$ and $\int w_m d\tau$:

$$\int w_e d\tau = \frac{1}{4\pi\epsilon_0}\frac{d}{\omega^2 a^2}|I_i|^2\left\{1-\frac{a^2}{4c^2}\omega^2\right\}\left\{1+\frac{a^2}{4c^2}\omega^2\right\} \approx \frac{1}{4\pi\epsilon_0}\frac{|I_i|^2 d}{\omega^2 a^2}$$

$$\int w_m d\tau = \frac{\mu_0 d}{32\pi}|I_i|^2\left\{1+\frac{a^2}{4c^2}\omega^2\right\}\left\{1-\frac{a^2}{6c^2}\omega^2\right\} \approx \frac{\mu_0 d}{4\pi}\frac{|I_i|^2}{8}\left\{1+\frac{a^2}{12c^2}\omega^2\right\}$$

(c) The reactance

$$X = \frac{4\omega}{|I_i|^2}\int (w_m - w_e)d\tau \approx \omega\frac{\mu_0 d}{8\pi} - \frac{1}{\omega}\frac{d}{\pi\epsilon_0 a^2}$$

equivalent to the reactance of an inductor $L = \mu_0 d/8\pi$ and a capacitor $C = \epsilon_0\pi a^2/d$ connected in series. The resonance frequency

$$\omega_{res} = \frac{1}{\sqrt{LC}} = 2\sqrt{2}\frac{c}{a}$$

Problem 7.2

(a) Choose a coordinate system such that the electric field is along the $x-$axis, the magnetic field along the $y-$axis and the wave propogates in $z-$direction. In medium n_1, the incident and reflected waves are described by:

$$\vec{E}^i = E^i e^{i(k_1 z-\omega t)}\hat{x}, \quad \vec{B}^i = \frac{E^i}{v_1}e^{i(k_1 z-\omega t)}\hat{y}$$

$$\vec{E}^r = E^r e^{i(-k_1 z-\omega t)}\hat{x}, \quad \vec{B}^r = -\frac{E^r}{v_1}e^{-i(k_1 z-\omega t)}\hat{y}$$

In medium n_2, there are both forward (denoted as +) and backward (−) propogating waves and are described by:

$$\vec{E}^+ = E^+ e^{i(k_2 z-\omega t)}\hat{x}, \quad \vec{B}^+ = \frac{E^+}{v_2}e^{i(k_2 z-\omega t)}\hat{y}$$

$$\vec{E}^- = E^- e^{i(-k_2 z-\omega t)}\hat{x}, \quad \vec{B}^- = -\frac{E^-}{v_2}e^{-i(k_2 z-\omega t)}\hat{y}$$

In medium n_3, there is only transmitted wave:

$$\vec{E}^t = E^t e^{i(k_3 z - \omega t)}\hat{x}, \qquad \vec{B}^t = \frac{E^t}{v_3} e^{i(k_3 z - \omega t)}\hat{y}$$

where $k_1 = \omega/v_1$, $k_2 = \omega/v_2$, and $k_3 = \omega/v_3$ are wave numbers in the three media. For nonpermeable media ($\mu_1 \approx \mu_2 \approx \mu_3 \approx \mu_0$), $\vec{E}_{||}$ and $\vec{B}_{||}$ are continuous at each interface ($x = 0, d$). At $x = 0$, one has:

$$E^i + E^r = E^+ + E^-; \qquad \frac{E^i - E^r}{v_1} = \frac{E^+ - E^-}{v_2}$$

At $x = d$, one has:

$$E^+ e^{ik_2 d} + E^- e^{-ik_2 d} = E^t e^{ik_3 d}; \qquad \frac{E^+ e^{ik_2 d} - E^- e^{-ik_2 d}}{v_2} = \frac{E^t}{v_3} e^{ik_3 d}$$

Let

$$\alpha \equiv \frac{v_1}{v_2} = \frac{n_2}{n_1}; \qquad \beta \equiv \frac{v_2}{v_3} = \frac{n_3}{n_2}$$

The four equations are then

$$E^i + E^r = E^+ + E^-; \qquad E^i - E^r = \alpha(E^+ - E^-)$$

$$E^+ e^{ik_2 d} + E^- e^{-ik_2 d} = E^t e^{ik_3 d}; \qquad E^+ e^{ik_2 d} - E^- e^{-ik_2 d} = \beta E^t e^{ik_3 d}$$

Solving for $E^+ e^{ik_2 d}$ and $E^- e^{-ik_2 d}$ from the last two equations:

$$E^+ e^{ik_2 d} = \frac{1}{2}(1+\beta) E^t e^{ik_3 d}, \qquad E^- e^{-ik_2 d} = \frac{1}{2}(1-\beta) E^t e^{ik_3 d}$$

Add the first two equations to eliminate E^r:

$$2E^i = (1+\alpha)E^+ + (1-\alpha)E^- = \frac{1}{2} E^t e^{ik_3 d}\left\{1+\alpha)(1+\beta)e^{-ik_2 d} + (1-\alpha)(1-\beta)e^{ik_2 d}\right\}$$

Solving for E^t in terms of E^i:

$$\frac{E^i}{E^t} = \frac{1}{2} e^{ik_3 d}\{(1+\alpha\beta)\cos(k_2 d) - 2i(\alpha+\beta)\sin(k_2 d)\}$$

Therefore,

$$4|\frac{E^i}{E^t}|^2 = (1+\alpha\beta)^2\cos^2(k_2 d) + (\alpha+\beta)^2\sin^2(k_2 d) = (1+\alpha\beta)^2 - (1-\alpha^2)(1-\beta^2)\sin^2(k_2 d)$$

The transmission coefficient T:

$$T = \frac{I^t}{I^i} = \frac{\epsilon_3 v_3 |E^t|^2}{\epsilon_1 v_1 |E^i|^2} = \frac{n_3}{n_1}|\frac{E^t}{E^i}|^2 = \frac{4\alpha\beta}{(1+\alpha\beta)^2 - (1-\alpha^2)(1-\beta^2)\sin^2(k_2 d)}$$

$$= \frac{4 n_1 n_2^2 n_3}{n_2^2(n_1+n_3)^2 + (n_2^2 - n_3^2)(n_2^2 - n_1^2)\sin^2(n_2 d\omega/c)}$$

It varies between the two extremum values

$$T_1 = \frac{4 n_1 n_2^2 n_3}{(n_2^2 + n_1 n_3)^2}, \qquad T_2 = \frac{4 n_1 n_3}{(n_1+n_3)^2}$$

as a function of ω for a fixed d or as a function of d for fixed ω. From the energy conservation, the reflection coefficient R is

$$R = 1 - T = \frac{n_2^2(n_1-n_3)^2 + (n_2^2-n_3^2)(n_2^2-n_1^2)\sin^2(n_2 d\omega/c)}{n_2^2(n_1+n_3)^2 + (n_2^2-n_3^2)(n_2^2-n_1^2)\sin^2(n_2 d\omega/c)}$$

In the special case of $d = 0$, the coefficients reduce to the familiar forms of two media.
(b) For $n_3 = 1$, the reflection coefficient

$$R = \frac{n_2^2(n_1-1)^2 + (n_2^2-1)(n_2^2-n_1^2)\sin^2(n_2 d\omega/c)}{n_2^2(n_1+1)^2 + (n_2^2-1)(n_2^2-n_1^2)\sin^2(n_2 d\omega/c)}$$

To have zero reflection at $\omega = \omega_0$, the following condition must be satisfied:

$$n_2^2(n_1-1)^2 + (n_2^2-1)(n_2^2-n_1^2)\sin^2(n_2 d\omega_0/c) = 0$$

Since $n_1 > 1, n_2 > 1$, this is only possible if $n_2 < n_1$. One set of possible solutions is given by

$$\sin^2(n_2 d\omega_0/c) = 1, \qquad \text{and} \qquad n_2^2(n_1-1)^2 + (n_2^2-1)(n_2^2-n_1^2) = 0$$

This leads to

$$n_2 = \sqrt{n_1} \qquad \text{and} \qquad d = (\ell + \frac{1}{2})\pi \frac{c}{\sqrt{n_1}\omega_0}$$

where ℓ is a non-zero integer.

Problem 7.3

Note: only need to consider the polarization perpendicular to the plane of incidence and assume $\mu = \mu_0$ in all media.
(a) Assuming the wave incident from left, let E_0 and $E_0^{''}$ be the incident and reflected waves on the left surface, E_+ and E_- be the right and left traveling waves on the left surface in the air gap, θ be the incident angle, and θ' be the refracted angle ($n \sin\theta = \sin\theta'$) in the gap on the left surface, the boundary conditions (parallel components of $\vec{E}$ and $\vec{H}$ continous) on the left surface lead to:

$$E_0 + E_0^{''} = E_+ + E_- \qquad n\cos\theta(E_0 - E_0^{''}) = \cos\theta'(E_+ - E_-)$$

On the right surface, the incident and reflected waves are $E_+ e^{ik\ell}$ and $E_- e^{-ik\ell}$ where $\ell = d/\cos\theta'$ is the path length between the two surfaces and $k = \omega/c$ is the wave number in air. Let E_0' be the transmitted wave, the boundary conditions on the right surface lead to:

$$E_+ e^{ik\ell} + E_- e^{-ik\ell} = E_0' \qquad \cos\theta'(E_+ e^{ik\ell} - E_- e^{-ik\ell}) = n\cos\theta E_0'$$

Defining

$$\alpha = e^{ik\ell} = e^{ikd/\cos\theta'}; \qquad \beta = \frac{n\cos\theta}{\cos\theta'},$$

the four boundary equations are:

$$E_+ + E_- = E_0 + E_0^{''} \qquad E_+ - E_- = \beta(E_0 - E_0^{''})$$

$$\alpha E_+ + \frac{E_-}{\alpha} = E_0^{'} \qquad \alpha E_+ - \frac{E_-}{\alpha} = \beta E_0^{'}$$

Solving these equations for $E_0^{''}$ and $E_0^{'}$:

$$E_0^{'} = \frac{4\alpha\beta}{(1+\beta)^2 - \alpha^2(1-\beta)^2} E_0 = \frac{4\beta e^{i\phi}}{(1+\beta)^2 - e^{2i\phi}(1-\beta)^2} E_0$$

$$E_0^{''} = \frac{(1-\beta^2)(\alpha^2-1)}{(1+\beta)^2 - (1-\beta)^2\alpha^2} E_0 = \frac{(1-\beta^2)(e^{2i\phi}-1)}{(1+\beta)^2 - e^{2i\phi}(1-\beta)^2} E_0$$

where $\phi = kd/\cos\theta' = \omega d/(c\cos\theta')$. The transmission coefficient

$$T = \frac{|E_0^{'}|^2}{|E_0|^2} = \frac{(4\beta)^2}{(1+\beta)^4 + (1-\beta)^4 - 2(1-\beta^2)^2\cos(2\phi)}$$

$$= \frac{(4n\cos\theta\cos\theta')^2}{(n\cos\theta+\cos\theta')^4 + (n\cos\theta-\cos\theta')^4 - 2(n^2\cos^2\theta - \cos^2\theta')^2\cos(2\omega d/(c\cos\theta'))}$$

The reflection coefficient

$$R = \frac{|E_0^{''}|^2}{|E_0|^2} = \frac{2(1-\beta^2)^2(1-\cos\phi)}{(1+\beta)^4 + (1-\beta)^4 - 2(1-\beta^2)^2\cos(2\phi)}$$

$$= \frac{4(n^2\cos^2\theta - \cos^2\theta')^2\sin^2(\omega d/c\cos\theta')}{(n\cos\theta+\cos\theta')^4 + (n\cos\theta-\cos\theta')^4 - 2(n^2\cos^2\theta - \cos^2\theta')^2\cos(2\omega d/(c\cos\theta'))}$$

It is easy to verify that $R + T = 1$.
(b) For $\theta > \theta_c = \sin^{-1}(1/n)$, $\cos\theta' = \sqrt{1-\sin^2\theta'} = \sqrt{1-n^2\sin^2\theta} = i\sqrt{n^2\sin^2\theta - 1} = i|\cos\theta'|$ is a pure imaginary. The transmission coefficient

$$T = \frac{|4\beta e^{i\phi}|^2}{|(1+\beta)^2 - e^{2i\phi}(1-\beta)^2|^2} = \frac{|4n\cos\theta\cos\theta'|^2 e^{2kd/|\cos\theta'|}}{|(n\cos\theta+\cos\theta')^2 - e^{2kd/|\cos\theta'|}(n\cos\theta-\cos\theta')^2|^2}$$

As $d \to 0$,

$$T \to \frac{|4n\cos\theta\cos\theta'|^2}{|(n\cos\theta+\cos\theta')^2 - (n\cos\theta-\cos\theta')^2|^2} = 1$$

As $d \to \infty$,

$$T \to \frac{|4n\cos\theta\cos\theta'|^2 e^{-2kd/|\cos\theta'|}}{(n^2\cos^2\theta + |\cos\theta'|^2)^2} \to 0$$

as expected.

Problem 7.4
(a) At normal incidence, the reflected wave $E_0^{''}$ is given by

$$\frac{E_0^{''}}{E_0} = \frac{1-n}{1+n}$$

where $n = c/v = c\sqrt{\mu\epsilon}$ is the index of refraction of the medium and E_0 is the incidence wave. For a conductor, $\epsilon \approx i\sigma/\omega$. Therefore,

$$n = c\sqrt{\mu\epsilon} = c\sqrt{i\frac{\mu\sigma}{\omega}} = (1+i)\frac{c}{\omega}\sqrt{\frac{\mu\sigma\omega}{2}} = (1+i)\frac{c}{\omega\delta}$$

where $\delta \equiv \sqrt{2/(\mu\sigma\omega)}$ is the skin depth. Therefore,

$$\frac{E_0^{''}}{E_0} = \frac{1-n}{1+n} = \frac{1-(1+i)c/(\omega\delta)}{1+(1+i)c/(\omega\delta)} = re^{i\phi}$$

where r and ϕ are the amplitude and the phase of the ratio respectively:

$$r = \frac{\sqrt{1+4c^4/(\omega^4\delta^4)}}{1+2c/(\omega\delta)+2c^2/(\omega^2\delta^2)} = \frac{\sqrt{\omega^4\delta^4+4c^4}}{2c^2+2c\omega\delta+\omega^2\delta^2}$$

$$\tan\phi = -\frac{2c/(\omega\delta)}{1 - 2c^2/(\omega^2\delta^2)} = -\frac{2c\omega\delta}{\omega^2\delta^2 - 2c^2}$$

For a perfect conductor, $\sigma \to \infty \;\Rightarrow\; \delta \to 0$, the amplitude and the phase

$$r \to 1 \quad \text{and} \quad \tan\phi \to 0^- \;\; (\phi \to \pi)$$

As expected the reflected wave has a 180° phase change with respect to the incident wave.
(b) The reflection coefficient

$$R = r^2 = \frac{\omega^4\delta^4 + 4c^4}{(2c^2 + 2c\omega\delta + \omega^2\delta^2)^2} \approx \frac{1 + (\omega\delta/c)^4/4}{(1 + \omega\delta/c)^2} \approx 1 - 2\frac{\omega\delta}{c}$$

7.1

I shall apply Eqs.(26), (27), and (28)

$$\sqrt{\frac{s_0+s_1}{2}} = a_1 \qquad \sqrt{\frac{s_0-s_1}{2}} = a_2 \qquad \delta_l = \delta_2 - \delta_1 = \sin^{-1}\left(\frac{s_3}{2a_1a_2}\right)$$

$$\sqrt{\frac{s_0+s_3}{2}} = a_+ \qquad \sqrt{\frac{s_0-s_3}{2}} = a_- \qquad \delta_c = \delta_- - \delta_+ = \sin^{-1}\left(\frac{s_2}{2a_+a_-}\right)$$

a) $s_0 = 3,\ s_1 = -1,\ s_2 = 2,\ s_3 = -2$

$$a_1 = 1,\ a_2 = \sqrt{2}$$

$$\delta_l = \sin^{-1}\left(\frac{-2}{2\sqrt{2}}\right) = -\frac{1}{4}\pi \text{ rad}$$

$$a_+ = \frac{1}{\sqrt{2}}, \quad a_- = \sqrt{\frac{5}{2}}$$

$$\delta_c = \sin^{-1}\left(\frac{2}{2\left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{\frac{5}{2}}\right)}\right) = 1.1071 \text{ rad}$$

b) $s_0 = 25,\ \ s_1 = 0,\ \ s_2 = 24,\ s_3 = 7$

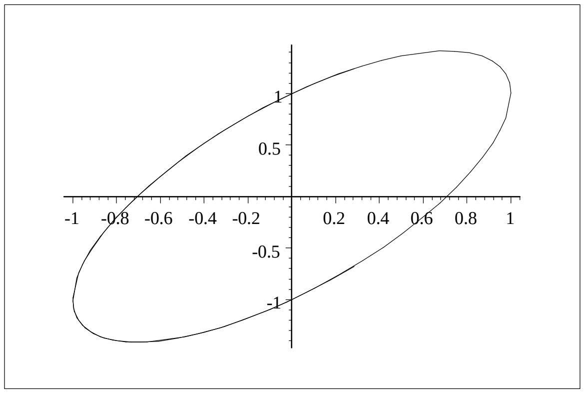
$$a_1 = \sqrt{\frac{25}{2}},\ a_2 = \sqrt{\frac{25}{2}}$$

$$\delta_l = \sin^{-1}\left(\frac{s_3}{2a_1a_2}\right) = \sin^{-1}\left(\frac{7}{2\left(\sqrt{\frac{25}{2}}\sqrt{\frac{25}{2}}\right)}\right) = 0.28379 \text{ rad}$$

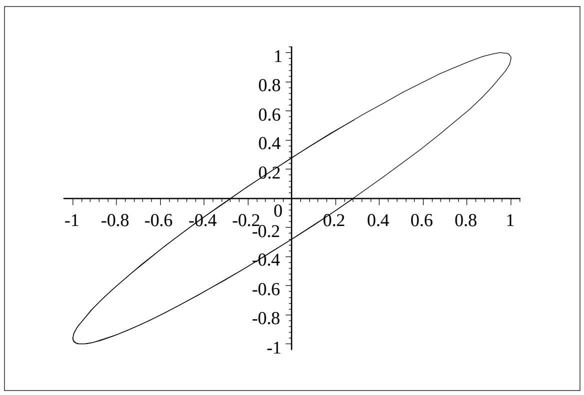
$$a_+ = \sqrt{\frac{32}{2}} = 4, \quad a_- = \sqrt{\frac{s_0-s_3}{2}} = 3$$

$$\delta_c = \delta_- - \delta_+ = \sin^{-1}\left(\frac{24}{2(4\times 3)}\right) = \frac{1}{2}\pi \text{ rad}$$

To plot the two cases $\mathrm{Re}E_x \equiv X = \cos x$, $\mathrm{Re}E_y \equiv Y = rcox(x-\delta_l)$, where $r = a_2/a_1$ and $x = \omega t$.
Case a) $\cos x$, $\sqrt{2}\cos(x+\frac{\pi}{4})$

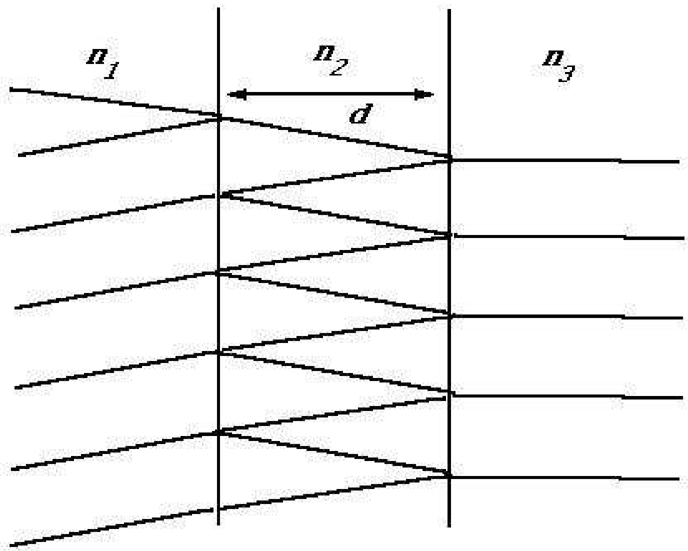


Case b) $\cos x,\ \cos(x - 0.28379)$



7.2

a) The figure describes the muliple internal reflections which interfere to give the overall reflection and refraction:



For the *ij* interface I shall use the notation

$$r_{ij} = \frac{E_0'}{E_0} = \frac{2n_i}{n_i + n_j}$$

$$R_{ij} = \frac{E_0''}{E_0} = \frac{n_i - n_j}{n_i + n_j}$$

Thus from the figure

$$E_0'' = E_0 R_{12} + r_{12} E_0 R_{23} r_{21} e^{i\phi} + r_{12} E_0 R_{23} R_{21} R_{23} r_{21} e^{i2\phi} + \ldots$$

$$E_0'' = E_0 R_{12} + r_{12} E_0 R_{23} r_{21} e^{i\phi} \sum_{n=0}^{\beta} (R_{21} R_{23} e^{i\phi})^n$$

$$E_0'' = E_0 \left(R_{12} + \frac{r_{12} r_{21} R_{23}}{(e^{-i\phi} - R_{21} R_{23})} \right)$$

Similarly

$$E_0' = E_0 r_{12} r_{23} + E_0 r_{12} R_{23} R_{21} r_{23} e^{i\phi} + \ldots$$

$$E_0' = E_0 \frac{r_{12} r_{23}}{1 - R_{21} R_{23} e^{i\phi}}$$

where the phase shift for the internally reflected wave is given by

$$\phi = \frac{2\pi(2d)}{\lambda_2} = \frac{\omega n_2 (2d)}{c}$$

Now for a plane wave

$$S_i = \frac{1}{2v_i} |E_{0i}|^2$$

Thus

$$R = \frac{S''}{S} = \frac{|E_0''|^2}{|E_0|^2}$$

$$T = \frac{v_1}{v_3} \frac{S'}{S} = \frac{n_3}{n_1} \frac{S'}{S}$$

From the above

$$R = \left[R_{12}^2 + \frac{2r_{12} r_{21} R_{23} R_{12} (\cos\phi - R_{21} R_{23}) + (R_{12} r_{21} R_{23})^2}{\left(1 + (R_{21} R_{23})^2 - 2R_{21} R_{23} \cos\phi\right)} \right]$$

$$T = \frac{n_3}{n_1} \frac{(r_{12} r_{23})^2}{\left(1 + (R_{21} R_{23})^2 - 2R_{21} R_{23} \cos\phi\right)}$$

Since these two equations are simple functions of ϕ, which is linearly proportional to the frequency, they are simple functions of frequency which you should plot.

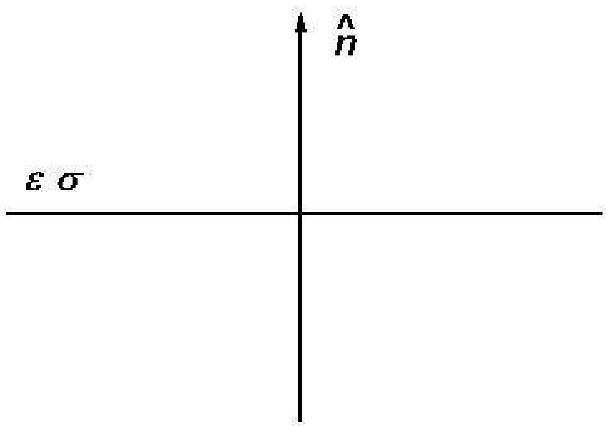
b) Since in part a) we used the convention that the incident wave is from the left, I will rephrase this question so that n_1 is are, n_2 is the coating, and n_3 is glass. In this case, we will have $n_1 < n_2 < n_3$, and $R_{21} R_{23} < 0$. Thus for T to be a maximum, from the above equation $\cos\phi = -1$, or $\phi = \pi$.

$$\phi = \frac{2\pi(2d)}{\lambda_2} = \pi \rightarrow d = \frac{\lambda_2}{4}$$

where λ_2 is the wavelength in the medium $= \frac{\lambda_1}{n_2}$

7.4

We have a nonpermeable conducting material, so $\mu = \mu_0$, and we have $J = \sigma E$, where σ is the conductivity. The following figure describes the system:



The two boundary conditions that we must satisfy for plane waves are

$$E_0 + E_0'' - E_0' = 0$$

$$k(E_0 - E_0'') - k'E_0' = 0$$

Or

$$\frac{E_0''}{E_0} = \frac{k - k'}{k + k'}$$

We must take into account the fact that $\vec{J} = \sigma\vec{E}$. Adding in this term in Maxwell's equations for a plane wave, we get

$$k = \frac{\omega}{c}$$

$$k'^2 = \varepsilon\mu\omega^2\left(1 + i\frac{\sigma}{\omega\varepsilon}\right)$$

Thus we can write

$$k' = \sqrt{\varepsilon\mu}\,\omega(\alpha + i\beta)$$

with

$$\alpha = \left(\frac{\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1}{2}\right)^{1/2}$$

$$\beta = \left(\frac{\sqrt{1 + (\frac{\sigma}{\omega\varepsilon})^2} - 1}{2} \right)^{1/2}$$

Thus

$$\frac{E_0''}{E_0} = \frac{1 - \sqrt{\varepsilon\mu_0}\, c\alpha - i\sqrt{\varepsilon\mu_0}\, c\beta}{1 + \sqrt{\varepsilon\mu_0}\, c\alpha + i\sqrt{\varepsilon\mu_0}\, c\beta}$$

1) For a very poor conductor σ is very small, so keeping only first order in σ

$$\alpha = \left(\frac{\sqrt{1 + (\frac{\sigma}{\omega\varepsilon})^2} + 1}{2} \right)^{1/2} \approx 1$$

$$\beta = \left(\frac{\sqrt{1 + (\frac{\sigma}{\omega\varepsilon})^2} - 1}{2} \right)^{1/2} \approx \frac{\sigma}{2\omega\varepsilon}$$

2) For the case of a very good conductor, $\frac{\sigma}{\omega\varepsilon} >> 1$, so

$$\alpha \approx \sqrt{\frac{\sigma}{2\omega\varepsilon}} = \sqrt{\frac{\frac{2}{\mu_0\omega\delta^2}}{2\omega\varepsilon}} = \frac{1}{\omega\delta\sqrt{\mu_0\varepsilon}}$$

$$\beta \approx \sqrt{\frac{\sigma}{2\omega\varepsilon}} = \frac{1}{\omega\delta\sqrt{\mu_0\varepsilon}}$$

where I have used (5.165) to relate the conductivity to the skin depth.

$$\sigma = \frac{2}{\mu_0\omega\delta^2}$$

$$\frac{E_0''}{E_0} = \frac{1 - \frac{c}{\omega\delta} - i\frac{c}{\omega\delta}}{1 + \frac{c}{\omega\delta} + i\frac{c}{\omega\delta}} = \frac{\delta - \frac{c}{\omega} - i\frac{c}{\omega}}{\delta + \frac{c}{\omega} + i\frac{c}{\omega}} \approx -1 + \frac{\omega}{c}\frac{2}{1+i}\delta = -1 + \frac{\omega}{c}(1-i)\delta$$

$$R = \left| \frac{E_0''}{E_0} \right|^2 = (-1 + \delta\omega/c)^2 + \left(\frac{\omega\delta}{c} \right)^2 \approx 1 - 2\delta\omega/c$$

Problem 7.12

(a) The Fourier transforms for charge density $\rho(\vec{r},t)$, current density $\vec{J}(\vec{r},t)$ and the electric field $\vec{E}(\vec{r},t)$ are

$$\rho(\vec{r},\omega)=\frac{1}{\sqrt{2\pi}}\int \rho(\vec{r},t)e^{i\omega t}dt, \qquad \rho(\vec{r},t)=\frac{1}{\sqrt{2\pi}}\int \rho(\vec{r},\omega)e^{-i\omega t}d\omega$$

$$\vec{J}(\vec{r},\omega)=\frac{1}{\sqrt{2\pi}}\int \vec{J}(\vec{r},t)e^{i\omega t}dt, \qquad \vec{J}(\vec{r},t)=\frac{1}{\sqrt{2\pi}}\int \vec{J}(\vec{r},\omega)e^{-i\omega t}d\omega$$

$$\vec{E}(\vec{r},\omega)=\frac{1}{\sqrt{2\pi}}\int \vec{E}(\vec{r},t)e^{i\omega t}dt, \qquad \vec{E}(\vec{r},t)=\frac{1}{\sqrt{2\pi}}\int \vec{E}(\vec{r},\omega)e^{-i\omega t}d\omega$$

Taking divergence of the Ohm's law:

$$\nabla\cdot\vec{J}(\vec{r},\omega)=\sigma(\omega)\nabla\cdot\vec{E}(\vec{r},\omega),$$

applying the Fourier transformed Gauss's law and the continuity equation:

$$\nabla\cdot\vec{E}(\vec{r},\omega)=\frac{\rho(\vec{r},\omega)}{\epsilon_0}, \qquad \nabla\cdot\vec{J}(\vec{r},\omega)=i\omega\rho(\vec{r},\omega)$$

we have

$$\frac{\sigma(\omega)}{\epsilon_0}\rho(\vec{r},\omega)-i\omega\rho(\vec{r},\omega)=0 \quad i.e. \quad (\sigma(\omega)-i\omega\epsilon_0)\rho(\vec{r},\omega)=0$$

(b) With

$$\sigma(\omega)=\frac{\sigma_0}{1-i\omega\tau}=\frac{\epsilon_0\omega_p^2\tau}{1-i\omega\tau}$$

we have

$$\left\{\frac{\epsilon_0\omega_p^2\tau}{1-i\omega\tau}-i\omega\epsilon_0\right\}\rho(\vec{r},\omega)=0$$

To have non-vanishing charge density, we must have:

$$\frac{\epsilon_0\omega_p^2\tau}{1-i\omega\tau}-i\omega\epsilon_0=0 \quad\Rightarrow\quad \omega_\pm=\frac{-i\pm\sqrt{4\omega_p^2\tau^2-1}}{2\tau}$$

In the approximation $\omega_p\tau\gg 1$:

$$\omega_\pm=\frac{-i}{2\tau}\pm\omega_p$$

Therefore,

$$\rho(\vec{r},\omega)=\rho_+(\vec{r})\delta(\omega-\omega_+)+\rho_-(\vec{r})\delta(\omega-\omega_-)$$

where ρ_+ and ρ_- are functions determined by initial conditions. The time-dependence of the charge density

$$\rho(\vec{r},t)=\frac{1}{\sqrt{2\pi}}\int\rho(\vec{r},\omega)e^{-i\omega t}d\omega=\frac{1}{\sqrt{2\pi}}e^{-t/(2\tau)}\left\{\rho_+(\vec{r})e^{-i\omega_p t}+\rho_-(\vec{r})e^{i\omega_p t}\right\}$$

Therefore, any initial charge distribution will oscillate with the plasma frequency ω_p and decay in amplitude with a decay constant 2τ.

Prob. 7.13

(a) The index of refraction of the ionosphere is

$$n = \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} = \frac{1}{\omega}\sqrt{\omega^2 - \omega_p^2}$$

The ratios between the amplitudes of reflected and incident wave are given by Eqs. (7.39) and (7.41) for the two polarizations. Note the Eqs. (7.41) and (7.39) have different sign conventions for E_0''.

$$\frac{E_0''}{E_0} = \frac{\cos\theta - n\sin\theta'}{\cos\theta + n\sin\theta'} \quad \text{for } \vec{E} \perp \text{ plane of incidence}$$

$$\frac{E_0''}{E_0} = \frac{n\cos\theta - \sin\theta'}{n\cos\theta + \sin\theta'} \quad \text{for } \vec{E} \,||\, \text{plane of incidence}$$

In both cases, the amplitude of the ratio is unity when $\sin\theta'$ is imaginary. This corresponds cases that the incidence angle (θ) is greater than the critical angle θ_c:

$$\theta_c = \sin^{-1} n = \sin^{-1}\left\{\frac{\sqrt{\omega^2 - \omega_p^2}}{\omega}\right\}$$

Therefore, the reflection is partial if $\theta < \theta_c$ and is total if $\theta > \theta_c$ for $\omega > \omega_p$.

(b) For simplicity, treat the ionosphere and the earth as flat surfaces and assume that the amateur can only receive distant stations when the wave is totally reflected. In this case,

$$\sin\theta_c = \frac{d}{\sqrt{4h^2 + d^2}} \quad \Rightarrow \quad \frac{\sqrt{\omega^2 - \omega_p^2}}{\omega} = \frac{d}{\sqrt{4h^2 + d^2}} \quad \Rightarrow \quad \omega_p = 2\pi\frac{c}{\lambda}\sqrt{\frac{4h^2}{d^2 + 4h^2}}$$

where $h = 300$ km is the effective height of the F layer, $d = 1000$ km is the distance between the station and the receiver and $\lambda = 21$ m is the wavelength. Plugging in the numbers, we get the plasma frequency

$$\omega_p = 2\pi\frac{3\times 10^8}{21}\sqrt{\frac{4\times(300)^2}{(1000)^2 + 4\times(300)^2}} = 4.6\times 10^7 \text{ Hz}$$

which corresponds to an electron density

$$n = \frac{m\epsilon_0\omega_p^2}{e^2} = 6.6\times 10^{11}/m^3$$

Note the day-night difference is due to the sunlight.

Prob. 7.28

Since the wave has a finite extent in x and y dimensions, the wave is not a plane wave. Assuming the wave is dominated by the transverse polarization, but have a small longitudinal part, the wave can be written as

$$\vec{E}(x,y,z,t) = \{E_0(x,y)(\vec{e}_1 \pm i\vec{e}_2) + F(x,y)\vec{e}_3\}\, e^{i(kz-\omega t)}$$

here $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ are unit vectors along $x-$, $y-$ and $z-$axes. The wave must satisfy Maxwell's equation

$$\nabla\cdot\vec{E}(x,y,z,t) = 0 = \left\{\frac{\partial E_0(x,y)}{\partial x} \pm i\frac{\partial E_0(x,y)}{\partial y} + F(x,y)ik\right\} e^{i(kz-\omega t)}$$

Therefore,

$$F(x,y) = \frac{i}{k}\left\{\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right\}$$

The electric field is then given by

$$\vec{E}(x,y,z,t) = \left\{E_0(x,y)(\vec{e}_1 \pm i\vec{e}_2) + \frac{i}{k}\{\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\}\vec{e}_3\right\} e^{i(kz-\omega t)}$$

The magnetic field can be derived from the Maxwell's equation:

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} = \nabla \times \left\{E_0(x,y)(\vec{e}_1 \pm i\vec{e}_2) + \frac{i}{k}\{\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\}\vec{e}_3\right\} e^{i(kz-\omega t)}$$

Since the amplitude modulation is slowly varying, $\partial E_0/\partial x$ and $\partial E_0/\partial y$ are generally small. Neglecting terms of $\partial^2 E_0/\partial x^2$ and $\partial^2 E_0/\partial y^2$, we have

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times \left\{E_0(x,y)(\vec{e}_1 \pm i\vec{e}_2)e^{i(kz-\omega t)}\right\}$$

Therefore,

$$\vec{B} = -\frac{i}{\omega}\left\{-\frac{\partial E_2}{\partial z}\vec{e}_1 + \frac{\partial E_1}{\partial z}\vec{e}_2 + (\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y})\vec{e}_3\right\}$$

$$= -\frac{i}{\omega}\left\{\mp iE_0(x,y)(ik)\vec{e}_1 + E_0(x,y)(ik)\vec{e}_2 + (\pm i\frac{\partial E_0}{\partial x} - \frac{\partial E_0}{\partial y})\vec{e}_3\right\}$$

$$= -\frac{i}{\omega}(\pm k)\left\{E_0(x,y)(\vec{e}_1 \pm i\vec{e}_2) + i(\frac{\partial E_0}{\partial x} \pm \frac{\partial E_0}{\partial y})\vec{e}_3\right\}$$

$$= \mp i\frac{k}{\omega}\vec{E} = \mp i\sqrt{\mu\epsilon}\vec{E}$$

Chapter 8 Problems

Problem 8.2

(a) In a cylindrical coordinate system with the $z-$axis along the axes of the two circular cylinders, the TEM mode has fields that vary as $e^{i(kz-\omega t)}$, where $k^2 = \omega^2/v^2 = \mu\epsilon\omega^2$. Therefore, the magnetic field has the form

$$\vec{B} = B_\phi \hat{\phi} e^{i(kz-\omega t)}$$

where B_ϕ is determined from Ampere's law:

$$\oint \vec{B} \cdot d\vec{\ell} = \mu I(z,t) = \mu I_0 e^{i(kz-\omega)t} \qquad \Rightarrow \qquad B_\phi = \frac{\mu I_0}{2\pi\rho}$$

Note that

$$H_0 = \frac{B_\phi|_{\rho=a}}{\mu} \qquad \Rightarrow \qquad I_0 = 2\pi a H_0$$

Therefore

$$\vec{B} = \mu H_0 \frac{a}{\rho} e^{i(kz-\omega t)} \hat{\phi} \qquad \text{and} \qquad \vec{H} = \frac{vecB}{\mu} = H_0 \frac{a}{\rho} e^{i(kz-\omega t)} \hat{\phi}$$

The electric field in between the two cylinders can be determined from the magnetic field through the Ampere-Maxwell's equation:

$$\nabla \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t} \qquad \Rightarrow \qquad \vec{E} = -\frac{\vec{k} \times \vec{B}}{\mu\epsilon\omega} = \sqrt{\frac{\mu}{\epsilon}} H_0 \frac{a}{\rho} e^{i(kz-\omega t)} \hat{\rho}$$

Here $\vec{k} = k\hat{z} = \omega\sqrt{\mu\epsilon}\hat{z}$ is the wave vector. The average Poynting vector

$$\langle \vec{S} \rangle = \frac{1}{2} \vec{E} \times \vec{H}^* = \frac{1}{2} \sqrt{\frac{\mu}{\epsilon}} |H_0|^2 \frac{a^2}{\rho^2} \hat{z}$$

The average power flow along the line (neglecting the wires) is

$$P = \int \langle \vec{S} \rangle \cdot d\vec{a} = \frac{1}{2} \sqrt{\frac{\mu}{\epsilon}} |H_0|^2 a^2 \int_a^b \frac{2\pi\rho d\rho}{\rho^2} = \sqrt{\frac{\mu}{\epsilon}} (\pi a^2 |H_0|^2) \ln(\frac{b}{a})$$

(b) The average power loss per unit area on the cylinder surfaces is given by Eq. (8.15):

$$\frac{dP}{da} = -\frac{1}{2\sigma\delta} |\vec{K}_{\text{eff}}|^2 = -\frac{1}{2\sigma\delta} |\vec{n} \times \vec{H}_{||}|^2 = -\frac{1}{2\sigma\delta} |H_\phi|^2 = -\frac{1}{2\sigma\delta} \frac{|B_\phi|^2}{\mu^2} = -\frac{1}{2\sigma\delta} |H_0|^2 \frac{a^2}{\rho^2}$$

The average power loss per unit length along the $z-$direction

$$\frac{dP}{dz} = \frac{dP}{da}|_{\rho=a}(2\pi a) + \frac{dP}{da}|_{\rho=b}(2\pi b) = -\frac{\pi a^2 |H_0|^2}{\sigma\delta} \left\{ \frac{1}{a} + \frac{1}{b} \right\}$$

From (a), one has

$$\pi a^2 |H_0|^2 = \sqrt{\frac{\epsilon}{\mu}} \frac{P}{\ln(b/a)}$$

Plugging into dP/dz:

$$\frac{dP}{dz} = -\frac{1}{\sigma\delta} \sqrt{\frac{\epsilon}{\mu}} \frac{\frac{1}{a} + \frac{1}{b}}{\ln(\frac{b}{a})} P = -2\gamma P$$

where

$$\gamma = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\frac{\frac{1}{a}+\frac{1}{b}}{\ln(\frac{b}{a})}$$

Integrating the above equation:

$$P(z) = P_0 e^{-2\gamma z}$$

(c) The characteristic impedance Z_0 is the ratio between the voltage and the current.

$$V = \int_a^b \vec{E}\cdot d\vec{\ell} = \sqrt{\frac{\mu}{\epsilon}}H_0 a\int_a^b \frac{d\rho}{\rho}e^{i(kz-\omega t)} = \sqrt{\frac{\mu}{\epsilon}}aH_0\ln(\frac{b}{a})e^{i(kz-\omega t)}$$

The current is given in (a) to be $I = 2\pi a H_0 e^{i(kz-\omega t)}$. The impedance is therefore

$$Z_0 = \frac{V}{I} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ln(\frac{b}{a})$$

(d) Series resistance per unit length can be calculated from the average power loss per unit length:

$$-\frac{1}{2}|I|^2 R = \frac{dP}{dz} \quad\Rightarrow\quad R = -\frac{2}{|I|^2}\frac{dP}{dz} = \frac{1}{2\pi\sigma\delta}\left\{\frac{1}{a}+\frac{1}{b}\right\}$$

The inductance per unit length can be calculated from the energy per unit length in the magnetic field:

$$\frac{1}{4}L|I|^2 = \int\frac{1}{4}\vec{B}\cdot\vec{H}^* da = \frac{1}{4}\left\{\int_0^a + \int_a^b + \int_b^\infty\right\}\vec{B}\cdot\vec{H}^*(2\pi\rho)d\rho$$

Note that inside the conductors,

$$\vec{H}(\xi,t) = \vec{H}_{||}e^{-(1-i)\xi/\delta}e^{-i\omega t} = H_\phi e^{-(1-i)\xi/\delta}e^{-i\omega t}\hat{\phi}$$

where ξ is the distance into the conductor and $\vec{H}_{||}$ is the tangential component of the field at the surface. Assuming $\delta \ll a$,

$$\int_0^a \vec{B}\cdot\vec{H}^* da = 2\pi\mu_c\int_0^a \mu_c|H_\phi|^2_{\rho=a}e^{-2\xi/\delta}\rho d\rho = 2\pi|H_0|^2\int_0^a e^{-2(a-\rho)/\delta}\rho d\rho \approx \pi\mu_c a\delta|H_0|^2$$

$$\int_a^b \vec{B}\cdot\vec{H}^* da = 2\pi\mu\int_a^b |H_\phi|^2\rho d\rho = 2\pi\mu a^2|H_0|^2\ln(\frac{b}{a})$$

$$\int_b^\infty \vec{B}\cdot\vec{H}^* da = 2\pi\int_b^\infty \mu_c|H_\phi|^2_{\rho=b}e^{-2\xi/\delta}\rho d\rho \approx \pi\mu_c\delta\frac{a^2}{b}|H_0|^2$$

The inductance per unit length

$$L = \frac{1}{|I|^2}\left\{\int_0^a + \int_a^b + \int_b^\infty\right\}\vec{B}\cdot\vec{H}^* da = \frac{1}{4\pi^2 a^2|H_0|^2}\left\{\int_0^a + \int_a^b + \int_b^\infty\right\}\vec{B}\cdot\vec{H}^* da = \frac{\mu}{2\pi}\ln(\frac{b}{a}) + \frac{\mu_c\delta}{4\pi}\left\{\frac{1}{a}+\frac{1}{b}\right\}$$

Problem 8.5

(a) Since the guide is a single conductor, there can be no TEM modes. To determine TM and TE modes, we choose a rectangular coordinate system with its origin at the middle of the side $\sqrt{2}a$ such that the three sides are described by $x = a/2$, $y = -a/2$, and $y = x$. For TM modes, the $\psi = 0$ on the surfaces. The boundary conditions at $x = a/2$ and $y = -a/2$ can be met by choosing $\psi_{mn}(x,y)$ to have the form:

$$\psi_{mn}(x,y) \sim \sin(\frac{m\pi(x-a/2)}{a})\sin(\frac{n\pi(y+a/2)}{a}) \sim \sin(\frac{m\pi(x+a/2)}{a})\sin(\frac{n\pi(y+a/2)}{a})$$

The boundary condition at $\psi(x,y)|_{y=x}=0$ can be met by requiring $\psi(x,y)$ be antisymmetric under the exchange of $x \leftrightarrow y$, i.e.,

$$\psi(x,y) \sim \sin(\frac{m\pi(x+a/2)}{a})\sin(\frac{n\pi(y+a/2)}{a}) - \sin(\frac{n\pi(x+a/2)}{a})\sin(\frac{m\pi(y+a/2)}{a})$$

Thus, the TM waves have the general form

$$\psi(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left\{\sin(\frac{m\pi(x+a/2)}{a})\sin(\frac{n\pi(y+a/2)}{a}) - \sin(\frac{n\pi(x+a/2)}{a})\sin(\frac{m\pi(y+a/2)}{a})\right\}$$

Here $A_{mn}=0$ for $m=n$ (ψ vanishes if $m=n$). The corresponding cutoff frequencies are given by Eq. (8.44):

$$\omega_{mn} = \frac{\pi}{\sqrt{\mu\epsilon}}\sqrt{\frac{m^2}{a^2}+\frac{n^2}{a^2}} = \frac{\pi c}{a}\sqrt{m^2+n^2} \quad \text{with} \quad m \neq n \text{ and } m,n>0$$

The dominant mode is $m=1, n=2$ or $m=2, n=1$:

$$\omega_{1,2} = \omega_{2,1} = \frac{\pi c}{a}\sqrt{5}$$

For TE waves, $\partial\psi/\partial n = 0$ on the surfaces. The conditions at $x=a/2$ and $y=-a/2$ are met by choosing $\psi_{mn}(x,y)$ to have the form:

$$\psi_{mn}(x,y) \sim \cos(\frac{m\pi(x+a/2)}{a})\cos(\frac{n\pi(y+a/2)}{a})$$

The boundary condition at $y=x$ is met if $\partial\psi/\partial n$

$$\frac{\partial\psi}{\partial n} = \vec{n}\cdot\nabla_t\psi = -\frac{1}{\sqrt{2}}(\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial y})$$

is antisymmetric under exchange $x \leftrightarrow y$, *i.e.*;

$$-\frac{1}{2}\left\{\frac{\partial\psi(x,y)}{\partial x} - \frac{\partial\psi(x,y)}{\partial y}\right\} = \frac{1}{2}\left\{\frac{\partial\psi(y,x)}{\partial y} - \frac{\partial\psi(y,x)}{\partial x}\right\}$$

Thus $\psi(x,y)$ must be symmetric under $x \leftrightarrow y$. Therefore, the TE waves have the general form

$$\psi(x,y) = \sum_{m,n=0}^{\infty} A_{mn}\left\{\cos(\frac{m\pi(x+a/2)}{a})\cos(\frac{n\pi(y+a/2)}{a}) + \cos(\frac{n\pi(x+a/2)}{a})\cos(\frac{m\pi(y+a/2)}{a})\right\}$$

The corresponding cutoff frequencies are

$$\omega_{mn} = \frac{\pi}{\sqrt{\mu\epsilon}}\sqrt{\frac{m^2}{a^2}+\frac{n^2}{a^2}} = \frac{\pi c}{a}\sqrt{m^2+n^2} \quad \text{with and } m,n \geq 0$$

Though m and n can be equal in this case, however they cannot be both zero. Otherwise, $H_z = \psi$ is a constant, which leads to vanishing $\vec{H}_t$ and $\vec{E}_t$. Consequently there is no wave. The dominant mode is therefore $m=1, n=0$ or $m=0, n=1$:

$$\omega_{1,0} = \omega_{0,1} = \frac{\pi c}{a}$$

Problem 8.8

(a) Assume $h \ll a$, $\delta_i, \delta_e \ll a$, the electric and magnetic fields are approximately

$$E_r = -\frac{ic^2}{\omega_\ell r}\ell(\ell+1)\frac{u_\ell(r)}{r}P_\ell(\cos\theta) \approx -\frac{ic^2}{\omega_\ell}\ell(\ell+1)\frac{u_\ell(a)}{a^2}P_\ell(\cos\theta) = -\frac{ic}{a}\sqrt{\ell(\ell+1)}u_\ell(a)P_\ell(\cos\theta)$$

$$E_\theta = -\frac{ic^2}{\omega r}\frac{du_\ell(r)}{dr}P_\ell^1(\cos\theta) \approx -\frac{ic^2}{\omega a}\frac{du_\ell(r)}{dr}|_{r=a}P_\ell^1(\cos\theta) = 0$$

$$B_\phi = \frac{u_\ell(r)}{r}P_\ell^1(\cos\theta) \approx \frac{u_\ell(a)}{a}P_\ell^1(\cos\theta)$$

where $\omega_\ell = \sqrt{\ell(\ell+1)}c/a$ is the resonance frequency and $u_\ell(r)$ is the solution of the radial equation (8.103). The average energy stored in the fields

$$U = \frac{1}{4}\int(\vec{D}\cdot\vec{E}^* + \vec{B}\cdot\vec{H}^*)d\tau = \frac{1}{4}\int(\epsilon|\vec{E}|^2 + \frac{|\vec{B}|^2}{\mu})d\tau$$

$$= \frac{1}{4}\int_a^{a+h} r^2 dr \int d\Omega \left\{\epsilon\frac{c^2}{a^2}\ell(\ell+1)u_\ell^2(a)[P_\ell(\cos\theta)]^2 + \frac{1}{\mu}\frac{u_\ell^2(a)}{a^2}[P_\ell^1(\cos\theta)]^2\right\}$$

$$= \frac{1}{4}\frac{u_\ell^2(a)}{a^2}\cdot\frac{1}{3}\{(a+h)^3 - a^3\}\int d\Omega\left\{\epsilon c^2\ell(\ell+1)[P_\ell(\cos\theta)]^2 + \frac{1}{\mu}[P_\ell^1(\cos\theta)]^2\right\}$$

$$\approx \frac{1}{4}\frac{u_\ell^2(a)}{a^2}\frac{1}{3}a^3\left\{1 + 3\frac{h}{a} - 1\right\}(2\pi)\int_0^\pi \sin\theta d\theta\left\{\frac{1}{\mu}\ell(\ell+1)[P_\ell(\cos\theta)]^2 + \frac{1}{\mu}[P_\ell^1(\cos\theta)]^2\right\}$$

$$= \frac{\pi h u_\ell^2(a)}{2\mu}\int_{-1}^{+1} dx\left\{\ell(\ell+1)[P_\ell(x)]^2 + [P_\ell^1(x)]^2\right\}$$

Note that

$$\int_{-1}^{+1}[P_\ell(x)]^2 dx = \frac{2}{2\ell+1}, \qquad \int_{-1}^{+1}[P_\ell^m(x)]^2 dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}$$

Plugging into U:

$$U = \frac{2\pi h u_\ell^2(a)}{\mu}\frac{\ell(\ell+1)}{2\ell+1}$$

(Note that the average energies in electric and magnetic fields are equal). The average power loss is given by Eq. (8.15):

$$\frac{dP}{da} = \frac{1}{2\sigma\delta}|\vec{K}_{\text{eff.}}|^2 = \frac{1}{2\sigma\delta}|\vec{n}\times\vec{H}_{||}|^2 = \frac{1}{2\sigma\delta}|\vec{H}_{||}|^2$$

Now note that

$$|\vec{H}_{||}| = \frac{B_\phi}{\mu} = \frac{u_\ell(a)}{\mu a}P_\ell^1(\cos\theta)$$

The average total power loss

$$P_{\text{loss}} = \int_{\text{interior}}\frac{dP}{da}da + \int_{\text{exterior}}\frac{dP}{da}da = \frac{\mu_\ell^2(a)}{2\mu^2}\left\{\frac{1}{\sigma_i\delta_i} + \frac{1}{\sigma_e\delta_e}\right\}\int[P_\ell^1(\cos\theta)]^2 d\Omega$$

$$= \frac{2\pi u_\ell^2(a)}{\mu^2}\left\{\frac{1}{\sigma_i\delta_i} + \frac{1}{\sigma_e\delta_e}\right\}\frac{\ell(\ell+1)}{2\ell+1}$$

Note that

$$\delta^2 = \frac{2}{\mu\omega_\ell\sigma} \qquad \Rightarrow \qquad \frac{1}{\sigma\delta} = \frac{\mu\omega_\ell\delta}{2}$$

Therefore,

$$P_{\text{loss}} = \frac{2\pi u_\ell^2(a)}{\mu^2}\left\{\frac{\mu\omega\delta_i}{2} + \frac{\mu\omega\delta_e}{2}\right\} = \frac{\pi u_\ell^2(a)\omega_\ell}{\mu}\frac{\ell(\ell+1)}{2\ell+1}(\delta_i+\delta_e)$$

The Schumann resonance Q value:

$$Q = \omega_\ell \frac{U}{P_{\text{loss}}} = \omega_\ell \frac{2\pi h u_\ell^2(a)}{\mu}\frac{\ell(\ell+1)}{2\ell+1}\frac{\mu}{\pi u_\ell^2(a)\omega_\ell}\frac{2\ell+1}{\ell(\ell+1)}\frac{1}{\delta_i+\delta_e} = \frac{2h}{\delta_i+\delta_e}$$

independent of ℓ and $N = 2$.

(b) For the lowest Schumann resonance,

$$\ell = 1 \;\Rightarrow\; \omega_1 = \sqrt{2}\frac{c}{a} = \sqrt{2}\frac{3\cdot 10^8}{6.4\cdot 10^6} = 66.3 \text{ Hz}$$

$$\delta_e = \sqrt{\frac{2}{\mu\omega_1\sigma_e}} = \sqrt{\frac{2}{4\pi\cdot 10^{-7}\times 66.3\times 0.1}} = 4.9\cdot 10^2 \text{ m}$$

$$\delta_i = \sqrt{\frac{2}{\mu\omega_1\sigma_i}} = \sqrt{\frac{2}{4\pi\cdot 10^{-7}\times 66.3\times 10^{-5}}} = 4.9\cdot 10^4 \text{ m}$$

$$Q = \frac{2h}{\delta_e+\delta_i} = \frac{2\times 10^5}{4.9\cdot 10^2(1+100)} = 4.0$$

(c) With $\sigma_i \approx 10^{-5}$ $(\Omega\text{m})^{-1}$, $\delta_i \approx 49$ km is not small compared with $h \approx 100$ km. However, the fields vary over distances of order a, at least for $\ell = 1$. Thus, the approximation of Section 8.1 are valid, at least for small ℓ values. When a/ℓ becomes of order of δ_i, these approximations won't be adequate. In this case, it occurs at $\ell \sim 100$.

Problem 8.18

(a) For TM modes, we have

$$(\nabla_t^2 + \gamma_\lambda^2)E_{z\lambda} = 0, \quad \text{and} \quad E_{z\lambda}|_C = 0$$

where the subscript C denotes boundary contour. Applying Green's theorem in two dimension:

$$\int_S (\phi\nabla_t^2\psi - \psi\nabla_t^2\phi)da = -\oint_C (\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n})d\ell$$

where the $-$ on the right side is due to the difference in the normal definition. With $\phi = E_{z\lambda}$ and $\psi = E_{z\mu}$, we gets

$$\int_S (E_{z\lambda}\nabla_t^2 E_{z\mu} - E_{z\mu}\nabla_t^2 E_{z\lambda})da = -\oint_C (E_{z\lambda}\frac{\partial E_{z\mu}}{\partial n} - E_{z\mu}\frac{\partial E_{z\lambda}}{\partial n})d\ell$$

The line integrals on the right-hand side vanish due to the boundary conditions. Therefore,

$$0 = \int_S (E_{z\lambda}\nabla_t^2 E_{z\mu} - E_{z\mu}\nabla_t^2 E_{z\lambda})da = \int_S \left\{E_{z\lambda}(-\gamma_\mu^2 E_{z\mu}) - E_{z\mu}(-\gamma_\lambda^2 E_{z\lambda})\right\} da = (\gamma_\lambda^2 - \gamma_\mu^2)\int_S E_{z\lambda}E_{z\mu}da$$

For the case $\gamma_\lambda \neq \gamma_\mu$, the integral must vanish:

$$\int_S E_{z\lambda}E_{z\mu}da = 0$$

Same argument applies to $H_{z\lambda}$ and $H_{z\mu}$, except in this case, the line integrals vanishes due to boundary conditions

$$\frac{\partial H_z}{\partial n}|_C = 0$$

(b) *Proof for TM modes only*
Applying Green's first identity

$$\int_S (\phi\nabla_t^2\psi + \nabla_t\phi\cdot\nabla_t\psi)da = -\oint_C \phi\frac{\partial\psi}{\partial n}d\ell$$

with $\phi = E_{z\lambda}$ and $\psi = E_{z\mu}$ for the TM modes, we get

$$\int_S (E_{z\lambda}\nabla_t^2 E_{z\mu} + \nabla_t E_{z\lambda}\cdot\nabla_t E_{z\mu})da = -\oint_C E_{z\lambda}\frac{\partial}{\partial n}E_{z\mu}d\ell$$

Again, the line integral on the right vanishes due to the boundary condition. $\nabla_t E_z$ and $\nabla_t^2 E_z$ are given by Eqs. (8.33, 8.34):

$$\nabla_t^2 E_{z\lambda} = -\gamma_\lambda^2 E_{z\lambda}, \qquad \nabla_t^2 E_{z\mu} = -\gamma_\mu^2 E_{z\mu}$$

$$\nabla_t E_{z\lambda} = -i\frac{\gamma_\lambda^2}{k_\lambda}\vec{E}_\lambda, \qquad \nabla_t E_{z\mu} = -i\frac{\gamma_\mu^2}{k_\mu}\vec{E}_\mu$$

where $\vec{E}_\lambda$ and $\vec{E}_\mu$ are transverse electric fields. The Green's first identify becomes

$$\gamma_\mu^2\int_S E_{z\lambda}E_{z\mu}da + \frac{\gamma_\lambda^2\gamma_\mu^2}{k_\lambda k_\mu}\int_S \vec{E}_\lambda\cdot\vec{E}_\mu da = 0 \qquad \Rightarrow \qquad \int_S E_{z\lambda}E_{z\mu}da = -\frac{\gamma_\mu^2}{k_\lambda k_\mu}\int_S \vec{E}_\lambda\cdot\vec{E}_\mu da$$

Assuming non-degeneracy and from (a), we obtain:

$$\text{For } \lambda \neq \mu; \quad \int_S \vec{E}_\lambda\cdot\vec{E}_\mu da = 0$$

By properly normalizing $\vec{E}_\lambda$, we have

$$\int_S \vec{E}_\lambda\cdot\vec{E}_\mu da = \delta_{\lambda\mu} \qquad \text{Eq.(8.131)}$$

$$\int_S E_{z\lambda}E_{z\mu}da = -\frac{\gamma_\mu^2}{k_\lambda k_\mu}\int_S \vec{E}_\lambda\cdot\vec{E}_\mu da = -\frac{\gamma_\lambda^2}{k_\lambda^2}\delta_{\lambda\mu} \qquad \text{Eq.(8.134) for TM waves}$$

Now turn into the relations of magnetic fields. Note that

$$\vec{H}_\lambda = \frac{\epsilon\omega}{k_\lambda}\hat{z}\times\vec{E}_\lambda = \frac{i\epsilon\omega}{\gamma_\lambda^2}\hat{z}\times\nabla_t E_{z\lambda}, \qquad \vec{H}_\mu = \frac{\epsilon\omega}{k_\mu}\hat{z}\times\vec{E}_\mu = \frac{i\epsilon\omega}{\gamma_\mu^2}\hat{z}\times\nabla_t E_{z\mu}$$

$$\vec{H}_\lambda\cdot\vec{H}_\mu = -\frac{(\epsilon\omega)^2}{\gamma_\lambda^2\gamma_\mu^2}(\hat{z}\times\nabla_t E_{z\lambda})\cdot(\hat{z}\times\nabla_t E_{z\mu}) = -\frac{(\epsilon\omega)^2}{\gamma_\lambda^2\gamma_\mu^2}\nabla_t E_{z\mu}\cdot\{(\hat{z}\times\nabla_t E_{z\lambda})\times\hat{z}\} = -\frac{(\epsilon\omega)^2}{\gamma_\lambda^2\gamma_\mu^2}\nabla_t E_{z\mu}\cdot\nabla_t E_{z\lambda}$$

Using Green's first identity with $\phi = E_{z\lambda}$ and $\psi = E_{z\mu}$, we have

$$\int \nabla_t E_{z\mu}\cdot\nabla_t E_{z\lambda}da = -\oint E_{z\mu}\frac{\partial E_{z\lambda}}{\partial n} - \int E_{z\mu}\nabla_t^2 E_{z\lambda}da = \gamma_\lambda^2\int E_{z\mu}E_{z\lambda}da = -\frac{\gamma_\lambda^4}{k_\lambda^2}\delta_{\lambda\mu}$$

Thus,

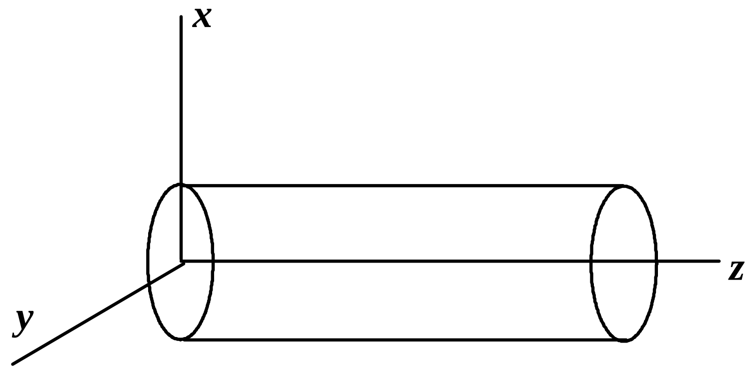
$$\int \vec{H}_\lambda\cdot\vec{H}_\mu da = -\frac{(\epsilon\omega)^2}{\gamma_\lambda^2\gamma_\mu^2}\int \nabla_t E_{z\mu}\cdot\nabla_t E_{z\lambda}da = -\frac{(\epsilon\omega)^2}{\gamma_\lambda^2\gamma_\mu^2}(-\frac{\gamma_\lambda^4}{k_\lambda^2}\delta_{\lambda\mu}) = \frac{(\epsilon\omega)^2}{k_\lambda^2}\delta_{\lambda\mu} = \frac{1}{Z_\lambda^2}\delta_{\lambda\mu} \qquad \text{Eq. (8.132)}$$

where $Z_\lambda = k_\lambda/(\epsilon\omega)$ is the wave impedance.

$$\frac{1}{2}\int(\vec{E}_\lambda\times\vec{H}_\mu)\cdot\hat{z}da = \frac{1}{2}\int\left\{\frac{ik_\lambda}{\gamma_\lambda^2}\nabla_t E_{z\lambda}\times\frac{i\epsilon\omega}{\gamma_\mu^2}(\hat{z}\times\nabla_t E_{z\mu})\right\}\cdot\hat{z}da$$

$$= -\frac{1}{2}\int\frac{k_\lambda\epsilon\omega}{\gamma_\lambda^2\gamma_\mu^2}(\nabla_t E_{z\lambda}\cdot\nabla_t E_{z\mu})da = -\frac{1}{2}\frac{k_\lambda\epsilon\omega}{\gamma_\lambda^2\gamma_\mu^2}(-\frac{\gamma_\lambda^4}{k_\lambda^2}\delta_{\lambda\mu}) = \frac{1}{2Z_\lambda}\delta_{\lambda\mu} \qquad \text{Eq. (8.133)}$$

8.3



a)

$$(\nabla_t^2 + \gamma^2)\psi = 0, \qquad \psi = E_z(TM) \text{ or } \psi = H_z(TE)$$

As in class, we will use cylindrical coordinates, and assume

$$\psi(\rho,\phi) = R(\rho)Q(\phi)$$

We get the two equations

$$\frac{\partial^2}{\partial\phi^2}Q(\phi) = -m^2 Q(\phi) \text{ with solns } Q(\phi) = e^{\pm im\phi},\ m = 0,1,2,\ldots$$

$$\frac{d^2}{dx^2}R(x) + \frac{1}{x}\frac{dR(x)}{dx} + \left(1 - \frac{m^2}{x^2}\right)R(x) \quad \left(\text{Bessel eqn.}\right)$$

with regular solutions $J_m(x)$, and singular solution (which we reject as nonphysical) $N_m(x)$. Here $x = \gamma\rho$.

Solutions:

TM: BC: $J_m(x_{mn}) = 0$, and

$$E_z(\rho,\phi) = E_0 J_m(\gamma_{mn}\rho)e^{\pm im\phi}, \quad m = 0,1,2,\ldots.;\ n = 1,2,3,\ldots;\ \gamma_{mn} = x_{mn}/R$$

Lowest cutoff frequencies:

$$\omega_{mn} = \frac{\gamma_{mn}}{\sqrt{\varepsilon\mu}} = \frac{x_{mn}}{R\sqrt{\varepsilon\mu}}$$

Using the results of Jackson, p. 114,

$$x_{0n} = 2.405, 5.52, 8.654, \ldots$$
$$x_{1n} = 3.832, 7.016, 10.173, \ldots.$$
$$x_{2n} = 5.136, 8.417, 11.620, \ldots.$$

TE: BC: $J'_m(x'_{mn}) = 0,$ and

$$E_z(\rho,\phi) = E_0 J_m(\gamma'_{mn}\rho)e^{\pm im\phi}, \quad m = 0,1,2,\ldots.; \; n = 1,2,3,\ldots; \; \gamma'_{mn} = x'_{mn}/R$$

Lowest cutoff frequencies:

$$\omega_{mn} = \frac{\gamma'_{mn}}{\sqrt{\varepsilon\mu}} = \frac{x'_{mn}}{R\sqrt{\varepsilon\mu}}$$

Using the results of Jackson, p. 370,

$$x'_{0n} = 3.832, 7.016, 10.173, \ldots$$
$$x'_{1n} = 1.841, 5.331, 8.536, \ldots.$$
$$x'_{2n} = 3.054, 6.706, 9.970, \ldots.$$

From the above we see the lowest cutoff frequency is the TE mode

$$\omega'_{11} = 1.841K, \text{ with } K = 1/\left(R\sqrt{\varepsilon\mu}\right)$$

The next four lowest cutoff frequencies are:

$$\omega_{01} = 2.405K = 1.31\omega'_{11}$$
$$\omega'_{21} = 3.054K = 1.66\omega'_{11}$$
$$\omega'_{01} = 3.832K = 2.08\omega'_{11}$$
$$\omega_{11} = 3.832K = 2.08\omega'_{11}$$

b) From Eq. (8.63) in the text

$$\beta_\lambda \propto \left(\frac{\omega}{1-\frac{\omega_\lambda^2}{\omega^2}}\right)^{1/2}\left[\xi_\lambda + \eta_\lambda\left(\frac{\omega_\lambda}{\omega}\right)^2\right]$$

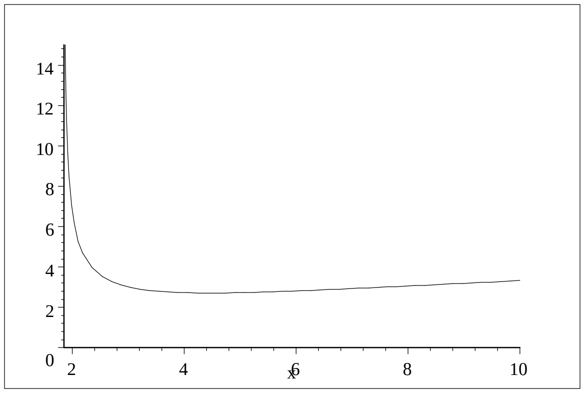
For TM modes, $\eta_\lambda = 0,$ and for TE mode, $\xi_\lambda + \eta_\lambda$ is of order unity. So for comparison purposes, I'll take

$$\beta_{11}(TE) = f_1(x) = \left(\frac{x}{1-\frac{1.841^2}{x^2}}\right)^{1/2}\left(1+\frac{1.841^2}{x^2}\right)$$

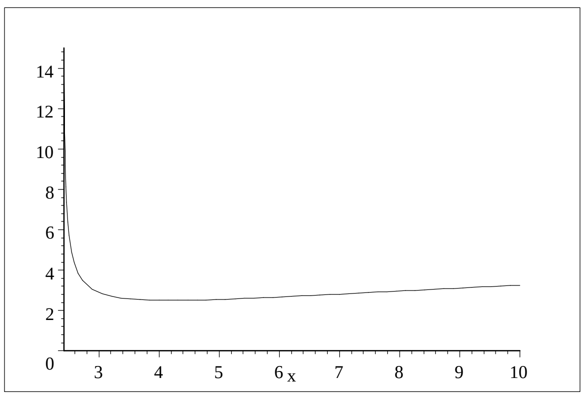
$$\beta_{01}(TM) = f_2(x) = \left(\frac{x}{1-\frac{2.405^2}{x^2}}\right)^{1/2}$$

where I've expressed the functions in terms of $x = \omega/K$.

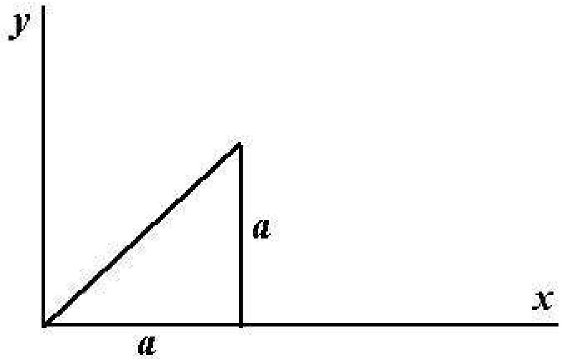
$$\left(1+\frac{1.841^2}{x^2}\right)\left(\frac{x}{1-\frac{1.841^2}{x^2}}\right)^{1/2}$$



$$\left(\frac{x}{1-\frac{2.405^2}{x^2}}\right)^{1/2}$$



8.4



a) TM:

$$(\nabla_t^2 + \gamma^2)\psi = 0; \quad \psi|_B = 0; \;\; E_z = \psi(x,y)e^{\pm ikz - i\omega t}; \quad B_z = 0$$

Since we have a node along $y = x$, then we just take the antisymmetrized version for the square waveguide, developed in class, ie,

$$\psi(x,y) = E_0\Big[\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{a}) - \sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{a})\Big]$$

Again

$$\gamma_{mn}^2 = \frac{c\pi^2}{a^2}(m^2 + n^2), \quad m,n = 1,2,3\ldots, \text{ but } m \neq n$$

TE:

$$(\nabla_t^2 + \gamma^2)\psi = 0; \quad \frac{\partial\psi}{\partial n}|_B = 0; \;\; H_z = \psi(x,y)e^{\pm ikz - i\omega t}; \quad E_z = 0$$

Now the BC require $\frac{\partial\psi}{\partial n}|_B = 0$, but using a 45^0 rotation of coordinates, we see

$$\frac{\partial}{\partial n} = \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$$

Thus the combination

$$\psi(x,y) = H_0\Big[\cos(\frac{m\pi x}{a})\cos(\frac{n\pi y}{a}) + \cos(\frac{n\pi x}{a})\cos(\frac{m\pi y}{a})\Big]$$

satisfies the above BC on the diagonal, as you can see by direct substitution.

$$\gamma_{mn}^2 = \frac{c\pi^2}{a^2}(m^2 + n^2), \quad m,n = 0,1,2,3\ldots, \text{ but } m \neq n = 0$$

b) The lowest cutoff freq. are: TM: ω_{12} or ω_{21}. TE: ω_{01} or ω_{10}. From Eq. (8.63) in the text

$$\beta_{12}(TM) \propto \left(\frac{\omega}{1-\omega_{12}^2/\omega^2} \right)^{1/2}$$

$$\beta_{01}(TE) \propto \left(\frac{\omega}{1-\omega_{12}^2/\omega^2} \right)^{1/2} \left(1 + \frac{\omega_{01}^2}{\omega^2} \right)$$

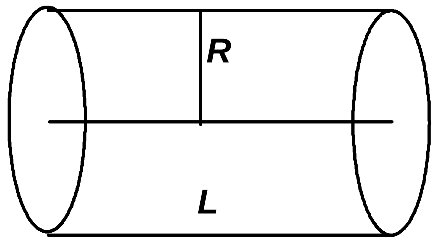
For the square wave guide, we don't have the antisymmetrization, but the formulas for the cutoff frequencies are the same without the present restrictions on *m* and *n*. So for the square guide, the cut off frequencies are

TM: ω_{11}

TE: ω_{01} (as before)

8.5

a)



For the TM modes, we saw in class the resonance frequencies are
TM:

$$\omega_{mnp} = \frac{1}{\sqrt{\varepsilon\mu}}\sqrt{\frac{x_{mn}^2}{R^2} + \frac{p^2\pi^2}{L^2}} \qquad \begin{aligned} p &= 0,1,2,\ldots. \\ m &= 0,1,2,\ldots. \\ n &= 1,2,3,\ldots. \end{aligned}$$

TE:

$$\omega'_{mnp} = \frac{1}{\sqrt{\varepsilon\mu}}\sqrt{\frac{x_{mn}'^2}{R^2} + \frac{p^2\pi^2}{L^2}} \qquad \begin{aligned} p &= 1,2,3,\ldots. \\ m &= 0,1,2,\ldots. \\ n &= 1,2,3,\ldots. \end{aligned}$$

Thus—

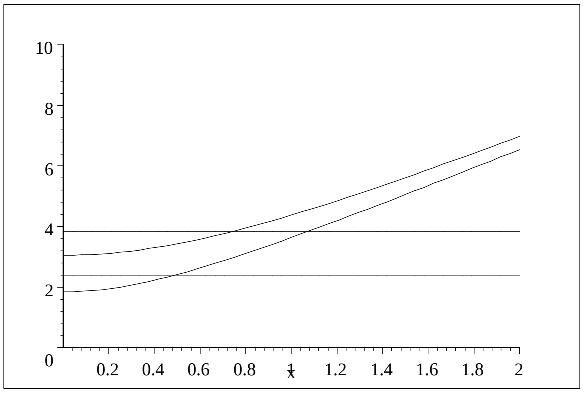
$$\frac{\omega_{mnp}}{\frac{1}{\sqrt{\varepsilon\mu}}} = \sqrt{\frac{x_{mn}^2}{R^2} + \frac{p^2\pi^2}{L^2}}$$

$$\frac{\omega'_{mnp}}{\frac{1}{\sqrt{\varepsilon\mu}}} = \sqrt{\frac{x_{mn}'^2}{R^2} + \frac{p^2\pi^2}{L^2}}$$

The lowest four frequencies are (in these units)

$$\begin{aligned} \omega_{010} &= 2.405 \\ \omega_{110} &= 3.832 \\ \omega'_{111} &= \sqrt{1.841^2 + \pi^2\left(\tfrac{R}{L}\right)^2} \\ \omega'_{211} &= \sqrt{3.054^2 + \pi^2\left(\tfrac{R}{L}\right)^2} \end{aligned}$$

$2.405, 3.832, \sqrt{1.841^2 + \pi^2 x^2}, \sqrt{3.054^2 + \pi^2 x^2}$



where $x = R/L$.

The answer is "No." ω'_{111} and ω_{010} cross when

$$\sqrt{1.841^2 + \pi^2 x^2} = 2.405$$

or $x = 0.49258$. For frequencies smaller than this cross over frequency, ω'_{111} is lowest, whereas for larger frequencies, ω_{010} is lowest.

More Problems for Chapter 8

Problem 8.2

(a) In a cylindrical coordinate system with the $z-$axis along the axes of the two circular cylinders, the TEM mode has fields that vary as $e^{i(kz-\omega t)}$, where $k^2 = \omega^2/v^2 = \mu\epsilon\omega^2$. Therefore, the magnetic field has the form

$$\vec{B} = B_\phi \hat{\phi} e^{i(kz-\omega t)}$$

where B_ϕ is determined from Ampere's law:

$$\oint \vec{B}\cdot d\vec{\ell} = \mu I(z,t) = \mu I_0 e^{i(kz-\omega)t} \qquad \Rightarrow \qquad B_\phi = \frac{\mu I_0}{2\pi\rho}$$

Note that

$$H_0 = \frac{B_\phi|_{\rho=a}}{\mu} \qquad \Rightarrow \qquad I_0 = 2\pi a H_0$$

Therefore

$$\vec{B} = \mu H_0 \frac{a}{\rho} e^{i(kz-\omega t)}\hat{\phi} \qquad \text{and} \qquad \vec{H} = \frac{vecB}{\mu} = H_0 \frac{a}{\rho} e^{i(kz-\omega t)}\hat{\phi}$$

The electric field in between the two cylinders can be determined from the magnetic field through the Ampere-Maxwell's equation:

$$\nabla \times \vec{B} = \mu\epsilon\frac{\partial \vec{E}}{\partial t} \qquad \Rightarrow \qquad \vec{E} = -\frac{\vec{k}\times\vec{B}}{\mu\epsilon\omega} = \sqrt{\frac{\mu}{\epsilon}} H_0 \frac{a}{\rho} e^{i(kz-\omega t)}\hat{\rho}$$

Here $\vec{k} = k\hat{z} = \omega\sqrt{\mu\epsilon}\hat{z}$ is the wave vector. The average Poynting vector

$$\langle\vec{S}\rangle = \frac{1}{2}\vec{E}\times\vec{H}^* = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}|H_0|^2\frac{a^2}{\rho^2}\hat{z}$$

The average power flow along the line (neglecting the wires) is

$$P = \int \langle\vec{S}\rangle\cdot d\vec{a} = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}|H_0|^2 a^2 \int_a^b \frac{2\pi\rho d\rho}{\rho^2} = \sqrt{\frac{\mu}{\epsilon}}(\pi a^2|H_0|^2)\ln(\frac{b}{a})$$

(b) The average power loss per unit area on the cylinder surfaces is given by Eq. (8.15):

$$\frac{dP}{da} = -\frac{1}{2\sigma\delta}|\vec{K}_{\text{eff}}|^2 = -\frac{1}{2\sigma\delta}|\vec{n}\times\vec{H}_{||}|^2 = -\frac{1}{2\sigma\delta}|H_\phi|^2 = -\frac{1}{2\sigma\delta}\frac{|B_\phi|^2}{\mu^2} = -\frac{1}{2\sigma\delta}|H_0|^2\frac{a^2}{\rho^2}$$

The average power loss per unit length along the $z-$direction

$$\frac{dP}{dz} = \frac{dP}{da}|_{\rho=a}(2\pi a) + \frac{dP}{da}|_{\rho=b}(2\pi b) = -\frac{\pi a^2|H_0|^2}{\sigma\delta}\left\{\frac{1}{a}+\frac{1}{b}\right\}$$

From (a), one has

$$\pi a^2|H_0|^2 = \sqrt{\frac{\epsilon}{\mu}}\frac{P}{\ln(b/a)}$$

Plugging into dP/dz:

$$\frac{dP}{dz} = -\frac{1}{\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\frac{\frac{1}{a}+\frac{1}{b}}{\ln(\frac{b}{a})}P = -2\gamma P$$

where

$$\gamma = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\frac{\frac{1}{a}+\frac{1}{b}}{\ln(\frac{b}{a})}$$

Integrating the above equation:

$$P(z) = P_0 e^{-2\gamma z}$$

(c) The characteristic impedance Z_0 is the ratio between the voltage and the current.

$$V = \int_a^b \vec{E}\cdot d\vec{\ell} = \sqrt{\frac{\mu}{\epsilon}}H_0 a\int_a^b \frac{d\rho}{\rho}e^{i(kz-\omega t)} = \sqrt{\frac{\mu}{\epsilon}}aH_0\ln(\frac{b}{a})e^{i(kz-\omega t)}$$

The current is given in (a) to be $I = 2\pi a H_0 e^{i(kz-\omega t)}$. The impedance is therefore

$$Z_0 = \frac{V}{I} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ln(\frac{b}{a})$$

(d) Series resistance per unit length can be calculated from the average power loss per unit length:

$$-\frac{1}{2}|I|^2 R = \frac{dP}{dz} \quad\Rightarrow\quad R = -\frac{2}{|I|^2}\frac{dP}{dz} = \frac{1}{2\pi\sigma\delta}\left\{\frac{1}{a}+\frac{1}{b}\right\}$$

The inductance per unit length can be calculated from the energy per unit length in the magnetic field:

$$\frac{1}{4}L|I|^2 = \int\frac{1}{4}\vec{B}\cdot\vec{H}^* da = \frac{1}{4}\left\{\int_0^a + \int_a^b + \int_b^\infty\right\}\vec{B}\cdot\vec{H}^*(2\pi\rho)d\rho$$

Note that inside the conductors,

$$\vec{H}(\xi,t) = \vec{H}_{||}e^{-(1-i)\xi/\delta}e^{-i\omega t} = H_\phi e^{-(1-i)\xi/\delta}e^{-i\omega t}\hat{\phi}$$

where ξ is the distance into the conductor and $\vec{H}_{||}$ is the tangential component of the field at the surface. Assuming $\delta \ll a$,

$$\int_0^a \vec{B}\cdot\vec{H}^* da = 2\pi\mu_c\int_0^a \mu_c|H_\phi|^2_{\rho=a}e^{-2\xi/\delta}\rho d\rho = 2\pi|H_0|^2\int_0^a e^{-2(a-\rho)/\delta}\rho d\rho \approx \pi\mu_c a\delta|H_0|^2$$

$$\int_a^b \vec{B}\cdot\vec{H}^* da = 2\pi\mu\int_a^b |H_\phi|^2\rho d\rho = 2\pi\mu a^2|H_0|^2\ln(\frac{b}{a})$$

$$\int_b^\infty \vec{B}\cdot\vec{H}^* da = 2\pi\int_b^\infty \mu_c|H_\phi|^2_{\rho=b}e^{-2\xi/\delta}\rho d\rho \approx \pi\mu_c\delta\frac{a^2}{b}|H_0|^2$$

The inductance per unit length

$$L = \frac{1}{|I|^2}\left\{\int_0^a + \int_a^b + \int_b^\infty\right\}\vec{B}\cdot\vec{H}^* da = \frac{1}{4\pi^2 a^2|H_0|^2}\left\{\int_0^a + \int_a^b + \int_b^\infty\right\}\vec{B}\cdot\vec{H}^* da = \frac{\mu}{2\pi}\ln(\frac{b}{a}) + \frac{\mu_c\delta}{4\pi}\left\{\frac{1}{a}+\frac{1}{b}\right\}$$

Problem 8.5

(a) Since the guide is a single conductor, there can be no TEM modes. To determine TM and TE modes, we choose a rectangular coordinate system with its origin at the middle of the side $\sqrt{2}a$ such that the three sides are described by $x = a/2$, $y = -a/2$, and $y = x$. For TM modes, the $\psi = 0$ on the surfaces. The boundary conditions at $x = a/2$ and $y = -a/2$ can be met by choosing $\psi_{mn}(x,y)$ to have the form:

$$\psi_{mn}(x,y) \sim \sin(\frac{m\pi(x-a/2)}{a})\sin(\frac{n\pi(y+a/2)}{a}) \sim \sin(\frac{m\pi(x+a/2)}{a})\sin(\frac{n\pi(y+a/2)}{a})$$

The boundary condition at $\psi(x,y)|_{y=x}=0$ can be met by requiring $\psi(x,y)$ be antisymmetric under the exchange of $x \leftrightarrow y$, i.e.,

$$\psi(x,y) \sim \sin(\frac{m\pi(x+a/2)}{a})\sin(\frac{n\pi(y+a/2)}{a}) - \sin(\frac{n\pi(x+a/2)}{a})\sin(\frac{m\pi(y+a/2)}{a})$$

Thus, the TM waves have the general form

$$\psi(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left\{\sin(\frac{m\pi(x+a/2)}{a})\sin(\frac{n\pi(y+a/2)}{a}) - \sin(\frac{n\pi(x+a/2)}{a})\sin(\frac{m\pi(y+a/2)}{a})\right\}$$

Here $A_{mn}=0$ for $m=n$ (ψ vanishes if $m=n$). The corresponding cutoff frequencies are given by Eq. (8.44):

$$\omega_{mn} = \frac{\pi}{\sqrt{\mu\epsilon}}\sqrt{\frac{m^2}{a^2}+\frac{n^2}{a^2}} = \frac{\pi c}{a}\sqrt{m^2+n^2} \quad \text{with} \quad m \neq n \text{ and } m,n>0$$

The dominant mode is $m=1, n=2$ or $m=2, n=1$:

$$\omega_{1,2} = \omega_{2,1} = \frac{\pi c}{a}\sqrt{5}$$

For TE waves, $\partial\psi/\partial n = 0$ on the surfaces. The conditions at $x=a/2$ and $y=-a/2$ are met by choosing $\psi_{mn}(x,y)$ to have the form:

$$\psi_{mn}(x,y) \sim \cos(\frac{m\pi(x+a/2)}{a})\cos(\frac{n\pi(y+a/2)}{a})$$

The boundary condition at $y=x$ is met if $\partial\psi/\partial n$

$$\frac{\partial\psi}{\partial n} = \vec{n}\cdot\nabla_t\psi = -\frac{1}{\sqrt{2}}(\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial y})$$

is antisymmetric under exchange $x \leftrightarrow y$, *i.e.*;

$$-\frac{1}{2}\left\{\frac{\partial\psi(x,y)}{\partial x} - \frac{\partial\psi(x,y)}{\partial y}\right\} = \frac{1}{2}\left\{\frac{\partial\psi(y,x)}{\partial y} - \frac{\partial\psi(y,x)}{\partial x}\right\}$$

Thus $\psi(x,y)$ must be symmetric under $x \leftrightarrow y$. Therefore, the TE waves have the general form

$$\psi(x,y) = \sum_{m,n=0}^{\infty} A_{mn}\left\{\cos(\frac{m\pi(x+a/2)}{a})\cos(\frac{n\pi(y+a/2)}{a}) + \cos(\frac{n\pi(x+a/2)}{a})\cos(\frac{m\pi(y+a/2)}{a})\right\}$$

The corresponding cutoff frequencies are

$$\omega_{mn} = \frac{\pi}{\sqrt{\mu\epsilon}}\sqrt{\frac{m^2}{a^2}+\frac{n^2}{a^2}} = \frac{\pi c}{a}\sqrt{m^2+n^2} \quad \text{with and } m,n \geq 0$$

Though m and n can be equal in this case, however they cannot be both zero. Otherwise, $H_z = \psi$ is a constant, which leads to vanishing $\vec{H}_t$ and $\vec{E}_t$. Consequently there is no wave. The dominant mode is therefore $m=1, n=0$ or $m=0, n=1$:

$$\omega_{1,0} = \omega_{0,1} = \frac{\pi c}{a}$$

Problem 8.8

(a) Assume $h \ll a$, $\delta_i, \delta_e \ll a$, the electric and magnetic fields are approximately

$$E_r = -\frac{ic^2}{\omega_\ell r}\ell(\ell+1)\frac{u_\ell(r)}{r}P_\ell(\cos\theta) \approx -\frac{ic^2}{\omega_\ell}\ell(\ell+1)\frac{u_\ell(a)}{a^2}P_\ell(\cos\theta) = -\frac{ic}{a}\sqrt{\ell(\ell+1)}u_\ell(a)P_\ell(\cos\theta)$$

$$E_\theta = -\frac{ic^2}{\omega r}\frac{du_\ell(r)}{dr}P_\ell^1(\cos\theta) \approx -\frac{ic^2}{\omega a}\frac{du_\ell(r)}{dr}|_{r=a}P_\ell^1(\cos\theta) = 0$$

$$B_\phi = \frac{u_\ell(r)}{r}P_\ell^1(\cos\theta) \approx \frac{u_\ell(a)}{a}P_\ell^1(\cos\theta)$$

where $\omega_\ell = \sqrt{\ell(\ell+1)}c/a$ is the resonance frequency and $u_\ell(r)$ is the solution of the radial equation (8.103). The average energy stored in the fields

$$U = \frac{1}{4}\int(\vec{D}\cdot\vec{E}^* + \vec{B}\cdot\vec{H}^*)d\tau = \frac{1}{4}\int(\epsilon|\vec{E}|^2 + \frac{|\vec{B}|^2}{\mu})d\tau$$

$$= \frac{1}{4}\int_a^{a+h} r^2dr\int d\Omega\left\{\epsilon\frac{c^2}{a^2}\ell(\ell+1)u_\ell^2(a)[P_\ell(\cos\theta)]^2 + \frac{1}{\mu}\frac{u_\ell^2(a)}{a^2}[P_\ell^1(\cos\theta)]^2\right\}$$

$$= \frac{1}{4}\frac{u_\ell^2(a)}{a^2}\cdot\frac{1}{3}\{(a+h)^3 - a^3\}\int d\Omega\left\{\epsilon c^2\ell(\ell+1)[P_\ell(\cos\theta)]^2 + \frac{1}{\mu}[P_\ell^1(\cos\theta)]^2\right\}$$

$$\approx \frac{1}{4}\frac{u_\ell^2(a)}{a^2}\frac{1}{3}a^3\left\{1 + 3\frac{h}{a} - 1\right\}(2\pi)\int_0^\pi \sin\theta d\theta\left\{\frac{1}{\mu}\ell(\ell+1)[P_\ell(\cos\theta)]^2 + \frac{1}{\mu}[P_\ell^1(\cos\theta)]^2\right\}$$

$$= \frac{\pi h u_\ell^2(a)}{2\mu}\int_{-1}^{+1} dx\left\{\ell(\ell+1)[P_\ell(x)]^2 + [P_\ell^1(x)]^2\right\}$$

Note that

$$\int_{-1}^{+1}[P_\ell(x)]^2dx = \frac{2}{2\ell+1}, \qquad \int_{-1}^{+1}[P_\ell^m(x)]^2dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}$$

Plugging into U:

$$U = \frac{2\pi h u_\ell^2(a)}{\mu}\frac{\ell(\ell+1)}{2\ell+1}$$

(Note that the average energies in electric and magnetic fields are equal). The average power loss is given by Eq. (8.15):

$$\frac{dP}{da} = \frac{1}{2\sigma\delta}|\vec{K}_{\text{eff.}}|^2 = \frac{1}{2\sigma\delta}|\vec{n}\times\vec{H}_{||}|^2 = \frac{1}{2\sigma\delta}|\vec{H}_{||}|^2$$

Now note that

$$|\vec{H}_{||}| = \frac{B_\phi}{\mu} = \frac{u_\ell(a)}{\mu a}P_\ell^1(\cos\theta)$$

The average total power loss

$$P_{\text{loss}} = \int_{\text{interior}}\frac{dP}{da}da + \int_{\text{exterior}}\frac{dP}{da}da = \frac{\mu_\ell^2(a)}{2\mu^2}\left\{\frac{1}{\sigma_i\delta_i} + \frac{1}{\sigma_e\delta_e}\right\}\int[P_\ell^1(\cos\theta)]^2d\Omega$$

$$= \frac{2\pi u_\ell^2(a)}{\mu^2}\left\{\frac{1}{\sigma_i\delta_i} + \frac{1}{\sigma_e\delta_e}\right\}\frac{\ell(\ell+1)}{2\ell+1}$$

Note that

$$\delta^2 = \frac{2}{\mu\omega_\ell\sigma} \qquad \Rightarrow \qquad \frac{1}{\sigma\delta} = \frac{\mu\omega_\ell\delta}{2}$$

Therefore,

$$P_{\text{loss}} = \frac{2\pi u_\ell^2(a)}{\mu^2}\left\{\frac{\mu\omega\delta_i}{2} + \frac{\mu\omega\delta_e}{2}\right\} = \frac{\pi u_\ell^2(a)\omega_\ell}{\mu}\frac{\ell(\ell+1)}{2\ell+1}(\delta_i+\delta_e)$$

The Schumann resonance Q value:

$$Q = \omega_\ell \frac{U}{P_{\text{loss}}} = \omega_\ell \frac{2\pi h u_\ell^2(a)}{\mu}\frac{\ell(\ell+1)}{2\ell+1}\frac{\mu}{\pi u_\ell^2(a)\omega_\ell}\frac{2\ell+1}{\ell(\ell+1)}\frac{1}{\delta_i+\delta_e} = \frac{2h}{\delta_i+\delta_e}$$

independent of ℓ and $N = 2$.
(b) For the lowest Schumann resonance,

$$\ell = 1 \;\Rightarrow\; \omega_1 = \sqrt{2}\frac{c}{a} = \sqrt{2}\frac{3\cdot 10^8}{6.4\cdot 10^6} = 66.3 \text{ Hz}$$

$$\delta_e = \sqrt{\frac{2}{\mu\omega_1\sigma_e}} = \sqrt{\frac{2}{4\pi\cdot 10^{-7}\times 66.3\times 0.1}} = 4.9\cdot 10^2 \text{ m}$$

$$\delta_i = \sqrt{\frac{2}{\mu\omega_1\sigma_i}} = \sqrt{\frac{2}{4\pi\cdot 10^{-7}\times 66.3\times 10^{-5}}} = 4.9\cdot 10^4 \text{ m}$$

$$Q = \frac{2h}{\delta_e+\delta_i} = \frac{2\times 10^5}{4.9\cdot 10^2(1+100)} = 4.0$$

(c) With $\sigma_i \approx 10^{-5}$ $(\Omega\text{m})^{-1}$, $\delta_i \approx 49$ km is not small compared with $h \approx 100$ km. However, the fields vary over distances of order a, at least for $\ell = 1$. Thus, the approximation of Section 8.1 are valid, at least for small ℓ values. When a/ℓ becomes of order of δ_i, these approximations won't be adequate. In this case, it occurs at $\ell \sim 100$.

Problem 8.18
(a) For TM modes, we have

$$(\nabla_t^2 + \gamma_\lambda^2)E_{z\lambda} = 0, \quad \text{and} \quad E_{z\lambda}|_C = 0$$

where the subscript C denotes boundary contour. Applying Green's theorem in two dimension:

$$\int_S(\phi\nabla_t^2\psi - \psi\nabla_t^2\phi)da = -\oint_C(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n})d\ell$$

where the $-$ on the right side is due to the difference in the normal definition. With $\phi = E_{z\lambda}$ and $\psi = E_{z\mu}$, we gets

$$\int_S(E_{z\lambda}\nabla_t^2 E_{z\mu} - E_{z\mu}\nabla_t^2 E_{z\lambda})da = -\oint_C(E_{z\lambda}\frac{\partial E_{z\mu}}{\partial n} - E_{z\mu}\frac{\partial E_{z\lambda}}{\partial n})d\ell$$

The line integrals on the right-hand side vanish due to the boundary conditions. Therefore,

$$0 = \int_S(E_{z\lambda}\nabla_t^2 E_{z\mu} - E_{z\mu}\nabla_t^2 E_{z\lambda})da = \int_S\left\{E_{z\lambda}(-\gamma_\mu^2 E_{z\mu}) - E_{z\mu}(-\gamma_\lambda^2 E_{z\lambda})\right\}da = (\gamma_\lambda^2 - \gamma_\mu^2)\int_S E_{z\lambda}E_{z\mu}da$$

For the case $\gamma_\lambda \neq \gamma_\mu$, the integral must vanish:

$$\int_S E_{z\lambda}E_{z\mu}da = 0$$

Same argument applies to $H_{z\lambda}$ and $H_{z\mu}$, except in this case, the line integrals vanishes due to boundary conditions

$$\frac{\partial H_z}{\partial n}|_C = 0$$

(b) *Proof for TM modes only*
Applying Green's first identity

$$\int_S (\phi\nabla_t^2\psi + \nabla_t\phi\cdot\nabla_t\psi)da = -\oint_C \phi\frac{\partial\psi}{\partial n}d\ell$$

with $\phi = E_{z\lambda}$ and $\psi = E_{z\mu}$ for the TM modes, we get

$$\int_S (E_{z\lambda}\nabla_t^2 E_{z\mu} + \nabla_t E_{z\lambda}\cdot\nabla_t E_{z\mu})da = -\oint_C E_{z\lambda}\frac{\partial}{\partial n}E_{z\mu}d\ell$$

Again, the line integral on the right vanishes due to the boundary condition. $\nabla_t E_z$ and $\nabla_t^2 E_z$ are given by Eqs. (8.33, 8.34):

$$\nabla_t^2 E_{z\lambda} = -\gamma_\lambda^2 E_{z\lambda}, \qquad \nabla_t^2 E_{z\mu} = -\gamma_\mu^2 E_{z\mu}$$

$$\nabla_t E_{z\lambda} = -i\frac{\gamma_\lambda^2}{k_\lambda}\vec{E}_\lambda, \qquad \nabla_t E_{z\mu} = -i\frac{\gamma_\mu^2}{k_\mu}\vec{E}_\mu$$

where $\vec{E}_\lambda$ and $\vec{E}_\mu$ are transverse electric fields. The Green's first identify becomes

$$\gamma_\mu^2\int_S E_{z\lambda}E_{z\mu}da + \frac{\gamma_\lambda^2\gamma_\mu^2}{k_\lambda k_\mu}\int_S \vec{E}_\lambda\cdot\vec{E}_\mu da = 0 \quad\Rightarrow\quad \int_S E_{z\lambda}E_{z\mu}da = -\frac{\gamma_\mu^2}{k_\lambda k_\mu}\int_S \vec{E}_\lambda\cdot\vec{E}_\mu da$$

Assuming non-degeneracy and from (a), we obtain:

$$\text{For } \lambda\neq\mu; \quad \int_S \vec{E}_\lambda\cdot\vec{E}_\mu da = 0$$

By properly normalizing $\vec{E}_\lambda$, we have

$$\int_S \vec{E}_\lambda\cdot\vec{E}_\mu da = \delta_{\lambda\mu} \qquad \text{Eq.(8.131)}$$

$$\int_S E_{z\lambda}E_{z\mu}da = -\frac{\gamma_\mu^2}{k_\lambda k_\mu}\int_S \vec{E}_\lambda\cdot\vec{E}_\mu da = -\frac{\gamma_\lambda^2}{k_\lambda^2}\delta_{\lambda\mu} \qquad \text{Eq.(8.134) for TM waves}$$

Now turn into the relations of magnetic fields. Note that

$$\vec{H}_\lambda = \frac{\epsilon\omega}{k_\lambda}\hat{z}\times\vec{E}_\lambda = \frac{i\epsilon\omega}{\gamma_\lambda^2}\hat{z}\times\nabla_t E_{z\lambda}, \qquad \vec{H}_\mu = \frac{\epsilon\omega}{k_\mu}\hat{z}\times\vec{E}_\mu = \frac{i\epsilon\omega}{\gamma_\mu^2}\hat{z}\times\nabla_t E_{z\mu}$$

$$\vec{H}_\lambda\cdot\vec{H}_\mu = -\frac{(\epsilon\omega)^2}{\gamma_\lambda^2\gamma_\mu^2}(\hat{z}\times\nabla_t E_{z\lambda})\cdot(\hat{z}\times\nabla_t E_{z\mu}) = -\frac{(\epsilon\omega)^2}{\gamma_\lambda^2\gamma_\mu^2}\nabla_t E_{z\mu}\cdot\{(\hat{z}\times\nabla_t E_{z\lambda})\times\hat{z}\} = -\frac{(\epsilon\omega)^2}{\gamma_\lambda^2\gamma_\mu^2}\nabla_t E_{z\mu}\cdot\nabla_t E_{z\lambda}$$

Using Green's first identity with $\phi = E_{z\lambda}$ and $\psi = E_{z\mu}$, we have

$$\int \nabla_t E_{z\mu}\cdot\nabla_t E_{z\lambda}da = -\oint E_{z\mu}\frac{\partial E_{z\lambda}}{\partial n} - \int E_{z\mu}\nabla_t^2 E_{z\lambda}da = \gamma_\lambda^2\int E_{z\mu}E_{z\lambda}da = -\frac{\gamma_\lambda^4}{k_\lambda^2}\delta_{\lambda\mu}$$

Thus,

$$\int \vec{H}_\lambda\cdot\vec{H}_\mu da = -\frac{(\epsilon\omega)^2}{\gamma_\lambda^2\gamma_\mu^2}\int \nabla_t E_{z\mu}\cdot\nabla_t E_{z\lambda}da = -\frac{(\epsilon\omega)^2}{\gamma_\lambda^2\gamma_\mu^2}\left(-\frac{\gamma_\lambda^4}{k_\lambda^2}\delta_{\lambda\mu}\right) = \frac{(\epsilon\omega)^2}{k_\lambda^2}\delta_{\lambda\mu} = \frac{1}{Z_\lambda^2}\delta_{\lambda\mu} \qquad \text{Eq. (8.132)}$$

where $Z_\lambda = k_\lambda/(\epsilon\omega)$ is the wave impedance.

$$\frac{1}{2}\int (\vec{E}_\lambda\times\vec{H}_\mu)\cdot\hat{z}da = \frac{1}{2}\int\left\{\frac{ik_\lambda}{\gamma_\lambda^2}\nabla_t E_{z\lambda}\times\frac{i\epsilon\omega}{\gamma_\mu^2}(\hat{z}\times\nabla_t E_{z\mu})\right\}\cdot\hat{z}da$$

$$= -\frac{1}{2}\int\frac{k_\lambda\epsilon\omega}{\gamma_\lambda^2\gamma_\mu^2}(\nabla_t E_{z\lambda}\cdot\nabla_t E_{z\mu})da = -\frac{1}{2}\frac{k_\lambda\epsilon\omega}{\gamma_\lambda^2\gamma_\mu^2}\left(-\frac{\gamma_\lambda^4}{k_\lambda^2}\delta_{\lambda\mu}\right) = \frac{1}{2Z_\lambda}\delta_{\lambda\mu} \qquad \text{Eq. (8.133)}$$

More Problems for Chapter 8

Problem 8.3

(a) Choose a rectangular coordinate system with x parallel to the strip along the side b, y perpendicular to the strip and z along the line. Let $K(\vec{z},t) = K_0 e^{i(kz-\omega t)}\hat{z}$ be the surface current density of the top strip. Thus, the magnetic field in between the two strips is given by

$$\vec{B} = \mu K\hat{x} = \mu K_0 e^{i(kz-\omega t)}\hat{x}, \quad \vec{H} = \frac{\vec{B}}{\mu} = K_0 e^{i(kz-\omega t)}\hat{x}$$

Therefore, $K_0 = H_0$. The electric field can be derived from the Maxwell's equation:

$$\nabla \times \vec{B} = \mu\epsilon\frac{\partial \vec{E}}{\partial t} = -i\mu\epsilon\omega\vec{E} \quad \Rightarrow \quad \vec{E} = -\frac{\nabla \times \vec{B}}{i\mu\epsilon\omega} = -\frac{\vec{k}\times\vec{B}}{\mu\epsilon\omega} = -\frac{kH_0}{\epsilon\omega}e^{i(kz-\omega t)}\hat{y}$$

The average Poynting vector

$$\vec{S} = \frac{1}{2}\vec{E}\times\vec{H}^* = \frac{1}{2}\left\{-\frac{kH_0}{\epsilon\omega}\right\}\hat{y}\times(H_0^*\hat{x}) = \frac{k|H_0|^2}{2\epsilon\omega}\hat{z} = \frac{\sqrt{\mu\epsilon}|H_0|^2}{2\epsilon}$$

The average power transmitted along the line

$$P = \int \vec{S}\cdot\hat{z}da = \frac{ab}{2}\sqrt{\frac{\mu}{\epsilon}}|H_0|^2$$

In terms of the power P,

$$|H_0|^2 = \frac{2P}{ab}\sqrt{\frac{\epsilon}{\mu}}$$

The power loss per unit area

$$\frac{dP}{da} = -\frac{1}{2\sigma\delta}|\vec{K}_{\text{eff}}|^2 = -\frac{1}{2\sigma\delta}|\vec{H}_{||}|^2 = -\frac{1}{2\sigma\delta}|H_0|^2$$

The power loss per unit length along the z:

$$\frac{dP}{dz} = 2b\frac{dP}{da} = -\frac{b}{\sigma\delta}|H_0|^2 = -2\left\{\frac{1}{a\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\right\}P = -2\gamma P$$

Thus

$$P(z) = P_0 e^{-2\gamma z} \qquad \text{with} \qquad \gamma = \frac{1}{a\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}$$

The potential difference between the two strips

$$V = \int \vec{E}\cdot d\vec{\ell} = \frac{kH_0}{\epsilon\omega}ae^{i(kz-\omega t)}$$

The wave impedance

$$Z = \frac{V}{I} = \frac{V}{Kb} = \frac{ka}{\epsilon\omega} = \frac{a}{b}\sqrt{\frac{\mu}{\epsilon}}$$

The series resistance per unit length

$$R = -\frac{2}{|I|^2}\frac{dP}{dz} = -\frac{2}{|H_0|^2b^2}\left(-\frac{b}{\sigma\delta}|H_0|^2\right) = \frac{2}{\sigma\delta b}$$

The inductance per unit length

$$L = \frac{1}{|I|^2}\int \vec{B}\cdot\vec{H}^* da = \frac{1}{|H_0|^2 b^2}\left\{ab\mu|H_0|^2 + 2\int_{\text{conductor}} \vec{B}\cdot\vec{H}^* da\right\}$$

where the integration of the second term taking into account the magnetic energy stored inside the conductors. Note that inside the conductors

$$\vec{H}(\xi,t) = H_0 e^{-(1-i)\xi/\delta} e^{-i\omega t}\hat{x}$$

Thus

$$\int_{\text{conductor}} \vec{B}\cdot\vec{H}^* da = \mu_c \int_0^\infty |H_0|^2 e^{-2\xi/\delta}(bd\xi) = \frac{1}{2}\mu_c\delta|H_0|^2 b$$

Thus

$$L = \frac{1}{|H_0|^2 b^2}(ab\mu|H_0|^2 + 2\cdot\frac{1}{2}\mu_c\delta|H_0|^2 b) = \frac{\mu a + \mu_c\delta}{b}$$

Alternative as suggested by Mr. Ben Burrington

Taking the results of Prob. 8.2 and making the following substitutions:

$$2\pi b_1 \Rightarrow b_2, \qquad b_1 - a_1 \Rightarrow a_2$$

where a_1, b_1 and a_2, b_2 are a, b's of Prob. 8.2 and Prob. 8.3 respectively. The substitutions are justified by the geometry. With

$$b_1 = \frac{b_2}{2\pi}, \qquad a_1 = \frac{b_2}{2\pi} - a_2$$

we have

$$\ln(\frac{b_1}{a_1}) = \ln(\frac{\frac{b_2}{2\pi}}{\frac{b_2}{2\pi} - a_2}) = -\ln(1 - \frac{2\pi a_2}{b_2}) \approx \frac{2\pi a_2}{b_2}, \quad \text{and} \quad \frac{1}{a_1} + \frac{1}{b_1} = \frac{1}{b_2/(2\pi) - a_2} + \frac{1}{b_2/(2\pi)} \approx \frac{4\pi}{b_2}$$

Thus

$$P = \sqrt{\frac{\mu}{\epsilon}}\pi a_1^2 |H_0|^2 \ln(\frac{b_1}{a_1}) \Rightarrow \sqrt{\frac{\mu}{\epsilon}}\pi(\frac{b_2}{2\pi})^2|H_0|^2\frac{2\pi a_2}{b_2} = \frac{a_2 b_2}{2}\sqrt{\frac{\mu}{\epsilon}}|H_0|^2$$

$$\gamma = \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\frac{1/a_1 + 1/b_1}{\ln(b_1/a_1)} \Rightarrow \frac{1}{2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}\frac{4\pi}{b_2}\cdot\frac{b_2}{2\pi a_2} = \frac{1}{a_2\sigma\delta}\sqrt{\frac{\epsilon}{\mu}}$$

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ln(\frac{b_1}{a_1}) \Rightarrow \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\frac{2\pi a_2}{b_2} = \frac{a_2}{b_2}\sqrt{\frac{\mu}{\epsilon}}$$

$$R = \frac{1}{2\pi\sigma\delta}(\frac{1}{a_1} + \frac{1}{b_1}) \Rightarrow \frac{1}{2\pi\sigma\delta}\frac{4\pi}{b_2} = \frac{2}{\sigma\delta b_2}$$

$$L = \frac{\mu}{2\pi}\ln(\frac{b_1}{a_1}) + \frac{\mu_c\delta}{4\pi}(\frac{1}{a_1} + \frac{1}{b_1}) \Rightarrow \frac{\mu}{2\pi}\frac{2\pi a_2}{b_2} + \frac{\mu_c\delta}{4\pi}\frac{4\pi}{b_2} = \frac{\mu a_2 + \mu_c\delta}{b_2}$$

(b) For the case $b \gg h$, the electric and magnetic fields are mostly confined in the region between the strip and the ground plane and are uniform within the region. Therefore, this case is very similar to part (a) with the slab and its mirror image. However the case $b \ll h$ is very different from (a). This case can be approximated by a wire above a grounding plane. The dielectric substrate should have little effect on the quantities calculated in (a) since both electric and magnetic fields extend mostly in the region without the substrate.

Problem 8.4
(a) The wave equation is

$$(\nabla_t^2 + \gamma^2)\psi = 0 \quad \text{with} \quad \gamma^2 = \mu\epsilon\omega^2 - k^2$$

Explicitly in polar coordinates, the equation has the form:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\psi}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \gamma^2\psi = 0$$

This is the Bessel's equation and has the following solutions:

$$\psi \sim A_m J_m(\gamma\rho)e^{im\phi}$$

For TM modes,

$$\psi|_{\rho=R} = 0, \quad \Rightarrow \quad J_m(\gamma R) = 0$$

Let x_{mn} be the n^{th} root of $J_m(x) = 0$, then

$$\gamma_{mn} = \frac{x_{mn}}{R}, \quad \Rightarrow \quad \omega = \sqrt{\frac{x_{mn}^2}{R^2\mu\epsilon} + \frac{k^2}{\mu\epsilon}}$$

Thus, the cutoff frequencies are

$$\omega_{mn}^{TM} = \frac{x_{mn}}{R\sqrt{\mu\epsilon}} = \frac{v}{R}x_{mn}$$

Here $v \equiv 1/\sqrt{\mu\epsilon}$. The four lowest cutoff frequencies are

$$\omega_1^{TM} = \frac{v}{R}x_{01} = 2.405\frac{v}{R}, \quad \omega_2^{TM} = \frac{v}{R}x_{11} = 3.832\frac{v}{R}, \quad \omega_3^{TM} = \frac{v}{R}x_{21} = 5.136\frac{v}{R}, \quad \omega_4^{TM} = \frac{v}{R}x_{02} = 5.520\frac{v}{R}$$

For TE modes,

$$\frac{\partial\phi}{\partial\rho}|_{\rho=R} = 0, \quad \Rightarrow \quad J_m'(\gamma R) = 0$$

Let y_{mn} be the n^{th} root of $J_m'(y) = 0$, then

$$\gamma_{mn} = \frac{y_{mn}}{R}, \quad \Rightarrow \quad \omega = \sqrt{\frac{y_{mn}^2}{R^2\mu\epsilon} + \frac{k^2}{\mu\epsilon}}$$

Thus the lowest four cutoff frequencies are

$$\omega_1^{TE} = \frac{v}{R}y_{11} = 1.841\frac{v}{R}, \quad \omega_2^{TE} = \frac{v}{R}y_{21} = 3.054\frac{v}{R}, \quad \omega_3^{TE} = \frac{v}{R}y_{01} = 3.832\frac{v}{R}, \quad \omega_4^{TE} = \frac{v}{R}y_{31} = 4.201\frac{v}{R}$$

Combining TM and TE modes, the mode with the lowest cutoff frequency is TE_{11}:

$$\omega_0 = 1.841\frac{v}{R}$$

The other four modes with the lowest frequencies are TM_{01}, TE_{21}, TE_{01} and TM_{11} with the ratios of frequencies given by

$$\frac{\omega_{01}^{TM}}{\omega_0} = \frac{2.405}{1.841} = 1.3, \quad \frac{\omega_{21}^{TE}}{\omega_0} = \frac{3.504}{1.841} = 1.9, \quad \frac{\omega_{01}^{TE}}{\omega_0} = \frac{3.832}{1.841} = 2.1, \quad \frac{\omega_{11}^{TM}}{\omega_0} = \frac{3.832}{1.841} = 2.1$$

Note that the modes TE_{01} and TM_{11} are degenerate.
(b) The lowest mode is TE_{11}. The longitudinal magnetic field of the mode has the form

$$\psi(\rho, \phi) = AJ_1(y_{11}\frac{\rho}{R})e^{i\phi} \quad \text{with the cutoff frequency} \quad \omega_{11} = y_{11}\frac{v}{R}$$

Here A is a constant describing the strength of the field and $y_{11} = 1.841$ is the first root of $J'(y) = 0$. The average power

$$P = \frac{\mu}{2\sqrt{\mu\epsilon}}(\frac{\omega}{\omega_{11}})^2\sqrt{1-\frac{\omega_{11}^2}{\omega^2}}\int|\psi|^2 da = \frac{\pi\mu|A|^2}{\sqrt{\mu\epsilon}}(\frac{\omega}{\omega_{11}})^2\sqrt{1-\frac{\omega_{11}^2}{\omega^2}}\int_0^R |J_1(y_{11}\frac{\rho}{R})|^2\rho d\rho$$

Note the identity

$$\int_0^R |J_m(y_{mn}\frac{\rho}{R})|^2\rho d\rho = \frac{R^2}{2}(1-\frac{m^2}{y_{mn}^2})|J_m(y_{mn})|^2$$

Thus

$$P = (1-\frac{1}{y_{11}^2})\frac{\pi\mu|A|^2R^2}{2\sqrt{\mu\epsilon}}(\frac{\omega}{\omega_{11}})^2\sqrt{1-\frac{\omega_{11}^2}{\omega^2}}|J_1(y_{11})|^2$$

To calculate the power loss, we note

$$|\vec{n}\times\nabla_t\psi| = |(-\hat{\rho})\times\left\{\hat{\rho}\frac{\partial\psi}{\partial\rho}+\hat{\phi}\frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\right\}| = \frac{1}{\rho}|\frac{\partial\psi}{\partial\phi}| = \frac{1}{\rho}|AJ_1(y_{11}\frac{\rho}{R})|$$

Thus the power loss per unit length

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}(\frac{\omega}{\omega_{11}})^2\left\{\frac{1}{\mu\epsilon\omega_{11}^2}(1-\frac{\omega_{11}^2}{\omega^2})\oint|\vec{n}\times\nabla_t\psi|^2 d\ell + (\frac{\omega_{11}}{\omega})^2\oint|\psi|^2 d\ell\right\}$$

$$= \frac{\pi R|A|^2}{\sigma\delta}(\frac{\omega}{\omega_{11}})^2\left\{\frac{1}{\mu\epsilon\omega_{11}^2}(1-\frac{\omega^2}{\omega_{11}^2})\frac{1}{R^2}+(\frac{\omega_{11}}{\omega})^2\right\}|J_1(y_{11})|^2$$

Thus the attenuation constant

$$\beta_{11}^{TE} = \frac{1}{2P}(-\frac{dP}{dz}) = \frac{y_{11}^2}{y_{11}^2-1}\frac{\sqrt{\mu\epsilon}}{\mu R}\frac{1}{\sqrt{1-\omega_{11}^2/\omega^2}}\frac{1}{\sigma\delta}\left\{\frac{1}{\mu\epsilon\omega_{11}^2}(1-\frac{\omega_{11}^2}{\omega^2})\frac{1}{R^2}+(\frac{\omega_{11}}{\omega})^2\right\}$$

Note that

$$\delta = \sqrt{\frac{2}{\mu\omega\sigma}} \qquad \Rightarrow \qquad \sigma = \frac{2}{\mu\omega\delta^2}$$

Thus

$$\beta_{11}^{TE} = \frac{y_{11}^2}{y_{11}^2-1}\frac{\sqrt{\mu\epsilon}\delta}{2R}\frac{\omega}{\sqrt{1-\omega_{11}^2/\omega^2}}\left\{\frac{1}{\mu\epsilon\omega_{11}^2}(1-\frac{\omega_{11}^2}{\omega^2})\frac{1}{R^2}+(\frac{\omega_{11}}{\omega})^2\right\}$$

$$= \frac{\delta}{2Rv}\frac{\omega^2}{\sqrt{\omega^2-\omega_{11}^2}}\left\{\frac{1}{y_{11}^2-1}+(\frac{\omega_{11}}{\omega})^2\right\} = \frac{1}{R}\sqrt{\frac{\epsilon}{2\sigma}}\sqrt{\frac{\omega^3}{\omega^2-\omega_{11}^2}}\left\{\frac{1}{y_{11}^2-1}+(\frac{\omega_{11}}{\omega})^2\right\}$$

The second lowest mode is TM_{01}. In this case, the longitudinal component of the electric field is given by

$$\psi(\rho,\phi) = AJ_0(x_{01}\frac{\rho}{R}) \quad \text{with the cutoff frequency} \quad \omega_{01} = x_{01}\frac{v}{R}$$

where A is a constant and $x_{01} = 2.405$ is the first root of $J_0(x)$. The average power and the power loss per unit length

$$P = \frac{\epsilon}{2\sqrt{\mu\epsilon}}(\frac{\omega}{\omega_{01}})^2\sqrt{1-\frac{\omega_{01}^2}{\omega^2}}\int|\psi|^2 da = \frac{\pi\epsilon|A|^2}{\sqrt{\mu\epsilon}}(\frac{\omega}{\omega_{01}})^2\sqrt{1-\frac{\omega_{01}^2}{\omega^2}}\int_0^R |J_0(x_{01}\frac{\rho}{R})|^2\rho d\rho$$

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}(\frac{\omega}{\omega_{01}})^2\frac{1}{\mu^2\omega_{01}^2}\oint|\frac{\partial\psi}{\partial n}|^2 d\ell = \frac{\pi|A|^2R}{\mu^2\sigma\delta\omega_{01}^2}(\frac{\omega}{\omega_{01}})^2\left\{|\frac{\partial}{\partial\rho}J_0(x_{01}\frac{\rho}{R})|^2\right\}_{\rho=R}$$

Applying the following identity equations for Bessel's functions

$$\int_0^R |J_m(x_{mn}\frac{\rho}{R})|^2 \rho d\rho = \frac{R^2}{2}|J_{m+1}(x_{mn})|^2 \quad \text{and} \quad \frac{d}{dx}\{x^m J_m(x)\} = x^m J_{m-1}(x)$$

we get

$$\int_0^R |J_0(x_{01}\frac{\rho}{R})|^2 \rho d\rho = \frac{R^2}{2}|J_1(x_{01})|^2$$

$$\left\{|\frac{\partial}{\partial\rho}J_0(x_{01}\frac{\rho}{R})|^2\right\}_{\rho=R} = \frac{x_{01}^2}{R^2}|J_1(x_{01})|^2$$

Thus the attenuation constant

$$\beta_{01}^{TM} = \frac{1}{2P}(-\frac{dP}{dz}) = \frac{\sqrt{\mu\epsilon}}{\mu^2\epsilon}\frac{x_{01}^2}{\sigma\delta R^3}\frac{1}{\omega_{01}^2\sqrt{1-\omega_{01}^2/\omega^2}} = \frac{x_{01}^2}{2\sqrt{\mu\epsilon}}\frac{\delta}{R^3}(\frac{\omega}{\omega_{01}})^2\frac{1}{\sqrt{\omega^2-\omega_{01}^2}}$$

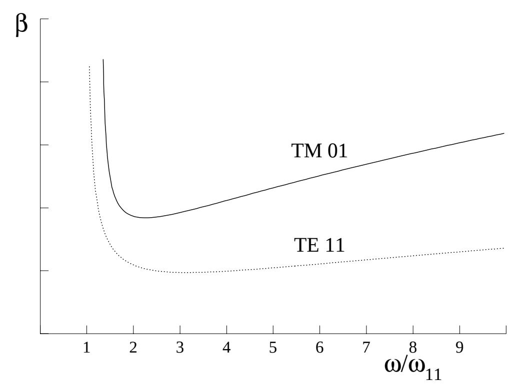
$$= \frac{\delta}{2Rv}\frac{\omega^2}{\sqrt{\omega^2-\omega_{01}^2}} = \frac{1}{R}\sqrt{\frac{\epsilon}{2\sigma}}\sqrt{\frac{\omega^3}{\omega^2-\omega_{01}^2}}$$

Note that for TM modes, β is always minimum at $\omega = \sqrt{3}\omega_0$, where ω_0 is the cutoff frequency. To facilitate the comparision, we rewrite the two constants as

$$\beta_{11}^{TE} = \frac{1}{R}\sqrt{\frac{\epsilon\omega_{11}}{2\sigma}}\sqrt{\frac{(\omega/\omega_{11})^3}{(\omega/\omega_{11})^2-1}}\left\{\frac{1}{y_{11}^2-1}+\frac{1}{(\omega/\omega_{11})^2}\right\}$$

$$\beta_{01}^{TM} = \frac{1}{R}\sqrt{\frac{\epsilon}{2\sigma}}\sqrt{\frac{\omega^3}{\omega^2-\omega_{01}^2}} = \frac{1}{R}\sqrt{\frac{\epsilon\omega_{11}}{2\sigma}}\sqrt{\frac{(\omega/\omega_{11})^3}{(\omega/\omega_{11})^2-(x_{01}/y_{11})^2}}$$

The two constants in the unit of $(1/R)\sqrt{(\epsilon\omega_{11})/(2\sigma)}$ are plotted in the figure below as functions of ω/ω_{11}.



Problem 8.16(a)

The eigenangle θ_p of the p^{th} mode is the solution of Eq. (8.123):

$$\tan(ka\sin\theta_p - \frac{p\pi}{2}) = \sqrt{\frac{2\Delta}{\sin^2\theta_p} - 1}$$

Note that

$$k_z = k\cos\theta_p \qquad \Rightarrow \qquad \sin\theta_p = \sqrt{1-\cos^2\theta_p} = \sqrt{1-\frac{k_z^2}{k^2}} = \frac{1}{k}\sqrt{k^2-k_z^2}$$

Thus

$$\tan\left\{a\sqrt{k^2-k_z^2}-\frac{p\pi}{2}\right\} = \sqrt{\frac{2\Delta k^2}{k^2-k_z^2}-1}$$

Differentiating the above equation with respect to k:

$$\frac{1}{\cos^2\left\{a\sqrt{k^2-k_z^2}-p\pi/2\right\}}\frac{a}{2}\frac{2k-2k_z(dk_z/dk)}{\sqrt{k^2-k_z^2}} = \frac{1}{2}\frac{\{(4\Delta k)(k^2-k_z^2)-(2\Delta k^2)(2k-2k_z(dk_z/dk))\}/(k^2-k_z^2)^2}{\sqrt{2\Delta k^2/(k^2-k_z^2)-1}}$$

After some algebra, the above equation can be written as

$$\frac{\sqrt{2\Delta k^2/(k^2-k_z^2)-1}}{\cos^2\left\{a\sqrt{k^2-k_z^2}-p\pi/2\right\}}\frac{a}{2}(k-k_z\frac{dk_z}{dk}) = \frac{\Delta k}{\sqrt{k^2-k_z^2}}-\frac{\Delta k^2}{(k^2-k_z^2)^{3/2}}(k-k_z\frac{dk_z}{dk})$$

We now note

$$k^2-k_z^2 = k_x^2 = k^2\sin^2\theta_p \quad \text{and} \quad \cos^2\left\{a\sqrt{k^2-k_z^2}-\frac{p\pi}{2}\right\} = \frac{1}{1+\tan^2\left\{a\sqrt{k^2-k_z^2}-p\pi/2\right\}} = \frac{\sin^2\theta_p}{2\Delta}$$

Plugging into the above equation:

$$\sqrt{\frac{2\Delta}{\sin^2\theta_p}-1}\left\{\frac{2\Delta}{\sin^2\theta_p}\right\}\frac{ka}{2}(1-\cos\theta_p\frac{dk_z}{dk}) = \frac{\Delta}{\sin\theta_p}-\frac{\Delta}{\sin^3\theta_p}(1-\cos\theta_p\frac{dk_z}{dk})$$

Solving for dk_z/dk:

$$\frac{dk_z}{dk} = \frac{1}{\cos\theta_p}\frac{\cos^2\theta_p+ka\sqrt{2\Delta-\sin^2\theta_p}}{1+ka\sqrt{2\Delta-\sin^2\theta_p}}$$

Note that $k = n_1\omega/c$, therefore the axial group velocity

$$v_g = \frac{d\omega}{dk_z} = \frac{d\omega}{dk}\frac{dk}{dk_z} = \frac{c\cos\theta_p}{n_1}\frac{1+ka\sqrt{2\Delta-\sin^2\theta_p}}{\cos^2\theta_p+ka\sqrt{2\Delta-\sin^2\theta_p}} = \frac{c\cos\theta_p}{n_1}\frac{1+\beta_p a}{\cos^2\theta_p+\beta_p a}$$

where $\beta_p = k\sqrt{2\Delta-\sin^2\theta_p}$. The group velocity is greater than the expected $c\cos\theta_p/n_1$. This is consistent with the Goos-Hanchen effect that the right ray is shifted forward after total internal reflection, resulting a greater group velocity.

Problem 8.20
(a) The field in the waveguide can be written as

$$\vec{E}^{(\pm)} = \sum_\lambda A_\lambda^{(\pm)}\vec{E}_\lambda^{(\pm)}$$

where the coefficients $A_\lambda^{(\pm)}$ are given by Eq. (8.146):

$$A_\lambda^{(\pm)} = -\frac{Z_\lambda}{2}\int_V \vec{J}\cdot\vec{E}_\lambda^{(\mp)}d\tau = -\frac{Z_\lambda}{2}\int \vec{I}\cdot\vec{E}_\lambda^{(\mp)}d\ell$$

Choose the bottom-left corner of the guide as the coordinate origin with the $x-$axis along the edge a and the $y-$axis along the edge b.

$$\vec{I} = I_0(-\sin\phi\hat{x} + \cos\phi\hat{y})$$

Here ϕ is the polar angle with respect to the center-of-the loop. Thus

$$A_\lambda^{(\pm)} = -\frac{Z_\lambda}{2}\int_{-\pi/2}^{\pi/2} I_0(-\sin\phi\hat{x} + \cos\phi\hat{y})\cdot\vec{E}_\lambda^{(\mp)}(Rd\phi) = -\frac{1}{2}RI_0Z_\lambda\int_{-\pi/2}^{\pi/2}\left\{-\sin\phi\{E_\lambda^{(\mp)}\}_x + \cos\phi\{E_\lambda^{(\mp)}\}_y\right\}d\phi$$

where $\{E_\lambda^{(\mp)}\}_x$ and $\{E_\lambda^{(\mp)}\}_y$ are $x-$ and $y-$ components of the eigen-field along the loop.
For TM waves, the electric field components are given by Eq. (8.135):

$$\{E_{mn}^{(\mp)}\}_x = \frac{2\pi m}{\gamma_{mn}a\sqrt{ab}}\cos\{\frac{m\pi(R\cos\phi)}{a}\}\sin\{\frac{n\pi(h+R\sin\phi)}{b}\}$$

$$\{E_{mn}^{(\mp)}\}_y = \frac{2\pi n}{\gamma_{mn}b\sqrt{ab}}\sin\{\frac{m\pi(R\cos\phi)}{a}\}\cos\{\frac{n\pi(h+R\sin\phi)}{b}\}$$

Here

$$\gamma_{mn}^2 = \pi^2\left\{\frac{m^2}{a^2} + \frac{n^2}{b^2}\right\}$$

Therefore,

$$A_{mn}^{(\pm)} = -\frac{\pi RI_0Z_{mn}}{\gamma_{mn}\sqrt{ab}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{-\frac{m}{a}\sin\phi\cos(\frac{m\pi R\cos\phi}{a})\sin(\frac{n\pi(h+R\sin\phi)}{b}) + \frac{n}{b}\cos\phi\sin(\frac{m\pi R\cos\phi}{a})\cos(\frac{n\pi(h+R\sin\phi)}{b})\right\}d\phi$$

$$= -\frac{\pi RI_0Z_{mn}}{\gamma_{mn}\sqrt{ab}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\frac{1}{\pi R}\frac{d}{d\phi}\{\sin(\frac{m\pi R\cos\phi}{a})\}\sin(\frac{n\pi(h+R\sin\phi)}{b}) + \frac{1}{\pi R}\sin(\frac{m\pi R\cos\phi}{a})\frac{d}{d\phi}\{\sin(\frac{n\pi(h+R\sin\phi)}{b})\}\right\}d\phi$$

$$= -\frac{I_0Z_{mn}}{\gamma_{mn}\sqrt{ab}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d}{d\phi}\left\{\sin(\frac{m\pi R\cos\phi}{a})\sin(\frac{n\pi(h+R\sin\phi)}{b})\right\}d\phi$$

$$= -\frac{I_0Z_{mn}}{\gamma_{mn}\sqrt{ab}}\sin(\frac{m\pi R\cos\phi}{a})\sin(\frac{n\pi(h+R\sin\phi)}{b})|_{\phi=-\pi/2}^{\phi=\pi/2} = 0$$

Therefore, no TM modes are excited. This is because that a circular current in the transverse plane will always result in a non-vanishing longitudinal component of $\vec{H}$, *i.e.*, $H_z \neq 0$.
(b) For TE waves,

$$\{E_{mn}^{(\mp)}\}_x = -\frac{2\pi n}{\gamma_{mn}b\sqrt{ab}}\cos(\frac{m\pi R\cos\phi}{a})\sin(\frac{n\pi(h+R\sin\phi)}{b})$$

$$\{E_{mn}^{(\mp)}\}_y = \frac{2\pi m}{\gamma_{mn}a\sqrt{ab}}\sin(\frac{m\pi R\cos\phi}{a})\cos(\frac{n\pi(h+R\sin\phi)}{b})$$

with the normalization reduced by a factor of $\sqrt{2}$ if $m = 0$ or $n = 0$. Thus

$$A_{mn}^{(\pm)} = -\frac{\pi RI_0Z_{mn}}{\gamma_{mn}\sqrt{ab}}\int_{-\pi/2}^{\pi/2}\left\{\frac{n}{b}\sin\phi\cos(\frac{m\pi R\cos\phi}{a})\sin(\frac{n\pi(h+R\sin\phi)}{b}) + \frac{m}{a}\cos\phi\sin(\frac{m\pi R\cos\phi}{a})\cos(\frac{n\pi(h+R\sin\phi)}{b})\right\}d\phi$$

The lowest modes ($m = 1, n = 0$):

$$A_{1,0} = -\frac{\pi RI_0Z_{1,0}}{\gamma_{1,0}\sqrt{2a^3b}}\int_{-\pi/2}^{\pi/2}\left\{\cos\phi\sin(\frac{\pi R\cos\phi}{a})\right\}d\phi = -\frac{\pi RI_0Z_{1,0}}{\gamma_{1,0}\sqrt{2a^3b}}\left\{\pi J_1(\frac{\pi R}{a})\right\}$$

where $\gamma_{1,0} = \pi/a$. Here we have used the integral representation of Bessel functions:

$$\int_0^{\pi} \sin\theta \sin(x\sin\theta)d\theta = \int_{-\pi/2}^{\pi/2} \cos\phi \sin(x\cos\phi)d\phi = J_1(x)$$

The amplitude is independent of the height h. For $R \ll a$,

$$J_1(\frac{\pi R}{a}) \approx \frac{\pi R}{2a}, \qquad \Rightarrow \qquad A_{1,0} \approx -\frac{\pi^3 R^2 I_0 Z_{1,0}}{\gamma_{1,0}\sqrt{8a^5 b}}$$

(c) The average power radiated in either direction

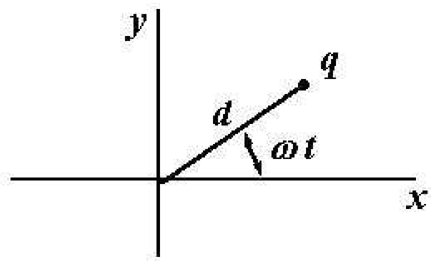
$$P = \frac{1}{2}\int (\vec{E} \times \vec{H}^*) \cdot \hat{z}da = \frac{1}{2}\int \left\{ (\sum_\lambda A_\lambda \vec{E}_\lambda) \times (\sum_\mu A_\mu^* \vec{H}^*{}_\mu) \right\} \cdot \hat{z}da = \frac{1}{2}\sum_{\lambda\mu} A_\lambda A_\mu^* \int (\vec{E}_\lambda \times \vec{H}^*{}_\mu) \cdot \hat{z}da = \frac{1}{2}\sum_\lambda \frac{|A_\lambda|^2}{Z_\lambda}$$

In this case,

$$P = \frac{1}{2}\frac{|A_{1,0}|^2}{Z_{1,0}} = \frac{1}{2Z_{1,0}}|A_{1,0}|^2 \approx \frac{I_0^2}{16} Z_{1,0}\frac{a}{b}(\frac{\pi R}{a})^4$$

9.1

a)



$$\rho(\vec{x},t) = q\delta(z)\delta(y - \sin\omega_0 t)\delta(x - d\cos\omega t)$$

To illustrate the equivalence of the two methods, I'll consider the lowest two moments.

$$n = 0 \,:\, Q(t) = \int \rho(\vec{x},t)d^3x = q = \mathrm{Re}(qe^{-i0\cdot\omega t})$$

$$n = 1 \,:\, \vec{p}(t) = \int \rho(\vec{x},t)\vec{x}d^3x = qd(\hat{\imath}\cos\omega t + \quad \sin\omega t) = \mathrm{Re}[qd(\hat{\imath} + i \quad)e^{-i1\cdot\omega t}]$$

So we identify $\vec{p} = qd(\hat{\imath} + i \quad)$ as the quantity to be used in Jackson's formulas.
Arbitrary n: The n'th multipoles will contribute with maximum frequencies of $\omega_n = n\omega$.

b) The proof that we can write

$$\rho(\vec{x},t) = \rho_0(\vec{x}) + \sum_{n=1}^{\beta} \mathrm{Re}[2\rho_n(\vec{x})e^{-in\omega t}]$$

with

$$\rho_n(\vec{x}) = \frac{1}{\tau}\int_0^{\tau} \rho(\vec{x},t)e^{in\omega t}dt$$

was presented in lecture and will not be repeated here.

c) We have already calculated the $n = 0, 1$ moments by the method of part a). Now we compute these moments by the method of part b).

$n = 0$:

$$\rho_0(\vec{x}) = \frac{1}{\tau}\int_0^{\tau}[q\delta(z)\delta(y - \sin\omega_0 t)\delta(x - d\cos\omega t)]dt$$

$$Q = \int \rho_0(\vec{x})d^3x = \frac{q}{\tau}\int_0^{\tau} dt \int d^3x[\delta(z)\delta(y - \sin\omega_0 t)\delta(x - d\cos\omega t)] = q$$

$n = 1$:

$$\rho_1(\vec{x}) = \frac{1}{\tau}\int_0^\tau [q\delta(z)\delta(y - \sin\omega_0 t)\delta(x - d\cos\omega t)]e^{i\omega t}dt$$

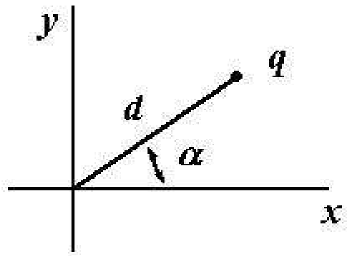
$$\vec{p}(\vec{x}) = \int d^3x\vec{x}(2\rho_1(\vec{x})) = \frac{2q}{\tau}\int_0^\tau dt\int d^3x\vec{x}[\delta(z)\delta(y - \sin\omega_0 t)\delta(x - d\cos\omega t)]$$

$$= \frac{2qd}{\tau}\int_0^\tau dte^{i\omega t}(\hat{\imath}\cos\omega t) + \quad \sin\omega t) = qd(\hat{\imath} + i \quad)$$

as before.

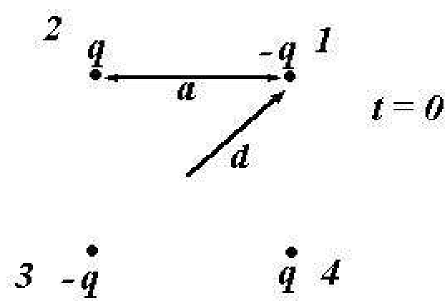
9.2

First consider a rotating charge which is at an angle α at time $t = 0$.



Compared to the lecture notes for this problem, where we assumed $\alpha = 0$, we should let $\omega t \to \omega t + \alpha$. Thus using the result developed in class, we can write for this problem

$$Q_\alpha(t) = \mathrm{Re}\left[\frac{3}{2}qd^2 \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} e^{-i2a} e^{-i2\omega t} \right]$$



From the figure

$$Q_{tot}(t) = Q_{\alpha 1}(t) + Q_{\alpha 2}(t) + Q_{\alpha 3}(t) + Q_{\alpha 4}(t)$$

$$= \mathrm{Re}\left[\frac{3}{2}qd^2 \left(-e^{-i\frac{\pi}{2}} + e^{-i\frac{3\pi}{2}} - e^{-i\frac{5\pi}{2}} + e^{-i\frac{7\pi}{2}} \right) \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} e^{-i2\omega t} \right]$$

$$Q_{tot} = \frac{3}{2}qd^2(4i) \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

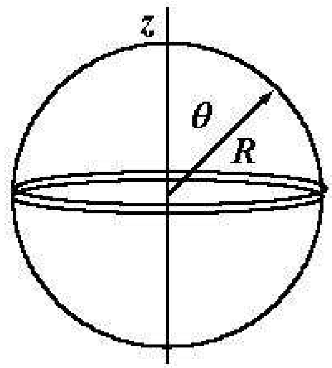
Thus from the class notes

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{1152\pi^2}\left(qd^2\frac{3}{2}\right)^2 16(1-\cos^4\theta) = \frac{1}{32\pi^2}c^2 Z_0 k^6 q^2 d^4 (1-\cos^4\theta)$$

$$P = \frac{c^2 Z_0 k^6}{360\pi}\left(qd^2\frac{3}{2}\right)^2 16 = \frac{1}{10\pi}c^2 Z_0 k^6 q^2 d^4$$

And, of course, the frequency of the radiation is 2ω.

9.3



Since the problem has azimuthal symmetry, we can expand $V(\vec{r},t)$ (in the radiation zone) in terms of Legendre polynomials:

$$V(\vec{r},t) = \sum_l b_l(t) r^{-l-1} P_l(\cos\theta)$$

Using the orthogonality of the Legendre polynomials, the leading term of the expansion in the radiation zone will be the $l = 1$ term.

$$b_1(t) = \frac{3}{2} R^2 \int_{-1}^{1} x V(\vec{r},t) dx = \frac{3}{2} V R^2 \cos\omega t$$

So,

$$V(\vec{r},t) = \left(\frac{3}{2} V R^2 \cos\omega t\right)/r^2 = \frac{\vec{p}\cdot\hat{r}}{r^2}\cos\omega t = \mathrm{Re}\left[\frac{\vec{p}\cdot\hat{r}}{r^2} e^{-i\omega t}\right]$$

with $\vec{p} = \frac{3}{2} V R^2 \hat{z},$ which should be used in the radiation formulas developed in lecture.

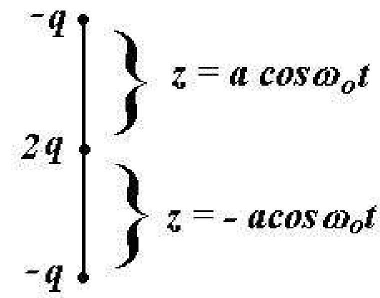
$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^4}{32\pi^2} |\vec{p}|^2 \sin^2\theta$$

$$P = \frac{c^2 Z_0 k^4}{32\pi^2}\frac{8\pi}{3} = \frac{1}{12\pi} c^2 Z_0 k^4 |\vec{p}|^2$$

with $\vec{p} = \frac{3}{2} V R^2 \hat{z}.$

9.11

We are working with small sources in the radiation zone.



From the notes and Eqs (9.170) and (9.172),

$$Q_{lm} = \int r^l Y_l^{m*} \rho d^3x$$

$$M_{lm} = -\frac{1}{l+1} \int r^l Y_l^{m*} \vec{\nabla} \cdot \left(\vec{r} \times \vec{J} \right) d^3x$$

a) Electric Dipole Radiation:

$$Q_{1m} = \int r Y_1^{m*} \rho d^3x$$

$$= \int r\delta(x)\delta(y)[-q\delta(z - a\cos\omega_0 t) - q\delta(z + a\cos\omega_0 t) + 2q\delta(z)] Y_1^{m*} dxdydz$$

$$= -qa\cos\omega_0 t[Y_1^{m*}(0,\phi) + Y_1^{m*}(\pi,\phi)] = -qa\delta_{m0}\cos\omega_0 t[Y_1^0(0) + Y_1^0(\pi)] = 0$$

b) Magnetic Dipole Radiation: Since the particles move in an orbit with no area,

$$\vec{r} \times \vec{J} = 0 \rightarrow M_{1m} = 0$$

c) Electric Quadrupole Radiation:

$$Q_{2m} = \int r^2 Y_2^{m*} \rho d^3x$$

$$= -q\delta_{m0}[a^2\cos^2\omega_0 t Y_2^0(\theta = 0) + a^2\cos^2\omega_0 t Y_2^0(\theta = \pi)]$$

$$= -q\delta_{m0} a^2 Y_2^0(0)(\cos 2\omega_0 t + 1)$$

where I have used $\cos^2\omega_0 t = (\cos 2\omega_0 t + 1)/2$. Thus the Fourier Series decomposition of this moment yields terms with frequency 0, and $2\omega_0$. The first term does not contribute to radiation, and the second can be written

$$Q_{20}(t) = \mathrm{Re}[-2qa^2 Y_2^0(0) e^{-2i\omega_0 t}]$$

so $Q_{20} = -2qa^2 Y_2^0(0)$ is the quantity that is used in the radiation formulas of Jackson. Using Eq. (9.151)

$$\frac{dP}{d\Omega}(2,0) = \frac{Z_0}{2k^2}|a(2,0)|^2 \left|\vec{X}_{20}\right|^2$$

and from Eq. (9.169)

$$a(2,0) = \frac{ck^4}{i(5\times 3)}\sqrt{\frac{3}{2}}\,Q_{20} = \frac{ck^4}{i(5\times 3)}\sqrt{\frac{3}{2}}\,(-2qa^2)\sqrt{\frac{5}{4\pi}}$$

where I have used $Y_2^0(0) = \sqrt{\frac{5}{4\pi}}$. Thus

$$|a(2,0)|^2 = \frac{1}{30\pi}c^2k^8q^2a^4$$

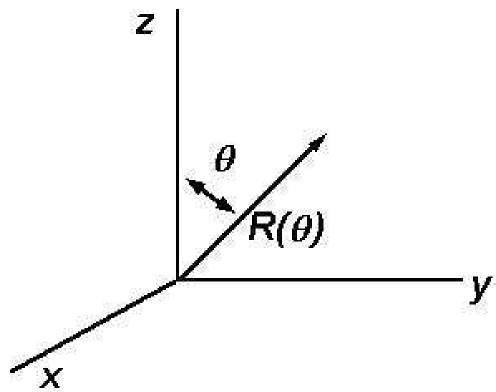
$$\frac{dP}{d\Omega}(2,0) = \frac{Z_0}{2k^2}\left(\frac{1}{30\pi}c^2k^8q^2a^4\right)\left(\frac{15}{8\pi}\sin^2\theta\cos^2\theta\right) = \frac{1}{32\pi^2}Z_0k^6c^2q^2a^4\sin^2\theta\cos^2\theta$$

$$P(2,0) = \frac{2\pi}{32\pi^2}Z_0k^6c^2q^2a^4\int_{-1}^{1}(1-x^2)x^2dx = \frac{2\pi}{32\pi^2}Z_0k^6c^2q^2a^4 \times \frac{4}{15}$$

$$P(2,0) = \frac{1}{60\pi}Z_0k^6c^2q^2a^4$$

9.12

The system is described by



and is azimuthally symmetric

$$R(\theta) = R_0[1+\beta(t)P_2(\cos\theta)]; \quad \beta(t) = \beta_0\cos\omega t; \quad kR << 1$$

$$Q = \int \rho r^2 dr d\phi d\cos\theta = 2\pi\int_{-1}^{1} d\cos\theta\rho\int_0^{R(\theta)} r^2 dr$$

$$= \frac{2\pi}{3}\rho\int_{-1}^{1} R_0^3(1+3\beta P_1 P_2)d\cos\theta + O(\beta^2) = \frac{4\pi}{3}\rho R_0^3 \rightarrow \rho = \frac{3}{4\pi R_0^3}Q$$

where I've used the fact that $1 = P_0$. Since the system is azimuthally symmetric, $Q_{lm} = \delta_{m0}Q_{l0}$.

$$Q_{lm} = 2\pi\rho\delta_{m0}\int_{-1}^{1} dx Y_l^0\int_0^{R(\theta)} r^{l+2}dr = \frac{2\pi\rho\delta_{m0}}{l+3}\int_{-1}^{1} dx R_0^{l+3}[1+(l+3)\beta P_2]Y_l^0$$

Using $Y_l^0 = \sqrt{\frac{2l+1}{4\pi}}P_l$ and $1 = P_0$,

$$Q_{lm} = \frac{2\pi\rho\delta_{m0}}{l+3}\sqrt{\frac{2l+1}{4\pi}}R_0^{l+3}\left[2\delta_{l0} + (l+3)\beta\frac{2}{2l+1}\delta_{l2}\right]$$

Notice that the $l = 0$ term is time independent and thus does not contribute to the radiation. Next consider the $l = 2$ term.

$$Q_{20}(t) = \frac{2}{5}\sqrt{\pi}\,\rho\sqrt{5}\,R_0^5\beta = \rho = \frac{3}{4\pi R_0^3}Q\frac{2}{5}\sqrt{\pi}\,\rho\sqrt{5}\,R_0^5\beta = \frac{3}{\sqrt{20\pi}}R_0^2 Q\beta(t)$$

$$Q_{20}(t) = \mathrm{Re}\left[\frac{3}{\sqrt{20\pi}}R_0^2 Q\beta_0 e^{-i\omega t}\right]$$

$$Q_{20} = \frac{3}{\sqrt{20\pi}} R_0^2 Q \beta_0$$

$$\frac{dP(2,0)}{d\Omega} = \frac{Z_0}{2k^2} |a(2,0)|^2 \left| \vec{X}_{20} \right|^2$$

$$a_E(2,0) = \frac{ck^4}{i(5 \times 3)} \sqrt{\frac{3}{2}} Q_{20}$$

$$\left| \vec{X}_{20} \right|^2 = \frac{15}{8\pi} \sin^2\theta \cos^2\theta$$

$$\frac{dP(2,0)}{d\Omega} = \frac{Z_0}{2k^2} \left| \frac{ck^4}{(5 \times 3)} \sqrt{\frac{3}{2}} Q_{20} \right|^2 \times \frac{15}{8\pi} \sin^2\theta \cos^2\theta$$

$$= \frac{1}{160} \frac{Z_0}{k^2} \frac{|ck^4 Q_{20}|^2}{\pi} \sin^2\theta \cos^2\theta = \frac{1}{160} \frac{Z_0}{k^2} \frac{(ck^4)^2 \left(\frac{3}{\sqrt{20\pi}} R_0^2 Q \beta_0 \right)^2}{\pi} \sin^2\theta \cos^2\theta$$

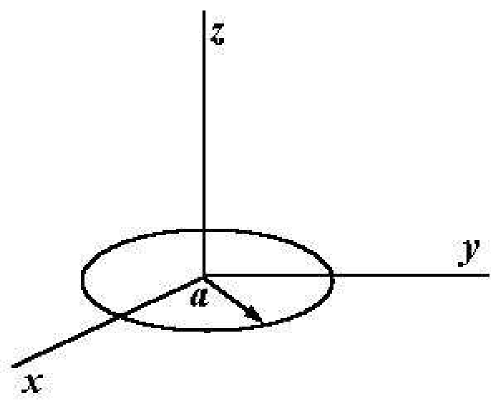
$$= \frac{9}{3200\pi^2} Z_0 k^6 c^2 R_0^4 Q^2 \beta_0^2 \sin^2\theta \cos^2\theta$$

$$P = \frac{9}{3200\pi^2} Z_0 k^6 c^2 R_0^4 Q^2 \beta_0^2 \times 2\pi \int_{-1}^{1} (1 - x^2) x^2 dx = \frac{9}{3200\pi^2} Z_0 k^6 c^2 R_0^4 Q^2 \beta_0^2 \times 2\pi \times \frac{4}{15}$$

$$P = \frac{3}{2000\pi} Z_0 k^6 c^2 R_0^4 Q^2 \beta_0^2$$

9.14

The system is described by



$$I(t) = I_0 \cos \omega t = \mathrm{Re}[I_0 e^{-i\omega t}]$$

$$\vec{J}(t) = \frac{1}{a} I(t) \delta(r-a)\delta(\cos\theta)\hat{\phi}$$

where I determined the normalization constant $\frac{1}{a}$ by the condition $\int \vec{J} \cdot d\vec{a} = I$

$$\vec{J}(t) = \mathrm{Re}\left[\frac{I_0}{a}\delta(r-a)\delta(\cos\theta)\hat{\phi}e^{-i\omega t}\right] \rightarrow \vec{J} = \frac{I_0}{a}\delta(r-a)\delta(\cos\theta)\hat{\phi}$$

We use the general expression for $\vec{H}$ and $\vec{E}$ in the radiation zone given by Eq.(9.149). Since this system has no net charge density and there is no intrisic magnetization, the the expansion coefficients in these equations are given by

$$a_E(l,m) = \frac{k^2}{i\sqrt{l(l+1)}} \int Y_l^{m*} ik\left(\vec{r} \cdot \vec{J}\right) j_l(kr) d^3x$$

$$a_M(l,m) = \frac{k^2}{i\sqrt{l(l+1)}} \int Y_l^{m*} \vec{\nabla} \cdot \left(\vec{r} \times \vec{J}\right) j_l(kr) d^3x$$

a) $\vec{r} \cdot \vec{J} = 0$ in the first equation, so there is no electric multipole radiation. In spherical coordinates

$$\vec{r} \times \vec{J} = -aJ\hat{\theta}$$

Using the formulas for $\vec{\nabla} \cdot \vec{A}$ in spherical coordinates given in the back of the book,

$$\vec{\nabla} \cdot \left(\vec{r} \times \vec{J}\right) = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(J\sin\theta J) = -\frac{\cos\theta}{\sin\theta}J - \frac{\partial}{\partial\theta}J$$

The first term does not contribute, because $\cos\theta = 0,$ while the second term can be written, using the chain rule,

$$\vec{\nabla}\cdot\left(\vec{r}\times\vec{J}\right) = \sin\theta\frac{\partial}{\partial\cos\theta}J$$

The problem has azimuthal symmetry, so $m = 0$. Realizing derivatives of δ –functions are defined by integration by parts,

$$a_M(l,m) = \frac{\delta_{m0}k^2}{i\sqrt{l(l+1)}}\int Y_l^{0*}\left(\sin\theta\frac{\partial}{\partial\cos\theta}J\right)j_l(kr)d^3x = \frac{ik^2}{\sqrt{l(l+1)}}\int\left[\frac{\partial}{\partial\cos\theta}(\sin\theta Y_l^{0*})\right]Jj_l(kr)d^3x$$

$$a_M(l,0) = \frac{i2\pi k^2}{\sqrt{l(l+1)}}\frac{I_0}{a}a^2 j_l(ka)\frac{\partial}{\partial\cos\theta}(\sin\theta Y_l^0)|_{\cos\theta=0}$$

$$a_M(l,0) = \frac{i2\pi k^2}{\sqrt{l(l+1)}}\frac{I_0}{a}a^2 j_l(ka)(1-x^2)^{1/2}\frac{d}{dx}Y_l^0(x)|_{x=0}$$

Since $Y_l^0(x)$ is either an even or odd polynomial in $x,$ then only odd l contribute to $a_M(l,0)$. This determines the expansion coefficients, and thus $\vec{H}$ and $\vec{E}$ in the radiation zone are known through Eq.(9.149). The power distribution is given by Eq. (9.151)

b) From our previous answers, we see $a_E(l,m) = 0,$ and that the lowest magnetic multipole contribution is $a_M(1,0)$.

$$a_M(1,0) = \frac{i2\pi k^2}{\sqrt{2}}I_0 a j_1(ka)(1-x^2)^{1/2}\frac{d}{dx}Y_1^0(x)|_{x=0}$$

Using

$$j_1(ka)\to\frac{ka}{3};\quad \frac{d}{dx}Y_1^0(x)|_{x=0} = \sqrt{\frac{3}{4\pi}}$$

$$a_M(1,0) = i2\pi k^3 I_0 a^2\sqrt{\frac{1}{24\pi}} = \frac{ik^3}{3}\sqrt{2}M_{l0}$$

$$M_{l0} = \frac{i2\pi k^3 I_0 a^2\sqrt{\frac{1}{24\pi}}}{\frac{ik^3}{3}\sqrt{2}} = \sqrt{\frac{3}{4\pi}}I_0\pi a^2$$

Note that you would get the same answer, if you used Eq. (9.172) directly.
From Eq. (9.151)

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left(2\pi k^3 Fa^2\sqrt{\frac{1}{24\pi}}\right)^2\frac{3}{8\pi}\sin^2\theta = \frac{1}{32\pi^2}Z_0k^4(I_0\pi a^2)^2\sin^2\theta$$

If we compare this result with the one that we get for an elementary magnetic dipole, which is given by Eq. (9.23)
with the substitution $\vec{p}\to\vec{m}/c,$

$$\frac{dP}{d\Omega} = \frac{1}{32\pi^2} Z_0 k^4 |\vec{m}|^2 \sin^2\theta$$

Thus we may identify

$$|\vec{m}| = I_0 \pi a^2$$

as would be expected.

More Problems for Chapter 9

Problem 9.1

Note that part (a) is merely the statement that the Fourier decomposition of the multipole moments gives the same result as the multipole expansion of the Fourier decomposed charge distribution, *i.e.*, it does not matter whether you expand first and then decompose or decompose first and then expand. This is obvious since the two integrations commute.

(b) Let T be the period and ω_0 be the characteristic frequency, thus:

$$\rho(\vec{r},t) = \rho(\vec{r},t+T)$$

Therefore,

$$\rho(\vec{r},t) = \int_0^T \rho(\vec{r},t')\delta(t-t')dt' = \int_0^T \rho(\vec{r},t')\left\{\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{-in\omega_0(t-t')}\right\}dt'$$

$$= \sum_{n=-\infty}^{\infty} e^{-in\omega_0 t}\frac{1}{T}\int_0^T \rho(\vec{r},t')e^{in\omega_0 t'}dt'$$

$$= \frac{1}{T}\int_0^T \rho(\vec{r},t')dt' + \sum_{n=1}^{\infty}\left\{e^{-in\omega_0 t}\frac{1}{T}\int_0^T \rho(\vec{r},t')e^{in\omega_0 t'}dt' + e^{in\omega_0 t}\frac{1}{T}\int_0^T \rho(\vec{r},t')e^{-in\omega_0 t'}dt'\right\}$$

$$= \rho_0(\vec{r}) + \sum_{n=1}^{\infty} 2Re\left\{\rho_n(\vec{r})e^{-in\omega_0 t}\right\}$$

Here

$$\rho_n(\vec{r}) = \frac{1}{T}\int_0^T \rho(\vec{r},t')e^{in\omega_0 t'}dt'$$

Note that $\rho(\vec{r},t)$ is assumed to be real.

(c) To facilitate the calculation of multipole moments later, we can write the charge distribution in spherical coordinates as

$$\rho(\vec{r},t) = \frac{q}{R^2}\delta(r-R)\delta(\cos\theta)\delta(\phi-\omega_0 t)$$

Therefore

$$\rho_n(\vec{r}) = \frac{1}{T}\int_0^T \rho(\vec{r},t)e^{in\omega_0 t}dt = \frac{q}{R^2 T}\delta(r-R)\delta(\cos\theta)\int_0^T \delta(\phi-\omega_0 t)e^{in\omega_0 t}\frac{d(\omega_0 t)}{\omega_0} = \frac{q}{2\pi R^2}\delta(r-R)\delta(\cos\theta)e^{in\phi}$$

Thus

$$\rho(\vec{r},t) = \rho_0(\vec{r}) + \sum_{n=1}^{\infty} Re\left\{2\rho_n(\vec{r})e^{-in\omega_0 t}\right\} = \frac{q}{2\pi R^2}\delta(r-R)\delta(\cos\theta)\left\{1 + 2\sum_{n=1}^{\infty}\cos(n\phi - n\omega_0 t)\right\}$$

The $\ell = 0, 1$ multiple moments using method (a) (before the Fourier decomposition) are:

$$q_{00} = \int Y_{00}^*\rho(\vec{r},t)d\tau = \frac{1}{\sqrt{4\pi}}\int \delta(\cos\theta)\delta(\phi-\omega_0 t)d\Omega\int \frac{q}{R^2}\delta(r-R)r^2 dr = \frac{q}{\sqrt{4\pi}}$$

$$q_{10} = \int rY_{10}^{*}\rho(\vec{r},t)d\tau = \sqrt{\frac{3}{4\pi}}\int \cos\theta\delta(\cos\theta)\delta(\phi-\omega_0 t)d\Omega \int \frac{q}{R^2}\delta(r-R)r^3 dr = 0$$

$$q_{11} = \int rY_{11}^{*}\rho(\vec{r},t)d\tau = -\sqrt{\frac{3}{8\pi}}\int \sin\theta e^{i\phi}\delta(\cos\theta)\delta(\phi-\omega_0 t)d\Omega \int \frac{q}{R^2}\delta(r-R)r^3 dr = -\sqrt{\frac{3}{8\pi}}qRe^{i\omega_0 t}$$

$$q_{1-1} = \int rY_{1-1}^{*}\rho(\vec{r},t)d\tau = -q_{11}^{*} = \sqrt{\frac{3}{8\pi}}qRe^{-i\omega_0 t}$$

Summarizing the $\ell = 1$ moments, the electric dipole moment is

$$\vec{p} = \sqrt{\frac{4\pi}{3}}\left\{\frac{q_{1-1}-q_{11}}{\sqrt{2}}\hat{x} - i\frac{q_{1-1}+q_{11}}{\sqrt{2}}\hat{y} + q_{10}\hat{z}\right\} = qR\left\{\cos(\omega_0 t)\hat{x} + \sin(\omega_0 t)\hat{y}\right\} = Re\left\{qR(\hat{x}+i\hat{y})e^{-i\omega_0 t}\right\}$$

The multipole moments using method (b) (after the Fourier transformation) are:

$$q_{00} = \int Y_{00}^{*}\rho(\vec{r},t)d\tau = \frac{1}{\sqrt{4\pi}}\int \frac{q}{2\pi R^2}\delta(r-R)r^2 dr\delta(\cos\theta)d(\cos\theta)\left\{1+2\sum_{n=1}^{\infty}\cos(n\phi-n\omega_0 t)\right\}d\phi = \frac{q}{\sqrt{4\pi}}$$

$$\vec{p} = \int \rho(\vec{r},t)\vec{r}d\tau = \frac{q}{2\pi R^2}\int \delta(r-R)\delta(\cos\theta)\left\{1+2\sum_{n=1}^{\infty}\cos(n\phi-n\omega_0 t)\right\}(r\sin\theta\cos\phi\hat{x} + r\sin\theta\sin\phi\hat{y} + r\cos\theta\hat{z})d\tau$$

$$= \frac{qR}{2\pi}\int_0^{2\pi}\left\{1+2\sum_{n=1}^{\infty}\cos(n\phi-n\omega_0 t)\right\}(\cos\phi\hat{x}+\sin\phi\hat{y})d\phi = qR\left\{\cos(\omega_0 t)\hat{x}+\sin(\omega_0 t)\hat{y}\right\} = Re\left\{qR(\hat{x}+i\hat{y})e^{-i\omega_0 t}\right\}$$

Here we have used the following two integrals:

$$\int_0^{2\pi}\left\{1+2\sum_{n=1}^{\infty}\cos(n\phi-n\omega_0 t)\right\}\cos\phi d\phi = \frac{1}{2}\int_0^{2\pi}\sum_{n=1}^{\infty}\left\{e^{in\phi}e^{-in\omega_0 t} + e^{-in\phi}e^{in\omega_0 t}\right\}(e^{i\phi}+e^{-i\phi})d\phi$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}(2\pi)\left\{(\delta_{n,-1}+\delta_{n,1})e^{-in\omega_0 t} + (\delta_{n,1}+\delta_{n,-1})e^{in\omega_0 t}\right\} = (2\pi)\frac{1}{2}(e^{-i\omega_0 t}+e^{i\omega_0 t}) = 2\pi\cos(\omega_0 t)$$

$$\text{Similarly}\quad \int_0^{2\pi}\left\{1+2\sum_{n=1}^{\infty}\cos(n\phi-n\omega_0 t)\right\}\sin\phi d\phi = 2\pi\sin(\omega_0 t)$$

Therefore, the two calculations agree with each other. There are high order multipole moments:

$$q_{\ell m} \sim \int r^{\ell}Y_{\ell m}^{*}(\theta,\phi)\rho(\vec{r},t)d\tau \sim \frac{q}{2\pi R^2}\int \delta(r-R)r^{\ell+2}dr\int Y_{\ell m}^{*}\delta(\cos\theta)\left\{1+2\sum_{n=1}^{\infty}\cos(n\phi-n\omega_0 t)\right\}d\Omega$$

$$\sim qR^{\ell}\int P_{\ell}^{m}(\cos\theta)\delta(\cos\theta)\ d(\cos\theta)\int_0^{2\pi}\left\{1+2\sum_{n=1}^{\infty}\cos(n\phi-n\omega_0 t)\right\}e^{im\phi}d\phi$$

Now note

$$P_{\ell}^{m}(\cos\theta) \sim (\sin\theta)^{|m|}(\cos\theta)^{\ell-|m|} \quad \Rightarrow \quad \int \delta(\cos\theta)P_{\ell}^{m}(\cos\theta)d(\cos\theta) \sim \delta_{\ell,|m|}$$

$$\int_0^{2\pi} \left\{1 + 2\sum_{n=1}^{\infty} \cos(n\phi - n\omega_0 t)\right\} e^{im\phi} d\phi = \int_0^{2\pi} \left\{1 + \sum_{n=1}^{\infty} (e^{i(n+m)\phi} e^{-in\omega_0 t} + e^{i(m-n)\phi} e^{in\omega_0 t})\right\} d\phi = 2\pi e^{im\omega_0 t}$$

Thus

$$q_{\ell m} \sim qR^\ell \delta_{\ell,|m|} e^{im\omega_0 t}$$

Thus multipole moments $q_{\ell m}$ is nonvanishing for $m = \ell$ or $m = -\ell$ with a frequency dependence of $\omega = \ell\omega_0$.

Problem 9.2

Let the rotational axis be the $z-$axis and the coordinate origin at the center of the square, properly choose $t = 0$ such that $\phi = \omega t$ for one of the $+q$ charge. In this case, the charge distribution is given by

$$\rho(\vec{r}, t) = q\delta(z) \left\{ \delta(x - \frac{a}{\sqrt{2}} \cos\omega t)\delta(y - \frac{a}{\sqrt{2}} \sin\omega t) - \delta(x + \frac{a}{\sqrt{2}} \cos\omega t)\delta(y - \frac{a}{\sqrt{2}} \sin\omega t) \right.$$

$$\left. +\delta(x + \frac{a}{\sqrt{2}} \cos\omega t)\delta(y + \frac{a}{\sqrt{2}} \sin\omega t) - \delta(x - \frac{a}{\sqrt{2}} \cos\omega t)\delta(y + \frac{a}{\sqrt{2}} \sin\omega t) \right\}$$

The quadrupole moments

$$Q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\rho(\vec{r}, t)d\tau$$

Now that

$$\int x^2\rho(\vec{r}, t)d\tau = qa^2 \cos(2\omega t), \quad \int y^2\rho(\vec{r}, t)d\tau = -qa^2 \cos(2\omega t), \quad \int z^2\rho(\vec{r}, t)d\tau = 0$$

$$\int xy\rho(\vec{r}, t)d\tau = qa^2 \sin(2\omega t), \quad \int xz\rho(\vec{r}, t)d\tau = 0, \quad \int yz\rho(\vec{r}, t)d\tau = 0$$

Therefore,

$$Q_{11} = \int (2x^2 - y^2 - z^2)\rho d\tau = 3qa^2 \cos(2\omega t) = 3qa^2 Re\left\{e^{-i(2\omega t)}\right\}$$

$$Q_{22} = \int (2y^2 - x^2 - z^2)\rho d\tau = -3qa^2 \cos(2\omega t) = -3qa^2 Re\left\{e^{-i(2\omega t)}\right\}$$

$$Q_{12} = Q_{21} = \int 3xy\rho d\tau = 3qa^2 \sin(2\omega t) = 3qa^2 Re\left\{ie^{-i(2\omega t)}\right\}$$

All other Q_{ij}'s vanish. Therefore, the quadrupole moment tensor is (with the understanding of $e^{-i(2\omega t)}$):

$$Q_{ij} = 3qa^2 \left\{ \begin{array}{ccc} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{array} \right\}$$

Evidently, the frequency of the radiation is 2ω as expected from the periodicity of the charge distribution $\rho(\vec{r}, t+T/2) = \rho(\vec{r}, t)$, here T is the rotation period. Thus $k = 2\omega/c$. The electric dipole moment vanishes for this configuration (two equal dipoles antiparallel to each other). The magnetic dipole moment also vanishes since the rotating square with net zero charge has zero net current flowing. Thus the radiation is dominated by the electric quadrupole. In the long wavelength limit, the radiation magnetic field is given by Eq. (9.44):

$$\vec{H} = -\frac{ick^3}{24\pi} \frac{e^{ikr}}{r} \vec{n} \times \vec{Q}(\vec{n})$$

where $\vec{n} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$ and

$$\vec{Q}(\vec{n}) = \sum_{i=1}^{3}(\hat{x}Q_{1i}n_i + \hat{y}Q_{2i}n_i + \hat{z}Q_{3i}n_i) = 3qa^2\sin\theta(\cos\phi + i\sin\phi)\hat{x} + 3qa^2\sin\theta(i\cos\phi - \sin\phi)\hat{y} = 3qa^2\sin\theta e^{i\phi}(\hat{x} + i\hat{y})$$

Therefore, the magnetic field

$$\vec{H} = -\frac{ick^3qa^2}{8\pi}\frac{e^{ikr}}{r}e^{i\phi}\sin\theta(-i\hat{x}\cos\theta + \hat{y}\cos\theta + \hat{z}i\sin\theta e^{i\phi})$$

and the electric field is given by

$$\vec{E} = \frac{iZ_0}{k}\nabla\times\vec{E} = Z_0\vec{H}\times\vec{n}$$

The angular distribution of the radiation is given by Eq. (9.45):

$$\frac{dP}{d\Omega} = \frac{c^2Z_0}{1152\pi^2}k^6|(\vec{n}\times\vec{Q}(\vec{n}))\times\vec{n}|^2 = \frac{c^2Z_0}{1152\pi^2}k^6\left\{|\vec{Q}(\vec{n})|^2 - |\vec{Q}(\vec{n})\cdot\vec{n}|^2\right\}$$

$$= \frac{c^2Z_0}{1152\pi^2}k^6\left\{(3qa^2\sin\theta)^2\times 2 - (3qa^2\sin\theta)^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi)\right\}$$

$$= \frac{c^2Z_0}{1152\pi^2}k^6(3qa^2)^2\sin^2\theta(2 - \sin^2\theta) = \frac{c^2Z_0k^6}{128\pi^2}(q^2a^4)(1 - \cos^4\theta)$$

$$= \frac{Z_0\omega^6}{2\pi^2c^4}(q^2a^4)(1 - \cos^4\theta)$$

Total power of radiation

$$P = \int d\Omega\frac{dP}{d\Omega} = \frac{Z_0\omega^6}{2\pi^2c^4}(q^2a^4)\int d\Omega(1 - \cos^4\theta) = \frac{8Z_0\omega^6}{5\pi c^4}q^2a^4$$

Problem 9.3

In the long wavelength limit, we can calculate the multipole moments from the static problem and keep only the lowest non-vanishing multipoles. From Eq. (3.36), the corresponding potential outside the shell is

$$\Phi(r,\theta) = V\left\{\frac{3}{2}(\frac{R}{r})^2P_1(\cos\theta) - \frac{7}{8}(\frac{R}{r})^4P_3(\cos\theta) + ...\right\}$$

The potential is dominated by the dipole term. Compared with the potential of an electric dipole p

$$\Phi(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}$$

we infer the dipole moment of the sphere to be:

$$\vec{p} = 6\pi\epsilon_0VR^2\hat{z}$$

Thus, the radiation fields are given by Eq. (9.19):

$$\vec{H} = \frac{ck^2}{4\pi}(\vec{n}\times\vec{p})\frac{e^{ikr}}{r} = -\frac{3}{2}(\frac{\omega R}{c})^2\frac{V}{Z_0}\sin\theta\frac{e^{i(\omega/c)r}}{r}\hat{\phi}$$

$$\vec{E} = Z_0\vec{H}\times\vec{n} = -\frac{3}{2}V(\frac{\omega R}{c})^2\sin\theta\frac{e^{i(\omega/c)r}}{r}\hat{\theta}$$

The radiation power per unit solid angle

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2}k^4|(\vec{n}\times\vec{P})\times\vec{n}|^2 = \frac{c^2 Z_0}{32\pi^2}k^4(|\vec{p}|^2 - |\vec{p}\cdot\vec{n}|^2) = \frac{9}{8}(\frac{\omega R}{c})^4\frac{V^2}{Z_0}\sin^2\theta$$

The total power

$$P = \int \frac{dP}{d\Omega}d\Omega = 3\pi(\frac{\omega R}{c})^4\frac{V^2}{Z_0}$$

Added note

There are charges and currents on the sphere. But the magnetic dipole moment vanishes. From the scalar potential, we can calculate the surface charge distribution

$$\sigma(\theta) = \epsilon_0 E_n|_{r=R} = -\epsilon_0\frac{\partial\Phi}{\partial n}|_{r=R} = \frac{3\epsilon_0 V}{R}\cos\theta$$

Therefore, the surface current density $\vec{K} = K\hat{\theta}$ (by symmetry, the current only flows in the θ direction) can be calculated from the continuity equation:

$$\nabla\cdot\vec{K} + \frac{\partial\sigma}{\partial t} = 0 \quad\Rightarrow\quad \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta K) = i\omega\frac{3\epsilon_0 V}{R}\cos\theta \quad\Rightarrow\quad K = \frac{3i}{2}\epsilon_0\omega V\sin\theta$$

The magnetic dipole moment

$$\vec{m} = \frac{1}{2}\int\vec{r}\times\vec{K}da = \frac{3i}{4}\epsilon_0\omega R^3 V\int_0^{\pi}\sin^2\theta d\theta\int_0^{2\pi}\hat{\phi}d\phi = 0$$

For the charge distribution of a dipole potential, all elements of the electric quadrupole moment tensors are zero.

$$Q_{11} = Q_{22} = \int(3x^2 - R^2)\sigma R^2 d\cos\theta d\phi \sim \int\sin^2\theta\cos\theta d\cos\theta\int\cos^2\phi d\phi = 0 \quad\Rightarrow\quad Q_{33} = 0$$

$$Q_{12} = Q_{21} = \int(3xy)\sigma R^2 d\cos\theta d\phi \sim \int\sin^2\theta\cos\theta d\cos\theta\int\sin\phi\cos\phi d\phi = 0$$

$$Q_{13} = Q_{31} = Q_{23} = Q_{32} = \int(3xz)\sigma R^2 d\cos\theta d\phi \sim \int\sin\theta\cos^2\theta d\cos\theta\int\cos\phi d\phi = 0$$

Problem 9.10

(b) The charge distribution given is not the one in usual sense since the total charge is nonvanishing and oscillating with time, against the teaching that the radiation does not having a monopole term. I guess that is why Jackson called it "transitional charge". Note that in a spherical coordinate system

$$z = r\cos\theta = \sqrt{\frac{4\pi}{3}}rY_{10}$$

Therefore, the charge distribution can be written

$$\rho(\vec{r},t) = \frac{2e}{\sqrt{6}a_0^4}re^{-3r/2a_0}Y_{00}Y_{10}e^{-i\omega_0 t} = \frac{\sqrt{2}e}{4\pi a_0^4}ze^{-3r/2a_0}e^{-i\omega_0 t}$$

Therefore, the electric dipole moment only has non-vanishing contribution in the $z-$ direction.

$$p_z = \int z\rho(\vec{r},t)d\tau = \frac{\sqrt{2}e}{3a_0^4}\int r^4 e^{-3r/2a_0}dr\int d\Omega Y_{10}Y_{10} = \frac{\sqrt{2}e}{3a_0^4}\frac{4!}{(3/2a_0)^5} = \frac{2^8\sqrt{2}}{3^5}ea_0$$

Here we have used the orthogonality condition of the spherical harmonics and

$$\int_0^\infty x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}}$$

The average power is given by Eq. (9.24):

$$P_{\text{quan.}} = \frac{c^2 Z_0 k^4}{12\pi} |\vec{p}|^2 = \frac{2^{15}}{3^{11}\pi} \frac{Z_0 \omega_0^4}{c^2} e^2 a_0^2 = (\frac{2}{3})^8 (\hbar\omega_0)(\frac{\alpha^4 c}{a_0})$$

(c) Let $\mathcal{N}(2p \to 1s)$ be the transition probability (this Jackson's probability is not the probability in usual sense, it is really the number of atoms making the transition from $2p \to 2s$ per unit time to yield the calculated power):

$$P_{\text{quan.}} = \hbar\omega_0 \mathcal{N}(2p \to 1s) \qquad \Rightarrow \qquad \mathcal{N}(2p \to 1s) = \frac{P_{\text{quan.}}}{\hbar\omega_0} = (\frac{2}{3})^8 \frac{\alpha^4 c}{a_0} \approx 6 \cdot 10^8 \text{ s}^{-1}$$

(d) Let the orbit plane be the $x - y$ plane, the electric dipole moment of the electron can be written as

$$\vec{p} = e(x\hat{x} + y\hat{y}) = 2ea_0 \left\{\cos(\omega_0 t)\hat{x} + \sin(\omega_0 t)\hat{y}\right\} = 2ea_0 Re\left\{(\hat{x} + i\hat{y})e^{-i\omega_0 t}\right\}$$

Or in complex form

$$\vec{p} = 2ea_0(\hat{x} + i\hat{y})e^{-i\omega_0 t} \qquad \Rightarrow \qquad |\vec{p}|^2 = 8e^2 a_0^2$$

The average power

$$P_{\text{class.}} = \frac{c^2 Z_0 k^4}{12\pi} |\vec{p}|^2 = \frac{9}{2^6} (\hbar\omega_0)(\frac{\alpha^4 c}{a_0})$$

The ratio of two powers

$$\frac{P_{\text{quan.}}}{P_{\text{class.}}} = \frac{(2/3)^8}{(9/2^6)} \approx 0.28$$

More Problems for Chapter 9

Problem 9.12

Since β is small and the charge distribution is uniform, we can approximate the charge distribution by

$$\rho(t) \approx \frac{Q}{4\pi R^3/3} = \frac{3Q}{4\pi R^3(\theta)} = \frac{3Q}{4\pi R_0^3}\frac{1}{(1+\beta(t)P_2(\cos\theta))^3}$$

where $\beta(t) = \beta_0 \cos\omega t$. Since the problem is spherical symmetric, all mutipole moments with $m \neq 0$ vanish. Therefore, the electric multipole moments (here we have ignored any currents on the sphere) are:

$$q_{\ell,0} = \int r^\ell Y^*_{\ell,0}(\theta,\phi)\rho(t)d\tau = \frac{3Q}{4\pi R_0^3}\int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \sqrt{\frac{2\ell+1}{4\pi}}\frac{P_\ell(\cos\theta)}{(1+\beta(t)P_2(\cos\theta))^3}\int_0^{R(\theta)} r^\ell r^2 dr$$

$$= \frac{3QR_0^\ell}{2(\ell+3)}\sqrt{\frac{2\ell+1}{4\pi}}\int_0^\pi P_\ell(\theta)(1+\beta(t)P_2(\theta))^\ell d(\cos\theta)$$

$$\approx \frac{3QR_0^\ell}{2(\ell+3)}\sqrt{\frac{2\ell+1}{4\pi}}\int_{-1}^{+1} P_\ell(x)\{1+\ell\beta(t)P_2(x)\}dx$$

$$= \frac{3QR_0^\ell}{2(\ell+3)}\sqrt{\frac{2\ell+1}{4\pi}}\frac{2}{2\ell+1}(\delta_{\ell,0}+\ell\beta(t)\delta_{\ell,2})$$

Thus, the only time-varying non-vanishing moment is the electric dipole moment ($\ell = 2$):

$$q_{2,0} = \frac{3}{5\sqrt{5\pi}}QR_0^2\beta(t) = \frac{3}{5\sqrt{5\pi}}QR_0^2 Re\{\beta_0 e^{-i\omega t}\}$$

For the long wavelength approximation,

$$a_E(\ell,m) \approx \frac{ck^{\ell+2}}{i(2\ell+1)!!}\sqrt{\frac{\ell+1}{\ell}}q_{\ell m} \quad \Rightarrow \quad a_E(2,0) = -i\frac{1}{25}\sqrt{\frac{3}{10\pi}}QR_0^2ck^4\beta_0$$

The angular distribution of the radiation

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}|a_E(2,0)|^2|\vec{X}_{2,0}|^2 = \frac{Z_0}{2k^2}\left\{\frac{3}{5^4\cdot 10\pi}Q^2R_0^4c^2k^8\beta_0^2\right\}\left\{\frac{15}{8\pi}\sin^2\theta\cos^2\theta\right\} = \frac{9c^2Z_0}{2\cdot 10^4\pi^2}Q^2R_0^4\beta_0^2k^6\sin^2\theta\cos^2\theta$$

The total power

$$P = \int d\Omega\frac{dP}{d\Omega} = \frac{9c^2Z_0}{2\cdot 10^4\pi^2}Q^2R_0^4\beta_0^2k^6\int\sin^2\theta\cos^2\theta d\Omega = \frac{3c^2Z_0}{12500\pi}Q^2R_0^4\beta_0^2k^6$$

Problem 9.16

Let the $z-$axis along the antenna so that the antenna spans between $-d/2 < z < d/2$. Therefore, the current density

$$\vec{J}(\vec{r}) = \hat{z}I\sin(kz)\delta(x)\delta(y) \quad \text{for} \quad |z| \leq \frac{d}{2}$$

where $kd = 2\pi$. The vector potential from Eq. (9.8):

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int e^{-ik\vec{n}\cdot\vec{r'}}\vec{J}(\vec{r'})d\tau' = \hat{z}\frac{\mu_0 I}{4\pi}\frac{e^{ikr}}{r}\int_{-d/2}^{d/2}\sin(kz')e^{-ikz'\cos\theta}dz' = \hat{z}\frac{\mu_0 I}{2\pi i}\frac{e^{ikr}}{kr}\frac{\sin(\pi\cos\theta)}{\sin^2\theta}$$

Here we have used the following integral

$$\int_{-\pi}^{\pi} \sin z e^{-iaz} dz = \frac{2i\sin(\pi a)}{a^2 - 1}$$

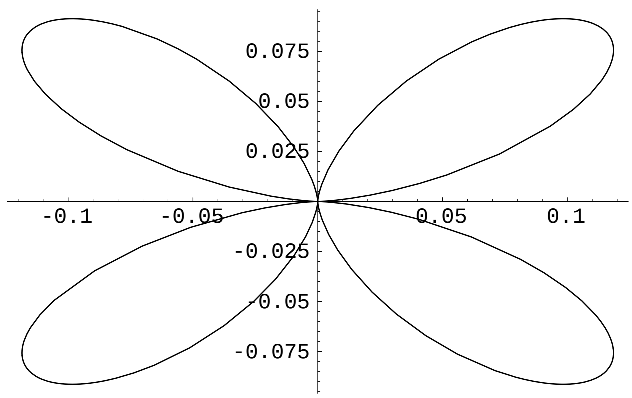
The magnetic field is given by

$$\vec{H} = \frac{1}{\mu_0}\nabla \times \vec{A} \approx \frac{ik}{\mu_0}\vec{n} \times \vec{A} = -\frac{I}{2\pi}\frac{e^{ikr}}{r}\frac{\sin(\pi\cos\theta)}{\sin\theta}\hat{\phi}$$

(a) The angular distribution of radiated power is

$$\frac{dP}{d\Omega} = \frac{r^2}{2}\vec{n}\cdot(\vec{E}\times\vec{H}^*) = \frac{Z_0 I^2}{8\pi^2}\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta} = \frac{Z_0 I^2}{8}\left\{\frac{\sin^2(\pi\cos\theta)}{\pi^2\sin^2\theta}\right\}$$

which is plotted below (in the unit of $Z_0 I^2/8$). The $\theta = 0$ direction is vertically up.



(b) The total power of radiation

$$P = \frac{Z_0 I^2}{8\pi^2}\int \frac{\sin^2(\pi\sin\theta)}{\sin^2\theta}d\Omega = \frac{Z_0 I^2}{4\pi}\int_{-1}^{+1}\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}d(\cos\theta) \approx \frac{1.56}{4\pi}Z_0 I^2$$

Now note

$$P = \frac{1}{2}I^2 R_{\text{rad}} \quad \Rightarrow \quad R_{\text{rad}} = \frac{2P}{I^2} = \frac{1.56}{2\pi}Z_0 = \frac{1.56}{2\pi}\cdot 377 = 93.6 \text{ Ohms}$$

Problem 9.17

The charge distribution can be calculated from the continuity equation:

$$\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0, \quad \Rightarrow \quad \rho(\vec{r}) = \frac{1}{i\omega}\nabla\cdot\vec{J} = -i\frac{I}{c}\cos(kz)\delta(x)\delta(y) \;\; \text{for} \;\; |z| \le d/2$$

(a) Exact calculations:
The dipole moment

$$\vec{p} = \int \vec{r}\rho(\vec{r})d\tau = -i\frac{I}{c}\int(x\hat{x} + y\hat{y} + z\hat{z})\delta(x)\delta(y)\cos(kz)dxdydz = -i\frac{I}{c}\int_{-d/2}^{d/2} z\cos(kz)dz(\hat{z}) = 0$$

The magnetic dipole moment

$$\vec{m} = \frac{1}{2}\int \vec{r}\times\vec{J}d\tau = \int I(x\hat{x} + y\hat{y} + z\hat{z})\times\hat{z}\delta(x)\delta(y)\sin(kz)\; dxdydz = 0$$

The only non-vanishing quadrupole moments are Q_{11}, Q_{22} and Q_{33}.

$$Q_{33} = \int(3z^2 - r^2)\rho(\vec{r})d\tau = -2i\frac{I}{c}\int_{-d/2}^{d/2} z^2\cos(kz)dz = i\frac{Id^3}{\pi^2 c}, \quad Q_{11} = Q_{22} = -\frac{1}{2}Q_{33}$$

Long wavelength limit:
In the long wavelength limit, we have

$$\vec{J}(\vec{r}) = I\sin(kz)\delta(x)\delta(y)\ \hat{z} \approx kIz\delta(x)\delta(y)\ \hat{z}; \qquad \rho(\vec{r}) = -i\frac{I}{c}\cos(kz)\delta(x)\delta(y) \approx -i\frac{I}{c}\delta(x)\delta(y)$$

The dipole moment

$$\vec{p} = \int \vec{r}\rho(\vec{r})d\tau = -i\frac{I}{c}\int \vec{r}\delta(x)\delta(y)d\tau = -i\frac{I}{c}\int_{-d/2}^{d/2} z\ dz\ \hat{z} = 0$$

The magmetic dipole moment

$$\vec{m} = \frac{1}{2}\int \vec{r}\times\vec{J}d\tau = \frac{1}{2}kI\int (x\hat{x}+y\hat{y}+z\hat{z})\times\hat{z}\ \delta(x)\delta(y)\ dxdydz = 0$$

The electric quadrupole moment

$$Q_{33} = \int (3z^2 - r^2)\rho(\vec{r})d\tau = -2i\frac{I}{c}\int_{-d/2}^{d/2} z^2 dz = -i\frac{Id^3}{6c}, \quad Q_{11} = Q_{22} = -\frac{1}{2}Q_{33}$$

Not surprising, the exact calculation and the long wavelength approximation yield very different values for the electric quadrupole moment tensor in this case. With $kd = 2\pi$, the approximation does not work.
(b) The angular power of radiation (of the exact calculation of the quadrupole moments):

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{1152\pi^2}\left\{|\vec{Q}(\vec{n})|^2 - |\vec{n}\cdot\vec{Q}(\vec{n})|^2\right\}$$

where

$$\vec{Q}(\vec{n}) = \sum_{i=1}^{3}\{Q_{1i}\hat{x} + Q_{2i}\hat{y} + Q_{3i}\hat{z}\}\, n_i = Q_{11}n_1\hat{x} + Q_{22}n_2\hat{y} + Q_{33}n_3\hat{z} = -\frac{1}{2}Q_{33}\{\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} - 2\cos\theta\hat{z}\}$$

Thus

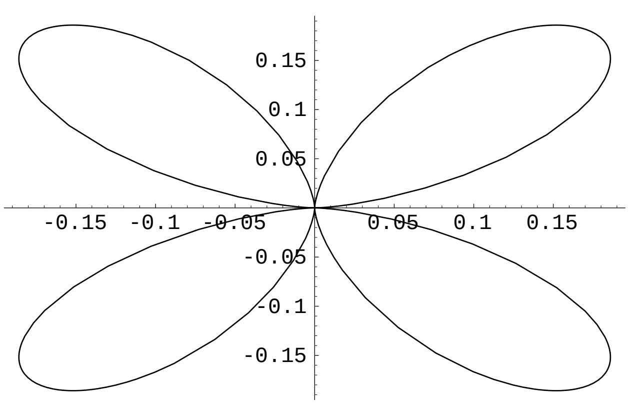
$$|\vec{Q}(\vec{n})|^2 = \frac{1}{4}|Q_{33}|^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + 4\cos^2\theta) = \frac{1}{4}|Q_{33}|^2(1+3\cos^2\theta)$$

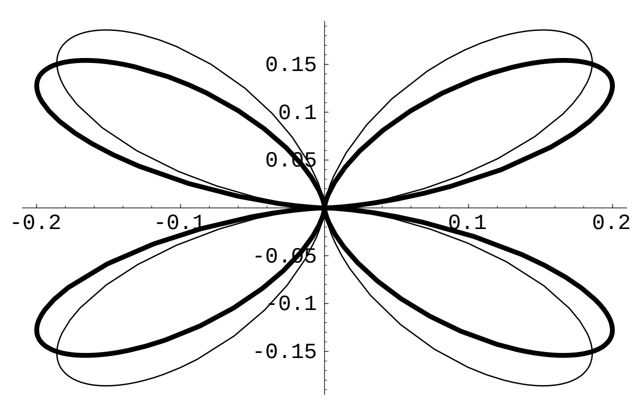
$$|\vec{n}\cdot\vec{Q}(\vec{n})|^2 = \frac{1}{4}|Q_{33}|^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi - 2\cos^2\theta)^2 = \frac{1}{4}|Q_{33}|^2(1-3\cos^2\theta)^2$$

Therefore,

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{1152\pi^2}\left\{\frac{1}{4}|Q_{33}|^2\right\}\left\{(1+3\cos^2\theta) - (1-3\cos^2\theta)^2\right\} = \frac{1}{8}Z_0 I^2\sin^2\theta\cos^2\theta$$

The left plot below shows graphically the angular distribution (in the unit of $Z_0 I^2/8$) of the quadrupole radiation. The right plot compares the shape of this distribution (thin line) with that (thick line) of the exact calculation scaled up by a factor of $157.9/93.6 = 1.69$ (see the discussion below). Evidently, apart from an overestimation of the radiation power, the angular distribution of the quadrupole agrees reasonablly well with that of the exact calculation.





(c) The total power of the exact calculation

$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{1}{8} Z_0 I^2 \int \sin^2\theta \cos^2\theta d\Omega = \frac{\pi}{15} Z_0 I^2$$

The corresponding radiation resistance:

$$R_{\text{rad}} = \frac{2\pi}{15} Z_0 = \frac{2\pi}{15} \cdot 377 = 157.9 \text{ Ohms}$$

The total power of the long wavelength approximation:

$$P_{\text{LW}} = \frac{c^2 Z_0 k^6}{1440\pi} \sum_{ij} |Q_{ij}|^2 = \frac{c^2 Z_0 k^6}{1440\pi} \cdot \frac{3}{2} |Q_{33}|^2 = \frac{\pi^5}{540} I^2 Z_0$$

The corresponding radiation resistance

$$R_{\text{rad}}^{\text{LW}} = \frac{\pi^5}{270} Z_0 = 427.3 \text{ Ohms}$$

Obviously the long wavelength approximation does not work in this case. In Problem 9.16, we have $R_{\text{rad}} = 93.4$ Ohms from the exact calculation without expansion. There is a puzzle here that the radiation by the electric dipole mode is greater than the sum of all modes. This is because the leading term in the expansion (electric $\ell = 2$ term or E2) is not a good approximation whenever the dimensions of the source are comparable to or larger than a wavelength.

Problem 9.22 (*Only TE modes are worked out*)
The general solutions to the Maxwell equations are given by Eq. (9.122):

$$\vec{H} = \sum_{\ell,m} \left\{ a_E(\ell,m) f_\ell(kr) \vec{X}_{\ell m} - \frac{i}{k} a_M(\ell,m) \nabla \times g_\ell(kr) \vec{X}_{\ell m} \right\}$$

$$\vec{E} = Z_0 \sum_{\ell,m} \left\{ \frac{i}{k} a_E(\ell,m) \nabla \times f_\ell(kr) \vec{X}_{\ell m} + a_M(\ell,m) g_\ell(kr) \vec{X}_{\ell m} \right\}$$

where $a_E(\ell,m)$ and $a_M(\ell,m)$ are the electric and magnetic multipoles respectively. $f_\ell(kr)$ and $g_\ell(kr)$ are linear combinations of spherical Bessel functions $j_\ell(kr)$ and $n_\ell(kr)$. Furthermore, the fields must be finite at $r = 0$. Thus, we have $f_\ell(kr) = j_\ell(kr)$ and $g_\ell(kr) = j_\ell(kr)$.

The fields of the TE modes are given by Eq. (9.116):

$$\vec{E}_{\ell m} = Z_0 j_\ell(kr) \vec{L} Y_{\ell m}(\theta,\phi); \qquad \vec{H}_{\ell m} = -\frac{i}{kZ_0} \nabla \times \vec{E}_{\ell m} \quad (\ell \neq 0)$$

The $\ell = 0$ case leads to null fields everywhere inside the cavity. The corresponding components are

$$\begin{aligned} (E_{\ell m})_r &= 0 & (H_{\ell m})_r &= \tfrac{1}{kr}\, \ell(\ell+1) Y_{\ell m}\, j_\ell(kr) \\ (E_{\ell m})_\theta &= i Z_0 \tfrac{1}{\sin\theta} \tfrac{\partial}{\partial\phi} Y_{\ell m}\, j_\ell(kr) & (H_{\ell m})_\theta &= \tfrac{1}{kr} \tfrac{\partial}{\partial\theta} Y_{\ell m}\, \tfrac{\partial}{\partial r}(r j_\ell(kr)) \\ (E_{\ell m})_\phi &= -i Z_0 \tfrac{\partial}{\partial\theta} Y_{\ell m}\, j_\ell(kr) & (H_{\ell m})_\phi &= \tfrac{1}{kr} \tfrac{1}{\sin\theta} \tfrac{\partial}{\partial\phi} Y_{\ell m}\, \tfrac{\partial}{\partial r}(r j_\ell(kr)) \end{aligned}$$

Similarly the TM fields are given by Eq. (9.118):

$$\vec{H}_{\ell m} = j_\ell(kr) \vec{L} Y_{\ell m}(\theta,\phi); \qquad \vec{E}_{\ell m} = \frac{iZ_0}{k} \nabla \times \vec{H}_{\ell m} \quad (\ell \neq 0)$$

with the following components

$$\begin{aligned} (H_{\ell m})_r &= 0 & (E_{\ell m})_r &= -\tfrac{Z_0}{kr}\, \ell(\ell+1) Y_{\ell m}\, j_\ell(kr) \\ (H_{\ell m})_\theta &= i \tfrac{1}{\sin\theta} \tfrac{\partial}{\partial\phi} Y_{\ell m}\, j_\ell(kr) & (E_{\ell m})_\theta &= -\tfrac{Z_0}{kr} \tfrac{\partial}{\partial\theta} Y_{\ell m}\, \tfrac{\partial}{\partial r}(r j_\ell(kr)) \\ (H_{\ell m})_\phi &= -i \tfrac{\partial}{\partial\theta} Y_{\ell m}\, j_\ell(kr) & (E_{\ell m})_\phi &= -\tfrac{Z_0}{kr} \tfrac{1}{\sin\theta} \tfrac{\partial}{\partial\phi} Y_{\ell m}\, \tfrac{\partial}{\partial r}(r j_\ell(kr)) \end{aligned}$$

(a) At $r = a$, the electric field must be perpendicular to the conducting surface and there must be no normal component of the magnetic field. Applying this boundary condition to the TE modes leads to $j_\ell(ka) = 0$. Let $x_{\ell n}$ be the n^{th} root of $j_\ell(x)$, the characteristic frequencies $\omega_{\ell n}^{TE}$ are therefore given by

$$\frac{\omega_{\ell n}^{TE}}{c} a = x_{\ell n} \qquad \Rightarrow \qquad \omega_{\ell n}^{TE} = \frac{c}{a} x_{\ell n}$$

For the TM modes, we have

$$\frac{\partial}{\partial r}\{r j_\ell(kr)\}|_{r=a} = 0$$

Let $y_{\ell n}$ be the n^{th} root of $\partial(x j_\ell(x))/\partial x$, the characteristic frequencies are then given by

$$\frac{\omega_{\ell n}^{TM}}{c} a = y_{\ell n} \qquad \Rightarrow \qquad \omega_{\ell n}^{TM} = \frac{c}{a} y_{\ell n}$$

In both cases, the characteristic frequencies are independent of m (degenerate in m), as result of the $\phi-$symmetry. We proceed with TE modes only.
(b) The four lowest roots of $j_\ell(x) = 0$ $(\ell \neq 0)$ are $x_{1,1} = 4.5, x_{2,1} = 5.8, x_{3,1} = 6.85$ and $x_{1,2} = 7.64$. Thus the corresponding wavelengths

$$\lambda_{\ell n} = \frac{2\pi}{x_{\ell n}} a \quad \Rightarrow \lambda_{1,1} = 1.4a,\ \lambda_{2,1} = 1.1a,\ \lambda_{3,1} = 0.9a,\ \lambda_{1,2} = 0.8a$$

(c) The lowest TE mode corresponds to $\ell = 1, n = 1$, independent of m. In this case, $k_{1,1} = \frac{2\pi}{\lambda_{1,1}} = 4.5a$. The fields for $m = 0$ are (apart from normalization constants):

$$(E_{1,0})_r = 0 \qquad (H_{1,0})_r = \sqrt{\tfrac{3}{\pi}}\frac{1}{(4.5r/a)} j_1(4.5r/a)\cos\theta$$

$$(E_{1,0})_\theta = 0 \qquad (H_{1,0})_\theta = -\sqrt{\tfrac{3}{4\pi}}\frac{1}{(4.5r/a)}\frac{\partial}{\partial r}(r j_1(4.5r/a))\sin\theta$$

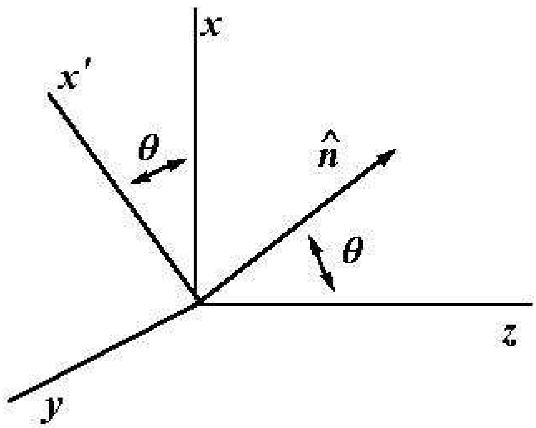
$$(E_{1,0})_\phi = i\sqrt{\tfrac{3}{4\pi}} Z_0 j_1(4.5r/a)\sin\theta \qquad (H_{1,0})_\phi = 0$$

10.1

a) Let us first simplify the expression we want to get for the cross section. Using $\hat{n}_0 = \hat{z}$,

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\hat{\varepsilon}_0 \cdot \hat{n}|^2 - \frac{1}{4} |\hat{n} \cdot (\hat{z} \times \hat{\varepsilon}_0)|^2 - \hat{z} \cdot \hat{n} \right]$$

Orienting the system as



and using

$$\hat{\varepsilon}_0 = \alpha_0 \hat{x} + \beta_0 \quad , \quad \text{with } |\alpha_0|^2 + |\beta_0|^2 = 1$$

$$\hat{n} = \cos\theta \hat{z} + \sin\theta \hat{x}$$

then

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\alpha_0|^2 \sin^2\theta - \frac{1}{4} |\beta_0|^2 \sin^2\theta - \cos\theta \right]$$

Using the result for the perfectly conducting sphere Eq. (10.14)

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}, \hat{\varepsilon}_0, \hat{n}_0) = k^4 a^6 \left| \hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 - \frac{1}{2} (\hat{z} \times \hat{\varepsilon}_0) \cdot (\hat{n} \times \hat{\varepsilon}^*) \right|^2$$

Using $\hat{\varepsilon}_\perp =$

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_\perp, \hat{\varepsilon}_0, \hat{n}_0) = k^4 a^6 \left| \beta_0 - \frac{1}{2} \beta_0 \cos\theta \right|^2 = k^4 a^6 |\beta_0|^2 \left(1 - \frac{1}{2} \cos\theta \right)^2$$

Similarly $\hat{\varepsilon}_\parallel = \hat{x}'$

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_\parallel, \hat{\varepsilon}_0, \hat{n}_0) = k^4 a^6 \left| \alpha_0 \cos\theta - \frac{1}{2} \alpha_0 \right|^2 = k^4 a^6 |\alpha_0|^2 \left(\cos\theta - \frac{1}{2} \right)^2$$

By definition

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = \frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_\perp, \hat{\varepsilon}_0, \hat{n}_0) + \frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_\parallel, \hat{\varepsilon}_0, \hat{n}_0)$$

$$= k^4 a^6 \left[|\beta_0|^2 \left(1 - \frac{1}{2}\cos\theta\right)^2 + |\alpha_0|^2 \left(\cos\theta - \frac{1}{2}\right)^2 \right]$$

which simplifies to

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\alpha_0|^2 \sin^2\theta - \frac{1}{4}|\beta_0|^2 \sin^2\theta - \cos\theta \right]$$

using $|\alpha_0|^2 + |\beta_0|^2 = 1$, and $\cos^2\theta = 1 - \sin^2\theta$.

b) If $\hat{\varepsilon}_0$ is linearly polarized making an angle ϕ with respect to the x axis, then

$$\hat{\varepsilon}_0 = \alpha_0 \hat{x} + \beta_0 \quad = \cos\phi \hat{x} + \sin\phi \quad , \text{ so } \alpha_0 = \cos\phi, \ \beta_0 = \sin\phi$$

Then from part a)

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\alpha_0|^2 \sin^2\theta - \frac{1}{4}|\beta_0|^2 \sin^2\theta - \cos\theta \right]$$

$$= k^4 a^6 \left[\frac{5}{4} - \cos^2\phi \sin^2\theta - \frac{1}{4}\sin^2\phi \sin^2\theta - \cos\theta \right]$$

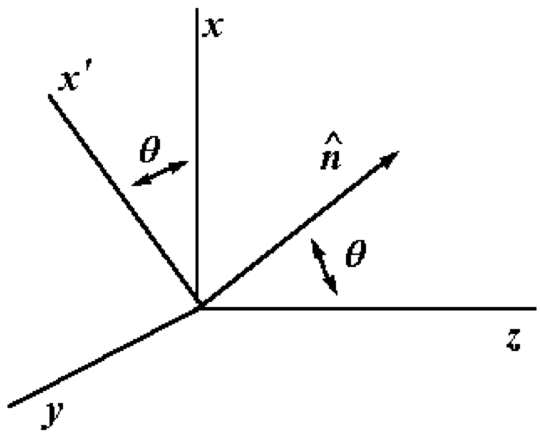
Using $\cos 2\phi = \cos^2\phi - \sin^2\phi$, this expression simplifies to

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{8}(1 + \cos^2\theta) - \frac{3}{8}\sin^2 \cos 2\phi - \cos\theta \right]$$

as desired.

10.2

Orienting the system as



Then

$$\hat{n} = \cos\theta\hat{z} + \sin\theta\hat{x}$$

Using the result for the perfectly conducting sphere Eq. (10.14) and writing $\hat{\varepsilon}_0 = \alpha_0\hat{x} + \beta_0\hat{y}$. where $|\alpha_0|^2 + |\beta_0|^2 = 1$,

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}, \hat{\varepsilon}_0, \hat{n}_0) = k^4a^6 \left|\hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 - \frac{1}{2}(\hat{z} \times \hat{\varepsilon}_0) \cdot (\hat{n} \times \hat{\varepsilon}^*)\right|^2$$

Using $\hat{\varepsilon}_\perp = \hat{y}$

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_\perp, \hat{\varepsilon}_0, \hat{n}_0) = k^4a^6 \left|\beta_0 - \frac{1}{2}\beta_0\cos\theta\right|^2 = k^4a^6 |\beta_0|^2 \left(1 - \frac{1}{2}\cos\theta\right)^2$$

Similarly $\hat{\varepsilon}_{||} = \hat{x}'$

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_{||}, \hat{\varepsilon}_0, \hat{n}_0) = k^4a^6 \left|\alpha_0\cos\theta - \frac{1}{2}\alpha_0\right|^2 = k^4a^6 |\alpha_0|^2 \left(\cos\theta - \frac{1}{2}\right)^2$$

By definition

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = \frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_\perp, \hat{\varepsilon}_0, \hat{n}_0) + \frac{d\sigma}{d\Omega}(\hat{n}, \hat{\varepsilon}_{||}, \hat{\varepsilon}_0, \hat{n}_0)$$

$$= k^4a^6 \left[|\beta_0|^2 \left(1 - \frac{1}{2}\cos\theta\right)^2 + |\alpha_0|^2 \left(\cos\theta - \frac{1}{2}\right)^2\right]$$

which simplifies to

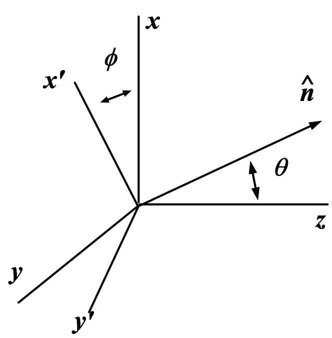
$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\alpha_0|^2 \sin^2\theta - \frac{1}{4}|\beta_0|^2 \sin^2\theta - \cos\theta\right]$$

using $|\alpha_0|^2 + |\beta_0|^2 = 1$, and $\cos^2\theta = 1 - \sin^2\theta$.

b) If $\hat{\varepsilon}_0$ is a linear combination of circular polarizations

$$\hat{\varepsilon}_0 = \frac{1}{\sqrt{1+r^2}\sqrt{2}}\left[\hat{x}' + i\hat{y}' + re^{i\alpha}(\hat{x}' - i\hat{y}')\right]$$

As is stated in problem 10.1, ϕ is measured with respect to the $\hat{x}'$ axis. For orientation, see the figure:



In term of the unit vectors in the x and y directions, respectively

$$\hat{x}' = \cos\phi\hat{x} + \sin\phi\hat{y}; \ \hat{y}' = \cos\phi\hat{y} - \sin\phi\hat{x}$$

corresponding to a rotation about the z axis of ϕ.Thus

$$\alpha_0 = \frac{1}{\sqrt{1+r^2}\sqrt{2}}\left[\cos\phi\left(1+re^{i\alpha}\right) - i\sin\phi\left(1-re^{i\alpha}\right)\right]$$

$$\beta_0 = \frac{1}{\sqrt{1+r^2}\sqrt{2}}\left[\sin\phi\left(1+re^{i\alpha}\right) + i\cos\phi\left(1-re^{i\alpha}\right)\right]$$

Notice that

$$|\alpha_0|^2 = \frac{1}{(1+r^2)\,2}\left[\cos^2\phi\left(1+r^2+2r\cos\alpha\right) + \sin^2\phi(1+r^2-2r\cos\alpha) - 4r\sin\phi\cos\phi\sin\alpha\right]$$

$$|\beta_0|^2 = \frac{1}{(1+r^2)\,2}\left[\sin^2\phi\left(1+r^2+2r\cos\alpha\right) + \cos^2\phi(1+r^2-2r\cos\alpha) + 4r\sin\phi\cos\phi\sin\alpha\right]$$

and

$$|\alpha_0|^2 + |\beta_0|^2 = 1$$

Plugging these results into

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\alpha_0|^2 \sin^2\theta - \frac{1}{4}|\beta_0|^2 \sin^2\theta - \cos\theta\right]$$

gives for the terms not linear with r,

$$\frac{d\sigma_1}{d\Omega} = k^4 a^6 \left[\frac{5}{8}\left(1 + \cos^2\theta\right) - \cos\theta\right]$$

whereas the terms linear in r contribute

$$\frac{d\sigma_2}{d\Omega} = k^4 a^6 \left[-\frac{3}{4}\left(\frac{r}{1+r^2}\right) \sin^2\theta \cos(2\phi - \alpha)\right]$$

where I have used

$$\cos(2\phi - \alpha) = \cos 2\phi \cos\alpha + \sin 2\phi \sin\alpha$$

and

$$\cos 2\phi = \cos^2\phi - \sin^2\phi$$

Adding the two contributions gives

$$\frac{d\sigma}{d\Omega}(\hat{\varepsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{8}\left(1 + \cos^2\theta\right) - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right) \sin^2\theta \cos(2\phi - \alpha)\right]$$

the desired result.

More Problems for Chapter 10

Problem 10.3

(a) Since $\lambda \gg R$, the fields are essentially constant over the size of the sphere. Furthermore the sphere can be treated as a perfectly conducting sphere since $\delta \ll R$. Therefore, to the 0^{th} order, the problem can be approximated as a perfect conducting sphere in a static field. Thus the tangential component of the electric field vanishes ($\vec{E}_{||} = 0$). To apply considerations of Sec. 8.1 to the power absorbed, we need to know the tangential component of the magnetic field $\vec{H}_{||}$. Consequently we need to solve the magnetostatic problem in field $\vec{H} = \vec{H}_0 e^{-i\omega t}$ of the plane wave. *Note that we can always project the field of an unpolarized beam into two independent polarizations.* In spherical coordinates with $\vec{H}$ pointing to the $+z$ axis, the magnetic scalar potential has the form:

$$\Phi_M = -H_0 r \cos\theta + \frac{m}{4\pi r^2}\cos\theta$$

Here we have chosen the center-of-sphere as the coordinate origin. The first term is due to the uniform external field while the second term is due to the included magnetic dipole moment $\vec{m}$ of the sphere. Note that $\vec{m}$ and $\vec{H}$ are in the same direction. Thus, the components of the magnetic field $\vec{H} = -\nabla\Phi_M$ are:

$$H_r = -\frac{\partial \Phi_M}{\partial r} = H_0 \cos\theta + \frac{m}{2\pi r^3}\cos\theta$$

$$H_\theta = -\frac{1}{r}\frac{\partial \Phi_M}{\partial \theta} = -H_0 \sin\theta + \frac{m}{4\pi r^3}\sin\theta$$

and $H_\phi = 0$. For a perfect conductor, $H_r = 0$ on the surface. Then,

$$H_r(r = R) = H_0\cos\theta + \frac{m}{2\pi R^3}\cos\theta = 0 \quad \Rightarrow \quad m = -2\pi R^3 H_0 \quad \Rightarrow \quad \vec{m} = -2\pi R^3 \vec{H}_0$$

The 0^{th} order magnetic field on the surface is thus

$$H_{||} = H_\theta = -\frac{3}{2}H_0 \sin\theta$$

(b) Since $\delta \ll R$, the power absorbed per unit area of surface is given by Eq. (8.15):

$$\frac{dP_{\text{loss}}}{da} = \frac{1}{2\sigma\delta}|\vec{K}_{\text{eff}}|^2 = \frac{1}{2\sigma\delta}|\vec{n} \times \vec{H}_{||}|^2 = \frac{9}{8\sigma\delta}|H_0|^2 \sin^2\theta$$

The total power absorbed is

$$P_{\text{abs}} = \int \frac{dP_{\text{abs}}}{da} R^2 d\Omega = \frac{3\pi R^2}{\sigma\delta}|H_0|^2$$

Now note that the incident flux

$$I = \langle \vec{S}\cdot\vec{n}\rangle = \frac{1}{2}\vec{n}\cdot(\vec{E}\times\vec{H}^*) = \frac{1}{2}\vec{n}\cdot\left\{(Z_0\vec{H}\times\vec{n})\times\vec{H}^*\right\} = \frac{Z_0}{2}|\vec{H}|^2 = \frac{1}{2}Z_0|H_0|^2$$

the absorption cross section is

$$\sigma_{\text{abs}} = \frac{P_{\text{abs}}}{I} = \frac{6\pi R^2}{\sigma\delta Z_0}$$

Furthermore,

$$\delta = \sqrt{\frac{2}{\mu_0\omega\sigma}} \qquad \Rightarrow \qquad \sigma = 6\pi R^2\sqrt{\frac{\epsilon_0\omega}{2\sigma}}$$

Therefore $\sigma_{\rm abs}$ is proportional to $\sqrt{\omega}$ if the conductivity σ is independent of frequency.

Problem 10.9(a)
Useful integral

$$\int_0^\infty \frac{j_1^2(z)}{z}dz = \frac{\pi}{2}\int_0^\infty \frac{J_{3/2}^2(z)}{z^2}dz = \frac{1}{4}$$

Starting with the 'Born' approximation formula Eq. (10.31)

$$\frac{\vec{\epsilon}^* \cdot \vec{A}_{\rm sc}^{(1)}}{D_0} = \frac{k^2}{4\pi}\int_{r\le a}(\epsilon_r - 1)\vec{\epsilon}^* \cdot \vec{\epsilon}_0 e^{i\vec{q}\cdot\vec{r}}d\tau = \frac{(\epsilon_r - 1)}{4\pi}k^2\vec{\epsilon}^* \cdot \vec{\epsilon}_0 \int_{r\le a} e^{i\vec{q}\cdot\vec{r}}d\tau$$

Define

$$I = \frac{1}{4\pi}\int e^{i\vec{q}\cdot\vec{r}}d\tau = \frac{1}{4\pi}\int_0^a r^2 dr \int d\Omega e^{iqr\cos\theta} = \int_0^a r^2 dr \frac{1}{2}\int_{-1}^{+1} e^{iqr\cos\theta}d(\cos\theta)$$

$$= \int_0^a r^2\left\{\frac{\sin(qr)}{qr}\right\}dr = \frac{1}{q^3}\int_0^{qa}\xi\sin\xi d\xi = \frac{1}{q^3}(\sin(qa) - qa\cos(qa)) = a^3\frac{j_1(qa)}{qa}$$

The differential scattering cross section, averaged over initial polarizations and summed over final polarizations, is

$$\frac{d\sigma}{d\Omega} = (ka)^4 a^2|\epsilon_r - 1|^2|\frac{j_1(qa)}{qa}|^2 \cdot \left\{\frac{1}{2}\sum_{pol.}|\vec{\epsilon}^* \cdot \vec{\epsilon}_0|^2\right\}$$

where the summation is over initial and final state polarizations:

$$\frac{1}{2}\sum_{pol.}|\vec{\epsilon}^* \cdot \vec{\epsilon}_0|^2 = \frac{1}{2}(1 + \cos^2\theta)$$

Now note

$$q^2 = k^2|\vec{n}_0 - \vec{n}|^2 = k^2(2 - 2\vec{n}_0 \cdot \vec{n}) = 2k^2(1 - \cos\theta)$$

For large $ka \gg 1$, $qa = ka\sqrt{2(1-\cos\theta)}$ can be large compared to unity. Since

$$|\frac{j_1(qa)}{qa}|^2 = \mathcal{O}\left\{\frac{1}{(qa)^4}\right\}$$

for large qa, the scattering is mainly confined to small qa. Small qa and $ka \gg 1$ imply $\theta \ll 1$, *i.e.* the differential cross section is sharply peaked in the forward direction.

$$d\Omega = d\phi d(\cos\theta) = d\phi\frac{1}{2k^2}d(q^2) = \frac{1}{2k^2a^2}d(q^2a^2)d\phi$$

Let $z = qa$, then

$$d\Omega = \frac{1}{2k^2a^2}(2zdz)d\phi = \frac{1}{(ka)^2}zdz$$

The total scattering cross section can be written

$$\sigma = \int\frac{d\sigma}{d\Omega}d\Omega = \frac{(ka)^4a^2|\epsilon_r - 1|^2}{(ka)^2}\int_0^{2\pi}d\phi\int_0^{2ka}zdz|\frac{j_1(z)}{z}|^2 \cdot \frac{1}{2}(1 + \cos^2\theta)$$

Since $\theta \ll 1$, $(1 + \cos^2\theta)/2 \to 1$ and $(qa)_{\max} = z_{\max} \to \infty$, then

$$\sigma \approx (ka)^2|\epsilon_r - 1|^2a^2 \times 2\pi\int_0^\infty \frac{j_1^2(z)}{z}dz = \frac{\pi}{2}(ka)^2|\epsilon_r - 1|^2a^2$$

Problem 10.12

(a) The diffracted field due to a plane surface is given by Eq. (10.101):

$$\vec{E}(\vec{r}) = \frac{1}{2\pi}\nabla \times \int_{\text{apertures}} \left\{\vec{n} \times \vec{E}(\vec{r'})\right\} \frac{e^{ik|\vec{r}-\vec{r'}|}}{|\vec{r}-\vec{r'}|} da'$$

Here $\vec{n}$ is a unit normal. In radiation zone, $|\vec{r}-\vec{r'}| \approx r - \vec{n}\cdot\vec{r'}$. Thus

$$\vec{E}(\vec{r}) \approx \frac{1}{2\pi}\nabla \times \left\{\frac{e^{ikr}}{r}\int_{\text{aperture}} (\vec{n}\times\vec{E}(\vec{r}))e^{-i\vec{k}\cdot\vec{r'}}da'\right\} \approx \frac{i}{2\pi}\frac{e^{ikr}}{r}\vec{k}\times\left\{\int_{\text{aperture}} (\vec{n}\times\vec{E}(\vec{r}))e^{-i\vec{k}\cdot\vec{r'}}da'\right\}$$

Choose a rectangular coordinate system with the $x-z$ plane as the plane of incidence and its origin at the center of the aperture, thus $\vec{n} = \hat{z}$ and the incident wave vector $\vec{k}_0 = k(\cos\alpha\hat{z} + \sin\alpha\hat{x})$. Let (θ,ϕ) be the spherical angles of the outgoing wave vector $\vec{k}$ and $(\rho',\beta',0)$ be the polar coordinates of $\vec{r'}$, thus

$$\vec{k} = k(\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}), \quad \vec{r'} = \rho'\cos\beta'\hat{x} + \rho'\sin\beta'\hat{y}$$

Consequently

$$\vec{k}\cdot\vec{r'} = k\rho'\sin\theta(\cos\phi\cos\beta' + \sin\phi\sin\beta') = k\sin\theta\ \rho'\cos(\phi-\beta'), \quad \text{and} \quad \vec{k}_0\cdot\vec{r'} = k\sin\alpha\ \rho'\cos\beta'$$

Therefore, the field

$$\vec{E}(\vec{r}) = \frac{i}{2\pi}\frac{e^{ikr}}{r}\vec{k}\times\int_{\text{aperture}} \vec{n}\times\left\{E_0 e^{i\vec{k}_0\cdot\vec{r'}}\hat{y}\right\}e^{-i\vec{k}\cdot\vec{r'}}da'$$

$$= \frac{i}{2\pi}\frac{e^{ikr}}{r}\vec{k}\times\int_{\text{aperture}} \vec{z}\times\left\{E_0 e^{ik\sin\alpha\ \rho'\cos\beta'}\hat{y}\right\}e^{-ik\sin\theta\ \rho'\cos(\phi-\beta')}\ \rho'd\rho'd\beta'$$

$$= -\frac{i}{2\pi}E_0\ \frac{e^{ikr}}{r}\ \vec{k}\times\hat{x}\int_0^a \rho'd\rho'\int_0^{2\pi} e^{ik\rho'(\sin\alpha\cos\beta' - \sin\theta\cos(\phi-\beta'))}d\beta'$$

Now note the exponent of the integrand

$$\sin\alpha\cos\beta - \sin\theta\cos(\phi-\beta) = \sin\alpha\cos\beta - \sin\theta\cos\phi\cos\beta - \sin\theta\sin\phi\sin\beta$$

$$= -\left\{(\sin\theta\cos\phi - \sin\alpha)\cos\beta + (\sin\theta\sin\phi)\sin\beta\right\}$$

$$= -\xi\left(\frac{\sin\theta\cos\phi - \sin\alpha}{\xi}\cos\beta + \frac{\sin\theta\sin\phi}{\xi}\sin\beta\right)$$

$$= -\xi(\cos\beta\cos\beta_0 + \sin\beta\sin\beta_0) = -\xi\cos(\beta-\beta_0)$$

Here

$$\xi^2 = (\sin\theta\cos\phi - \sin\alpha)^2 + (\sin\theta\sin\phi)^2 = \sin^2\theta + \sin^2\alpha - 2\sin\alpha\sin\theta\cos\phi$$

and

$$\cos\beta_0 = \frac{\sin\theta\cos\phi - \sin\alpha}{\xi}; \quad \sin\beta_0 = \frac{\sin\theta\sin\phi}{\xi}$$

Applying the integral

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\pm ix\sin\theta}d\theta = J_0(x)$$

we get the electric field

$$\vec{E}(\vec{r}) = -\frac{iE_0}{2\pi}\ \frac{e^{ikr}}{r}\ \vec{k}\times\hat{x}\ \int_0^a \rho' d\rho' \int_0^{2\pi} e^{-ik\rho'\xi\cos(\beta'-\beta_0)}d\beta'$$

$$= -iE_0\ \frac{e^{ikr}}{r}\ \vec{k}\times\hat{x}\ \int_0^a \rho' J_0(k\rho'\xi)d\rho'$$

$$= -ia^2E_0\ \frac{e^{ikr}}{r}\ \vec{k}\times\hat{x}\frac{J_1(ka\xi)}{ka\xi} \equiv \mathcal{A}(\vec{k}\times\hat{x})$$

Now that

$$\vec{H} = \frac{1}{Z_0}\hat{r}\times\vec{E} = \frac{\mathcal{A}}{Z_0}\hat{r}\times(\vec{k}\times\hat{x})$$

The average Poynting vector

$$\langle\vec{S}\rangle = \frac{1}{2}Re\left\{\vec{E}\times\vec{H}^*\right\} = \frac{1}{2Z_0}|\vec{E}|^2\hat{r} = \frac{1}{2Z_0}|\mathcal{A}|^2\cdot|\vec{k}\times\hat{x}|^2\hat{r}$$

Note that $|\vec{k}\times\hat{x}|^2 = k^2(\sin^2\theta\sin^2\phi+\cos^2\theta)$. The time-averaged diffracted power per unit solid angle

$$\frac{dP}{d\Omega} = r^2\langle\vec{S}\rangle\cdot\hat{r} = \frac{r^2}{2\mu_0 c}|\mathcal{A}|^2\cdot|\vec{k}\times\hat{x}|^2 = \frac{1}{2Z_0}k^2a^4|E_0|^2|\frac{J_1(ka\xi)}{ka\xi}|^2(\sin^2\theta\sin^2\phi+\cos^2\theta)$$

The explicit dependence on ϕ of the differential power is the result of the polarization of the incoming wave. Now that the incident power P_i:

$$P_i = \int\frac{1}{2}(\vec{E}\times\vec{H}^*)\cdot\vec{n}_0\,da' = \frac{\pi a^2}{2Z_0}|E_0|^2\cos\alpha \quad\Rightarrow\quad |E_0|^2 = \frac{2Z_0}{\pi a^2\cos\alpha}P_i$$

Thus

$$(\frac{dP}{d\Omega})_\perp = \frac{P_i}{\cos\alpha}\frac{(ka)^2}{4\pi}(\sin^2\theta\sin^2\phi+\cos^2\theta)|\frac{2J_1(ka\xi)}{ka\xi}|^2$$

(b) For the polarization in the plane of incidence, the differential diffractive power is given by Eq. (10.114):

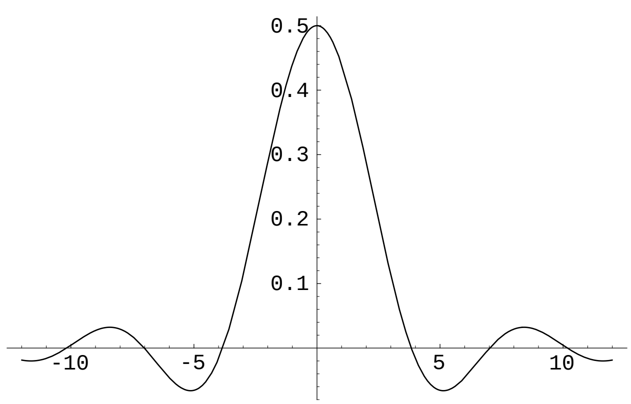
$$(\frac{dP}{d\Omega})_{||} = P_i\cos\alpha\frac{(ka)^2}{4\pi}(\sin^2\theta\cos^2\phi+\cos^2\theta)|\frac{2J_1(ka\xi)}{ka\xi}|^2$$

For normal incidence ($\alpha = 0$), the $\perp$ case with $\phi = 0$ and $||$ case with $\phi = \pi/2$ should be identical. Indeed the two formula are indeed the same. Furthermore for $\alpha = 0$, the diffractive power for an unpolarized beam is

$$\frac{dP}{d\Omega} = \frac{1}{2}\left\{(\frac{dP}{d\Omega})_\perp + (\frac{dP}{d\Omega})_{||}\right\}$$

$$= P_i\frac{(ka)^2}{4\pi}\ \frac{1}{2}(1+\cos^2\theta)\ |\frac{2J_1(ka\xi)}{ka\xi}|^2$$

As expected, the diffraction pattern of an unpolarized beam is independent of the azimuthal angle and is determined by the function $J_1(x)/x$, which is plotted below.



The vector results above are very similar to

$$\frac{dP}{d\Omega} = \frac{P_i}{\cos\alpha}\frac{(ka)^2}{4\pi}(\frac{\cos\alpha + \cos\theta}{2})^2|\frac{2J_1(ka\xi)}{ka\xi}|^2$$

of the scalar Kirchhoff approximation apart from the angular factor resulting from polarization.

Problem 10.18

(a) In the long wavelength limit, the small circular hole can be viewed as electric and magnetic dipoles with moments

$$\vec{p}_{\text{eff.}} = \frac{4\epsilon_0}{3}a^3\vec{E}_0; \qquad \vec{m}_{\text{eff.}} = -\frac{8}{3}a^3\vec{H}_0$$

Therefore, the diffracted electric field in the Fraunhofer zone is given by Eq. (10.2):

$$\vec{E}(\vec{r}) = \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr}}{r}(\vec{n}\times\vec{p}_{\text{eff.}})\times\vec{n} - \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr}}{r}(\vec{n}\times\frac{\vec{m}_{\text{eff.}}}{c})$$

where $\vec{n} = \vec{k}/k$. Inserting the effective dipole moments, we have

$$\vec{E}(\vec{r}) = \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\frac{4\epsilon_0 a^3}{3}(\vec{n}\times\vec{E}_0)\times\vec{n} + \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\frac{8a^3}{3c}(\vec{n}\times\vec{H}_0)$$

$$= \frac{k^2a^3}{4\pi\epsilon_0}\frac{e^{ikr}}{r}(\vec{n}\times\vec{E}_0)\times\vec{n} + \frac{2k^2a^3}{3\pi}\frac{e^{ikr}}{r}(\vec{n}\times c\vec{B}_0)$$

$$= \frac{k^2a^3}{3\pi}\frac{e^{ikr}}{r}\left\{\vec{n}\times(\vec{E}_0\times\vec{n}) + 2c\vec{n}\times\vec{B}_0\right\}$$

With explicit time-dependence, the field can be written as

$$\vec{E} = \frac{e^{ikr-i\omega t}}{3\pi r}k^2a^3\left\{2c\frac{\vec{k}}{k}\times\vec{B}_0 + \frac{\vec{k}}{k}\times(\vec{E}_0\times\frac{\vec{k}}{k})\right\}$$

(b) Choose a coordinate system such that $\vec{E}_0$ is along the $z-$axis, $\vec{B}_0$ is along the $x-$axis and let $\vec{k} = k(\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta)$, the time-averaged radiation power per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{r^2}{2Z_0}|\vec{E}|^2 = \frac{1}{18\pi^2 Z_0}k^4a^6|2c\frac{\vec{k}}{k}\times\vec{B}_0 + \frac{\vec{k}}{k}\times(\vec{E}_0\times\frac{\vec{k}}{k})|^2$$

Note that

$$|2c\vec{n}\times\vec{B}_0 + \vec{n}\times(\vec{E}_0\times\vec{n})|^2 = |2c\vec{n}\times\vec{B}_0 + \vec{E}_0 - (\vec{n}\cdot\vec{E}_0)\vec{n}|^2$$

$$= 4c^2|\vec{n}\times\vec{B}_0|^2 + 4cRe\left\{(\vec{n}\times\vec{B}_0)\times\vec{E}_0^*\right\} + |\vec{E}_0|^2 - |\vec{n}\cdot\vec{E}_0|^2$$

$$= 4c^2|B_0|^2(\sin^2\theta\sin^2\phi + \cos^2\theta) - 4c\sin\theta\sin\phi\ Re(\vec{E}_0^*\cdot\vec{B}_0) + |E_0|^2\sin^2\theta$$

Thus, the differential power

$$\frac{dP}{d\Omega} = \frac{1}{18\pi^2 Z_0}k^4a^6\left\{4c^2|B_0|^2(\sin^2\theta\sin^2\phi + \cos^2\theta) - 4c\sin\theta\sin\phi\ Re(\vec{E}_0^*\cdot\vec{B}_0) + |E_0|^2\sin^2\theta\right\}$$

The total power transmitted

$$P = \int\frac{dP}{d\Omega}d\Omega = \frac{1}{18\pi^2 Z_0}k^4a^6\int_0^{\pi/2}\sin\theta d\theta\int_0^{2\pi}d\phi\left\{4c^2|B_0|^2(\sin^2\theta\sin^2\phi + \cos^2\theta) - 4cRe(\vec{E}_0^*\cdot\vec{B}_0)\sin\theta\sin\phi + |E_0|^2\sin^2\theta\right\}$$

$$= \frac{2}{27\pi Z_0}k^4a^6(4c^2|B_0|^2 + |E_0|^2)$$

More Problems for Chapter 10

Problem 10.15

From Prob. 8.2, the TEM fields in this case are

$$\vec{E} = \frac{V}{\ln(b/a)}\frac{\hat{\rho}}{\rho}, \qquad c\vec{B} = \frac{V}{\ln(b/a)}\frac{\hat{\phi}}{\rho}$$

In Kirchhoff approximation, the problem can be simplified as a plane wave incident on a conducting plane sheet with a ring cut out of it. Therefore, the radiated field is given by Eq. (10.109):

$$\vec{E}(\vec{r}) = \frac{ie^{ikr}}{2\pi r}\vec{k}\times\int \hat{z}\times\vec{E}(\vec{r'})e^{-i\vec{k}\cdot\vec{r'}}da' = \frac{ie^{ikr}}{2\pi r}\frac{V}{\ln(b/a)}\vec{k}\times\int\frac{e^{-i\vec{k}\cdot\vec{r'}}}{\rho'}\hat{\phi}'da'$$

Let (θ,ϕ) be the spherical angles of $\vec{k}$, ϕ' be the polar angle of $\vec{r'}$, then

$$\vec{k} = k(\sin\theta\cos\phi\hat{x}+\sin\theta\sin\phi\hat{y}+\cos\theta\hat{z}), \qquad \vec{r'} = \rho'(\cos\phi'\hat{x}+\sin\phi'\hat{y})$$

$$\vec{k}\cdot\vec{r'} = k\rho'\sin\theta\cos(\phi'-\phi), \qquad \hat{\phi}' = -\sin\phi'\hat{x}+\cos\phi'\hat{y}$$

The electric field

$$\vec{E}(\vec{r}) = \frac{ie^{ikr}}{2\pi r}\frac{V}{\ln(b/a)}\vec{k}\times\int_a^b d\rho'\int_0^{2\pi}d\phi' e^{-ik\rho'\sin\theta\cos(\phi'-\phi)}(-\sin\phi'\hat{x}+\cos\phi'\hat{y})$$

Using the identities

$$\int_0^{2\pi}d\phi e^{i(x\cos\phi - m\phi)} = 2\pi i^m J_m(x); \quad J_{-m}(x) = (-1)^m J_m(x); \quad J_m(-x) = (-1)^m J_m(x)$$

the integrals over ϕ' can be carried out:

$$\int_0^{2\pi} e^{-ik\rho'\sin\theta\cos(\phi'-\phi)}\sin\phi' d\phi' = \int_0^{2\pi} e^{-ik\rho'\sin\theta\cos\phi'}\sin(\phi+\phi')d\phi'$$

$$= \frac{1}{2i}\int_0^{2\pi} e^{-ik\rho'\sin\theta\cos\phi'}\left\{e^{i(\phi+\phi')} - e^{-i(\phi+\phi')}\right\}$$

$$= \frac{1}{2i}\int_0^{2\pi} d\phi'\left\{e^{i\phi}e^{i(-k\rho'\sin\theta\cos\phi'+\phi')} - e^{-i\phi}e^{i(-k\rho'\sin\theta\cos\phi'-\phi')}\right\}$$

$$= \frac{1}{2i}\left\{(2\pi)i^{-1}e^{i\phi}J_{-1}(-k\rho'\sin\theta) - (2\pi)i^{+1}e^{-i\phi}J_1(-k\rho'\sin\theta)\right\}$$

$$= -2\pi i\sin\phi J_1(k\rho'\sin\theta)$$

Similarly

$$\int_0^{2\pi} e^{-ik\rho'\sin\theta\cos(\phi'-\phi)}\cos\phi' d\phi' = \int_0^{2\pi} e^{-ik\rho'\sin\theta\cos\phi'}\cos(\phi+\phi')d\phi'$$

$$= \frac{1}{2}\int_0^{2\pi} e^{-ik\rho'\sin\theta\cos\phi'}\left\{e^{i(\phi+\phi')}+e^{-i(\phi+\phi')}\right\}$$

$$= \frac{1}{2}\int_0^{2\pi} d\phi'\left\{e^{i\phi}e^{i(-k\rho'\sin\theta\cos\phi'+\phi')}+e^{-i\phi}e^{i(-k\rho'\sin\theta\cos\phi'-\phi')}\right\}$$

$$= \frac{1}{2}\left\{(2\pi)i^{-1}e^{i\phi}J_{-1}(-k\rho'\sin\theta)+(2\pi)i^{+1}e^{-i\phi}J_1(-k\rho'\sin\theta)\right\}$$

$$= -2\pi i\cos\phi J_1(k\rho'\sin\theta)$$

Noting that

$$-\sin\phi\hat{x}+\cos\phi\hat{y}=\hat{\phi}, \quad \text{and} \quad \int_a^b J_1(x)dx=\frac{1}{2}(J_0(b)-J_0(a))$$

we get the electric field:

$$\vec{E}(\vec{r}) = \frac{ie^{ikr}}{2\pi r}\frac{V}{\ln(b/a)}\vec{k}\times\int_a^b d\rho'(-2\pi i)J_1(k\rho'\sin\theta)(-\sin\phi\hat{x}+\cos\phi\hat{y})$$

$$= \frac{e^{ikr}}{r}\frac{V}{\ln(b/a)}\vec{k}\times\hat{\phi}\int_a^b d\rho' J_1(k\rho'\sin\theta)$$

$$= \frac{e^{ikr}}{r}\frac{V}{\ln(b/a)}\hat{k}\times\hat{\phi}\frac{J_0(kb\sin\theta)-J_0(ka\sin\theta)}{2\sin\theta}$$

Now that $\hat{k}=\hat{r}$ and thus $\hat{k}\times\hat{\phi}=\hat{r}\times\hat{\phi}=-\hat{\theta}$

$$\vec{E}(\vec{r}) = -\frac{e^{ikr}}{r}\frac{V}{\ln(b/a)}\frac{J_0(kb\sin\theta)-J_0(ka\sin\theta)}{2\sin\theta}\hat{\theta}$$

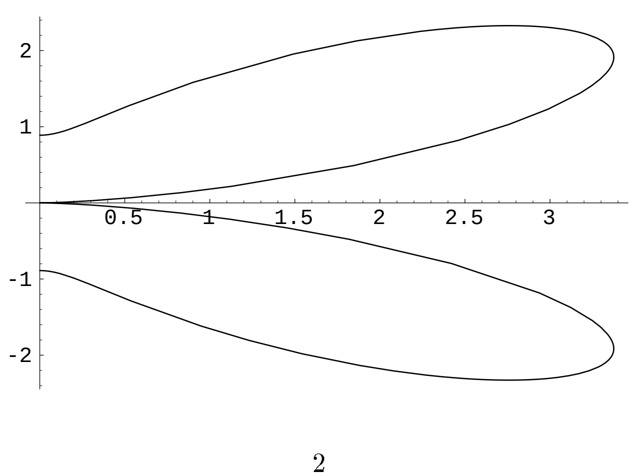
The average Poynting vector

$$\langle\vec{S}\rangle = \frac{|\vec{E}|^2}{2Z_0}\hat{k} = \frac{V^2}{8Z_0\ln^2(b/a)}\frac{\{J_0(kb\sin\theta)-J_0(ka\sin\theta)\}^2}{\sin^2\theta}\frac{\hat{k}}{r^2}$$

The average distributions of the radiation

$$\frac{dP}{d\Omega} = r^2\langle\vec{S}\rangle\cdot\hat{k} = \frac{V^2}{8Z_0\ln^2(b/a)}\frac{\{J_0(kb\sin\theta)-J_0(ka\sin\theta)\}^2}{\sin^2\theta}$$

The distribution for $kb=4$ and $ka=1$ in the unit of $V^2/(8Z_0\ln^2(b/a))$ is plotted below. The horizontal axis is the $z-$direction. As expected from the functional form, there is no radiation in the forward direction ($\theta=0$). For large b/a values, the distribution has many local maxima and minima as θ is varied from 0 to $\pi/2$.



The total power

$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{\pi V^2}{4Z_0 \ln^2(b/a)} \int_0^{\pi/2} \frac{\{J_0(kb\sin\theta) - J_0(ka\sin\theta)\}^2}{\sin\theta} d\theta$$

The integral does not have a simple analytical form and has to be carried out numerically.

Note to the grader: the following discussion should not be part of the grading

Long wavelength limit: $(kb \ll 1)$

$$J_0(kb\sin\theta) \sim 1 - \frac{1}{4}(kb\sin\theta)^2; \qquad J_0(ka\sin\theta) \sim 1 - \frac{1}{4}(ka\sin\theta)^2$$

Thus

$$\frac{dP}{d\Omega} \approx \frac{k^4 V^2 (b^2 - a^2)^2}{128 Z_0 \ln^2(b/a)} \sin^2\theta$$

The total radiated power is then

$$P_{\text{rad.}} = \int \frac{dP}{d\Omega} d\Omega = \frac{k^4 V^2 (b^2 - a^2)^2}{128 Z_0 \ln^2(b/a)} (2\pi) \int_0^{\pi/2} \sin^2\theta d(\cos\theta) = \frac{k^4 V^2 (b^2 - a^2)^2}{96 Z_0 \ln^2(b/a)}$$

This is to be compared with the power flow along an infinite coaxial line:

$$P_{\text{trans.}} = \frac{1}{2} \int (\vec{E} \times \vec{H}) \cdot \hat{z} da = \frac{V^2}{2Z_0 \ln^2(b/a)} \int_0^{2\pi} d\phi \int_a^b \frac{d\rho}{\rho} = \frac{\pi V^2}{Z_0 \ln(b/a)}$$

$$\frac{P_{\text{rad.}}}{P_{\text{trans.}}} = \frac{k^4 (b^2 - a^2)^2}{96\pi \ln(b/a)} \ll 1$$

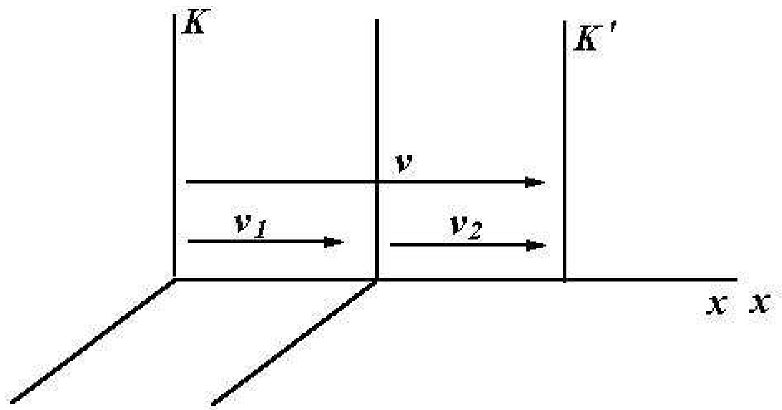
Therefore, most of the power is reflected back. The fields inside the coaxial cable is very similar to those of an "open" transmission line. Note that in this case, the coaxial cable can only operate in its TEM mode. All other modes are cut off.

Short wavelength limit: $(ka \gg 1)$
The radiation can be appreciable and higher modes are excited. The fields in the plane at $z = 0$ are far from the simple TEM fields. The Smythe-Kirchhoff approximation has only qualitative validity.

[redacted]

11.3

Let us just focus on the 0, 1 component transformation, since the 2, 3 components remain unchanged, if we take the relative velocities between Lorentz frames to be along the x -direction. We want to relate a single Lorentz transformation to two sequential transformations as described by



Thus we require

$$A = A_2 A_1$$

where A is a Lorentz transformation. Rewritten explicitly, the above equation reads

$$\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} = \begin{pmatrix} \gamma_2 & -\beta_2\gamma_2 \\ -\beta_2\gamma_2 & \gamma_2 \end{pmatrix}\begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 \\ -\beta_1\gamma_1 & \gamma_1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_2\gamma_1 + \beta_2\gamma_2\beta_1\gamma_1 & -\gamma_2\beta_1\gamma_1 - \beta_2\gamma_2\gamma_1 \\ -\gamma_2\beta_1\gamma_1 - \beta_2\gamma_2\gamma_1 & \gamma_2\gamma_1 + \beta_2\gamma_2\beta_1\gamma_1 \end{pmatrix}$$

So

$$\gamma = \gamma_2\gamma_1 + \beta_2\gamma_2\beta_1\gamma_1$$

$$\beta\gamma = \gamma_2\beta_1\gamma_1 + \beta_2\gamma_2\gamma_1$$

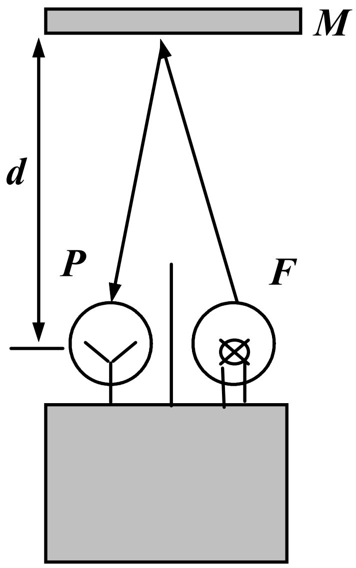
$$\beta = \frac{\beta\gamma}{\gamma} = \frac{\gamma_2\beta_1\gamma_1 + \beta_2\gamma_2\gamma_1}{\gamma_2\gamma_1 + \beta_2\gamma_2\beta_1\gamma_1} = \frac{\beta_1 + \beta_2}{1 + \beta_2\beta_1}$$

Or

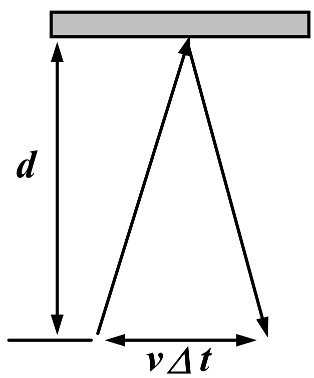
$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

as required.

11.4 The "clock" is shown in the figure



a) To the observer the pulse travels the trajectory



Thus, if the speed of light is c in both reference frames,

$$c\Delta t = 2\sqrt{d^2 + \left(\frac{v\Delta t}{2}\right)^2}$$

or

$$\Delta t = \frac{2d}{c\sqrt{1 - v^2/c^2}} = \gamma \Delta\tau$$

b) Now let us assume the clock-mirror system is moving away from the observer with speed v. Assume the fixed and moving frames coincide when a light pulse is given off. In the moving frame the time required for the light wave to move to the mirror and then to the phototube detector is given by

$$\frac{2d}{c} = \Delta t'$$

In the rest frame, the light hits the mirror in a time determined by

$$c\Delta t_1 = \frac{d}{\gamma} + v\Delta t_1$$

where $\frac{d}{\gamma}$ comes from the fact that the moving distance d is "length contracted." Solving for Δt_1

$$\Delta t_1 = \frac{d}{\gamma\,(c - v)}$$

Similarly the time for the light to travel from the mirror to the detector is determined by

$$c\Delta t_2 = \frac{d}{\gamma} - v\Delta t_2$$

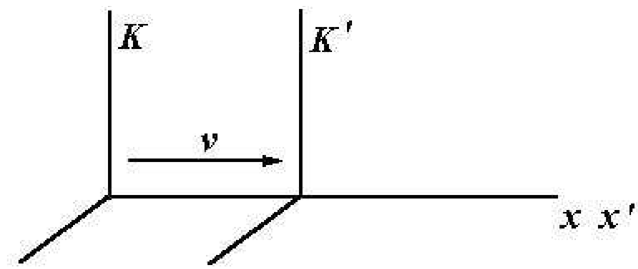
or

$$\Delta t_2 = \frac{d}{\gamma\,(c + v)}$$

So the total time in the fixed frame is given by

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{d}{\gamma}\left(\frac{1}{(c+v)} + \frac{1}{(c-v)}\right) = \frac{2d}{c\gamma\,(1 - v^2/c^2)} = \gamma\frac{2d}{c} = \gamma\Delta t'$$

11.5



From the class notes, and Eq. (11.31), we know

$$u_\parallel = \frac{u'_\parallel + v}{1 + \frac{vu'_\parallel}{c^2}}; \qquad u_\perp = \frac{u'_\perp}{\gamma(1 + \frac{vu'_\parallel}{c^2})}$$

and

$$dt = dt'\gamma(1 + \frac{vu'_\parallel}{c^2})$$

Thus taking the differential of the first equation above and using the second equation for *dt*,

$$\frac{du_\parallel}{dt} \equiv a_\parallel = \frac{(1 + \frac{vu'_\parallel}{c^2})a'_\parallel - (u'_\parallel + v)\frac{v}{c^2}a'_\parallel}{\gamma(1 + \frac{vu'_\parallel}{c^2})^3} = \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{(1 + \frac{vu'_\parallel}{c^2})^3}a'_\parallel$$

Similarly,

$$\frac{du_\perp}{dt} \equiv a_\perp = \frac{\left(1 + \frac{vu'_\parallel}{c^2}\right)a'_\perp - u'_\perp\frac{v}{c^2}a'_\parallel}{\gamma^2(1 + \frac{vu'_\parallel}{c^2})^3}$$

This is equal to the expression we want to prove,

$$\frac{du_\perp}{dt} \equiv a_\perp = \frac{\left(1 - \frac{v^2}{c^2}\right)}{(1 + \frac{vu'_\parallel}{c^2})^3}\left(a'_\perp + \frac{\vec{v}}{c^2} \times (\vec{a}' \times \vec{u}')\right)$$

since the BAC - CAB theorem shows that

$$a'_\perp + \frac{\vec{v}}{c^2} \times (\vec{a}' \times \vec{u}') = \left(1 + \frac{vu'_\parallel}{c^2}\right)a'_\perp - u'_\perp\frac{v}{c^2}a'_\parallel$$

11.6

Background:

$$dt = \gamma(\tau)d\tau, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

where dt is measured in the K_0 frame and $d\tau$ is the proper time. Using the Lorentz transformation for acceleration

$$\vec{a}_{||} = \frac{(1-\frac{v^2}{c^2})^{3/2}}{(1-\frac{\vec{v}\cdot\vec{u}'}{c^2})}\vec{a}'_{||}, \text{ but in this case } \vec{u}' = 0$$

$a = \dfrac{dv}{dt} = (1-v^2/c^2)^{3/2}\dfrac{dv'}{d\tau}$, where a_0 is acceleration in K_0 and $a' = \dfrac{dv'}{d\tau}$

So

$$dv = (1-v^2/c^2)^{3/2}a'dt = (1-v^2/c^2)a'd\tau$$

$$\int_0^v \frac{dv}{(1-v^2/c^2)} = a'\int_0^\tau d\tau$$

$$-\frac{1}{2}c\ln(c-v) + \frac{1}{2}c\ln(c+v) = a'\tau = \frac{1}{2}c\ln\left(\frac{c+v}{c-v}\right)$$

$$e^{\frac{2a'\tau}{c}} = \left(\frac{c+v}{c-v}\right) \to v(\tau) = c\frac{e^{\frac{2}{c}a'\tau}-1}{1+e^{\frac{2}{c}a'\tau}} = c\left(\frac{e^{\frac{1}{c}a'\tau}-e^{-\frac{1}{c}a'\tau}}{e^{\frac{1}{c}a'\tau}+e^{-\frac{1}{c}a'\tau}}\right) = c\tanh(\frac{a'\tau}{c})$$

$$\beta(\tau) = \tanh(\frac{a'\tau}{c})$$

$$\frac{dx}{dt} = v(t) \to dx = v(\tau)\gamma(\tau)d\tau$$

Or

$$x_{12} \equiv \int dx = \int_{\tau_1}^{\tau_2} v(\tau)\gamma(\tau)\,d\tau = c\int_{\tau_1}^{\tau_2}\frac{\tanh(\frac{a'\tau}{c})}{\sqrt{1-\tanh^2(\frac{a'\tau}{c})}}d\tau = c\int_{\tau_1}^{\tau_2}\tanh(\frac{a'\tau}{c})\cosh(\frac{a'\tau}{c})d\tau$$

$$= c\int_{\tau_1}^{\tau_2}\left(\sinh\frac{1}{c}a'\tau\right)d\tau = \frac{c^2}{a}\cosh(\frac{a'\tau}{c})|_{\tau_1}^{\tau_2}$$

Let's work part b) first:

b) The 10–year time frame going out is divided into two parts:

1st 5 years: $\tau_1 = 0$, $\tau_2 = 5$ yrs, $a' = g$

$$x_{02} = \frac{c^2}{a'}\left[\cosh(\frac{g\tau_2}{c}) - 1\right] = c\left[\tau_2\left(\frac{c}{a'\tau_2}\right)\left(\cosh(\frac{a'\tau_2}{c}) - 1\right)\right]$$

$$\frac{a'\tau_2}{c} = \frac{9.81 \cdot 5 \cdot 365 \cdot 24 \cdot 3600}{3 \times 10^8} = 5.16$$

$$= c\left[\tau_2\left(\frac{c}{a'\tau_2}\right)\left(\cosh(\frac{a'\tau_2}{c}) - 1\right)\right] = \text{c} \cdot 5 \cdot \text{yrs}\frac{1}{5.16}\left[\cosh(5.16) - 1\right]$$

$$= 83.4 \text{ light-years}$$

2^{nd} 5 years: $\tau_1 = 5$ yrs, $\tau_2 = 10$ yrs, $a' = -g$
By symmetry, this is the same as the first five years, 83.4 light-years.

Total distance after 10 years:

$$x_{Total} = (83.4 + 83.4) \text{ light-years} = 166.8 \text{ light-years}$$

a) Working out the time that elapses in the K_0 frame.

1^{st}5 years:

$$dt = \gamma d\tau \rightarrow t = \int_0^\tau \frac{1}{\sqrt{1 - \tanh^2(a'\tau/c)}} d\tau = \int_0^\tau \cosh(a'\tau/c) d\tau$$

$$= \frac{c}{a'}\sinh(\frac{a'\tau}{c}) = \tau\left(\frac{1}{a'\tau/c}\right)\sinh(\frac{a'\tau}{c})$$

$$\rightarrow t = 5\text{yrs} \cdot \frac{1}{5.16}\sinh(5.16) = 84.4 \text{ yrs}$$

2^{nd} 5 years:

$$t = 84.4 \text{ yrs by symmetry}$$

By symmetry, the return trip takes as long as the trip out.

$$\rightarrow t_{tot} = 2 \cdot (84.4 + 84.4) \text{ yrs} = 337.6 \text{ yrs}$$

It is the year $2100 + 337 = 2437$ on earth and the twin on earth is 357 yrs old!

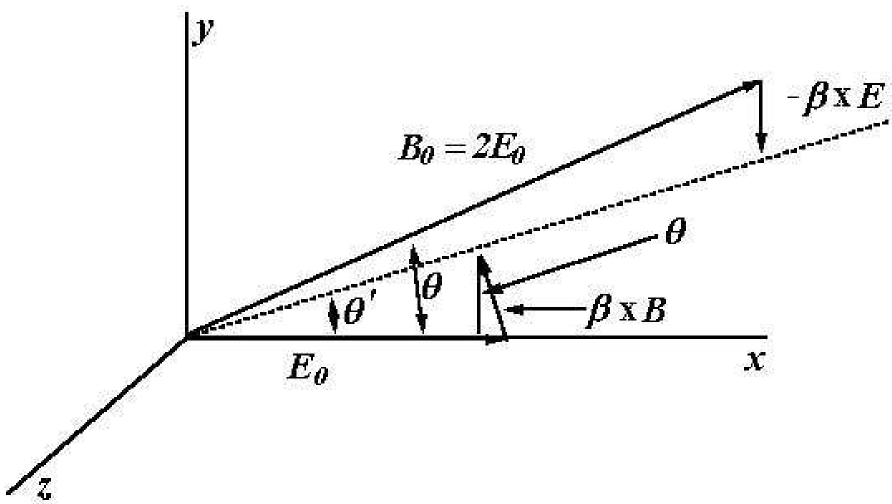
11.5

From Eq. (11.149), it is clear that we should take $\vec{\beta} \parallel \hat{z}$, so $\vec{\beta} \cdot \vec{E} = \vec{\beta} \cdot \vec{B} = 0$. Then

$$E' = \gamma(\vec{E} + \vec{\beta} \times \vec{B})$$

$$\vec{B}' = \gamma(\vec{B} - \vec{\beta} \times \vec{E})$$

The vectors in parentheses should make the same angle wrt the x axis θ' if they are to be parallel. This can best be seen from the figure,



From the figure

$$\vec{E}' = \gamma(E_0\hat{\imath} - \beta(2E_0)\sin\theta\hat{\imath} + \beta 2E_0\cos\theta\hat{\jmath})$$

$$\vec{B}' = \gamma(\cos\theta 2E_0\hat{\imath} + \sin\theta 2E_0\hat{\jmath} - \beta E_0\hat{\jmath})$$

Thus

$$\tan\theta' = \frac{2\beta\cos\theta}{1 - 2\beta\sin\theta} = \frac{2\sin\theta - \beta}{2\cos\theta}$$

$$(2\cos\theta) \cdot (2\beta\cos\theta) - (1 - 2\beta\sin\theta) \cdot (2\sin\theta - \beta) = 0$$

or

$$2\beta^2\sin\theta - 5\beta + 2\sin\theta = 0$$

This quadratic equation has the solution

$$\beta = \frac{1}{4\sin\theta}\left(5 - \sqrt{\left(25 - 16\sin^2\theta\right)}\right)$$

where I've chosen the solution which give $\beta = 0$ if $\theta = 0$.

If $\theta << 1$, then $\beta \to 0$, and the original fields are parallel.

If $\theta \to \pi/2$ then $\beta = \frac{1}{4}(5-3) = 1/2$. $\gamma = \frac{2}{\sqrt{3}}$

$$\vec{E}' = 0 + O(\theta - \frac{\pi}{2})\hat{\jmath}$$

$$\vec{B}' = \vec{B}' = \gamma(2E_0\hat{\jmath} - \frac{1}{2}E_0\hat{\jmath}) = \gamma E_0 \frac{3}{2}\hat{\jmath}$$

So in these two limits, the fields are parallel to the *x* and *y* axes, respectively.

More Problems for Chapter 11

Problem 11.3
Let the frame K' be moving with velocity $v_1\hat{z}$ with respect to K, and let K'' be moving with velocity $v_2\hat{z}$ with respect to K'. Then,

$$ct' = \gamma_1(ct - \beta_1 z), \quad z' = \gamma_1(z - \beta_1 ct), \quad x' = x, \quad y' = y$$

$$ct'' = \gamma_2(ct' - \beta_2 z'), \quad z'' = \gamma_2(z' - \beta_2 ct'), \quad x'' = x', \quad y'' = y'$$

with

$$\beta_1 = \frac{v_1}{c}, \quad \gamma_1 = \frac{1}{\sqrt{1-\beta_1^2}}; \qquad \beta_2 = \frac{v_2}{c}, \quad \gamma_2 = \frac{1}{\sqrt{1-\beta_2^2}}$$

Then $x'' = x$, $y'' = y$ and

$$z'' = \gamma_1\gamma_2\left\{(z - \beta_1 ct) - \beta_2(ct - \beta_1 z)\right\} = \gamma_1\gamma_2\left\{(1+\beta_1\beta_2)z - (\beta_1+\beta_2)ct\right\}$$

$$ct'' = \gamma_1\gamma_2\left\{(ct - \beta_1 z) - \beta_2(z - \beta_1 ct)\right\} = \gamma_1\gamma_2\left\{(1+\beta_1\beta_2)ct - (\beta_1+\beta_2)z\right\}$$

Thus

$$z'' = \gamma(z - \beta ct), \qquad ct'' = \gamma(ct - \beta z)$$

with

$$\gamma = \gamma_1\gamma_2(1+\beta_1\beta_2), \qquad \beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}$$

It is easy to show that $\gamma = 1/\sqrt{1-\beta^2}$:

$$\gamma^2 = \gamma_1^2\gamma_2^2(1+\beta_1\beta_2)^2 = \frac{(1+\beta_1\beta_2)^2}{(1+\beta_1\beta_2)^2-(\beta_1+\beta_2)^2} = \left\{1 - \frac{(\beta_1+\beta_2)^2}{(1+\beta_1\beta_2)^2}\right\}^{-1} = \frac{1}{1-\beta^2}$$

Thus two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation in that direction with velocity

$$v = c\beta = \frac{v_1+v_2}{1+v_1v_2/c^2}$$

<u>Problem 11.5</u>
Begins with Eq. (11.31):

$$u_{||} = \frac{u'_{||}+v}{1+\frac{v}{c^2}u'_{||}}, \qquad \vec{u}_\perp = \frac{\vec{u}'_\perp}{\gamma_v(1+\frac{v}{c^2}u'_{||})}$$

Note that

$$\frac{d}{dt} = \frac{1}{\gamma_v(1+\frac{v}{c^2}u'_{||})}\frac{d}{dt'}$$

Therefore

$$a_{||} = \frac{1}{\gamma_v(1+\frac{v}{c^2}u'_{||})}\left\{\frac{a'_{||}}{(1+\frac{v}{c^2}u'_{||})} - \frac{u'_{||}+v}{c^2}\frac{va'_{||}}{(1+\frac{v}{c^2}u'_{||})^2}\right\} = \frac{1}{\gamma_v^3}\frac{1}{(1+\vec{v}\cdot\vec{u}'/c^2)^3}a'_{||}$$

Similarly

$$\vec{a}_\perp = \frac{1}{\gamma_v^2(1+\vec{v}\cdot\vec{u}'/c^2)}\left\{\frac{\vec{a}'_\perp}{(1+\vec{v}\cdot\vec{u}'/c^2)} - \frac{u'_\perp v}{c^2}\frac{a'_{||}}{(1+\vec{v}\cdot\vec{u}'/c^2)^2}\right\}$$

$$= \frac{1}{\gamma_v^2(1+\vec{v}\cdot\vec{u}'/c^2)^3}\left\{\vec{a}'_\perp(1+\frac{\vec{v}\cdot\vec{u}'}{c^2}) - \vec{u}'_\perp a'_{||}\frac{v}{c^2}\right\}$$

$$= \frac{1}{\gamma_v^2(1+\vec{v}\cdot\vec{u}'/c^2)^3}\left\{\vec{a}'_\perp + \frac{1}{c^2}\{(\vec{v}\cdot\vec{u}')\vec{a}'_\perp - (\vec{v}\cdot\vec{a}')\vec{u}'_\perp\}\right\}$$

$$= \frac{1}{\gamma_v^2(1+\vec{v}\cdot\vec{u}'/c^2)^3}\left\{\vec{a}'_\perp + \frac{\vec{v}}{c^2}\times(\vec{a}'\times\vec{u}')\right\}$$

Problem 11.6

Let $v(t)$ be the instantaneous velocity of the rocket with respect to the earth. At a given time t, consider the rocket's motion in an inertial frame moving with (constant) velocity $v(t)$ with respect to the earth.

(a) The rocket's velocity in this frame is $u' = 0$, while its acceleration is $a'_{||} = g$ and $a'_{\perp} = 0$. Then by the Problem 11.5, we know that an observer in the earth's frame would see the rocket to have an acceleration

$$a_{||} = (1 - \frac{v(t)^2}{c^2})^{3/2} g$$

such an observer measures the acceleration by using

$$a_{||} = \frac{dv(t)}{dt} = (1 - \frac{v(t)^2}{c^2})^{3/2} g$$

Therefore $v(t)$ can be solved from the above differential equation. The initial condition for the 1st part of the journey (the five years of acceleration) is $v(0) = 0$:

$$\int_0^v \frac{dv}{(1 - \frac{v^2}{c^2})^{3/2}} = \int_0^t g dt \quad \Rightarrow \quad gt = \frac{v}{(1 - v^2/c^2)^{1/2}} \quad \Rightarrow \quad v^2 = \frac{g^2 t^2}{1 + g^2 t^2/c^2}$$

Thus

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \sqrt{1 + \frac{g^2 t^2}{c^2}}$$

The relationship between dt in the earth's frame and dt' in the instantaneous rest frame is

$$dt = \gamma(v) \left\{ dt' + \frac{\vec{v} \cdot d\vec{r'}}{c^2} \right\}$$

But in the instantaneous rest frame, $\vec{u'} = 0$, which leads to $d\vec{r'} = 0$. Therefore $dt = \gamma(v) dt'$. Thus

$$t' = \int \frac{dt}{\gamma(t)} = \int \frac{dt}{\sqrt{1 + g^2 t^2/c^2}} = \frac{c}{g} \sinh^{-1}(\frac{gt}{c}), \quad \Rightarrow \quad t = \frac{c}{g} \sinh(\frac{gt'}{c})$$

For the first leg of the journey

$$t \approx \frac{3 \cdot 10^8}{10} \sinh(\frac{3 \cdot 10^8 \times 3 \cdot 10^7}{10}) \approx 75 \times (3 \cdot 10^7) \text{ s} \approx 75 \text{ years}$$

The total journey takes four times of the first leg:

$$t_{\text{total}} \sim 4 \times 75 \sim 300 \text{ years}$$

Therefore, the year on earth is 2400 when the twin returns to visit his/her sibling's grave.

(b) The furtherest distance the rocket ship traveled

$$s = 2 \int_0^{75\text{years}} v(t) dt = \int \frac{gt}{\sqrt{1 + g^2 t^2/c^2}} = \frac{c^2}{g} \sqrt{1 + \frac{g^2 t^2}{c^2}} |_0^{75 \text{ years}} \approx 148 \text{ light} - \text{years}$$

Problem 11.8

(a) In frame K' in which the fluid is at rest, the frequency and wave vector are related by $ck' = n(\omega')\omega'$ because only in this frame can we define the index of refraction. We assume the speed of light in the fluid is

$$v'_p = \frac{\omega'}{k'} = \frac{c}{n(\omega')}$$

Applying velocity addition formula for parallel velocities, we get

$$v_p \equiv u = \frac{v + v'_p}{1 + vv'_p/c^2} = \frac{c}{n(\omega')}\left\{\frac{1+\beta n(\omega')}{1+\beta/n(\omega')}\right\}$$

where $\beta = v/c$. Expanding in powers of β and keeping only first order in β:

$$u \approx \frac{c}{n(\omega')} + v\left\{1 - \frac{1}{n^2(\omega')}\right\} + \mathcal{O}(\beta^2 c)$$

To find the correction for dispersion, we must relate ω' to the lab frequency ω. We now note that both ω'/c and k' are the time and space components of a 4-vector (because the phase of a wave is a Lorentz invariant). Thus

$$\frac{\omega}{c} = \gamma(\frac{\omega'}{c} + \beta k'), \quad \text{and} \quad k = \gamma(k' + \beta\frac{\omega'}{c})$$

Since $ck' = \omega' n(\omega')$, solving ω and ck in terms of ω', we get

$$\omega = \gamma(1 + \beta n(\omega'))\omega', \qquad ck = \gamma(n(\omega') + \beta)\omega'$$

In passing, we note that the lab phase velocity is

$$\frac{\omega}{k} = \frac{c}{n(\omega')}\left\{\frac{1+\beta n(\omega')}{1+\beta/n(\omega')}\right\}$$

just as we found above from the velocity addition formula. And the index of refraction for the moving fluid can be defined as

$$n(\beta,\omega) = c\frac{k}{\omega} = \frac{n(\omega')+\beta}{1+\beta n(\omega')}$$

It depends not only on the frequency, but also on the speed of the fluid.
To the first order in β, we have

$$\omega' \approx \omega(1 - \beta n(\omega)) \qquad \Rightarrow \qquad \omega' - \omega \approx -\beta\omega n(\omega)$$

Taylor expanding $n(\omega')$ at $\omega' = \omega$:

$$n(\omega') = n(\omega) + \frac{dn}{d\omega}(\omega' - \omega) + \mathcal{O}((\omega' - \omega)^2)$$

$$= n(\omega) - \beta\omega n(\omega)\frac{dn}{d\omega} + \mathcal{O}(\beta^2) \approx n(\omega)\left\{1 - \beta\omega\frac{dn}{d\omega}\right\}$$

Thus

$$\frac{1}{n(\omega')} \approx \frac{1}{n(\omega)}\left\{1 + \beta\omega\frac{dn}{d\omega}\right\}; \qquad \frac{1}{n^2(\omega')} \approx \frac{1}{n^2(\omega)}\left\{1 + 2\beta\omega\frac{dn}{d\omega}\right\}$$

The velocity formula therefore becomes

$$u = \frac{c}{n(\omega)}\left\{1 + \beta\omega\frac{dn}{d\omega}\right\} + v\left\{1 - \frac{1}{n^2(\omega)}(1 + 2\beta\omega\frac{dn}{d\omega})\right\}$$

$$\approx \frac{c}{n(\omega)} + v\left\{1 - \frac{1}{n^2(\omega)} + \frac{\omega}{n(\omega)}\frac{dn(\omega)}{d\omega}\right\}$$

Similarly the velocity formula for the antiparallel case is

$$u = \frac{c}{n(\omega)} + v\left\{1 + \frac{1}{n^2(\omega)} + \frac{\omega}{n(\omega)}\frac{dn(\omega)}{d\omega}\right\}$$

Problem 11.13

(a) In the wire's rest frame K', the wire has a constant linear charge density q_0. In this frame, the electric and magnetic fields in Gaussian units are given by (in cylindrical coordinates):

$$\vec{E}' = \frac{2q_0}{r}\hat{r} \qquad \vec{B}' = 0$$

Here we used r instead of ρ to denote the polar radius to avoid confusion (see below). Lorentz-transform along the $z-$axis using the inverse of Eq. (11.149) to get the fields in the lab frame ($\vec{\beta} = \frac{v}{c}\hat{z}$):

$$\vec{E} = \gamma(\vec{E}' - \vec{\beta} \times \vec{B}') - \frac{\gamma^2}{1+\gamma}\vec{\beta}(\vec{\beta}\cdot\vec{E}') = \gamma\vec{E}' = \frac{2\gamma q_0}{r}\hat{r}$$

$$\vec{B} = \gamma(\vec{B}' + \vec{\beta} \times \vec{E}') - \frac{\gamma^2}{1+\gamma}\vec{\beta}(\vec{\beta}\cdot\vec{B}') = \gamma\vec{\beta} \times \vec{E}' = \frac{2\gamma\beta q_0}{r}\hat{\phi}$$

Note that the radial (r) and angular (ϕ) lengths (coordinates) are the same in both frames since the relative motion is in the $z-$direction.

(b) In the rest frame K', the current and charge densities are:

$$\vec{J}' = 0, \qquad \rho' = \frac{q_0}{2\pi r}\delta(r)$$

Note that it is easy to verify that the charge per unit length is q_0. Since $(c\rho', \vec{J}')$ transform as a four-vector, we have in the lab frame:

$$c\rho = \gamma(c\rho' + \beta J_z') = \gamma c\rho' \qquad \Rightarrow \qquad \rho = \gamma\rho' = \frac{\gamma q_0}{2\pi r}\delta(r)$$

i.e., the line charge density in the lab frame is γq_0, consistent with the Lorentz contraction of the wire in $z-$direction.

$$J_z = \gamma(J_z' + \beta c\rho') = \gamma\beta c\rho' \qquad \Rightarrow \qquad \vec{J} = \frac{\gamma q_0 v}{2\pi r}\delta(r)\hat{z} = \rho v\hat{z} = \rho\vec{v}$$

This is the current density of a line current $\gamma q_0 v$.

(c) An observer in the laboratory frame sees a line charge of density γq_0 and a line current $\gamma q_0 v$. Therefore, the electric and magnetic fields can be readily calculated from Gauss's and Ampere's laws to be:

$$\vec{E} = \frac{2\gamma q_0}{r}\hat{r}; \qquad \vec{B} = \frac{4\pi}{c}\,(\gamma q_0 v)\,\frac{1}{2\pi r}\hat{\phi} = \frac{2\gamma\beta q_0}{r}\hat{\phi}$$

in agreement with those of (a).

Problem 11.16

(a) Since the equation

$$J^\alpha - \frac{1}{c^2}(U_\beta J^\beta)U^\alpha = \frac{\sigma}{c}F^{\alpha\beta}U_\beta$$

is a covariant equation, it is valid in all inertial frames if it is valid in one of them. In the rest frame of the conducting medium, $U^\alpha = (c, \vec{0})$, so that in this frame we have

$$\alpha = 0: \quad c\rho - \frac{1}{c^2}(c\cdot c\rho)c = \frac{\sigma}{c}F^{00}\cdot c \qquad \Rightarrow \qquad c\rho - c\rho = 0$$

$$\alpha = i: \quad J^i - \frac{1}{c^2}(c \cdot c\rho)\cdot 0 = \frac{\sigma}{c} F^{i0}\cdot c \quad \Rightarrow \quad J^i = \sigma E^i$$

The equation gives Ohm's law in the rest frame and therefore valid in all frames.
(b) If the medium has a velocity $\vec{v} = c\vec{\beta}$, then $U^\alpha = \gamma c(1, \vec{\beta})$ and the equation becomes to:

$$\alpha = 0: \quad c\rho - \frac{(\gamma c)^2}{c^2}(c\rho - \vec{\beta}\cdot\vec{J}) = \frac{\sigma}{c}F^{0i}U_i = \gamma\sigma\vec{\beta}\cdot\vec{E} \quad \Rightarrow \quad \gamma^2(c\rho - \vec{\beta}\cdot\vec{J}) = c\rho - \gamma\sigma\vec{\beta}\cdot\vec{E}$$

$$\alpha = i: \quad J^i - \frac{(\gamma c)^2}{c^2}(c\rho - \vec{\beta}\cdot\vec{J})\beta^i = \frac{\sigma}{c}(F^{i0}\gamma c - \sum_j F^{ij}\gamma c\beta^j) \quad \Rightarrow \quad \vec{J} - \gamma^2(c\rho - \vec{\beta}\cdot\vec{J})\vec{\beta} = \gamma\sigma(\vec{E} + \vec{\beta}\times\vec{B})$$

Here we have used the following identity:

$$\sum_j F^{1j}\beta^j = F^{12}\beta^2 + F^{13}\beta^3 = -B_z\beta_y + B_y\beta_z = -(\vec{\beta}\times\vec{B})_x$$

and similar ones for $\sum_j F^{2j}\beta^j$ and $\sum_j F^{3j}\beta^j$. Therefore

$$\vec{J} = \gamma\sigma(\vec{E} + \vec{\beta}\times\vec{B}) + \gamma^2(c\rho - \vec{\beta}\cdot\vec{J})\vec{\beta} = \gamma\sigma(\vec{E} + \vec{\beta}\times\vec{B}) + (c\rho - \gamma\sigma\vec{\beta}\cdot\vec{E})\vec{\beta}$$

$$= \gamma\sigma\left\{\vec{E} + \vec{\beta}\times\vec{B} - \vec{\beta}(\vec{\beta}\cdot\vec{E})\right\} + \rho\vec{v}$$

(c) Since $(c\rho, \vec{J})$ is a four-vector,

$$0 = c\rho' = \gamma(c\rho - \vec{\beta}\cdot\vec{J}) \quad \Rightarrow \quad c\rho = \vec{\beta}\cdot\vec{J} \quad \Rightarrow \quad \rho\vec{v} = \vec{\beta}(\vec{\beta}\cdot\vec{J})$$

Thus Ohm's law generalizes to:

$$\vec{J} = \gamma\sigma\left\{\vec{E} + \vec{\beta}\times\vec{B} - \vec{\beta}(\vec{\beta}\cdot\vec{E})\right\} + \vec{\beta}(\vec{\beta}\cdot\vec{J}) \quad \Rightarrow \quad \vec{J} - \vec{\beta}(\vec{\beta}\cdot\vec{J}) = \gamma\sigma\left\{\vec{E} - \vec{\beta}(\vec{\beta}\cdot\vec{E}) + \vec{\beta}\times\vec{B}\right\}$$

More Problems for Chapter 11

Problem 11.23

(a) Let $\mathcal{P}$ and $\mathcal{P}'$ be 4-vectors in lab and CM frame respectively, then we have

$$\mathcal{P}_1 = (E_1, \vec{p}_{\text{LAB}}), \quad \mathcal{P}_2 = (m_2, \vec{0}); \qquad \mathcal{P}'_1 = (E'_1, \vec{p'}), \quad \mathcal{P}'_2 = (E'_2, -\vec{p'})$$

From the energy and momentum conservation in the lab frame, we have

$$\mathcal{P}_1 + \mathcal{P}_2 = \mathcal{P}_3 + \mathcal{P}_4$$

The total center-of-mass energy W:

$$W^2 = (E'_1 + E'_2)^2 = (E'_1 + E'_2)^2 - (\vec{p'}_1 + \vec{p'}_2)^2 = (\mathcal{P}'_1 + \mathcal{P}'_2)^2$$

Now note $(\mathcal{P}'_1 + \mathcal{P}'_2)^2$ is Lorentz invariant, we have

$$W^2 = (\mathcal{P}'_1 + \mathcal{P}'_2)^2 = (\mathcal{P}_1 + \mathcal{P}_2)^2 = \mathcal{P}_1^2 + \mathcal{P}_2^2 + 2\mathcal{P}_1 \cdot \mathcal{P}_2 = m_1^2 + m_2^2 + 2m_2 E_1$$

To find $\vec{p'}$, we consider $(\mathcal{P}_1 \cdot \mathcal{P}_2)^2$ and $(\mathcal{P}'_1 \cdot \mathcal{P}'_2)^2$:

$$(\mathcal{P}_1 \cdot \mathcal{P}_2)^2 = (m_2 E_1)^2 = m_2^2(p_1^2 + m_1^2) = m_2^2 p_1^2 + m_1^2 m_2^2$$

$$(\mathcal{P}'_1 \cdot \mathcal{P}'_2)^2 = (E'_1 E'_2 + p'^2)^2 = E'^2_1 E'^2_2 + 2E'_1 E'_2 p'^2 + p'^4$$

$$= (p'^2 + m_1^2)(p'^2 + m_2^2) + 2E'_1 E'_2 p'^2 + p'^4$$

$$= 2p'^4 + (m_1^2 + m_2^2)p'^2 + 2E'_1 E'_2 p'^2 + m_1^2 m_2^2$$

$$= p'^2(2p'^2 + m_1^2 + m_2^2 + 2E'_1 E'_2) + m_1^2 m_2^2$$

$$= p'^2(E'^2_1 + 2E'_1 E'_2 + E'^2_2) + m_1^2 m_2^2 = p'^2 W^2 + m_1^2 m_2^2$$

From Lorentz invariance, we have

$$(\mathcal{P}_1 \cdot \mathcal{P}_2)^2 = (\mathcal{P}'_1 \cdot \mathcal{P}'_2)^2 \quad \Rightarrow \quad m_2^2 p_1^2 = p'^2 W^2 \quad \Rightarrow p' = \frac{m_2}{W} p_1$$

Since $\vec{p}_1$ and $\vec{p'}$ are in the same direction (the Lorentz boost is along $\vec{p}_1$), therefore we have

$$\vec{p'} = \frac{m_2}{W} \vec{p}_1$$

(b) We can also obtain $\vec{p'}$ from Lorentz transformation of $\vec{p}_1$ (and $-\vec{p'}$ from $\vec{p}_2$):

$$p' = \gamma_{\text{cm}}(p_1 - \beta_{\text{cm}} E_1); \qquad (-p') = \gamma_{\text{cm}}(-\beta_{\text{cm}} m_2)$$

Thus

$$\beta_{\text{cm}} = \frac{p_1}{m_2 + E_1}, \qquad \Rightarrow \qquad \vec{\beta}_{\text{cm}} = \frac{\vec{p}_1}{m_2 + E_1}$$

$$\gamma_{\text{cm}} = \frac{1}{\sqrt{1 - \beta_{\text{cm}}^2}} = \frac{m_2 + E_1}{\sqrt{(m_2 + E_1)^2 - p_1^2}} = \frac{m_2 + E_1}{\sqrt{m_2^2 + 2m_2 E_1 + E_1^2 - p_1^2}} = \frac{m_2 + E_1}{W}$$

(c) In the non-relativistic limit,

$$E_1 \approx m_1 + \frac{p_1^2}{2m_1}$$

therefore,

$$W^2 \approx m_1^2 + m_2^2 + 2m_2(m_1 + \frac{p_1^2}{2m_1}) = (m_1 + m_2)^2 + \frac{m_2}{m_1}p_1^2 = (m_1 + m_2)^2 \left\{ 1 + \frac{m_2}{(m_1 + m_2)^2} \frac{p_1^2}{m_1} \right\}$$

$$W = (m_1 + m_2)\sqrt{1 + \frac{m_2}{(m_1 + m_2)^2} \frac{p_1^2}{m_1}} \approx (m_1 + m_2) \left\{ 1 + \frac{m_2}{(m_1 + m_2)^2} \frac{p_1^2}{2m_1} \right\} = m_1 + m_2 + \frac{m_2}{m_1 + m_2} \frac{p_1^2}{2m_1}$$

Similarly

$$\vec{p'} = \frac{m_2}{W}\vec{p}_1 \approx \frac{m_2}{m_1 + m_2}\vec{p}_1$$

$$\vec{\beta}_{\rm cm} = \frac{\vec{p}_1}{m_2 + E_1} \approx \frac{\vec{p}_1}{m_1 + m_2}$$

These are the familiar Galilean relativity results.

[redacted]

12.1

a)

$$\mathsf{L} = -\frac{1}{2}mu_\alpha u^\alpha - \frac{q}{c}u_\alpha A^\alpha \quad \text{(invariant Lagrangian)}$$

Show this Lagrangian gives the correct eqn. of motion, ie, Eq. (12.2)

$$\frac{du^\alpha}{d\tau} = \frac{e}{mc}F^{\alpha\beta}u_\beta$$

The Action is $A = \int_{\tau_1}^{\tau_2} \mathsf{L}\, d\tau$.

$\delta A = 0$ yields the Lagrange equations of motion

$$\frac{d}{d\tau}\frac{\partial \mathsf{L}}{\partial u_\alpha} - \frac{\partial \mathsf{L}}{\partial x_\alpha} = 0$$

$$\frac{d}{d\tau}\frac{\partial \mathsf{L}}{\partial u_\alpha} = -m\frac{d}{d\tau}u^\alpha - \frac{q}{c}\frac{\partial A^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}$$

$$\frac{\partial \mathsf{L}}{\partial x_\alpha} = -\frac{q}{c}u_\beta \partial^\alpha A^\beta$$

So $\frac{d}{d\tau}\frac{\partial \mathsf{L}}{\partial u_\alpha} - \frac{\partial \mathsf{L}}{\partial x_\alpha} = 0$ yields

$$m\frac{d}{d\tau}u^\alpha = \frac{q}{c}u_\beta \partial^\alpha A^\beta - \frac{q}{c}\partial_\mu A^\alpha u^\mu = \frac{q}{c}u_\beta(\partial^\alpha A^\beta - \partial^\beta A^\alpha)$$

or

$$\frac{d}{d\tau}u^\alpha = \frac{q}{mc}F^{\alpha\beta}u_\beta$$

12.2

(a)

$$L' = L + \frac{d}{dt}\lambda(t,\vec{x})$$

$\delta\int_{t_1}^{t_2}(L' - L)dt = [\delta(\lambda(t_2,\vec{x}) - \lambda(t_1,\vec{x})] = 0 \rightarrow L$ and L' yield the same Euler-Lagrange Eqs. of Mot.

where the last equality follows from the fact that variation at the end points is zero since the end points are held fixed.

(b) For simplicity of notation in this part, I'm going to set $c = 1$.

$$L = -m\sqrt{1-u^2} + e\vec{u}\bullet\vec{A} - e\phi \text{ with } A^\mu = (\phi,\vec{A})$$

If $A^\alpha \rightarrow A^\alpha + \partial^\alpha\Lambda,$ then

$$\phi \rightarrow \phi + \partial^0\Lambda$$

$$\vec{A} \rightarrow \vec{a} - \vec{\nabla}\Lambda$$

where the minus sign in the second equation should be noticed. Thus

$$L \rightarrow -m\sqrt{1-u^2} + e\vec{u}\bullet\vec{A} - e\phi - e\vec{u}\bullet\vec{\nabla}\Lambda - e\partial^0\Lambda$$

Now

$$\frac{d}{dt}\Lambda(t,\vec{x}) = \frac{\partial}{\partial x^\mu}\Lambda(t,\vec{x})\frac{\partial x^\mu}{\partial t} = \frac{\partial}{\partial t}\Lambda + \vec{\nabla}\Lambda\bullet\vec{u}$$

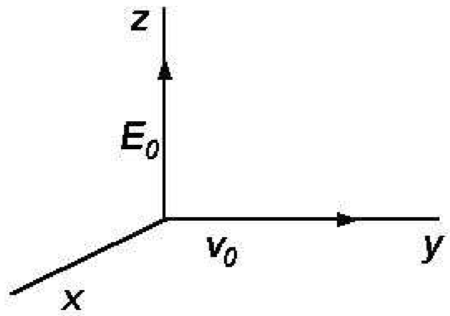
Or

$$L \rightarrow -m\sqrt{1-u^2} + e\vec{u}\bullet\vec{A} - e\phi - e\frac{d}{dt}\Lambda(t,\vec{x})$$

By the argument of part (a), this Lagrangian gives the same equations of motion as the original Lagrangian.

12.3

a) Take $\vec{E}_0$ along z, $\vec{v}_0$ along y



Generally

$$\frac{d\vec{p}}{dt} = e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right); \quad \frac{dE}{dt} = e\vec{v} \cdot \vec{E}$$

Since $\vec{B} = 0,$ and $\vec{E} = E_0\hat{z}$

$$\frac{dp_z}{dt} = eE_0$$

$$\frac{dp_y}{dt} = 0$$

The initial condition $\vec{p}(0) = mv_0$ and the above equations show the subsequent motion is in the $y-z$ plane. Consistent with this initial condition, we have

$$p_y(t) = mv_0; \; p_z(t) = eE_0t$$

$$E(t) = \sqrt{\vec{p}^2(t)c^2 + m^2c^4} = \sqrt{m^2v_0^2c^2 + m^2c^4 + (ceE_0t)^2} = \sqrt{\omega_0^2 + (ceE_0t)^2}$$

Using

$$\vec{p} = m\gamma\vec{v} = \frac{E}{c^2}\vec{v}$$

$$v_y(t) = \frac{p_y(t)}{E(t)/c^2} = \frac{mv_0c^2}{\sqrt{\omega_0^2 + (ceE_0t)^2}}$$

$$y(t) = \frac{mv_0c^2}{ceE_0}\int_0^t \frac{dt}{\sqrt{\rho^2 + t^2}} = \frac{mv_0c}{eE_0}\sinh^{-1}(t/\rho)$$

where $\rho \equiv \frac{\omega_0}{ceE_0}$. So

$$y(t) = \frac{mv_0c}{eE_0}\sinh^{-1}\left(\frac{tceE_0}{\omega_0}\right) \qquad \text{Eq. (1)}$$

Similarly

$$v_z(t) = \frac{p_z(t)}{E(t)/c^2} = \frac{eE_0tc^2}{\sqrt{\omega_0^2 + (ceE_0t)^2}}$$

Thus

$$z(t) = \frac{eE_0c^2}{ceE_0}\int_0^t \frac{tdt}{\sqrt{\rho^2+t^2}} = c\left(\sqrt{(\rho^2+t^2)} - \rho\right) \qquad \text{Eq.(2)}$$

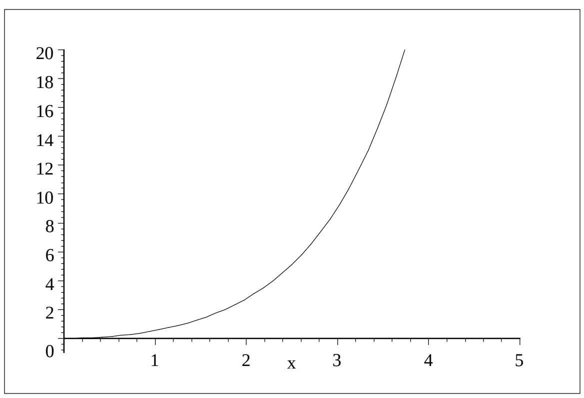
b) From Eq. (1)

$$t = \frac{\omega_0}{ceE_0}\sinh\left(\frac{eE_0y}{mv_0c}\right) = \rho\sinh(ky),\ \text{with } k = \frac{eE_0}{mv_0c}$$

Then from Eq.(2)

$$z = c\rho\left(\sqrt{\sinh^2(ky)+1} - 1\right) \qquad \text{Eq.(3)}$$

Let us plot $\left(\sqrt{\sinh^2(x)+1} - 1\right)$



For small t : $t/\rho << 1$, and $ky << 1$. Thus we can taylor expand Eq.(3) and get

$$z = c\rho k^2y^2/2$$

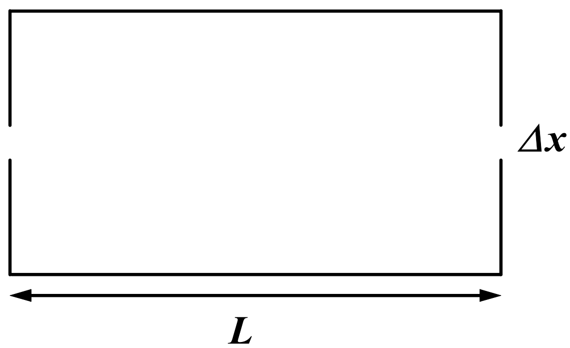
which is quadratic in y giving a parabolic shape.

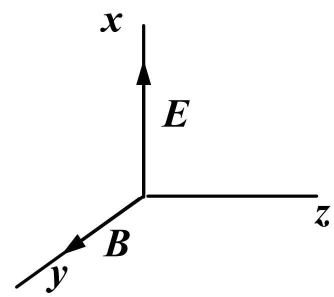
For large t : $t/\rho >> 1$, and we see the sinh term dominates in Eq.(3) and we get

$$z \sim \frac{c\rho e^{ky}}{2}$$

which is an exponential shape.

12.4 The velocity selector and coordinate system are described as





We begin with the Lorentz force

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

With the choice of directions of the field, the requirement that $\frac{d\vec{p}}{dt} = 0$ so that the particle is undeflected yields from the above that

$$\vec{E} + \frac{\vec{u}}{c} \times \vec{B} = 0$$

or from the figure

$$\vec{E} = E\hat{x}$$

$$\vec{B} = B\hat{y}$$

Then $\vec{u} = c\frac{E}{B}\hat{z}$. Now assume $\vec{v} = \vec{u} + \Delta v\hat{z}$, then

$$\frac{d\vec{p}}{dt} = -q\frac{\Delta v}{c}B\hat{x}$$

Or, taking the x component of the above equation

$$\frac{dp_x(t)}{dt} = q\frac{\Delta v}{c}B$$

where I've dropped the minus sign since the sign of the deflection of the particle is unimportant.

$$p_x(t) = m\gamma\frac{dx(t)}{dt} = q\frac{\Delta v}{c}Bt$$

Thus

$$m\gamma\Delta x = q\frac{\Delta v}{c}Bt^2/2$$

but $t = L/u = \frac{LB}{cE}$.

$$\Delta v = \Delta x\frac{2m\gamma c}{qBt^2} = \Delta x\frac{2m\gamma c}{qB\left(\frac{LB}{cE}\right)^2} = \Delta x\frac{2m\gamma c^3E^2}{qB^3L^2}$$

Let us assume that L, u, and E are given. In term of these variables, using $B = E\frac{c}{u}$

$$\Delta v = \Delta x\frac{2m\gamma c^3E^2}{q\left(E\frac{c}{u}\right)^3L^2} = \frac{\Delta x}{L}\left(\frac{m\gamma c^2}{qEL}\right)\left(\frac{u}{c}\right)^2u$$

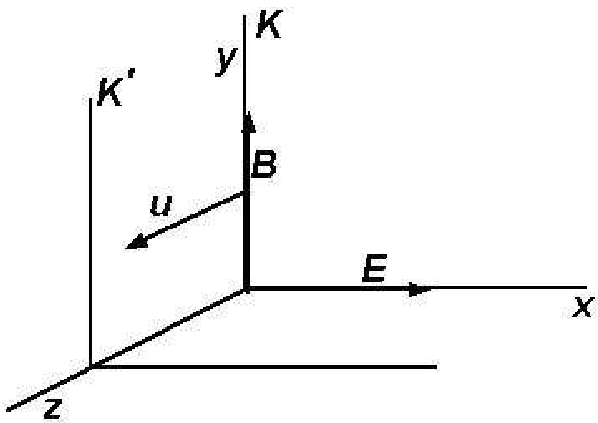
For a numerical example let us take an electron with, $u = c/2, \gamma = 2/\sqrt{3}, L =$ 2m, $E = 3\times10^6$V/m, $\Delta x = 0.5\times10^{-3}$m, $m = 9.1\times10^{-31}$kg, $q = 1.6\times10^{-19}$C.

$$\Delta v = \frac{0.5\times10^{-3}}{2}\left(\frac{9.1\times10^{-31}\cdot(2/\sqrt{3})\cdot\left(3\times10^8\right)^2}{1.6\times10^{-19}\cdot3\times10^6\cdot2}\right)\left(\frac{1}{4}\right)u$$

$$\Delta v = 6.2\times10^{-6}u$$

12.5

a) The system is described by



Background: particle having m, e. Choose $\vec{u} \perp$ to $\vec{B}$ and $\vec{E}$. We want $\vec{E}'_{\perp} = 0 = \gamma(E + \frac{\vec{u}}{c} \times \vec{B})$. Thus $\frac{\vec{u}}{c} \times \vec{B} = -\vec{E}$; now $\vec{B} \times \left(\vec{u} \times \vec{B}\right) = c\vec{E} \times \vec{B}$. Using BAC - CAB on the lhs of the equation gives $\vec{u} = c\frac{\vec{E}\times\vec{B}}{B^2}$. Thus from the figure,

$$\vec{u} = c\frac{\vec{E} \times \vec{B}}{B^2} = c\frac{E}{B}\hat{z}$$

Then, using Eq. (11.149)

$$\vec{E}_{\parallel} = 0; \quad \vec{E}'_{\perp} = 0; \quad \vec{B}'_{\parallel} = 0; \quad \vec{B}'_{\perp} = \frac{1}{\gamma}\vec{B} = \sqrt{1 - \left(\frac{u}{c}\right)^2}\ \vec{B}$$

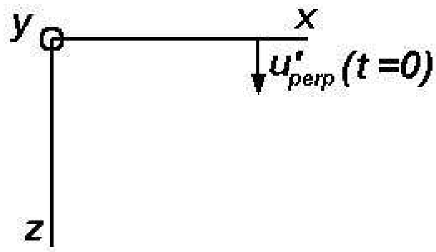
So

$$\vec{B}'_{\perp} = \sqrt{1 - \left(\frac{E}{B}\right)^2}\,\vec{B} = \sqrt{\frac{B^2 - E^2}{B^2}}\,B\hat{\varepsilon}_2$$

Now from the class notes,

$$\frac{d\vec{u}'}{dt'} = \vec{u}' \times \vec{\omega}_{B'}, \quad \text{where } \vec{\omega}_{B'} = \frac{eB'}{E'}, \quad \text{where in this case } E' \text{ is the energy of the particle.}$$

I’ll choose the same boundary conditions as in class, decribed in the figure.



So

$$\vec{u}'_\perp = \omega_{B'} a[\cos(\omega_{B'} t')\hat{\varepsilon}_3 - \sin(\omega_{B'} t')\hat{\varepsilon}_1]$$

where $u'_\perp(t = 0) = \omega_{B'} a$ (ie, the BC determine a).

$$\vec{x}'(t') = u'_\| t' \hat{\varepsilon}_2 + a(\hat{\varepsilon}_3 \sin\omega_{B'} t' + \hat{\varepsilon}_1 \cos\omega_{B'} t')$$

Consider the inverse Lorentz transformation between the frames,

$$\begin{pmatrix} ct \\ z \\ x \\ y \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ z' \\ x' \\ y' \end{pmatrix}$$

or

$$t = (\gamma t' + \beta\gamma z')/c = (\gamma t' + \beta\gamma a \sin\omega_{B'} t')/c \equiv f(t') \to t' = f^{-1}(t)$$

So

$$z(t) = \beta\gamma c f^{-1}(t) + \gamma a \sin\omega_{B'} f^{-1}(t)$$

$$x(t) = a\cos\omega_{B'} f^{-1}(t)$$

$$y(t) = u'_{\|0} f^{-1}(t)$$

b) If $|E| > |B|$, one can transform to a frame where the field is a static $\vec{E}$ field alone. Then the solution is as we found in section 12.3 of the text, with the above transformation taking you to the unprimed frame.

12.14

a) We are given

$$\mathsf{L} = -\frac{1}{8\pi}\partial_\alpha A_\beta \partial^\alpha A^\beta - \frac{1}{c}J_\alpha A^\alpha$$

which can be rewritten

$$\mathsf{L} = -\frac{1}{8\pi}\partial_\beta A^\alpha \partial^\beta A_\alpha - \frac{1}{c}J_\alpha A^\alpha$$

Using the Euler-Lagrange equations of motion,

$$\partial^\beta \frac{\partial \mathsf{L}}{\partial(\partial^\beta A_\alpha)} - \frac{\partial \mathsf{L}}{\partial A_\alpha} = 0$$

Noting

$$\frac{\partial \mathsf{L}}{\partial(\partial^\beta A_\alpha)} = -\frac{1}{4\pi}\partial_\beta A^\alpha$$

$$\frac{\partial \mathsf{L}}{\partial A_\alpha} = -\frac{1}{c}J^\alpha$$

The Euler-Lagrange equations of motion are

$$\partial^\beta(\partial_\beta A^\alpha) = \partial_\beta(\partial^\beta A^\alpha) = \frac{4\pi}{c}J^\alpha$$

or

$$\partial_\beta(\partial^\beta A^\alpha - \partial^\alpha A^\beta + \partial^\alpha A^\beta) = \partial_\beta F^{\beta\alpha} + \partial_\beta \partial^\alpha A^\beta = \frac{4\pi}{c}J^\alpha$$

If we assume the Lorentz gauge, $\partial_\beta A^\beta = 0,$ then the above reduces to

$$\partial_\beta F^{\beta\alpha} = \frac{4\pi}{c}J^\alpha$$

Maxwell's equations, given by Eq. (11.141).

b) Eq. (12.85) gives

$$-\frac{1}{16\pi}(F_{\alpha\beta}F^{\alpha\beta}) - \frac{1}{c}J_\alpha A^\alpha$$

The term in parentheses can be written

$$F_{\alpha\beta}F^{\alpha\beta} = 2\partial_\alpha A_\beta \partial^\alpha A^\beta - 2\partial_\alpha(A_\beta \partial^\beta A^\alpha) + 2A_\beta \partial^\beta \partial_\alpha A^\alpha$$

The last term vanishes if we choose the Lorentz gauge, and the second term is of the form of a 4-divergence. Thus the Lagrangian of this problem differs from the usual one, of Eq. (12.85) by a 4-divergence $\partial_\alpha(A_\beta \partial^\beta A^\alpha)$.

The 4-divergence does not change the euations of motion since the fields vanish at the limits of integration given by the action. Using the generalized Gauss's theorem or by integrating by parts, we see the 4-divergence gives zero contribution to the action.

More Problems for Chapter 12

Problem 12.2

(a) Let the Lagrangian L be replaced by

$$L' = L + \frac{d}{dt}\Omega(x_\alpha),$$

with Ω a given function of the coordinates x_α. The action is

$$A = \int_{t_1}^{t_2} L dt, \qquad \Rightarrow \qquad A' = \int_{t_1}^{t_2} L dt + \int_{t_1}^{t_2} \frac{d\Omega}{dt} dt = A + \Omega(x_\alpha)|_{t_1}^{t_2}$$

The variation of the action

$$\delta A' = \delta A + \delta \left\{ \Omega(x_\alpha)|_{t_1}^{t_2} \right\} = \delta A$$

since, under the variation of the paths $x_\alpha(t)$, the end points remain fixed. Thus L and L' yield the same Euler-Lagrange equations.

(b)

$$A^\alpha \to A^\alpha + \frac{\partial \Lambda}{\partial x_\alpha}$$

The Lagrangian is

$$L = -mc^2 \sqrt{1 - \frac{u^2}{c^2}} + \frac{e}{c} \vec{u} \cdot \vec{A} - e\Phi$$

Under the gauge transformation, we have

$$\vec{A} \to \vec{A} - \nabla\Lambda, \qquad \Phi \to \Phi + \frac{1}{c}\frac{\partial \Lambda}{\partial t}$$

then

$$L \to L' = -mc^2 \sqrt{1 - \frac{u^2}{c^2}} + \frac{e}{c} \vec{u} \cdot \vec{A} - e\Phi - \frac{e}{c} \vec{u} \cdot \nabla\Lambda - \frac{e}{c}\frac{\partial \Lambda}{\partial t}$$

$$= -mc^2\sqrt{1-\frac{u^2}{c^2}} + \frac{e}{c}\vec{u}\cdot\vec{A} - e\Phi - \frac{e}{c}\left\{\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right\}\Lambda = L - \frac{e}{c}\frac{d\Lambda}{dt}$$

Since L' and L differ by a total time derivative, the two Lagrangians yield the same equations of motion.

Problem 12.3

(a) The motion of the particle is governed by Eq. (11.144):

$$\frac{dU^\alpha}{d\tau} = \frac{e}{mc}F^{\alpha\beta}U_\beta$$

Rewriting it in terms of familiar particle velocity $\vec{v}$ and electric field $\vec{E}$, we get

$$\frac{d(\gamma c)}{d\tau} = \frac{\gamma e}{mc}\vec{E}\cdot\vec{v}; \qquad \frac{d(\gamma\vec{v})}{d\tau} = \frac{\gamma e}{m}\vec{E}$$

Let $\eta \equiv eE/mc$, $v_{||}$ and $v_\perp$ be the parallel and perpendicular components of the velocity defined by the direction of $\vec{E}$, we then have the following three equations:

$$\frac{d(\gamma c)}{d\tau} = \frac{eE}{mc}(\gamma v_{||}) = \eta(\gamma v_{||}); \qquad \frac{d(\gamma v_{||})}{d\tau} = \frac{eE}{mc}(\gamma c) = \eta(\gamma c); \qquad \frac{d(\gamma v_\perp)}{d\tau} = 0$$

Integrating the last equation, $\gamma v_\perp = \text{constant} \equiv \alpha$, thus $v_\perp = \alpha/\gamma$. From the remaining equations, we get

$$\frac{d^2}{d\tau^2}(\gamma v_{||}) = \eta^2(\gamma v_{||}) \qquad \Rightarrow \qquad \gamma v_{||} = A\sinh(\eta\tau) + B\cosh(\eta\tau)$$

$$\frac{d^2}{d\tau^2}(\gamma c) = \eta^2(\gamma c) \qquad \Rightarrow \qquad \gamma c = A\cosh(\eta\tau) + B\sinh(\eta\tau)$$

where A and B are the same constants due to $d(\gamma v_{||})/d\tau = \eta(\gamma c)$. The three constants $(\alpha,\ A,\ B)$ are determined by the initial condition:

$$\text{At } \tau = 0:\ v_{||} = 0,\ \ v_\perp = v_0 \qquad \Rightarrow \qquad \alpha = \gamma_0 v_0,\ A = \gamma_0 c,\ B = 0$$

where $\gamma_0 = 1/\sqrt{1-v_0^2/c^2}$. Thus

$$\gamma = \gamma_0\cosh(\eta\tau); \qquad v_{||} = c\tanh(\eta\tau); \qquad v_\perp = \frac{v_0}{\cosh(\eta\tau)}$$

These results are expressed in terms of proper time. To rewrite them as functions of laboratory time, we use $dt = \gamma d\tau$:

$$t = \int_0^\tau \gamma d\tau = \int_0^\tau \gamma_0\cosh(\eta\tau)d\tau = \frac{\gamma_0}{\eta}\sinh(\eta\tau)$$

Thus

$$\sinh(\eta\tau) = \frac{\eta t}{\gamma_0}; \qquad \cosh(\eta\tau) = \sqrt{1+\eta^2t^2/\gamma_0^2}; \qquad \tanh(\eta\tau) = \frac{\eta t}{\gamma_0}\frac{1}{\sqrt{1+\eta^2t^2/\gamma_0^2}}$$

Therefore,

$$\gamma = \gamma_0\sqrt{1+\frac{\eta^2t^2}{\gamma_0^2}}; \qquad v_{||} = \frac{\eta ct}{\gamma_0\sqrt{1+\eta^2t^2/\gamma_0^2}}, \qquad v_\perp = \frac{v_0}{\sqrt{1+\eta^2t^2/\gamma_0^2}}$$

In the coordinate system defined by $\vec{E} = E\hat{z}$ and $\vec{v_0} = v_0\hat{x}$, and assuming the particle is at the origin initially, the the position of the particle is given by

$$x = \int_0^t v_\perp dt = \int_0^t \frac{v_0 dt}{\sqrt{1+\eta^2t^2/\gamma_0^2}} = \frac{v_0\gamma_0}{\eta}\sinh^{-1}(\frac{\eta t}{\gamma_0})$$

$$z = \int_0^t v_{||} dt = \int_0^t \frac{\eta c t dt}{\gamma_0 \sqrt{1 + \eta^2 t^2/\gamma_0^2}} = \frac{c\gamma_0}{\eta} \left\{ \sqrt{1 + \frac{\eta^2 t^2}{\gamma_0^2}} - 1 \right\}$$

We could also get this result by starging from the Lorentz force equations (11.124):

$$\frac{d\vec{p}}{dt} = e\vec{E} = eE\hat{z}, \qquad \Rightarrow \qquad \vec{p} = \vec{p}_0 + eEt\hat{z}$$

In perpendicular and parallel components:

$$p_{||} = \gamma m v_{||} = eEt, \qquad p_{\perp} = \gamma m v_{\perp} = \gamma_0 m v_0$$

Then

$$\gamma v_{||} = \frac{eE}{m} t = \eta c t, \qquad \gamma v_{\perp} = \gamma_0 v_0$$

so

$$\gamma^2 v^2 = \gamma_0^2 v_0^2 + \eta^2 c^2 t^2 \qquad \Rightarrow \qquad v^2 = \frac{\gamma_0^2 v_0^2 + \eta^2 c^2 t^2}{\gamma_0^2 + \eta^2 t^2} \text{ or } \gamma = \sqrt{\gamma_0^2 + \eta^2 t^2}$$

(b) To determine the trajectory, we need to eliminate the time-dependence. From the equation for x, we get

$$\frac{\eta t}{\gamma_0} = \frac{\gamma_0}{\eta} \sinh(\frac{\eta x}{\gamma_0 v_0})$$

Pluggint it into the equation for z:

$$z = \frac{c\gamma_0}{\eta} \left\{ \sqrt{1 + \sinh^2(\frac{\eta x}{\gamma_0 v_0})} - 1 \right\} = \frac{c\gamma_0}{\eta} \left\{ \cosh(\frac{\eta x}{\gamma_0 v_0}) - 1 \right\}$$

For $t \ll \gamma_0/\eta$ (*i.e.* $x \ll \eta/(\gamma_0 v_0)$):

$$\cosh(\frac{\eta x}{\gamma_0 v_0}) \approx 1 + \frac{1}{2}(\frac{\eta x}{\gamma_0 v_0})^2$$

$$z \approx \frac{c\gamma_0}{\eta}(\frac{\eta^2 x^2}{2\gamma_0^2 v_0^2}) = \frac{1}{2}\frac{\eta c}{\gamma_0 v_0^2} x^2$$

It is a parabola. In terms of t, we have

$$x \approx v_0 t, \qquad z \approx \frac{c\eta}{2\gamma_0} t^2 = \frac{eE}{2m\gamma_0} t^2$$

For $t \gg \gamma_0/\eta$:

$$x \approx \frac{\gamma_0 v_0}{\eta} \ln(\frac{2\eta t}{\gamma_0}), \qquad z \approx ct$$

Eliminating t:

$$z \approx \frac{c\gamma_0}{2\eta} e^{\eta x} \gamma_0 v_0$$

The particle moves along the $z-$direction with a speed close to c with a gradual motion in $x-$direction.

Problem 12.6(b)

Choose the $z-$axis along the $\vec{E}$ and $\vec{B}$ direction, we have

$$F^{03} = -E, \; F^{12} = -B, \; F^{21} = B, \; F^{30} = E, \text{ and the rest } F^{\alpha\beta} = 0$$

The equation:

$$\frac{dU^\alpha}{d\tau} = \frac{e}{mc} F^{\alpha\beta} U_\beta$$

becomes

$$\frac{dU^0}{d\tau} = -\frac{eE}{mc}U_3, \quad \frac{dU^1}{d\tau} = -\frac{eB}{mc}U_2, \quad \frac{dU^2}{d\tau} = \frac{eB}{mc}U_1, \quad \frac{dU^3}{d\tau} = \frac{eE}{mc}U_0$$

Use $U^\alpha = dx^\alpha/d\tau$, the above four equations become to:

$$\frac{d^2(ct)}{d\tau^2} = \frac{eE}{mc}\frac{dz}{d\tau}, \quad \frac{d^2x}{d\tau^2} = \frac{eB}{mc}\frac{dy}{d\tau}, \quad \frac{d^2y}{d\tau^2} = -\frac{eB}{mc}\frac{dx}{d\tau}, \quad \frac{d^2z}{d\tau^2} = \frac{eE}{mc}\frac{d(ct)}{d\tau}$$

Integrating over proper time,

$$\frac{d(ct)}{d\tau} = \frac{eE}{mc}z, \quad \frac{dx}{d\tau} = \frac{eB}{mc}y, \quad \frac{dy}{d\tau} = -\frac{eB}{mc}x, \quad \frac{dz}{d\tau} = \frac{eE}{mc}(ct)$$

Let $\omega \equiv eB/mc$ and $\eta \equiv eE/mc$, the second and the third equations are coupled and can be solved

$$\frac{d^2x}{d\tau^2} = -\omega^2 x, \; \frac{d^2y}{d\tau^2} = -\omega^2 y \quad \Rightarrow \quad x \sim \sin(\omega\tau), \;\; y \sim \cos(\omega\tau) \text{ (by an appropriate choice of axes)}$$

Note that

$$x\frac{dx}{d\tau} + y\frac{dy}{d\tau} = 0 \quad \Rightarrow \quad x^2 + y^2 = \text{constant} \equiv \mathcal{A}^2 R^2$$

Therefore,

$$x = \mathcal{A}R\sin\phi, \quad y = \mathcal{A}R\cos\phi \quad \text{with } \phi = \omega\tau$$

Also

$$\frac{d^2z}{d\tau^2} = \eta^2 z, \quad \frac{d^2(ct)}{d\tau^2} = \eta^2(ct) \quad \Rightarrow \quad z \sim \cosh(\eta\tau), \quad ct \sim \sinh(\eta\tau)$$

Note that

$$ct\frac{d}{d\tau}(ct) - z\frac{d}{d\tau}z = 0 \quad \Rightarrow \quad z^2 - c^2t^2 = \text{constant} \equiv \mathcal{B}^2$$

Therefore,

$$z = \mathcal{B}\cosh(\rho\phi), \quad ct = \mathcal{B}\sinh(\rho\phi) \quad \text{with } \; \rho\phi = \eta\tau \;\; (\rho = \frac{\eta}{\omega} = \frac{E}{B})$$

Thus the position and velocity 4-vectors are

$$x^\alpha = (ct, x, y, z) = (\mathcal{B}\sinh(\rho\phi), \mathcal{A}R\sin\phi, \mathcal{A}R\cos\phi, \mathcal{B}\cosh(\rho\phi))$$

$$U^\alpha = (\eta z, \omega y, -\omega x, \eta(ct)) = (\mathcal{B}\eta\cosh(\rho\phi), \mathcal{A}\omega R\cos\phi, -\mathcal{A}\omega R\sin\phi, \mathcal{B}\eta\sinh(\rho\phi))$$

From $U^\alpha U_\alpha = c^2$, we get

$$\mathcal{B}^2\eta^2\cosh^2(\rho\phi) - \mathcal{A}^2\omega^2R^2\cos^2\phi - \mathcal{A}^2\omega^2R^2\sin^2\phi - \mathcal{B}^2\eta^2\sinh^2(\rho\phi) = c^2 \quad \Rightarrow \quad \mathcal{B}^2\eta^2 - \omega^2\mathcal{A}^2R^2 = c^2$$

which leads to

$$\mathcal{B} = \frac{1}{\eta}\sqrt{c^2 + \omega^2\mathcal{A}^2R^2} = \frac{\omega R}{\eta}\sqrt{\mathcal{A}^2 + \frac{c^2}{\omega^2R^2}} = \frac{R}{\rho}\sqrt{1+\mathcal{A}^2}$$

Therefore we have

$$x = \mathcal{A}R\sin\phi, \quad y = \mathcal{A}R\cos\phi, \quad z = \frac{R}{\rho}\sqrt{1+\mathcal{A}^2}\cosh(\rho\phi), \quad ct = \frac{R}{\rho}\sqrt{1+\mathcal{A}^2}\sinh(\rho\phi)$$

More Problems for Chapter 12

Problem 12.5

(a) For $|\vec{E}| < |\vec{B}|$, we can also find a frame K' in which $\vec{E'} = 0$. In this frame, the particle is moving in a uniform magnetic field $\vec{B'}$. Let $\vec{E}$ pointing to $+x$ and $\vec{B}$ pointing to $+y$ direction, the velocity of frame K' in frame K can be obtained from Eq. 12.43 to be

$$\vec{u} = c\frac{\vec{E}\times\vec{B}}{B^2}.$$

Thus,

$$\vec{E'} = 0, \qquad \vec{B'} = \frac{1}{\gamma}\vec{B} = \sqrt{B^2 - E^2}\frac{\vec{B}}{B}$$

In frame K' with a Cartesian coordinate system, the motion will be helix, *i.e.*, an uniform motion along $\vec{B'}$ and gyration in the transverse plane. With a properly chosen origin, the position of the particle can be written as

$$x' = a\cos(\omega_B t'); \qquad y' = v_{||}t'; \qquad z' = a\sin(\omega_B t')$$

where a is the gyration radius determined by particle's transverse momentum ($cp_\perp = eB'a$) and $v'_{||}$ is the velocity component along the $\vec{B'}$ in frame K', $\omega_B = eB'/(\gamma' mc)$ and γ' is the Lorentz boost factor of the particle in frame K'. Translating back to frame K:

$$\begin{aligned} x =&\ x' = a\cos(\omega_B t') \\ y =&\ y' = v_{||}t' \\ z =&\ \gamma(z' + ut') = \gamma\{a\sin(\omega_B t') + ut'\} = \tfrac{B}{\sqrt{B^2-E^2}}\{a\sin(\omega_B t') + \tfrac{E}{B}(ct')\} \end{aligned}$$

These are explicit parametric equations for the particle's trajectory in terms of parameter t'. (b) For the case of $|\vec{E}| > |\vec{B}|$, the magnetic field $\vec{B'}$ vanishes in the frame K' moving with a velocity

$$\vec{u} = c\frac{\vec{E}\times\vec{B}}{E^2}.$$

In this frame, the particle moves in a uniform electric field $\vec{E'}$:

$$\vec{E'} = \frac{\vec{E}}{\gamma'} = \sqrt{E^2 - B^2}\frac{\vec{E}}{E}$$

Thus

$$\frac{d\vec{p'}}{dt'} = q\vec{E'}$$

which leads to

$$\frac{d}{dt'}(\gamma' m v'_x) = qE', \quad \frac{d}{dt'}(\gamma' m v'_y) = 0, \quad \frac{d}{dt'}(\gamma' m v'_z) = 0$$

The differential equations for the most general case of initial velocities are difficult to integrate. Assuming the particle is at rest for simplicity, integrating the above equations

$$v'_x = \frac{dx'}{dt'} = c\frac{\alpha t'}{\sqrt{1 + \alpha^2 t'^2}}, \quad v'_y = 0, \quad v'_z = 0$$

Here $\alpha = qE'/mc$. Integrating the above equations (with a properly chosen origin):

$$x'(t') = \frac{c}{\alpha}\left\{\sqrt{1+\alpha^2 t'^2} - 1\right\}; \qquad y'(t') = 0; \qquad z'(t') = 0$$

Translating back to frame K, the parametric equations for the particle's trajectory are

$$\begin{aligned} x =& \; x' = \tfrac{c}{\alpha}\left\{\sqrt{1+\alpha^2 t'^2} - 1\right\} \\ y =& \; y' = 0 \\ z =& \; \gamma(z' + ut') = \gamma u t' = \tfrac{B}{\sqrt{E^2 - B^2}}(ct') \end{aligned}$$

Problem 12.9

(a) Let $z-$axis points from south to north, in this case, $\vec{M} = -M\ \hat{z}$. The vector potential of the earth's magnetic dipole moment

$$\vec{A}(\vec{r}) = \frac{\vec{M} \times \vec{r}}{r^3} = -\frac{M \sin\theta}{r^2}\hat{\phi} \equiv A_\phi \hat{\phi}$$

Thus the magnetic field $\vec{B}$:

$$\vec{B}(\vec{r}) = \nabla \times \vec{A} = -\frac{2M\cos\theta}{r^3}\ \hat{r} - \frac{M\sin\theta}{r^3}\ \hat{\theta}$$

Let $d\vec{s}$ be the small displacement

$$d\vec{s} = \hat{r}dr + \hat{\theta}r d\theta + \hat{\phi} r \sin\theta d\phi$$

and let $d\vec{s}$ point to the direction of the magnetic field, we get

$$\frac{dr}{B_r} = \frac{r d\theta}{B_\theta} \qquad \Rightarrow \qquad \frac{dr}{r} = \frac{2\cos\theta}{\sin\theta} d\theta$$

Integrating the above equation yields the equation for a line magnetic force to be:

$$r = r_0 \sin^2\theta$$

The magnetic field as a function of θ

$$B = \sqrt{B_r^2 + B_\theta^2} = \frac{M}{r^3}\sqrt{4\cos^2\theta + \sin^2\theta} = \frac{M}{r_0^3}\frac{\sqrt{1+3\cos^2\theta}}{\sin^6\theta}$$

(b) The gradient drift velocity is given by Eq. (12.55):

$$\vec{V}_G = \omega_B a \cdot \frac{a}{2B^2}\vec{B} \times \nabla_\perp B$$

where $\vec{B}$ is the field at the equator:

$$\vec{B} = -\frac{M}{r^3}|_{r=R}\hat{\theta} = -\frac{M}{R^3}\hat{\theta}$$

and B is its magnitude. Since the problem is azimuthal symmetric, we have

$$\nabla_\perp B = \frac{\partial B}{\partial r}|_{r=R}\hat{r} = -\frac{3M}{R^4}\hat{r}$$

Thus

$$\vec{V}_G = \frac{\omega_B a^2}{2B^2}(-B\hat{\theta}) \times \left\{-\frac{3M}{R^4}\hat{r}\right\} = -\frac{3\omega_B a^2}{2R}\hat{\phi}$$

Now note that

$$\vec{V}_G = R\dot{\phi}\hat{\phi}, \quad \Rightarrow \quad R\dot{\phi} = -\frac{3a^2}{2R}\omega_B$$

Integrating the above equation of motion,

$$\phi(t) = \phi_0 - \frac{3a^2}{2R^2}\omega_0(t - t_0)$$

(c) Let $\theta = \pi/2 + \alpha$, the magnetic field along the line of the force is then given by

$$B(\alpha) = \frac{M}{R^3}\frac{\sqrt{1 + 3\sin^2\alpha}}{\cos^6\alpha}$$

For small α values,

$$B(\alpha) \approx \frac{M}{R^3}\frac{\sqrt{1+3\alpha^2}}{(1-\alpha^2/2)^6} \approx \frac{M}{R^3}(1 + \frac{9}{2}\alpha^2 + ...)$$

Note that from Eq. (12.72), we have

$$v_{||}^2(\alpha) = v^2(0) - v_\perp^2(0)\frac{B(z)}{B_0} = v^2(0) - v_\perp^2(0)(1 + \frac{9}{2}\alpha^2) = v_{||}^2(0) - \frac{9}{2}v_\perp^2(0)\alpha^2$$

Note that $v_\perp(0) = \omega_B a$ and $v_{||}^2(\alpha) = (R\dot{\alpha})^2$. Plugging these into the above equation, we get

$$\dot{\alpha}^2 + \frac{9\omega_B^2 a^2}{2R^2}\alpha^2 = \frac{v_{||}^2(0)}{R^2}$$

This is the "energy equation" of a harmonic oscillator with the corresponding frequency given by

$$\Omega = \frac{3}{\sqrt{2}}\frac{a}{R}\omega_B$$

The change in azimuth in one period of oscillation is

$$\Delta\phi = \frac{3}{2}(\frac{a}{R})^2\omega_B \times \frac{2\pi\sqrt{2}}{3\omega_B}(\frac{R}{a}) = \sqrt{2}\pi\frac{a}{R}$$

independent of M.

(d) For $R = 3 \times 10^9$ cm ($\sim 5R_{\mathrm{earth}}$), $M = 8.1 \times 10^{25}$ gauss-cm^3, we have

$$B = B_\theta = \frac{8.1 \times 10^{25}}{27 \times 10^{27}} = 3 \times 10^{-3} \text{ gauss}$$

$$\omega_B = \frac{eB}{\gamma mc} = \frac{e}{mc}\frac{B}{\gamma} = 1.76 \times 10^7 \text{ s}^{-1}\text{gauss}^{-1}\ \frac{3 \times 10^{-3}\text{ gauss}}{\gamma} = \frac{5.3 \times 10^4}{\gamma}\ \text{s}^{-1}$$

and $a = v/\omega_B$. The time to drift once around the earth (in azimuth) is

$$T_\phi = 2\pi\frac{2R^2}{3\omega_B a^2}$$

and the time for one oscillation in latitude is

$$T_\theta = \frac{2\pi}{\Omega} = \frac{2\pi\sqrt{2}R}{3\omega_B a} = \frac{2\pi\sqrt{2}}{3}\frac{R}{v}$$

For a 10 MeV electron, we have

$$\gamma = \frac{E}{m} = \frac{10 + 0.511}{0.511} \approx 20.5, \quad v \approx c, \quad \omega_B = 2.57 \times 10^3 \text{ s}^{-1}, \quad a = 117 \text{ km}, \quad T_\phi = 107 \text{ s}, \quad T_\theta = 0.30 \text{ s}$$

For a 10 keV electron, we have

$$\gamma = \frac{E}{m} = \frac{0.511 + 0.010}{0.511} \approx 1.02, \quad v \approx 0.2c, \quad \omega_B = 5.2 \times 10^4 \text{ s}^{-1}, \quad a = 1.2 \text{ km}, \quad T_\phi = 5.5 \times 10^4 \text{ s}, \quad T_\theta = 1.5 \text{ s}$$

Problem 12.11

(a) The Thomas precession formula is

$$(\frac{d\vec{s}}{dt})_{\rm lab} = \frac{1}{\gamma}(\frac{d\vec{s}}{d\tau})_{\rm rest} + \vec{\omega}_T \times \vec{s}$$

$\vec{\omega}_T$ given by Eq. (11.119):

$$\vec{\omega}_T = \frac{\gamma^2}{1+\gamma}\frac{\vec{a}\times\vec{v}}{c^2}$$

From the Lorentz force and Newton's second law, we have

$$\frac{d\vec{p}}{dt} = \frac{e}{c}\vec{v}\times\vec{B}$$

where $\vec{p}$ is muon momentum and e is the muon charge. Since the magnetic field does not do any work, γ is a constant of the motion. Therefore, the above equation can be written as

$$\frac{d\vec{v}}{dt} = \frac{e}{\gamma mc}\vec{v}\times\vec{B} = \left\{-\frac{e}{\gamma mc}\vec{B}\right\}\times\vec{v} = \vec{\omega}_B \times \vec{v}$$

Here $\vec{\omega}_B = -(e\vec{B})/(\gamma mc)$ is the orbital gyration frequency. Therefore the acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \vec{\omega}_B\times\vec{v} = \frac{e}{\gamma mc}\vec{v}\times\vec{B}$$

Therefore

$$\vec{\omega}_T = \frac{\gamma^2}{1+\gamma}\frac{1}{c^2}\frac{e}{\gamma mc}(\vec{v}\times\vec{B})\times\vec{v} = \frac{\gamma^2}{1+\gamma}\frac{e}{\gamma mc}\frac{v^2}{c^2}\vec{B} = \frac{\gamma-1}{\gamma}\frac{e\vec{B}}{mc}$$

The precession in the rest frame is given by Eq. (11.101):

$$(\frac{d\vec{s}}{d\tau})_{\rm rest} = \vec{\mu}\times\vec{B\prime} = \frac{ge}{2mc}\vec{s}\times\vec{B\prime}$$

where

$$\vec{B\prime} = \gamma(\vec{B} - \frac{\vec{v}}{c}\times\vec{E}) - \frac{\gamma^2}{1+\gamma}\frac{\vec{v}}{c}(\frac{\vec{v}\cdot\vec{B}}{c}) = \gamma\vec{B}$$

Then

$$(\frac{d\vec{s}}{dt})_{\rm lab} = \frac{1}{\gamma}(\frac{ge}{2mc}\vec{s})\times(\gamma\vec{B}) + (\frac{\gamma-1}{\gamma}\frac{e\vec{B}}{mc})\times\vec{s} = \frac{e}{mc}\left\{\frac{\gamma-1}{\gamma} - \frac{g}{2}\right\}\vec{B}\times\vec{s} = \vec{W}\times\vec{s}$$

where the spin precession frequency is

$$\vec{W} = (1 - \frac{g}{2} - \frac{1}{\gamma})\frac{e}{mc}\vec{B}$$

The difference between the spin precession and the orbital gyration frequencies is

$$\vec{\Omega} = \vec{W} - \vec{\omega}_B = (1 - \frac{g}{2} - \frac{1}{\gamma})\frac{e\vec{B}}{mc} + \frac{e\vec{B}}{\gamma mc} = \frac{e\vec{B}}{mc}\frac{2-g}{2} = \Omega\frac{\vec{B}}{B} \quad \Rightarrow \quad \Omega = \frac{eBa}{mc}$$

(b) Newton's first law on the centripetal motion,

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{e}{c}\vec{v} \times \vec{B} \quad \Rightarrow \quad \gamma m \frac{v^2}{R} = \frac{evB}{c}$$

Thus, the muon momentum

$$p = \gamma m v = \frac{eRB}{c} = 1.28 \cdot 10^3 \text{ MeV/c}$$

The Lorentz boost factor

$$\gamma = \frac{E}{mc^2} = \frac{\sqrt{p^2c^2 + m^2c^4}}{mc^2} = 12.1$$

The number of periods of precession per observed laboratory mean lifetime is

$$\frac{\gamma\tau_0}{T} = \frac{\gamma\tau_0\Omega}{2\pi} = \frac{eBa\gamma\tau_0}{2\pi mc} = \frac{eB\alpha\gamma\tau_0}{(2\pi)^2 mc} = 7.12$$

(c)

$$\Omega = \frac{eBa}{mc}, \text{ and } \omega_B = \frac{eB}{\gamma mc}, \quad \Rightarrow \quad \Omega = a\gamma\omega_B = \frac{\alpha\gamma}{2\pi}\omega_B$$

(i) E=300 MeV, $m = m_\mu = 106$ MeV,

$$\gamma = \frac{E}{mc^2} = 2.83, \quad \Omega = (\frac{\alpha\gamma}{2\pi})\omega_B = 0.0033\omega_B$$

(ii) E=300 MeV, $m = m_e = 0.511$ MeV,

$$\gamma = \frac{E}{mc^2} = 587, \quad \Omega = (\frac{\alpha\gamma}{2\pi})\omega_B = 0.682\omega_B$$

(iii) E=5 GeV, $m = m_e = 0.511$ GeV,

$$\gamma = \frac{E}{mc^2} = 9.78 \cdot 10^3, \quad \Omega = (\frac{\alpha\gamma}{2\pi})\omega_B = 11.4\omega_B$$

Problem 12.14

$$\mathcal{L} = -\frac{1}{8\pi}\partial_\alpha A_\beta \partial^\alpha A^\beta - \frac{1}{c}J_\alpha A^\alpha$$

(a)

$$\frac{\partial \mathcal{L}}{\partial A^\beta} = -\frac{1}{c}J_\beta$$

$$\partial^\alpha \frac{\partial \mathcal{L}}{\partial(\partial^\alpha A^\beta)} = \partial^\alpha(-\frac{1}{8\pi})(2\partial_\alpha A_\beta) = -\frac{1}{4\pi}\partial_\alpha\partial^\alpha A_\beta$$

Thus, the Euler-Lagrange equations are

$$\partial_\alpha\partial^\alpha A^\beta = \frac{4\pi}{c}J^\beta$$

These are Maxwell's equations in the Lorentz gauge:

$$\partial_\alpha A^\alpha = 0, \quad i.e. \quad \nabla \cdot \vec{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0$$

(b) From Eq. (12.85), we have

$$\mathcal{L}' = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha$$

then

$$\mathcal{L}' - \mathcal{L} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{8\pi} \partial_\alpha A_\beta \partial^\alpha A^\beta = -\frac{1}{16\pi} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha) + \frac{1}{8\pi} \partial_\alpha A_\beta \partial^\alpha A^\beta$$

$$= \frac{1}{8\pi} \partial_\alpha A_\beta \partial^\beta A^\alpha = \frac{1}{8\pi} \left\{ \partial_\alpha (A_\beta \partial^\beta A^\alpha) - A_\beta \partial_\alpha \partial^\beta A^\alpha \right\}$$

The second term vanishes in the Lorentz gauge, and the 1st term is the divergence of a four-vector:

$$\mathcal{L}' - \mathcal{L} = \partial_\alpha \Lambda^\alpha, \quad \text{with} \quad \Lambda^\alpha = \frac{1}{8\pi} A_\beta \partial^\beta A^\alpha$$

The two actions differ by

$$A' - A = \int (\mathcal{L}' - \mathcal{L}) d^4x = \int \partial_\alpha \Lambda^\alpha d^4x = \int_S \Lambda^\alpha d^3x$$

where the surface integral is over the surface in four-dimension. Now note that since Λ is not varied on the surface, we have

$$\delta(A' - A) = \delta \int_S \Lambda^\alpha d^3x = 0$$

Thus the equations of motion are unchanged.

Chapter 13 Problems

Problem 13.1

(a) Let $\vec{v} = v\hat{x}$ be the velocity of the incident particle (of mass M). Since electron is much light ($m \ll M$), $\vec{v}$ is also the velocity of the center-of-mass frame. In this frame, the electron moves at a velocity $-\vec{v}$ before the scattering and therefore its 4-momentum is given by $\mathcal{P}^i_{CM} = (\gamma mc; -\gamma mv, 0, 0)$. After the scattering, the electron energy remains the same, but the momentum is deflected by a scattering angle θ. Thus, the 4-momentum after the scattering is $\mathcal{P}^f_{CM} = (\gamma mc; -\gamma mv\cos\theta, \gamma mv\sin\theta, 0)$, here we have chosen the $x-y$ plane as the scattering plane. The invariant 4-momentum transfer squared is

$$(\delta\mathcal{P})^2 = (\mathcal{P}^f_{CM} - \mathcal{P}^i_{CM})^2 = -(-\gamma mv\cos\theta + \gamma mv)^2 - (\gamma mv\sin\theta)^2 = -2(\gamma mv)^2(1-\cos\theta)$$

$(\delta\mathcal{P})^2$ can also be calculated in the laboratory frame. In this regard, the electron 4-momenta before and after the scattering are given respectively by

$$\mathcal{P}^i_{LAB} = (mc; \vec{0}); \qquad \mathcal{P}^f_{LAB} = (\frac{E}{c}; \vec{p})$$

where E and $\vec{p}$ are electron's energy and momentum after the scattering. The 4-momentum transfer squared calculated using the laboratory variables is

$$(\delta\mathcal{P})^2 = (\mathcal{P}^f_{LAB} - \mathcal{P}^i_{LAB})^2 = (\frac{E}{c} - mc)^2 - p^2 = -2m(E - mc^2)$$

Equating the two 4-momentum transfer squared, we get the energy transfer

$$T(b) \equiv E - mc^2 = -\frac{(\delta\mathcal{P})^2}{2m} = \gamma^2 mv^2(1-\cos\theta) = 2\gamma^2 mv^2\sin^2\frac{\theta}{2}$$

The angle factor can be calculated from the relationship between b and θ,

$$b = \frac{ze^2}{\gamma mv^2}\cot\frac{\theta}{2} \quad\Rightarrow\quad \sin^2\frac{\theta}{2} = \frac{1}{1 + (\gamma mv^2 b)^2/(ze^2)^2} = (\frac{ze^2}{\gamma mv^2})^2\frac{1}{b^2 + b^2_{min}}$$

where $b_{min} = ze^2/\gamma mv^2$. Thus the energy transfer is

$$T(b) = 2\gamma^2 mv^2\sin^2\frac{\theta}{2} = \frac{2z^2e^4}{mv^2}\frac{1}{b^2 + b^2_{min}}$$

(b) The transverse electric field is

$$E_\perp = E_2 = \frac{\gamma zeb}{(b^2 + \gamma^2v^2t^2)^{3/2}}$$

The transverse momentum impulse

$$\Delta p = \int F_\perp dt = e\int E_\perp dt = \gamma ze^2 b\int_{-\infty}^{\infty}\frac{dt}{(b^2+\gamma^2v^2t^2)^{3/2}} = \frac{2ze^2}{vb}$$

The energy transfer

$$T \approx \frac{(\Delta p)^2}{2m} = \frac{2z^2e^4}{mv^2}\frac{1}{b^2}$$

Withe the exception of the cutoff b_{min} in the exact classical calculation, the two results are the same. Note that the energy transfer diverges without the cutoff b_{min}. This is because the two particles can get infinitely close to each other with the assumption we made in (b). In practice, this cannot be the case.

Problem 13.11

Fields of a magnetic monopole g are the same as for a charge q, with the exchanges $\vec{E} \to \vec{B}$, $\vec{B} \to -\vec{E}$ and $q \to g$. For a magnetic particle moving in the x direction, there is only an electric field in the z direction (if the observation point is on the $y-$axis). Following Eqs. (11.152) we have,

$$B_1 = -\frac{g\gamma vt}{(b^2+\gamma^2v^2t^2)^{3/2}}; \quad B_2 = \frac{\gamma gb}{(b^2+\gamma^2v^2t)^{3/2}}; \quad E_3 = -\beta B_2 = -\frac{\gamma(\beta g)b}{(b^2+\gamma^2v^2t^2)^{3/2}}$$

Since a magnetic field does no work and therefore does not cause energy transfer, the energy loss is mainly caused by the action of the electric field of the passing particle on the atomic electrons. The momentum transfer can be calculated in exactly the same way as in Prob. 13.1(b) with the following replacement for the electric field:

$$E_\perp = \frac{\gamma zeb}{(b^2+\gamma^2v^2t^2)^{3/2}} \quad \Rightarrow \quad E_\perp = \frac{\gamma(\beta g)b}{(b^2+\gamma^2v^2t^2)^{3/2}}$$

Therefore, the momentum and energy transfer can be obtained with the replacement $ze \to \beta g$:

$$\Delta p_g = \frac{2e(\beta g)}{bv} = \frac{2eg}{bc}; \qquad T_g(b) = \frac{2e^2(\beta g)^2}{mv^2} = \frac{2g^2e^2}{mc^2}\frac{1}{b^2}$$

Since the limits on $b_{\max}$ and $b_{\min}$ are essentially the same, having to do with the electrons binding frequency and the electron's Compton wavelength, the whole calculation proceeds as before. The Bethe formula thus has the following analog for energy loss by a magnetic monopole:

$$\frac{dE}{dx} \approx 4\pi NZ\frac{g^2e^2}{mc^2}\ln\left\{\frac{2\gamma^2mv^2}{\hbar\langle\omega\rangle}\right\} = 4\pi NZ\frac{g^2e^2}{mc^2}\ln\left\{\frac{2\gamma^2\beta^2mc^2}{\hbar\langle\omega\rangle}\right\} = 4\pi NZ\frac{g^2e^2}{mc^2}\left\{2\ln(\gamma\beta) + \ln\{\frac{2mc^2}{\hbar\langle\omega\rangle}\}\right\}$$

We have omitted the v^2/c^2 term, because its presence in the monopole situation is not clear. It comes in part from close collisions of the electrons with nuclei and involves the electron's spin. Evidently, the loss is linear in $\ln(\gamma\beta)$. At high energies, the dE/dx energy loss by a monopole is identical to that of a charged particle. The difference is at low energies where dE/dx is more or less flat for monoples. However it should be noted that the formula above is not valid for an extremely slow monopole.

(b) Dirac quantization condition is

$$\frac{ge}{\hbar c} = \frac{n}{2} \quad \Rightarrow \quad g = \frac{n}{2}e\,\frac{\hbar c}{e^2} = \frac{137}{2}ne$$

Thus the losses in the two cases can be written as

$$(\frac{dE}{dx})_{ze} = 4\pi NZ\frac{z^2e^4}{mc^2\beta^2}\ln\left\{\frac{2\gamma^2\beta^2mc^2}{\hbar\langle\omega\rangle}\right\} = 4\pi NZ\frac{e^4}{mc^2}\left\{\frac{z^2}{\beta^2}\right\}\ln\left\{\frac{2\gamma^2\beta^2mc^2}{\hbar\langle\omega\rangle}\right\}$$

$$(\frac{dE}{dx})_g = 4\pi NZ\frac{g^2e^2}{mc^2\beta^2}\ln\left\{\frac{2\gamma^2\beta^2mc^2}{\hbar\langle\omega\rangle}\right\} = 4\pi NZ\frac{e^4}{mc^2}\left\{\frac{137n}{2}\right\}^2\ln\left\{\frac{2\gamma^2\beta^2mc^2}{\hbar\langle\omega\rangle}\right\}$$

For $\beta \sim 1$, the charged particle will lose energy at the same rate as a monopole provided $z = 137n/2$. For $n = 1$, $z = 68.5$. For $n = 2$, $z = 137$. A Dirac monopole is thus expected to ionize and lose energy like a relativistic heavy nucleus. At low energies, the log-term $\ln(2mc^2/\hbar\langle\omega\rangle)$ dominates and therefore the loss is more or less constant.

14.2

Background. In the nonrelativistic approximaion the Lienard-Wiechert potenials are

$$\phi(\vec{x},t) = \frac{e}{R}|_{ret}, \qquad \vec{A}(\vec{x},t) = \frac{e\vec{\beta}}{R}|_{ret}$$

Let us assume that we observe the radiation close enough to source so $R/c << 1$ and $t' \cong t$. Then in the radiation zone

$$\vec{B} = \vec{\nabla}\times\vec{A} = \hat{n}\frac{\partial}{\partial R}\times\vec{A} = -\hat{n}\frac{\partial}{c\partial t'}\times\vec{A} = \frac{e\dot{\vec{\beta}}\times\hat{n}}{cR}$$

$$\vec{E} = \vec{B}\times\hat{n}$$

$$\frac{dP(t)}{d\Omega} = \frac{c}{4\pi}\left|R\vec{B}\right|^2 = \frac{e^2}{4\pi c}\left|\dot{\beta}\times\hat{n}\right|^2 = \frac{e^2}{4\pi c^3}|\dot{v}|^2\sin^2\theta$$

where θ is the angle between $\hat{n}$ and $\dot{\beta}$ (assuming here the particle is moving linearly)

$$P(t) = \frac{2e^2}{3c^3}|\dot{v}|^2$$

Let the time-average be defined by

$$\langle f(t)\rangle \equiv \frac{1}{\tau}\int_0^\tau f(t)dt$$

Then

$$\left\langle\frac{dP(t)}{d\Omega}\right\rangle = \frac{e^2}{4\pi c^3}\sin^2\theta\left\langle|\dot{v}|^2\right\rangle$$

$$\langle P(t)\rangle = \frac{2e^2}{3c^3}\left\langle|\dot{v}|^2\right\rangle$$

a) Suppose $\vec{x}(t) = \hat{z}a\cos\omega_0 t$. Then $\dot{v} = \frac{d^2z}{dt^2} = -a\omega_0^2\cos\omega_0 t$

$$\left\langle|\dot{v}|^2\right\rangle = (a\omega_0^2)^2\frac{1}{\tau}\int_0^\tau\cos^2\omega_0 t dt = (a\omega_0^2)^2/2$$

So

$$\left\langle\frac{dP(t)}{d\Omega}\right\rangle = \frac{e^2}{8\pi c^3}(a\omega_0^2)^2\sin^2\theta$$

$$\langle P(t)\rangle = \frac{e^2}{3c^3}(a\omega_0^2)^2$$

b) Suppose $\vec{x}(t) = R(\hat{\imath}\cos\omega_0 t + \quad \sin\omega_0 t)$. Then

$$\dot{v}(t) = -R\omega_0^2(\hat{\imath}\cos\omega_0 t + \quad \sin\omega_0 t)$$

$$\hat{n} \times \dot{v} = \begin{vmatrix} \hat{\imath} & & \hat{k} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -R\omega_0^2\cos\omega_0 t & -R\omega\sin\omega_0 t & 0 \end{vmatrix}$$

$$\frac{dP(t)}{d\Omega} = \frac{e^2}{4\pi c^3}(R\omega_0^2)^2\left[\cos^2\theta\left(\sin^2\omega_0 t + \cos^2\omega_0 t\right) + \sin^2\theta\left(\sin^2(\omega_0 t + \phi)\right)\right]$$

$$\left\langle \frac{dP(t)}{d\Omega} \right\rangle = \frac{e^2}{4\pi c^3}(R\omega_0^2)^2\frac{(1+\cos^2\theta)}{2}$$

$$\langle P(t) \rangle = \frac{e^2}{4\pi c^3}(R\omega_0^2)^2 2\pi \int_{-1}^{1}\frac{(1+x^2)}{2}dx = \frac{2e^2}{3c^3}(R\omega_0^2)^2$$

14.4 Background. In the nonrelativistic approximation the Lienard-Wiechert potenials are

$$\phi(\vec{x},t) = \frac{e}{R}|_{ret}, \qquad \vec{A}(\vec{x},t) = \frac{e\vec{\beta}}{R}|_{ret}$$

Let us assume that we observe the radiation close enough to source so $R/c << 1$ and $t' \cong t$. Then in the radiation zone

$$\vec{B} = \vec{\nabla} \times \vec{A} = \hat{n}\frac{\partial}{\partial R} \times \vec{A} = -\hat{n}\frac{\partial}{c\partial t'} \times \vec{A} = \frac{e\dot{\vec{\beta}} \times \hat{n}}{cR}$$

$$\vec{E} = \vec{B} \times \hat{n}$$

$$\frac{dP(t)}{d\Omega} = \frac{c}{4\pi}\left|R\vec{B}\right|^2 = \frac{e^2}{4\pi c}\left|\dot{\vec{\beta}} \times \hat{n}\right|^2 = \frac{e^2}{4\pi c^3}|\dot{v}|^2 \sin^2\theta$$

where θ is the angle between $\hat{n}$ and $\dot{\vec{\beta}}$ (assuming here the particle is moving linearly)

$$P(t) = \frac{2e^2}{3c^3}|\dot{v}|^2$$

Let the time-average be defined by

$$\langle f(t) \rangle \equiv \frac{1}{\tau}\int_0^\tau f(t)dt$$

Then

$$\left\langle \frac{dP(t)}{d\Omega} \right\rangle = \frac{e^2}{4\pi c^3}\sin^2\theta \left\langle |\dot{v}|^2 \right\rangle$$

$$\langle P(t) \rangle = \frac{2e^2}{3c^3}\left\langle |\dot{v}|^2 \right\rangle$$

a) Suppose $\vec{x}(t) = \hat{z}a\cos\omega_0 t$. Then $\dot{v} = \frac{d^2z}{dt^2} = -a\omega_0^2\cos\omega_0 t$

$$\left\langle |\dot{v}|^2 \right\rangle = \left(a\omega_0^2\right)^2 \frac{1}{\tau}\int_0^\tau \cos^2\omega_0 t dt = \left(a\omega_0^2\right)^2/2$$

So

$$\left\langle \frac{dP(t)}{d\Omega} \right\rangle = \frac{e^2}{8\pi c^3}\left(a\omega_0^2\right)^2 \sin^2\theta$$

$$\langle P(t) \rangle = \frac{e^2}{3c^3}\left(a\omega_0^2\right)^2$$

b) Suppose $\vec{x}(t) = R(\hat{\imath}\cos\omega_0 t + \hat{\jmath}\sin\omega_0 t)$. Then

$$\dot{v}(t) = -R\omega_0^2\left(\hat{\imath}\cos\omega_0 t + \hat{\jmath}\sin\omega_0 t\right)$$

$$\hat{n} \times \dot{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -R\omega_0^2\cos\omega_0 t & -R\omega\sin\omega_0 t & 0 \end{vmatrix}$$

$$\frac{dP(t)}{d\Omega} = \frac{e^2}{4\pi c^3}\left(R\omega_0^2\right)^2\left[\cos^2\theta\left(\sin^2\omega_0 t + \cos^2\omega_0 t\right) + \sin^2\theta\left(\sin^2(\omega_0 t + \phi)\right)\right]$$

$$\left\langle \frac{dP(t)}{d\Omega} \right\rangle = \frac{e^2}{4\pi c^3}\left(R\omega_0^2\right)^2\frac{\left(1+\cos^2\theta\right)}{2}$$

$$\langle P(t)\rangle = \frac{e^2}{4\pi c^3}\left(R\omega_0^2\right)^2 2\pi\int_{-1}^{1}\frac{\left(1+x^2\right)}{2}dx = \frac{2e^2}{3c^3}\left(R\omega_0^2\right)^2$$

14.5 This is a one-dimensional problem in dimension r.

a) Let $q = ze$. We know that nonrelativistically, we can use Larmor's formula

$$P = \frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{dp}{dt}\right)^2 = \frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{dV}{dr}\right)^2$$

where I've used Newton's second law,

$$\frac{dp}{dt} = -\frac{dV}{dr}$$

The total energy radiated is

$$\Delta W = \int_{-\infty}^{\infty} P dt = 2\int_0^{\infty} P dt$$

Using the fact that

$$dt = dr\left(\frac{1}{dr/dt}\right) = \frac{dr}{v(t)}$$

and from conservation of energy

$$\frac{v_0^2 m}{2} = \frac{v^2(r)m}{2} + V(r) = V(r_{\min})$$

$$v(r) = \sqrt{\frac{2}{m}}\sqrt{V(r_{\min}) - V(r)}$$

$$\Delta W = 2\cdot\frac{2}{3}\frac{q^2}{m^2c^3}\sqrt{\frac{m}{2}}\int_{r_{\min}}^{\infty}\left(\frac{dV}{dr}\right)^2\frac{dr}{\sqrt{V(r_{\min}) - V(r)}}$$

b) If $V(r) = zZe^2/r$, we can most easily do the integral by changing variables from r to $V(r)$

$$dr = \frac{dV(r)}{|dV(r)/dr|} = zZe^2\frac{dV}{V^2}$$

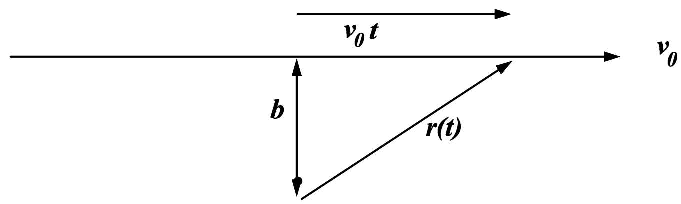
$$\left(\frac{dV}{dr}\right)^2 = \frac{(zZe^2)^2}{r^4} = \frac{V^4}{(zZe^2)^2}$$

$$\Delta W = \frac{4}{3}\frac{(ze)^2}{m^2c^3}\sqrt{\frac{m}{2}}\left(\frac{1}{zZe^2}\right)\int_{V_m}^{0}\frac{V^2}{\sqrt{V_m - V}}dV = \frac{4}{3}\frac{(ze)^2}{m^2c^3}\sqrt{\frac{m}{2}}\left(\frac{1}{zZe^2}\right)\frac{16}{15}V_m^{\frac{5}{2}}$$

where $V_m = V(r_{\min}) = \frac{mv_0^2}{2}$.

$$\Delta W = \frac{4}{3}\frac{(ze)^2}{m^2c^3}\sqrt{\frac{m}{2}}\left(\frac{1}{zZe^2}\right)\frac{16}{15}\left(\frac{mv_0^2}{2}\right)^{5/2} = \frac{8}{45}\frac{zmv_0^5}{Zc^3}$$

14.7 The system is described by the figure



a) From Larmor's formula

$$P = \frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{dp}{dt}\right)^2 = \frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{dV}{dr}\right)^2 = \frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{zZe^2}{r(t)^2}\right)^2$$

$$\Delta W = \int_{-\infty}^{\infty} Pdt = 2\int_0^{\infty} Pdt = 2\cdot\frac{2}{3}\frac{q^2}{m^2c^3}\left(zZe^2\right)^2\int_0^{\infty}\frac{1}{\left(b^2+(v_0t)^2\right)^2}dt$$

$$\Delta W = 2\cdot\frac{2}{3}\frac{(ze)^2}{m^2c^3}\left(zZe^2\right)^2\frac{\pi}{4v_0b^3} = \frac{1}{3}\frac{\pi z^4Z^2e^6}{m^2c^3v_0}\frac{1}{b^3}$$

b) Using the result for $r_{\min}$ from problem 14.5 for b,

$$\Delta W = \frac{1}{3}\frac{\pi z^4Z^2e^6}{m^2c^3v_0}\frac{1}{\left(\frac{2zZe^2}{mv_0^2}\right)^3} = \frac{\pi}{24}\frac{zmv_0^5}{Zc^3}$$

which compares to $\Delta W = \frac{8}{45}\frac{zmv_0^5}{Zc^3}$ for a head-on collision.

c) Following the book we define the radiation cross-section χ as

$$\chi = \frac{1}{3}\frac{\pi z^4Z^2e^6}{m^2c^3v_0}\int_{b_m}^{\infty}\frac{1}{b^3}2\pi b db = \frac{1}{3}\frac{\pi z^4Z^2e^6}{m^2c^3v_0}\cdot\frac{2}{b_m}\pi$$

Using the uncertainty relation to estimate b_m as

$$b_m = \frac{\hbar}{mv_0}$$

$$\chi = \frac{1}{3}\frac{\pi z^4Z^2e^6}{m^2c^3v_0}\cdot\frac{2\pi mv_0}{\hbar} = \left(\frac{2\pi^2}{3}\right)Z\left(\frac{Ze^2}{\hbar c}\right)\frac{z^4e^4}{mc^2}$$

Compare this to eq (15.30)/N.

$$\text{Eq.(15.30)}/\!/N = \frac{16}{3}\cdot Z\left(\frac{Ze^2}{\hbar c}\right)\frac{z^4e^4}{mc^2}$$

14.11 (a)

a) Using Eq. (14.24) for the relativistic power radiated and letting $e \to ze$,

$$P = -\frac{2}{3}\frac{z^2e^2}{m^2c^3}\left(\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}\right); \quad d\tau = dt/\gamma$$

And from Eq. (11.125)

$$\frac{d\vec{p}}{d\tau} = \frac{ze}{c}\left(U_0\vec{E} + \vec{U} \times \vec{B}\right)$$

$$\frac{dp_0}{d\tau} = \frac{ze}{c}\vec{U} \cdot \vec{E}$$

where

$$U = (\gamma c, \gamma\vec{v}) = p^\alpha/m$$

Or

$$\frac{d\vec{p}}{d\tau} = ze\gamma\left(\vec{E} + \vec{\beta} \times \vec{B}\right)$$

$$\frac{dp_0}{d\tau} = ze\gamma\vec{\beta} \cdot \vec{E}$$

$$P = \frac{2}{3}\frac{z^2e^2}{m^2c^3}\left(\left(\frac{d\vec{p}}{d\tau}\right)^2 - \left(\frac{dp_0}{d\tau}\right)^2\right)$$

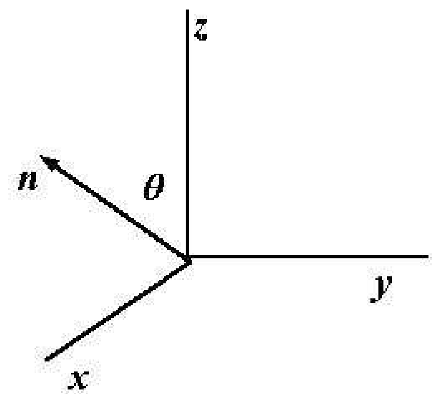
$$P = \frac{2}{3}\frac{z^4e^4}{m^2c^3}\gamma^2\left[\left(\vec{E} + \vec{\beta} \times \vec{B}\right)^2 - \left(\vec{\beta} \cdot \vec{E}\right)^2\right]$$

14.2

a) From Jackson, Eq (14.38)

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}\frac{\left\{\hat{n}\times\left[(\hat{n}-\vec{\beta})\times\dot{\beta}\right]\right\}^2}{(1-\hat{n}\cdot\vec{\beta})^5}$$

Using azimuthal symmetry, we can choose $\hat{n}$ in the x-z plane.



From the figure

$$\hat{n} = \cos\theta\hat{z} + \sin\theta\hat{x}$$

$$\vec{\beta}(t') = -\frac{a}{c}\omega_0\sin\omega_0 t'\hat{z} = -\beta\sin\omega_0 t'\hat{z}$$

$$\dot{\beta}(t') = -\frac{a\omega_0^2}{c}\cos\omega_0 t'\hat{z} = -\omega_0\beta\cos\omega_0 t'\hat{z}$$

Using $\vec{\beta}\times\mathring{\beta} = 0$

$$\left\{\hat{n}\times\left[(\hat{n}-\vec{\beta})\times\dot{\beta}\right]\right\}^2 = \left(\hat{n}\times\dot{\beta}\right)^2$$

and

$$\hat{n}\times\dot{\beta} = \quad \omega_0\beta\sin\theta\cos\omega_0 t'$$

So

$$\frac{dP}{d\Omega}(t') = \frac{e^2 c\beta^4}{4\pi a^2}\frac{\sin^2\theta\cos^2\omega_0 t'}{(1+\beta\cos\theta\sin\omega_0 t')^5}$$

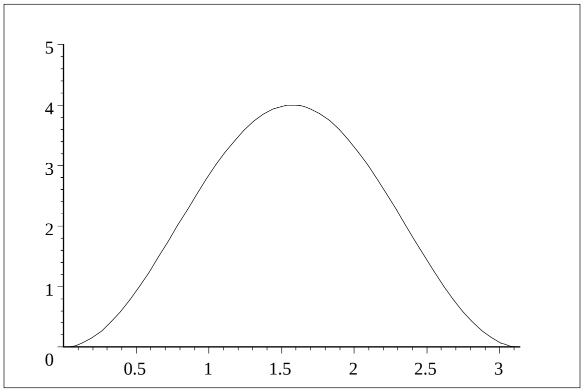
Defining $\phi = \omega_0 t'$,

$$\langle \frac{dP}{d\Omega}(t') \rangle = \frac{e^2 c \beta^4}{4\pi a^2} \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin^2\theta \cos^2\phi}{(1+\beta\cos\theta\sin\phi)^5} d\phi$$

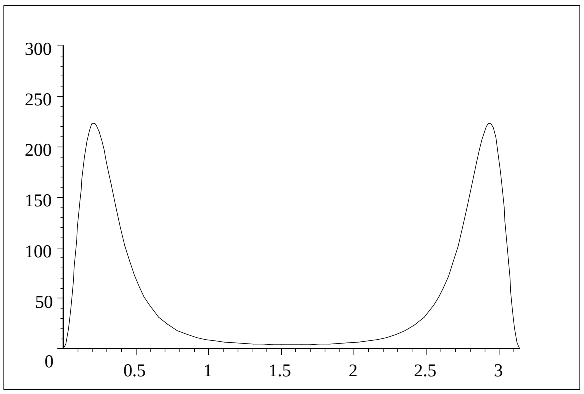
Or, doing the integral,

$$\langle \frac{dP}{d\Omega} \rangle = \frac{e^2 c \beta^4}{32\pi a^2} \frac{4+\beta^2\cos^2\theta}{(1-\beta^2\cos^2\theta)^{7/2}} \sin^2\theta$$

c) $\frac{4+.05^2\cos^2\theta}{(1-.05^2\cos^2\theta)^{7/2}} \sin^2\theta$



$\frac{4+.95^2\cos^2\theta}{(1-.95^2\cos^2\theta)^{7/2}} \sin^2\theta$



14.14

a) We can start with the result derived in problem 14.13. For simplicity of notation, I'm going to use ω, rather than ω_0, for the fundamental frequency

$$\frac{dP_m}{d\Omega} = \frac{e^2\omega^4 m^2}{(2\pi c)^3}\left|\int_0^{2\pi/\omega} \vec{v}(t)\times\hat{n}e^{im\omega\left(t-\frac{\hat{n}\cdot\vec{x}(t)}{c}\right)}dt\right|^2$$

I will choose the coordinate system so that the particle moves in the $\hat{z}$ direction and azimuthal symmetry allows me to choose $\hat{n}$ in the x - z plane, so $\hat{n}\cdot\hat{z} = \cos\theta$. Also I'm going to choose the zero of time by requiring that the particle be at the origin at $t = 0$. Thus

$$\vec{x}(t) = \hat{z}a\sin\omega t$$

$$\hat{n}\cdot\vec{x}(t) = \hat{z}a\cos\theta\sin\omega t$$

$$\vec{v}(t)\times\hat{n} = -\omega a\sin\theta\cos\omega t\hat{y}$$

$$\left|\int_0^{2\pi/\omega} \vec{v}(t)\times\hat{n}e^{im\omega\left(t-\frac{\hat{n}\cdot\vec{x}(t)}{c}\right)}dt\right| = \omega a\sin\theta\frac{1}{\omega}\left|\int_0^{2\pi}\cos x e^{imx-im\alpha\sin x}dx\right|$$

where $x = \omega t$ and $\alpha = \frac{\omega a}{c}\cos\theta = \beta\cos\theta$. Note the identity

$$\left|\int_0^{2\pi}\cos x e^{imx-im\alpha\sin x}dx\right| = \frac{2\pi}{\alpha}J_m(m\alpha).$$

Then

$$\left|\int_0^{2\pi/\omega} \vec{v}(t)\times\hat{n}e^{im\omega\left(t-\frac{\hat{n}\cdot\vec{x}(t)}{c}\right)}dt\right| = \frac{2\pi a}{\beta}\tan\theta J_m(m\alpha).$$

So

$$\frac{dP_m}{d\Omega} = \frac{e^2\omega^4 m^2}{(2\pi c)^3}\left(\frac{2\pi a}{\beta}\right)^2\tan^2\theta J_m^2(m\beta\cos\theta)$$

Since $\omega = c\beta/a$, the above can be written

$$\frac{dP_m}{d\Omega} = \frac{e^2c\beta^2}{2\pi a^2}m^2\tan^2\theta J_m^2(m\beta\cos\theta)$$

b) We remember that

$$J_m(x) = \frac{x^m}{2^m m!} + H.O.T.$$

If $x = m\beta\cos\theta$, then only the lowest m ($m = 1$) will dominate as $\beta \to 0$. So

$$\frac{dP_{tot}}{d\Omega} = \frac{dP_1}{d\Omega} = \frac{e^2c\beta^2}{2\pi a^2}\tan^2\theta J_1^2(\beta\cos\theta)$$

Or

$$P_{tot} = \frac{e^2 c\beta^2}{2\pi a^2} 2\pi \int_{-1}^{1} \frac{(1-x^2)}{x^2} J_1^2(\beta x) dx$$

Using

$$J_1^2(\beta x) = \left(\frac{\beta x}{2}\right)^2$$

$$P_{tot} = \frac{e^2 c\beta^2}{2\pi a^2} \frac{2\pi}{4} \beta^2 \frac{4}{3}$$

Letting $\beta = \frac{\omega a}{c}$

$$P_{tot} = \frac{2e^2\omega^4}{3c^3} \frac{1}{2} a^2$$

noting that

$$\bar{a}^2 = \frac{1}{T} a^2 \int_0^T \sin^2 \omega t dt = \frac{1}{2} a^2$$

then

$$\frac{2e^2\omega^4}{3c^3} \bar{a}^2$$

More Problems for Chapter 14

Problem 14.4

(a) The instantaneous power radiated per unit solid angle for $\beta \ll 1$ is given by Eq. (14.20):

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}|\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})|^2$$

$$\vec{r}(t) = a\cos(\omega_0 t)\hat{z}, \quad \dot{\vec{\beta}} = \frac{1}{c}\ddot{\vec{r}} = -a\frac{\omega_0^2}{c}\cos(\omega_0 t)\hat{z}$$

Therefore

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}\frac{a^2\omega_0^4}{c^2}\cos^2(\omega_0 t)|\hat{n} \times (\hat{n} \times \hat{z})|^2 = \frac{e^2}{4\pi c}\frac{a^2\omega_0^4}{c^2}\cos^2(\omega_0 t)\sin^2\theta$$

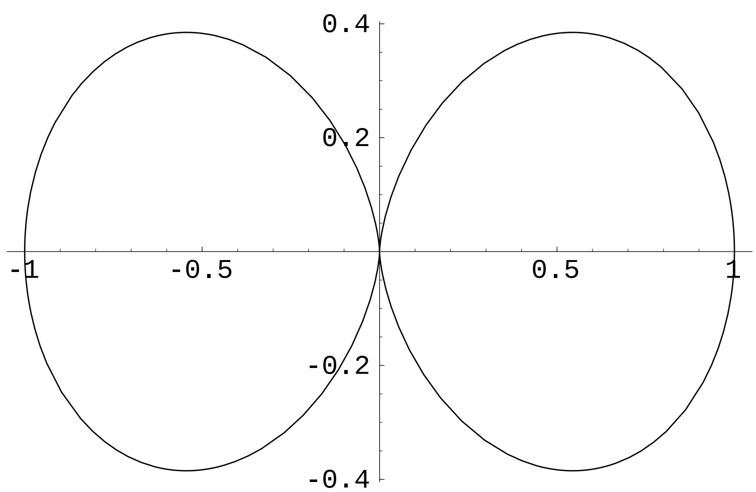
The average power per unit solid angle

$$\langle\frac{dP}{d\Omega}\rangle = \frac{e^2}{4\pi c}\frac{a^2\omega_0^4}{c^2}\sin^2\theta\langle\cos^2(\omega_0 t)\rangle = \frac{e^2}{8\pi c}\frac{a^2\omega_0^4}{c^2}\sin^2\theta$$

The total average power

$$P = \int\langle\frac{dP}{d\Omega}\rangle d\Omega = 2\pi\int_{-1}^{+1}\langle\frac{dP}{d\Omega}\rangle d(\cos\theta) = \frac{e^2}{3c^3}a^2\omega_0^4$$

The average power in the unit of $(e^2a^2\omega_0^4)/(8\pi c^3)$ is plotted below. The positive vertical axis defines $\theta = 0$.



(b)

$$\vec{r}(t) = R\cos(\omega_0 t)\hat{x} + R\sin(\omega_0 t)\hat{y}, \quad \dot{\vec{\beta}} = \frac{1}{c}\ddot{\vec{r}} = -\frac{\omega_0^2 R}{c}\{\cos(\omega_0 t)\hat{x} + \sin(\omega_0 t)\hat{y}\}$$

The problem is azimuthal symmetric and therefore, the differential power radiated is independent of ϕ. Without loosing generality, we can choose $\hat{n}$ in the $x - z$ plane. In this case, $\hat{n} = \cos\theta\hat{z} + \sin\theta\hat{x}$ and

$$|\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})| = |\hat{n}(\hat{n}\cdot\dot{\vec{\beta}}) - \dot{\vec{\beta}}| = \frac{\omega_0^2 R}{c}| - \cos^2\theta\cos(\omega_0 t)\hat{x} - \sin(\omega_0 t)\hat{y} + \sin\theta\cos\theta\cos(\omega_0 t)\hat{z}|$$

$$|\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})|^2 = \frac{\omega_0^4 R^2}{c^2} \left\{ \cos^2\theta \cos^2(\omega_0 t) + \sin^2(\omega_0 t) \right\}$$

The differential potential

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c} \frac{\omega_0^4 R^2}{c^2} \left\{ \cos^2\theta \cos^2(\omega_0 t) + \sin^2(\omega_0 t) \right\}$$

and the average

$$\langle \frac{dP}{d\Omega} \rangle = \frac{e^2}{4\pi c} \frac{\omega_0^4 R^2}{c^2} \left\{ \frac{1}{2} \cos^2\theta + \frac{1}{2} \right\} = \frac{e^2}{8\pi c} \frac{\omega_0^4 R^2}{c^2} (1 + \cos^2\theta)$$

The total power

$$P = \int \langle \frac{dP}{d\Omega} \rangle d\Omega = \frac{e^2}{8\pi c} \frac{\omega_0^4 R^2}{c^2} (2\pi) \int_{-1}^{+1} (1 + \cos^2\theta) d(\cos\theta) = \frac{2e^2}{3c^3} \omega_0^4 R^2$$

Alternate approach

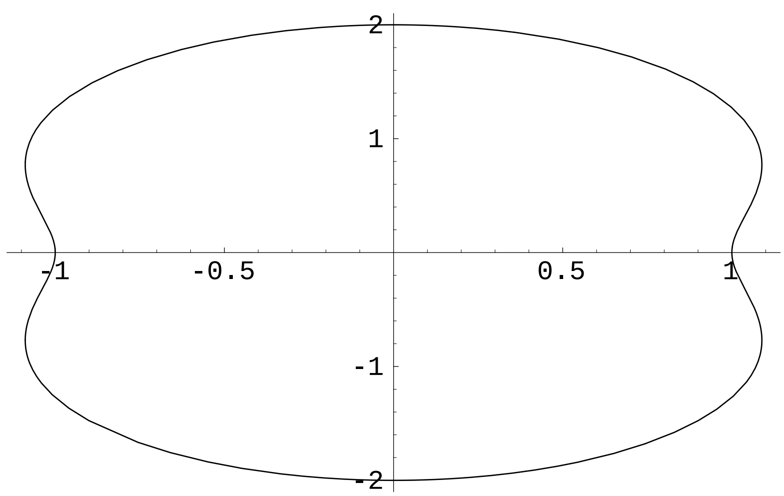
$$\vec{r}(t) = R(\hat{x} + i\hat{y}) e^{-i\omega_0 t}, \qquad \dot{\vec{\beta}} = -\frac{\omega_0^2 R}{c} (\hat{x} + i\hat{y})$$

The average power

$$\langle \frac{dP}{d\Omega} \rangle = \frac{c}{8\pi} r^2 Re \left\{ \vec{E} \times \vec{B}^* \right\} = \frac{e^2}{8\pi c} |\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})|^2$$

$$|\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})|^2 = \left\{ \hat{n}(\hat{n} \cdot \dot{\vec{\beta}}) - \dot{\vec{\beta}} \right\} \left\{ \hat{n}(\hat{n} \cdot \dot{\vec{\beta}}^*) - \dot{\vec{\beta}}^* \right\} = \dot{\vec{\beta}} \cdot \dot{\vec{\beta}}^* - (\hat{n} \cdot \dot{\vec{\beta}})(\hat{n} \cdot \dot{\vec{\beta}}^*) = \frac{\omega_0^4 R^2}{c^2} (1 + \cos^2\theta)$$

The $\langle dP/d\Omega \rangle$ (in $(e^2 \omega_0^4 R^2)/(8\pi c^3)$ unit) vs θ is plotted below, again $\theta = 0$ is defined by the upper vertical axis.



Problem 14.5

(a) For a non-relativistic particle with charge ze, the power radiated is

$$P = \frac{2}{3} \frac{(ze)^2}{c^3} \dot{v}^2$$

where $\dot{v}$ is the acceleration, given by

$$m\dot{v} = F = -\frac{dV}{dr}, \quad \Rightarrow \quad \dot{v} = -\frac{1}{m} \frac{dV}{dr}$$

Assuming the amount of energy radiated is small, we have approximately

$$\frac{1}{2}mv^2 + V(r) = E_{tot} = V(r_{min}), \quad \Rightarrow \quad v = \sqrt{\frac{2}{m}}\sqrt{V(r_{min}) - V(r)}$$

Note that the particle has zero velocity at the position of the closest approach. The total energy radiated is the time integral of power radiated

$$\Delta W = \int_{-\infty}^{\infty} P dt = 2\int_{r_{min}}^{\infty} P\frac{dr}{v} = \frac{4}{3}\frac{(ze)^2}{m^2c^3}\sqrt{\frac{m}{2}}\int_{r_{min}}^{\infty}\frac{|dV/dr|^2}{\sqrt{V(r_{min}) - V(r)}}dr$$

(b) For a Coulomb potential,

$$V(r) = \frac{zZe^2}{r},$$

the energy radiated is

$$\Delta W = \frac{4}{3}\frac{(ze)^2}{m^2c^3}\sqrt{\frac{m}{2}}(zZe^2)^{3/2}\int_{r_{min}}^{\infty}\frac{1}{\sqrt{1/r_{min} - 1/r}}\frac{dr}{r^4} = \frac{4}{3}\frac{(ze)^2}{m^2c^3}\sqrt{\frac{m}{2}}(zZe^2)^{3/2}\frac{16}{15}(\frac{1}{r_{min}})^{5/2}$$

In terms of v_0, we have

$$\frac{1}{2}mv_0^2 = V(r_{min}) = \frac{zZe^2}{r_{min}} \quad \Rightarrow \quad \frac{1}{r_{min}} = \frac{mv_0^2}{2zZe^2}$$

Therefore,

$$\Delta W = \frac{4}{3}\frac{(ze)^2}{m^2c^3}\sqrt{\frac{m}{2}}(zZe^2)^{3/2}\frac{16}{15}(\frac{mv_0^2}{2zZe^2})^{5/2} = \frac{8}{45}\frac{zmv_0^5}{Zc^3}$$

Problem 14.12

(a)

$$\vec{r}(t) = \hat{z}a\cos(\omega_0 t'), \quad \vec{\beta} = \dot{\vec{r}}(t) = -\frac{\omega_0 a}{c}\sin(\omega_0 t')\hat{z}, \quad \dot{\vec{\beta}} = -\frac{\omega_0^2 a}{c}\cos(\omega_0 t')\hat{z}$$

The differential power in particle's own time is given by Eq. (14.38):

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c}\frac{|\hat{n}\times\left\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right\}|^2}{(1-\hat{n}\cdot\beta)^5} = \frac{e^2}{4\pi c}\frac{|\dot{\vec{\beta}}|^2\sin^2\theta}{(1-\hat{n}\cdot\vec{\beta})^5}$$

$$= \frac{e^2a^2\omega_0^4}{4\pi c^3}\frac{\sin^2\theta\cos^2(\omega_0 t')}{\{1+\frac{\omega_0 a}{c}\cos\theta\sin(\omega_0 t')\}^2} = \frac{e^2c\beta_0^4}{4\pi a^2}\frac{\sin^2\theta\cos^2(\omega_0 t')}{(1+\beta_0\cos\theta\sin(\omega_0 t'))^5}$$

where $\beta_0 \equiv a\omega_0/c$. Here I used β_0 instead of β for the constant to avoid confusions.

(b) The average power

$$\langle\frac{dP}{d\Omega}\rangle = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0}\frac{dP(t')}{d\Omega}dt' = \frac{e^2c\beta_0^4}{8\pi^2a^2}\sin^2\theta\int_0^{2\pi/\omega_0}\frac{\cos^2(\omega_0 t')}{(1+\beta_0\cos\theta\sin(\omega_0 t'))^5}d(\omega_0 t')$$

$$= \frac{e^2c\beta_0^4}{8\pi^2a^2}\sin^2\theta\int_0^{2\pi}\frac{\cos^2\phi}{(1+\beta_0\cos\theta\sin\phi)^5}d\phi = \frac{e^2c\beta_0^4}{8\pi^2a^2}\sin^2\theta\left\{\frac{\pi}{4}\frac{4+\beta_0^2\cos^2\theta}{(1-\beta_0^2\cos^2\theta)^{7/2}}\right\}$$

Thus

$$\langle\frac{dP}{d\Omega}\rangle = \frac{e^2c\beta_0^4}{32\pi a^2}\frac{4+\beta_0^2\cos^2\theta}{(1-\beta_0^2\cos^2\theta)^{7/2}}\sin^2\theta$$

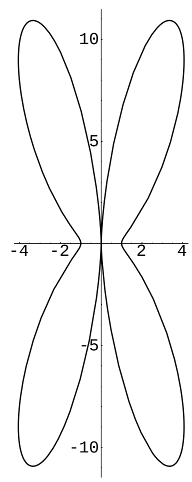
(c) For non-relativistic case, $\beta_0 \ll 1$, therefore

$$\langle \frac{dP}{d\Omega} \rangle \rightarrow \frac{e^2 c \beta_0^4}{8\pi a^2} \sin^2\theta$$

In the relativistic case, $\beta_0 \rightarrow 1$,

$$\langle \frac{dP}{d\Omega} \rangle \rightarrow \frac{e^2 c \beta_0^4}{32\pi a^2} \frac{4 + \cos^2\theta}{(1-\cos^2\theta)^{7/2}} \sin^2\theta$$

As β_0 approaches 1, $\langle dP/d\Omega \rangle$ develops peaks close to $\theta = 0, \pi$. The $\langle dP/d\Omega \rangle$ distribution for the non-relativistic case ($\beta_0 = 0$) is the same as Prob. 14.4(a). For $\beta_0 = 0.9$, $\langle dP/d\Omega \rangle$ is plotted below:



As β_0 increases, the four lobeds become narrower. The upper two are increasing clustered together, forming a strong peak in the forward direction. Similarly, the lower two lobeds form a strong peak in the backward direction.

Problem 14.13

From Eq. (14.67), we have

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \vec{n} \times (\vec{n} \times \vec{\beta}(t)) e^{i\omega(t - \vec{n}\cdot\vec{r}(t)/c)}\ dt \right|^2$$

If the charge is in periodic motion with period T, the integrand almost repeat itself (except for a phase factor) each period. We can thus break the integral over time into a sum of terms, times a common integral over one cycle. If the charge has actually been in periodic motion always, the total radiated energy is infinite. To keep track of things and to avoid square of delta functions, we make the integral from $-NT$ to $+NT$ where N is a large integer, thus

$$A_N = \sum_{n=-N}^{N-1} \int_{nT}^{(n+1)T} dt\ \vec{n} \times (\vec{n} \times \vec{\beta}) e^{i\omega(t - \vec{n}\cdot\vec{r}(t)/c)}$$

Changing variables to $t' = t - nT$ and using the factor that $\vec{r}(t)$ and $\vec{\beta}(t)$ are periodic, we have

$$A_N = \sum_{n=-N}^{N-1} e^{in\omega T} A_0(\omega), \qquad \text{where} \qquad A_0(\omega) = \int_0^T dt'\ \vec{n} \times (\vec{n} \times \vec{\beta}(t')) e^{i\omega(t' - \vec{n}\cdot\vec{r}(t')/c)}$$

The sum of the phase factor

$$\mathcal{S}_N = \sum_{n=-N}^{N-1} e^{in\omega T} = \sum_{n=0}^{N-1} e^{in\omega T} + \sum_{n=0}^{N-1} e^{-i(n+1)\omega T} = \frac{1 - e^{iN\omega T}}{1 - e^{i\omega T}} + e^{-i\omega T} \frac{1 - e^{-iN\omega T}}{1 - e^{-i\omega T}}$$

Therefore,

$$\frac{d^2I}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}|\mathcal{S}_N(\omega)|^2|A_0(\omega)|^2$$

Multiplying both sides of $\mathcal{S}_N$ by $e^{i\omega T/2}$, we get

$$\mathcal{S}_N e^{i\omega T/2} = e^{i\omega T/2}\frac{1-e^{iN\omega T}}{1-e^{i\omega T}} + c.c.$$

where $c.c.$ is a short-hand for complex conjugate. This is a standard diffraction pattern function that peaks up strongly at $\omega = (2\pi/T)m$ if N is large. Here m is an integer. Let $\omega T = 2\pi m + x$ and assume $x \ll 1$, then

$$\mathcal{S}_N e^{i\omega T/2} = e^{im\pi}e^{ix/2}\frac{1-e^{i2\pi Nm}e^{iNx}}{1-e^{i2\pi m}e^{ix}} + c.c. \approx (-1)^m\frac{1-e^{iNx}}{-ix} + c.c.$$

$$= (-1)^m e^{iNx/2}\frac{e^{iNx/2}-e^{-iNx/2}}{ix} + c.c. = 2(-1)^m e^{iNx/2}\frac{\sin(Nx/2)}{x} + cc.$$

$$= (-1)^m\frac{\sin(Nx/2)}{x}\times 2(e^{iNx/2}+e^{-iNx/2}) = 2(-1)^m\frac{\sin(Nx)}{x}$$

Thus for frequencies near $\omega = m(2\pi/T) \equiv m\omega_0$, the frequency spectrum is sharply peaked. Evidently as $N \to \infty$ the frequency spectrum becomes a series of lines at $\omega = m\omega_0$. The integral over frequency of $|\mathcal{S}_N|^2$ near $\omega = m\omega_0$ is

$$\int_{\omega\sim m\omega_0} d\omega|\mathcal{S}_N|^2|A_0(\omega)|^2 \approx \frac{4}{T}|A_0(m\omega_0)|^2\int_{-\infty}^{\infty}\frac{\sin^2(Nx)}{x^2}dx = \frac{8N}{T}\int_0^{\infty}\frac{\sin^2 t}{t^2}dt = \frac{4\pi}{T}N = 2\omega_0 N$$

The radiated energy in each line is proportional to N. Since the total time interval is $2NT = 4\pi N/\omega_0$, the power radiated in each harmonic is

$$\frac{dP_m}{d\Omega} = \frac{e^2\omega_0^2 m^2}{4\pi^2 c}|A_0(m\omega_0)|^2 \times 2\omega_0 N \times \frac{\omega_0}{4\pi N} = \frac{e^2\omega_0^4}{8\pi^3 c}m^2|A_0(m\omega_0)|^2$$

or

$$\frac{dP_m}{d\Omega} = \frac{e^2\omega_0^4 m^2}{(2\pi c)^3}|\int_0^{2\pi/\omega_0} dt\ \vec{n}\times(\vec{n}\times\vec{v}(t))e^{im\omega_0(t-\vec{n}\cdot\vec{r}(t)/c)}|^2$$

$$= \frac{e^2\omega_0^4 m^2}{(2\pi c)^3}|\int_0^{2\pi/\omega_0} dt\ \vec{n}\times\vec{v}(t)\ e^{im\omega_0(t-\vec{n}\cdot\vec{r}(t)/c)}|^2$$

Alternate Approach

The energy distribution is given by Eq. (14.70):

$$\frac{d^2I}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}|\int_{-\infty}^{\infty}\vec{n}\times(\vec{n}\times\vec{\beta})e^{i\omega(t-\vec{n}\cdot\vec{r}(t)/c)}\ dt|^2$$

Expanding the integrand in Fourier series

$$\vec{n}\times(\vec{n}\times\vec{\beta})e^{-i\vec{n}\cdot\vec{r}(t)/c} = \sum_{m=-\infty}^{\infty}A_m(\omega)e^{-im\omega_0 t}$$

Then we have

$$\frac{d^2I}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\sum_m\sum_{m'}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dt'\ A_m(\omega)A^*_{m'}(\omega)e^{i\omega t}e^{-im\omega_0 t}e^{-i\omega t'}e^{im'\omega_0 t'}$$

$$= \frac{e^2\omega^2}{4\pi^2 c}\sum_m\sum_{m'}\int_{-\infty}^{\infty} dt\ A_m(\omega)A^*_{m'}(\omega)e^{i(\omega-m\omega_0)t}\left\{2\pi\delta(\omega-m'\omega_0)\right\}$$

The above equation shows that the frequency spectrum is discrete. Integrating over ω, the total energy radiated per unit solid angle is

$$\frac{dI}{d\Omega} = \int \frac{d^2I}{d\omega d\Omega}d\omega = \frac{e^2}{2\pi c}\sum_m\sum_{m'}(m'\omega_0)^2 A_m(m'\omega_0)A^*_{m'}(m'\omega_0)\ \int_{-\infty}^{\infty} e^{i(m'-m)\omega_0 t}\ dt$$

To facilitate the power calculation, we replace the time interval $(-\infty,\infty)$ with $(-NT, NT)$ where N is a large integer and $T = 2\pi/\omega_0$ is the period. In this case, the energy radiated per unit solid angle in the time interval $2NT$ is

$$\frac{dI}{d\Omega}(-NT,NT) = \frac{e^2}{2\pi c}\sum_m\sum_{m'}(m'\omega_0)^2 A_m(m'\omega_0)A^*_{m'}(m'\omega_0)\ \int_{-NT}^{NT} e^{i(m'-m)\omega_0 t}\ dt$$

The average power per unit solid angle is therefore

$$\frac{dP}{d\Omega} = \frac{1}{2NT}\frac{dI}{d\Omega}(-NT,NT) = \frac{e^2}{2\pi c}\sum_m\sum_{m'}(m'\omega_0)^2 A_m(m'\omega_0)A^*_{m'}(m'\omega_0)\ \left\{\frac{1}{2NT}\int_{-NT}^{NT} e^{i(m'-m)\omega_0 t}\ dt\right\}$$

Note that

$$\frac{1}{2NT}\int_{-NT}^{NT} e^{i(m'-m)\omega_0 t}\ dt = \frac{1}{T}\int_0^T e^{i(m'-m)\omega_0 t}dt = \delta_{m'm}$$

Thus

$$\frac{dP}{d\Omega} = \frac{e^2}{2\pi c}\sum_m\sum_{m'}(m'\omega_0)^2 A_m(m'\omega_0)A^*_{m'}(m'\omega_0)\ \{\delta_{mm'}\} = \frac{e^2}{2\pi c}\sum_m (m\omega_0)^2|A_m(m\omega_0)|^2$$

This is the total power for all harmonics. For m^{th} harmonics, we have

$$\langle\frac{dP}{d\Omega}\rangle_{m^{th}} = \frac{m^2e^2\omega_0^2}{2\pi c}|A_m(m\omega_0)|^2$$

$A_m(\omega)$ given by reverse Fourier integration:

$$A_m(\omega) = \frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} \vec{n}\times(\vec{n}\times\vec{\beta})e^{-i\omega\hat{n}\cdot\vec{r}/c}e^{im\omega_0 t}\ dt$$

Therefore

$$\langle\frac{dP}{d\Omega}\rangle_{m^{th}} = \frac{m^2e^2\omega_0^4}{(2\pi c)^3}|\int_0^{2\pi/\omega_0} \vec{n}\times(\vec{n}\times\vec{v})e^{im\omega_0(t-\hat{n}\cdot\vec{r}/c)}\ dt|^2$$

$$= \frac{m^2e^2\omega_0^4}{(2\pi c)^3}|\int_0^{2\pi/\omega_0} \vec{n}\times\vec{v}\ e^{im\omega_0(t-\hat{n}\cdot\vec{r}/c)}\ dt|^2$$

More Problems for Chapter 14

Problem 14.14
(a)

$$\vec{r} = a\cos(\omega_0 t)\hat{z}, \quad \vec{\beta} = -\frac{a\omega_0}{c}\sin(\omega_0 t)\hat{z}$$

$$\hat{n}\cdot\vec{r} = a\cos\theta\cos(\omega_0 t), \quad \hat{n}\times(\hat{n}\times\vec{\beta}) = -\frac{a\omega_0}{c}\sin(\omega_0 t)\hat{n}\times(\hat{n}\times\hat{z})$$

Plugging into the result of Prob. (14.13):

$$A_m(\omega) = -\frac{a\omega_0}{c}\hat{n}\times(\hat{n}\times\hat{z})\frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} dt\sin(\omega_0 t)e^{im\omega_0 t}e^{-i\frac{\omega a}{c}\cos\theta\cos(\omega_0 t)}$$

$$= -\beta_0\hat{n}\times(\hat{n}\times\hat{z})\int_0^{2\pi}\frac{d\phi}{2\pi}e^{-i\beta_0\cos\theta\cos\phi}\frac{1}{2i}\left\{e^{i(m+1)\phi} - e^{i(m-1)\phi}\right\}$$

where $\beta_0 = a\omega_0/c$.

$$A_m(m\omega_0) = i\frac{\beta_0}{2}\hat{n}\times(\hat{n}\times\hat{z})\int_0^{2\pi}\frac{d\phi}{2\pi}\left\{e^{i(-m\beta_0\cos\theta\cos\phi+(m+1)\phi)} - e^{i(-m\beta_0\cos\theta\cos\phi+(m-1)\phi)}\right\}$$

Let $\phi' = \pi - \phi$ and using the identity

$$J_m(x) = \frac{1}{i^m}\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i(x\cos\phi - m\phi)}$$

we have

$$\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i(-m\beta_0\cos\theta\cos\phi+(m+1)\phi)} = (-1)^{m+1}\int_0^{2\pi}\frac{d\phi'}{2\pi}e^{i(m\beta_0\cos\theta\cos\phi'-(m+1)\phi')} = (-i)^{m+1}J_{m+1}(m\beta_0\cos\theta)$$

$$\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i(-m\beta_0\cos\theta\cos\phi+(m-1)\phi)} = (-1)^{m-1}\int_0^{2\pi}\frac{d\phi'}{2\pi}e^{i(m\beta_0\cos\theta\cos\phi'-(m-1)\phi')} = (-i)^{m-1}J_{m-1}(m\beta_0\cos\theta)$$

Consequently, we get

$$A_m(m\omega_0) = i\frac{\beta_0}{2}\hat{n}\times(\hat{n}\times\hat{z})\left\{(-1)^{m+1}J_{m+1}(m\beta_0\cos\theta) - (-i)^{m-1}J_{m-1}(m\beta_0\cos\theta)\right\}$$

$$= (-i)^m\frac{\beta_0}{2}\hat{n}\times(\hat{n}\times\hat{z})\left\{J_{m+1}(m\beta_0\cos\theta) + J_{m-1}(m\beta_0\cos\theta)\right\} = (-i)^m\frac{\hat{n}\times(\hat{n}\times\hat{z})}{\cos\theta}J_m(m\beta_0\cos\theta)$$

Here we have used a Bessel identity:

$$J_{m+1}(x) + J_{m-1}(x) = \frac{2m}{x}J_m(x)$$

Then

$$|A_m(m\omega_0)|^2 = \frac{|\hat{n}\times(\hat{n}\times\hat{z})|^2}{\cos^2\theta}J_m^2(m\beta_0\cos\theta) = \frac{\sin^2\theta}{\cos^2\theta}J_m^2(m\beta_0\cos\theta)$$

Therefore, the average power for the mth harmonics:

$$\langle\frac{dP}{d\Omega}\rangle_{m^{th}} = \frac{m^2e^2\omega_0^2}{2\pi c}\tan^2\theta J_m^2(m\beta_0\cos\theta) = \frac{e^2c\beta_0^2}{2\pi a^2}m^2\tan^2\theta J_m^2(m\beta_0\cos\theta)$$

Here we used $\beta_0 = \omega_0 a/c$ instead of β to avoid confusions.
(b) The total power radiated in the m^{th} harmonic is

$$P_m = \int\langle\frac{dP}{d\Omega}\rangle d\Omega = \frac{e^2\beta_0^2 c}{2\pi a^2}m^2(2\pi)\int_{-1}^{+1} d(\cos\theta)\tan^2\theta J_m^2(m\beta_0\cos\theta)$$

In the non-relativistic limit, $\beta_0 \ll 1$, the contribution from large m will be negligible since

$$J_m^2(m\beta_0\cos\theta) \approx \frac{1}{(m!)^2}(\frac{m\beta_0\cos\theta}{2})^{2m}$$

The radiation is dominated by $m = 1$ harmonic:

$$P_1 = \frac{e^2\beta_0^2 c}{a^2}\int_{-1}^{+1} d(\cos\theta)\tan^2\theta\frac{1}{4}\beta_0^2\cos^2\theta = \frac{e^2\beta_0^4 c}{3a^2} = \frac{e^2a^2\omega_0^4}{3c^3} \equiv \frac{2}{3}\frac{e^2}{c^3}\omega_0^4\overline{a^2}$$

where

$$\overline{a^2} = \langle a^2\cos^2(\omega_0 t)\rangle = \frac{1}{2}a^2$$

Problem 14.26
(a) The radius of the orbit can be calculated using the numerical form Eq. (12.42):

$$\rho = \frac{p\ (\text{MeV/c})}{3.0\times10^{-4}\ B(\text{gauss})} = \frac{10^{13}\cdot10^{-6}}{3.0\times10^{-4}\cdot3\times10^{-4}} = 1.1\times10^{14}\ \text{cm}$$

The natural frequency of the motion

$$\omega_0 = \frac{c}{\rho} = 2.7\times10^{-4}\ \text{s}^{-1}$$

and the critical frequency

$$\omega_c = \frac{3}{2}\omega_0\gamma^3 = \frac{3}{2}\omega_0(\frac{E}{mc^2})^3 \approx 3\cdot10^{18}\ \text{s}^{-1}$$

$$\hbar\omega_c \approx 6.6\times10^{-22}\ \text{MeV s}\cdot3\times10^{18}\ \text{s}^{-1} = 2\ \text{keV}$$

(b) The average observable power is given by

$$P(\omega, E) = \frac{1}{T}\frac{dI}{d\omega} = \frac{\omega_0}{2\pi}\frac{dI}{d\omega}$$

At low frequencies ($\omega \ll \omega_c$), $dI/d\omega$ is given by Eq. (14.89). Therefore the average power spectrum has the form

$$P(\omega, E) \sim \omega_0\frac{e^2}{c}(\frac{\omega}{\omega_0})^{1/3} \sim (\omega\omega_0^2)^{1/3}$$

At high frequencies ($\omega \gg \omega_c$), $dI/d\omega$ is given by Eq. (14.90). Thus

$$P(\omega, E) \sim \gamma\omega_0(\frac{\omega}{\omega_c})^{1/2}e^{-\omega/\omega_c} \sim \frac{1}{\gamma^2}(\omega\omega_c)^{1/2}e^{-\omega/\omega_c}$$

Note that

$$\omega_0 = \frac{c}{\rho} = \frac{ceB}{pc} \sim \frac{1}{E}, \quad \gamma = \frac{E}{mc^2} \sim E, \quad \omega_c = \frac{3}{2}\gamma^3\omega_0 \sim E^2$$

The average power

$$P(\omega, E) \sim (\frac{\omega}{E^2})^{1/3} \text{ for } \omega \ll \omega_c \qquad \text{and} \qquad P(\omega, E) \sim (\frac{\omega}{E^2})^{1/2} e^{-\omega/\omega_c} \text{ for } \omega \gg \omega_c$$

It can be written in the form

$$P(\omega, E) = \text{const } (\frac{\omega}{E^2})^{1/3} f(\frac{\omega}{\omega_c})$$

where $f(x) = 1$ for $x \ll 1$ and $f(x) \sim x^{1/6} e^{-x}$ for $x \gg 1$.
(c) Now with

$$N(E)dE \sim E^{-n} dE$$

we have

$$P(\omega) = \frac{\int_0^\infty P(E,\omega)N(E)dE}{\int_0^\infty N(E)dE} = \text{const} \int P(E,\omega)N(E)dE = \text{const} \int \frac{\omega^{1/3}}{E^{2/3}} f(\frac{\omega}{\omega_c}) E^{-n} dE$$

Approximate f as a step function such that $f(x) = 1$ for $x < 1$ and $f(x) = 0$ for $x > 1$. Given that $\omega_c \sim E^2 = \delta E^2$ (here δ is a constant, see below), the non-vanishing contribution to the integral is therefore from $E > \sqrt{\omega/\delta}$.

$$P(\omega) = \text{const } \omega^{1/3} \int_{\sqrt{\omega/\delta}}^\infty E^{-(n+2/3)} dE = \text{const } \omega^{1/3} \left\{ \sqrt{\frac{\omega}{\delta}} \right\}^{-(n-1/3)}$$

$$= \text{const } \frac{\omega^{1/3}}{\omega^{n/2-1/6}} = \text{const } \omega^{-(n-1)/2} = \text{const } \omega^{-\alpha}$$

where $\alpha = (n-1)/2$.
(d) The critical frequency ω_c of the radiation is related to electron energies through

$$\omega_c = \frac{3}{2}\gamma^3 \omega_0 = \frac{3}{2}\gamma^3 \frac{c}{\rho} = \frac{3}{2}\gamma^3 \frac{ceB}{cp} \approx \frac{3}{2}\gamma^3 \frac{ceB}{E} = \frac{3}{2}\gamma^2 \frac{eB}{mc} = \frac{3}{2} \frac{eB}{m^3 c^5} E^2 (= \delta E^2)$$

Thus

$$E = \gamma mc^2 = mc^2 \sqrt{\frac{2}{3} \omega_c \frac{mc}{e} \frac{1}{B}}$$

Taking the cutoff frequency 10^{18} Hz as the critical frequency and note that $e/mc = 1.76 \times 10^7$ s^{-1} gauss^{-1}, we have

$$E = mc^2 \sqrt{\frac{2}{3} \cdot 10^{18} \cdot \frac{1}{1.76 \times 10^7} \cdot \frac{1}{3 \times 10^{-4}}} \approx 1.1 \times 10^7 mc^2 \approx 5.7 \times 10^{12} \text{ eV}$$

consistent with the electron energy (and therefore all other numbers) of part (a). In this frequency region, we have $n = 2\alpha + 1 = 1.70$.
(e) The half-life is given by

$$t_{1/2} = \frac{3m^3c^5}{2e^4B^2} \frac{1}{\gamma} = \frac{3}{2} (\frac{mc}{e})^2 \frac{1}{B^2} \frac{mc^2}{e^2} \frac{mc^2}{E} c$$

Again note that

$$\frac{e}{mc} = 1.76 \times 10^7 \text{ s}^{-1} \text{ gauss}^{-1}; \qquad \frac{e^2}{mc^2} = 2.82 \times 10^{-13} \text{ cm}$$

Therefore, the half-life can be expressed in terms of B in milli-gauss and E in GeV as

$$t_{1/2} = \frac{3}{2} \cdot \frac{1}{(1.76 \times 10^7)^2} \cdot \frac{10^6}{B^2} \cdot \frac{1}{2.82 \times 10^{-13}} \frac{0.511 \times 10^{-3}}{E} \cdot 3 \times 10^{10} = \frac{2.63 \times 10^{11}}{EB^2} \text{ s}$$

For the numbers in part (a),

$$t_{1/2} = \frac{2.63 \times 10^{11}}{10^4 \cdot 0.3^2} = 2.92 \times 10^8 \text{ s} \sim 9.3 \text{ years}$$

The Crab nebula was observed in year 1054, more than 900 years ago. Therefore, initial energetic electrons are probably long gone. However, my astrophysics colleagues told me that there is not much trouble making energetic electrons from the pulsar at the center. Electrons and positrons can be pair produced from the energetic photons from the pulsar and they are accelerated by the rapidly rotating magnetic field associated with the neutron star. It is interesting to note that earlier editions of Jackson had $E = 10^{12}$ eV and $B = 10^{-4}$ gauss, which results a half-life about 834 years. Presumably the change is due to recent progresses made in this area.

[redacted]

16.1

It’s useful to apply in this case the Virial Theorem, familiar from classical mechanics:

$$\langle T\rangle = \frac{1}{2}\langle\frac{dV}{dr}\rangle r$$

If $V = ar^n$, then

$$\langle T\rangle = \frac{n}{2}\langle V\rangle$$

In our case $V = \frac{1}{2}kr^2$, with $k = m\omega_0^2$, so $n = 2$ and

$$\langle T\rangle = \langle V\rangle$$

Or,

$$\langle\frac{dV}{dr}\rangle = \frac{E}{r}$$

We are given

$$\frac{dE}{dt} = -\frac{\tau}{m}\langle\left(\frac{dV}{dr}\right)^2\rangle$$

This can be rewritten

$$\frac{dE}{dt} = -\frac{\tau}{m}kE$$

So

$$E = E_0e^{-\frac{\tau}{m}kt} = E_0e^{-\tau\omega_0^2t} = E_0e^{-\Gamma t}$$

Similarly,

$$\frac{d\vec{L}}{dt} = -\frac{\tau}{m}\langle\frac{1}{r}\frac{dV}{dr}\rangle\vec{L}$$

But $\frac{1}{r}\frac{dV}{dr} = m\omega_0^2$, so

$$L = L_0e^{-\tau\omega_0^2t} = L_0e^{-\Gamma t}$$

16.2

$V = -, \; q = -e$

$$\frac{dE}{dt} = -\frac{\tau}{m}\langle\left(\frac{dV}{dr}\right)^2\rangle$$

If $V = ar^n$, then the Virial theorem tells us

$$\langle T\rangle = \frac{n}{2}\langle V\rangle$$

In the present case, $n = -1$, so

$$E = \langle T\rangle + \langle V\rangle = \frac{1}{2}\langle V\rangle = -\frac{Ze^2}{2r}$$

$$\frac{dV}{dr} = \frac{Ze^2}{r^2}$$

Now

$$\frac{dE}{dt} = -\frac{\tau}{m}\langle\left(\frac{dV}{dr}\right)^2\rangle$$

gives

$$\frac{d}{dr}\frac{1}{r(t)} = \frac{2Ze^2\tau}{mr^4(t)}$$

or

$$r^2 dr = -2Ze^2\tau dt/m$$

But $\tau = \frac{2}{3}\frac{e^2}{c^3 m}$, so

$$r^2 dr = -3Z(c\tau)^3\frac{t}{\tau}$$

Integrating both sides gives

$$r^3(t) = r_0^3 - 9Z(c\tau)^3\frac{t}{\tau}$$

b) At this point, for simplicity of notation, I'm going to take $c = \hbar = 1$. Then from problem 14.21,

$$\frac{1}{T} = \frac{2}{3}e^2(Ze^2)^4\frac{m}{n^5}$$

We are given

$$r = \frac{n^2 a_0}{Z}$$

Where $a_0 = \frac{1}{me^2}$ = Bohr radius, and $\tau = \frac{2}{3}\frac{e^2}{m}$

$$-\frac{dn}{dt} = -\frac{Z}{2a_0 n}\frac{dr}{dt} = \frac{Z}{2a_0 n}\frac{3}{r^2}Z\tau^2 = \frac{Z}{2a_0 n}3\left(\frac{Z}{a_0 n^2}\right)^2 Z\left(\frac{2}{3}\frac{e^2}{m}\right)^2 = \frac{2}{3}\frac{Z^4}{m}\frac{e^6}{n^5}$$

in agreement with the result of problem 14.21.

c) From part b)

$$t = \frac{r_0^3 - r^3(t)}{9Z\tau^2}$$

But $r(t) = \frac{n_f^2 a_0}{Z},\ r_0 = \frac{n_i^2 a_0}{Z}$

$$t = \frac{\left(\frac{n_i^2 a_0}{Z}\right)^3 - \left(\frac{n_f^2 a_0}{Z}\right)^3}{9Z\tau^2} = \frac{1}{9}a_0^3\frac{n_i^6 - n_f^6}{Z^4\tau^2}$$

In our present case $Z = 1,$ so

$$t = \frac{1}{9}\left(\frac{1}{me^2}\right)^3\frac{n_i^6 - n_f^6}{\left(\frac{2}{3}\frac{e^2}{m}\right)^2} = \frac{1}{4m}e^{-10}(n_i^6 - n_f^6)$$

In these units, (from the particle data book) $\text{MeV}^{-1} = 6.6 \times 10^{-22}\text{s}.$ $e^2 = \alpha = 1/137,$ and $m = 207 \times .511\text{MeV}.$

$$t = \frac{1 \times 6.6 \times 10^{-22}\text{s}}{4 \times 207 \times .511(1/137)^5}(n_i^6 - n_f^6) = 7.53 \times 10^{-14}(n_i^6 - n_f^6)\text{s}$$

For the cases desired,

$$t_1 = 7.53 \times 10^{-14}(10^6 - 4^6)\text{s} = 7.5 \times 10^{-8}\text{s}$$

$$t_2 = 7.53 \times 10^{-14}(10^6 - 1^6)\text{s} = 7.5300 \times 10^{-8}\text{s}$$

Just as a check on working with these units, notice

$$\tau = \frac{2}{3}\frac{e^2}{m} = \frac{2}{3}\frac{1}{.511 \times 137}6.6 \times 10^{-22}\text{s} = 6.29 \times 10^{-24}\text{s}$$

in agreement with what we found before.