JACKSON

3RD EDITION

143a–Reminder of Linear Algebra–12 September 2001

It is expected that you have learned linear algebra somewhere along the line. Here we launch into a very simple example using real matrices. The matrices that we will encounter in quantum mechanics are complex. The concepts are the same.

Consider the coupled pair of equations

$$\frac{dv}{dt} = 4v - 5w, \; v = 8 \; at \; t = 0 \tag{1}$$

$$\frac{dw}{dt} = 2v - 3w, \; w = 5 \; at \; t = 0 \tag{2}$$

$$\tag{3}$$

This initial value problem can be written in matrix form.

$$\vec{u}(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}, \; \vec{u}_0 = \begin{bmatrix} 8 \\ 5 \end{bmatrix}, \; A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \tag{4}$$

$$\frac{d\vec{u}}{dt} = A\vec{u}, \; \vec{u} = \vec{u}_0 \; at \; t = 0 \tag{5}$$

If $\vec{u}$ in the above equation were a scalar the above equation then the form would be

$$\frac{du}{dt} = au, \; u = u_0 \; at \; t = 0 \tag{6}$$

and the solution would be

$$u(t) = u_0 e^{at} \tag{7}$$

With the vector form we will try as a solution

$$\vec{u} = e^{\lambda t}\vec{x} \tag{8}$$

This results in

$$\lambda e^{\lambda t}\vec{x} = Ae^{\lambda t}\vec{x} \tag{9}$$

$$\lambda\vec{x} = A\vec{x} \tag{10}$$

This last equation is just the fundamental equation for for the eigenvalue λ and eigenvector $\vec{x}$. We can find eigenvalues by solving

$$\det(A - \lambda I) = 0 \tag{11}$$

In our example,

$$det \begin{bmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{bmatrix} = \lambda^2 - \lambda - 2 \tag{12}$$

$$= (\lambda+1)(\lambda-2) \tag{13}$$

and

$$\lambda_1 = -1, \ \lambda_2 = 2 \tag{14}$$

For

$$\lambda_{-1}, \ (A - \lambda_1 I)\vec{x} = \begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_{1a} \\ x_{1b} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{15}$$

This and the similar equation for λ_2 result in

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \vec{x}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \tag{16}$$

resulting in solutions

$$\vec{u}_1 = e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad and \quad \vec{u}_2 = e^{2t} \begin{bmatrix} 5 \\ 2 \end{bmatrix} \tag{17}$$

The differential equation that we are trying to solve is linear and homogeneous so

$$\vec{u} = c_1 e^{\lambda_1 t} \vec{x}_1 + c_2 e^{\lambda_2 t} \vec{x}_2 \tag{18}$$

and c_1 and c_2 can be determined by the initial conditions to be $c_1 = 3$ and $c_2 = 1$, resulting in a final, complete solution of

$$\vec{u}(t) = 3e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 5 \\ 2 \end{bmatrix} \tag{19}$$

The point of this simple exercise is to show that the eigenvalues and eigenvectors are the "key" to solving the problem. The eigenvectors are the normal modes. The behaviour of the system is a linear combination of these normal modes.

Each of the following conditions is necessary and sufficient for the number λ to be an eigenvalue of A.

1) There exists a *nonzero vector* such that $A\vec{x} = \lambda\vec{x}$

2) The matrix $A - \lambda I$ is singular

3) $det(A - \lambda I) = 0$

An interesting property of eigenvalues is that the $trace(A) = \Sigma\lambda$. Also, suppose that the $n \times n$ matrix A has n linearly independent eigenvectors. Then if these vectors are chosen to be the columns of the matrix S, it follows that $S^{-1}AS$ is a diagonal matrix Λ with is eigenvalues along its diagonal. Another interesting property is that if $AB = BA$, then A and B share the same eigenvectors. you can see these easily via the following steps.

$$Ax = \lambda\vec{x}, AB\vec{x} = BA\vec{x} = B\lambda\vec{x} = \lambda B\vec{x} \tag{20}$$

One last thing to remember, if nonzero eigenvectors $\vec{x}_1, \vec{x}_2, \ldots \vec{x}_k$ correspond to different eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_k$, then those eigenvectors are linearly independent.

The Complex Case

The complex space $\mathbf{C}^n$ contains all vectors $\vec{x}$ with n complex components.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}, x_j = a_j + ib_j \tag{21}$$

Working with complex marices and vectors is very similar to working with real ones. There are, however, important differences. For instance, the length of the vector is

$$||\vec{x}||^2 = |x_1|^2 + \ldots |x_n|^2. \tag{22}$$

Closely connected with the concept of length is inner product. Now the inner product of two vectors is $\bar{\vec{x}}^T\vec{y}$ where the bar over $\vec{x}$ means complex conjugate and the T means transpose. For real vectors this inner product was just $\vec{x}^T\vec{y}$. This combination of conjugating plus transposing is called the Hermitian transpose and the bar and the T are, together, replaced by *a superscript H*.

Matrices that are equal to their conjugate transpose are called **Hermitian**. Here are some of the important properties of **Hermitian** matrices.

1) if $A = A^H$ (A is Hermitian) then for all complex vectors $\vec{x}, \vec{x}^H A\vec{x}$ is real.

2) every eigenvalue of a Hermitian matrix is real.

3) the eigenvectors of a Hermitan matrix are orthonormal to each other.

4) for a Hermitian matrix A, there exists a diagonalizing unitary matrix U and $U^{-1}AU = U^HAU = \Lambda$.

A matrix with orthonormal columns is called a unitary matrix and

$$U^HU = I, U^H = U^{-1} \tag{23}$$

Any Hermitian matrix can be decomposed into

$$A = U\Lambda U^H = \lambda \vec{x}_1 \vec{x}_1^H + \lambda_2 \vec{x}_2 \vec{x}_2^H + \dots \tag{24}$$

In every case an Hermitian matrix A can be diagonalized by a unitary U.

Real versus Complex

$$R^n \iff C^n \tag{25}$$

$$\text{length} : \|x\|^2 = \ x_1^2 + \dots x_n^n \iff \|x\|^2 = |\ x_1|^2 + \dots |x_n|^n \tag{26}$$

$$\text{transpose} : A_{ij}^T = A_{ji} \iff \text{Hermitiantranspose} : A_{ij}^H = \overline{A_{ji}} \tag{27}$$

$$(AB)^T = B^T A^T \iff (AB)^H = B^H A^H \tag{28}$$

$$\text{inner product} : x^T y = x_1 y_1 + \dots x_n y_n \iff x^H y = \overline{x_1} y_1 + \dots \overline{x_n} y_n \tag{29}$$

$$\text{Symmetric matrices} : A^T = A \iff \text{Hermitian matrices} : A^H = A \tag{30}$$

$$x^H Ax \text{ is real, every eigenvalue is real, and} A = U\Lambda U^{-1} = U\Lambda U^H \tag{31}$$

$$\text{Orthogonal matrices} : Q^T Q = I \iff \text{Unitary matrices} : U^H U = I \tag{32}$$

Exercises

1) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}. \tag{33}$$

Verify that the trace equals the sum of the eigenvalues, and that the determinant equals their product.

2) With the same matrix as in excercise 1, solve the differential equation $\frac{du}{dt} = Au.u_0 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$. What are the two exponential solutions.

3) Find all the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{34}$$

and write down two different diagonalizing matrices S.

4) If A and B share the same eigenvector matrix S, so that $A = S\Lambda_1 S^{-1}$ and $B = S\Lambda_2 S^{-1}$, prove that $AB = BA$.

5) For A, below, compute the eigenvalues, eigenvectors. Find the unitary U that diagonalizes A and complete the spectral decomposition (into three matrices $\lambda_i x_i x_i^H$.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{35}$$

PY522 Homework No. 1
Solutions

1. (Jackson 6.11)
A transverse plane wave is incident normally in vacuum on a perfectly absorbing flat screen.
(a) From the law of conservation of linear momentum, show that the pressure (called *radiation pressure) exerted on the screen is equal to the field energy per unit volume in* the wave.

Let our wave travel in the x direction, with $\mathbf{E}$ in the y direction and $\mathbf{B}$ in the z direction

$$\mathbf{E} = E_2\hat{\mathbf{y}}$$
$$\mathbf{B} = B_3\hat{\mathbf{z}}$$

From Jackson 6.121 we have

$$\frac{d(\mathbf{P}_{mech})_\alpha}{dt} = \sum_\beta \int_V d^3r\, \partial_\beta T_{\alpha\beta} - \frac{d(\mathbf{P}_{field})_\alpha}{dt}$$
$$= \sum_\beta \int_V d^3r\, \partial_\beta T_{\alpha\beta} - \varepsilon_0 \frac{d}{dt}\int_V d^3r\, (\mathbf{E}\times\mathbf{B})_\alpha$$

The left-hand-side is the force on the screen and we want the pressure so we will take the force per volume and then integrate in the x direction

$$\sum_\alpha \frac{d}{dV}\frac{d(\mathbf{P}_{mech})_\alpha}{dt} = \sum_{\alpha,\beta} \partial_\beta T_{\alpha\beta} - \varepsilon_0 \sum_\alpha \frac{d(\mathbf{E}\times\mathbf{B})_\alpha}{dt}$$

Now evaluate the stress tensor:

$$T_{\alpha\beta} = \varepsilon_0\left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}\left(E^2 + c^2B^2\right)\delta_{\alpha\beta}\right]$$

T has only diagonal components

$$T_{11} = -\frac{\varepsilon_0}{2}\left[E_2^{\ 2} + c^2 B_3^{\ 2}\right]$$
$$T_{22} = \frac{\varepsilon_0}{2}\left[E_2^{\ 2} - c^2 B_3^{\ 2}\right]$$
$$T_{33} = \frac{\varepsilon_0}{2}\left[-E_2^{\ 2} + c^2 B_3^{\ 2}\right]$$

The force is only in the x direction because E and B do not depend on y or z and the cross product (Poynting vector) is also in the x direction.

$$\frac{d(\mathbf{P}_{mech})_1}{dt} = \int_V d^3r\, \partial_1 T_{11} - \varepsilon_0 \frac{d}{dt}\int_V d^3r\, (\mathbf{E}\times\mathbf{B})_1$$

$$= -\int_V d^3r \left[\partial_1 \frac{\varepsilon_0}{2}\left[E_2^{\,2} + c^2 B_3^{\,2}\right] + \varepsilon_0 \frac{d(E_2 B_3)}{dt} \right]$$

The pressure is

$$P = \int_{-\infty}^{0} dx \frac{d}{dV}\frac{d(\mathbf{P}_{mech})_1}{dt} = -\frac{\varepsilon_0}{2}\left[E_2^{\,2} + c^2 B_3^{\,2}\right] + \varepsilon_0 \int_0^{\infty} dx\, \frac{d(E_2 B_3)}{dt}$$

Now we must take the time average. The last term contributes zero because its time average is zero.

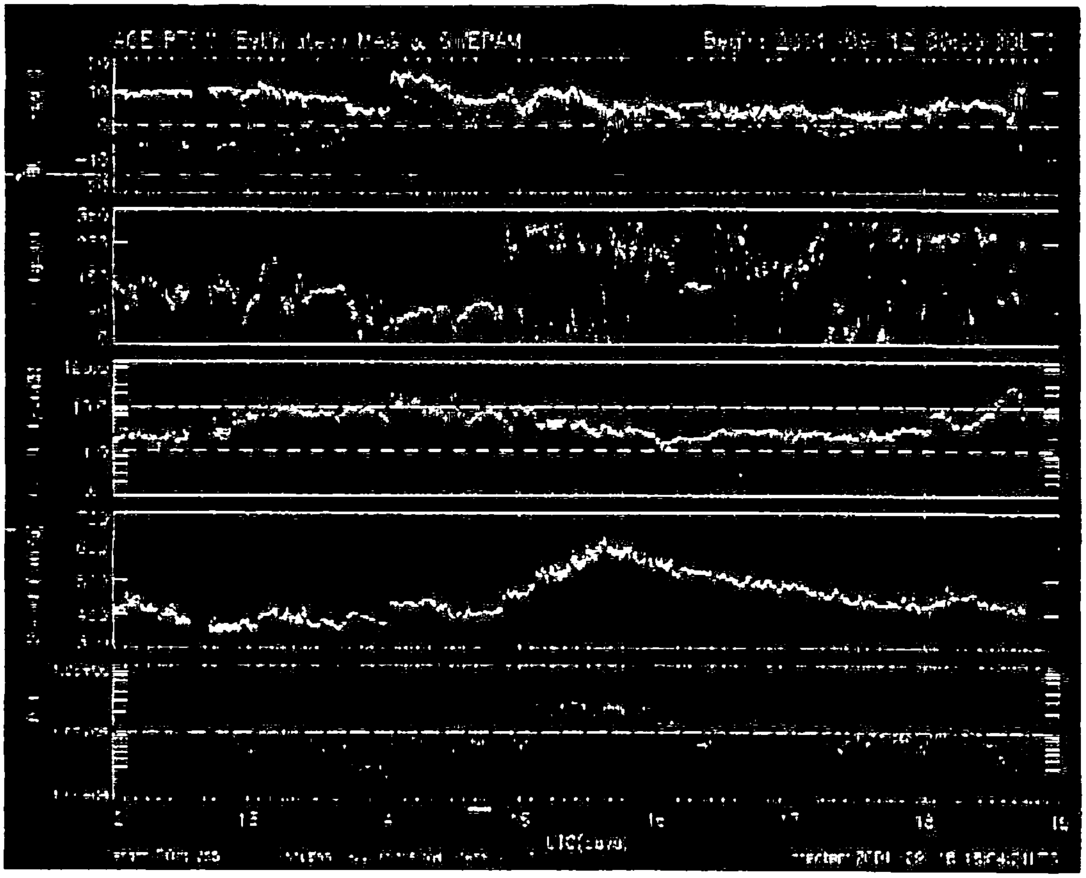
$$\overline{P} = \frac{\varepsilon_0}{2}\left[\overline{E_2}^{\,2} + c^2 \overline{B_3}^{\,2}\right] = \frac{\varepsilon_0}{2}\overline{E_2}^{\,2} + \frac{1}{2\mu_0}\overline{B_3}^{\,2} = \overline{u}$$

(b) In the neighborhood of the earth the flux of electromagnetic energy from the sun is approximately 1.4 kW/m^2. If an interplanetary "sailplane" had a sail of mass 1 g/ m^2 of area and negligible other weight, what would be its maximum acceleration in meters per second squared due to the solar radiation pressure? How does this answer compare with the acceleration due to the solar "wind" (corpuscular radiation)?

$$\overline{P} = \overline{u} = \frac{S}{c} = \frac{1.4\times10^3\,\text{W/m}^2}{3\times10^8\,\text{m/s}} \approx 5\times10^{-6}\,\text{N/m}^2$$

$$a = \frac{\overline{P}A}{m} = \frac{5\times10^{-6}\,\text{N/m}^2}{10^{-3}\,\text{kg}/\text{m}^2} \approx 5\times10^{-3}\,\text{m/s}^2$$

Compare to solar wind:
At http://sec.noaa.gov/ace/MAG_SWEPAM_7d.html
I found the following plot of the solar wind for the last 7 days beginning 9/12/01:



From this plot we see that the solar wind (in the vicinity of the earth) is made of protons traveling at

$$v \sim (4-6)\times 10^5\,\mathrm{m/s}$$

with a density of

$$\rho \sim \left(10^6 - 10^7\right)/\mathrm{m}^3$$

Thus, the solar wind flux is

$$N \approx \left(5\times 10^5\,\mathrm{m/s}\right)\left(3\times 10^7/\mathrm{m}^3\right) \approx 2\times 10^{12}\,\mathrm{m}^{-2}\mathrm{s}^{-1}$$

The corresponding acceleration on our sail plane is

$$a \approx \frac{\left(2\times 10^{12}\,\mathrm{m}^{-2}\mathrm{s}^{-1}\right)m_p v}{10^{-3}\,\mathrm{kg/m}^2} \approx \frac{\left(2\times 10^{12}\,\mathrm{m}^{-2}\mathrm{s}^{-1}\right)\left(1.6\times 10^{-27}\,\mathrm{kg}\right)\left(7\times 10^5\,\mathrm{m/s}\right)}{10^{-3}\,\mathrm{kg/m}^2} \approx 2\times 10^{-6}\,\mathrm{m/s}^2$$

significantly smaller than the acceleration due to radiation.

2. (Jackson 6.16)
(a) Calculate the force in newtons acting on a Dirac monopole of the minimum magnetic charge located a distance 0.05 nm from and in the median plane of a magnetic dipole with dipole moment equal to one nuclear magneton ($e\hbar/2m_p$).

The force on monopole in dipole field

$$F = gH = \frac{gB}{\mu_0}$$

Dipole field,

$$B = \frac{m\mu_0}{2\pi r^3}$$

where

$$g = \frac{2\pi\hbar}{e}n$$

$$m = \frac{e\hbar}{2m_p}$$

This gives

$$F = \left(\frac{2\pi\hbar}{e}n\right)\left(\frac{e\hbar}{2m_p}\right)\left(\frac{1}{2\pi r^3}\right) = \frac{\hbar}{r^3 e}n\frac{e\hbar}{2m_p}$$

Numerically (for $n = 1$)

$$F = \frac{(1.05\times10^{-34}\,\mathrm{J\cdot s})(5.0\times10^{-27}\,\mathrm{J/T})}{(5\times10^{-11}\,\mathrm{m})^3(1.6\times10^{-19}\,\mathrm{C})} \approx 2.6\times10^{-11}\,\mathrm{N}$$

(b) Compare the force in part a with atomic forces such as the direct electrostatic force between charges (at the same separation), the spin-orbit force, the hyperfine interaction. Comment on the question of binding of magnetic monopoles to nuclei with magnetic moments. Assume the monopole mass is at least that of a proton.
Reference: D. Sivers, *Phys. Rev.* **D2**, 2048 (1970).

For the direct electrostatic force

$$F = \frac{ke^2}{r^2} = \frac{(1.44\,\mathrm{eV\cdot nm})(10^{-9}\,\mathrm{m/nm})(1.6\times10^{-19}\,\mathrm{J/eV})}{(5\times10^{-11}\,\mathrm{m})^2} \approx 9\times10^{-8}\,\mathrm{N}$$

The force on the monopole is about 3000 times smaller than the direct electrostatic force.

The spin orbit force is α^2 times smaller than the direct electrostatic force, putting it about the same order-of-magnitude as the monopole force. The hyperfine interaction (electron spin with proton spin) is about 6 orders of magnitude smaller than direct electrostatic or about 100 times smaller than monopole.

PY522 Homework No. 2
Due Friday, September 21, 2001

1. Consider the problem of a thin spherical shell of charge Q and radius a rotating with constant angular frequency ω (Jackson 5.13). Take the z-axis to be the rotation axis. The solution for the static magnetic field is:

Outside:

$$\mathbf{B}_{out} = \frac{\mu_0 Q\omega}{12\pi}\frac{a^2}{r^3}\left[2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right]$$

Inside:

$$\mathbf{B}_{in} = \frac{\mu_0 Q\omega}{6\pi a}\hat{z}$$

Imagine now that ω is not constant but is varying slowly such that **B** *is still given by the above expressions.*

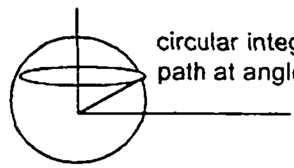
a) Use Faraday's law to find the induced electric field inside the shell.

Note that in the static case we have

$$\mathbf{E}_{out} = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$$

$$\mathbf{E}_{in} = 0$$

Using Faraday's law,



the induced field satisfies

$$\oint d\mathbf{l}\cdot\mathbf{E} = -\frac{\partial}{\partial t}\int_S d\mathbf{a}\cdot\mathbf{B}$$

or

$$2\pi a\sin\theta E = -\frac{\partial B}{\partial t}\pi(a\sin\theta)^2$$

$$\mathbf{E} = -\frac{a\sin\theta}{2}\frac{\partial B}{\partial t}\hat{\phi} = -\frac{\mu_0 Q\sin\theta}{12\pi}\dot{\omega}\hat{\phi}$$

b) Evaluate the rate that the induced field does work on the charge using

$$\frac{dU_{mech}}{dt} = \int_V d^3r\, \mathbf{E}\cdot\mathbf{J}$$

This term represents the rate of change of mechanical energy. The induced electric field is going to cause a reduction (increase) in energy if the acceleration is positive (negative).

$$\mathbf{J} = \frac{Q\omega \sin\theta\, \delta(r-a)}{4\pi a}\hat{\phi}$$

$$\frac{dU_{mech}}{dt} = \int_V d^3r\, \mathbf{E}\cdot\mathbf{J} = \int_V d^3r \left(-\frac{\mu_0 Q \sin\theta}{12\pi}\dot{\omega}\right)\frac{Q\omega \sin\theta\, \delta(r-a)}{4\pi a}$$

$$\frac{dU_{mech}}{dt} = 2\pi \int dr r^2 \int_{-1}^{1} d\cos\theta \left(-\frac{\mu_0 Q \sin\theta}{12\pi}\dot{\omega}\right)\frac{Q\omega \sin\theta\, \delta(r-a)}{4\pi a}$$

$$= -2\pi a^2 \frac{\mu_0 Q\dot{\omega}}{12\pi}\frac{Q\omega}{4\pi a}\int_{-1}^{1} d\cos\theta \sin^2\theta = -\frac{\mu_0 a Q^2 \omega\dot{\omega}}{24\pi}\left(\frac{4}{3}\right)$$

Note we have used:

$$\int_{-1}^{1} d\cos\theta \sin\theta \int_{-1}^{1} d\cos\theta\left[1-\cos^2\theta\right] = 2 - \frac{2}{3} = \frac{4}{3}$$

$$\frac{dU_{mech}}{dt} = -\frac{\mu_0 a Q^2 \omega\dot{\omega}}{18\pi}$$

This equation says that if we give the sphere a positive acceleration, it will radiate electromagnetic energy and thus lose mechanical energy.

c) Evaluate the rate of change of stored electromagnetic energy using

$$\frac{dU_{field}}{dt} = \frac{d}{dt}\int_V d^3r \left[\frac{E^2}{2\varepsilon_0} + \frac{B^2}{2\mu_0}\right]$$

This term represents the rate of change of stored electromagnetic energy. If the acceleration is positive (negative), there is more (less) stored energy.

$$\frac{dU_{field}}{dt} = \frac{d}{dt}\int_V d^3r \left[\frac{E^2}{2\varepsilon_0} + \frac{B^2}{2\mu_0}\right]$$

The induced electric field is time dependent but it contributes to order $(d\omega/dt)^2$ so neglect it.

$$\frac{dU_{field}}{dt} = \frac{d}{dt}\frac{1}{2\mu_0}\int_V d^3r\, B^2 = \frac{d}{dt}\frac{1}{2\mu_0}\left(\frac{\mu_0 Q\omega}{6\pi a}\right)^2 \frac{4\pi a^3}{3} = \frac{\mu_0 a Q^2 \omega\dot{\omega}}{27\pi}$$

d) Evaluate rate that energy enters the spherical volume by integrating the Poynting vector over the surface of the shell and demonstrate explicitly that

$$\frac{d}{dt}\left[U_{field} + U_{mech}\right] = -\oint_S d\mathbf{a}\cdot(\mathbf{E}\times\mathbf{H})$$

$$(\mathbf{E}\times\mathbf{B})\cdot\hat{\mathbf{r}} = \left(E_\phi\hat{\phi}\times B_\theta\hat{\theta}\right)\cdot\hat{\mathbf{r}} = -E_\phi B_\theta$$

$$= -\left(-\frac{\mu_0 Q\sin\theta}{12\pi}\dot{\omega}\right)\left(\frac{\mu_0 Q\omega}{12\pi}\frac{\sin\theta}{a}\right) = \frac{\mu_0{}^2 Q^2\dot{\omega}\omega\sin^2\theta}{144\pi^2 a}$$

$$-\frac{1}{\mu_0}\oint_S d\mathbf{a}\cdot(\mathbf{E}\times\mathbf{B}) = -\frac{2\pi a^2}{\mu_0}\int_{-1}^{1} d\cos\theta\,\frac{\mu_0{}^2 Q^2\dot{\omega}\omega\sin^2\theta}{144\pi^2 a}$$

$$= -\frac{\mu_0 a Q^2\dot{\omega}\omega}{72\pi}\frac{4}{3} = -\frac{\mu_0 a Q^2\dot{\omega}\omega}{54\pi}$$

This is the energy flowing into the spherical volume, which causes a change in mechanical and/or field energy inside the volume. Note that the volume we are talking about contains the shell itself and its interior.

Finally, check that

$$-\frac{\mu_0 a Q^2\omega\dot{\omega}}{18\pi} + \frac{\mu_0 a Q^2\omega\dot{\omega}}{27\pi} = -\frac{\mu_0 a Q^2\dot{\omega}\omega}{54\pi}$$

2. (Jackson 7.2)
A plane wave is incident upon a layered interface as shown in the figure. The indices of refraction of the three nonpermeable media are n_1, n_2, n_3. The thickness of the intermediate layer is d. Each of the other media is semi-infinite.
(a) Calculate the transmission and reflection coefficients (ratios of transmitted and reflected Poynting's flux to the incident flux), and sketch their behavior as a function of frequency for $n_1 = 1$, $n_2 = 2$, $n_3 = 3$; $n_1 = 3$, $n_2 = 2$, $n_3 = 1$; and $n_1 = 2$, $n_2 = 4$, $n_3 = 1$.

The Poynting vector goes as

$$S \sim \sqrt{\varepsilon}\, E^2 \sim n\, E^2$$

where E is the amplitude of the electric field (Jackson 7.13 with $\mu = 1$).

Describe the fields in the different regions as
Incident (medium 1),

$$\mathbf{E}_i = \mathbf{E}_0 e^{i\mathbf{k}_1\cdot\mathbf{r}-i\omega t}$$

Reflected (medium 1),

$$\mathbf{E}_r = \mathbf{E}_1 e^{-i\mathbf{k}_1\cdot\mathbf{r}-i\omega t}$$

intermediate (medium 2)

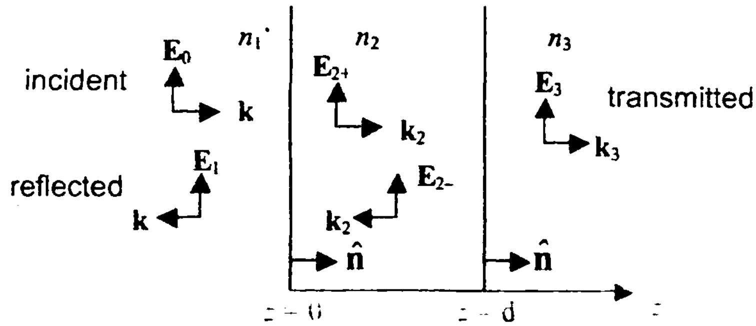
$$\mathbf{E}_2 = \mathbf{E}_{2+} e^{i\mathbf{k}_2\cdot\mathbf{r}-i\omega t} + \mathbf{E}_{2-} e^{-i\mathbf{k}_2\cdot\mathbf{r}-i\omega t}$$

transmitted (medium 3)

$$\mathbf{E}_t = \mathbf{E}_3 e^{i\mathbf{k}_3\cdot\mathbf{r}-i\omega t}$$

The boundary conditions are given by Jackson 7.37 (last two equations). The incident angle is zero and thus, so is the refraction angle.

$$\theta_i = \theta_r = 0$$



At the 1-2 interface ($z = 0$),
Tangential E condition gives

$$(\mathbf{E}_i + \mathbf{E}_r) \times \hat{\mathbf{n}} = \mathbf{E}_2 \times \hat{\mathbf{n}}$$

or

$$E_0 + E_1 = E_{2+} + E_{2-} \quad (1)$$

Tangential H condition gives

$$(\mathbf{k}_1 \times \mathbf{E}_i + \mathbf{k}_1 \times \mathbf{E}_r) \times \hat{\mathbf{n}} = (\mathbf{k}_2 \times \mathbf{E}_2) \times \hat{\mathbf{n}}$$

or

$$n_1 E_0 - n_1 E_1 = n_2 E_{2+} - n_2 E_{2-} \quad (2)$$

and at the 2-3 interface ($z = d$),
Tangential E condition gives

$$\mathbf{E}_2 \times \hat{\mathbf{n}} = \mathbf{E}_t \times \hat{\mathbf{n}}$$

or

$$E_{2+} e^{ik_2 d} + E_{2-} e^{-ik_2 d} = E_3 e^{ik_3 d} \quad (3)$$

Tangential H condition gives

$$(\mathbf{k}_2 \times \mathbf{E}_2) \times \hat{\mathbf{n}} = (\mathbf{k}_3 \times \mathbf{E}_t) \times \hat{\mathbf{n}}$$

or

$$n_2 E_{2+} e^{ik_2 d} - n_2 E_{2-} e^{-ik_2 d} = n_3 E_3 e^{ik_3 d} \quad (4)$$

The last two boundary conditions at $z = d$, (3) and (4), give

$$E_{2+} = \frac{(n_2 + n_3)}{2n_2} E_3 e^{ik_3 d - ik_2 d}$$

$$E_{2-} = \frac{(n_2 - n_3)}{2n_2} E_3 e^{ik_3 d + ik_2 d}$$

while the two at $z = 0$, (1) and (2), give

$$2n_1 E_0 = (n_1 + n_2) E_{2+} + (n_1 - n_2) E_{2-}$$

or

$$E_0 = \frac{(n_1 + n_2)}{2n_1} E_{2+} + \frac{(n_1 - n_2)}{2n_1} E_{2-}$$

$$= \frac{(n_1 + n_2)}{2n_1} \frac{(n_2 + n_3)}{2n_2} E_3 e^{ik_3 d - ik_2 d} + \frac{(n_1 - n_2)}{2n_1} \frac{(n_2 - n_3)}{2n_2} E_3 e^{ik_3 d + ik_2 d}$$

and solving for E_3 we get

$$E_3 = \frac{4e^{-ik_3 d}}{\left(1 + \frac{n_2}{n_1}\right)\left(1 + \frac{n_3}{n_2}\right) e^{-ik_2 d} + \left(1 - \frac{n_2}{n_1}\right)\left(1 - \frac{n_3}{n_2}\right) e^{ik_2 d}} E_0$$

Now solve for E_1, again using (1) and (2)

$$E_1 = \frac{(n_1 - n_2)}{2n_1} E_{2+} + \frac{(n_1 + n_2)}{2n_1} E_{2-}$$

$$= \frac{(n_1 - n_2)}{2n_1} \frac{(n_2 + n_3)}{2n_2} E_3 e^{ik_3 d - ik_2 d} + \frac{(n_1 + n_2)}{2n_1} \frac{(n_2 - n_3)}{2n_2} E_3 e^{ik_3 d + ik_2 d}$$

or

$$E_1 = \frac{\left(1 - \frac{n_2}{n_1}\right)\left(1 + \frac{n_3}{n_2}\right) + \left(1 + \frac{n_2}{n_1}\right)\left(1 - \frac{n_3}{n_2}\right) e^{2ik_2 d}}{\left(1 + \frac{n_2}{n_1}\right)\left(1 + \frac{n_3}{n_2}\right) + \left(1 - \frac{n_2}{n_1}\right)\left(1 - \frac{n_3}{n_2}\right) e^{2ik_2 d}} E_0$$

The reflection coefficient is

$$R = \left| \frac{E_1}{E_0} \right|^2$$

Numerically, for $n_1 = 1, n_2 = 2, n_3 = 3$,

$$R = \left| \frac{\left(-\frac{5}{2}\right) + \left(-\frac{3}{2}\right) e^{2ik_2d}}{\left(\frac{15}{2}\right) + \left(\frac{1}{2}\right) e^{2ik_2d}} \right|^2 = \left| \frac{-5 - 3e^{2ik_2d}}{15 + e^{2ik_2d}} \right|^2$$

The reflection is minimum when

$$e^{2ik_2d} = 1$$

corresponding to

$$d = \frac{\lambda}{2}$$

and

$$\omega = \frac{2\pi c}{n_2 \lambda} = \frac{\pi c}{2d}$$

For $n_1 = 3, n_2 = 2, n_3 = 1$,

$$R = \left| \frac{\left(\frac{1}{2}\right) + \left(\frac{5}{6}\right) e^{2ik_2d}}{\left(\frac{5}{2}\right) + \left(\frac{1}{6}\right) e^{2ik_2d}} \right|^2 = \left| \frac{3 + 5e^{2ik_2d}}{15 + e^{2ik_2d}} \right|^2$$

The reflection is minimum when

$$e^{2ik_2d} = -1$$

corresponding to

$$d = \frac{\lambda}{4}$$

and

$$\omega = \frac{2\pi c}{n_2 \lambda} = \frac{\pi c}{4d}$$

For $n_1 = 2$, $n_2 = 4$, $n_3 = 1$,

$$R = \left| \frac{\left(-\frac{5}{4}\right) + \left(\frac{9}{4}\right) e^{2ik_2 d}}{\left(\frac{15}{4}\right) + \left(-\frac{3}{4}\right) e^{2ik_2 d}} \right|^2 = \left| \frac{-5 + 9e^{2ik_2 d}}{15 - 3e^{2ik_2 d}} \right|^2$$

The reflection is minimum when

$$e^{2ik_2 d} = 1$$

corresponding to

$$d = \frac{\lambda}{2}$$

and

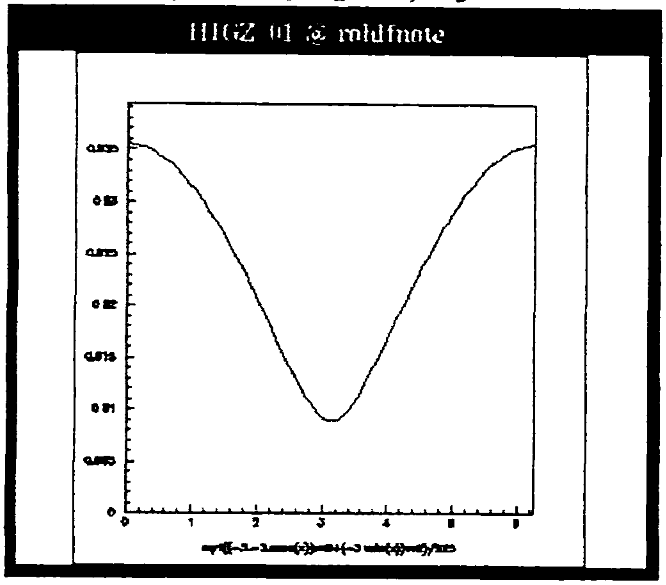
$$\omega = \frac{2\pi c}{n_2 \lambda} = \frac{\pi c}{4d}$$

The transmission coefficient is

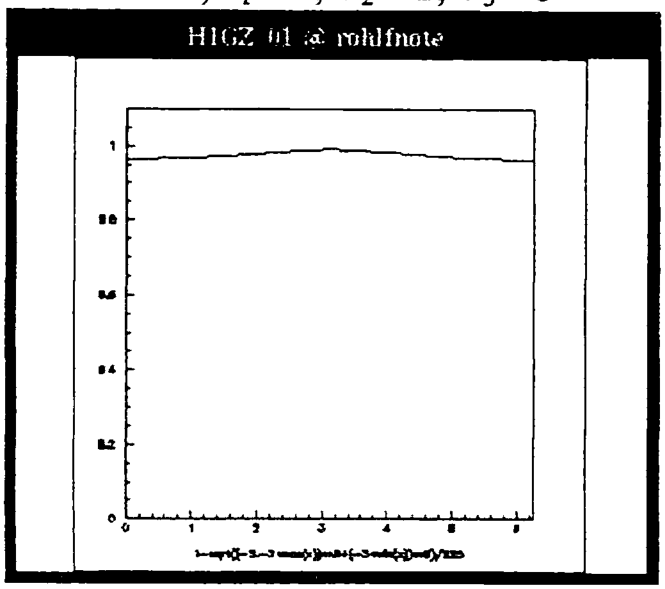
$$T = \frac{n_3}{n_1} \left| \frac{E_3}{E_0} \right|^2$$

Sketches of the solution:

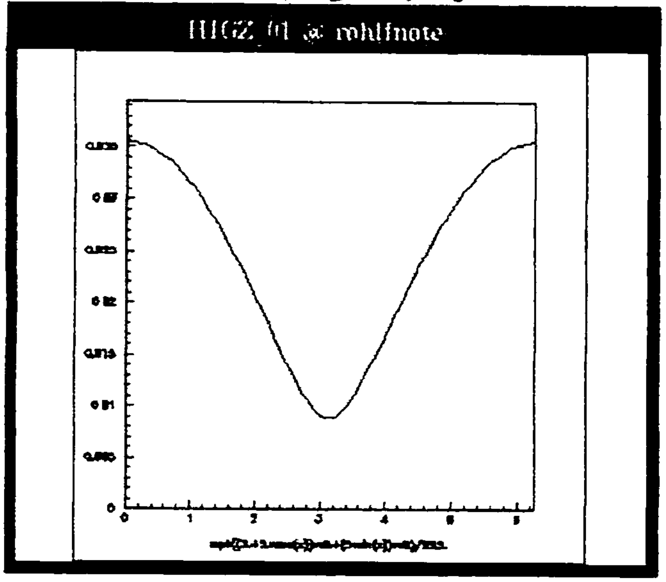
Reflection, $n_1 = 1, n_2 = 2, n_3 = 3$



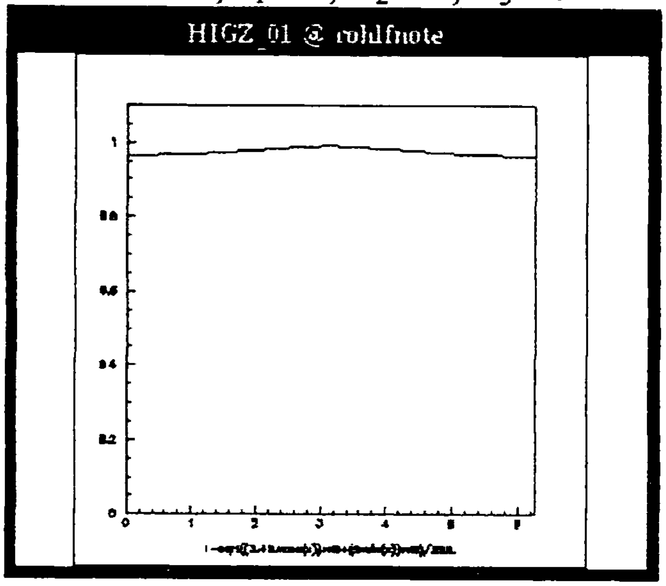
Tranmission, $n_1 = 1, n_2 = 2, n_3 = 3$



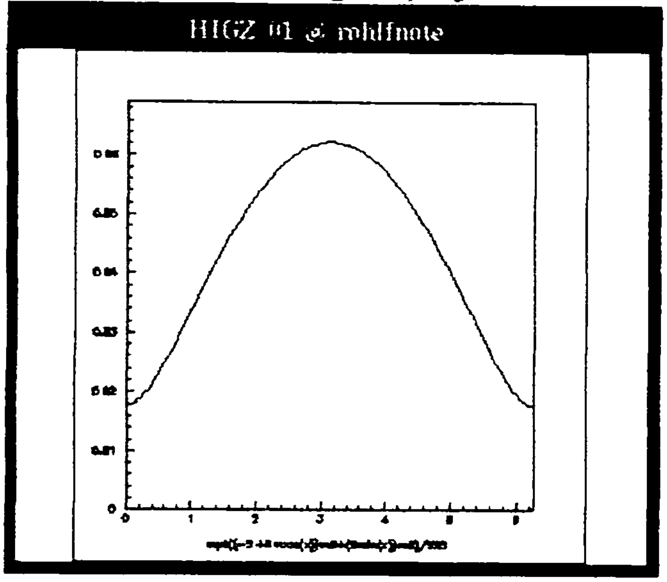
Reflection, $n_1 = 3, n_2 = 2, n_3 = 1$



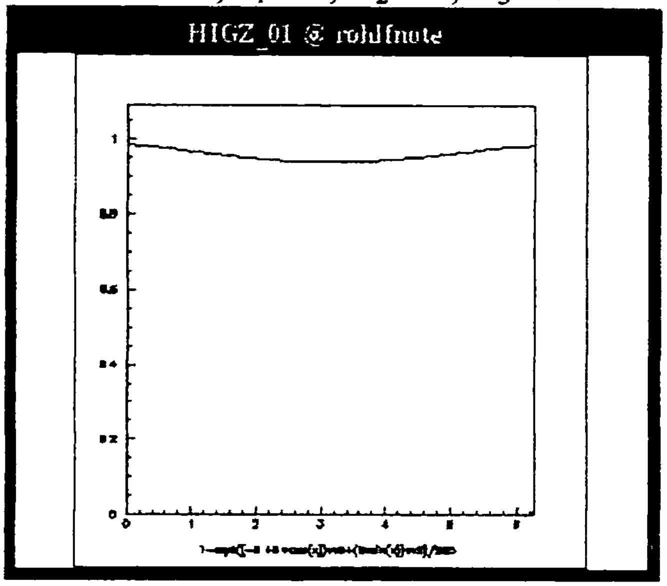
Transmission, $n_1 = 3, n_2 = 2, n_3 = 1$



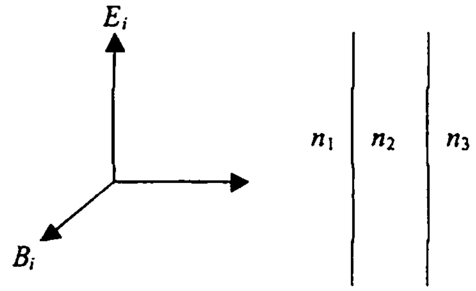
Reflection, $n_1 = 2, n_2 = 4, n_3 = 1$



Transmission, $n_1 = 2, n_2 = 4, n_3 = 1$



(b) The medium n_1 is part of an optical system (e.g., a lens); medium n_3 is air ($n_3 = 1$). It is desired to put an optical coating (medium n_2) on the surface so that there is no reflected wave for a frequency ω_0. What thickness d and index of refraction n_2 are necessary?



For $n_3 = 1$,

$$R = \left| \frac{\left(1-\frac{n_2}{n_1}\right)\left(1+\frac{1}{n_2}\right)+\left(1+\frac{n_2}{n_1}\right)\left(1-\frac{1}{n_2}\right)e^{2ik_2d}}{\left(1+\frac{n_2}{n_1}\right)\left(1+\frac{1}{n_2}\right)+\left(1-\frac{n_2}{n_1}\right)\left(1-\frac{1}{n_2}\right)e^{2ik_2d}} \right|^2$$

The reflection is minimum when

$$e^{2ik_2d} = -1$$

or

$$2k_2 d = \pi$$

or

$$d = \frac{\pi}{2k_2} = \frac{\lambda}{4}$$

To completely eliminate reflection ($R = 0$), we must also have

$$\left(1-\frac{n_2}{n_1}\right)\left(1+\frac{1}{n_2}\right)-\left(1+\frac{n_2}{n_1}\right)\left(1-\frac{1}{n_2}\right) = 0$$

Solve for n_2 to get

$$n_2 = \sqrt{n_1}$$

PY522 Fall 2001
Homework No. 3

Read the relevant sections of Jackson, Chapters 7 and 8

1. (Jackson 7.6)
A plane wave of frequency ω is incident normally from vacuum on a semi-infinite slab of material with a *complex* index of refraction $n(\omega)$ $[n^2(\omega) = \varepsilon(\omega)/\varepsilon_0]$.
(a) Show that the ratio of reflected power to incident power is

$$R = \left|\frac{1-n(\omega)}{1+n(\omega)}\right|^2$$

while the ratio of transmitted into the medium to the incident power is

$$T = \frac{4\,\mathrm{Re}\,n(\omega)}{|1+n(\omega)|^2}$$

Look at the derivation of Jackson 7.39 and see that n may be complex. Then with

$$n = 1$$
$$n' = n(\omega)$$
$$i = 0$$

we have

$$\frac{E_0''}{E_0} = \frac{1-n(\omega)}{1+n(\omega)}$$

which immediately gives

$$R = \left|\frac{1-n(\omega)}{1+n(\omega)}\right|^2$$

The transmitted power ratio is

$$T = \mathrm{Re}\,n(\omega)\left|\frac{2}{1+n(\omega)}\right|^2 = \frac{4\,\mathrm{Re}\,n(\omega)}{|1+n(\omega)|^2}$$

(b) Evaluate $\mathrm{Re}[i\omega(\mathbf{E}\cdot\mathbf{D}^* - \mathbf{B}\cdot\mathbf{H}^*)/2]$ as a function of (x, y, z). Show that this rate of change of energy per unit volume accounts for the relative transmitted power T.

$$\mathrm{Re}[i\omega \mathrm{E}\cdot \mathrm{D}^* - \mathrm{B}\cdot \mathrm{H}] = \mathrm{Re}\left[i\omega\left(\frac{4(n^2)^*}{|1+n|^2} - \frac{4|n|^2}{|1+n|^2} \right) E_0{}^2 e^{-2\,\mathrm{Im}(n)\omega z/c} \right]$$

$$= 8\omega E_0{}^2 \frac{\mathrm{Re}(n)\mathrm{Im}(n)}{|1+n|^2} e^{-2\,\mathrm{Im}(n)\omega z/c}$$

The Poynting vector gives

$$S_z = \frac{1}{2}\mathrm{Re}[\mathrm{E}\times \mathrm{H}] = 2cE_0{}^2 \frac{\mathrm{Re}(n)}{|1+n|^2} e^{-2\,\mathrm{Im}(n)\omega z/c}$$

The complex version of Poynting's theorem is

$$i\omega \int_V d^3r\, [\mathbf{E}\cdot \mathbf{D}^* - \mathbf{B}\cdot \mathbf{H}^*] + \int_V d^3r\, \mathbf{E}\cdot \mathbf{J}^* = -\oint_S d\mathbf{a}\cdot (\mathbf{E}\times \mathbf{H}^*)$$

With no current,

$$\nabla \cdot \mathbf{S} + i\omega[\mathbf{E}\cdot \mathbf{D}^* - \mathbf{B}\cdot \mathbf{H}^*] = 0$$

or

$$\frac{\partial S_z}{\partial z} = -i\omega[\mathbf{E}\cdot \mathbf{D}^* - \mathbf{B}\cdot \mathbf{H}^*]$$

evaluating this derivative, we get

$$\frac{\partial S_z}{\partial z} = -8\omega E_0{}^2 \frac{\mathrm{Re}(n)\mathrm{Im}(n)}{|1+n|^2} e^{-2\,\mathrm{Im}(n)\omega z/c}$$

in agreement.

(c) For a conductor, with $n^2 = 1 + i(\sigma/\omega\varepsilon_0)$, σ real, write out the results of parts a and b in the limit $\varepsilon_0\omega << \sigma$. Express your answer in terms of δ as much as possible. Calculate ½ $\mathrm{Re}(\mathbf{J}^*\cdot\mathbf{E})$ and compare with the result of part b. Do both enter the complex form of Poynting's theorem?

$$n^2 = 1 + i\frac{\sigma}{\omega\varepsilon_0}$$

and

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$$

For

$$\omega\varepsilon_0 << \sigma$$

we have

$$n^2 \cong i\frac{\sigma}{\omega\varepsilon_0}$$

This gives

$$n = \frac{1+i}{\sqrt{2}}\sqrt{\frac{\sigma}{\omega\varepsilon_0}} = (1+i)\sqrt{\frac{\sigma}{2\omega\varepsilon_0}} = \frac{(1+i)c}{\omega}\sqrt{\frac{\omega\sigma\mu}{2}} = \frac{(1+i)c}{\omega\delta}$$

so that

$$|1+n|^2 = \left|1+\frac{(1+i)c}{\omega\delta}\right|^2 = \left(1+\frac{c}{\omega\delta}\right)^2 + \left(\frac{c}{\omega\delta}\right)^2$$

and

$$|1-n|^2 = \left|1-\frac{(1+i)c}{\omega\delta}\right|^2 = \left(1-\frac{c}{\omega\delta}\right)^2 + \left(\frac{c}{\omega\delta}\right)^2$$

This gives

$$R = \left|\frac{1-n}{1+n}\right|^2 = \frac{\left(1-\frac{c}{\omega\delta}\right)^2 + \left(\frac{c}{\omega\delta}\right)^2}{\left(1+\frac{c}{\omega\delta}\right)^2 + \left(\frac{c}{\omega\delta}\right)^2} = 1 - \frac{4\frac{\omega\delta}{c}}{1+\left(1+\frac{\omega\delta}{c}\right)^2}$$

$$T = \frac{4\operatorname{Re} n}{|1+n|^2} = \frac{\left(\frac{4c}{\omega\delta}\right)}{\left(1+\frac{c}{\omega\delta}\right)^2 + \left(\frac{c}{\omega\delta}\right)^2} = \frac{\left(\frac{4\omega\delta}{c}\right)}{1+\left(1+\frac{\omega\delta}{c}\right)^2}$$

For

$$\frac{\omega\delta}{c} << 1$$

we have

$$R \cong 1 - \frac{2\omega\delta}{c}$$

$$T \cong \frac{2\omega\delta}{c}$$

Now evaluate

$$\frac{1}{2}\mathrm{Re}\,2(\mathrm{J}\cdot\mathrm{E}^*) = \frac{1}{2}\mathrm{Re}\left[\sigma|\mathrm{E}|^2 e^{-2\,\mathrm{Im}(n)z\omega/c}\right] = \frac{4\sigma E_0^{\ 2} e^{-2z/\delta}}{2|1+n|^2} = \frac{8\omega E_0^{\ 2} e^{-2z/\delta}}{1+\left(1+\frac{\omega\delta}{c}\right)^2}$$

Compare with

$$\mathrm{Re}\left[i\omega\mathrm{E}\cdot\mathrm{D}^* - \mathrm{B}\cdot\mathrm{H}\right] = \frac{8\omega E_0^{\ 2}\,\mathrm{Re}(n)\,\mathrm{Im}(n) e^{-2\,\mathrm{Im}(n)\omega/c}}{|1+n|^2} = \frac{8\omega E_0^{\ 2} e^{-2z/\delta}}{1+\left(\frac{\omega\delta}{c}\right)^2}$$

These terms are equal and do not enter the complex form of Poynting's theorem.

2. Consider a TE_{11} wave propagating in a rectangular waveguide of dimension *a* by *b*.

a) Calculate the power transported by this mode.

The TE_{11} wave has

$$\mathbf{E}_z = 0$$

$$\mathbf{H}_{0z} = H_0 \cos\frac{\pi x}{a}\cos\frac{\pi y}{b}$$

The transverse fields are given by

$$\mathbf{H}_{0t} = \frac{ik}{\gamma^2} \nabla_t H_{0z}$$

$$\mathbf{E}_{0t} = -\frac{i\omega\mu}{\gamma^2} \hat{\mathbf{z}} \times \nabla_t H_{0z}$$

where

$$\gamma^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$

For the Poynting vector we need

$$\mathbf{E}_t \times \mathbf{H}_t^* = \frac{\omega\mu k}{\gamma^4} (\hat{\mathbf{z}} \times \nabla_t H_z) \times \nabla_t H_z^*$$

$$\mathbf{E}_t \times \mathbf{H}_t^* = \frac{\omega\mu k}{\gamma^4} \left[\left(\hat{\mathbf{z}} \cdot \nabla_t H_z^* \right) \nabla_t H_z - \hat{\mathbf{z}} \left(\nabla_t H_z^* \cdot \nabla_t H_z \right) \right]$$

$$\mathbf{E}_t \times \mathbf{H}_t^* = \frac{\omega\mu k}{\gamma^4} \hat{\mathbf{z}} \left| \nabla_t H_z \right|^2$$

The time-average Poynting vector is

$$\mathbf{S} = \frac{\omega\mu k H_0^{\ 2}}{2\gamma^4} \hat{\mathbf{z}} \left[\left(\frac{\pi}{a}\right)^2 \sin^2 \frac{\pi x}{a} \cos^2 \frac{\pi y}{b} + \left(\frac{\pi}{b}\right)^2 \cos^2 \frac{\pi x}{a} \cos^2 \frac{\pi y}{b} \right]$$

We integrate this over the cross section of the guide to get the power transported. Using the average of sin and cos squared equal to 1/2, we get:

$$P = \int_0^a dx \int_0^b dy\, S = \frac{\omega\mu k H_0^{\ 2}}{2\gamma^4}\frac{ab}{4}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right] = \frac{\omega\mu k H_0^{\ 2} ab}{8\gamma^2}$$

b) Estimate the power loss in the walls due to ohmic heating.

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\oint_C d\ell\, \left|\hat{\mathbf{n}}\times\mathbf{H}\right|^2$$

To get the power dissipated we need the tangential **H** field at the walls. For the wall at $y = 0$,

$$\mathbf{H}_{\tan} = H_0\left[\cos\frac{\pi x}{a}\hat{\mathbf{z}} - \frac{ik\pi}{\gamma^2 a}\sin\frac{\pi y}{a}\hat{\mathbf{i}}\right]$$

The contribution from this wall is

$$\left(-\frac{dP}{dz}\right)_{y=0} = \frac{H_0^{\ 2}}{2\sigma\delta}\int_0^a dx\left[\cos^2\frac{\pi x}{a} + \frac{k^2\pi^2}{\gamma^4 a^2}\sin^2\frac{\pi x}{a}\right]$$

$$= \frac{H_0^{\ 2}}{2\sigma\delta}\frac{a}{2}\left[1 + \frac{k^2\pi^2}{\gamma^4 a^2}\right]$$

The wall at $y = b$ gives an identical result.

For the wall at $x = 0$,

$$\mathbf{H}_{\tan} = H_0\left[\cos\frac{\pi y}{b}\hat{\mathbf{z}} - \frac{ik\pi}{\gamma^2 b}\sin\frac{\pi y}{b}\hat{\mathbf{j}}\right]$$

The contribution from this wall is

$$\left(-\frac{dP}{dz}\right)_{x=0} = \frac{H_0^{\ 2}}{2\sigma\delta}\int_0^b dy\left[\cos^2\frac{\pi y}{b} + \frac{k^2\pi^2}{\gamma^4 b^2}\sin^2\frac{\pi y}{b}\right]$$

$$= \frac{H_0^{\ 2}}{2\sigma\delta}\frac{b}{2}\left[1 + \frac{k^2\pi^2}{\gamma^4 b^2}\right]$$

The wall at $x = a$ gives an identical result.

The net result is

$$-\frac{dP}{dz} = \frac{H_0^{\ 2}}{2\sigma\delta}\left[a + b + \frac{k^2\pi^2}{\gamma^4}\left(\frac{1}{a} + \frac{1}{b}\right)\right]$$

3. Consider a TM_{01} wave propagating in a circular waveguide of radius a. Find the cutoff wavelength.

TM modes

$$H_z = 0$$

and the E_z vanishes on the wall ($\rho = a$)

$$E_z = AJ_0\left(\frac{x_{01}}{a}\rho\right)e^{\pm i\phi}$$

x_{01} is the first zero of J_0

$$\gamma_{01}a = x_{01}$$

The expression for k is

$$k_{01}^{\ 2} = \varepsilon\mu\omega^2 - \gamma_{01}^{\ 2} = \varepsilon\mu\omega^2 - \frac{x_{01}^{\ 2}}{a^2}$$

The cutoff frequency is

$$\omega_{cutoff} = \sqrt{\frac{x_{01}^{2}}{\varepsilon\mu a^{2}}}$$

The cutoff wavelength is

$$\lambda_{cutoff} = \frac{2\pi}{\omega_{cutoff}\sqrt{\mu\varepsilon}} = \frac{2\pi}{\sqrt{\mu\varepsilon}}\sqrt{\frac{\varepsilon\mu a^{2}}{x_{01}^{2}}} = \frac{2\pi a}{x_{01}} = \frac{2\pi a}{2.405} = 2.61\,a$$

PY522 Fall 2001
Homework No. 4 Solutions

3. Jackson 8.2, parts (a),(b),(c)
A transmission line consisting of two concentric circular cylinders of metal with conductivity σ and skin depth δ is filled with a uniform lossless dielectric (μ,ε). A TEM mode is propagated along this line.

(a) Show that the time-averaged power flow along the line is

$$P = \sqrt{\frac{\mu}{\varepsilon}}\pi a^2 |H_0|^2 \ln\left(\frac{b}{a}\right)$$

where H_0 is the peak value of the azimuthal magnetic field at the surface of the inner conductor.

For the TEM mode

$$\nabla_t \times \mathbf{E}_{\text{TEM}} = 0$$
$$\nabla_t \cdot \mathbf{E}_{\text{TEM}} = 0$$

so our solution is

$$\mathbf{E}_{\text{TEM}} = -\nabla\Phi$$
$$\nabla^2\Phi = 0$$

We have azimuthal symmetry so our solution cannot depend on ϕ or z:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) = 0$$

The solution is

$$\Phi = A\ln\rho$$

(where A is a constant)

$$\mathbf{E}_{TEM} = -\nabla\Phi = -\frac{A}{\rho}\hat{\rho}$$

$$\mathbf{H}_{TEM} = \frac{\sqrt{\mu\varepsilon}}{\mu}\hat{\mathbf{z}}\times\mathbf{E}_{TEM} = -\sqrt{\frac{\varepsilon}{\mu}}\frac{A}{\rho}\hat{\phi}$$

We also have

$$H_0 \equiv \sqrt{\frac{\varepsilon}{\mu}}\frac{A}{a}$$

so the solution is

$$\mathbf{H}_{TEM} = -H_0\frac{a}{\rho}\hat{\phi}$$

$$\mathbf{E}_{TEM} = -\sqrt{\frac{\mu}{\varepsilon}}\frac{H_0}{\rho}\hat{\rho}$$

The time-averaged energy flow is given by

$$\mathbf{S} = \frac{1}{2}\mathbf{E}_{TEM}\times\mathbf{H}_{TEM}^{*} = \frac{1}{2}H_0^2\sqrt{\frac{\mu}{\varepsilon}}\frac{a^2}{\rho^2}\hat{\mathbf{z}}$$

and the power flow along the line is

$$P = \int_A da\,\mathbf{S}\cdot\hat{\mathbf{z}} = \frac{1}{2}H_0^2\sqrt{\frac{\mu}{\varepsilon}}\frac{a^2}{\rho^2}\int_a^b d\rho\,2\pi\rho\frac{1}{\rho^2} = \pi a^2 H_0^2\sqrt{\frac{\mu}{\varepsilon}}\ln\frac{b}{a}$$

(b) Show that the transmitted power is attenuated along the line as

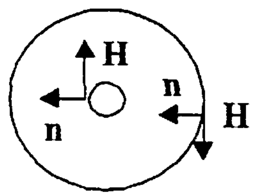
$$P(z) = P_0 e^{-2\gamma z}$$

where

$$\gamma = \frac{1}{2\sigma\delta}\sqrt{\frac{\varepsilon}{\mu}}\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln\left(\frac{b}{a}\right)}$$

The power loss is given by (Jackson 8.58)

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\oint_C dl\left|\hat{\mathbf{n}}\times\mathbf{H}\right|^2$$



$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\left[\oint_{\text{outer}} dl\left|H_0\frac{a}{b}\right|^2 + \oint_{\text{inner}} dl\left|H_0\right|^2\right]$$

$$= \frac{1}{2\sigma\delta}\left[2\pi b H_0^{\,2}\frac{a^2}{b^2} + 2\pi a H_0^{\,2}\right] = \frac{\pi a^2 H_0^{\,2}}{\sigma\delta}\left[\frac{1}{a}+\frac{1}{b}\right]$$

Now write in terms of P:

$$-\frac{dP}{dz} = \frac{P}{\sigma\delta\sqrt{\frac{\mu}{\varepsilon}}\ln\frac{b}{a}}\left[\frac{1}{a}+\frac{1}{b}\right] = -2\gamma P$$

where

$$\gamma \equiv \frac{\left[\frac{1}{a}+\frac{1}{b}\right]}{2\sigma\delta\sqrt{\frac{\mu}{\varepsilon}}\ln\frac{b}{a}}$$

Therefore (integrating),

$$P = P_0 e^{-2\gamma z}$$

(P_0 is the power at $z = 0$)

(c) The characteristic impedance Z_0 of the line is defined as the ratio of the voltage between the cylinders to the axial current flowing in one of them at any position z. Show that for this line

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\ln\left(\frac{b}{a}\right)$$

The voltage difference between the cylinders is

$$\Delta V = \int_b^a d\ell \cdot \mathbf{E}_{\text{TEM}} = \sqrt{\frac{\mu}{\varepsilon}}H_0\int_a^b d\rho\frac{1}{\rho} = \sqrt{\frac{\mu}{\varepsilon}}H_0\ln\frac{b}{a}$$

The current is given by (Ampere's law around the inner conductor)

$$I = \oint d\ell \cdot \mathbf{H}_{\text{TEM}} = 2\pi a H_0$$

We have

$$\frac{\Delta V}{I} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\ln\frac{b}{a}$$

PY522 Fall 2001
Homework No. 5 Solutions

1. Jackson 8.20

An infinitely long rectangular waveguide has a coaxial line terminating in the short side of the guide with the thin central conductor forming a semicircular loop of radius R whose center is a height h above the floor of the guide. The half-loop is in the plane $z = 0$ and its radius R is sufficiently small that the current can be taken as having a constant value I_0 everywhere on the loop.

(a) Prove that to the extent that the current is constant around the half-loop, the TM modes are not excited. Give a physical explanation of this lack of excitation.

The coefficients for the various modes are given by Jackson 8.146:

$$A_\lambda = -\frac{Z_\lambda}{2}\int_V d^3x\, \mathbf{J}\cdot\mathbf{E}_\lambda$$

The current is

$$\mathbf{J} = \frac{I_0}{A}\left[-\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}\right]$$

which gives

$$A_\lambda = -\frac{Z_\lambda}{2} I_0 \int_{-\pi/2}^{+\pi/2} d\theta\, R\left[-\sin\theta E_{x\lambda} + \cos\theta E_{y\lambda}\right]$$

The field components are given by Jackson 8.135 (TM waves)

$$E_{xmn} = \frac{2\pi m}{\gamma_{mn} a\sqrt{ab}}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$E_{ymn} = \frac{2\pi n}{\gamma_{mn} b\sqrt{ab}}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

so

$$A_\lambda = -\frac{Z_\lambda}{2} I_0 \int_{-\pi/2}^{+\pi/2} d\theta\, R \left[-\sin\theta \frac{2\pi m}{\gamma_{mn} a\sqrt{ab}} \cos\left(\frac{m\pi R\cos\theta}{a}\right) \sin\left(\frac{n\pi(h+R\sin\theta)}{b}\right)\right]$$

$$-\frac{Z_\lambda}{2} I_0 \int_{-\pi/2}^{+\pi/2} d\theta\, R \left[\cos\theta \frac{2\pi n}{\gamma_{mn} b\sqrt{ab}} \sin\left(\frac{m\pi R\cos\theta}{a}\right) \cos\left(\frac{n\pi(h+R\sin\theta)}{b}\right)\right]$$

Expand the sine and cosine

$$A_\lambda = -\frac{Z_\lambda}{2} \frac{2\pi R I_0}{\gamma_{mn}\sqrt{ab}} \frac{m}{a} \cos\left(\frac{n\pi h}{b}\right) \int_{-\pi/2}^{+\pi/2} d\theta \left[\sin\theta \cos\left(\frac{m\pi R\cos\theta}{a}\right) \sin\left(\frac{n\pi R\sin\theta}{b}\right)\right]$$

$$-\frac{Z_\lambda}{2} \frac{2\pi R I_0}{\gamma_{mn}\sqrt{ab}} \frac{n}{b} \cos\left(\frac{n\pi h}{b}\right) \int_{-\pi/2}^{+\pi/2} d\theta \left[\cos\theta \sin\left(\frac{m\pi R\cos\theta}{a}\right) \cos\left(\frac{n\pi R\sin\theta}{b}\right)\right]$$

Notice that we have a perfect differential,

$$\frac{d}{d\theta}\left[\sin\left(\frac{m\pi R\cos\theta}{a}\right) \sin\left(\frac{n\pi R\sin\theta}{b}\right)\right]$$

$$= \frac{m\pi R\sin\theta}{a} \cos\left(\frac{m\pi R\cos\theta}{a}\right) \sin\left(\frac{n\pi R\sin\theta}{b}\right) + \sin\left(\frac{m\pi R\cos\theta}{a}\right) \frac{n\pi R\cos\theta}{b} \cos\left(\frac{n\pi R\sin\theta}{b}\right)$$

which gives zero:

$$\left[\sin\left(\frac{m\pi R\cos\theta}{a}\right) \sin\left(\frac{n\pi R\sin\theta}{b}\right)\right]_{\theta=-\pi/2}^{\theta=+\pi/2} = 0$$

so

$$A_\lambda = 0$$

for all TM modes.

Let's examine why there can be no TM modes. Since the current is constant, we have

$$A_\lambda \sim \int d\ell \cdot \mathbf{E}_\lambda \sim \int da \cdot (\nabla \times \mathbf{E}_\lambda) \sim \int da B_{z\lambda}$$

But $B_{z\lambda}$ is zero for TM modes.

(b) Determine the amplitude for the lowest TE mode in the guide and show that its value is independent of the height h.

The TE field components are (Jackson 8.136) for the 1,0 (lowest) mode are

$$E_{x10} = 0$$

$$E_{y10} = \frac{\sqrt{2}}{\sqrt{ab}} \sin\left(\frac{\pi x}{a}\right)$$

This gives

$$A_{10} = -\frac{Z_{10}}{2} I_0 \int_{-\pi/2}^{+\pi/2} d\theta\, R \left[\cos\theta \frac{\sqrt{2}}{\sqrt{ab}} \sin\left(\frac{\pi R\cos\theta}{a}\right)\right]$$

Now for small R, we have

$$A_{10} = -\frac{Z_{10}}{2} I_0 \frac{\sqrt{2}}{\sqrt{ab}} \frac{\pi R}{a} R \int_{-\pi/2}^{+\pi/2} d\theta \cos^2\theta = -\frac{\sqrt{2} Z_{10} I_0 \pi^2 R^2}{4a\sqrt{ab}}$$

(c) Show that the power radiated in either direction in the lowest TE mode is

$$P = \frac{I_0^2}{16} Z \frac{a}{b} \left(\frac{\pi R}{a}\right)^4$$

where Z is the impedance of the TE_{10} mode. Here assume $R << a,b$.

The transverse magnetic field is given by (Jackson 8.31)

$$\mathbf{H}_t = \frac{1}{Z}\hat{\mathbf{z}} \times \mathbf{E}_t$$

which gives

$$H_{x10} = -\frac{1}{Z} E_{y10}$$

Therefore, the time-averaged transmitted power is

$$P = \frac{A_{10}{}^2}{2Z} = \frac{Z I_0{}^2 \pi^4 R4}{16 a^3 b}$$

PY522 Fall 2001
Homework No. 6 **SOLUTIONS**

1. Derive the expression for the electric dipole field (Jackson 9.18)

$$\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\left\{k^2(\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}}\frac{e^{ikr}}{r}+[3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p})-\mathbf{p}]\left(\frac{e^{ikr}}{r^3}-\frac{ike^{ikr}}{r^2}\right)\right\}$$

by direct calculation of

$$\mathbf{E}=\frac{i}{k}\sqrt{\frac{\mu_0}{\varepsilon_0}}\nabla\times\left[\frac{ck^2}{4\pi}\hat{\mathbf{n}}\times\mathbf{p}\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\right].$$

$$\mathbf{E}=\frac{i}{\omega\varepsilon_0}\nabla\times\mathbf{H}=\frac{i}{\omega\varepsilon_0}\frac{ck^2}{4\pi}\nabla\times(f\hat{\mathbf{n}}\times\mathbf{p})$$
$$=\frac{ik}{4\pi\varepsilon_0}[\nabla f\times(\hat{\mathbf{n}}\times\mathbf{p})+f\nabla\times(\hat{\mathbf{n}}\times\mathbf{p})]$$

where

$$f=\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)$$

and

$$\nabla f=\left(\frac{ike^{ikr}}{r}-\frac{2e^{ikr}}{r^2}+\frac{2e^{ikr}}{ikr^3}\right)\hat{\mathbf{n}}$$

The triple cross-product is

$$\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{p})=\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p})-\mathbf{p}$$

and

$$\nabla\times(\hat{\mathbf{n}}\times\mathbf{p})=\hat{\mathbf{n}}(\nabla\cdot\mathbf{p})+(\mathbf{p}\cdot\nabla)\hat{\mathbf{n}}-\mathbf{p}(\nabla\cdot\hat{\mathbf{n}})-(\hat{\mathbf{n}}\cdot\nabla)\mathbf{p}$$
$$=(\mathbf{p}\cdot\nabla)\hat{\mathbf{n}}-\mathbf{p}(\nabla\cdot\hat{\mathbf{n}})$$

which further simplifies according to

$$\nabla\cdot\hat{\mathbf{n}}=\frac{2}{r}$$

$$(\mathbf{p}\cdot\nabla)\hat{\mathbf{n}} = \frac{\mathbf{p}-\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p})}{r}$$

so that

$$\nabla\times(\hat{\mathbf{n}}\times\mathbf{p}) = \frac{-\mathbf{p}-\hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}})}{r}$$

Putting this together, we get

$$\mathbf{E} = \frac{ik}{4\pi\varepsilon_0}\left[\frac{ike^{ikr}}{r}\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{p})+\left(-\frac{2e^{ikr}}{r^2}+\frac{2e^{ikr}}{ikr^3}\right)\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p})-\mathbf{p}\right]$$

$$+\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\left(\frac{-\mathbf{p}-(\mathbf{p}\cdot\hat{\mathbf{n}})}{r}\right)$$

or

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left\{k^2(\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}}\frac{e^{ikr}}{r}+[3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p})-\mathbf{p}]\left(\frac{e^{ikr}}{r^3}-\frac{ike^{ikr}}{r^2}\right)\right\}$$

2. Jackson 9.5

(a) Show that for harmonic time variation at frequency ω the electric dipole scalar potential is

$$\Phi = \frac{e^{ikr}}{4\pi\varepsilon_0 r^2}\hat{\mathbf{n}}\cdot\mathbf{p}(1-ikr)$$

where $k=\omega/c$, $\hat{\mathbf{n}}$ is a unit vector in the radial direction, $\mathbf{p}$ is the dipole moment, and the time-dependence $\exp(-i\omega t)$ is understood. (The vector potential $\mathbf{A}$ is given by Jackson 9.16, you do not need to derive this.)

$$\mathbf{A}(\mathbf{r}) = -i\omega\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\mathbf{p}$$

The Lorentz condition is

$$\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}=0$$

so

$$\frac{\partial\Phi}{\partial t} = -i\omega\Phi = -c^2\nabla\cdot\mathbf{A}$$

this gives

$$\Phi = \frac{c^2}{i\omega}\left(-i\omega\frac{\mu_0}{4\pi}\right)\nabla\cdot\left(\frac{e^{ikr}}{r}\mathbf{p}\right) = -\frac{\mu_0 c^2}{4\pi}\mathbf{p}\cdot\nabla\left(\frac{e^{ikr}}{r}\right)$$

$$= -\frac{\mu_0 c^2}{4\pi}\mathbf{p}\cdot\hat{\mathbf{n}}\left(\frac{ike^{ikr}}{r}-\frac{e^{ikr}}{r^2}\right) = \frac{e^{ikr}}{4\pi\varepsilon_0 r^2}\hat{\mathbf{n}}\cdot\mathbf{p}(1-ikr)$$

(b) Calculate the electric field *from the potentials* and show that it is given by Jackson 9.18,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left\{k^2(\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}}\frac{e^{ikr}}{r}+[3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p})-\mathbf{p}]\left(\frac{e^{ikr}}{r^3}-\frac{ike^{ikr}}{r^2}\right)\right\}$$

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\nabla\left[\frac{e^{ikr}}{r^2}\hat{\mathbf{n}}\cdot\mathbf{p}(1-ikr)\right] + \frac{\mu_0\omega^2}{4\pi}\frac{e^{ikr}}{r}\mathbf{p}$$

$$= \frac{ik}{4\pi\varepsilon_0}\nabla\left[\hat{\mathbf{n}}\cdot\mathbf{p}\left(\frac{e^{ikr}}{r} - \frac{e^{ikr}}{ikr^2}\right)\right] + \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\mathbf{p}$$

So we have

$$\mathbf{E} = \frac{ik}{4\pi\varepsilon_0}\nabla[f\hat{\mathbf{n}}\cdot\mathbf{p}] + \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\mathbf{p}$$

where

$$f = \frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)$$

The first term has a part

$$\nabla[f\hat{\mathbf{n}}\cdot\mathbf{p}] = (\mathbf{p}\cdot\nabla)f\hat{\mathbf{n}} = \frac{f}{r}[\mathbf{p} - \hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}})] + \hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}})\frac{df}{dr}$$

Putting this together, we have

$$\mathbf{E} = \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\mathbf{p} + \frac{ik}{4\pi\varepsilon_0}\left[\hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}})\left(\frac{df}{dr} - \frac{f}{r}\right) + \frac{f}{r}\mathbf{p}\right]$$

$$= \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}[\mathbf{p} - \hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}})] + \frac{1}{4\pi\varepsilon_0}[3\hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}}) - \mathbf{p}]\left(\frac{e^{ikr}}{r^3} - \frac{ike^{ikr}}{r^2}\right)$$

$$= \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}(\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}} + \frac{1}{4\pi\varepsilon_0}[3\hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}}) - \mathbf{p}]\left(\frac{e^{ikr}}{r^3} - \frac{ike^{ikr}}{r^2}\right)$$

3. Jackson 9.3

Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulating gap. An alternating potential is applied between the two halves of the sphere so that the potentials are

$$\Phi = \pm V\cos(\omega t) \ .$$

In the long-wavelength limit, find the radiation fields, the angular distribution of radiated power, and the total radiated power from the sphere.

For the static case, the potential is given by Jackson 2.27. The leading term is

$$\Phi = \frac{3VR^2}{2r^2}\cos\theta$$

Since the potential is given by (previous problem)

$$\Phi = \frac{e^{ikr}}{4\pi\varepsilon_0 r^2}\hat{\mathbf{n}}\cdot\mathbf{p}(1-ikr) \approx \frac{e^{ikr}}{4\pi\varepsilon_0 r^2}\hat{\mathbf{n}}\cdot\mathbf{p}$$

we see that

$$\frac{3VR^2}{2r^2}\cos\theta = \frac{|\mathbf{p}|\cos\theta}{4\pi\varepsilon_0 r^2}$$

or

$$|\mathbf{p}| = 6\pi\varepsilon_0 VR^2$$

The direction is the z direction:

$$\mathbf{p} = 6\pi\varepsilon_0 VR^2\hat{\mathbf{k}}$$

The radiation fields are

$$\mathbf{H} = \frac{ck^2}{4\pi}\hat{\mathbf{n}}\times\mathbf{p}\frac{e^{ikr}}{r} = \frac{ck^2}{4\pi}6\pi\varepsilon_0 VR^2\frac{e^{ikr}}{r}\hat{\mathbf{n}}\times\hat{\mathbf{k}}$$

$$= \frac{3ck^2\varepsilon_0 VR^2}{2}\frac{e^{ikr}}{r}\sin\theta(-\hat{\phi})$$

and

$$\mathbf{E} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\mathbf{H}\times\hat{\mathbf{n}} = \frac{3k^2VR^2}{2}\frac{e^{ikr}}{r}\sin\theta\left(-\hat{\theta}\right)$$

The angular distribution of radiated power is

$$\frac{dP}{d\Omega} = \langle\mathbf{S}\rangle\cdot\hat{\mathbf{n}}\,r^2 = \frac{\omega^4\mu_0}{32\pi^2 c}|\mathbf{p}|^2\sin^2\theta$$

$$= \frac{\omega^4\mu_0}{32\pi^2 c}\left(6\pi\varepsilon_0 VR^2\right)^2\sin^2\theta = \frac{9\omega^4V^2R^4\mu_0\varepsilon_0{}^2}{8c}\sin^2\theta$$

$$= \frac{9c\varepsilon_0V^2k^4R^4}{8}\sin^2\theta$$

The total radiated power is

$$P = \frac{\omega^4\mu_0\left(6\pi\varepsilon_0 VR^2\right)^2}{12\pi^2 c} = 3\pi c\varepsilon_0 V^2k^4R^4$$

PY522 Fall 2000
Homework No. 7

1. It is impossible for a spherically symmetric distribution of charge oscillating radially to radiate. Prove this by the following method. Take the current to be

$$\mathbf{J}(\mathbf{r}', t_{\mathrm{R}}) = \mathbf{r}' f(r') e^{-i\omega t_{\mathrm{R}}}$$

where f is a function only of r' and t_{R} is the retarded time. Choose an observation point along the z-axis to calculate **A** and the corresponding electromagnetic fields.

The radiation part will be the $1/r$ part corresponding to

$$t_{\mathrm{R}} \approx t - \frac{r}{c} + \frac{\mathbf{r}' \cdot \hat{\mathbf{n}}}{c}$$

and

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{1}{r} \int d^3r' \, \mathbf{J}\left(\mathbf{r}', t - \frac{r}{c} + \frac{\mathbf{r}' \cdot \hat{\mathbf{n}}}{c}\right) + Order\left(\frac{1}{r^2}\right)$$

so

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{1}{r} \int d^3r' \, \mathbf{r}' f(r') e^{-i\omega t} e^{ikr} e^{-ikr'\cos\theta}$$

where as usual, θ is the angle between **r** and **r'**.

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} e^{-i\omega t} \int d^3r' \, \mathbf{r}' f(r') e^{-ikr'\cos\theta}$$

or

$$A_x = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} e^{-i\omega t} \int_0^\infty dr' \, r'^2 \int_{-1}^{1} d\cos\theta' \int_0^{2\pi} d\phi' \sin\theta' \cos\phi' \, r' f(r') \, e^{-ikr'\cos\theta}$$

$$A_y = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} e^{-i\omega t} \int_0^\infty dr' \, r'^2 \int_{-1}^{1} d\cos\theta' \int_0^{2\pi} d\phi' \sin\theta' \sin\phi' \, r' f(r') \, e^{-ikr'\cos\theta}$$

$$A_z = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} e^{-i\omega t} \int_0^\infty dr' \, r'^2 \int_{-1}^{1} d\cos\theta' \int_0^{2\pi} d\phi' \cos\theta' \, r' f(r') \, e^{-ikr'\cos\theta}$$

Now we choose the observation point to be along the z axis, *i.e.*

$$\hat{\mathbf{n}} = \hat{\mathbf{z}}$$

$$\cos\theta' = \cos\theta$$

$$A_x = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}e^{-i\omega t}\int_0^\infty dr'\, r'^3 f(r')\int_0^{2\pi} d\phi' \cos\phi' \int_{-1}^{1} d\cos\theta' \sin\theta'\, e^{-ikr'\cos\theta'} = 0$$

$$A_y = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}e^{-i\omega t}\int_0^\infty dr'\, r'^3 f(r')\int_0^{2\pi} d\phi' \sin\phi' \int_{-1}^{1} d\cos\theta' \sin\theta'\, e^{-ikr'\cos\theta'} = 0$$

so **A** is only in the z direction (radial). The fields are

$$\mathbf{H} = \frac{1}{\mu_0}\nabla\times\mathbf{A} \sim \mathbf{A}\times\hat{\mathbf{n}} = 0$$

$$\mathbf{E} = \frac{i}{\omega\varepsilon_0}\nabla\times\mathbf{H} \sim (\mathbf{A}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}} = 0$$

2. A charge q has uniform motion in a circle of radius a plane with an angular frequency ω. Using the electric dipole approximation,
(a) Find the fields **E** and **H** in the radiation zone, and

$$\mathbf{p} = q\mathbf{r} = qa(\cos\omega t\,\hat{\mathbf{x}} + \sin\omega t\,\hat{\mathbf{y}})$$

which we may write as

$$\mathbf{p} = qa\,\mathrm{Re}\left[(\hat{\mathbf{x}} + i\,\hat{\mathbf{y}})e^{-i\omega t}\right]$$

The two components do not have the same phase. This will effect the angular distribution.

$$\mathbf{H} = \frac{ck^2}{4\pi}\hat{\mathbf{n}}\times\mathbf{p}\frac{e^{ikr}}{r}$$

$$\mathbf{E} = \frac{k^2}{4\pi\varepsilon_0}(\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}}\frac{e^{ikr}}{r}$$

(b) Find the angular distribution of the radiated power.

$$\frac{dP}{d\Omega} = \langle\mathbf{S}\rangle\cdot\hat{\mathbf{n}}\,r^2 = \frac{\omega^4\mu_0}{32\pi^2 c}|\hat{\mathbf{n}}\times\mathbf{p}|^2 = \frac{\omega^4\mu_0}{32\pi^2 c}\left[|\mathbf{p}|^2 - (\hat{\mathbf{n}}\cdot\mathbf{p})(\hat{\mathbf{n}}\cdot\mathbf{p})^*\right]$$

we have

$$\hat{\mathbf{n}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

so

$$\hat{\mathbf{n}}\cdot\mathbf{p} = qa\sin\theta(\cos\phi + i\sin\phi) = qa\sin\theta e^{i\phi}$$

and

$$\frac{dP}{d\Omega} = \frac{\omega^4\mu_0}{32\pi^2 c}\left[2q^2a^2 - q^2a^2\sin^2\theta\right] = \frac{\omega^4\mu_0 q^2a^2}{32\pi^2 c}\left(2 - \sin^2\theta\right)$$

(c) Is there magnetic dipole radiation? Why or why not?

The current density is

$$\mathbf{J}(\mathbf{r}') = \rho \mathbf{v}$$

The velocity is

$$\mathbf{v} = \frac{2\pi a}{T}\hat{\phi} = \omega a \hat{\phi}$$

(where T is the period). The charge density is

$$\rho = \frac{q}{a}\delta(z)\delta(\rho - a)\delta(\phi - \omega t)$$

because

$$\int d\phi \int dz \int dr\, r\, \rho = q$$

Therefore

$$\mathbf{J}(\mathbf{r}') = q\omega\delta(z)\delta(\rho - a)\delta(\phi - \omega t)\hat{\phi}$$

The dipole moment is

$$\mathbf{m} = \frac{1}{2}\int d^3r' \mathbf{r} \times \mathbf{J}(\mathbf{r}') = \frac{q\omega a^2}{2}\hat{\mathbf{z}}$$

The magnetic moment does not depend on time (because there is no phi-dependence to the current density. Therefore, there is no magnetic dipole radiation.

(d) Is there electric quadrupole radiation? Why or why not?

The quadrupole moment is

$$Q_{ij} = \int d^3r' \rho \left(3x_i' x_j' - \delta_{ij} r'^2\right)$$

$$= \int d^3r' \left(3x_i' x_j' - \delta_{ij} r'^2\right) \frac{q}{a}\delta(z)\delta(\rho - a)\delta(\phi - \omega t)$$

$$= q\left(3x_i x_j - \delta_{ij} a^2\right)$$

where

$$x_1 = a\cos\omega t$$

$$x_2 = a \sin \omega t$$

$$x_3 = 0$$

$$Q_{ij} = qa^2 \begin{bmatrix} 3\cos^2 \omega t - 1 & 3\cos \omega t \sin \omega t & 0 \\ 3\cos \omega t \sin \omega t & 3\sin^2 \omega t - 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

This depends on time so there is electric quadrupole radiation.

3. Two identical charges separated by a distance d are rotating in the x-y plane with an angular frequency ω.

(a) Find the electric dipole radiation fields **E** and **H**.

$$\mathbf{p}=\sum q\mathbf{r}=qa(\cos\omega t\ \hat{\mathbf{x}}+\sin\omega t\ \hat{\mathbf{y}})+qa(-\cos\omega t\ \hat{\mathbf{x}}-\sin\omega t\ \hat{\mathbf{y}})=0$$

(b) Find the angular distribution of the radiated power.

$$\frac{dP}{d\Omega}=0$$

(c) Calculate the magnetic dipole moment and the corresponding angular distribution of radiated power.

$$\rho=\frac{q}{a}\delta(z)\delta(\rho-a)[\delta(\phi-\omega t)+\delta(\phi-\omega t+\pi)]$$

$$\mathbf{J}(\mathbf{r}')=q\omega\delta(z)\delta(\rho-a)[\delta(\phi-\omega t)+\delta(\phi-\omega t+\pi)]\hat{\phi}$$

The dipole moment is

$$\mathbf{m}=\frac{1}{2}\int d^3r'\mathbf{r}\times\mathbf{J}(\mathbf{r}')=q\omega a^2\hat{\mathbf{z}}$$

The magnetic dipole is just twice as big as in the previous problem. There is no time dependence and no magnetic-dipole radiation.

(d) Calculate the electric quadrupole moment and the corresponding angular distribution of radiated power.

The quadrupole moment is

$$Q_{ij}=\int d^3r'\rho\left(3x_i'x_j'-\delta_{ij}r'^2\right)$$

$$=\int d^3r'\left(3x_i'x_j'-\delta_{ij}r'^2\right)\frac{q}{a}\delta(z)\delta(\rho-a)[\delta(\phi-\omega t)+\delta(\phi-\omega t+\pi)]$$

$$=\sum_{n=1}^{2}q\left(3x_i^{(n)}x_j^{(n)}-\delta_{ij}a^2\right)$$

where

$$x_1^{(1)}=a\cos\omega t$$

$$x_2^{(1)} = a \sin \omega t$$
$$x_3^{(1)} = 0$$
$$x_1^{(2)} = -a \cos \omega t$$
$$x_2^{(2)} = -a \sin \omega t$$
$$x_3^{(2)} = 0$$

$$Q_{ij} = qa^2 \begin{bmatrix} 6\cos^2 \omega t - 2 & 6\cos \omega t \sin \omega t & 0 \\ 6\cos \omega t \sin \omega t & 6\sin^2 \omega t - 2 & 0 \\ 0 & 0 & -2` \end{bmatrix}$$

Write as cosine and sine of double angles.

$$6\cos^2 \omega t - 2 = 4\cos^2 \omega t - 2\sin^2 \omega t = 1 + 3\cos 2\omega t$$

and

$$6\cos \omega t \sin \omega t = 3\sin 2\omega t$$

to get

$$Q_{ij} = qa^2 \begin{bmatrix} 1+3\cos 2\omega t & 3\sin 2\omega t & 0 \\ 3\sin 2\omega t & 1+3\cos 2\omega t & 0 \\ 0 & 0 & -2` \end{bmatrix}$$

Note the time-independent part does not effect radiation. So the relevant part of $\mathbf{Q}$ is

$$Q_{ij} = qa^2 e^{-2i\omega t} \begin{bmatrix} 3 & 3i & 0 \\ 3i & 3 & 0 \\ 0 & 0 & 0` \end{bmatrix}$$

The angular distribution is

$$\frac{dP}{d\Omega} = \frac{\mu_0 k^6 c^3}{1152\pi^2} \left| (\hat{\mathbf{n}} \times \mathbf{Q}) \times \hat{\mathbf{n}} \right|^2$$

where the vector Q is defined as

$$Q_i = Q_{ij} n_j = \begin{pmatrix} 3n_x + 3in_y \\ 3in_x + 3n_y \\ 0 \end{pmatrix}$$

Note: n_x, etc. are components (projections) of the unit vector.
We have

$$\left|(\hat{\mathbf{n}} \times \mathbf{Q}) \times \hat{\mathbf{n}}\right|^2 = \mathbf{Q}\mathbf{Q}^* - (\hat{\mathbf{n}} \cdot \mathbf{Q})(\hat{\mathbf{n}} \cdot \mathbf{Q})^*$$
$$= 9n_x^{\ 2} + 9n_y^{\ 2} + 9n_x^{\ 2} + 9n_y^{\ 2} - \left[\left(3n_x^{\ 2} + 3n_y^{\ 2}\right)^2\right]$$
$$= 9\left(1 - n_z^{\ 2}\right)\left[2 - \left(1 - n_z^{\ 2}\right)\right] = 9\sin^2\theta\left[2 - \left(1 - \cos^2\theta\right)\right]$$
$$= 9\sin^2\theta\left(1 + \cos^2\theta\right)$$

Finally,

$$\frac{dP}{d\Omega} = \frac{\mu_0 k^6 c^3}{128\pi^2} \sin^2\theta\left(1 + \cos^2\theta\right)$$

PY522 Fall 2001
Homework No. 8

1. Show that you may write the vector potential in the radiation zone as

$$\mathbf{A} = \frac{\mu_0}{4\pi r}[\dot{\mathbf{p}}]_{t_R} + \frac{\mu_0}{4\pi c r}[\dot{\mathbf{m}} \times \hat{\mathbf{n}}]_{t_R} + \frac{\mu_0}{24\pi c r}[\ddot{\mathbf{Q}}]_{t_R}$$

The dipole approximation is

$$t_R \approx t - \frac{r}{c}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\, \mathbf{J}\left(\mathbf{r}', t - \frac{r}{c}\right)$$

Integrate by parts (as done in class)

$$\mathbf{A}(\mathbf{r},t) = -\frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\, \mathbf{r}'\nabla \cdot \mathbf{J}\left(\mathbf{r}', t - \frac{r}{c}\right) = \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\, \mathbf{r}'\frac{\partial}{\partial t_R}\rho\left(\mathbf{r}', t - \frac{r}{c}\right)$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r}\frac{\partial}{\partial t_R}\int d^3r'\, \mathbf{r}'\rho\left(\mathbf{r}', t - \frac{r}{c}\right) = \frac{\mu_0}{4\pi}\frac{1}{r}\left[\frac{\partial \mathbf{p}}{\partial t_R}\right]_{t_R} = \frac{\mu_0}{4\pi}\frac{1}{r}\left[\frac{\partial \mathbf{p}}{\partial t}\right]_{t_R}$$

where the last step is valid because

$$dt_R \approx dt$$

The next part of the potential is given by looking at

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\, \mathbf{J}\left(\mathbf{r}', t - \frac{r}{c} + \frac{\mathbf{r}' \cdot \hat{\mathbf{n}}}{c}\right)$$

Now we break this into components

$$\mathbf{A}_\omega = \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\, \mathbf{J}(\mathbf{r}')e^{-i\omega(t - r/c)}e^{i\omega \mathbf{r}' \cdot \hat{\mathbf{n}}/c}$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\, \mathbf{J}\left(\mathbf{r}', t - \frac{r}{c}\right)e^{i\omega \mathbf{r}' \cdot \hat{\mathbf{n}}/c}$$

and expand the exponential

$$\mathbf{A}_\omega \approx \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\,\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\left(1+\frac{i\omega\mathbf{r}'\cdot\hat{\mathbf{n}}}{c}\right)$$

$$=\frac{\mu_0}{4\pi}\frac{1}{r}\left\{\int d^3r'\,\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)+\frac{\partial}{\partial t}\int d^3r'\,\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\frac{\mathbf{r}'\cdot\hat{\mathbf{n}}}{c}\right\}$$

The first term is the electric dipole as discussed above and the second term will give the magnetic dipole and electric quadrupole parts.

Now we integrate by parts (exactly like we did in class) using

$$\mathbf{J}(\mathbf{r}'\cdot\hat{\mathbf{n}})=\frac{1}{2}[\mathbf{J}(\mathbf{r}'\cdot\hat{\mathbf{n}})+\mathbf{r}'(\hat{\mathbf{n}}\cdot\mathbf{J})+(\mathbf{r}'\times\mathbf{J})\times\hat{\mathbf{n}}]$$

and

$$\hat{\mathbf{n}}\cdot\mathbf{J}x_i'+(\mathbf{r}'\cdot\hat{\mathbf{n}})J_i=\nabla'\cdot[(\mathbf{r}'\cdot\hat{\mathbf{n}})x_i'\mathbf{J}]-(\mathbf{r}'\cdot\hat{\mathbf{n}})x_i'\nabla'\cdot\mathbf{J}$$

to get for the second term

$$\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi cr}\frac{\partial}{\partial t}\left\{\mathbf{m}\times\hat{\mathbf{n}}+\frac{1}{2}\int d^3r'\,\mathbf{r}'(\mathbf{r}'\cdot\hat{\mathbf{n}})\frac{\partial\rho}{\partial t_R}\right\}_{t_R}$$

$$\approx\frac{\mu_0}{4\pi cr}[\dot{\mathbf{m}}\times\hat{\mathbf{n}}]_{t_R}+\left\{\frac{1}{6}\frac{\partial^2}{\partial t^2}\int d^3r'\,3\mathbf{r}'(\mathbf{r}'\cdot\hat{\mathbf{n}})\rho\right\}_{t_R}$$

$$=\frac{\mu_0}{4\pi cr}[\dot{\mathbf{m}}\times\hat{\mathbf{n}}]_{t_R}+\frac{\mu_0}{24\pi cr}[\ddot{\mathbf{Q}}]_{t_R}$$

where again

$$t_R\approx t-\frac{r}{c},\quad \text{and}\quad dt_R\approx dt$$

2. A quadrupole oscillator consists of charges $-q$, $+2q$, and $-q$ in a straight line along the z direction. The positive charges are stationary at the origin while the negative charges have oscillating positions given by

$$z_1 = a\cos\frac{\omega t}{2}, \quad z_2 = -a\cos\frac{\omega t}{2}$$

a) Write down explicit expressions for the charge and current densities and show that the electric and magnetic dipole moments are both zero.

$$\rho = 2q\delta(x)\delta(y)\delta(z) - q\delta(x)\delta(y)\delta(z - z_1) - q\delta(x)\delta(y)\delta(z - z_2)$$

so

$$\mathbf{p} = \int d^3r'\mathbf{r}'\rho(\mathbf{r}') = 0 - z_1\hat{\mathbf{z}} - z_2\hat{\mathbf{z}} = 0$$

$$\mathbf{J} = \{-q\delta(x)\delta(y)\delta(z - z_1)\dot{z}_1 - q\delta(x)\delta(y)\delta(z - z_2)\dot{z}_2\}\hat{\mathbf{z}}$$

so

$$\mathbf{m} = \frac{1}{2}\int d^3r'\mathbf{r}' \times \mathbf{J}(\mathbf{r}') = 0$$

because **r'** and **J** are both in the z direction.

b) Determine the components of the vector **Q**:

$$Q_i = n_j Q_{ij}$$

$$Q_{11} = \int d^3r'\rho(3x'^2 - r'^2) = \int d^3r'\rho(2x'^2 - y'^2 - z'^2) = -\int d^3r'\rho z'^2$$

where the last step is valid because of the delta functions in x and y. We have

$$Q_{11} = -q(-z_1^{\ 2} - z_2^{\ 2}) = 2qa^2\cos^2\frac{\omega t}{2} = qa^2(1 + \cos\omega t)$$

$$Q_{22} = Q_{11} = qa^2(1 + \cos\omega t)$$

$$Q_{33} = \int d^3r'\rho(3z'^2 - r'^2) = 2\int d^3r'\rho z'^2 = -2qa^2(1 + \cos\omega t)$$

$$Q_{12} = Q_{13} = Q_{23} = 0$$

so

$$Q_{ij} = qa^2 \begin{pmatrix} 1+\cos\omega t & 0 & 0 \\ 0 & 1+\cos\omega t & 0 \\ 0 & 0 & -2-2\cos\omega t \end{pmatrix}$$

The vector **Q** is

$$\mathbf{Q} = qa^2 \begin{Bmatrix} (1+\cos\omega t)n_1 \\ (1+\cos\omega t)n_2 \\ -2(1+\cos\omega t)n_3 \end{Bmatrix}$$

c) Determine the fields **E** and **H**.

$$\mathbf{H} = \frac{1}{\mu_0 c}\dot{\mathbf{A}} \times \hat{\mathbf{n}} = \frac{1}{\mu_0 c}\frac{\mu_0}{24\pi c r}\left[\dddot{\mathbf{Q}} \times \hat{\mathbf{n}}\right]_{t_R} = \frac{1}{24\pi c^2 r}\left[\dddot{\mathbf{Q}} \times \hat{\mathbf{n}}\right]_{t_R}$$

$$\dddot{\mathbf{Q}} = qa^2\omega^3 \begin{Bmatrix} n_1 \sin\omega t \\ n_2 \sin\omega t \\ -2n_3 \sin\omega t \end{Bmatrix}$$

$$H_1 = \frac{1}{24\pi c^2 r}\left[\dddot{Q}_2 n_3 - \dddot{Q}_3 n_2\right]_{t_R} = \frac{qa^2\omega^3 \sin\omega t_R}{24\pi c^2 r}\left[n_2 n_3 + 2n_3 n_2\right]$$

$$= \frac{qa^2\omega^3 \sin\omega t_R}{8\pi c^2 r} n_2 n_3$$

$$H_2 = \frac{1}{24\pi c^2 r}\left[\dddot{Q}_3 n_1 - \dddot{Q}_1 n_3\right]_{t_R} = \frac{qa^2\omega^3 \sin\omega t_R}{24\pi c^2 r}\left[-2n_3 n_1 - n_1 n_3\right]$$

$$= -\frac{qa^2\omega^3 \sin\omega t_R}{8\pi c^2 r} n_1 n_3$$

$$H_3 = \frac{1}{24\pi c^2 r}\left[\dddot{Q}_1 n_2 - \dddot{Q}_2 n_1\right]_{t_R} = \frac{qa^2\omega^3 \sin\omega t_R}{24\pi c^2 r}\left[n_1 n_2 - n_2 n_1\right] = 0$$

and

$$\mathbf{E} = \mu_0 c\,\mathbf{H}\times\hat{\mathbf{n}}$$

d) Determine the time-averaged angular distribution of radiated power.

The angular distribution (not yet time averaged) is

$$\frac{dP}{d\Omega} = \left(\mathbf{E}\times\mathbf{H}^*\right)\cdot\hat{\mathbf{n}}\,r^2 = \mu_0 c\left|\mathbf{H}\right|^2 r^2 = \mu_0 c r^2\left(H_1^{\,2} + H_2^{\,2}\right)$$

$$= \mu_0 c\left(\frac{qa^2\omega^3 \sin\omega t_R}{8\pi c^2}\right)^2\left(n_2^{\,2} n_3^{\,2} + n_1^{\,2} n_3^{\,2}\right)$$

$$= \frac{\mu_0 q^2 a^4 \omega^6 \sin^2\omega t_R}{64\pi^2 c^3}\sin^2\theta\cos^2\theta$$

Now time average:

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^4 \omega^6}{128\pi^2 c^3}\sin^2\theta\cos^2\theta$$

This is a typical quadrupole angular distribution.

e) Determine the total power radiated.

We can integrate

$$\int d\Omega\sin^2\theta\cos^2\theta = \frac{8\pi}{15}$$

or using Jackson 9.49

$$P = \frac{\mu_0 k^6 c^3}{1440\pi} Q_{ij} Q_{ij} = \frac{\mu_0}{1440\pi}\frac{2}{c^3}\left\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\right\rangle$$

where the bracket denotes time average. Note that the Jackson formula already has the time average and that the triple time derivative gives a factor of k^3c^3.

$$P = \frac{\mu_0}{1440\pi}\frac{2}{c^3}q^2a^4\left\langle(1+1+4)\sin^2\omega t_R\right\rangle = \frac{\mu_0 k^6 c^3 q^2 a^4}{240\pi}$$

3. For the ideal antenna discussed in class,

$$\mathbf{J} = I_0\delta(x)\delta(y)\theta\left(\frac{\ell}{2}-z\right)\theta\left(\frac{\ell}{2}+z\right)\sin\omega t\,\hat{\mathbf{z}}$$

consider the limit

$$\frac{\pi\ell}{\lambda}\to\infty$$

a) Calculate the total power radiated.

$$P = \frac{\mu_0 c I_0^{\,2}}{8\pi^2}2\pi\int_0^{\pi} d\theta\sin\theta\frac{\sin^2\theta}{\cos^2\theta}\sin^2\left(\frac{\pi\ell\cos\theta}{\lambda}\right)$$

make change of variables:

$$\theta' = \frac{\pi}{2}-\theta$$
$$\sin\theta' = \cos\theta$$
$$\cos\theta' = \sin\theta$$

$$P = \frac{\mu_0 c I_0^{\,2}}{4\pi}\int_{-\pi/2}^{\pi/2} d\theta'\frac{\cos^3\theta'}{\sin^2\theta'}\sin^2\left(\frac{\pi\ell\sin\theta'}{\lambda}\right)$$

Now let

$$z \equiv \frac{\pi\ell\sin\theta'}{\lambda}$$
$$dz = \frac{\pi\ell\cos\theta'}{\lambda}d\theta'$$
$$\cos^2\theta' = 1-\frac{z^2\lambda^2}{\pi^2\ell^2}$$

to get

$$P = \frac{\mu_0 c I_0{}^2}{4\pi} \int_0^{\pi\ell/\lambda} dz \left(\frac{\lambda}{\pi\ell}\right) \frac{1 - \frac{z^2\lambda^2}{\pi^2\ell^2}}{\frac{z^2\lambda^2}{\pi^2\ell^2}} \sin^2 z$$

$$= \frac{\mu_0 c I_0{}^2}{4\pi} \frac{\pi\ell}{\lambda} \int_0^{\pi\ell/\lambda} dz \frac{\sin^2 z}{z^2} \left(1 - \frac{z^2\lambda^2}{\pi^2\ell^2}\right)$$

For the case

$$\frac{\pi\ell}{\lambda} >> 1$$

we have

$$\mathrm{P} \approx \frac{\mu_0 c I_0{}^2}{4} \frac{\ell}{\lambda} \int_0^{\infty} dz \frac{\sin^2 z}{z^2} = \frac{\mu_0 c I_0{}^2 \ell}{4\lambda} \int_0^{\infty} dz \frac{2 \sin z \cos z}{z}$$

$$= \frac{\mu_0 c I_0{}^2 \ell}{4\lambda} \int_0^{\infty} dz \frac{\sin 2z}{z} = \frac{\mu_0 c I_0{}^2 \ell}{4\lambda} \frac{\pi}{2}$$

Notice that the intensity goes as ℓ but the angular distribution at $\pi/2$ goes as ℓ^2. Therefore, as ℓ increases, a larger and larger fraction of the radiated power is concentrated at $\pi/2$. The antenna becomes highly directional.

b) Show that the fraction of radiation emitted in the main lobe is

$$\frac{P_{main}}{P_{total}} = 1 - \frac{1}{\pi^2} + \frac{1}{2\pi^4} + \ldots \approx 0.9$$

Hint: You will come across an integral

$$\int_0^{2\pi} dx \frac{\sin x}{x} = \int_0^{\infty} dx \frac{\sin x}{x} - \int_{2\pi}^{\infty} dx \frac{\sin x}{x}$$

Expand the last integral in powers of $1/\pi$ by integrating by parts.

$$P_{main} = \frac{\mu_0 c I_0^{\ 2}}{4\pi} \int_0^{\lambda/\ell} d\theta' \frac{\cos^3\theta'}{\sin^2\theta'} \sin^2\left(\frac{\pi\ell\sin\theta'}{\lambda}\right)$$

$$\approx \frac{\mu_0 c I_0^{\ 2}}{4\pi} \int_0^{\lambda/\ell} d\theta' \frac{1}{\theta'^2} \sin^2\left(\frac{\pi\ell\theta'}{\lambda}\right)$$

$$= \frac{\mu_0 c I_0^{\ 2}}{4\pi} \frac{\pi\ell}{\lambda} \int_0^{\pi} dz \frac{\sin^2 z}{z^2}$$

$$\frac{P_{main}}{P_{total}} = \frac{\dfrac{\mu_0 c I_0^{\ 2}\ell}{4\lambda}}{\dfrac{\mu_0 c I_0^{\ 2}\ell\pi}{8\lambda}} \int_0^{\pi} dz \frac{\sin^2 z}{z^2} = \frac{2}{\pi}\int_0^{\pi} dz \frac{\sin^2 z}{z^2} = \frac{2}{\pi}\int_0^{\pi} dz \frac{\sin 2z}{z}$$

$$= \frac{2}{\pi}\int_0^{2\pi} dx \frac{\sin x}{x}$$

$$\frac{P_{main}}{P_{total}} = \frac{2}{\pi}\int_0^{\infty} dx \frac{\sin x}{x} - \frac{2}{\pi}\int_{2\pi}^{\infty} dx \frac{\sin x}{x} = 1 - \frac{2}{\pi}\int_{2\pi}^{\infty} dx \frac{\sin x}{x}$$

$$\frac{P_{main}}{P_{total}} = 1 + \frac{2}{\pi}\int_{2\pi}^{\infty} dx \frac{1}{x}\frac{d}{dx}\cos x$$

Now integrate by parts.

$$\frac{P_{main}}{P_{total}} = 1 + \frac{2}{\pi}\left[\frac{\cos x}{x}\right]_{2\pi}^{\infty} + \frac{2}{\pi}\int_{2\pi}^{\infty} dx \frac{1}{x^2}\cos x$$

$$= 1 - \frac{1}{\pi^2} + \frac{2}{\pi}\int_{2\pi}^{\infty} dx \frac{\cos x}{x^2}$$

by parts again:

$$\frac{P_{main}}{P_{total}} = 1 - \frac{1}{\pi^2} + \frac{2}{\pi}\left[\frac{\sin x}{x^2}\right]_{2\pi}^{\infty} + \frac{4}{\pi}\int_{2\pi}^{\infty} dx \frac{\sin x}{x^3}$$

$$\frac{P_{main}}{P_{total}} = 1 - \frac{1}{\pi^2} + 0 + \frac{4}{\pi}\int_{2\pi}^{\infty} dx \frac{\sin x}{x^3}$$

by parts again:

$$\frac{P_{main}}{P_{total}} = 1 - \frac{1}{\pi^2} - \frac{4}{\pi}\left[\frac{\cos x}{x^3}\right]_{2\pi}^{\infty} + \ldots = 1 - \frac{1}{\pi^2} + \frac{1}{2\pi^4} + \ldots$$

PY522 Fall 2001
Homework No. 9

1. (Jackson 11.3)
Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with a velocity

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

This is an alternate way to derive the parallel-velocity addition law.

Let

$$\beta \equiv \frac{v_1}{c}, \qquad \gamma \equiv \frac{1}{\sqrt{1 - \frac{v_1^2}{c^2}}}$$

$$\beta' \equiv \frac{v_2}{c}, \qquad \gamma' \equiv \frac{1}{\sqrt{1 - \frac{v_2^2}{c^2}}}$$

The Lorentz transformations are

$$\Lambda = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \Lambda' = \begin{pmatrix} \gamma' & \beta'\gamma' & 0 & 0 \\ \beta'\gamma' & \gamma' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so

$$\Lambda\Lambda' = \begin{pmatrix} \gamma\gamma'(1+\beta\beta') & \gamma\gamma'(\beta+\beta') & 0 & 0 \\ \gamma\gamma'(\beta+\beta') & \gamma\gamma'(1+\beta\beta') & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let

$$\Lambda'' = \Lambda\Lambda' = \begin{pmatrix} \gamma'' & \beta''\gamma'' & 0 & 0 \\ \beta''\gamma'' & \gamma'' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so

$$\gamma'' = \gamma\gamma'(1 + \beta\beta')$$

and

$$\beta''\gamma'' = \gamma\gamma'(\beta + \beta')$$

which gives

$$\beta'' = \frac{\beta + \beta'}{1 + \beta\beta'}$$

and

$$\gamma'' = \gamma\gamma'(1 + \beta\beta') = \frac{1 + \beta\beta'}{\sqrt{1 - \beta^2}\sqrt{1 - \beta'^2}} = \frac{1}{\sqrt{1 - \left(\frac{\beta + \beta'}{1 + \beta\beta'}\right)^2}}$$

$$= \frac{1}{\sqrt{1 - \beta''^2}}$$

This is the velocity addition rule.

2. (Jackson 11.4)
A possible clock is shown in the figure. It consists of a flashtube *F* and a photocell *P* shielded so that each views only the mirror *M*, located a distance *d* away, and mounted rigidly with respect to the flashtube-photocell assembly. The electronic innards of the box are such that when the photocell responds to a light flash from the mirror, the flashtube is triggered with a negligible delay and emits a short flash toward the mirror. The clock thus "ticks" once every $(2d/c)$ seconds when at rest.

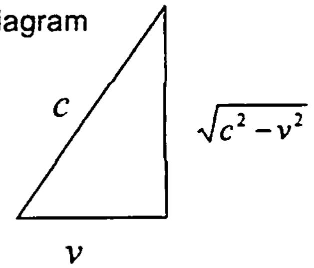
(a) Suppose that the clock moves with a uniform velocity v, perpendicular to the line from *PF* to *M*, relative to an observer. Using the second postulate of relativity, show by explicit geometrical or algebraic construction that the observer sees the relativistic time dilation as the clock moves by.

Let K be the frame in which the clock is at rest. One tick takes

$$\Delta t = \frac{2d}{c}$$

In frame K′, as the clock moves by, the light beam must travel a greater distance (each way) in the time $\Delta t'$. Therefore, the vertical component of velocity is less than c.

Velocity Vector Diagram



$$\Delta t' = \frac{2d'}{\sqrt{c^2 - v^2}} = \frac{2d}{c}\frac{1}{\sqrt{c^2 - v^2}} = \gamma \Delta t$$

The time interval is longer in the frame where the clock is moving.

(b) Suppose that the clock moves with a velocity v parallel to the line from *PF* to *M*. Verify here, too, the clock is observed to tick more slowly, by the same time dilation factor.

Along one path, say *FM*, the light travels a longer path (because the mirror moves away), but along the other path, say *MP*, the light travels a shorter path (because the flasher is approaching).

Let t_1' be the time taken for light to arrive at the mirror.

$$ct_1' = d' + vt_1'$$

This gives

$$t_1' = \frac{d'}{c-v}$$

Let t_2' be the time taken for light to arrive back at the flasher box.

$$ct_2' = d' - vt_1'$$

This gives

$$t_2' = \frac{d'}{c+v}$$

The time interval in K′ is

$$\Delta t' = t_1' + t_2' = \frac{d'}{c+v} = \frac{2cd'}{c^2 - v^2}$$

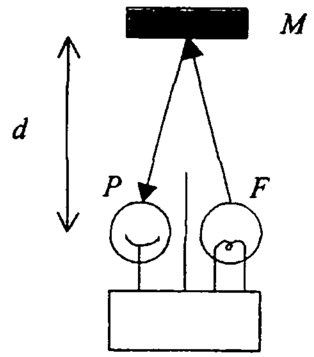
The length d' is contracted,

$$d' = d\sqrt{1-\frac{v^2}{c^2}}$$

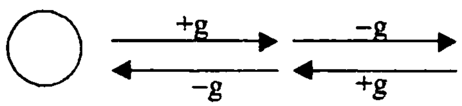
which gives

$$\Delta t' = \frac{2cd\sqrt{1-\frac{v^2}{c^2}}}{c^2\left(1-\frac{v^2}{c^2}\right)} = \frac{\Delta t}{\sqrt{1-\frac{v^2}{c^2}}}$$

Again, the time interval is longer in the frame where the clock is moving.



3. (Jackson 11.6)
Assume that a rocket ship leaves the earth in the year 2100. One of a set of twins born in 2080 remains on earth; the other rides in the rocket. The rocket ship is so constructed that it has an acceleration g in its own rest frame (this makes the occupants feel at home). It accelerates in a straight-line path for 5 years (by its own clocks), decelerates at the same rate for 5 more years, turns around, accelerates for 5 years, decelerates for 5 years, and lands on earth. The twin in the rocket is 40 years old.
(a) What year is it on earth?



Let K be the earth frame. In this frame the rocket has a speed $v(t)$. Let K' be a frame that moves with the rocket. Each segment of the flight has a duration of 5 years in this frame.

The trick to this problem is that we can do a Lorentz transformation at each INSTANT.

Consider the time t when the speed of the rocket is v in frame K. In frame K',

$$v'(t') = 0$$

Consider an *infinitesimal* change in speed,

$$\Delta v' = g\Delta t'$$

In frame K, the new rocket velocity is (by velocity addition)

$$v_{new} = \frac{\Delta v' + v}{1 + \frac{v\Delta v'}{c^2}} \approx (\Delta v' + v)\left(1 - \frac{v\Delta v'}{c^2}\right)$$

$$= \Delta v' + v - \frac{v^2 \Delta v'}{c^2} + order(\Delta v'^2)$$

therefore,

$$v_{new} - v = \Delta v = \left(1 - \frac{v^2}{c^2}\right)\Delta v'$$

or

$$dv = \left(1 - \frac{v^2}{c^2}\right)dv'$$

and

$$\frac{dv}{dt'} = \left(1 - \frac{v^2}{c^2}\right)\frac{dv'}{dt'} = \left(1 - \frac{v^2}{c^2}\right)g$$

This gives

$$\frac{dv}{\left(1 - \frac{v^2}{c^2}\right)} = g\,dt'$$

Integrate to get

$$t' = \frac{1}{g}\int dv \frac{1}{\left(1 - \frac{v^2}{c^2}\right)} = \frac{c}{g}\tanh^{-1}\left(\frac{v}{c}\right)$$

or

$$v = c\tanh\left(\frac{gt'}{c}\right)$$

This is the speed of the spacecraft observed on earth as a function of spacecraft time as it accelerates from rest.

Time measured on earth is related to spacecraft time by

$$dt = \frac{dt'}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{dt'}{\sqrt{1-\tanh^2\left(\frac{gt'}{c}\right)}} = \cosh\left(\frac{gt'}{c}\right)dt'$$

or

$$t = \int dt' \cosh\left(\frac{gt'}{c}\right) = \frac{g}{c}\sinh\left(\frac{gt'}{c}\right)$$

Now plug in, for $t' = 5$ years, we get $t = 84$ years.

For all 4 segments, the amount of time passing on earth is 336 years. The year is 2436.

(b) How far away from the earth did the rocket ship travel?

$$\frac{dx}{dt} = v = c\tanh\left(\frac{gt'}{c}\right) = c\frac{\sinh\left(\frac{gt'}{c}\right)}{\cosh\left(\frac{gt'}{c}\right)}$$

$$= c\frac{\sinh\left(\frac{gt'}{c}\right)}{\sqrt{1+\sinh\left(\frac{gt'}{c}\right)}} = \frac{gt}{\sqrt{1+\frac{g^2t^2}{c^2}}}$$

so

$$x = \int dt \frac{gt}{\sqrt{1+\frac{g^2t^2}{c^2}}} = \frac{c^2}{g}\left(\sqrt{1+\frac{g^2t^2}{c^2}} - 1\right)$$

which for $t = 5$ years gives $x = 7.95 \times 10^{17}$ m.

The rocket travels twice that distance away from the earth, or 1.59×10^{18} m.

PY522 Fall 2001
Homework No. 10

1. Quarks were discovered inside the proton by scattering high energy (E_i) electrons from a hydrogen target and measuring the energy (E_f) and angle (θ) of the scattered electrons. The process is electron-quark elastic scattering,

$$e + q \rightarrow e + q.$$

In this process we may neglect the masses of both the quark and the electron. Let the variable x represent the proton momentum fraction of the quark that scatters, *i.e.*, if p^*_p is the momentum of the proton in the electron-quark center of mass system, then the quark momentum is $p^*_q = xp^*_p$. Derive an expression for x in terms E_i, E_f, θ, and the proton mass (M).

In the center-of-mass system, the electron and quark have momenta of equal magnitudes and opposite directions. Let β and γ correspond to the proton in the center-of-mass frame. The proton momentum (times c) in the center-of-mass frame is given by the Lorentz transformation

$$p_{\mathrm{p}}{}^*c = \gamma\beta Mc^2,$$

where M is the mass of the proton. The quark momentum (times c) is smaller than the proton momentum by the factor x,

$$p_{\mathrm{q}}{}^*c = p_{\mathrm{p}}{}^*cx = \gamma\beta Mc^2x.$$

For energetic electrons we may neglect the electron mass. The electron momentum (times c) in the center-of-mass frame is

$$p_{\mathrm{e}}{}^*c = \gamma E_{\mathrm{i}} - \gamma\beta E_{\mathrm{i}}.$$

By definition of the center of mass frame, the electron and the quark have equal magnitudes of momentum

$$p_{\mathrm{e}}{}^* = p_{\mathrm{q}}{}^*.$$

Therefore,

$$\gamma E_i - \gamma\beta E_i = \gamma\beta M c^2 x \, .$$

Solving for x, we get

$$x = \frac{E_i(1-\beta)}{\beta M c^2} \, .$$

In the center-of-mass frame the electron energy is not changed after the scatter. The energy of the scattered electron in the center-of-mass frame is

$$E_f{}^* = \gamma E_f - \beta\gamma E_f \cos\theta \, .$$

Conservation of energy (neglecting the electron mass) gives

$$\gamma E_i - \gamma\beta E_i = \gamma E_f - \beta\gamma E_f \cos\theta \, .$$

We may solve this expression for β,

$$\beta = \frac{E_i - E_f}{E_i - E_f \cos\theta} \, .$$

Now we substitute this expression for β into our expression for the proton momentum fraction, to get

$$x = \frac{E_i(1-\beta)}{\beta M c^2} = \frac{E_i E_f (1-\cos\theta)}{(E_i - E_f) M c^2} \, .$$

2. Obtain the general expression for fields of a uniformly moving charge,

$$\mathbf{v} = v\,\hat{\mathbf{z}}$$

by making a Lorentz transformation on the static Coulomb field,

$$\mathbf{E} = \frac{e\mathbf{r}}{r^3}, \qquad \mathbf{B} = 0$$

Let the charge be at rest in the frame K'.

$$F'^{\mu\nu} = \begin{pmatrix} 0 & -\frac{ex'}{r'^3} & -\frac{ey'}{r'^3} & -\frac{ez'}{r'^3} \\ \frac{ex'}{r'^3} & 0 & 0 & 0 \\ \frac{ey'}{r'^3} & 0 & 0 & 0 \\ \frac{ez'}{r'^3} & 0 & 0 & 0 \end{pmatrix}$$

$$F^{\mu\nu}(x') = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F'^{\alpha\beta}(x)$$

where

$$\Lambda^{\alpha}{}_{\beta} \equiv \frac{\partial x'^{\alpha}}{\partial x^{\beta}} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

so the electric field pieces are

$$F^{10} = \Lambda^{1}{}_{\alpha}\Lambda^{0}{}_{\beta}F'^{\alpha\beta} = \Lambda^{0}{}_{\beta}F'^{1\beta}$$

$$= \gamma F'^{10} + \beta\gamma F'^{13} = \frac{\gamma e x'}{r'^3}$$

$$F^{20} = \Lambda^{0}{}_{\beta}F'^{2\beta} = \gamma F'^{20} + \beta\gamma F'^{23} = \frac{\gamma e y'}{r'^3}$$

$$F^{30} = \Lambda^3{}_\alpha \Lambda^0{}_\beta F'^{\alpha\beta} = \Lambda^3{}_\alpha \Lambda^0{}_\beta F'^{\alpha\beta}$$

$$= \Lambda^0{}_\beta \left(\gamma\beta F'^{0\beta} + \gamma F'^{3\beta}\right) = \gamma\beta\gamma\beta F'^{03} + \gamma^2 F'^{30} = \frac{ez'}{r'^3}$$

and the magnetic field pieces are

$$F^{21} = \Lambda^2{}_\alpha \Lambda^1{}_\beta F'^{\alpha\beta}(x) = F'^{21} = 0$$

$$F^{31} = \Lambda^3{}_\alpha \Lambda^1{}_\beta F'^{\alpha\beta} = \Lambda^3{}_\alpha F'^{\alpha 1}$$

$$= \gamma\beta F'^{01} + \gamma F'^{21} = -\frac{\beta\gamma e x'}{r'^3}$$

$$F^{32} = \Lambda^3{}_\alpha \Lambda^2{}_\beta F'^{\alpha\beta} = \Lambda^3{}_\alpha F'^{\alpha 2}$$

$$= \gamma\beta F'^{02} + \gamma F'^{22} = -\frac{\beta\gamma e y'}{r'^3}$$

Now transform the coordinates

$$x' = x$$

$$y' = y$$

$$z' = \gamma(z - vt)$$

$$r'^3 = \left(x'^2 + y'^2 + z'^2\right)^{3/2} = \left[x^2 + y^2 + \gamma^2 (z - vt)^2\right]^{3/2}$$

The fields are

$$E_x = F^{10} = \frac{\gamma e x}{\left[x^2 + y^2 + \gamma^2 (z - vt)^2\right]^{3/2}}$$

$$E_y = F^{20} = \Lambda^0{}_\beta F'^{2\beta} = \gamma F'^{20} + \beta\gamma F'^{23}$$

$$= \frac{\gamma e y}{\left[x^2 + y^2 + \gamma^2 (z - vt)^2\right]^{3/2}}$$

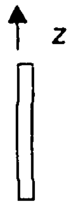
$$E_z = \frac{e\gamma(z - vt)}{\left[x^2 + y^2 + \gamma^2 (z - vt)^2\right]^{3/2}}$$

$$B_x = F^{32} = -\frac{\beta\gamma e y}{\left[x^2 + y^2 + \gamma^2 (z - vt)^2\right]^{3/2}}$$

$$B_y = -F^{31} = \frac{\beta\gamma e x}{\left[x^2 + y^2 + \gamma^2 (z - vt)^2\right]^{3/2}}$$

$$B_z = F^{21} = 0$$

3. (Jackson 11.13)
An infinitely long straight wire of negligible cross-sectional area is at rest and has a uniform linear charge density q_0 in the inertial frame K'. The frame K' (and the wire) move with a velocity v parallel to the direction of the wire with respect to the laboratory frame K.
(a) Write down the electric and magnetic fields in cylindrical coordinates in the rest frame of the wire. Using the Lorentz transformation properties of the fields, find the components of the electric and magnetic fields in the laboratory.



In the frame K' (using Gauss's law)

$$E'_\rho 2\pi\rho L = 4\pi q_0 L$$

$$E' = \frac{2q_0}{\rho'}\hat{\rho}$$

or

$$E'_x = \frac{2q_0}{\sqrt{x'^2 + y'^2}}\cos\phi'$$

$$E'_y = \frac{2q_0}{\sqrt{x'^2 + y'^2}}\sin\phi'$$

$$E'_z = 0$$

$$B' = 0$$

In the frame K

$$x = x'$$

$$y = y'$$

and so

$$\rho = \rho' = \sqrt{x^2 + y^2}$$
$$\phi = \phi'$$

The fields transform as

$$E_x = \gamma E'_x = \frac{2\gamma q_0}{\sqrt{x^2 + y^2}} \cos\phi'$$

$$E_y = \gamma E'_y = \frac{2\gamma q_0}{\sqrt{x^2 + y^2}} \sin\phi'$$

$$E_z = 0$$

$$B_x = -\beta\gamma E'_y = -\frac{2\beta\gamma q_0}{\sqrt{x^2 + y^2}} \sin\phi'$$

$$B_y = \beta\gamma E'_x = \frac{2\beta\gamma q_0}{\sqrt{x^2 + y^2}} \cos\phi'$$

$$B_z = 0$$

Note that **B** is in the phi-direction.

$$\mathbf{B} = \frac{2\beta\gamma q_0}{\rho} \hat{\phi}$$

(b) What are the charge and current densities associated with the wire in its rest frame? In the laboratory?

In frame K'

$$\rho' = q_0$$
$$\mathbf{J}' = 0$$

$$J^\mu = \begin{pmatrix} cq_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In frame K

$$J'^\mu = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} cq_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma c q_0 \\ 0 \\ 0 \\ \beta\gamma c q_0 \end{pmatrix}$$

or

$$\rho = \gamma q_0$$
$$\mathbf{J} = \gamma q_0 \mathbf{v}$$

(c) From the laboratory charge and current densities, calculate directly the electric and magnetic fields in the laboratory. Compare with the results of part a.

$$\mathbf{E} = \frac{2\gamma q_0}{\rho}\hat{\rho}$$

and from Ampere's law

$$2\pi\rho B_\phi = \frac{4\pi}{c}\gamma q_0 v$$

or

$$\mathbf{B} = \frac{2\beta\gamma q_0}{\rho}\hat{\phi}$$

in agreement with part a.

4. (a) Evaluate

$$F^{\mu\nu}F_{\mu\nu}$$

Use your result to convince your self that if the magnitudes of **E** and **B** are equal in one frame, then they are equal in all frames and if one exceeds the other in some frame then it will do so in all frames.

Note:

$$F^{i0} = E_i$$
$$F_{i0} = -E_i$$
$$F^{ij} = -\varepsilon_{ijk}B_k = F_{ij}$$

$$F^{\mu\nu}F_{\mu\nu} = F^{0\nu}F_{0\nu} + F^{i\nu}F_{i\nu}$$
$$= F^{00}F_{00} + F^{0j}F_{0j} + F^{i0}F_{i0} + F^{ij}F_{ij} = -2E^2 + 2B^2$$

This quantity is invariant, thus…

(b) Evaluate

$$\Im^{\mu\nu}F_{\mu\nu}$$

where

$$\Im^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$

Use your result to convince your self that if **E** and **B** are perpendicular in one frame, then they perpendicular in all frames.

Note:

$$\Im^{ij} = \varepsilon_{ijk}E_k$$
$$\Im^{0i} = -B_i$$

$$
\begin{aligned}
\Im^{\mu\nu} F_{\mu\nu} &= \Im^{0\nu} F_{0\nu} + \Im^{i\nu} F_{i\nu} \\
&= \Im^{00} F_{00} + \Im^{0j} F_{0j} + \Im^{i0} F_{i0} + \Im^{ij} F_{ij} \\
&= 2\Im^{0i} F_{0i} + \left(\varepsilon_{ij\ell} E_\ell\right)\left(-\varepsilon_{ijk} B_k\right) = -2E_i B_i - 2B_k E_k \\
&= -4E_i B_i
\end{aligned}
$$

This quantity is invariant, thus…

PY522 Fall 2001
Homework No. 11

1. (Jackson 13.4)
Note: mu meson is the old name for what we now call the muon. (It is actually NOT a meson as originally assumed!)

(a) Taking $\hbar\langle\omega\rangle = 12Z$ eV in the quantum-mechanical energy-loss formula, calculate the rate of energy loss (in MeV/cm) in air at NTP, aluminum, copper, and lead for a proton and a muon, each with kinetic energies of 10, 100, 1000 MeV.
(b) Covert your results to energy loss in units of MeV·(cm^2/g) and compare the values obtained in different materials. Explain why all the energy losses in MeV·(cm^2/g) are within a factor of 2 of each other, whereas the values in MeV/cm differ greatly.

The Jackson formula is

$$\frac{dE}{dx} = \left(0.30\,\mathrm{MeV}\cdot\mathrm{cm}^{-1}\right)\frac{Z}{\beta^2 A}\left[\ln\left(\frac{2mc^2\gamma^2\beta^2}{(12\,\mathrm{eV})Z}\right)-\beta^2\right]$$

mass (MeV)	T (MeV)	E (MeV)	beta	gamma	Z	A	rho (g/cm**3)	dE/dx (MeV/cm)	dE/dx (cm**2/g)
proton in air									
940	10	950	0.145	1.011	7	14	0.0012	0.0476	39.69
940	100	1040	0.428	1.106	7	14	0.0012	0.0076	6.33
940	1000	1940	0.875	2.064	7	14	0.0012	0.0023	1.93
proton in Al									
940	10	950	0.145	1.011	13	27	2.7	91.6556	33.95
940	100	1040	0.428	1.106	13	27	2.7	15.1444	5.61
940	1000	1940	0.875	2.064	13	27	2.7	4.6907	1.74
proton in Cu									
940	10	950	0.145	1.011	29	63.6	8.93	240.2818	26.91
940	100	1040	0.428	1.106	29	63.6	8.93	42.0812	4.71
940	1000	1940	0.875	2.064	29	63.6	8.93	13.4114	1.50
pion in air									
105	10	115	0.408	1.095	7	14	0.0012	0.0083	6.88
105	100	205	0.859	1.952	7	14	0.0012	0.0024	1.97
105	1000	1105	0.995	10.524	7	14	0.0012	0.0024	1.98
pion in Al									
105	10	115	0.408	1.095	13	27	2.7	16.4330	6.09
105	100	205	0.859	1.952	13	27	2.7	4.8025	1.78
105	1000	1105	0.995	10.524	13	27	2.7	4.9173	1.82
pion in Cu									
105	10	115	0.408	1.095	29	63.6	8.93	45.5794	5.10
105	100	205	0.859	1.952	29	63.6	8.93	13.7137	1.54
105	1000	1105	0.995	10.524	29	63.6	8.93	14.4130	1.61

2. Estimate the range of a 5 MeV alpha particle in air.

The speed of a 5 MeV alpha particle is given by

$$E_k = \frac{mv^2}{2},$$

or

$$\frac{v}{c} = \sqrt{\frac{2E_k}{mc^2}} \approx \sqrt{\frac{10\text{MeV}}{3700\text{MeV}}} \approx 0.05.$$

The alpha particle is heavily ionizing because of its small speed. The rate of energy loss by ionization is

$$-\frac{dE}{dx} = (30.7\,\text{keV}\cdot\text{m}^2/\text{kg})\frac{Z\rho}{A\beta^2}\left[\ln\left(\frac{2mc^2\gamma^2\beta^2}{I}\right) - \beta^2\right].$$

The logarithmic factor is

$$\ln\left(\frac{2mc^2\gamma^2\beta^2}{I}\right) = \ln\left[\frac{(2)(0.5\text{MeV})(1)^2(0.05)^2}{(16\text{eV})(7)^{0.9}}\right] \approx 3.$$

The density of air is about one kilogram per cubic meter. For air,

$$\frac{Z}{A} = 0.5.$$

The initial rate of ionization loss is

$$-\frac{dE}{dx} = (30.7\,\text{keV}\cdot\text{m}^2/\text{kg})(0.5)\left[\frac{1\text{kg/m}^3}{(0.05)^2}\right](3) \approx 20\text{MeV/m}.$$

The rate of ionization loss increases rapidly as the alpha particle loses energy. Therefore the range of the alpha particle in air is only a few centimeters.

3. A relativistic muon travels a distance of 1 m water. Make an estimate of the number of visible photons emitted as Cerenkov radiation. Use your result to show that the energy loss by Cerenkov radiation $(-dE/dx)_C$ is very small compared to the ionization loss $(-dE/dx)$. (See Figure 7.9 on p. 315 for the frequency dependence of n.)

$$\beta \approx 1$$

$$\cos\theta_C = \frac{1}{\beta n} \approx 0.75$$

$$\frac{d^2N}{dEdx} \approx 370\sin^2\theta_c(E)\mathrm{eV}^{-1}\mathrm{cm}^{-1} \approx 160/\mathrm{cm/eV}$$

The visible photon window covers

$$\Delta\mathrm{E} \approx 1.3\,\mathrm{eV}$$

so the number of visible photons per cm is

$$\frac{dN}{dx} \approx (160/\mathrm{cm/eV})(1.3\,\mathrm{eV}) \approx 210/\mathrm{cm}$$

The average photon energy is 2.3 eV, so

$$\left(\frac{dE}{dx}\right)_C \approx (210/\mathrm{cm})(2.3\,\mathrm{eV}) \approx 480\mathrm{eV/cm}$$

and for 1 m the energy into radiation is

$$E_C \approx (480\mathrm{eV/cm})(100\mathrm{cm}) \approx 48\mathrm{keV}$$

We have only counted visible photons. There are more outside this region (especially UV). However, the Cerenkov radiation is still much smaller than ionization loss which is measured in MeV/cm.

4. Jackson 13.9

Assuming that Lucite has an index of refraction of 1.50 in the visible region, compute the angle of emission of Cerenkov radiation for electrons and protons as a function of their kinetic energies in MeV. Determine how many quanta with wavelengths between 400 and 600 nm are emitted per cm of path in Lucite by a 1 MeV electron, a 500 MeV proton, and 5 GeV proton.

$$\cos\theta_C = \frac{1}{\beta n}$$

for $\beta = 1$, this corresponds to

$$\theta = 48^\circ$$

Since

$$\beta = \frac{pc}{E} = \frac{\sqrt{(mc^2+K)^2 - (mc^2)^2}}{mc^2+K} = \frac{\sqrt{K^2+2mc^2K}}{mc^2+K}$$

We have

$$\cos\theta_C = \frac{mc^2+K}{n\sqrt{K^2+2mc^2K}}$$

mc**2	K (MeV)	beta	cos theta-C	theta-C
0.511	1	0.941079	0.7084067	44.9
0.511	10	0.998818	0.6674559	48.2
0.511	100	0.999987	0.6666753	48.2
940	500	0.757549	0.8800306	28.4
940	1000	0.874771	0.762104	40.4
940	5000	0.987399	0.6751744	47.6

$$\frac{d^2N}{dEdx} \approx 370\sin^2\theta_c(E)\,\text{eV}^{-1}\text{cm}^{-1}$$

The energy window is 2.1 eV to 3.1 eV, so

$$\frac{dN}{dx} \approx 370\sin^2\theta_c(E)\,\text{cm}^{-1}$$

mc**2	K (MeV)	beta	cos theta-C	theta-C	sin theta-C	N per cm
0.511	1	0.941079	0.7084067	44.9	0.7058045	184
940	500	0.757549	0.8800306	28.4	0.4749171	83
940	5000	0.987399	0.6751744	47.6	0.7376581	201

Physics 6346, Electromagnetic Theory I
Fall 2000
Homework 1 Solutions (Revised 9/5/00)

1. **Review of vector calculus.** Prove the following identities from vector calculus:

   (a) $\nabla \times \nabla\psi = 0$

   (b) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

   (c) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

   (d) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

   Solution. This is standard material. If you had trouble, please review a textbook on mathematical methods, such as *Arfken* or *Boas.*

2. **Curvilinear coordinates.** We will often work in coordinate systems other that rectangular coordinates—for instance, in cylindrical or spherical coordinates. Explicit forms of vector operations in these coordinate systems are on the inside back cover of *Jackson.* Here we'll review how this is done. A good discussion can be found in *Morse and Feshbach*, §1.3.

   We want to go from the rectangular coordinates (x, y, z) to a new set of coordinates (ξ_1, ξ_2, ξ_3). An infinitesimal displacement $d\mathbf{r}$ along the curve $\mathbf{r}(\xi_1, \xi_2, \xi_3)$ is given by

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \xi_1} d\xi_1 + \frac{\partial \mathbf{r}}{\partial \xi_2} d\xi_2 + \frac{\partial \mathbf{r}}{\partial \xi_3} d\xi_3. \tag{1}$$

   If we vary ξ_1 while holding ξ_2 and ξ_3 fixed, then $\partial \mathbf{r}/\partial \xi_1$ is tangent to $\mathbf{r}$ along this curve; likewise for ξ_2 and ξ_3. If $\mathbf{e}_1$ is a unit vector along this direction, then we can write $\partial \mathbf{r}/\partial \xi_1 = h_1 \mathbf{e}_1$, with $h_1 = |\partial \mathbf{r}/\partial \xi_1|$; similarly, $\partial \mathbf{r}/\partial \xi_2 = h_2 \mathbf{e}_2$, with $h_2 = |\partial \mathbf{r}/\partial \xi_2|$, and $\partial \mathbf{r}/\partial \xi_3 = h_1 \mathbf{e}_3$, with $h_3 = |\partial \mathbf{r}/\partial \xi_3|$. The quantities (h_1, h_2, h_3) are called *scale factors.* If $(\mathbf{e}_1 . \mathbf{e}_2, \mathbf{e}_3)$ are orthogonal at each point in space, then we have an *orthogonal* coordinate system. In this coordinate system an element of arclength ds is given by

$$ds^2 = dx^2 + dy^2 + dz^2 = h_1^2 d\xi_1^2 + h_2^2 d\xi_2^2 + h_3^2 d\xi_3^2, \tag{2}$$

   where the scale factors h_n are given by

$$h_n^2 = \left(\frac{\partial x}{\partial \xi_n}\right)^2 + \left(\frac{\partial y}{\partial \xi_n}\right)^2 + \left(\frac{\partial z}{\partial \xi_n}\right)^2. \tag{3}$$

   In terms of the scale factors, the volume element in the new coordinate system is

$$dV = h_1 h_2 h_3 d\xi_1 \, d\xi_2 \, d\xi_3, \tag{4}$$

the gradient of a scalar function Φ is

$$\nabla\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial\xi_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial\Phi}{\partial\xi_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial\Phi}{\partial\xi_3}\mathbf{e}_3, \tag{5}$$

and the Laplacian of Φ is

$$\nabla^2\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\xi_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial\xi_1}\right) + \frac{\partial}{\partial\xi_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial\xi_2}\right) + \frac{\partial}{\partial\xi_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial\xi_3}\right)\right]. \tag{6}$$

Find dV, $\nabla\Phi$, and $\nabla^2\Phi$ in the following coordinate systems:

(a) Cylindrical: $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$.

(b) Spherical: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

(c) Parabolic cylindrical coordinates: $x = (1/2)(u^2 - v^2)$, $y = uv$, $z = z$.
Solution. The results for cylindrical and spherical coordinates can be found in textbooks and *Jackson*. For parabolic cylinder coordinates, we have $(\xi_1, \xi_2, \xi_3) = (u, v, z)$, with the scale factors $h_1 = h_2 = \sqrt{u^2 + v^2}$, $h_3 = 1$, and orthogonal unit vectors $(\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_z)$. Then $dV = (u^2 + v^2)du\,dv\,dz$, and

$$\nabla\Phi = \frac{1}{\sqrt{u^2+v^2}}\left(\frac{\partial\Phi}{\partial u}\mathbf{e}_u + \frac{\partial\Phi}{\partial v}\mathbf{e}_v\right) + \frac{\partial\Phi}{\partial z}\mathbf{e}_z, \tag{7}$$

$$\nabla^2\Phi = \frac{1}{u^2+v^2}\left(\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2}\right) + \frac{\partial^2\Phi}{\partial z^2}. \tag{8}$$

3. **Dirac delta function.**

(a) *Jackson's* Problem 1.2 gives a possible representation of the Dirac δ function, in terms of a Gaussian. Give at least two other representations (you may have already encountered the square box, the Lorentzian, and the sinc function, but you can also be creative). Make sure to normalize your function properly.
Solution. For the Gaussian, we have

$$\delta(x) = \frac{1}{\sqrt{2\pi}\epsilon}e^{-x^2/2\epsilon^2}, \tag{9}$$

in the limit that $\epsilon \to 0$. If you plot this you'll see that the function becomes narrower and higher as $\epsilon \to 0$. You can also try the Lorentzian,

$$\delta(x) = \frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2}, \tag{10}$$

or the sinc function,

$$\delta(x) = \frac{1}{\pi}\frac{\sin(x/\epsilon)}{x}, \tag{11}$$

with $\epsilon \to 0$ in both cases.

(b) Using the result of *Jackson's* 1.2, express the three dimensional δ function in cylindrical and spherical coordinates.

Solution. If you work through *Jackson* 1.2, you'll find that in an orthogonal coordinate system (ξ_1, ξ_2, ξ_3), the Dirac delta function can be written as

$$\delta^{(3)}(\mathbf{x}-\mathbf{x}') = \frac{1}{h_1 h_2 h_3}\delta(\xi_1-\xi_1')\delta(\xi_2-\xi_2')\delta(\xi_3-\xi_3'). \tag{12}$$

In cylindrical coordinates, $(\xi_1, \xi_2, \xi_3) = (\rho, \phi, z)$, and $h_1 = 1$, $h_2 = \rho$, and $h_3 = 1$, so that

$$\delta^{(3)}(\mathbf{x}-\mathbf{x}') = \frac{\delta(\rho-\rho')\,\delta(\phi-\phi')\,\delta(z-z')}{\rho}. \tag{13}$$

In spherical coordinates, $(\xi_1, \xi_2, \xi_3) = (r, \theta, \phi)$, and $h_1 = 1$, $h_2 = r$, and $h_3 = r\sin\theta$, so that

$$\delta^{(3)}(\mathbf{x}-\mathbf{x}') = \frac{\delta(r-r')\,\delta(\theta-\theta')\,\delta(\phi-\phi')}{r^2\sin\theta}. \tag{14}$$

This is often written in the following form:

$$\delta^{(3)}(\mathbf{x}-\mathbf{x}') = \frac{\delta(r-r')\,\delta(\cos\theta-\cos\theta')\,\delta(\phi-\phi')}{r^2}. \tag{15}$$

See p. 120 in *Jackson*.

4. **Simple applications of Gauss's law.** The following charge distributions are highly symmetric; the electric fields produced by them are easily calculated using Gauss's law in integral form. You should be able to do these in your sleep (of course, as a first year graduate student you're probably not sleeping much). Calculate the electric field (magnitude and direction) in each case, being careful to spell out all steps.

   (a) A point charge q.

   Solution. These are all standard exercises, which can be found in any introductory text, so I'll simply quote the result. If you had difficulty, please review and see me. For a point charge,

$$\mathbf{E}(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{e}_r}{r^2}. \tag{16}$$

   (b) A sphere of radius R with a uniform charge density ρ (find the field both inside and outside the sphere).

   Solution.

$$\mathbf{E}(\mathbf{x}) = \begin{cases} (\rho r/3\epsilon_0)\mathbf{e}_r & r < R \\ (\rho R^3/3\epsilon_0 r^2)\mathbf{e}_r & r > R. \end{cases} \tag{17}$$

   (c) An infinite line of charge with charge per unit length λ.

   Solution.

$$\mathbf{E}(\mathbf{x}) = \frac{\lambda}{2\pi\epsilon_0 r}\mathbf{e}_r. \tag{18}$$

(d) An infinite cylinder of radius R with a uniform charge density ρ (find the field both inside and outside the cylinder).

Solution.

$$\mathbf{E}(\mathbf{x}) = \begin{cases} (\rho r/2\epsilon_0)\mathbf{e}_r & r < R \\ (\rho R^2/2\epsilon_0 r)\mathbf{e}_r & r > R. \end{cases} \tag{19}$$

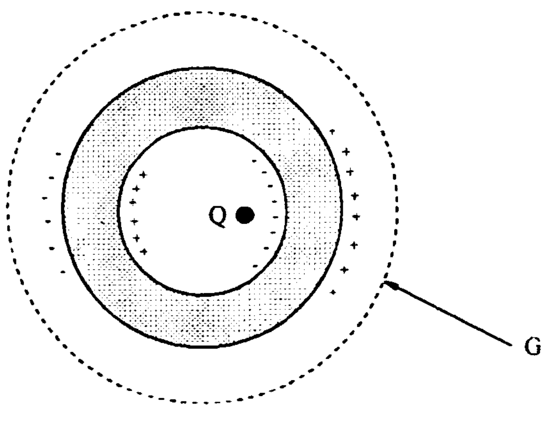
5. Use Gauss's law to prove the following statements about conductors (essentially *Jackson* 1.1):

(a) Any excess charge placed on a conductor must lie entirely on its surface.

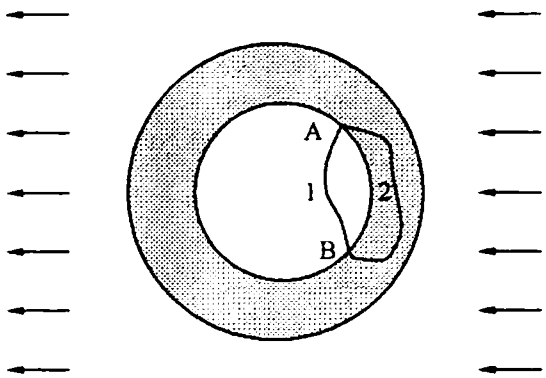
Solution. First, recall that in equilibrium the electric field inside a conductor is zero. Why? Suppose that the field were initially nonzero; any charges in the interior would then move in response to the field (since this is a conductor). After some relaxation time this process stops, since the moving charges produce currents which dissipate energy. The final configuration is one in which the charges have been arranged so that the field in the interior is zero. Since $\mathbf{E} = 0$ everywhere inside the conductor, from Gauss's Law the charge density $\rho = 0$ everywhere in the interior. Therefore any excess charge can only reside on the surface of the conductor.

(b) A closed, hollow conductor shields its interior from fields due to charges outside, but does not shield its exterior from the fields due to charges placed within it.

Solution. Let's start with the second part. Consider a positive charge Q placed inside a hollow conductor as shown in the figure below. The charge induces a charge density on the interior surface of the conductor in such a way that the electric field in the interior of the conductor is zero (the net charge on the interior surface must be $-Q$). Assuming that the conductor is charge neutral, this means that there is an induced charge density on the exterior surface of total charge Q. If we apply Gauss's Law to the Gaussian surface G surrounding the conductor, the total charge enclosed is still Q, and there is therefore an electric field outside the conductor.



Next, consider some charge exterior to the conductor, which produces an electric field, as shown sketched in the figure. The electric field in the conductor is zero, with induced charge densities on the exterior and interior surfaces of the conductor. Now imagine moving a charge on the interior surface from point A to point B along path 2 which goes through the conductor itself. Since $\mathbf{E} = 0$ in the conductor, $\int_2 \mathbf{E} \cdot d\mathbf{l} = 0$ along this path. Next, move the same charge from A to B along path 1, in the interior cavity of the conductor. Since the electrostatic field is conservative, the line integral $\int_1 \mathbf{E} \cdot d\mathbf{l} = 0$ along this path also. In fact, this must be true for any path which we chose in the interior, so we have quite generally $\mathbf{E} = 0$ in the interior—the conductor shields its interior from fields due to charges placed outside. This is the principle behind the *Faraday cage.*



(c) The electric field at the exterior surface of a charged conductor is normal to the surface and has a magnitude $\sigma(\mathbf{x})/\epsilon_0$, where $\sigma(\mathbf{x})$ is the local surface charge density.

Solution. First, we note that in equilibrium the field at exterior surface must be normal to the surface—a component tangent to the surface would cause charges to move on the surface, until they had arranged themselves so that the tangential component is zero. Once we've established this result, the magnitude of the field is derived by using Gauss's Law with a Gaussian "pillbox" which cuts through the surface. The electric field is zero on the conducting side of the pillbox, so $\oint \mathbf{E} \cdot \mathbf{n} da = EA$, with A the area on the surface (note that for a sheet of charge this becomes $2EA$, where the factor of two comes from the two side of the sheet). Setting this equal to Q/ϵ_0, and then dividing by A, we obtain the desired result.

Physics 6346, Electromagnetic Theory I
Fall 2000
Homework 2 Solutions

1. **Potential for the hydrogen atom.** This is essentially *Jackson* 1.5 done in reverse. Start with the time averaged electron charge density

$$\rho(r) = -q\frac{\alpha^3}{8\pi}e^{-\alpha r}, \tag{1}$$

and a point charge q at the origin, compute the electric field using Gauss's law, and then find $\Phi(r)$ by integration. You should find Jackson's $\Phi(r)$.

What is the numerical magnitude of E at the typical atomic distance of 1 Å?

Solution. The proton at the origin contributes a charge density of $q\delta^{(3)}(\mathbf{x})$, so the total charge density is

$$\rho(\mathbf{x}) = q\delta^{(3)}(\mathbf{x}) - q\frac{\alpha^3}{8\pi}e^{-\alpha r}. \tag{2}$$

This charge density is spherically symmetric, so the electric field is radial: $\mathbf{E}(\mathbf{x}) = E_r\mathbf{e}_r$. By using Gauss's Law in integral form, applied to a spherical Gaussian surface of radius r, we find

$$\begin{aligned} \oint_S \mathbf{E}\cdot\mathbf{n}\,da &= 4\pi r^2 E_r \\ &= \frac{1}{\epsilon_0}\left[q - \int_0^r \left(q\frac{\alpha^3}{8\pi}e^{-\alpha r'}\right)4\pi r'^2\,dr'\right]. \end{aligned} \tag{3}$$

Performing the integral, we find

$$E_r = \frac{q}{4\pi\epsilon_0}\left(\frac{\alpha^2}{2} + \frac{\alpha}{r} + \frac{1}{r^2}\right)e^{-\alpha r}. \tag{4}$$

Since $E_r = -\partial\Phi/\partial r$, we can integrate the electric field to obtain the potential,

$$\Phi(r) = \frac{q}{4\pi\epsilon_0}\left(\frac{\alpha}{2} + \frac{1}{r}\right)e^{-\alpha r}. \tag{5}$$

To estimate the value of the electric field at the atomic scale, use $\alpha^{-1} \approx 1$ Å. Putting in the numbers, I find that the magnitude of the electric field is about 10^{11} V/m (a BIG number!).

2. **Charged ring.** A circular ring of radius R carries a uniformly distributed charge q. The ring is centered at the origin of the $x-y$ plane, so that the z-axis is the symmetry axis.

(a) Find the electrostatic potential on the z-axis of the ring.

Solution. This is straightforward. The result is

$$\Phi(z) = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{R^2+z^2}}. \tag{6}$$

(b) Find the potential at any point in space. You can express the integral as an elliptic integral or a hypergeometric function, or as an expansion in Legendre functions (see *Jackson*, p. 91).

Solution. Let's work in cylindrical coordinates (although spherical coordinates would work fine). The charge density is

$$\rho(\mathbf{x}) = q\frac{\delta(\rho-R)\delta(z)}{2\pi\rho}. \tag{7}$$

Recalling that in cylindrical coordinates

$$|\mathbf{x}-\mathbf{x}'| = \sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z-z')^2}, \tag{8}$$

we have for the potential

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{2\pi}\int_0^{2\pi} d\phi'\,\frac{1}{\sqrt{\rho^2+R^2+z^2-2R\rho\cos(\phi-\phi')}}\\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{2\pi}\int_0^{2\pi} d\alpha\,\frac{1}{\sqrt{\rho^2+R^2+z^2-2R\rho\cos\alpha}},\end{aligned} \tag{9}$$

where in the last line we've substituted $\alpha = \phi-\phi'$ and have shifted the limits of integration.

At this point we're basically done; the integral can't be expressed in terms of elementary functions. However, with a bit more work we can express it in terms of an ***elliptic integral***, the properties of which are discussed in textbooks on mathematical physics. To do this, first notice that the integral can be written as twice the integral from 0 to π; then change the integration variable to $\beta = (\pi-\alpha)/2$, with the result

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\frac{2}{\pi}\int_0^{\pi/2}\frac{d\beta}{\sqrt{\rho^2+z^2+R^2+2R\rho\cos 2\beta}}. \tag{10}$$

Next, use $\cos 2\beta = 1-2\sin^2\beta$ to write the integral as

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\frac{2}{\pi}\frac{1}{\sqrt{(\rho+R)^2+z^2}}\int_0^{\pi/2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}}, \tag{11}$$

with

$$k^2 = \frac{4\rho R}{(\rho + R)^2 + z^2}. \tag{12}$$

The integral is now in the standard form; it is the *complete elliptic integral of the first kind*, $K(k)$ (see I. S. Gradshteyn and M. Ryzhik, *Table of Integrals, Series, and Products, Fifth Edition*, Sec. 8.11), so our potential can be written as

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\frac{2}{\pi}\frac{1}{\sqrt{(\rho + R)^2 + z^2}}K(k). \tag{13}$$

The results for spherical coordinates can be obtained by substituting $\rho = r\sin\theta$, $z = r\cos\theta$, into these expressions.

(c) A positive test charge is located at the center of the ring. Is this a position of stable equilibrium? (Consider displacements both in the plane of the ring and normal to the plane of the ring.) For stable displacements, find the frequency of small oscillations of the test charge.

Solution. Let's first consider displacements normal to the plane of the ring. Then the potential is given by Eq. (6); expand the potential about $z = 0$:

$$\Phi(\rho = 0, z) = \frac{q}{4\pi\epsilon_0}\frac{1}{R}\left(1 - \frac{1}{2}\frac{z^2}{R^2} + \ldots\right), \tag{14}$$

so we see that $z = 0$ is an extremal point. If we place a test charge q' near $z = 0$, then the potential energy is

$$U(\rho = 0, z) = q'\Phi(\rho = 0, z) = \frac{qq'}{4\pi\epsilon_0}\frac{1}{R}\left(1 - \frac{1}{2}\frac{z^2}{R^2} + \ldots\right), \tag{15}$$

so $z = 0$ is a stable equilibrium point for displacements normal to the plane if $qq' < 0$, and unstable if $qq' > 0$. The frequency of small oscillations for a particle of mass m is obtained by setting

$$U(\rho = 0, z) = U(0,0) + \frac{1}{2}m\omega^2 z^2, \tag{16}$$

so that

$$\omega^2 = -\frac{qq'}{4\pi\epsilon_0 m R^3}. \tag{17}$$

Next, we want consider displacements in the plane of the ring. Expanding the potential near $\rho = 0$, we obtain

$$\Phi(\rho, z = 0) = \frac{q}{4\pi\epsilon_0}\frac{1}{R}\left(1 + \frac{\rho^2}{4R^2} + \ldots\right). \tag{18}$$

The potential energy of a test charge of charge q' is then

$$U(\rho, z=0) = q'\Phi(\rho, z=0) = \frac{qq'}{4\pi\epsilon_0}\frac{1}{R}\left(1 + \frac{\rho^2}{4R^2} + \ldots\right). \tag{19}$$

Therefore there is a stable equilibrium point for in-plane displacements when $qq' > 0$, and an unstable equilibrium when $qq' < 0$. The frequency of small oscillations is

$$\omega^2 = \frac{qq'}{8\pi\epsilon_0 m R^3}. \tag{20}$$

Notice that motion which is stable in one direction is unstable in the other, so that the origin is a saddle point (recall that in a charge-free region of space the potential cannot have minima or maxima).

(d) Suppose that the ring is now replaced with an ellipse. How are your results from part (c) changed?

Solution. The results are essentially unchanged. To see this, consider the expansion of the potential near the origin:

$$\begin{aligned}\Phi(\mathbf{r}) &= \Phi(0) + \mathbf{r}\cdot\nabla\Phi + \frac{1}{2}\sum_{\alpha,\beta} x_\alpha x_\beta \left(\frac{\partial^2\Phi}{\partial x_\alpha \partial x_\beta}\right)_{\mathbf{r}=0} + \ldots \\ &= \Phi(0) - \mathbf{r}\cdot\mathbf{E}(0) - \frac{1}{2}\sum_{\alpha,\beta} x_\alpha x_\beta \left(\frac{\partial E_\alpha}{\partial x_\beta}\right)_{\mathbf{r}=0} + \ldots.\end{aligned} \tag{21}$$

For any sufficiently symmetrical charge distribution, the electric field at the origin will vanish, so this will be a saddle point of the potential. Of course, the frequency of small oscillations will depend on the details of the charge distribution, but the existence of the saddle point only depends on the symmetry.

3. **Capacitance.** *Jackson* Problem 1.6.

Solution. These are standard, and the results can be found in elementary textbooks. The capacitance of a parallel plate capacitor is

$$C = \frac{\epsilon_0 A}{d}, \tag{22}$$

with A the area of the plates and d their separation. The capacitance of two concentric spheres of radii a, b ($b > a$) is

$$C = 4\pi\epsilon_0 \frac{ab}{b-a}. \tag{23}$$

The capacitance of two concentric conducting cylinders of length L large compared to their radii a, b ($b > a$) is

$$C = \frac{2\pi\epsilon_0 L}{\ln(b/a)}. \tag{24}$$

If the capacitance per unit length for a coaxial cable is 3×10^{-11} F/m, then for $a = 1$ mm I find that we need $b = 6.4$ mm.

4. **Force between conductors.** *Jackson* Problem 1.9. You only need to consider the parallel plate geometry.

   Solution. First, let's assume that the charge on each conductor is held fixed (so that the conductors are electrically isolated). The energy stored in the capacitor is

$$W = \frac{Q^2}{2C} = \frac{Q^2 d}{2\epsilon_0 A}. \tag{25}$$

   The force on one of the conductors is then

$$F = -\left(\frac{\partial W}{\partial d}\right)_Q = -\frac{Q^2}{2\epsilon_0 A}. \tag{26}$$

   We see that the force is attractive, and that the force per unit area is $\sigma^2/2\epsilon_0$ (with $\sigma = Q/A$ the charge per unit area). This agrees with the derivation on p. 42 of *Jackson.*

   Now let's assume that the potential difference between the conductors is held fixed. This can only be arranged by attaching the conductors to a source of charge (a battery). In calculating the work done in moving the capacitor, we need to account for the work done by the battery in moving charge on and off the plates. After accounting for this (see the lecture notes), we find that

$$W = \frac{1}{2}CV^2 = \frac{\epsilon_0 A}{2d}V^2, \tag{27}$$

   and

$$F = \left(\frac{\partial W}{\partial d}\right)_V = -\frac{\epsilon_0 A V^2}{2d^2}. \tag{28}$$

   By using $Q = CV$, we can show that results in Eqs. (26) and (28) are equivalent.

5. **Poisson's equation.** In the jellium model of a metal surface, the ionic charge density can be taken as

$$\rho_{\text{ion}} = \begin{cases} \rho_0 & \text{if } z < 0 \\ 0 & \text{if } z > 0 \end{cases} \tag{29}$$

   where the metal occupies the half-space $z < 0$. A plausible model for the electron density is

$$\rho_{\text{electron}} = -\rho_0 \frac{1}{e^{z/\alpha} + 1}. \tag{30}$$

   (a) What are the physical interpretations of the constants ρ_0 and α? Plot the total charge density for several different values of these constants.

   Solution. The total charge density is the sum of the ionic and electronic contributions:

$$\rho(z) = \begin{cases} \rho_0 \frac{e^{z/\alpha}}{e^{z/\alpha}+1} & \text{if } z < 0 \\ -\rho_0 \frac{1}{e^{z/\alpha}+1} & \text{if } z > 0. \end{cases} \tag{31}$$

We can write this in the more compact form

$$\rho(z) = -\rho_0 \frac{\text{sgn}(z)}{e^{|z|/\alpha} + 1}. \tag{32}$$

The prefactor ρ_0 is the free electron charge density in the bulk of the metal, and α is the distance over which the electrons "leak" out of the metal and into the vacuum. We expect α to be of order 1 Å.

(b) Solve Poisson's equation for this model charge density. [Hint: note that since the charge density only varies in the z-direction, Poisson's equation is one dimensional. You will need to solve it for $z < 0$ and $z > 0$, and match the solutions appropriately.]

Solution. We need to integrate the one-dimensional Poisson's equation, which is

$$\frac{d^2\Phi}{dz^2} = -\frac{\rho(z)}{\epsilon_0}. \tag{33}$$

Integrating once, we have

$$\frac{d\Phi}{dz} = -\frac{\alpha\rho_0}{\epsilon_0} \ln\left(e^{-|z|/\alpha} + 1\right), \tag{34}$$

where the integration constants are chosen to be zero to guarantee that the electric field vanishes at $z \pm \infty$. We need to integrate this again in order to find the potential; there will be another integration constant, which we will choose so that $\Phi(-\infty) = 0$ (the potential is chosen to be zero in the bulk of the metal). With this choice, we have

$$\Phi(z) = -\frac{\alpha\rho_0}{\epsilon_0} \int_{-\infty}^{z} \ln\left(e^{-|z'|/\alpha} + 1\right) dz'. \tag{35}$$

(c) Calculate the quantity $\Phi(\infty) - \Phi(-\infty)$. Use this to find the surface dipole density on the surface. [Hint: you should find that $D = -\pi^2\alpha^2\rho_0/6$.]

Solution. From our solution above, we see that

$$\begin{aligned} \Phi(\infty) - \Phi(-\infty) &= -\frac{\alpha\rho_0}{\epsilon_0} \int_{-\infty}^{\infty} \ln\left(e^{-|z'|/\alpha} + 1\right) dz' \\ &= -\frac{2\alpha^2\rho_0}{\epsilon_0} \int_0^{\infty} \ln\left(e^{-u} + 1\right) du \\ &= -\frac{2\alpha^2\rho_0}{\epsilon_0} \left(\frac{\pi^2}{12}\right). \end{aligned} \tag{36}$$

At length scales larger than α, it appears that the potential is discontinuous upon crossing the surface. This discontinuity can be attributed to a "double layer," with surface dipole moment density D given by [see *Jackson*, p. 34, Eq. (1.27)]

$$\begin{aligned} D &= \epsilon_0[\Phi(\infty) - \Phi(-\infty)] \\ &= -\frac{\pi^2\alpha^2\rho_0}{6}. \end{aligned} \tag{37}$$

This is the same result which we would have obtained by calculating the dipole moment density directly:

$$
\begin{aligned}
D &= \int_{-\infty}^{\infty} z\rho(z)\,dz \\
&= -2\rho_0 \int_0^{\infty} \frac{z}{e^{z/\alpha}+1}\,dz \\
&= -\frac{\pi^2\alpha^2\rho_0}{6}.
\end{aligned}
\tag{38}
$$

Surface dipole layers make an important contribution to the work function of metal surfaces; for further discussion, see A. Zangwill, *Physics at Surfaces* (Cambridge University Press 1988), pp. 57–63.

Physics 6346, Electromagnetic Theory I
Fall 2000
Homework 3 Solutions (Revised 9/27/00)

1. *Jackson* 2.1. A point charge q is brought to a position a distance d away from an infinite plane conductor held at zero potential. Using the method of images, find:

   (a) the surface-charge density induced on the plane, and plot it;

   (b) the force between the plane and the charge by using Coulomb's law for the force between the charge and its image;

   (c) the total force acting on the plane by integrating $\sigma^2/2\epsilon_0$ over the whole plane;

   (d) the work necessary to remove the charge q from its position to infinity;

   (e) the potential energy between the charge q and its image [compare the answer to part d and discuss].

   (f) Find the answer to part d in electron volts for an electron originally one Angstrom from the surface.

   Solution. All of these results are worked out in some detail in Chapter 4 of the lecture notes.

2. Two infinite, grounded conducting planes are located at $x = -a$ and $x = a$. A point charge q is placed at a point (x', y', z') between the plates.

   (a) Find the positions of all point charges needed to satisfied the boundary conditions on the potential.

   Solution. We start by placing image charges of strength $-q$ at positions $2a - x'$ and $-2a - x'$ in an attempt to make the potential vanish at $x = \pm a$. However, the image charge at $2a - x'$ contributes to the potential at $x = -a$, so in order to cancel this contribution we need another image charge of strength $+q$ at $-4a + x'$; likewise, we need $+q$ at $4a + x'$ to cancel the contribution at $x = a$ from the negative image charge at $-2a - a'$. Of course, these two image charges will require corrections, in the form of image charges of strength $-q$ located at $-6a - x'$ and $6a - x'$, and so on. So the result is that we require an infinite number of image charges, with $+q$ placed at $4na + x'$ (with n an integer: $n = 0$ is the source charge), and $-q$ placed at $2(2n + 1)a - x'$. The potential is then

$$\Phi(x, y, z) = \frac{q}{4\pi\epsilon_0}\left[\sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{(x - x' - 4na)^2 + (y - y')^2 + (z - z')^2}} - \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{(x + x' - 2(2n + 1)a)^2 + (y - y')^2 + (z - z')^2}}\right]. \quad (1)$$

You can verify that $\Phi(x = a, y, z) = 0$ by changing variables to $m = -n$ in the second sum; it is then identical to the first sum. Likewise, to show that $\Phi(x = -a, y, z) = 0$ change variables to $m = -(n+1)$ in the second sum, and show that it is equal to the first sum.

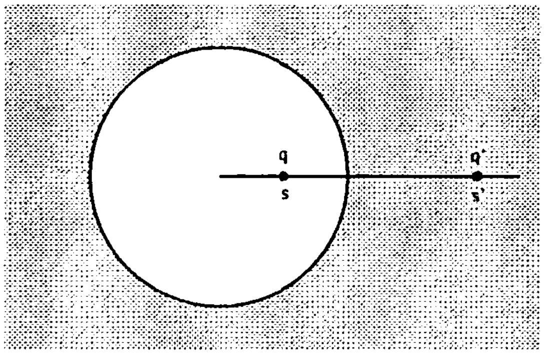
(b) Find the Green's function $G(\mathbf{x}, \mathbf{x}')$ between the plates.

Solution. The Green's function is the solution of Poisson's equation for a point charge of strength $q/4\pi\epsilon_0 = 1$, which satisfies the Dirichlet boundary condition that $G = 0$ on $x = \pm a$. But we've already solved this problem above—the Green's function is just Φ in Eq. (1) with $q/4\pi\epsilon_0 = 1$.

3. *Jackson* 2.2. Using the method of images, discuss the problem of a point charge *inside* a hollow, grounded, conducting sphere of inner radius a. Find

(a) the potential inside the sphere;

Solution. The charge q is inside the sphere of radius a, at a position s from the center (taken to be on the z-axis). The image charge q' is a distance s' from the center, as shown in the figure. This essentially identical to the *exterior* problem;



with $s' = a^2/s$ and $q' = -(a/s)q$, one can show that the potential

$$\begin{aligned}\Phi &= \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+y^2+(z-s)^2}} + \frac{q'}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+y^2+(z-s')^2}} \\ &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+s^2-2sr\cos\theta}} - \frac{(a/s)}{\sqrt{r^2+s'^2-2s'r\cos\theta}}\right] \qquad (2)\end{aligned}$$

is zero on the sphere $x^2 + y^2 + z^2 = a^2$.

(b) the induced surface-charge density;

Solution. We need to calculate the normal component of the electric field; in this case we want the *inwardly* directed normal, so

$$\sigma(\theta) = -\epsilon_0 E_r(r = a, \theta)$$

$$
\begin{aligned}
&= \epsilon_0 \left(\frac{\partial \Phi}{\partial r}\right)_{r=a} \\
&= -\frac{qa}{4\pi}\frac{a^2 - s^2}{a^2}\frac{1}{[s^2 + a^2 - 2sa\cos\theta]^{3/2}}. \qquad (3)
\end{aligned}
$$

By integrating this around the sphere, we find that the total induced charge density is $-q$.

(c) the magnitude and direction of the force acting on q.

Solution. The force can be obtained from the force that the image charge exerts on the source charge,

$$
F_z = -\frac{1}{4\pi\epsilon_0}\frac{qq'}{(s - s')^2} = \frac{1}{4\pi\epsilon_0}\frac{q^2 as}{(s^2 - a^2)^2}. \qquad (4)
$$

(d) Is there any change in the solution if the sphere is kept at a fixed potential V? If the sphere has a total charge Q on its inner and outer surfaces?

Solution. We can imagine adding total charge Q to the conductor in two steps: we first ground the conductor and allow the surface charge density to adjust so that the potential on the surface is zero. We then remove the ground wire and add the charge; since the existing surface charge is already in equilibrium with the point charge, the added charge will distribute itself uniformly over the surface. Such a spherical symmetric charge distribution doesn't contribute to the electric field *inside* the sphere, so the field inside is the same as before. However, we choose the potential to be zero infinitely far from the sphere, the surface of the sphere is now at a potential $V = (Q - q)/(4\pi\epsilon_0 a)$, so the potential inside the sphere is shifted by this constant. Likewise, if the sphere is raised to a potential V, the potential inside the sphere is increased by the constant V, which doesn't effect the electric field inside the sphere.

4. *Jackson* 2.10. A large parallel plate capacitor is made up of two plane conducting sheets with separation D, one of which has a small hemispherical boss of radius a on its inner surface ($D \gg a$). The conductor with the boss is kept at zero potential, and the other conductor is at a potential such that far from the boss the electric field between the plates is E_0.

(a) Calculate the surface-charge densities at an arbitrary point on the plane and on the boss, and sketch their behavior as a function of the distance (or angle).

Solution. The trick here is to realize that the field between the capacitor plates is equivalent to that of a grounded conducting sphere placed in a uniform electric field. The potential is therefore

$$
\Phi(r, \theta) = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta, \qquad (5)
$$

for $0 \leq \theta \leq \pi/2$. We can write this in rectangular coordinates as

$$\Phi(x,y,z) = -E_0 z + \frac{E_0 a^3 z}{(x^2+y^2+z^2)^{3/2}}. \tag{6}$$

The surface charge density σ is obtained from the normal component of the electric field at the conducting surface. On the hemispherical boss, we have

$$\begin{aligned} \sigma(\theta) &= \epsilon_0 E_r(r=a,\theta) \\ &= -\epsilon_0 \left(\frac{\partial \Phi}{\partial r}\right)_{r=a} \\ &= 3\epsilon_0 E_0 \cos\theta. \end{aligned} \tag{7}$$

On the remaining, flat portion of the conductor, we have for the surface charge density

$$\begin{aligned} \sigma(x,y) &= \epsilon_0 E_z(x,y,z=0) \\ &= -\epsilon_0 \left(\frac{\partial \Phi}{\partial z}\right)_{z=0} \\ &= \epsilon_0 E_0 \left[1 - \frac{a^3}{(x^2+y^2)^{3/2}}\right]. \end{aligned} \tag{8}$$

(b) Show that the total charge on the boss has the magnitude $3\pi\epsilon_0 E_0 a^2$.
Solution. Now we integrate $\sigma(\theta)$ over the hemisphere:

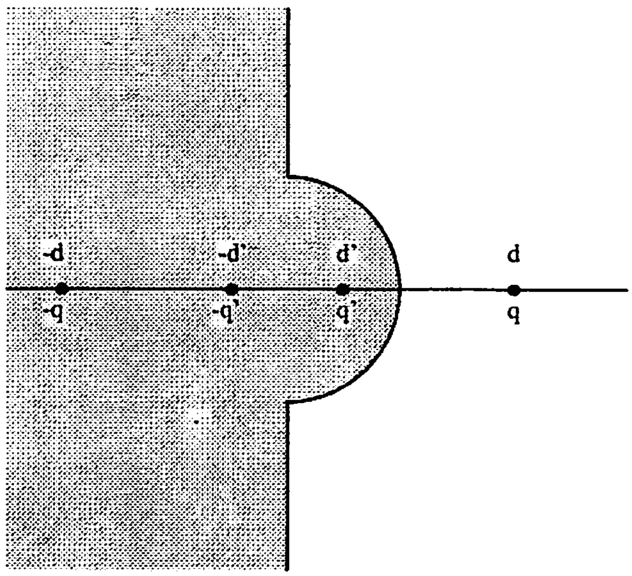
$$\begin{aligned} q_{\text{induced}} &= \int_0^{\pi/2} \sigma(\theta)(2\pi a^2 \sin\theta) d\theta \\ &= 3\pi\epsilon_0 E_0 a^2. \end{aligned} \tag{9}$$

(c) If, instead of the other conducting sheet at a different potential, a point charge q is placed a distance d from its center, show that the charge induced on the boss is

$$q' = -q\left[1 - \frac{d^2 - a^2}{d\sqrt{d^2+a^2}}\right]. \tag{10}$$

Solution. Referring to the figure below, to take care of the boundary condition on the boss, we place an image charge at $d' = a^2/d$ of strength $q' = -q(a/d)$. Then, to make the potential zero on the $z = 0$ plane, we place an image charge of strength $-q'$ at $-d'$ and another of strength $-q$ at $-d$.
The potential on the vacuum side is then

$$\begin{aligned} \Phi(x,y,z) = \frac{q}{4\pi\epsilon_0}\Bigg[&\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}} \\ &- \frac{(d/s)}{\sqrt{x^2+y^2+(z-d')^2}} + \frac{(d/s)}{\sqrt{x^2+y^2+(z+d')^2}}\Bigg]. \end{aligned} \tag{11}$$



In spherical coordinates this becomes

$$
\begin{aligned}
\Phi(r,\theta) &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+d^2-2dr\cos\theta}} - \frac{1}{\sqrt{r^2+d^2+2dr\cos\theta}}\right. \\
&\quad \left. - \frac{(a/d)}{\sqrt{r^2+d'^2-2d'r\cos\theta}} + \frac{(a/d)}{\sqrt{r^2+d'^2+2d'r\cos\theta}}\right]. \qquad (12)
\end{aligned}
$$

Next, calculate the induced surface charge density on the boss:

$$
\begin{aligned}
\sigma(\theta) &= -\epsilon_0\left(\frac{\partial\Phi}{\partial r}\right)_{r=a} \\
&= -\frac{qa}{4\pi}\frac{d^2-a^2}{a^2}\left[\frac{1}{(a^2+d^2-2da\cos\theta)^{3/2}}\right. \\
&\quad \left. - \frac{1}{(a^2+d^2+2da\cos\theta)^{3/2}}\right]. \qquad (13)
\end{aligned}
$$

The induced charge on the boss is

$$
\begin{aligned}
q_{\text{boss}} &= \int_0^{\pi/2} \sigma(\theta)(2\pi a^2\sin\theta\,d\theta) \\
&= -q\left[1 - \frac{d^2-a^2}{d\sqrt{d^2+a^2}}\right]. \qquad (14)
\end{aligned}
$$

I did the integrals using *Maple*, but they could be easily performed by changing variables to $u = \cos\theta$.

5. **Conducting cylinder in a uniform electric field.**

(a) Use the method of images to find the potential outside of an infinite conducting cylinder of radius a placed in a uniform electric field perpendicular to its axis. [Hint: you'll need to use infinite *line* charges rather than *point* charges.]

Solution. Place source line charges $\pm\lambda$ at $x = \mp x_0$, and image line charges $\pm\lambda$ inside the cylinder at $x = \pm a^2/x_0$. Recall that the potential due to an infinite line charge located at (x_0, y_0) is

$$\Phi(x,y) = -\frac{\lambda}{2\pi\epsilon_0}\ln\left[\sqrt{(x-x_0)^2+(y-y_0)^2}/R\right], \tag{15}$$

where R is a scale factor with the dimensions of length. The potential due to the two source line charges is

$$\begin{aligned}
\Phi_{\text{source}}(x,y) &= -\frac{\lambda}{2\pi\epsilon_0}\left\{\ln\left[\sqrt{(x+x_0)^2+(y-y_0)^2}/R\right]\right. \\
&\qquad \left. -\ln\left[\sqrt{(x-x_0)^2+(y-y_0)^2}/R\right]\right\} \\
&= -\frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{(x+x_0)^2+y^2}{(x-x_0)^2+y^2}\right] \\
&= -\frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{1+2(\rho/x_0)\cos\phi+(\rho/x_0)^2}{1-2(\rho/x_0)\cos\phi+(\rho/x_0)^2}\right],
\end{aligned} \tag{16}$$

where in the last line we've introduced polar coordinates (ρ, ϕ) such that $x = \rho\cos\phi$, $y = \rho\sin\phi$. Now take $x_0 \gg \rho$, and expand the logarithm:

$$\Phi_{\text{source}} \approx -\left(\frac{4\lambda}{4\pi\epsilon_0 x_0}\right)\rho\cos\phi. \tag{17}$$

This is the potential for a uniform electric field $\mathbf{E} = E_0\mathbf{e}_x$, with $E_0 = 4\lambda/(4\pi\epsilon_0 x_0)$. The potential for the two image line charges is

$$\begin{aligned}
\Phi_{\text{image}}(x,y) &= -\frac{\lambda}{2\pi\epsilon_0}\left\{\ln\left[\sqrt{(x-a^2/x_0)^2+(y-y_0)^2}/R\right]\right. \\
&\qquad \left. -\ln\left[\sqrt{(x+a^2/x_0)^2+(y-y_0)^2}/R\right]\right\} \\
&= -\frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{(x-a^2/x_0)^2+y^2}{(x+a^2/x_0)^2+y^2}\right] \\
&= -\frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{1-(2a^2/x_0\rho)\cos\phi+(a^4/x_0^2\rho^2)}{1+(2a^2/x_0\rho)\cos\phi+(a^4/x_0^2\rho^2)}\right].
\end{aligned} \tag{18}$$

Now take $(a^2/x_0\rho) \ll 1$, and expand the logarithm, to obtain

$$\begin{aligned} \Phi_{\text{image}} &\approx \left(\frac{4\lambda}{4\pi\epsilon_0 x_0}\right)\frac{a^2\cos\phi}{\rho} \\ &= E_0\frac{a^2\cos\phi}{\rho}. \end{aligned} \tag{19}$$

Adding this to the potential for the sources, we have

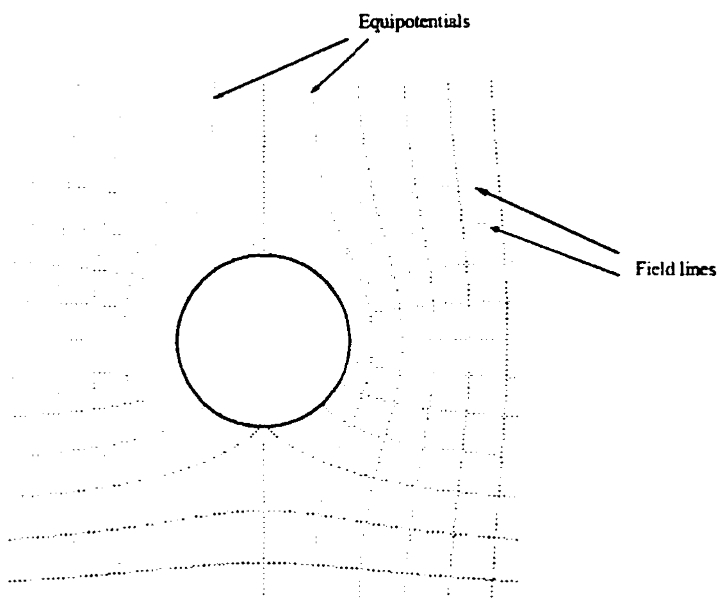
$$\Phi(\rho,\phi) = -E_0\left(\rho - \frac{a^2}{\rho}\right)\cos\phi, \tag{20}$$

which is the sum of a uniform field and a two dimensional point dipole.

(b) Calculate the induced surface charge density on the cylinder.
Solution. The surface charge density is

$$\sigma(\phi) = \epsilon_0 E_\rho(\rho = a, \phi) = 2\epsilon_0 E_0 \cos\phi. \tag{21}$$

(c) Carefully plot the equipotentials and field lines.
Solution. I produced the figure below using the contour plot package in *Maple V*, which produced a .eps file, which was then edited using xfig to make it look nice.



Physics 6346, Electromagnetic Theory I
Fall 2000
Homework 4 Solutions

1. **Fourier series.** Consider the function

$$f(x) = x(\pi - x), \qquad 0 \le x \le \pi. \tag{1}$$

(a) Periodically extend this function as an even function and calculate the Fourier cosine series for the function.

Solution. The Fourier series is

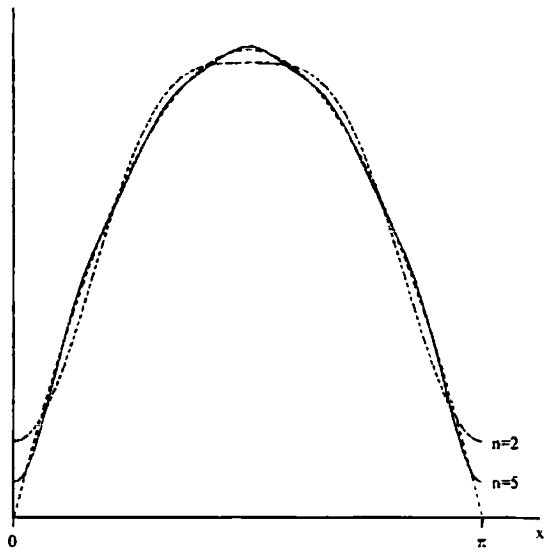
$$f(x) = \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{\cos(2nx)}{n^2}. \tag{2}$$

(b) Plot the Fourier coefficients as a function of n.

Solution. Straightforward—they decrease as n^{-2}.

(c) Plot the first few terms to see how fast the series converges to the function (you may want to use *Maple* or plotting software).

Solution. I've used *Maple V* to plot $f(x)$, along with the two term and five term Fourier series below.



(d) By evaluating $f(x)$ at $\pi/2$, derive a series expression for π^2.

Solution. From Eq. (1), we have $f(\pi/2) = \pi^2/4$; evaluating the series at the same point, and equating the two, we find

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}. \tag{3}$$

2. **Fourier transform.** The Fourier transform of $f(x)$ is

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx, \tag{4}$$

and the inverse transform is

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{ikx} \frac{dk}{2\pi}. \tag{5}$$

(a) Assuming that $f(x)$ and all of its derivatives vanish at $\pm\infty$, show that the Fourier transform of df/dx is $ikF(k)$. Generalize to show that the Fourier transform of $d^n f/dx^n$ is $(ik)^n F(k)$. [Hint: integrate by parts.]
Solution. Using the hint, this is straightforward.

(b) The *convolution* of two functions $f(x)$ and $g(x)$ is defined by

$$h(x) = \int_{-\infty}^{\infty} g(x-y) f(y)\, dy,$$

with $h(x)$ the convolution. Show that

$$H(k) = F(k)G(k),$$

with $H(k)$, $F(k)$, and $G(k)$ the Fourier transforms of $h(x)$, $f(x)$, and $g(x)$, respectively. In words, the Fourier transform of a convolution is the product of the Fourier transforms.

Solution.

$$\begin{aligned} H(k) &= \int_{-\infty}^{\infty} dx\, e^{-ikx} h(x) \\ &= \int_{-\infty}^{\infty} dx\, e^{-ikx} \int_{-\infty}^{\infty} dy\, g(x-y) f(y) dy \\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \left[f(y)e^{-iky}\right]\left[g(x-y)e^{-ik(x-y)}\right] \\ &= \int_{-\infty}^{\infty} dy\, f(y)e^{-iky} \int_{-\infty}^{\infty} dx\, g(x-y)e^{-ik(x-y)} \\ &= F(k)G(k). \end{aligned} \tag{6}$$

3. **Fourier transforms and Green's functions.** Fourier transforms are very useful for solving certain types of differential equations. Consider, for instance, the equation of motion for the damped, forced harmonic oscillator of mass m:

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{f(t)}{m}, \tag{7}$$

where ω_0 is the natural oscillation frequency, γ is a damping constant, and $f(t)$ is some general forcing function ($f(t) = F_0\cos(\omega t)$, for instance). We can find the general steady-state solution of this differential equation as follows:

(a) Defining the Fourier transforms of $x(t)$ and $f(t)$ as

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\,dt, \tag{8}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt, \tag{9}$$

show, using your results from Prob. 2, that

$$X(\omega) = F(\omega)G(\omega). \tag{10}$$

What is $G(\omega)$?

Solution. The Green's function is

$$G(\omega) = \frac{1}{m(-\omega^2 + i\gamma\omega + \omega_0^2)}. \tag{11}$$

(b) Now we need to invert the Fourier transform to obtain $x(t)$. Again using your results from Prob. 2, show that

$$x(t) = \int_{-\infty}^{\infty} g(t-s)f(s)\,ds, \tag{12}$$

where $g(t)$ is the inverse Fourier transform of $G(\omega)$:

$$g(t) = \int_{-\infty}^{\infty} G(\omega)e^{i\omega t}\frac{d\omega}{2\pi}. \tag{13}$$

This gives a particular solution of the differential equation; we can always add a solution of the homogeneous equation (i.e., with $f = 0$). However, the particular solution will generally give us the steady-state behavior. The function $g(t)$ is the *Green's function* for this differential equation.

Solution. Again, a straightforward application of the results from Problem 2.

(c) *Bonus.* Calculate $g(t)$ explicitly using contour integration.

Solution. Using contour integration, one finds for the underdamped case ($\omega_0 > \gamma/2$)

$$g(t) = \frac{1}{m\sqrt{\omega_0^2 - (\gamma/2)^2}} e^{-\gamma t/2} \sin\left[\sqrt{\omega_0^2 - (\gamma/2)^2}t\right]\theta(t), \tag{14}$$

where $\theta(t) = 1$ for $t > 0$ and $\theta(t) = 0$ for $t < 0$. Note that the Green's function is *causal*—there is only a response *after* the force is applied.

In the critically damped case ($\omega_0 = \gamma/2$)

$$g(t) = \frac{t}{m}e^{-\gamma t/2}\theta(t), \tag{15}$$

and in the overdamped case ($\gamma/2 > \omega_0$)

$$g(t) = \frac{1}{m\sqrt{(\gamma/2)^2 - \omega_0^2}} e^{-\gamma t/2} \sinh\left[\sqrt{(\gamma/2)^2 - \omega_0^2}t\right]\theta(t). \tag{16}$$

4. *Jackson* 2.14. A variant of the preceding problem is a long hollow conducting cylinder that is divided into equal quarters, alternate segments being held at potential $+V$ and $-V$.

(a) Solve by means of the series solution (2.71) and show that the potential inside the cylinder is

$$\Phi(\rho,\phi) = \frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{\sin\left[(4n+2)\phi\right]}{2n+1}. \tag{17}$$

Solution. Start by expressing the boundary condition as a Fourier series. We have

$$\Phi(b,\phi) = \begin{cases} V & 0<\phi<\pi/2 \\ -V & \pi/2<\phi<\pi \\ V & \pi<\phi<3\pi/2 \\ -V & 3\pi/2<\phi<2\pi. \end{cases} \tag{18}$$

We can express this as a Fourier series:

$$\Phi(b,\phi) = \frac{4V}{\pi}\left[\sin 2\phi + \frac{\sin 6\phi}{3} + \frac{\sin 10\phi}{5} + \ldots\right]. \tag{19}$$

In the *interior* region $\rho < b$, the general solution of Laplace's equation in polar coordinates is

$$\Phi(\rho,\phi) = A_0 + \sum_{n=1}^{\infty}\rho^n(A_n\cos n\phi + B_n\sin n\phi). \tag{20}$$

Evaluating this on the cylinder, and comparing to the boundary condition in Eq. (19) above, we find that $A_n = 0$ for all n, and

$$B_n = \begin{cases} \frac{4V}{\pi}\frac{2}{b^n n} & n = 2,6,10,\ldots \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Substituting this back into Eq. (20), we obtain Eq. (17).

(b) Sum the series and show that

$$\Phi(\rho,\phi) = \frac{2V}{\pi}\tan^{-1}\left(\frac{2\rho^2b^2\sin 2\phi}{b^4-\rho^4}\right). \tag{22}$$

Solution. The potential can be written as

$$\Phi(\rho,\phi) = \frac{4V}{\pi}\mathrm{Im}\left[\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{\rho^2}{b^2}e^{i2\phi}\right)^{2n+1}\right]. \tag{23}$$

Now

$$\sum_{n=0}^{\infty}\frac{z^{2n+1}}{2n+1} = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right), \tag{24}$$

so that

$$\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{\rho^2}{b^2}e^{i2\phi}\right)^{2n+1} = \frac{1}{2}\ln\left[\frac{1+(\rho^2/b^2)e^{i2\phi}}{1-(\rho^2/b^2)e^{i2\phi}}\right]$$

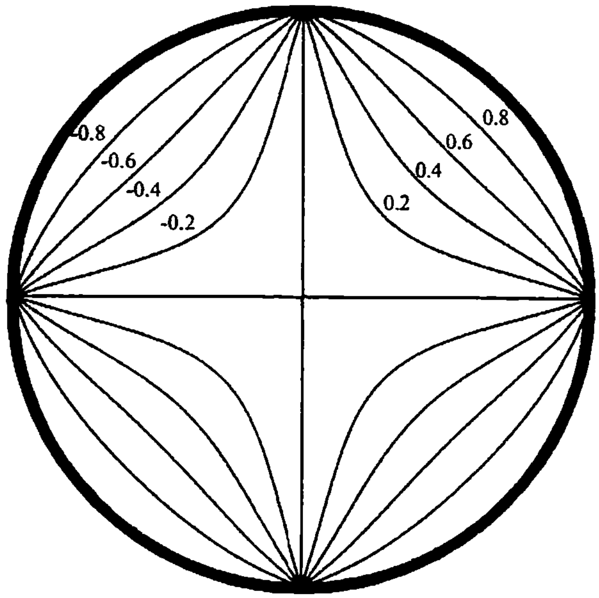
$$= \frac{1}{2}\ln\left\{\frac{\left[(b^4-\rho^4)^2+4b^4\rho^4\sin^2 2\phi\right]^{1/2}}{b^4+\rho^4-2b^2\rho^2\cos 2\phi}\right\}$$

$$+\frac{i}{2}\tan^{-1}\left(\frac{2b^2\rho^2\sin 2\phi}{b^4-\rho^4}\right). \qquad (25)$$

Taking the imaginary part, we obtain Eq. (22).

(c) Sketch the field lines and equipotentials.

Solution. The equipotentials, labeled in units of V, are shown plotted below. The field lines will be orthogonal to the equipotentials.



5. *Jackson* 2.23. A hollow cube has conducting walls defined by six planes $x = 0$, $y = 0$, $z = 0$, and $x = a$, $y = a$, $z = a$. The walls at $z = 0$ and $z = a$ are held at a constant potential V. The other four sides are at zero potential.

   (a) Find the potential $\Phi(x, y, z)$ at any point inside the cube.

   Solution. Let's start with the simpler problem of wall at $z = 0$ being held at zero potential and the wall at $z = a$ held at potential V. Then the solution is

$$\Phi_{\text{top}}(x, y, z) = \sum_{m,n} A_{mn}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh\gamma_{mn}z, \qquad (26)$$

with

$$\gamma_{mn} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2}, \tag{27}$$

and

$$\begin{aligned} A_{mn} &= \frac{4}{a^2 \sinh(\gamma_{mn} a)} \int_0^a dx \int_0^a dy\, V \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \\ &= \frac{16V}{\pi^2 mn \sinh(\gamma_{mn} a)}, \quad m, n \text{ odd}, \end{aligned} \tag{28}$$

with $A_{mn} = 0$ if either m or n is even. Next, solve the problem with the wall at $z = a$ held at zero potential and the wall at $z = 0$ held at potential V. The solution is the same, with $z \to a - z$:

$$\Phi_{\text{bottom}}(x, y, z) = \sum_{m,n \text{ odd}} A_{mn} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \sinh \gamma_{mn}(a - z), \tag{29}$$

with A_{mn} and γ_{mn} the same as before. To find the solution when the potential at $z = 0$ and $z = a$ is V, we simply add these two solutions:

$$\begin{aligned} \Phi(x, y, z) &= \Phi_{\text{top}}(x, y, z) + \Phi_{\text{bottom}}(x, y, z) \\ &= \sum_{m,n \text{ odd}} A_{mn} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \\ &\quad \times \left[\sinh \gamma_{mn} z + \sinh \gamma_{mn}(a - z)\right]. \end{aligned} \tag{30}$$

(b) (optional) Evaluate the potential at the center of the cube numerically, accurate to three significant figures.

Solution. I used *Maple V* to evaluate the series. If I keep 1 term in each sum, then I find that $\Phi(a/2, a/2, a/2) = .3329577152\,V$; keeping 10 terms in each sum gives $\Phi(a/2, a/2, a/2) = .3333333331\,V$. This looks a little suspicious; if we use the results of Problem 2.28, we see that the potential in the center should be the average value over the sides, which is $2V/6 = 0.33\bar{3}\,V$!

(c) Find the surface-charge density on the surface $z = a$.

Solution. The surface charge density $\sigma = \epsilon_0 \mathbf{E} \cdot \mathbf{n}$, where $\mathbf{n}$ is the unit normal directed toward the center of the cube. Therefore, we have

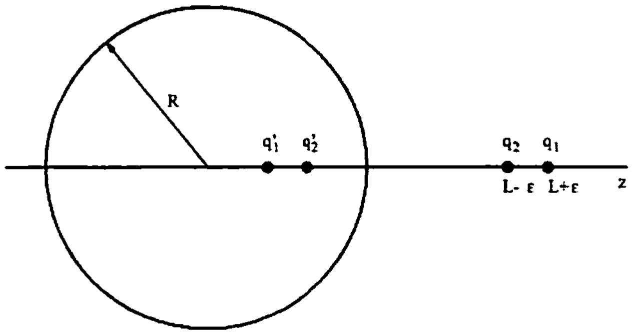
$$\begin{aligned} \sigma(x, y, a) &= \epsilon_0 \left(\frac{\partial \Phi}{\partial z}\right)_{z=a} \\ &= \epsilon_0 \sum_{m,n \text{ odd}} A_{mn} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \gamma_{mn}(\cosh \gamma_{mn} a - 1). \end{aligned} \tag{31}$$

Physics 6346, Electromagnetic Theory 1
Fall 2000
Homework 5 Solutions

1. **Image Dipoles.** A point dipole of dipole moment $\mathbf{p}$ is placed a distance L from the center of a grounded conducting sphere of radius R. For simplicity, assume that the dipole points away from the center of the sphere.

   (a) Use the method of images to find the potential in the region exterior to the sphere. [Hint: consider the dipole to be two point charges, and then take the limit that their separation tends to zero.]

   Solution. The two source charges $q_1 = q$ and $q_2 = -q$ are located at $z = L + \epsilon$ and $z = L - \epsilon$, respectively. Their images are $q_1' = -qR/(L+\epsilon)$ at $z = R^2/(L+\epsilon)$ and $q_2' = qR/(L-\epsilon)$ at $z = R^2/(L-\epsilon)$, as shown in the figure below.



The potential for this arrangement of charges is

$$\begin{aligned}\Phi(x,y,z) &= \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{x^2+y^2+(z-L-\epsilon)^2}} - \frac{1}{\sqrt{x^2+y^2+(z-L+\epsilon)^2}}\right. \\ &\quad - \frac{R/(L+\epsilon)}{\sqrt{x^2+y^2+[z-R^2/(L+\epsilon)]^2}} \\ &\quad \left. + \frac{R/(L-\epsilon)}{\sqrt{x^2+y^2+[z-R^2/(L-\epsilon)]^2}}\right\}. \qquad (1)\end{aligned}$$

We now take the limit $\epsilon \rightarrow 0$, while keeping the dipole moment $p = q(2\epsilon)$ fixed. The result is (I used *Maple* to help with the algebra)

$$\begin{aligned}\Phi(x,y,z) &= \frac{p}{4\pi\epsilon_0}\left\{\frac{z-L}{[x^2+y^2+(z-L)^2]^{3/2}}\right. \\ &\quad \left. + \frac{(R^3/L^3)(z-R^2/L)}{[x^2+y^2+(z-R^2/L)^2]^{3/2}} + \frac{(R/L^2)}{[x^2+y^2+(z-R^2/L)^2]^{1/2}}\right\}. \qquad (2)\end{aligned}$$

The first term is the potential due to the dipole located at $z = L$, pointing in the positive z direction, and the second term is the image dipole with dipole moment $p' = p(R^3/L^3)$ located at $z = R^2/L$, also pointing in the positive z direction. The third term, which is somewhat unexpected, is an image *point charge* of charge pR/L^2 located at $z = R^2/L$.

(b) Find the force that the sphere exerts upon the dipole.

Solution. Returning to the two point charges shown in the figure, you can just add the forces that the two images exert on each source charge. I again used *Maple* to help me with the algebra; the result is

$$F_z = -\frac{p^2}{4\pi\epsilon_0}\frac{2LR(2R^2+L^2)}{(R^2-L^2)^4}, \tag{3}$$

so the dipole is attracted to the sphere. If we take the limit $R \gg L$ (while holding $d = L - R$ fixed), then this becomes

$$F_z = -\frac{1}{4\pi\epsilon_0}\frac{3p^2}{8d^4}, \tag{4}$$

which is the correct result for a point dipole a distance d from a grounded conducting sheet.

(c) Find the induced surface charge density on the sphere.

Solution. The surface charge density is $\sigma(\theta) = \epsilon_0 E_r(r = R, \theta)$, where

$$E_r(R,\theta) = -\left(\frac{\partial\Phi}{\partial r}\right)_{r=R} = \frac{p}{4\pi\epsilon_0}\frac{L^3 - 5LR^2 + (L^2R + 3R^3)\cos\theta}{[R^2 + L^2 - 2RL\cos\theta]^{5/2}}. \tag{5}$$

2. **Two Dimensional Electric Quadrupole** (similar to *Jackson* Problem 2.20). An electric quadrupole focusing field can be constructed by using four symmetrically placed line charges with charge per unit length $\pm\lambda$, as shown in the figure below. We would like to solve this problem using complex variable methods.

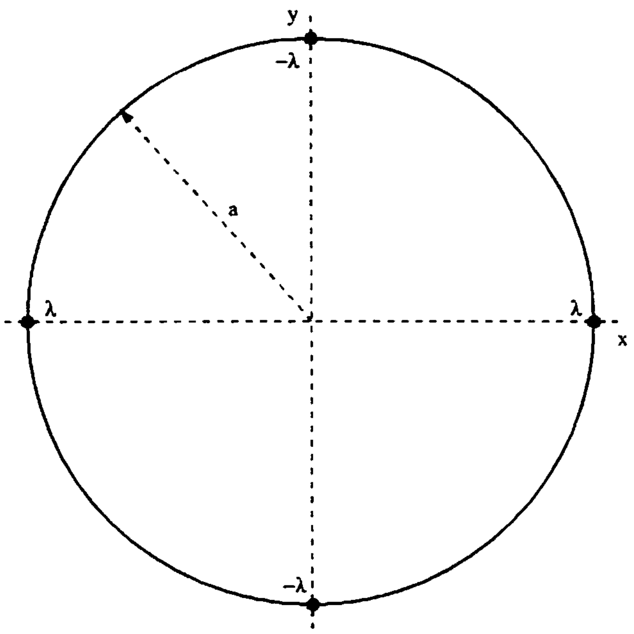
(a) Show that the complex potential $w(z)$ which solves this problem is

$$w(z) = \frac{2\lambda}{4\pi\epsilon_0}\ln\left[\frac{(z-ia)(z+ia)}{(z-a)(z+a)}\right]. \tag{6}$$

Solution. As explained in class, the complex potential for a line charge with charge per unit length λ located at z_0 is $-(\lambda/2\pi\epsilon_0)\ln(z - z_0)$. For the arrangement of charges above, we have

$$w(z) = -\frac{\lambda}{2\pi\epsilon_0}\ln(z-a) - \frac{\lambda}{2\pi\epsilon_0}\ln(z+a) + \frac{\lambda}{2\pi\epsilon_0}\ln(z-ia) + \frac{\lambda}{2\pi\epsilon_0}\ln(z+ia), \tag{7}$$

which is exactly Eq. (6) above.



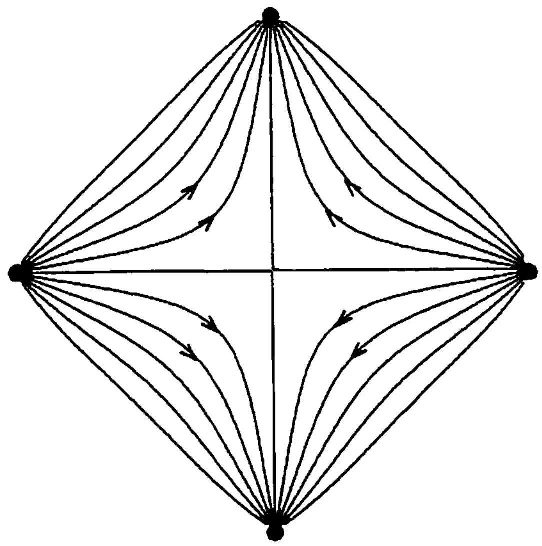
(b) Using $z = x + iy$, separate the complex potential into real and imaginary parts $\Phi(x,y)$ and $\Psi(x,y)$. Use these functions to plot the equipotentials and field lines (make separate plots). You only need to consider the region inside the circle $x^2 + y^2 = a^2$. Your plot of the field lines should look like the figure on p. 91 of *Jackson.*

Solution. Separating the real and imaginary parts, we find

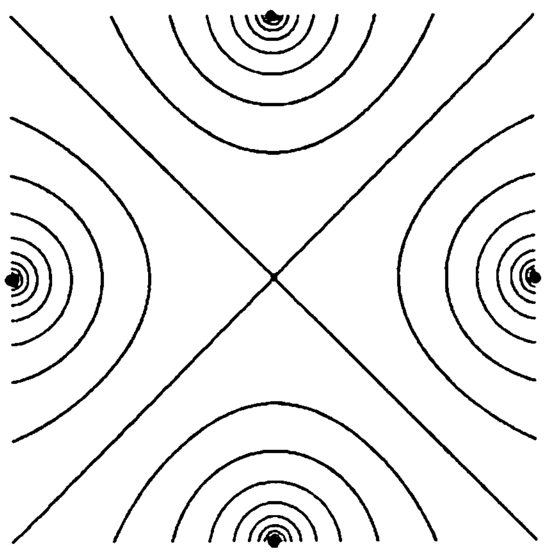
$$\Phi(x,y) = \frac{\lambda}{2\pi\epsilon_0} \ln\left\{ \frac{\sqrt{[(x^2+y^2)^2 - a^4]^2 + 16a^4x^2y^2}}{(x^2-y^2-a^2)^2 + 4x^2y^2} \right\}, \tag{8}$$

$$\Psi(x,y) = \frac{\lambda}{2\pi\epsilon_0} \tan^{-1}\left[\frac{4a^2xy}{a^4 - (x^2+y^2)^2} \right]. \tag{9}$$

The field lines (lines of constant Ψ) are



and the equipotentials (lines of constant Φ) are



3. **Parallel Plate Capacitor.** Consider the analytic function $z(w)$ given by

$$z = ia(w/V_0) + \frac{a}{2\pi}\left[-1 + e^{-2\pi i(w/V_0)}\right], \tag{10}$$

with a and V_0 real constants. Here $w = \Phi + i\Psi$ is the complex potential and $z = x + iy$: in this particular case $w(z)$ is defined implicitly through the equation above.

(a) Show that $\Phi = 0$ on the line $y = 0$, $x > 0$, and that $\Phi = V_0$ on the line $y = a$, $x > 0$. What electrostatic problem does this function solve?

Solution. Let's first separate this into real and imaginary parts

$$x = -\frac{a}{V_0}\Psi - \frac{a}{2\pi}\left[1 - e^{2\pi\Psi/V_0}\cos(2\pi\Phi/V_0)\right], \tag{11}$$

$$y = \frac{a}{V_0}\Phi - \frac{a}{2\pi}e^{2\pi\Psi/V_0}\sin(2\pi\Phi/V_0). \tag{12}$$

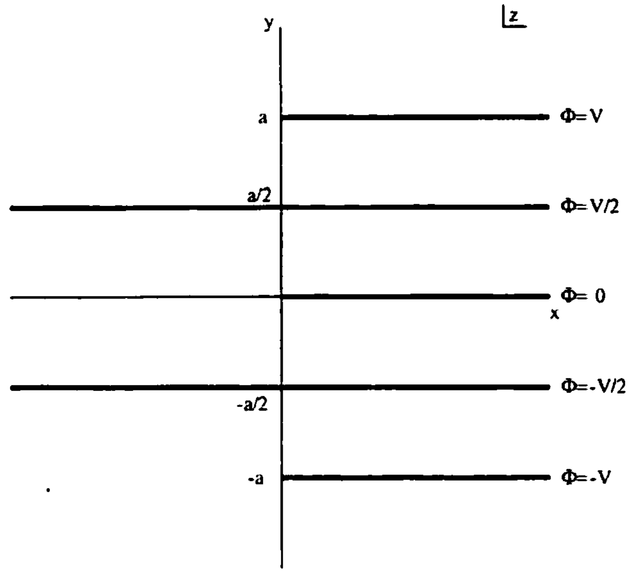
When $\Phi = nV_0$, with n an integer, these equations become

$$x = -\frac{a}{V_0}\Psi - \frac{a}{2\pi}\left[1 - e^{2\pi\Psi/V_0}\right], \quad y = an. \tag{13}$$

As we vary Ψ between $-\infty$ and ∞, x varies between 0 and ∞; therefore, we see that the lines $x \geq 0$, $y = na$ are equipotential surfaces, with potential nV_0. We can also show that when $\Phi = nV_0/2$, we have

$$x = -\frac{a}{V_0}\Psi - \frac{a}{2\pi}\left[1 + e^{2\pi\Psi/V_0}\right], \quad y = (a/2)n. \tag{14}$$

As we vary Ψ from $-\infty$ to ∞ along these equipotentials, x varies between $-\infty$ and ∞; we have identified a second set of equipotentials, with $-\infty < x < \infty$ and $y = (a/2)n$, at potentials $nV_0/2$. These are shown in the figure below.



We can think of this either as a periodic array of semi-infinite conducting plates at $y = na$, held at potentials nV_0, or as an infinite parallel plate capacitor, with plates located at $y = \pm a/2$, with a semi-infinite conductor stuck halfway in between. We'll stick with the former interpretation, and confine attention to values of the potential between 0 and V_0 (essentially ignoring the periodicity).

(b) Calculate the complex electric field $E = E_x - iE_y = -dw/dz$. [Hint: $dw/dz = 1/(dz/dw)$; your field E will depend on w, and therefore on z implicitly through $z(w)$.] Show that near the point $x = y = 0$ the electric field takes the approximate form $E(z) \sim Az^{-1/2}$, and find the constant A.

Solution. The complex electric field is

$$E = -\frac{dw}{dz} = -\frac{1}{(dz/dw)} = \frac{iV_0}{a}\frac{1}{1 - e^{-2\pi i w/V_0}}. \tag{15}$$

Now when z is small, w is also small; expanding $z(w)$ for small w, we obtain

$$z \approx -\pi a \frac{w^2}{V_0^2}. \tag{16}$$

Likewise, expanding Eq. (15), we obtain

$$E \approx \frac{V_0^2}{2\pi a w}. \tag{17}$$

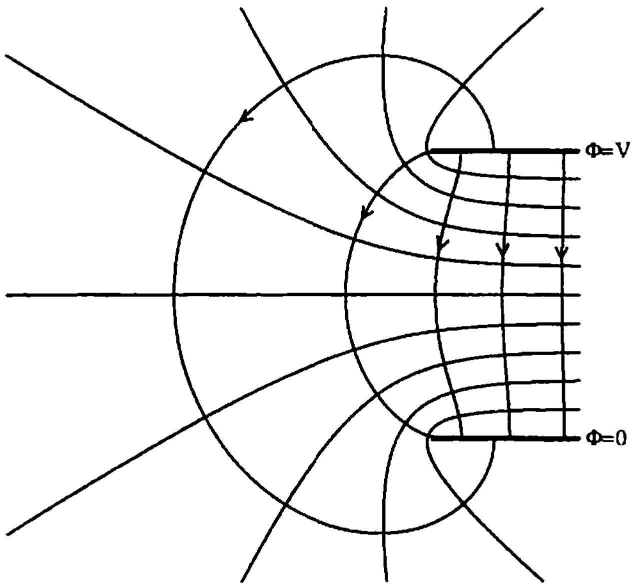
Combining these expressions, we see that at the edge of the conductor

$$E \approx \frac{V_0}{(-4\pi a z)^{1/2}}, \tag{18}$$

so that the field behaves as $z^{-1/2}$, as we would expect near the edge of a conducting sheet.

(c) Sketch the field lines and equipotential surfaces for this problem. As an aid, you might want to consider what an equipotential looks like for Φ very small; i.e., $\Phi = \epsilon V_0$ with ϵ small.

Solution. Here I meant for you to *sketch* these curves—a freehand drawing is sufficient. However, I went ahead and used *Maple* to generate the equipotentials (parallel to the x axis inside the capacitor) and field lines shown in the figure below. This shows how the field "fringes" at the edge of a parallel plate capacitor.



(d) *Bonus.* Calculate the charge density on the conducting surfaces. Use your result to find the induced charge on the conductor (of finite length), and thus the capacitance of this arrangement. Compare your result to the "infinite" parallel plate capacitor and comment on the influence of "fringing" fields on the calculation of the capacitance.

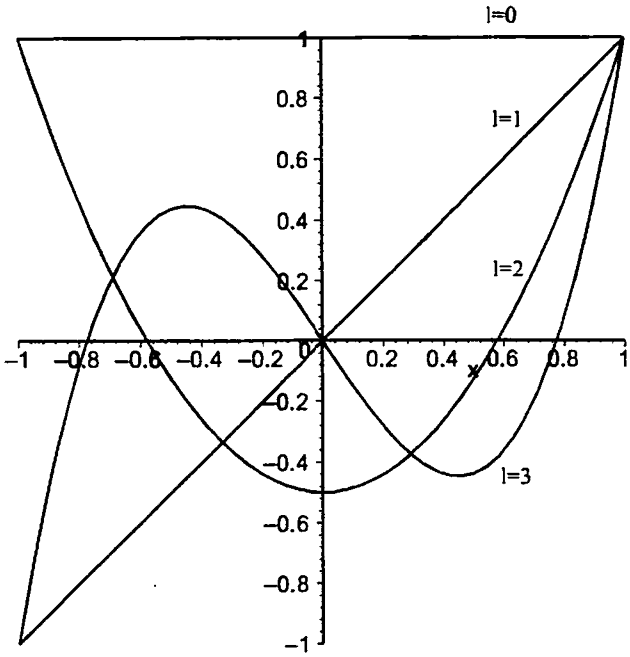
Solution. This one is tough. Come see me if you want some hints.

Physics 6346, Electromagnetic Theory I
Fall 2000
Homework 6 Solutions (Revised 10/20/00)

1. **Legendre polynomials.** In this problem you will explore some of the properties of Legendre polynomials.

   (a) Start by carefully plotting the first four Legendre polynomials (you may use *Maple* if you like).

   Solution. The Legendre polynomials $P_l(x)$ for $l = 0, 1, 2, 3$ are shown plotted below.



   (b) Starting with the power series for $P_l(x)$, derive *Rodrigues' formula*,

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l. \tag{1}$$

   Solution. Recalling that $P_l(1) = 1$, we have

$$\begin{aligned} P_l(x) &= \frac{(2l-1)(2l-3)\cdots 3\cdot 1}{l!} \left[x^l - \frac{l(l-1)}{2(2l-1)} x^{l-2} \right. \\ &\left. + \frac{l(l-1)(l-2)(l-3)}{2\cdot 4(2l-1)(2l-3)} x^{l-4} + \ldots \right]. \end{aligned} \tag{2}$$

Now integrate both sides l times; the right hand side is then (ignoring all of the integration constants)

$$
\begin{aligned}
\text{RHS} &= \frac{(2l-1)(2l-3)\cdots 3\cdot 1}{l!}\left[\frac{l!}{(2l)!}x^{2l} - \frac{l(l-1)}{2(2l-1)}\frac{(l-2)!}{(2l-2)!}x^{2l-2} + \ldots\right] \\
&= \frac{(2l-1)(2l-3)\cdots 3\cdot 1}{(2l)!}\left[x^{2l} - lx^{2l-2} + \frac{l(l-1)}{2!}x^{2l-4} + \ldots\right] \\
&= \frac{(2l-1)(2l-3)\cdots 3\cdot 1}{(2l)!}(x^2-1)^l \\
&= \frac{1}{2^l l!}(x^2-1)^l. \qquad (3)
\end{aligned}
$$

We now differentiate this l times (the integration constants will drop out), and we obtain Rodrigues' formula.

(c) From Rodrigues' formula, derive the normalization integral for the Legendre polynomials,

$$N_l = \int_{-1}^{1} [P_l(x)]^2\, dx = \frac{2}{2l+1}. \qquad (4)$$

Solution.

$$
\begin{aligned}
N_l &= \int_{-1}^{1} P_l^2(x)\, dx \\
&= \int_{-1}^{1} \frac{1}{(2^l l!)^2}\frac{d^l}{dx^l}(x^2-1)^l \frac{d^l}{dx^l}(x^2-1)^l\, dx \\
&= \frac{(-1)^l}{(2^l l!)^2}\int_{-1}^{1}(x^2-1)^l \frac{d^{2l}}{dx^{2l}}(x^2-1)^l\, dx \quad \text{(integrate by parts)} \\
&= \frac{(-1)^l(2l)!}{(2^l l!)^2}\int_{-1}^{1}(x^2-1)^l\, dx \quad \text{(carry out the differentiation)} \\
&= \frac{(2l)!}{(2^l l!)^2}\frac{2^{2l+1}(l!)^2}{(2l+1)!} \quad \text{(do the integral)} \\
&= \frac{2}{2l+1}. \qquad (5)
\end{aligned}
$$

(d) Using Rodrigues' formula, derive the recurrence formula

$$\frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx} - (2l+1)P_l = 0. \qquad (6)$$

Solution. Follows from a straightforward application of Rodrigues' formula:

$$
\begin{aligned}
\frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx} - (2l+1)P_l &= \frac{1}{2^l l!}\frac{d^l}{dx^l}\left[\frac{1}{2(l+1)}\frac{d^2}{dx^2}(x^2-1)^{l+1}\right. \\
&\qquad \left. -2l(x^2-1)^{l-1} - (2l+1)(x^2-1)^l\right] \\
&= 0. \qquad (7)
\end{aligned}
$$

2. *Jackson* 3.1. Two concentric spheres have radii a, b ($b > a$) and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential V. The other hemispheres are at zero potential. Determine the potential in the region $a \leq r \leq b$ as a series in Legendre polynomials. Include terms at least up to $l = 4$. Check your solution against known results in the limiting cases $b \to \infty$, and $a \to 0$.

Solution. The boundary conditions are

$$\Phi(r = b, \theta) = \begin{cases} 0 & 0 < \theta < \pi/2 \quad (0 \leq u \leq 1) \\ V & \pi/2 < \theta < \pi \quad (-1 \leq u \leq 0) \end{cases} \tag{8}$$

$$\Phi(r = a, \theta) = \begin{cases} V & 0 < \theta < \pi/2 \quad (0 \leq u \leq 1) \\ 0 & \pi/2 < \theta < \pi \quad (-1 \leq u \leq 0) \end{cases} \tag{9}$$

where $u = \cos\theta$. We want to express these as an expansion in Legendre polynomials, as

$$\Phi(b, \theta) = \sum_{l=0}^{\infty} \beta_l P_l(\cos\theta), \tag{10}$$

$$\Phi(a, \theta) = \sum_{l=0}^{\infty} \alpha_l P_l(\cos\theta), \tag{11}$$

with the expansion coefficients

$$\begin{aligned} \alpha_l &= \frac{2l+1}{2} \int_0^{\pi} \Phi(a, \theta) P_l(\theta) \sin\theta \, d\theta \\ &= \frac{2l+1}{2} \int_{-1}^{1} \Phi(a, u) P_l(u) \, du \\ &= \frac{V(2l+1)}{2} \int_0^1 P_l(u) \, du. \end{aligned} \tag{12}$$

Likewise, for the β_l's we have

$$\begin{aligned} \beta_l &= \frac{2l+1}{2} \int_0^{\pi} \Phi(b, \theta) P_l(\theta) \sin\theta \, d\theta \\ &= \frac{2l+1}{2} \int_{-1}^{1} \Phi(b, u) P_l(u) \, du \\ &= \frac{V(2l+1)(-1)^l}{2} \int_0^1 P_l(u) \, du, \end{aligned} \tag{13}$$

where in the last line we've used $P_l(-u) = (-1)^l P_l(u)$. We need to calculate the integrals; the results are given in Sec. 3.2 of *Jackson*. We have

$$\int_0^1 P_l(u) \, du = \begin{cases} 1 & l = 0, \\ 0 & l \text{ even}, \neq 0, \\ \left(-\frac{1}{2}\right)^{(l-1)/2} \frac{(l-2)!!}{2[(l+1)/2]!!} & l \text{ odd}, \end{cases} \tag{14}$$

where we take $(-1)!! = 1$.

This problem has azimuthal symmetry, so the general expansion is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta). \tag{15}$$

Applying the boundary condition at $r = a$, we have

$$\Phi(a,\theta) = \sum_{l=0}^{\infty} \left[A_l a^l + B_l a^{-(l+1)}\right] P_l(\cos\theta) = \sum_{l=0}^{\infty} \alpha_l P_l(\cos\theta), \tag{16}$$

so that

$$A_l a^l + B_l a^{-(l+1)} = \alpha_l. \tag{17}$$

Likewise, applying the boundary condition at $r = b$, we obtain

$$A_l b^l + B_l b^{-(l+1)} = \beta_l. \tag{18}$$

Solving for A_l and B_l, we find

$$A_l = \frac{1}{b^{2l+1} - a^{2l+1}}[-a^{l+1}\alpha_l + b^{l+1}\beta_l], \tag{19}$$

$$B_l = \frac{(ab)^{l+1}}{b^{2l+1} - a^{2l+1}}[b^l\alpha_l - a^l\beta_l]. \tag{20}$$

Combining all of these results, we find that $A_0 = V/2$, $B_0 = 0$, and $A_l = B_l = 0$ for l even and not equal to zero. For l odd, we have

$$A_l = -V\frac{a^{l+1} + b^{l+1}}{b^{2l+1} - a^{2l+1}}\left(-\frac{1}{2}\right)^{(l-1)/2} \frac{(2l+1)(l-2)!!}{4\left(\frac{l+1}{2}\right)!!}, \tag{21}$$

$$B_l = V\frac{(ab)^{l+1}(a^l + b^l)}{b^{2l+1} - a^{2l+1}}\left(-\frac{1}{2}\right)^{(l-1)/2} \frac{(2l+1)(l-2)!!}{4\left(\frac{l+1}{2}\right)!!}. \tag{22}$$

Therefore,

$$\begin{aligned}\Phi(r,\theta) \quad = \quad & V\left\{\frac{1}{2} + \frac{3}{4}\frac{1}{b^3 - a^3}\left[-(a^2+b^2)r + a^2b^2(a+b)\frac{1}{r^2}\right]\cos\theta\right. \\ & +\frac{7}{16}\frac{1}{b^7 - a^7}\left[(a^4+b^4)r^3 - a^4b^4(a^3+b^3)\frac{1}{r^4}\right]\frac{1}{2}(5\cos^3\theta - 3\cos\theta) \\ & \left. + \ldots\right\}. \end{aligned} \tag{23}$$

For $b \to \infty$, we have

$$\Phi(r,\theta) = V\left\{\frac{1}{2} + \frac{3}{4}\frac{a^2}{r^2}\cos\theta - \frac{7}{16}\frac{a^4}{r^4}\frac{1}{2}(5\cos^3\theta - 3\cos\theta) + \ldots\right\}. \tag{24}$$

which agrees with Eq. (2.27) in *Jackson* (let $V \to V/2$ there and then superimpose constant $V/2$ everywhere so that the boundary conditions match). In the limit $a \to 0$, we have

$$\Phi(r,\theta) = V\left\{\frac{1}{2} - \frac{3}{4}\frac{r}{b}\cos\theta + \frac{7}{16}\frac{r^3}{b^3}\frac{1}{2}(5\cos^3\theta - 3\cos\theta) + \ldots\right\}, \tag{25}$$

which agrees with *Jackson* 3.36 (let $V \to -V/2$ there and then superimpose constant $V/2$ everywhere).

3. **Charged disc.** A thin disc of radius R is located in the $x-y$ plane with its center at the origin. It has a charge Q which is uniformly distributed on its top and bottom surfaces.

   (a) Find the potential along the z axis for this charge distribution.

   Solution. We do this by considering concentric rings of radius ρ, area $2\pi\rho\, d\rho$, and charge $(Q/\pi R^2)2\pi\rho\, d\rho$; the distance from a point on the ring to a point on the z-axis is $\sqrt{\rho^2+z^2}$. The potential is then

$$\begin{aligned}\Phi(z) &= \frac{1}{4\pi\epsilon_0}\int_0^R \frac{Q}{\pi R^2}\frac{2\pi\rho\, d\rho}{\sqrt{\rho^2+z^2}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{2Q}{R^2}\left[\sqrt{R^2+z^2} - z\right]. \end{aligned}\tag{26}$$

   (b) Find the potential for $r < R$ and $r > R$ as an expansion in Legendre polynomials.

   Solution. Let's first consider $r > R$. Take the potential in Eq. (26) and expand for small R/z:

$$\begin{aligned}\Phi(z) &= \frac{1}{4\pi\epsilon_0}\frac{2Q}{R}\frac{z}{R}\left[\sqrt{1+(R/z)^2} - 1\right] \\ &= \frac{1}{4\pi\epsilon_0}\frac{2Q}{R}\frac{z}{R}\left[\frac{1}{2}\left(\frac{R}{z}\right)^2 - \frac{1}{2^2 2!}\left(\frac{R}{z}\right)^4 + \frac{3\cdot 1}{2^3 3!}\left(\frac{R}{z}\right)^6 + \ldots\right]. \end{aligned}\tag{27}$$

   If we compare this to the general expansion in Eq. (15), then we see that for $r > R$ the potential is

$$\begin{aligned}\Phi(r,\theta) &= \frac{1}{4\pi\epsilon_0}\frac{Q}{R}\left[\frac{R}{r} - \frac{1}{2\cdot 2!}\left(\frac{R}{r}\right)^3 P_2(\cos\theta) + \frac{3\cdot 1}{2^2 3!}\left(\frac{R}{r}\right)^5 P_4(\cos\theta) + \ldots\right] \\ &= \frac{Q}{4\pi\epsilon_0 R}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2n-3)!!}{2^{n-1}n!}\left(\frac{R}{r}\right)^{2n-1} P_{2n-2}(\cos\theta). \end{aligned}\tag{28}$$

   We see that the leading term in the expansion is $Q/(4\pi\epsilon_0 r)$, which is the potential of a point charge Q centered at the origin, as expected.

We can perform an analogous expansion for $r < R$, with the result

$$\begin{aligned}\Phi(r,\theta) &= \frac{1}{4\pi\epsilon_0}\frac{2Q}{R}\left[1-\left(\frac{r}{R}\right)P_1(\cos\theta)+\frac{1}{2}\left(\frac{r}{R}\right)^2P_2(\cos\theta)\right. \\ &\left. -\frac{1}{2^2 2!}\left(\frac{r}{R}\right)^4P_4(\cos\theta)+\frac{3\cdot 1}{2^3 3!}\left(\frac{r}{R}\right)^6P_6(\cos\theta)+\dots\right]. \quad (29)\end{aligned}$$

4. *Jackson* 3.6. Two point charges q and $-q$ are located on the z axis at $z = a$ and $z = -a$, respectively.

   (a) Find the electrostatic potential as an expansion in spherical harmonics and powers of r for both $r > a$ and $r < a$.

   Solution. Start with the potential along the z-axis, which is

$$\Phi(z) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|z-a|}-\frac{1}{|z+a|}\right). \quad (30)$$

For $z > a$,

$$\begin{aligned}\Phi(z) &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{z-a}-\frac{1}{z+a}\right) \\ &= \frac{2qa}{4\pi\epsilon_0 z^2}\frac{1}{1-(a/z)^2} \\ &= \frac{2qa}{4\pi\epsilon_0 z^2}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^{2n}. \quad (31)\end{aligned}$$

This problem has azimuthal symmetry, so the general expansion for the potential is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right]P_l(\cos\theta). \quad (32)$$

On the positive z-axis, $\theta = 0$, and since $P_l(1) = 1$, we have

$$\Phi(r=z,\theta=0) = \sum_{l=0}^{\infty}\left[A_l z^l + B_l z^{-(l+1)}\right]. \quad (33)$$

Comparing Eqs. (31) and (33), we see that $A_l = 0$, and $B_l = 2qa^l/4\pi\epsilon_0$ (for l odd). Therefore, for $r > a$ we have

$$\Phi(r,\theta) = \frac{2q}{4\pi\epsilon_0 r}\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^{2n+1}P_{2n+1}(\cos\theta). \quad (34)$$

We can do the same calculation for $r < a$; simply swap r and a. The two expressions can be combined as

$$\Phi(r,\theta) = \frac{2q}{4\pi\epsilon_0 r_>}\sum_{n=0}^{\infty}\left(\frac{r_<}{r_>}\right)^{2n+1}P_{2n+1}(\cos\theta), \quad (35)$$

where $r_>$ ($r_<$) is the larger (smaller) of r and a.

(b) Keeping the product $qa \equiv p/2$ constant, take the limit of $a \to 0$ and find the potential for $r = 0$. This is by definition a dipole along the z axis and its potential.

Solution. In this limit only the $n = 0$ term in the sum in Eq. (34) survives, so we have

$$\Phi(r, \theta) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}. \tag{36}$$

(c) (Optional.) Suppose now that the dipole of part (b) is surrounded by a grounded spherical shell of radius b concentric with the origin. By linear superposition find the potential everywhere inside the shell.

Solution. In the limit of a point dipole we see that only the $l = 1$ Legendre polynomial enters. In order to make the potential vanish on $r = b$, we'll need another $l = 1$ term; since we're only concerned with the *interior* problem with $r < b$, the term of the for $A_1 r P_1(\cos\theta)$ is acceptable. Choosing $A_1 = -p/(4\pi\epsilon_0 b^3)$, we have

$$\Phi(r, \theta) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}\left(1 - \frac{r^3}{b^3}\right). \tag{37}$$

This corresponds to the sum of the potentials due to a point dipole and a uniform electric field.

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Homework 7 Solutions (Revised 10/27/00)

1. **Spherical harmonics.** A non-conducting spherical shell of radius a carries a charge density such that the electrical potential at the surface of the shell is $\Phi = V_0 x/a$.

   (a) Express the potential on the surface of the sphere in terms of spherical harmonics $Y_l^m(\theta, \phi)$.
   Solution.

$$\Phi(a,\theta,\phi) = V_0(x/a) \tag{1}$$
$$= V_0 \sin\theta\cos\phi \tag{2}$$
$$= V_0\sqrt{\frac{2\pi}{3}}\left[-Y_1^1(\theta,\phi) + Y_1^{-1}(\theta,\phi)\right]. \tag{3}$$

   (b) Find the potential at all points inside and outside the sphere, expressing your result in terms of spherical harmonics.
   Solution. From the form of the potential at the surface, we see that only the $l = 1$, $m = \pm 1$ spherical harmonics will enter. Therefore, we have inside the sphere

$$\Phi_{\text{in}}(r,\theta,\phi) = V_0\left(\frac{r}{a}\right)\sqrt{\frac{2\pi}{3}}\left[-Y_1^1(\theta,\phi) + Y_1^{-1}(\theta,\phi)\right]$$
$$= V_0\left(\frac{r}{a}\right)\sin\theta\cos\phi, \tag{4}$$

   and outside of the sphere

$$\Phi_{\text{out}}(r,\theta,\phi) = V_0\left(\frac{a}{r}\right)^2\sqrt{\frac{2\pi}{3}}\left[-Y_1^1(\theta,\phi) + Y_1^{-1}(\theta,\phi)\right] \tag{5}$$
$$= V_0\left(\frac{a}{r}\right)^2\sin\theta\cos\phi. \tag{6}$$

   (c) Calculate the components of the electric field outside of the shell in spherical coordinates.
   Solution.

$$E_r = -\frac{\partial\Phi_{\text{out}}}{\partial r} = 2V_0\frac{a^2}{r^3}\sin\theta\cos\phi, \tag{7}$$

$$E_\theta = -\frac{1}{r}\frac{\partial\Phi_{\text{out}}}{\partial\theta} = -V_0\frac{a^2}{r^3}\cos\theta\cos\phi. \tag{8}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial\Phi_{\text{out}}}{\partial\theta} = V_0\frac{a^2}{r^3}\sin\phi. \tag{9}$$

(d) Now suppose that the shell is surrounded by and concentric with a grounded spherical conductor of radius $b > a$. Assuming that the potential on the inner sphere is the same as in part (a), what is the potential in the region between the two spheres?

Solution. In the region between the spheres the potential can be expanded in spherical harmonics,

$$\Phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm} r^l + B_{lm} r^{-(l+1)} \right] Y_l^m(\theta, \phi). \tag{10}$$

To match the boundary condition on the inner sphere we only need the $l = 1$, $m = \pm 1$ terms. Therefore on $r = a$ we have

$$\begin{aligned} \Phi(a, \theta, \phi) &= V_0 \sqrt{\frac{2\pi}{3}} \left[-Y_1^1 + Y_1^{-1} \right] & (11) \\ &= (A_{11} a + B_{11} a^{-2}) Y_1^1 + (A_{1-1} a + B_{1-1} a^{-2}) Y_1^{-1}. & (12) \end{aligned}$$

At $r = b$, we have

$$\begin{aligned} \Phi(b, \theta, \phi) &= 0 & (13) \\ &= (A_{11} b + B_{11} b^{-2}) Y_1^1 + (A_{1-1} b + B_{1-1} b^{-2}) Y_1^{-1}. & (14) \end{aligned}$$

Matching coefficients of the Y_l^m's, we we obtain four equations and four unknowns; solving, we obtain

$$\begin{aligned} A_{11} &= V_0 \sqrt{\frac{2\pi}{3}} \frac{a^2}{b^3 - a^3} & (15) \\ B_{11} &= -V_0 \sqrt{\frac{2\pi}{3}} \frac{a^2 b^3}{b^3 - a^3} & (16) \\ A_{1,-1} &= -V_0 \sqrt{\frac{2\pi}{3}} \frac{a^2}{b^3 - a^3} & (17) \\ B_{1,-1} &= V_0 \sqrt{\frac{2\pi}{3}} \frac{a^2 b^3}{b^3 - a^3}. & (18) \end{aligned}$$

2. *Jackson* 3.9. A hollow right circular cylinder of radius b has its axis coincident with the z-axis and its ends at $z = 0$ and $z = L$. The potential on the end faces is zero, while the potential on the cylindrical surface is given as $V(\phi, z)$. Using the appropriate separation of variables in cylindrical coordinates, find a series solution for the potential anywhere inside the cylinder.

Solution. Assume a separable solution $\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$, and substitute into Laplace's equation. The equation for Z is

$$\frac{d^2 Z}{dz^2} + \kappa^2 Z = 0. \tag{19}$$

where κ^2 is the separation constant. The solution is

$$Z(z) = A\cos\kappa z + B\sin\kappa z, \tag{20}$$

with A and B constants. The boundary conditions are $Z(0) = Z(L) = 0$, so $A = 0$ and $\kappa = \kappa_n = n\pi/L$, with $n = 1, 2, \ldots$.

For $Q(\phi)$ we have

$$\frac{d^2Q}{d\phi^2} + \nu^2 Q = 0. \tag{21}$$

with ν^2 a second separation constant. The solution is

$$Q(\phi) = e^{\pm i\nu\phi}. \tag{22}$$

Since we are allowing for the full range of the azimuthal angle ϕ, we have $Q(\phi + 2\pi) = Q(\phi)$, so that $\nu = m$, with m an integer.

The radial function $R(\rho)$ is the solution of

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \left(\kappa^2 + \frac{m^2}{\rho^2}\right) R = 0. \tag{23}$$

Changing variables to $\xi = \kappa\rho$, we find

$$\frac{d^2R}{d\xi^2} + \frac{1}{\xi}\frac{dR}{d\xi} - \left(1 + \frac{m^2}{\xi^2}\right) R = 0. \tag{24}$$

The solutions are the Bessel functions of imaginary argument:

$$R(\xi) = C_m K_m(\xi) + D_m I_m(\xi), \tag{25}$$

with C_m and D_m constants. Since we want the field *inside* the cylinder, $C_m = 0$ (since $K_m(\xi)$ diverges for small ξ).

Pulling all of this together, we have

$$\Phi(\rho, \phi, z) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} C_{mn} I_m(n\pi\rho/L) e^{im\phi} \sin(n\pi z/L). \tag{26}$$

On the surface of the cylinder the potential is $V(\phi, z)$:

$$\Phi(b, \phi, z) = V(\phi, z) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} C_{mn} I_m(n\pi b/L) e^{im\phi} \sin(n\pi z/L). \tag{27}$$

To determine the expansion coefficients C_{mn}, we multiply both sides by $e^{-im'\phi}\sin(n'\pi z/L)$, and integrate ϕ from 0 to 2π and z from 0 to L, with the result that

$$C_{mn} = \frac{1}{\pi L\, I_m(n\pi b/L)} \int_0^{2\pi} d\phi \int_0^L dz\, e^{-im\phi} \sin(n\pi z/L)\, V(\phi, z). \tag{28}$$

This completes the solution—any further calculation requires that we specify $V(\phi, z)$.

3. *Jackson* 3.10. For the cylinder in Problem 3.9 the cylindrical surface is made of two equal half-cylinders, one at potential V and the other at potential $-V$, so that

$$V(\phi, z) = \begin{cases} V & \text{for} \quad -\pi/2 < \phi < \pi/2 \\ -V & \text{for} \quad \pi/2 < \phi < 3\pi/2 \end{cases} \tag{29}$$

(a) Find the potential inside the cylinder.

Solution. We need to substitute Eq. (29) into (28) and do the integrals: the result is

$$C_{mn} = V \frac{8 \sin(m\pi/2)}{\pi^2 mn I_m(n\pi b/L)}, \tag{30}$$

for m and n odd; $C_{mn} = 0$ if either m or n is even.

(b) Assuming that $L \gg b$, consider the potential at $z = L/2$ as a function of ρ and ϕ and compare it with two-dimensional Problem 2.13.

Solution. First, for $L \gg b$, we have for $I_m(n\pi b/L)$ the approximate behavior

$$I_m(n\pi b/L) \approx \frac{1}{m!} \left(\frac{n\pi b}{2L} \right)^m . \tag{31}$$

In the sum there are also terms involving $I_m(n\pi\rho/L)$; however, $\rho < b \ll L$, so we can also expand these terms using

$$I_m(n\pi\rho/L) \approx \frac{1}{m!} \left(\frac{n\pi\rho}{2L} \right)^m . \tag{32}$$

Therefore the potential is approximately

$$\Phi(\rho, z) = \frac{16V}{\pi^2} \sum_{n=1,3,5,\ldots} \sum_{m=1,3,5,\ldots} \frac{\sin(m\pi/2)}{mn} \left(\frac{\rho}{b} \right)^m \cos m\phi \sin(n\pi z/L). \tag{33}$$

Now set $z = L/2$:

$$\Phi(\rho, z = L/2) = \frac{16V}{\pi^2} \sum_{n=1,3,5,\ldots} \sum_{m=1,3,5,\ldots} \frac{\sin(m\pi/2) \sin(n\pi/2)}{mn} \left(\frac{\rho}{b} \right)^m \cos m\phi. \tag{34}$$

We can perform the sum on n:

$$\begin{aligned} \sum_{n=1,3,5,\ldots} \frac{\sin(n\pi/2)}{n} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \\ &= \frac{\pi}{4}, \end{aligned} \tag{35}$$

so that

$$\Phi(\rho, z = L/2) = \frac{4V}{\pi} \sum_{m=1,3,5,\ldots} \frac{\sin(m\pi/2)}{m} \left(\frac{\rho}{b} \right)^m \cos m\phi. \tag{36}$$

This is equal to the result in Problem (2.13) after setting $V_1 = V$ and $V_2 = -V$ (show this!).

4. Similar to *Jackson* 3.20. A point charge q is placed at a position $z = z_0$ between two infinite parallel conducting plates which are at $z = 0$ and $z = L$. The plates are both grounded.

   (a) Use separation of variables in cylindrical coordinates to show that the potential between the plates is

$$\Phi(\rho, z) = \frac{q}{\pi\epsilon_0 L}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi z_0}{L}\right)\sin\left(\frac{n\pi z}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right). \tag{37}$$

Solution. Separate variables as usual. In the z-direction we have functions of the form $\sin(n\pi z/L)$. The problem has azimuthal symmetry, so that $Q(\phi) = 1$ (i.e., $m = 0$). In the radial variable, $R(\rho) = AK_0(n\pi\rho/L) + BI_0(n\pi\rho/L)$, but since $\Phi \to 0$ as $\rho \to \infty$, we must set $B = 0$. Therefore, the expansion is

$$\Phi(\rho, z) = \sum_{n=1}^{\infty} A_n K_0\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right). \tag{38}$$

How do we determine the expansion coefficients A_n? Note that we are actually solving Poisson's equation,

$$\nabla^2\Phi = -\frac{\rho(\mathbf{x})}{\epsilon_0}, \tag{39}$$

where the charge density is

$$\rho(\mathbf{x}) = q\frac{\delta(\rho)}{2\pi\rho}\delta(z - z_0). \tag{40}$$

Using the completeness of the functions $\sin(n\pi z/L)$, we can represent the delta function as

$$\delta(z - z_0) = \frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z_0}{L}\right)\sin\left(\frac{n\pi z}{L}\right). \tag{41}$$

Next, calculate $\nabla^2\Phi$:

$$\begin{aligned}\nabla^2\Phi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{\partial^2\Phi}{\partial z^2} \\ &= \sum_{n=1}^{\infty} A_n\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial K_0(n\pi\rho/L)}{\partial\rho}\right]\right. \\ &\qquad \left. - \left(\frac{n\pi}{L}\right)^2 K_0(n\pi\rho/L)\right\}\sin\left(\frac{n\pi z}{L}\right).\end{aligned} \tag{42}$$

Also,

$$-\frac{\rho(\mathbf{x})}{\epsilon_0} = -\frac{q}{\pi\epsilon_0 L}\frac{\delta(\rho)}{\rho}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z_0}{L}\right)\sin\left(\frac{n\pi z}{L}\right). \tag{43}$$

Comparing these two expressions, we see that if we set

$$A_n = (q/\pi\epsilon_0 L)\sin(n\pi z_0/L)a_n, \tag{44}$$

then

$$a_n\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial K_0(n\pi\rho/L)}{\partial\rho}\right]-\left(\frac{n\pi}{L}\right)^2 K_0(n\pi\rho/L)\right\} = -\frac{\delta(\rho)}{\rho}. \tag{45}$$

Multiplying both sides by ρ, and integrating, we find that

$$a_n \lim_{\rho\to 0}\rho\frac{\partial K_0(n\pi\rho/L)}{\partial\rho} = a_n = 1, \tag{46}$$

so that we obtain Eq. (37).

(b) Calculate the induced surface charge density on both the upper and lower plates.

Solution. The charge densities are

$$\begin{aligned}\sigma_L(\rho) &= -\epsilon_0\left(-\frac{\partial\Phi}{\partial z}\right)_{z=L}\\ &= \frac{q}{\pi L}\sum_{n=1}^{\infty}(n\pi/L)K_0(n\pi\rho/L)\sin(n\pi z_0/L)(-1)^n,\end{aligned} \tag{47}$$

$$\begin{aligned}\sigma_0(\rho) &= -\epsilon_0\left(\frac{\partial\Phi}{\partial z}\right)_{z=0}\\ &= -\frac{q}{\pi L}\sum_{n=1}^{\infty}(n\pi/L)K_0(n\pi\rho/L)\sin(n\pi z_0/L).\end{aligned} \tag{48}$$

(c) By integrating over the area of the plates, and summing the resulting Fourier series, find the total induced charge on each plate.

Solution. To calculate the total induced charge, integrate over the area of each plate:

$$\begin{aligned}q_L &= \int_0^\infty \sigma(\rho)\,2\pi\rho\,d\rho\\ &= \frac{2q}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi z_0}{L}\right),\end{aligned} \tag{49}$$

$$\begin{aligned}q_0 &= \int_0^\infty \sigma(\rho)\,2\pi\rho\,d\rho\\ &= -\frac{2q}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi z_0}{L}\right).\end{aligned} \tag{50}$$

The sums can be calculated as follows:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\left(\frac{n\pi z_0}{L}\right) &= \operatorname{Im}\left\{\sum_{n=1}^{\infty} \frac{\left(-e^{i\pi z_0/L}\right)^n}{n}\right\} \\
&= \operatorname{Im}\left\{-\ln\left(1+e^{i\pi z_0/L}\right)\right\} \\
&= -\frac{\pi z_0}{2L}.
\end{aligned} \tag{51}$$

Using the same trick,

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi z_0}{L}\right) = \frac{\pi}{2}(1 - z_0/L). \tag{52}$$

The result is that $q_L = -q(z_0/L)$, $q_0 = -q(1 - z_0/L)$. Note that $q_0 + q_L = -q$.

(d) Fix z_0 and let $L \to \infty$; what is the total induced charge on the lower plate? Is it what you expect?

Solution. The total induced charge on the lower plate will be $-q$, which is what we would expect from the method of images applied to the infinite grounded conducting plane.

Physics 6346, Electromagnetic Theory I
Fall 2000
Homework 8 Solutions (Revised 11/27/00)

1. *Jackson* 4.5. A localized charge density is placed in an external electrostatic field described by a potential $\Phi^{(0)}(x, y, z)$. The external potential varies slowly in space over the region where the charge density is different from zero.

   (a) From first principles calculate the total *force* acting on the charge distribution as an expansion in multipole moments times derivatives of the electric field, up to and including the quadrupole moments. Show that the force is

$$\mathbf{F} = q\mathbf{E}^{(0)} + \{\nabla[\mathbf{p}\cdot\mathbf{E}^{(0)}(\mathbf{x})]\}_0 + \left\{\nabla\left[\frac{1}{6}\sum_{j,k} Q_{jk}\frac{\partial E_j^{(0)}}{\partial x_k}(\mathbf{x})\right]\right\}_0 + \ldots. \quad (1)$$

   Compare this to the expansion (4.24) of the *energy* W. Note that (4.24) is a number—it is not a function of $\mathbf{x}$ that can be differentiated! What is the connection to $\mathbf{F}$?

   Solution. The force on a charge distribution with charge density $\rho(\mathbf{x})$ placed in an external field $\mathbf{E}^{(0)}$ is

$$\mathbf{F} = \int \rho(\mathbf{x})\mathbf{E}^{(0)}(\mathbf{x})\, d^3x, \quad (2)$$

   or in component form,

$$F_i = \int \rho(\mathbf{x})E_i^{(0)}(\mathbf{x})\, d^3x. \quad (3)$$

   Assume that the field varies slowly in the vicinity of $\mathbf{x} = 0$, and expand the field:

$$E_i^{(0)}(\mathbf{x}) = E_i^{(0)}(0) + \sum_j x_j\left(\frac{\partial E_i^{(0)}(\mathbf{y})}{\partial y_j}\right)_{\mathbf{y}=0} + \frac{1}{2}\sum_{j,k} x_j x_k\left(\frac{\partial^2 E_i^{(0)}(\mathbf{y})}{\partial y_j \partial y_k}\right)_{\mathbf{y}=0} + \ldots. \quad (4)$$

   Now multiply by $\rho(\mathbf{x})$ and integrate:

$$\begin{aligned} F_i &= E_i^{(0)}(0)\int \rho(\mathbf{x})d^3x + \sum_j\left(\frac{\partial E_i^{(0)}(\mathbf{y})}{\partial y_j}\right)_{\mathbf{y}=0}\int x_j\rho(\mathbf{x})d^3x \\ &\quad + \frac{1}{2}\sum_{j,k}\left(\frac{\partial^2 E_i^{(0)}(\mathbf{y})}{\partial y_j \partial y_k}\right)_{\mathbf{y}=0}\int x_j x_k \rho(\mathbf{x})d^3x + \ldots. \end{aligned} \quad (5)$$

   The first term is

$$E_i^{(0)}(0)\int \rho(\mathbf{x})d^3x = qE_i^{(0)}(0), \quad (6)$$

with q the total charge. The second term is

$$\sum_j \left(\frac{\partial E_i^{(0)}(\mathbf{y})}{\partial y_j}\right)_{\mathbf{y}=0} \int x_j \rho(\mathbf{x}) d^3x = \sum_j p_j \left(\frac{\partial E_i^{(0)}(\mathbf{y})}{\partial y_j}\right)_{\mathbf{y}=0}$$
$$= \left[\mathbf{p}\cdot\nabla_{\mathbf{y}} E_i^{(0)}(\mathbf{y})\right]_{\mathbf{y}=0}. \qquad (7)$$

Returning to vector notation, this term is

$$\left[(\mathbf{p}\cdot\nabla_{\mathbf{y}})\mathbf{E}^{(0)}(\mathbf{y})\right]_{\mathbf{y}=0} = \left[\nabla(\mathbf{p}\cdot\mathbf{E}^{(0)}))\right]_{\mathbf{x}=0}, \qquad (8)$$

where we've used a vector identity and the fact that $\nabla\times\mathbf{E}^{(0)} = 0$ (remember that $\mathbf{p}$ doesn't depend upon $\mathbf{x}$). The third term is

$$\frac{1}{2}\sum_{j,k}\left(\frac{\partial^2 E_i^{(0)}(\mathbf{y})}{\partial y_j \partial y_k}\right)_{\mathbf{y}=0} \int x_j x_k \rho(\mathbf{x}) d^3x = \frac{\partial}{\partial y_i}\frac{1}{2}\sum_{j,k}\left(\frac{\partial E_j^{(0)}(\mathbf{y})}{\partial y_k}\right)_{\mathbf{y}=0} \int x_j x_k \rho(\mathbf{x}) d^3x, \qquad (9)$$

where we've used the fact that $\nabla\times\mathbf{E}^{(0)} = 0$ to write $\partial E_i^{(0)}/\partial y_j = \partial E_j^{(0)}/\partial y_i$. In the integrand we can add the term $-(1/3)\delta_{jk}r^2\rho(\mathbf{x})$, since $\nabla\cdot\mathbf{E}^{(0)} = 0$; the integral is then $(1/3)Q_{jk}$. Returning to vector notation, we have

$$\nabla_{\mathbf{y}}\left[\frac{1}{2}\sum_{j,k}\left(\frac{\partial E_j^{(0)}(\mathbf{y})}{\partial y_k}\right)_{\mathbf{y}=0}\right] = \nabla_{\mathbf{y}}\left[\frac{1}{6}\sum_{j,k}Q_{jk}\left(\frac{\partial E_j^{(0)}(\mathbf{y})}{\partial y_k}\right)_{\mathbf{y}=0}\right]. \qquad (10)$$

Collecting all of these results, we obtain the desired result, Eq. (1).
Note that the force is

$$\mathbf{F} = -\nabla_{\mathbf{y}} W(\mathbf{y}), \qquad (11)$$

where the work is

$$W(\mathbf{y}) = q\Phi^{(0)}(\mathbf{y}) - \mathbf{p}\cdot\mathbf{E}^{(0)}(\mathbf{y}) - \frac{1}{6}\sum_{jk}Q_{jk}\frac{\partial E_j^{(0)}(\mathbf{y})}{\partial y_k} + \ldots. \qquad (12)$$

(b) Repeat the calculation of (a) for the total *torque*. For simplicity evaluate only one Cartesian component of the torque, say N_1. Show that this component is

$$N_1 = [\mathbf{p}\times\mathbf{E}^{(0)}(0)]_1 + \frac{1}{3}\left[\frac{\partial}{\partial x_3}\left(\sum_j Q_{2j}E_j^{(0)}\right) - \frac{\partial}{\partial x_2}\left(\sum_j Q_{3j}E_j^{(0)}\right)\right]_0 + \ldots. \qquad (13)$$

Solution. Start with the torque on the charge distribution,

$$\mathbf{N} = \int \mathbf{x}\times\left[\rho(\mathbf{x})\mathbf{E}^{(0)}(\mathbf{x})\right] d^3x, \qquad (14)$$

and follow the same steps as above.

2. *Jackson* 4.7. A localized charge distribution of charge has a charge density

$$\rho(\mathbf{x}) = \frac{1}{64\pi} r^2 e^{-r} \sin^2\theta. \tag{15}$$

(a) Make a multipole expansion of the potential due to this charge density and determine all of the nonvanishing multipole moments. Write down the potential at large distances as a finite expansion in Legendre polynomials.

Solution. This charge density has azimuthal symmetry since it is independent of ϕ, so all of the multipole moments with $m \neq 0$ vanish. To determine the other moments, first rewrite the charge density in terms of the Legendre polynomials, using

$$\sin^2\theta = 1 - \cos^2\theta = \frac{2}{3}\left[P_0(\cos\theta) - P_2(\cos\theta)\right]. \tag{16}$$

so that

$$\begin{aligned} \rho(\mathbf{x}) &= \frac{1}{96\pi} r^2 e^{-r} \left[P_0(\cos\theta) - P_2(\cos\theta)\right] \\ &= \frac{1}{96\pi} r^2 e^{-r} \left[\sqrt{4\pi} Y_0^0(\theta,\phi) - \sqrt{\frac{4\pi}{5}} Y_2^0(\theta,\phi)\right]. \end{aligned} \tag{17}$$

The multipole moments are then

$$\begin{aligned} q_{l0} &= \int Y_l^{0*}(\theta',\phi') r'^l \rho(\mathbf{x}')\, d^3x' \\ &= \sqrt{\frac{2l+1}{4\pi}} \frac{1}{96\pi} \int_0^{2\pi} d\phi' \int_{-1}^{1} d(\cos\theta')\, P_l(\cos\theta')[P_0(\cos\theta') - P_2(\cos\theta')] \\ &\quad \times \int_0^\infty (r'^2 dr') r'^{l+2} e^{-r'}. \end{aligned} \tag{18}$$

Due to the orthogonality of the P_l's, only the $l = 0$ and $l = 2$ terms are non-zero. The result is

$$q_{00} = \frac{1}{\sqrt{4\pi}}, \quad q_{20} = -6\sqrt{\frac{5}{4\pi}}. \tag{19}$$

The potential is then

$$\Phi(\mathbf{x}) = \frac{e}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{6a_0^2 P_2(\cos\theta)}{r^3}\right], \tag{20}$$

where I've reinstated the units of length (a_0) and charge (e). Notice that this charge distribution has a monopole and quadrupole term, but no dipole term.

(b) Determine the potential explicitly at any point in space, and show that near the origin, correct to r^2 inclusive,

$$\Phi(\mathbf{r}) \approx \frac{1}{4\pi\epsilon_0}\left[\frac{1}{4} - \frac{r^2}{120} P_2(\cos\theta)\right]. \tag{21}$$

Solution. The potential is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d^3x'. \tag{22}$$

The Coulomb potential can be expanded in spherical harmonics as

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_l^{m*}(\theta',\phi')Y_l^m(\theta,\phi), \tag{23}$$

where $r_>$ ($r_<$) is the larger (smaller) of $r = |\mathbf{x}|$ and $r' = |\mathbf{x}'|$. Substituting Eqs. (23) and (17) into (22), we have

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{96\pi}\int_0^\infty (r'^2dr')\frac{r_<^l}{r_>^{l+1}}r'^2e^{-r'} \\ &\quad\times\frac{4\pi}{2l+1}\int d\Omega' Y_l^{m*}(\theta',\phi')\left[\sqrt{4\pi}Y_0^0(\theta',\phi') - \sqrt{\frac{4\pi}{5}}Y_2^0(\theta',\phi')\right]. \end{aligned}\tag{24}$$

Carrying out the angular integrals using the orthogonality of the spherical harmonics, we find

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left[\tilde{I}_0(r)P_0(\cos\theta) - \tilde{I}_2(r)P_2(\cos\theta)\right], \tag{25}$$

with

$$\begin{aligned}\tilde{I}_0(r) &\equiv \frac{1}{24}\int_0^\infty \frac{r'^4}{r_>}e^{-r'}dr' \\ &= \frac{1}{24}\left[\int_0^r \frac{r'^4}{r}e^{-r'}dr' + \int_r^\infty \frac{r'^4}{r'}e^{-r'}dr'\right] \\ &= \frac{1}{r} - \frac{1}{24}\left(\frac{24}{r} + 18 + 6r + r^2\right)e^{-r} \\ &\approx \frac{1}{4} - \frac{r^4}{480} + O(r^5), \end{aligned}\tag{26}$$

where I've used *Maple* to do the integrals and perform the small r expansion in the last line; likewise,

$$\begin{aligned}\tilde{I}_2(r) &\equiv \frac{1}{120}\int_0^\infty r'^4\frac{r_<^2}{r_>^3}e^{-r'}dr' \\ &= \frac{1}{120}\left[\int_0^r r'^4\frac{r'^2}{r^3}e^{-r'}dr' + \int_r^\infty r'^4\frac{r^2}{r'^3}e^{-r'}dr'\right] \\ &= \frac{6}{r^3} - \frac{1}{24}\left(\frac{144}{r^3} + \frac{144}{r^2} + \frac{72}{r} + 24 + 6r + r^2\right)e^{-r} \\ &\approx \frac{r^2}{120} - \frac{r^4}{336} + O(r^5). \end{aligned}\tag{27}$$

Substituting Eqs. (26) and (27) into (25), we obtain the advertised result, Eq. (21).

(c) If there exists at the origin a nucleus with a quadrupole moment $Q = 10^{-28}$ m^2, determine the magnitude of the interaction energy, assuming that the unit of charge in $\rho(\mathbf{r})$ above is the electronic charge and the unit of length is the hydrogen Bohr radius $a_0 = 4\pi\epsilon_0\hbar^2/me^2 = 0.529 \times 10^{-10}$ m. Express your answer as a frequency by dividing by Planck's constant h.

The charge density in this problem is that for the $m = \pm 1$ state of the $2p$ level in hydrogen, while the quadrupole interaction is of the same order as found in molecules.

Solution. Consider the nucleus as a localized charge distribution in the "external" potential $\Phi(\mathbf{x})$ provided by the electron and calculated above. The Cartesian quadrupole moment tensor for the nucleus is diagonal (due to the cylindrical symmetry), with $Q_{11} = Q_{22} = -eQ/2$, $Q_{33} = eQ$ (according to Sec. 4.3 of *Jackson*, "the quadrupole moment of a nuclear state is defined as the value of $(1/e)Q_{33}\ldots$"). The energy of interaction is then

$$\begin{aligned} W &= -\frac{1}{6}\sum_{i,j} Q_{ij}\frac{\partial E_j(0)}{\partial x_i} \\ &= -\frac{eQ}{6}\left[-\frac{1}{2}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right) + \frac{\partial E_z}{\partial z}\right]_0 \\ &= -\frac{eQ}{4}\frac{\partial E_z(0)}{\partial z}, \end{aligned} \tag{28}$$

where in the last line we've used $\nabla \cdot \mathbf{E} = 0$.

Next, we need to calculate the field gradient near the nucleus. Using Eq. (21), we have

$$\begin{aligned} \Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{4} - \frac{r^2}{120}P_2(\cos\theta)\right] \\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{4} - \frac{1}{240}(2z^2 - x^2 - y^2)\right], \end{aligned} \tag{29}$$

so that

$$\begin{aligned} \left(\frac{\partial E_z}{\partial z}\right)_0 &= -\left(\frac{\partial^2\Phi}{\partial z^2}\right)_0 \\ &= \frac{1}{60}\frac{e}{4\pi\epsilon_0 a_0^3}, \end{aligned} \tag{30}$$

where in the last line the units have been re-instated. The energy of interaction is then

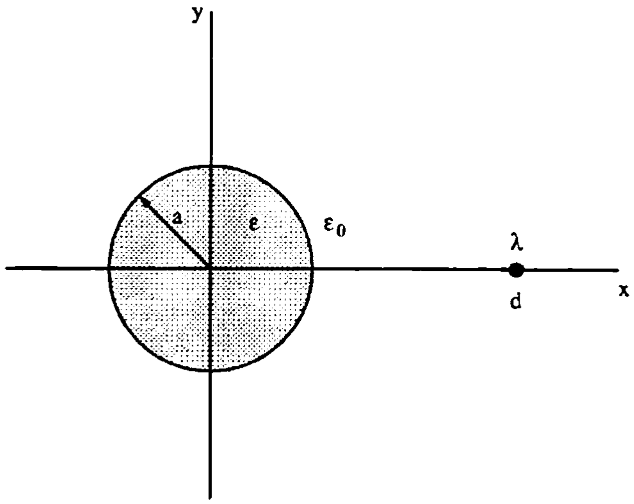
$$|W| = \frac{1}{240}\frac{e^2}{4\pi\epsilon_0 a_0}\frac{Q}{a_0^2}. \tag{31}$$

Expressed as a frequency, this is

$$\begin{aligned}\nu &= \frac{|W|}{h} \\ &= \frac{1}{240}\frac{e^2}{4\pi\epsilon_0 h a_0}\frac{Q}{a_0^2} \\ &= 0.982 \text{ MHz},\end{aligned} \tag{32}$$

which is a radio frequency signal (in the AM band).

3. **Dielectric cylinder.** An infinite line charge, with charge per unit length λ, is parallel to and a distance d from the center of an infinite dielectric cylinder of radius $a < d$ and dielectric constant ϵ (the dielectric constant outside of the cylinder is ϵ_0).



(a) Find the potential inside and outside of the cylinder.

Solution. The boundary conditions at the surface of the cylinder are

$$-\epsilon_0\left(\frac{\partial\Phi_{\text{out}}}{\partial\rho}\right)_{\rho=a} = -\epsilon\left(\frac{\partial\Phi_{\text{in}}}{\partial\rho}\right)_{\rho=a}, \tag{33}$$

$$-\frac{1}{a}\left(\frac{\partial\Phi_{\text{out}}}{\partial\phi}\right)_{\rho=a} = -\frac{1}{a}\left(\frac{\partial\Phi_{\text{in}}}{\partial\phi}\right)_{\rho=a}. \tag{34}$$

The potential outside of the cylinder is that of the line charge plus the potential due to the charge induced in the cylinder. Write this as

$$\begin{aligned}\Phi_{\text{out}}(\rho,\phi) &= -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\sqrt{\rho^2+d^2-2d\rho\cos\phi}/R_0\right) \\ &\quad +A_0\ln(\rho/R_0)+\sum_{n=1}^{\infty}A_n\left(\frac{a}{\rho}\right)^n\cos n\phi,\end{aligned} \tag{35}$$

where R_0 is a scale factor needed to make the argument of the logarithm dimensionless. Inside the cylinder we have

$$\Phi_{\text{in}}(\rho, \phi) = \sum_{n=1}^{\infty} B_n \left(\frac{\rho}{a}\right)^n \cos n\phi + B_0. \tag{36}$$

In order to implement the boundary conditions, we first need to expand the logarithm:

$$\begin{aligned} \ln\left(\sqrt{\rho^2 + d^2 - 2d\rho\cos\phi}/R_0\right) &= \ln(d/R_0) + \ln\sqrt{1 + (\rho/d)^2 - 2(\rho/d)\cos\phi} \\ &= \ln(d/R_0) - \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\rho}{d}\right)^n \cos n\phi. \end{aligned} \tag{37}$$

Applying the boundary condition in Eq. (33), we have

$$\epsilon_0 \left(\frac{\lambda}{2\pi\epsilon_0}\frac{a^{n-1}}{d^n} - \frac{n}{a}A_n\right) = \epsilon\frac{n}{a}B_n, \tag{38}$$

and applying the boundary condition in Eq. (34), we have

$$\frac{\lambda}{2\pi\epsilon_0}\frac{1}{n}\left(\frac{a}{d}\right)^n + A_n = B_n, \tag{39}$$

along with $A_0 = 0$ and $B_0 = -(\lambda/2\pi\epsilon_0)\ln(d/R_0)$. Solving for the expansion coefficients, we obtain

$$A_n = -\frac{\lambda}{2\pi\epsilon_0}\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 1}\frac{1}{n}\left(\frac{a}{d}\right)^n, \tag{40}$$

$$B_n = \frac{\lambda}{2\pi\epsilon_0}\frac{2}{\epsilon/\epsilon_0 + 1}\frac{1}{n}\left(\frac{a}{d}\right)^n. \tag{41}$$

With these coefficients we can evaluate the sums:

$$\begin{aligned} \sum_{n=1}^{\infty} A_n \left(\frac{a}{\rho}\right)^n \cos n\phi &= -\frac{\lambda}{2\pi\epsilon_0}\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 1}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{a^2}{d\rho}\right)^n \cos n\phi \\ &= \frac{\lambda}{2\pi\epsilon_0}\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 1}\ln\sqrt{1 + (a^2/d\rho)^2 - 2(a^2/d\rho)\cos\phi}. \end{aligned} \tag{42}$$

Therefore, the potential outside the cylinder is

$$\begin{aligned} \Phi_{\text{out}}(\rho, \phi) &= -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\sqrt{\rho^2 + d^2 - 2d\rho\cos\phi}/R_0\right) \\ &\quad + \frac{\lambda}{2\pi\epsilon_0}\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 1}\ln\sqrt{1 + (a^2/d\rho)^2 - 2(a^2/d\rho)\cos\phi}. \end{aligned} \tag{43}$$

The potential inside the cylinder is

$$\Phi_{\rm in}(\rho,\phi) = -\frac{\lambda}{2\pi\epsilon_0}\frac{2}{\epsilon/\epsilon_0+1}\ln\sqrt{1+(\rho/d)^2-2(\rho/d)\cos\phi}. \tag{44}$$

The same result could be obtained using the method of images in two dimensions.

(b) Find the force per unit length that the line charge exerts upon the cylinder. Is there a simple interpretation of your result?

Solution. To find the force on the line charge, calculate the radial component of the electric field outside of the cylinder:

$$\begin{aligned} E_\rho &= -\frac{\partial\Phi}{\partial\rho} \\ &= \frac{\lambda}{2\pi\epsilon_0}\frac{\rho-d\cos\phi}{\rho^2+d^2-2\rho d\cos\phi} \\ &\quad -\frac{\lambda}{2\pi\epsilon_0}\frac{\epsilon/\epsilon_0-1}{\epsilon/\epsilon_0+1}\frac{-(d'^2/\rho^3)+(d'/\rho^2)\cos\phi}{1+(d'/\rho)^2-2(d'/\rho)\cos\phi}. \end{aligned} \tag{45}$$

The first term is the field due to the line charge itself, and the second term is the field due to the charge induced on the cylinder—this is the field which produces the force on the line charge. Multiplying this part by λ, we obtain the force per unit length on the line charge as

$$F/l = -\frac{\lambda^2}{2\pi\epsilon_0}\frac{\epsilon/\epsilon_0-1}{\epsilon/\epsilon_0+1}\frac{a^2}{d(d^2-a^2)}, \tag{46}$$

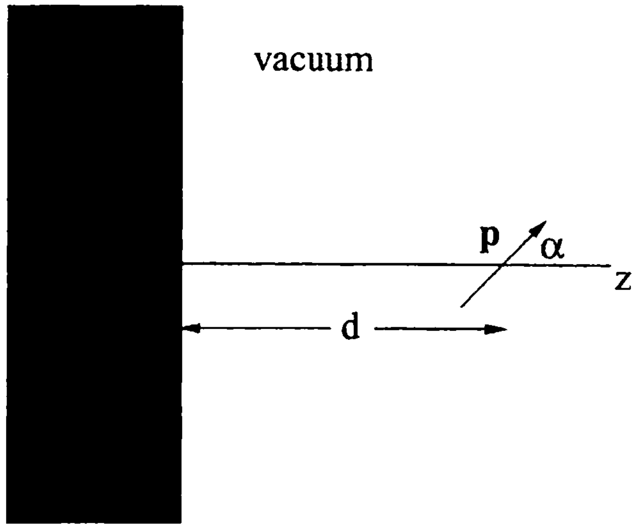
which is always attractive. For large d we see that the force goes as a^2/d^3; the induced dipole moment in the cylinder goes as $p \propto a^2/d$, so the two dimensional dipole field experienced by the line charge behaves as $p/d^2 \sim a^2/d^3$.

4. **Dipoles and dielectrics.** A point dipole of dipole moment $\mathbf{p}$ is placed a distance d in front of a semi-infinite dielectric slab, with dielectric constant ϵ. We can choose the axes such that the dipole is in the $x-z$ plane, making an angle α with the z-axis as shown in the figure below; the dielectric occupies the region $z<0$.

   (a) Use the method of images to find the potential for both $z>0$ and $z<0$.

   Solution. This is completely analogous to the problem of the point charge in front of the dielectric. For $z>0$ we assume that in addition to the dipole at $z=d$, there is an image dipole $\mathbf{p}'$ at $z=-d$; for $z<0$, we assume that there is a dipole $\mathbf{p}''$ at $z=d$. By matching $\partial\Phi/\partial y$ and $\epsilon\partial\Phi/\partial z$ at $z=0$, we obtain for $z>0$

$$\Phi_> = \frac{1}{4\pi\epsilon_0}\left\{\frac{p_x x+p_z(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}}+\frac{\epsilon/\epsilon_0-1}{\epsilon/\epsilon_0+1}\frac{-p_x x+p_z(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right\}, \tag{47}$$



while for $z < 0$,

$$\Phi_< = \frac{1}{4\pi\epsilon}\frac{2\epsilon/\epsilon_0}{\epsilon/\epsilon_0+1}\frac{p_x x + p_z(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}}. \tag{48}$$

(b) Find the force on the dipole.

Solution. First find the energy of interaction between the dipole and the field produced by its image:

$$W = -\frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{3(\mathbf{p}'\cdot\mathbf{n})(\mathbf{p}\cdot\mathbf{n})-\mathbf{p}\cdot\mathbf{p}'}{|\mathbf{x}_1-\mathbf{x}_2|^3} \tag{49}$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{\epsilon/\epsilon_0-1}{\epsilon/\epsilon_0+1}\frac{3p_z(p_z)-(-p_x^2+p_z^2)}{2(2d)^3} \tag{50}$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{\epsilon/\epsilon_0-1}{\epsilon/\epsilon_0+1}\frac{p^2(1+\cos^2\alpha)}{16d^3}. \tag{51}$$

The factor of 1/2 arises from the fact that the field $\mathbf{E}'$ acting on the dipole is proportional to the dipole moment $\mathbf{p}$; since $dU = -\mathbf{E}'\cdot d\mathbf{p}$, upon integration $U = -(1/2)\mathbf{p}\cdot\mathbf{E}'$. To obtain the force, we differentiate:

$$F_z = -\frac{\partial U}{\partial d} \tag{52}$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{3}{16}\frac{\epsilon/\epsilon_0-1}{\epsilon/\epsilon_0+1}\frac{p^2(1+\cos^2\alpha)}{d^4}. \tag{53}$$

Since $\epsilon/\epsilon_0 > 1$, the force is always attractive.

(c) Find the torque on the dipole. Assuming that the dipole is free to rotate about its axis, in what direction does it point in equilibrium?

Solution. The torque is given by

$$N = -\frac{\partial U}{\partial \alpha} \tag{54}$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 1}\frac{p^2\sin(2\alpha)}{16d^3}. \tag{55}$$

The torque is zero when $\alpha = 0,\ \pi/2,\ \pi,\ 3\pi/2$; however, by examining the energy we see that only $\alpha = 0$ and π are stable, so the dipole prefers to align perpendicular to the dielectric.

(d) Take a suitable limit of your result in (a) to find the potential for $z > 0$ for a dipole in front of a grounded conducting slab.

Solution. Take the limit $\epsilon \to \infty$ to obtain the results for the dipole in front of the grounded conductor: $\Phi_< = 0$, and

$$\Phi_> = \frac{1}{4\pi\epsilon_0}\left\{\frac{p_x x + p_z(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{-p_x x + p_z(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right\}. \tag{56}$$

5. **Debye-Hückel screening in plasmas** (bonus). A *plasma* is a neutral ionized gas which is produced at high temperatures. We want to determine the electrostatic potential in the vicinity of a test charge which is placed in the plasma. The mean density of electrons of charge $-e$ is n_0, and the mean density of ions of charge Ze is N_0; electrical neutrality implies that $ZN_0 = n_0$.

(a) If the ions and electrons are in thermal equilibrium at a temperature T, then the particle densities are given in terms of the Boltzmann weights as

$$N(\mathbf{x}) = N_0 \exp\left[-Ze\Phi(\mathbf{x})/k_BT\right], \quad n(\mathbf{x}) = n_0 \exp\left[e\Phi(\mathbf{x})/k_BT\right], \tag{57}$$

where k_B is Boltzmann's constant and $\Phi(\mathbf{x})$ is the electrostatic potential. Use these expressions to derive a closed equation for the potential Φ.

Solution. The charge density is

$$\rho(\mathbf{x}) = ZeN(\mathbf{x}) - en(\mathbf{x}) \tag{58}$$

$$= en_0\left[e^{-Ze\Phi(\mathbf{x})/k_BT} - e^{e\Phi(\mathbf{x})/k_BT}\right]. \tag{59}$$

The potential is determined by solving Poisson's equation,

$$\nabla^2\Phi = -\rho/\epsilon \tag{60}$$

$$= -\frac{en_0}{\epsilon}\left[e^{-Ze\Phi(\mathbf{x})/k_BT} - e^{e\Phi(\mathbf{x})/k_BT}\right]. \tag{61}$$

This equation is nonlinear.

(b) The equation which you derived in part (a) is nonlinear, and generally difficult to solve. By assuming that $|Ze\Phi/k_BT| \ll 1$, (the high temperature limit), show that this equation reduces to

$$\nabla^2\Phi = \kappa^2\Phi, \tag{62}$$

and determine κ in terms of the other parameters. You may assume that the background dielectric constant is ϵ.

Solution. At high temperatures we can linearize the charge density, by expanding the exponentials to lowest order in Φ. We then obtain the result quoted above, with

$$\kappa^2 = \frac{e^2 n_0(Z+1)}{\epsilon k_B T}. \tag{63}$$

(c) Now suppose that a test charge q is placed at $\mathbf{x} = 0$ in the plasma. Modify the linearized equation derived above to account for the test charge, and show that in three dimensions this equation has the solution

$$\Phi(\mathbf{x}) = C\frac{e^{-r/\lambda}}{r}, \tag{64}$$

and determine the constants C and λ. The length λ is the *Debye screening length* and is the length scale over which the potential of the test charge is screened by the electrons and ions.

Solution. If we add a point charge at the origin, there is an additional term $-q\delta(\mathbf{x})$ in the charge density, so that our equation becomes

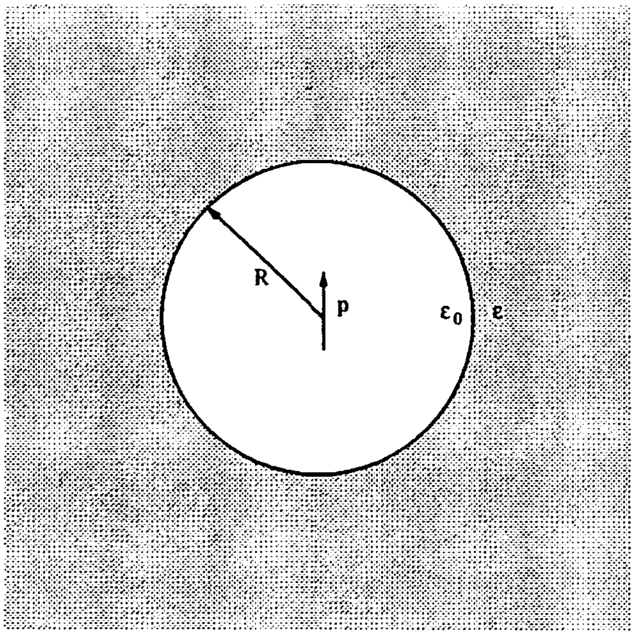
$$\nabla^2\Phi = \kappa^2\Phi - (q/\epsilon)\delta(\mathbf{x}). \tag{65}$$

By substituting in the trial solution suggested above, we see that $\lambda = 1/\kappa$; close to the origin the charge density is dominated by the point charge, so that $C = q/4\pi\epsilon$. Therefore, the potential is

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\epsilon}\frac{e^{-\kappa r}}{r}. \tag{66}$$

Physics 6346, Electromagnetic Theory I
Fall 2000
Homework 9 Solutions

1. **Spherical void in a dielectric.** A spherical void of radius R is in an otherwise homogeneous dielectric medium of permittivity ϵ. At the center of the void is a point dipole $\mathbf{p}$. Find the electric field inside and outside the void.



Solution. From our previous experience with these types of problems, we expect that the potential inside the void will be that due to a dipole and the potential of a uniform field, and outside the void we'll just have a dipole field. So the potential is

$$\Phi_{\text{in}} = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2} - E_{\text{in}} r\cos\theta, \tag{1}$$

and

$$\Phi_{\text{out}} = \frac{1}{4\pi\epsilon}\frac{p_{\text{out}}\cos\theta}{r^2}, \tag{2}$$

with E_{in} and p_{out} constants to be determined from the boundary conditions. The boundary conditions are that the tangential component of $\mathbf{E}$ is continuous, so that

$$-\frac{1}{a}\left(\frac{\partial\Phi_{\text{in}}}{\partial\theta}\right)_{r=a} = -\frac{1}{a}\left(\frac{\partial\Phi_{\text{out}}}{\partial\theta}\right)_{r=a}, \tag{3}$$

and that the normal component of D is continuous, so that

$$-\epsilon_0 \left(\frac{\partial \Phi_{\text{in}}}{\partial r} \right)_{r=a} = -\epsilon \left(\frac{\partial \Phi_{\text{out}}}{\partial r} \right)_{r=a} . \tag{4}$$

Applying these two conditions, we obtain the following equations:

$$\frac{1}{4\pi\epsilon_0} \frac{p}{a^3} - E_{\text{in}} = \frac{1}{4\pi\epsilon} \frac{p_{\text{out}}}{a^3}, \tag{5}$$

$$\frac{1}{4\pi} \frac{2p}{a^3} + \epsilon_0 E_{\text{in}} = \frac{1}{4\pi} \frac{2p_{\text{out}}}{a^3}. \tag{6}$$

Solving for E_{in} and p_{out}, we obtain

$$E_{\text{in}} = \frac{1}{4\pi\epsilon_0} \frac{2(\epsilon - \epsilon_0)}{2\epsilon + \epsilon_0} \frac{p}{a^3}, \tag{7}$$

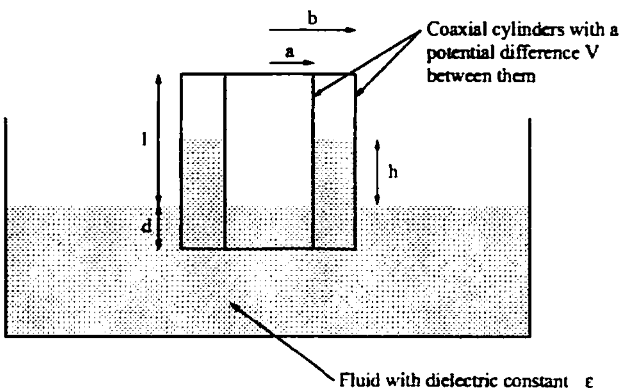
$$p_{\text{out}} = \frac{3\epsilon}{2\epsilon + \epsilon_0} p. \tag{8}$$

2. *Jackson* 4.13. Two long, coaxial, cylindrical conducting surfaces of radii a and b are lowered vertically into a liquid dielectric. If the liquid rises an average height h between the electrodes when a potential difference V is established between them, show that the susceptibility of the liquid is

$$\chi_e = \frac{(b^2 - a^2)\rho g h \ln(b/a)}{\epsilon_0 V^2}, \tag{9}$$

where ρ is density of the liquid, g is the acceleration due to gravity, and the susceptibility of air is neglected.

Solution. A cross section of the set up is sketched in the figure below.



We first need to calculate the electric field between the cylinders. We know that $\mathbf{E} = -\nabla\Phi$, and that $\nabla \cdot \mathbf{D} = 0$. Since the liquid is assume to respond linearly to the electric field between the cylinders, we also have $\mathbf{D} = \epsilon\mathbf{E}$, so that $\nabla^2\Phi = 0$. Let's assume that the cylinders are long so that we can ignore fringing effects at the ends of the cylinders. Then the potential is only a function of the radial distance ρ from the axis of the cylinders, so that Laplace's equation becomes

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) = 0, \tag{10}$$

the solution of which is

$$\Phi(\rho) = A\ln(\rho/R), \tag{11}$$

with A and R integration constants which will be determined from the boundary conditions. We have

$$\Phi(b) - \Phi(a) = V = A\ln(b/a), \tag{12}$$

so $A = V/\ln(b/a)$. The electric field is then

$$\begin{aligned} \mathbf{E} &= -\frac{\partial\Phi}{\partial\rho}\mathbf{e}_\rho \\ &= -\frac{V}{\ln(b/a)}\frac{\mathbf{e}_\rho}{\rho} \end{aligned} \tag{13}$$

between the cylinders, and zero outside. The electric displacement field is $\mathbf{D} = \epsilon\mathbf{E}$ for the region between the cylinders which contains the liquid, and is $\mathbf{D} = \epsilon_0\mathbf{E}$ for the region between the cylinders above the liquid.

Having determined $\mathbf{E}$ and $\mathbf{D}$, we can now calculate the work required to bring the liquid between the cylinders to a height z above the background (we'll set $z = h$ at the end of the calculation):

$$\begin{aligned} W &= \frac{1}{2}\int \mathbf{E}\cdot\mathbf{D}\,d^3x \\ &= \frac{1}{2}\int_a^b 2\pi\rho\,d\rho\left(\int_{-d}^{z}\epsilon E^2\,dz + \int_z^l \epsilon E^2\,dz\right) \\ &= \frac{\pi V^2}{\ln(b/a)}\left[\epsilon(z+d) + \epsilon_0(l-z)\right]. \end{aligned} \tag{14}$$

To find the electrostatic force acting upon the liquid, we need to differentiate W with respect to z. The only trick here is to remember that the liquid is moving up in the presence of a constant potential, so that charge must be supplied from an external source (a battery, say), and we must account for the work done by the battery in supplying the charge. This is is discussed in *Jackson*; the end result is a change in the sign of the force, with the final result that

$$
\begin{aligned}
F_z &= \left(\frac{\partial W}{\partial z}\right)_V \\
&= \frac{(\epsilon - \epsilon_0)\pi V^2}{\ln(b/a)}. \qquad (15)
\end{aligned}
$$

This has the right sign—we see that the electrostatic force is upward. This is balanced by the weight of the liquid, mg. The mass of liquid which is between the cylinders and above the background is

$$m = \rho\pi(b^2 - a^2)h, \qquad (16)$$

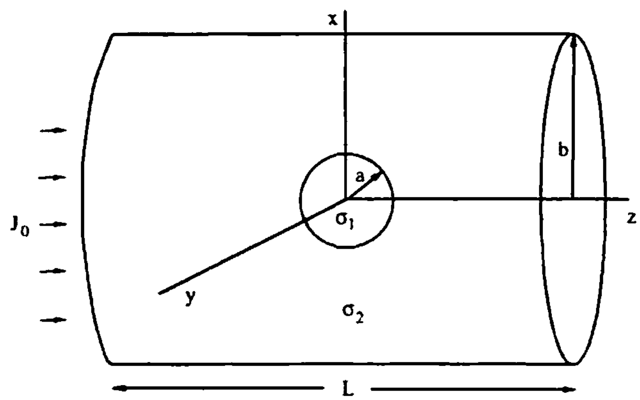
with ρ the mass density of the liquid (not to be confused with the coordinate ρ!). We then have

$$\rho\pi(b^2 - a^2)gh = \frac{(\epsilon - \epsilon_0)\pi V^2}{\ln(b/a)}. \qquad (17)$$

The susceptibility is $\chi_e = \epsilon/\epsilon_0 - 1$, so we finally have

$$\chi_e = \frac{(b^2 - a^2)\rho g h \ln(b/a)}{\epsilon_0 V^2}, \qquad (18)$$

3. **Defects in conductors** (from the 1999 Comprehensive Exam). A cylindrical conductor, of conductivity σ_2, radius b, and length L, contains a spherical defect of radius a and conductivity σ_1 (see the Figure below). The cylinder and the defect have the same permeability μ and permittivity ϵ. A *steady*, uniform current density $J_0\,\mathbf{e}_z$ flows parallel to the axis of the cylinder far from the defect. For simplicity, assume that $b, L \gg a$, so that you can consider the limit $L \to \infty$ and $b \to \infty$ (but with J_0 fixed).



(a) Write down, with a few sentences of explanation, the equations which will determine the current density $\mathbf{J}$ and electric field $\mathbf{E}$ in the conductors. Also, write down the boundary conditions on the electrostatic potential Φ at the interface between the two conductors.

Solution. If the current is steady (no time dependence), then $\nabla \cdot \mathbf{J} = 0$. The electric fields which are established are static fields, so we also know that $\nabla \times \mathbf{E} = 0$, which is satisfied by $\mathbf{E} = -\nabla\Phi$. If we assume that the current transport in each region is linear, so that Ohm's Law is satisfied, $\mathbf{J} = \sigma\mathbf{E}$ in each region, with σ the conductivity in that region. Therefore, we see that the potential is the solution of Laplace's equation, $\nabla^2\Phi = 0$, both inside and outside the defect.

From the basic differential equations, we can derive the boundary conditions (by analogy with dielectrics) that the normal component of $\mathbf{J}$ is continuous and that the tangential component of $\mathbf{E}$ is continuous. For the problem at hand, this translates into the following two conditions:

$$-\sigma_1 \left(\frac{\partial \Phi_1}{\partial r}\right)_{r=a} = -\sigma_2 \left(\frac{\partial \Phi_2}{\partial r}\right)_{r=a}, \tag{19}$$

$$-\frac{1}{a} \left(\frac{\partial \Phi_1}{\partial \theta}\right)_{r=a} = -\frac{1}{a} \left(\frac{\partial \Phi_2}{\partial \theta}\right)_{r=a}. \tag{20}$$

(b) By solving the boundary value problem, show that the electric field in the cylinder consists of a uniform field plus a dipole field due to the defect, and calculate the dipole moment.

Solution. We need to solve Laplace's equation,

$$\nabla^2\Phi = 0, \tag{21}$$

with the boundary conditions above, along with the requirement that the current density outside of the defect becomes uniform ($\mathbf{J} = J_0\mathbf{e}_z$) far from the defect; in terms of the potential, this becomes

$$\Phi_2 = -\frac{J_0}{\sigma_2} r\cos\theta, \quad \text{as } r \to \infty. \tag{22}$$

Mathematically, the problem is completely equivalent to the problem of a dielectric sphere placed in a uniform electric field, so I'll just quote the result:

$$\Phi_1 = -\frac{3J_0}{\sigma_1 + 2\sigma_2} r\cos\theta, \tag{23}$$

$$\Phi_2 = -\frac{J_0}{\sigma_2} r\cos\theta + \frac{(\sigma_1 - \sigma_2)a^3 J_0}{\sigma_2(\sigma_1 + 2\sigma_2)} \frac{\cos\theta}{r^2}. \tag{24}$$

We see that outside of the defect the electric field consists of a uniform field and a dipole field, with dipole moment

$$p = 4\pi\epsilon \frac{(\sigma_1 - \sigma_2)a^3 J_0}{\sigma_2(\sigma_1 + 2\sigma_2)}. \tag{25}$$

(c) Find the surface charge density Σ which has accumulated on the boundary between the two conductors.

Solution. The surface charge density is equal to the discontinuity of the normal component of the $\mathbf{D}$ field across the boundary. Therefore we have

$$\begin{aligned} \Sigma &= (\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n} \\ &= \epsilon[E_{2,r}(a,\theta) - E_{1,r}(a,\theta)] \\ &= \frac{3\epsilon(\sigma_1 - \sigma_2)J_0}{\sigma_2(\sigma_1 + 2\sigma_2)} \cos\theta. \end{aligned} \tag{26}$$

4. **Biot-Savart Law.**

(a) Use the Biot-Savart Law to calculate the magnetic field produced by a straight, infinitely long wire which carries a current I.

(b) Use the Biot-Savart Law to calculate the magnetic field produced by a circular current loop of radius a. You only need to find the field along the axis of the loop.

Solutions. Both of these problems are elementary applications of the Biot-Savart Law, so if you had difficulty with them you need to review an introductory physics text such as *Halliday and Resnick*. The result for the infinitely long wire, with the current in the z-direction, is

$$\mathbf{B}(\rho) = \frac{\mu_0 I}{2\pi\rho}\mathbf{e}_\phi. \tag{27}$$

with ρ the distance from the wire (the field lines form concentric circles around the wire). For the current loop, in the $x-y$ plane with the center at the origin, the field along the z-axis is

$$\mathbf{B}(z) = \frac{\mu_0}{4\pi}\frac{2(\pi a^2 I)}{(z^2+a^2)^{3/2}}\mathbf{e}_z. \tag{28}$$

The quantity $\pi a^2 I$ is the magnitude of the magnetic moment of the loop.

Physics 6346, Electromagnetic Theory I
Fall 2000
Homework 10 Solutions (Revised 11/27/00)

1. **Simple applications of Ampère's Law.** Use Ampere's Law in integral form to find the magnetic field in the following situations.

   (a) A long cylinder of radius a carries a current I which is uniformly distributed throughout its interior. Find the magnetic field inside and outside the cylinder.

   Solution. These are all rather elementary applications, so I'll just quote the results. Let's assume that the axis of the cylinder coincides with the z-axis, and that the current is in the z-direction. Then for $\rho < a$, we have

   $$\mathbf{B}(\rho) = \frac{\mu_0 I}{2\pi}\frac{\rho}{a^2}\mathbf{e}_\phi, \tag{1}$$

   and for $\rho > a$ we have

   $$\mathbf{B}(\rho) = \frac{\mu_0 I}{2\pi\rho}\mathbf{e}_\phi. \tag{2}$$

   (b) A long solenoid of radius a is constructed by by wrapping wire around a cylinder, with N turns per unit length. If the wire carries a current I, find the magnetic field inside the solenoid.

   Solution. If we let the z-axis coincide with the axis of the solenoid, with the currents circulating in a counter-clockwise direction in the $x - y$ plane, then the field inside the solenoid is

   $$\mathbf{B} = \mu_0 I N \mathbf{e}_z. \tag{3}$$

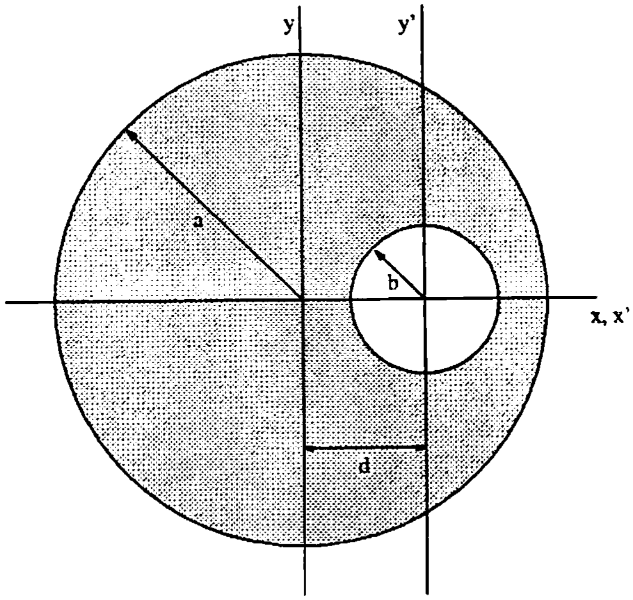
   (c) A thin conducting sheet, of thickness d, carries a current density $\mathbf{J} = J_0\mathbf{e}_x$. Find the magnetic field just above the sheet.

   Solution. If we let the sheet reside in the $x - y$ plane, then the field is

   $$\mathbf{B} = -\frac{\mu_0 J d}{2}\mathrm{sign}(z)\mathbf{e}_y, \tag{4}$$

   so the field is in $-y$ direction above the sheet and in the y direction below the sheet.

2. *Jackson* 5.6. A cylindrical conductor of radius a has a a hole of radius b bored parallel to, and a distance d from, the cylinder axis ($d + b < a$). The current density is uniform throughout the remaining metal of the cylinder and is parallel to the axis. Use Ampere's law and the principle of linear superposition to find the magnitude and the direction of the magnetic-flux density in the hole.



Solution. First, let's forget about the hole and calculate the magnetic field produced by a uniform current density $\mathbf{J} = J\mathbf{e}_z$:

$$\begin{aligned} \mathbf{B} &= \frac{\mu_0 J}{2}\rho\mathbf{e}_\phi \\ &= \frac{\mu_0 J}{2}(-y\mathbf{e}_x + x\mathbf{e}_y). \end{aligned} \tag{5}$$

Next, let's calculate the magnetic field which would be produced by a current density $\mathbf{J} = -J\mathbf{e}_z$ which is flowing in the hole; using the primed coordinates centered on the hole, we have

$$\mathbf{B}' = -\frac{\mu_0 J}{2}(-y'\mathbf{e}_x + x'\mathbf{e}_y). \tag{6}$$

We now superimpose these two fields; the result is the field inside the hole which is produced by a current density $\mathbf{J}$ flowing outside the hole, with no current inside the hole:

$$\mathbf{B}_{\text{total}} = \mathbf{B} + \mathbf{B}' = \frac{\mu_0 J}{2}[-(y - y')\mathbf{e}_x + (x - x')\mathbf{e}_y]. \tag{7}$$

From the figure we see that $y = y'$ and $x - x' = d$, so we have

$$\mathbf{B}_{\text{total}} = \frac{\mu_0 J d}{2}\mathbf{e}_y, \tag{8}$$

which is a constant magnetic field inside the hole. The current density J times the cross sectional area $\pi(a^2 - b^2)$ is the total current I, so we can also write our result as

$$\mathbf{B}_{\text{total}} = \frac{\mu_0 I}{2\pi}\frac{d}{a^2 - b^2}\mathbf{e}_y. \tag{9}$$

3. **Analogies with fluid flow.** There are many analogies between problems in electromagnetism and fluid flow, and this problem is designed to lead you through some of them. As background reading, I highly recommend Chapters 12 and 40 of *The Feynman Lectures on Physics*, Vol. II.

   (a) First, consider a fluid with mass density $\rho(\mathbf{x}, t)$; the velocity field of the fluid is $\mathbf{v}(\mathbf{x}, t)$. Show that conservation of mass implies that the density and velocity satisfy the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{10}$$

   Solution. This is identical to the derivation in class for the equation of continuity which connects the charge density and the current density; here we have a mass density ρ and a mass current $\mathbf{J}_M = \rho \mathbf{v}$ (a mass per unit time per unit area).

   (b) *Incompressible* fluid flow occurs in situations where the density of the fluid is constant (in both space and time). These flows occur when the characteristic fluid velocity V is small compared to the speed of sound c in the fluid (so that the *Mach number* $\mathrm{Ma} = V/c \ll 1$). Under these conditions $\nabla \cdot \mathbf{v} = 0$. In general a fluid may swirl around; this motion is characterized by the *vorticity* $\boldsymbol{\omega} = \nabla \times \mathbf{v}$. As you can see, there is a clear analogy with the equations of magnetostatics, $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. Use this analogy to derive the Biot-Savart law for fluids; i.e., for a prescribed vorticity $\boldsymbol{\omega}(\mathbf{x})$ the fluid velocity is

$$\mathbf{v}(\mathbf{x}) = \frac{1}{4\pi} \int \boldsymbol{\omega}(\mathbf{x}') \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \, d^3x'. \tag{11}$$

   Solution. We can just use our analogy to come up with the result. We have $\mathbf{v} \Leftrightarrow \mathbf{B}$ and $\boldsymbol{\omega} \Leftrightarrow \mu_0 \mathbf{J}$. The Biot-Savart law in magnetostatics is

$$\mathbf{B}(\mathbf{x}) = \frac{1}{4\pi} \int d^3x' \mu_0 \mathbf{J}(\mathbf{x}') \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}, \tag{12}$$

   so making the necessary replacements we immediately obtain Eq. (11).

   (c) The kinetic energy density of a fluid is $(1/2)\rho v^2$, so that the total kinetic energy is

$$E_K = \int \frac{1}{2} \rho v^2 \, d^3x. \tag{13}$$

   Show that this can be written in terms of the vorticity as

$$E_K = \frac{\rho}{8\pi} \int d^3x \int d^3x' \, \frac{\boldsymbol{\omega}(\mathbf{x}) \cdot \boldsymbol{\omega}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}. \tag{14}$$

   The easiest way to show this is to first realize that $\nabla \cdot \mathbf{v} = 0$ can be enforced by writing $\mathbf{v} = \nabla \times \mathbf{A}$, where $\mathbf{A}$ is a "vector potential." Find the relationship between

$\mathbf{A}$ and $\boldsymbol{\omega}$ (you can use the magnetostatics analogy). Then write $v^2 = \mathbf{v}\cdot(\nabla\times\mathbf{A})$, integrate by parts, and write everything in terms of the vorticity.

Solution. First, writing $\mathbf{v} = \nabla\times\mathbf{A}$ and using our analogy, we have

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi}\int d^3x'\,\frac{\boldsymbol{\omega}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}. \tag{15}$$

Next, a few vector identities:

$$\begin{aligned} v^2 &= \mathbf{v}\cdot\mathbf{v} \\ &= \mathbf{v}\cdot(\nabla\times\mathbf{A}) \\ &= \nabla\cdot(\mathbf{A}\times\mathbf{v}) + \mathbf{A}\cdot(\nabla\times\mathbf{v}) \\ &= \nabla\cdot(\mathbf{A}\times\mathbf{v}) + \mathbf{A}\cdot\boldsymbol{\omega}. \end{aligned} \tag{16}$$

We now substitute this into the expression for the energy, Eq. (13); the term involving the volume integral of $\nabla\cdot(\mathbf{A}\times\mathbf{v})$ can be converted into a surface integral (using the divergence theorem), which vanishes as the surface is taken to infinity. We're left with

$$\begin{aligned} E_K &= \frac{\rho}{2}\int d^3x\,\boldsymbol{\omega}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x}) \\ &= \frac{\rho}{8\pi}\int d^3x\int d^3x'\,\frac{\boldsymbol{\omega}(\mathbf{x})\cdot\boldsymbol{\omega}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}, \end{aligned} \tag{17}$$

where in the last line we've used Eq. (15).

(d) In some cases the vorticity is produced by *vortex lines*, which can be thought of as isolated singularities in the fluid velocity. If we integrate the vorticity in the vortex over the area perpendicular to the vortex, we obtain the *circulation* in the vortex:

$$\begin{aligned} \int_A \boldsymbol{\omega}\cdot\mathbf{n}\,dS &= \int_A(\nabla\times\mathbf{v})\cdot\mathbf{n}\,dS \\ &= \oint_C \mathbf{v}\cdot d\mathbf{l} \\ &= \kappa \end{aligned} \tag{18}$$

where in the second line we've used Stokes's theorem and κ is the circulation in the vortex. So a vortex is analogous to a current-carrying wire. Show that for a vortex the results above can be written as

$$\mathbf{v}(\mathbf{x}) = \frac{\kappa}{4\pi}\int d\mathbf{l}'\times\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}, \tag{19}$$

and

$$E_K = \frac{\rho\kappa^2}{8\pi}\int\int\frac{d\mathbf{l}'\cdot d\mathbf{l}}{|\mathbf{x}-\mathbf{x}'|}. \tag{20}$$

Solution. We only need to make the replacement

$$\omega(\mathbf{x}')\, d^3x' \to \kappa\, d\mathbf{l}' \tag{21}$$

in the results in previous sections to obtain Eqs. (19) and (20).

4. **Vortex rings in superfluid helium** (extra credit). When cooled to temperatures below about 2 K liquid ^{4}He becomes a *superfluid*—for all practical purposes it is an ideal fluid with *zero* viscosity. As a result, vortices are stable entities (in a normal fluid the vorticity eventually diffuses away due to viscous effects). Even more amazingly, it turns out that the circulation of vortices in superfluid helium is quantized in units of h/m_4, where h is Planck's constant and m_4 is the mass of a ^{4}He atom (that the circulation should be quantized was first predicted by Onsager and independently by Feynman). This quantization was first convincingly demonstrated by Rayfield and Reif [Physical Review **136**, 1194 (1964)], who produced vortex rings (like smoke rings) in superfluid helium and measured their velocity and kinetic energy.

   (a) Use your results from the previous problem to show that the speed of a vortex ring of radius r and circulation κ is

$$v = \frac{\kappa}{4\pi r} \ln\left(\frac{C_1 r}{a}\right), \tag{22}$$

   and the kinetic energy is

$$E_K = \frac{\rho\kappa^2 r}{2} \ln\left(\frac{C_2 r}{a}\right), \tag{23}$$

   where C_1 and C_2 are numerical constants of order 1 and a is the size of the "core" of the vortex. You might want to have a look at *Jackson* Problem 5.32, which calculates the self inductance of a current loop.

   Solution. Let's start with the energy of the vortex ring. Let the ring be in the $x-y$ plane, with radius r. Then $\mathbf{x} = r\mathbf{e}_\rho$, $\mathbf{x}' = r\mathbf{e}_{\rho'}$, $d\mathbf{l} = r d\phi\, \mathbf{e}_\phi$, $d\mathbf{l}' = r d\phi'\, \mathbf{e}_{\phi'}$; we also have $\mathbf{e}_\rho \cdot \mathbf{e}_{\rho'} = \cos(\phi - \phi')$ and $\mathbf{e}_\phi \cdot \mathbf{e}_{\phi'} = \cos(\phi - \phi')$. The kinetic energy is then

$$E_K = \frac{\rho\kappa^2 r^2}{8\pi} \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \frac{\cos(\phi - \phi')}{\sqrt{2r^2[1 - \cos(\phi - \phi')]}}. \tag{24}$$

   Changing variables to $\alpha = \phi - \phi'$, we can perform one of the angular integrals, with the result

$$\begin{aligned} E_K &= \frac{\rho\kappa^2 r}{4} \int_{-\pi}^{\pi} d\alpha \frac{\cos\alpha}{\sqrt{2(1-\cos\alpha)}} \\ &= \frac{\rho\kappa^2 r}{2} \int_0^{\pi} d\alpha \frac{\cos\alpha}{2\sin(\alpha/2)} \\ &= \frac{\rho\kappa^2 r}{2} \int_0^{\pi/2} d\beta \frac{\cos 2\beta}{\sin\beta}, \end{aligned} \tag{25}$$

where in the last line we've changed variables to $\alpha = 2\beta$. The problem now is that the integral is divergent, since the integrand goes as β^{-1} for small β. Let's introduce a "cut-off" ϵ in the integral:

$$\int_{\epsilon}^{\pi/2} d\beta \frac{\cos 2\beta}{\sin\beta} \approx \ln\left(\frac{2e^{-2}}{\epsilon}\right) + O(\epsilon^2), \tag{26}$$

where the result has been expanded for small ϵ. The physical origin of our cut-off is the fact that our model for the vortex ring breaks down close to the ring—the vortex ring has a "core" of linear dimension a, which we assume is much smaller than r. Therefore, we expect that ϵ is of order a/r, so that the kinetic energy is

$$E_K \approx \frac{\rho\kappa^2 r}{2} \ln(C_2 r/a), \tag{27}$$

with C_2 a constant whose precise value will depend upon the details of the model of the core.

The motion of the vortex ring is self-induced, with the motion of the ring occurring in the direction normal to the plane of the ring (the z-direction). The velocity of the ring will be equal to the fluid velocity in the core of the vortex ring, so as before we take $\mathbf{x} = r\mathbf{e}_\rho$, $\mathbf{x}' = r\mathbf{e}_{\rho'}$, $d\mathbf{l} = rd\phi\,\mathbf{e}_\phi$, $d\mathbf{l}' = rd\phi'\,\mathbf{e}_{\phi'}$. We use Eq. (19) to calculate the velocity; we'll need the following quantities:

$$\begin{aligned} d\mathbf{l}' \times (\mathbf{x} - \mathbf{x}') &= r\mathbf{e}_{\phi'} d\phi' \times (r\mathbf{e}_\rho - r\mathbf{e}_{\rho'}) \\ &= r^2[1 - \cos(\phi - \phi')]\, d\phi'\, \mathbf{e}_z, \end{aligned} \tag{28}$$

$$|\mathbf{x} - \mathbf{x}'|^3 = r^3 \left\{2[1 - \cos(\phi - \phi')]\right\}^{3/2}. \tag{29}$$

The velocity on the vortex ring is then

$$\begin{aligned} \mathbf{v}_r &= \frac{\kappa}{8\pi r}\mathbf{e}_z \int_{-\pi}^{\pi} \frac{d\alpha}{\sqrt{2(1-\cos\alpha)}} \\ &= \frac{\kappa}{4\pi r}\mathbf{e}_z \int_0^{\pi/2} \frac{d\beta}{\sin\beta} \end{aligned} \tag{30}$$

where we've changed variables to $\alpha = \phi - \phi'$ in the second line and $\alpha = 2\beta$ in the third. Once again, the integral is divergent, and a cut-off must be introduced. We then have

$$\mathbf{v}_r \approx \frac{\kappa}{4\pi r} \ln(C_1 r/a)\mathbf{e}_z, \tag{31}$$

with C_1 another numerical constant which depends on the details of the model of the vortex core.

(b) Ignore the weak logarithmic dependence on the radius, and show that the velocity of the vortex ring is inversely proportional to the energy of the ring. How could you use this relationship to extract the circulation of the vortex ring? Look up Rayfield and Reif's paper, and use their data to infer a value of κ. How does it compare to h/m_4?

Solution. As suggested, let's ignore the logs. Then combining our results, we find

$$v_r \approx \frac{\rho\kappa^3}{8\pi E_K}, \tag{32}$$

so that the speed of the vortex ring is inversely proportional to its kinetic energy! The data in Rayfield and Reif agree with this prediction. The constant of proportionality goes as $\rho\kappa^3$, so once the fluid density is known the circulation can be determined. To obtain precise values of κ one needs a model of the core, and more detailed calculations are required, along the lines of *Jackson* Problem 5.32.

Physics 6346, Electromagnetic Theory I
Fall 2000
Homework 11 Solutions

1. *Jackson* 5.13. A sphere of radius a carries a uniform surface-charge distribution σ. The sphere is rotated about a diameter with constant angular velocity ω. Find the vector potential and magnetic-flux density both inside and outside the sphere.

   Solution. As a result of the rotating surface charge density we have an effective current density J which is localized on the surface of the sphere; with $\mathbf{r} = a\mathbf{e}_r$ and $\boldsymbol{\omega} = \omega\mathbf{e}_z$, we have

$$\begin{aligned}\mathbf{J}(\mathbf{x}) &= [\sigma\delta(r-a)]\,\boldsymbol{\omega}\times\mathbf{r} \\ &= \sigma a\omega\delta(r-a)\sin\theta\,\mathbf{e}_\phi \\ &= \sigma a\omega\delta(r-a)\sin\theta[-\sin\phi\,\mathbf{e}_x + \cos\phi\,\mathbf{e}_y]. \end{aligned} \tag{1}$$

   The vector potential is

$$\begin{aligned}\mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \\ &= \frac{\mu_0\sigma a^3\omega}{4\pi}\int\frac{[-\sin\phi'\mathbf{e}_x + \cos\phi'\mathbf{e}_y]\sin\theta'}{|\mathbf{x}-\mathbf{x}'|}d\Omega', \end{aligned} \tag{2}$$

   where we've carried out the integration on the radial coordinate r' using the delta function. The integral can be calculated by first expressing the terms in the integrand in terms of spherical harmonics:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_l^{m*}(\theta',\phi')Y_l^m(\theta,\phi). \tag{3}$$

$$\sin\theta\sin\phi = \frac{i}{2}\sqrt{\frac{8\pi}{3}}(Y_1^1 + Y_1^{-1}), \quad \sin\theta\cos\phi = \frac{1}{2}\sqrt{\frac{8\pi}{3}}(-Y_1^1 + Y_1^{-1}). \tag{4}$$

   Substituting into Eq. (2), and using the fact that the spherical harmonics are orthonormal on the unit sphere, we have

$$\begin{aligned}\mathbf{A} &= \frac{\mu_0\sigma a^3\omega}{3}\frac{r_<}{r_>^2}\sin\theta\,[-\sin\phi\,\mathbf{e}_x + \cos\phi\,\mathbf{e}_y] \\ &= \frac{\mu_0\sigma a^3\omega}{3}\frac{r_<}{r_>^2}\sin\theta\,\mathbf{e}_\phi, \end{aligned} \tag{5}$$

   where $r_>$ ($r_<$) is the larger (smaller) of r and a. Inside the sphere, this is

$$\mathbf{A}_{\text{in}} = \frac{\mu_0\sigma\omega a}{3}r\sin\theta\,\mathbf{e}_\phi, \tag{6}$$

so that the magnetic flux density is

$$\mathbf{B}_{\text{in}} = \frac{2}{3}\mu_0\sigma\omega a\,\mathbf{e}_z. \tag{7}$$

Outside the sphere the vector potential is

$$\mathbf{A}_{\text{out}} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3}, \tag{8}$$

where the magnetic dipole moment is

$$\mathbf{m} = \frac{4\pi a^3}{3}\sigma\omega a\,\mathbf{e}_z. \tag{9}$$

The field exterior to the sphere is therefore a dipole field:

$$\mathbf{B}_{\text{out}} = \frac{\mu_0}{4\pi}\frac{3(\mathbf{m}\cdot\mathbf{n})\mathbf{n}-\mathbf{m}}{|\mathbf{x}|^3}. \tag{10}$$

2. *Jackson* 5.17. A current distribution $\mathbf{J}(\mathbf{x})$ exists in a medium of unit relative permeability adjacent to a semi-infinite slab of material having relative permeability μ, and filling the half-space, $z < 0$.

   (a) Show that for $z > 0$ the magnetic induction can be calculated by replacing the medium of permeability μ, by an image current distribution $\mathbf{J}^*$, with components

$$\left(\frac{\mu_r-1}{\mu_r+1}\right)J_x(x,y,-z),\quad \left(\frac{\mu_r-1}{\mu_r+1}\right)J_y(x,y,-z),\quad -\left(\frac{\mu_r-1}{\mu_r+1}\right)J_z(x,y,-z). \tag{11}$$

   (b) Show that for $z < 0$ the magnetic induction appears to be due to a current distribution $[2\mu_r/(\mu_r+1)]\mathbf{J}$ in a medium of unit relative permeability.

   Solution. Let's define the image currents $\mathbf{J}_1$, which is zero for $z > 0$, and $\mathbf{J}_2$, which is zero for $z < 0$. The magnetic field for $z > 0$ is then

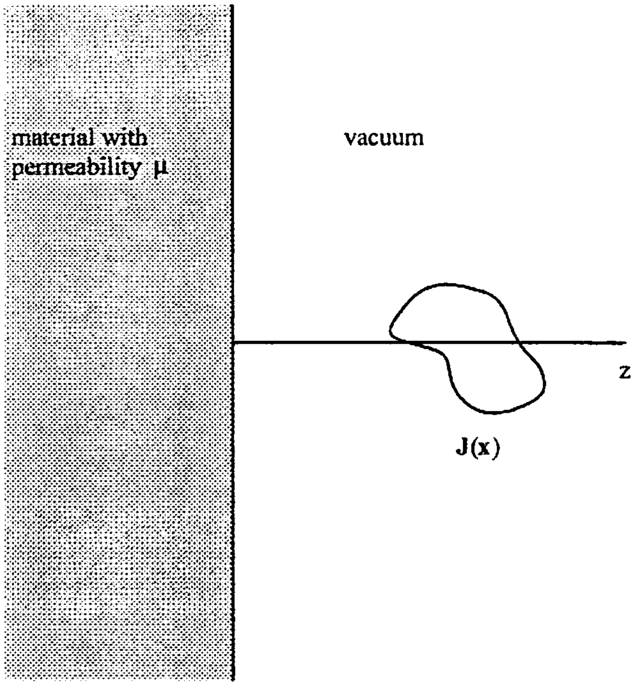
$$\mathbf{B} = \frac{\mu_0}{4\pi}\int[\mathbf{J}(\mathbf{x}')+\mathbf{J}_1(\mathbf{x}')]\times\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\,d^3x', \tag{12}$$

   and for $z < 0$ the field is

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int\mathbf{J}_2(\mathbf{x}')\times\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\,d^3x'. \tag{13}$$

   The normal component of B must be continuous at $z = 0$. We have

$$B_z(x,y,z=0^+) = \frac{\mu_0}{4\pi}\int\frac{(J_x+J_{1x})(y-y')-(J_y+J_{1y})(x-x')}{[(x-x')^2+(y-y')^2+z'^2]^{3/2}}\,d^3x', \tag{14}$$



$$B_z(x, y, z = 0^-) = \frac{\mu_0}{4\pi} \int \frac{J_{2x}(y - y') - J_{2y}(x - x')}{[(x - x')^2 + (y - y')^2 + z'^2]^{3/2}} \, d^3x'. \tag{15}$$

Taking $B_z(x, y, z = 0^+) = B_z(x, y, z = 0^-)$, we have

$$\int d^3x' \frac{1}{[(x - x')^2 + (y - y')^2 + z'^2]^{3/2}} \times [(J_x + J_{1x} - J_{2x})(y - y') - (J_y + J_{1y} - J_{2y})(y - y')] = 0. \tag{16}$$

We need to be careful here—$\mathbf{J}_1$ and $\mathbf{J}_2$ are non-zero in different regions. However, we can take $z' \to -z'$ in J_{1x} without changing the integral, so that $J_x(x, y, z)$, $J_{1x}(x, y, -z)$, and $J_{2x}(x, y, z)$ are all non-zero in the same region of space. A sufficient condition for the integral to vanish is that the integrand vanish, so we have

$$J_x(x, y, z) + J_{1x}(x, y, -z) - J_{2x}(x, y, z) = 0. \tag{17}$$

$$J_y(x, y, z) + J_{1y}(x, y, -z) - J_{2y}(x, y, z) = 0. \tag{18}$$

Next, we need to make sure that the tangential component of $\mathbf{H}$ is continuous across the interface. Using $\mathbf{H} = \mathbf{B}/\mu$, we have

$$H_y(x, y, z = 0^+) = \frac{1}{4\pi} \int \frac{(x - x')(J_z + J_{1z}) - (-z')(J_x + J_{1x})}{[(x - x')^2 + (y - y')^2 + z'^2]^{3/2}} \, d^3x', \tag{19}$$

$$H_y(x,y,z=0^-) = \frac{1}{4\pi\mu_r}\int \frac{(x-x')J_{2z}-(-z')J_{2x}}{[(x-x')^2+(y-y')^2+z'^2]^{3/2}}\,d^3x', \tag{20}$$

with similar expressions for H_x. Setting $H_y(z=0^+) = H_y(z=0^-)$, and taking $z \to -z$ in the integrals involving $\mathbf{J}_1$, we find

$$\mu_r J_z(x,y,z) + \mu_r J_{1z}(x,y,-z) - J_{2z}(x,y,z) = 0, \tag{21}$$

$$\mu_r J_x(x,y,z) - \mu_r J_{1x}(x,y,-z) - J_{2x}(x,y,z) = 0, \tag{22}$$

$$\mu_r J_y(x,y,z) - \mu_r J_{1y}(x,y,-z) - J_{2y}(x,y,z) = 0. \tag{23}$$

Solving Eqs. (17), (18), (22), and (23), we find

$$J_{1x}(x,y,z) = \frac{\mu_r - 1}{\mu_r + 1}J_x(x,y,-z), \quad J_{1y}(x,y,z) = \frac{\mu_r - 1}{\mu_r + 1}J_y(x,y.-z), \tag{24}$$

$$J_{2x}(x,y,z) = \frac{2\mu_r}{\mu_r + 1}J_x(x,y,z), \quad J_{2y}(x,y,z) = \frac{2\mu_r}{\mu_r + 1}J_y(x.y,z). \tag{25}$$

To find the z-components, we need another equation. The current density $\mathbf{J}_2$ must have zero divergence, so that

$$\begin{aligned}
\nabla\cdot\mathbf{J}_2 &= \frac{\partial J_{2x}}{\partial x} + \frac{\partial J_{2y}}{\partial y} + \frac{\partial J_{2z}}{\partial z} \\
&= \frac{2\mu_r}{\mu_r+1}\left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y}\right) + \mu_r\frac{\partial}{\partial z}\left[J_z + J_{1z}(-z)\right] \\
&= -\frac{2\mu_r}{\mu_r+1}\frac{\partial J_z}{\partial z} + \mu_r\frac{\partial}{\partial z}\left[J_z + J_{1z}(-z)\right] \\
&= 0,
\end{aligned} \tag{26}$$

where we've used Eqs. (21), (24), and (25), along with $\nabla\cdot\mathbf{J} = 0$. Integrating this last equation, we have

$$J_z(x,y,z) = \frac{\mu_r + 1}{2}[J_z(x,y,z) + J_{1z}(x,y,-z)]. \tag{27}$$

We can finally solve for the z-components, with the result

$$J_{1z}(x,y,z) = -\frac{\mu_r - 1}{\mu_r + 1}J_z(x,y,-z), \quad J_{2z}(x,y,z) = \frac{2\mu_r}{\mu_r + 1}J_z(x,y,z). \tag{28}$$

To summarize,

$$\mathbf{J}_1(x,y,z) = \frac{\mu_r - 1}{\mu_r + 1}\left[J_x(x,y,-z)\mathbf{e}_x + J_y(x,y,-z)\mathbf{e}_y - J_z(x,y,-z)\mathbf{e}_z\right], \tag{29}$$

$$\mathbf{J}_2 = \frac{2\mu_r}{\mu_r + 1}\mathbf{J}(x,y,z). \tag{30}$$

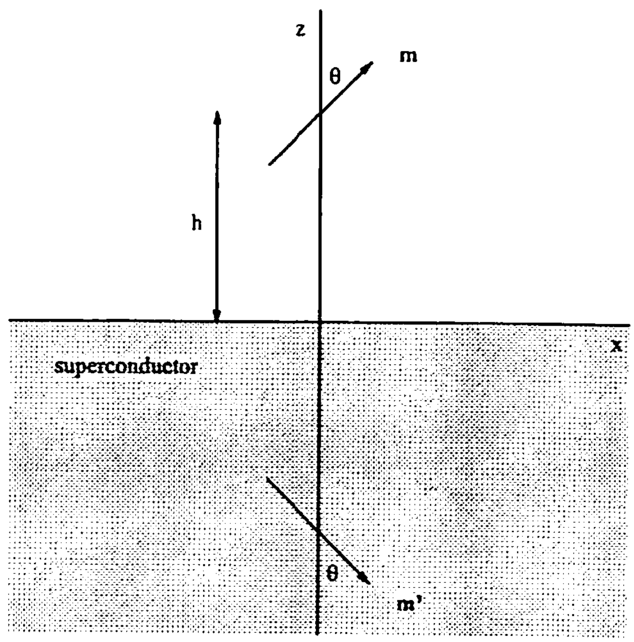
3. **Superconducting levitation.** A now common lecture demonstration involves levitating a permanent magnet above a superconducting disc (usually YBCO, one of the new high temperature superconductors). We'll idealize the problem by assuming that (a) $\mathbf{B} = 0$ inside the superconductor; (b) the superconductor is infinite in extent, occupying the half-space $z < 0$; (c) the permanent magnet can be approximated by a point magnetic dipole with dipole moment $\mathbf{m}$, which makes an angle θ with the z-axis; (d) the magnet can be considered to be a point mass of mass M.

   (a) This problem can be solved using the method of images. What image dipole is needed?

   Solution. Using the results of the previous problem, one can show quite generally that for a linear magnetic material with relative permeability μ_r, the image dipole has components

$$m'_x = -\left(\frac{\mu_r - 1}{\mu_r + 1}\right) m_x, \quad m'_z = \left(\frac{\mu_r - 1}{\mu_r + 1}\right) m_z. \tag{31}$$

   For the purposes of this problem we can think of a superconductor as a material with $\mu_r = 0$, so that the image dipole is reflected through the x-axis, as shown in the figure below. This will insure that $\mathbf{B} \cdot \mathbf{n} = 0$ on the surface of the superconductor.



   (b) If the dipole is a distance h above the superconductor, what is the magnetic force on the dipole?

Solution. The magnetic force on the dipole is the force between the dipole and its image. This is

$$\mathbf{F} = -\nabla U = \nabla(\mathbf{m}\cdot\mathbf{B}'). \tag{32}$$

where $\mathbf{B}'$ is the magnetic field produced by the image dipole,

$$\mathbf{B}' = \frac{\mu_0}{4\pi}\frac{3(\mathbf{m}'\cdot\mathbf{e}_z)\mathbf{e}_z - \mathbf{m}'}{(z+h)^3}. \tag{33}$$

Therefore,

$$\begin{aligned} U &= -\frac{\mu_0}{4\pi}\frac{3(\mathbf{m}'\cdot\mathbf{e}_z)(\mathbf{m}'\cdot\mathbf{e}_z) - \mathbf{m}\cdot\mathbf{m}'}{(z+h)^3} \\ &= \frac{\mu_0}{4\pi}\frac{m^2(1+\cos^2\theta)}{(z+h)^3}, \end{aligned} \tag{34}$$

where we've used $m'_x = m_x$, $m'_z = -m_z$, and $m_z = m\cos\theta$. The force is then

$$F_z = -\left(\frac{\partial U}{\partial z}\right)_{z=h} = \frac{\mu_0}{4\pi}\frac{3m^2(1+\cos^2\theta)}{16h^4}. \tag{35}$$

(c) What is the torque on the dipole? For fixed h, what θ is needed for stable equilibrium?

Solution. The torque $\mathbf{N}$ is

$$\begin{aligned} \mathbf{N} &= \mathbf{m}\times\mathbf{B}' \\ &= \frac{\mu_0}{4\pi}\frac{m^2}{8h^3}\sin\theta\cos\theta\,\mathbf{e}_y. \end{aligned} \tag{36}$$

The torque is zero when $\theta = 0$, $\pi/2$, and π. By examining the potential energy U calculated above, we see that $\theta = \pi/2$ is a stable equilibrium point—the dipole is aligned parallel to the surface of the superconductor.

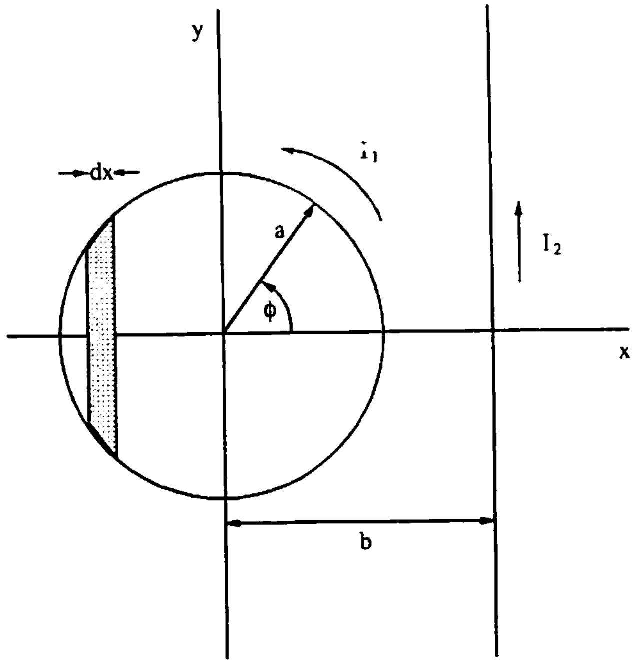
(d) Calculate the equilibrium height h^* of the dipole above the superconductor.

Solution. Setting $\theta = \pi/2$ in the magnetic force, and equating it with the weight Mg, we find that the equilibrium height is

$$h^* = \left(\frac{\mu_0}{4\pi}\frac{3m^2}{16Mg}\right)^{1/4}. \tag{37}$$

4. **Induction.** A long, straight wire is in the same plane as a circular wire loop of radius a and resistance R; the straight wire is a distance $b > a$ from the center of the loop. A current $I(t) = I_0\cos\omega t$ flows in the straight wire. Find the force on the loop.

Solution. Let's place the loop and the straight wire in the $x-y$ plane, with the center of the loop at the origin and the straight wire at $x = b$, as shown in the figure below. The



changing current I_2 in the straight wire changes the flux through the loop, inducing a current I_1 in the loop. If $dI_2/dt < 0$, then Lenz's law tells us that the induced current is counterclockwise, as shown in the figure. With this in mind, the force on the loop is

$$\mathbf{F}_{12} = I_1 \oint d\mathbf{l}_1 \times \mathbf{B}, \tag{38}$$

where $\mathbf{B}$ is the magnetic field produced by the straight wire. We have $d\mathbf{l}_1 = a\,d\phi\,\mathbf{e}_\phi$, and the magnetic field produced by the wire is $\mathbf{B} = (\mu_0 I_2/2\pi R)\mathbf{e}_z$, where the distance from the wire to the loop is $R = b - a\cos\phi$. Using $\mathbf{e}_\phi \times \mathbf{e}_z = \mathbf{e}_\rho = \cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y$, and putting all of this together, we have

$$F_x = \frac{\mu_0 I_1 I_2}{2\pi} \int_0^{2\pi} \frac{\cos\phi}{(b/a) - \cos\phi}\, d\phi \tag{39}$$

$$= \mu_0 I_1 I_2 \left[\frac{1}{\sqrt{1 - (a/b)^2}} - 1 \right]. \tag{40}$$

The integral for F_y vanishes by symmetry, as expected. The force on the loop is in the positive x direction.

We now need to find the induced current I_1. To do this, let's first find the flux through the loop due to the current I_2 (see the figure for the area element of width dx and height $2y = 2\sqrt{a^2 - x^2}$):

$$\Phi_{12} = \int_{\text{loop}} B_z\, da$$

$$= \int_{-a}^{a} \frac{\mu_0 I_2}{2\pi(b-x)} 2\sqrt{a^2 - x^2}\, dx. \tag{41}$$

We can put the integral into a more convenient form by changing variables as $x = a\cos\phi$, so that

$$\begin{aligned} \Phi_{12} &= \frac{\mu_0 I_2 a^2}{\pi} \int_0^{\pi} \frac{\sin^2\phi}{b - a\cos\phi} d\phi \\ &= \mu_0 I_2 b \left[1 - \sqrt{1-(a/b)^2}\right], \end{aligned} \tag{42}$$

where the integral can be found in tables (see Gradshteyn and Ryzhik Eq. 3.644.4, for instance). As a check, note that when $b \gg a$, the flux is

$$\Phi_{12} \approx \mu_0 I_2 b \left(\frac{a^2}{2b^2}\right) = \frac{\mu_0 I_2}{2\pi b}(\pi a^2), \tag{43}$$

which is the magnetic field a distance b from the loop ($\mu_0 I_2 / 2\pi b$) times the area of the loop (πa^2). We see that the mutual inductance for this arrangement of currents is

$$M = \mu_0 b \left[1 - \sqrt{1-(a/b)^2}\right]. \tag{44}$$

From Faraday's law, we have for the induced emf $\mathcal{E} = -d\Phi/dt = I_1 R$, with Φ the *total* flux through the loop,

$$\Phi = L I_1 + M I_2, \tag{45}$$

where L is the self-inductance of the loop. We therefore have a simple differential equation for the current I_1,

$$\frac{L}{R}\frac{dI_1}{dt} + I_1 = -\frac{M}{R}\frac{dI_2}{dt}. \tag{46}$$

Let's assume that the self-inductance is negligible (or more precisely, that $L\omega/R \ll 1$, with ω the frequency). Then

$$I_1 = -\frac{\mu_0 b}{R}\frac{dI_2}{dt}\left[1 - \sqrt{1-(a/b)^2}\right]. \tag{47}$$

Putting all of this together, the force is

$$\begin{aligned} F_x &= -\frac{\mu_0^2}{R} I_2 \frac{dI_2}{dt} b \left[\frac{2-(a/b)^2}{\sqrt{1-(a/b)^2}} - 2\right] \\ &= \frac{\mu_0^2 I_0^2 \omega b}{R} \sin\omega t \cos\omega t \left[\frac{2-(a/b)^2}{\sqrt{1-(a/b)^2}} - 2\right]. \end{aligned} \tag{48}$$

For $b \gg a$, this becomes

$$F_x \approx -\frac{\mu_0^2 I_0^2 \omega}{4R}\frac{a^4}{b^3}\cos 2\omega t. \tag{49}$$

You should check and make sure that you understand why this behaves as b^{-3}, why it is proportional to a^4, and so on.

What happens when the self-inductance is *not* negligible?

5. **Eddy currents.** A wire loop, of negligible thickness, has a radius R and carries a counter-clockwise current I. The loop is centered at the origin of the $x-y$ plane, so that the z axis passes through the center of the loop.

   (a) Find the magnetic field in the $x-y$ plane. Express your result in terms of elliptic integrals.

   Solution. The vector potential for a wire loop of radius R is (in cylindrical coordinates (ρ, ϕ, z))

$$A_\phi(\rho, z) = \frac{\mu_0 I}{4\pi}\frac{4R}{\sqrt{(\rho+R)^2+z^2}}\left[\frac{(2-k^2)K(k)-2E(k)}{k^2}\right], \tag{50}$$

   where

$$k^2 \equiv \frac{4R\rho}{(\rho+R)^2+z^2}, \tag{51}$$

   and where K and E are the complete elliptic integrals of the first and second kind. To calculate the magnetic field, we have to take the curl of the vector potential:

$$B_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi), \quad B_\rho = -\frac{\partial A_\phi}{\partial z}. \tag{52}$$

   In the $x-y$ plane the field will only have a z-component, so we only need to take the ρ derivative of A_ϕ. This is a little complicated; there are identities which can be used to simplify the algebra, but I chose to use *Maple* instead. The result for the field in the plane is

$$B_z(\rho, z=0) = \frac{\mu_0 I}{2\pi}\left[\frac{K(k)}{R+\rho}+\frac{E(k)}{R-\rho}\right], \tag{53}$$

   with k evaluated at $z=0$. For $\rho \ll R$, this can be expanded as

$$B_z(\rho, z=0) = \frac{\mu_0 I}{2R}\left[1+\frac{3\rho^2}{4R^2}+O(\rho^4/R^4)\right]. \tag{54}$$

   (b) A small coin, of thickness d, radius a, and conductivity σ is placed at the center of, and coplanar with, the loop. If the current in the loop is $I(t) = I_0\cos(\omega t)$, find the time-averaged power loss, assuming that $d \ll a \ll R$ and that the loop is a perfect conductor.

Solution. The changing field will induce eddy currents in the coin. To determine these currents, we first need to find the flux through a loop of radius ρ centered on the coin. Since the radius of the coin is small compared to the radius of the wire loop, we can assume that the magnetic field is uniform across the coin. Therefore, we have for the flux

$$\begin{aligned} \Phi &= \int_0^\rho B_z 2\pi\rho'\, d\rho' \\ &= \int_0^\rho \frac{\mu_0 I}{2R} 2\pi\rho'\, d\rho' \\ &= \frac{\pi\mu_0 I\rho^2}{2R}. \end{aligned} \tag{55}$$

The emf generated in this loop is

$$\begin{aligned} \mathcal{E} &= -\frac{d\Phi}{dt} \\ &= -\frac{\pi\mu_0\rho^2}{2R}\frac{dI}{dt}. \end{aligned} \tag{56}$$

However, the emf is

$$\begin{aligned} \mathcal{E} &= \oint \mathbf{E}\cdot d\mathbf{l} \\ &= 2\pi\rho E_\phi, \end{aligned} \tag{57}$$

so the induced electric field is

$$E_\phi = -\frac{\mu_0\rho}{4R}\frac{dI}{dt}. \tag{58}$$

The eddy current is then $J_\phi = \sigma E_\phi$, so that the power dissipated per unit volume is $\mathbf{J}\cdot\mathbf{E} = \sigma E_\phi^2$. To find the total power dissipated, we need to integrate this over the volume of the coin; the volume element is $d \times 2\pi\rho\, d\rho$, so we have

$$\begin{aligned} P &= \int_0^a (\sigma E_\phi^2)\, d(2\pi\rho d\rho) \\ &= \frac{\pi}{32}\frac{\mu_0^2\sigma d a^4}{R^2}\left(\frac{dI}{dt}\right)^2. \end{aligned} \tag{59}$$

Finally, we need to time average the power over a cycle, with the result that $\langle (dI/dt)^2\rangle = I_0^2\omega^2/2$. Our final result is then

$$P_{\text{average}} = \frac{\pi}{64}\mu_0^2 I_0^2\sigma\omega^2 d a^4 R^2. \tag{60}$$

(c) Assuming that there is negligible heat transfer between the coin and the surrounding medium, find the time rate of change of the temperature of the coin. You can take the specific heat per unit volume c_p to be approximately temperature independent.

Solution. Using $P_{\text{ave}} = \dot{Q} = V c_p \dot{T}$, with $V = \pi a^2 d$ the volume of the coin, we have

$$\dot{T} = \frac{\mu_0^2 I_0^2 \sigma \omega^2}{64 c_p} \frac{a^2}{R^2}. \tag{61}$$

(d) Put some reasonable numbers into your expression above, and evaluate the efficacy of this inductive heater.

Solution. From my copy of the ***Physicist's Desk Reference***, I found that the specific heat of copper is 0.092 cal/g K, the mass density is 8.96 g/cm^3, and the electrical conductivity is $6.67 \times 10^5\ \Omega^{-1}\text{cm}^{-1}$. Putting these numbers together, and taking a frequency of 1 MHz and $a/R = 0.1$, I find that $\dot{T} \approx 2$ K/s, which is a fairly rapid temperature raise. However, it should be noted that at these frequencies the skin depth for copper is about 10^{-5} m, so that the approximations which we've used above don't hold, and a different analysis is needed (we need to solve the diffusion equation for the magnetic field). At a frequency of 100 Hz the skin depth is about 1 mm, so our analysis might be applicable. Of course, at this frequency $\dot{T} \approx 2 \times 10^{-8}$ K/s, so there is a negligible temperature increase (a good thing given our use of 60 Hz power supply).

Induction "furnaces" are often used in materials processing applications. One can also purchase kitchen ranges which use induction heating.

Physics 6346 Exam I Solutions

1. **Short answer** (30 points).

   (a) Write down the basic differential equations which determine the electrostatic field **E**.

   Solution.
   $$\nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{E} = \rho/\epsilon_0. \tag{1}$$

   (b) Write down an expression for the *energy density* stored in an electrostatic field **E**.
   Solution. The energy density is $u = \epsilon_0 |\mathbf{E}|^2/2$.

   (c) A hollow conducting sphere of radius R has a charge Q placed on its surface. What is the electric field *inside* the sphere? The potential?

   Solution. The electric field inside is zero. Assuming that we continue with our convention that the potential is zero infinitely far from a localized charge distribution, then the potential at the surface, and at all points within, is $\Phi = Q/4\pi\epsilon_0 R$.

   (d) What is the electrostatic potential of a dipole of dipole moment $\mathbf{p}$? What is the electric field?

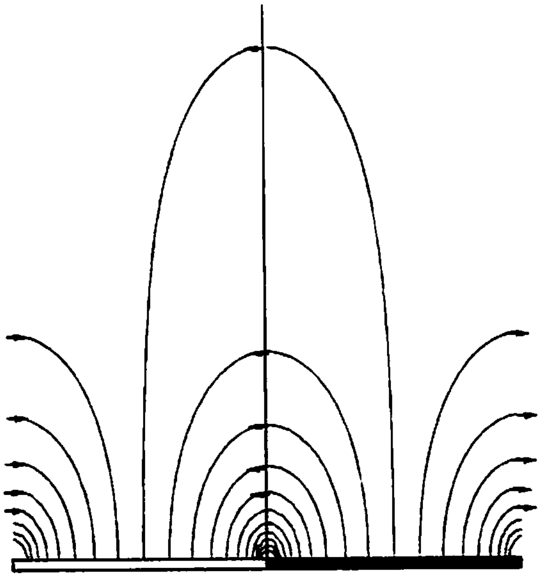
   Solution. If the dipole is at the origin, the potential is
   $$\Phi(\mathbf{x}) = \frac{\mathbf{p} \cdot \mathbf{x}}{4\pi\epsilon_0 |\mathbf{x}|^3}, \tag{2}$$
   and the electric field is
   $$\mathbf{E}(\mathbf{x}) = \frac{3\mathbf{n}(\mathbf{p} \cdot \mathbf{n}) - \mathbf{p}}{4\pi\epsilon_0 |\mathbf{x}|^3}, \tag{3}$$
   where $\mathbf{n}$ is a unit vector directed from the origin to $\mathbf{x}$.

   (e) A point charge of charge $q > 0$ is outside a conducting sphere which carries a total charge $Q > 0$. Is the force between the point charge and the sphere attractive, repulsive, or both? Explain.

   Solution. Sufficiently far from the sphere the force will be repulsive. As the charge is brought closer to the sphere, eventually the force due to the image charge will dominate, and the net force will become attractive.

2. **Fourier series and separation of variables** (35 points). A series of thin conducting strips, each of width L and infinitely long, are held at potentials $-V$ and V (there are small gaps between the conductors which can be ignored) as shown in the figure below. The pattern repeats itself along the x-axis; you can ignore the thickness of the conductors.

   (a) On the figure above, sketch the field lines (no calculation required). You need only consider the region $-L < x < L$ and $y > 0$.



(b) Express the potential along the x-axis, $\Phi(x, 0)$, as a Fourier series.

Solution.

$$\Phi(x,0) = \frac{4V}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left[(2n+1)\pi x/L\right]}{2n+1}. \tag{4}$$

(c) Use the method of separation of variables in rectangular coordinates to find the potential $\Phi(x, y)$. You can leave your result in the form of an infinite series. You may want to treat $y > 0$ and $y < 0$ separately.

Solution. The method is standard. Assume $\Phi(x, y) = X(x)Y(y)$ and substitute into Laplace's equation to obtain

$$\frac{d^2X}{dx^2} + k^2X = 0, \quad \frac{d^2Y}{dy^2} - k^2Y = 0, \tag{5}$$

where k^2 is the separation constant. Solving the equation for Y, and requiring that the potential be bounded for $|y| \to \infty$, we find (taking $k > 0$)

$$Y(y) = Ce^{-k|y|}. \tag{6}$$

The general solutions for $X(x)$ are linear combinations of $\sin(kx)$ and $\cos(kx)$; however, comparing with the boundary condition in Eq. (4), we see that the coefficient of the $\cos(kx)$ term must be zero, and that $k = (2n+1)\pi/L$, so that

$$\Phi(x,y) = \frac{4V}{\pi}\sum_{n=0}^{\infty}\exp[-(2n+1)\pi|y|/L]\frac{\sin\left[(2n+1)\pi x/L\right]}{2n+1}. \tag{7}$$

(d) Sum the series for the potential to obtain a more compact expression. A possibly useful result is

$$\sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1} = \frac{1}{2} \ln \left(\frac{1+z}{1-z} \right). \tag{8}$$

Solution. Write the solution as

$$\Phi(x.y) = \frac{4V}{\pi} \mathrm{Im} \sum_{n=0}^{\infty} \frac{\zeta^{2n+1}}{2n+1}, \tag{9}$$

where

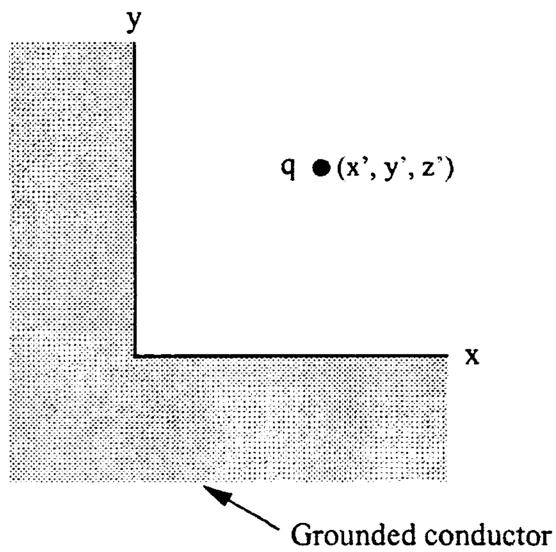
$$\zeta = e^{(i\pi/L)(x+i|y|)}. \tag{10}$$

Using the hint, we find

$$\begin{aligned} \Phi(x,y) &= \frac{2V}{\pi} \mathrm{Im} \ln \left(\frac{1+\zeta}{1-\zeta} \right) \\ &= \frac{2V}{\pi} \tan^{-1} \left[\frac{\sin(\pi x/L)}{\sinh(\pi |y|/L)} \right]. \end{aligned} \tag{11}$$

The solution is a periodic extension of the conducting "trough" problem in *Jackson.*

3. **Method of images** (35 points). A point charge q is placed at a position (x', y', z'), near a grounded conducting "corner," which occupies the region $x < 0$, $y < 0$, $-\infty < z < \infty$, as shown in the figure below.



(a) Find the electrostatic potential $\Phi(\mathbf{x})$ in the region $x > 0$, $y > 0$.

Solution. This problem can be solved by introducing three image charges: $-q$ at positions $(-x', y', z')$ and $(x', -y', z')$, and $+q$ at $(-x', -y', z')$. The potential is then

$$\begin{aligned}\Phi(x,y,z) &= \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}}\right. \\ &\quad+\frac{1}{[(x+x')^2+(y+y')^2+(z-z')^2]^{1/2}} \\ &\quad-\frac{1}{[(x+x')^2+(y-y')^2+(z-z')^2]^{1/2}} \\ &\quad\left.-\frac{1}{[(x-x')^2+(y+y')^2+(z-z')^2]^{1/2}}\right\}.\end{aligned} \tag{12, 13}$$

You can verify that this is zero on $x = 0$ and on $y = 0$.

(b) What is the Green's function $G(x, y, z; x', y', z')$ in the region $x > 0$, $y > 0$, for these boundary conditions?

Solution. Remember that the Green's function is the solution for the unit charge (set $(q/4\pi\epsilon_0) = 1$ in the potential):

$$\begin{aligned}G(x,y,z;x',y',z') &= \frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}} \\ &\quad+\frac{1}{[(x+x')^2+(y+y')^2+(z-z')^2]^{1/2}} \\ &\quad-\frac{1}{[(x+x')^2+(y-y')^2+(z-z')^2]^{1/2}} \\ &\quad-\frac{1}{[(x-x')^2+(y+y')^2+(z-z')^2]^{1/2}}.\end{aligned} \tag{14}$$

(c) Suppose that the point charge is replaced by a localized charge distribution in the region $x > 0$, $y > 0$, with charge density $\rho(\mathbf{x})$. What is the electrostatic potential now?

Solution. With the Dirichlet boundary conditions, the potential is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int G(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}')\,d^3x', \tag{15}$$

with the Green's function given above.

(d) Let's return to the point charge; assuming that it is located at $x' = a$, $y' = a$, and $z' = 0$, find the net force acting on the charge (magnitude and direction).

Solution. The force is determined by calculating the force between the point charge and all of its image charges. The resultant is

$$F_x = -\frac{q^2}{4\pi\epsilon_0}\frac{4-\sqrt{2}}{16a^2}, \tag{16}$$

$$F_y = -\frac{q^2}{4\pi\epsilon_0}\frac{4-\sqrt{2}}{16a^2}. \tag{17}$$

We see that the charge is attracted to the corner.

Physics 6346 Exam II Solutions

1. **Short answer** (30 points).

   (a) Write down the fundamental equations of electrostatics in a dielectric medium.
   Solution.
   $$\nabla \cdot \mathbf{D} = \rho, \quad \nabla \times \mathbf{E} = 0. \tag{1}$$
   We also would need a constitutive relation which relates **E** and **D**.

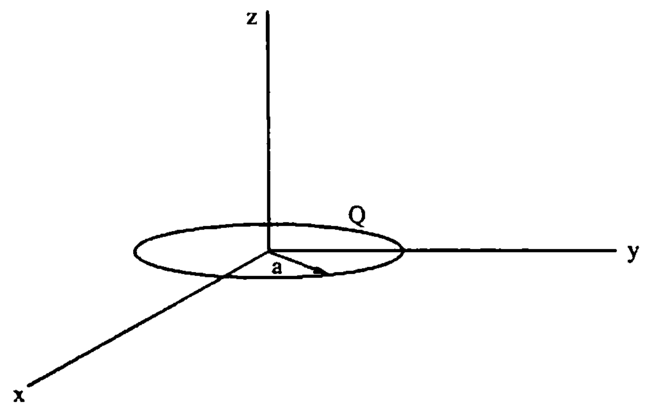
   (b) If a complex function $f(z) = u + iv$ is analytic, what are the equations satisfied by u and v?
   Solution. If $f(z)$ is analytic, then its real and imaginary parts satisfy the Cauchy-Riemann equations,
   $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2}$$
   We can combine these equations to show that $\nabla^2 u = \nabla^2 v = 0$.

   (c) If we expand a function $f(\theta, \phi)$ on the unit sphere in spherical harmonics,
   $$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y_l^m(\theta, \phi), \tag{3}$$
   what are the expansion coefficients?
   Solution. Using the fact that the spherical harmonics are orthonormal on the unit sphere, we have
   $$A_{lm} = \int f(\theta, \phi) Y_l^{m*}(\theta, \phi)\, d\Omega. \tag{4}$$

   (d) Of the functions $J_1(x)$, $N_1(x)$, $K_1(x)$, and $I_1(x)$, which oscillate with x? Which diverge for small x? Which diverge for large x?
   Solution. $J_1(x)$ and $N_1(x)$ oscillate with x, $N_1(x)$ and $K_1(x)$ diverge as $x \to 0$, and $I_1(x)$ diverges as $x \to \infty$.

   (e) If a localized charge distribution with charge density $\rho(\mathbf{x})$ (which is localized around $\mathbf{x} = 0$) is placed in an *external* potential $\Phi(\mathbf{x})$, what is the electrostatic energy W in terms of the total charge q, the dipole moment $\mathbf{p}$, and the quadrupole moment tensor Q_{ij}?
   Solution. The expansion is
   $$W = q\Phi(0) - \mathbf{p} \cdot \mathbf{E}(0) - \frac{1}{6} \sum_{ij} Q_{ij} \frac{\partial E_i}{\partial x_j}(0) + \ldots \tag{5}$$

2. **Charged ring.** (35 points). A charge Q is uniformly distributed on a ring of radius a: the ring is in the $x - y$ plane with its center at the origin.



(a) Find the electrostatic potential along the z-axis.
Solution. The potential is

$$\Phi(z) = \frac{Q}{4\pi\epsilon_0}\frac{1}{\sqrt{z^2+a^2}}. \tag{6}$$

(b) Now calculate the potential $\Phi(r,\theta,\phi)$ for $r > a$ and for $r < a$ as an expansion in Legendre polynomials. You only need to find the first two nonvanishing terms in the expansion.
Solution. Recall that for a problem with azimuthal symmetry,

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right]P_l(\cos\theta). \tag{7}$$

Along the positive z-axis, $\cos\theta = 1$, and using $P_l(1) = 1$, we have

$$\Phi(r=z,\theta=0) = \sum_{l=0}^{\infty}\left[A_l z^l + B_l z^{-(l+1)}\right]. \tag{8}$$

The strategy is to expand our potential obtained in part (a) to determine the expansion coefficients. Expanding Eq. (6) for $z > a$, we have

$$\Phi(z) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{z} - \frac{1}{2}\frac{a^2}{z^3} + \ldots\right]. \tag{9}$$

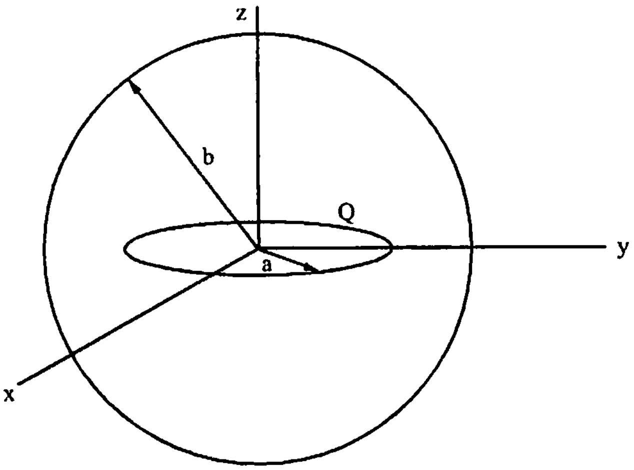
Comparing with Eq. (8), we see that $A_l = 0$, and $B_0 = Q/4\pi\epsilon_0$. $B_1 = 0$, and $B_2 = -(Q/4\pi\epsilon_0)(a^2/2)$, so that for $r > a$ we have

$$\Phi(r,\theta) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{1}{2}\frac{a^2}{r^3}P_2(\cos\theta) + \ldots\right]. \tag{10}$$

Likewise, for $r < a$ we have

$$\Phi(r,\theta) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{2}\frac{r^2}{a^3}P_2(\cos\theta) + \ldots\right]. \tag{11}$$

(c) Now suppose that the ring is placed inside a grounded, conducting sphere of radius $b > a$. Calculate the potential for $a < r < b$. You need only find the first two nonvanishing terms in the series.



Solution. The expansion is now

$$\Phi(r,\theta) = \left(\frac{Q}{4\pi\epsilon_0}\frac{1}{r} + A_0\right) + \left(-\frac{1}{2}\frac{Q}{4\pi\epsilon_0}\frac{a^2}{r^3} + A_2 r^2\right)P_2(\cos\theta) + \ldots. \tag{12}$$

We need to impose the boundary condition that $\Phi(b,\theta) = 0$; this determines the expansion coefficients as $A_0 = -(Q/4\pi\epsilon_0)(1/b)$ and $A_2 = (Q/4\pi\epsilon_0)(a^2/b^5)$, so we have

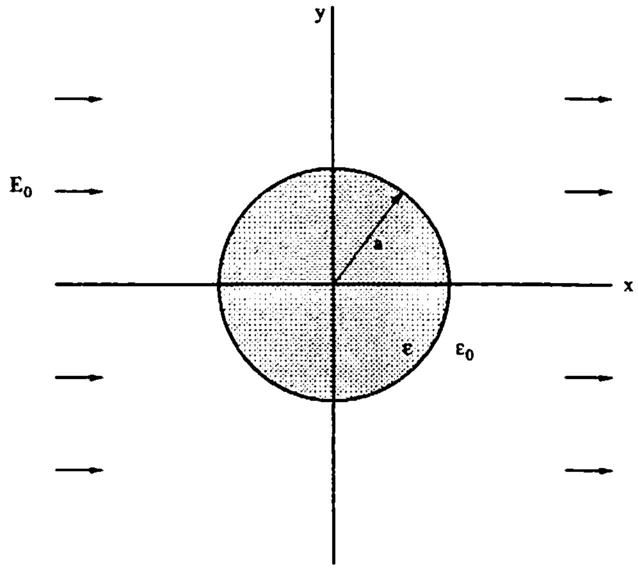
$$\Phi(r,\theta) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{1}{b} - \frac{1}{2}\frac{a^2}{r^3}\left(1 - \frac{r^5}{b^5}\right)P_2(\cos\theta) + \ldots\right]. \tag{13}$$

3. **Dielectric cylinder in a uniform field** (35 points). A long dielectric cylinder of radius a and dielectric constant ϵ is placed in a uniform electric field $\mathbf{E}_0 = E_0\mathbf{e}_x$ (see the figure below).

   (a) Working in polar coordinates (ρ, ϕ), write down the equations and boundary conditions which will determine the potential inside and outside the cylinder.

   Solution. We have

$$\mathbf{E} = -\nabla\Phi, \quad \nabla\cdot\mathbf{D} = 0. \tag{14}$$



Since we're considering an isotropic, linear material, $\mathbf{D} = \epsilon\mathbf{E}$, so that $\nabla^2\Phi = 0$ both inside and outside the cylinder. The boundary conditions are that the tangential components of $\mathbf{E}$ are continuous across the interface, so that

$$\left(\frac{\partial\Phi_{\text{in}}}{\partial\phi}\right)_{\rho=a} = \left(\frac{\partial\Phi_{\text{out}}}{\partial\phi}\right)_{\rho=a}, \tag{15}$$

and that the normal component of $\mathbf{D}$ is continuous across the interface, so that we have

$$\epsilon\left(\frac{\partial\Phi_{\text{in}}}{\partial\rho}\right)_{\rho=a} = \epsilon_0\left(\frac{\partial\Phi_{\text{out}}}{\partial\rho}\right)_{\rho=a}. \tag{16}$$

We also need the exterior potential to approach that of a uniform field,

$$\Phi_{\text{out}} \to -E_0\rho\cos\phi \quad \text{as } \rho \to \infty. \tag{17}$$

(b) Solve these equations and find the potential inside and outside the cylinder.

Solution. The potential can only depend on $\cos\phi$, so that the only acceptable solution inside the cylinder is

$$\Phi_{\text{in}}(\rho,\phi) = A_1\rho\cos\phi. \tag{18}$$

Outside the cylinder the solution must be of the form

$$\Phi_{\text{out}}(\rho,\phi) = (B_1\rho + C_1\rho^{-1})\cos\phi = -E_0\rho\cos\phi + C_1\rho^{-1}\cos\phi. \tag{19}$$

Applying the two boundary conditions, we find that

$$A_1 = -\frac{2\epsilon_0}{\epsilon+\epsilon_0}E_0, \quad C_1 = \frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}E_0a^2. \tag{20}$$

(c) Use your results to find **E**, **D**, and **P** inside the cylinder.

Solution. The potential inside the cylinder is

$$\Phi_{\text{in}}(\rho, \phi) = -\frac{2\epsilon_0}{\epsilon + \epsilon_0} E_0 \rho \cos\phi, \tag{21}$$

so that the electric field is

$$\mathbf{E}_{\text{in}} = \frac{2\epsilon_0}{\epsilon + \epsilon_0} E_0 \mathbf{e}_x, \tag{22}$$

the electric displacement is

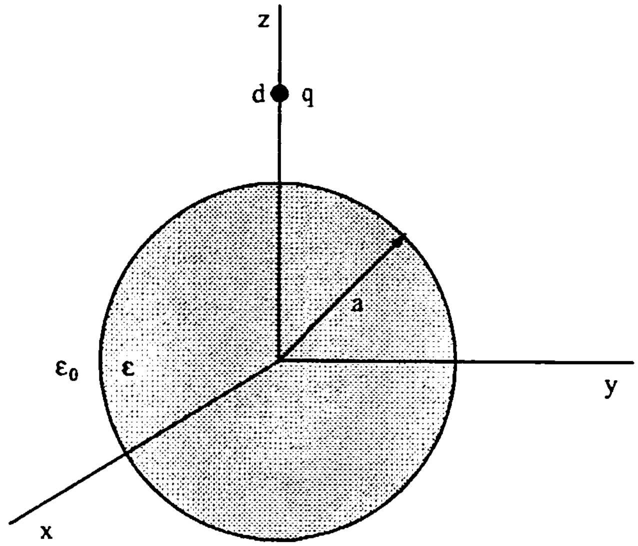
$$\mathbf{D}_{\text{in}} = \epsilon \mathbf{E}_{\text{in}} = \frac{2\epsilon_0 \epsilon}{\epsilon + \epsilon_0} E_0 \mathbf{e}_x, \tag{23}$$

and the polarization is

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E} = \frac{2\epsilon_0(\epsilon - \epsilon_0)}{\epsilon + \epsilon_0} E_0 \mathbf{e}_x. \tag{24}$$

Physics 6346 Final Exam Solutions

1. **Point charge outside of a dielectric sphere** (25 points). A point charge of charge q is placed a distance d from the center of a neutral dielectric sphere of radius a and dielectric constant ϵ. In solving this problem, let the origin of the coordinate system be at the center of the sphere, and put the point charge on the z-axis. Assume that the dielectric constant of the surrounding medium is ϵ_0.



   (a) Write down the equations of electrostatics and the boundary conditions which are needed to solve this problem.

   Solution. We need to solve Poisson's equation,

$$\nabla^2\Phi = -(q/\epsilon_0)\delta(\mathbf{x} - d\hat{z}) \tag{1}$$

   outside of the sphere, and Laplace's equation $\nabla^2\Phi = 0$ inside the sphere. The boundary conditions are that

$$-\epsilon_0\left(\frac{\partial\Phi}{\partial r}\right)_{r=a^+} = -\epsilon\left(\frac{\partial\Phi}{\partial r}\right)_{r=a^-}, \tag{2}$$

$$-\frac{1}{a}\left(\frac{\partial\Phi}{\partial\theta}\right)_{r=a^+} = -\frac{1}{a}\left(\frac{\partial\Phi}{\partial\theta}\right)_{r=a^-}. \tag{3}$$

(b) Calculate the potential inside and outside of the sphere.

Solution. This was done in some detail in the lecture notes. The result is that outside the sphere

$$\Phi = \frac{q}{4\pi\epsilon_0|\mathbf{x} - d\hat{z}|} + \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta), \tag{4}$$

while inside the sphere,

$$\Phi = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta). \tag{5}$$

Using the boundary conditions,

$$A_l = \frac{q}{4\pi\epsilon_0} \frac{2l+1}{d^{l+1}[(\epsilon/\epsilon_0 + 1)l + 1]}. \tag{6}$$

$$B_l = -\frac{q}{4\pi\epsilon_0} \frac{a^{2l+1} l(\epsilon/\epsilon_0 - 1)}{d^{l+1}[(\epsilon/\epsilon_0 + 1)l + 1]}. \tag{7}$$

(c) Calculate the force on the point charge.

Solution. The force is due to the electric field produced by the polarization of the sphere. The result is (see the notes)

$$F_z = -\frac{q^2}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{a^{2l+1}}{d^{2l+3}} \frac{(\epsilon/\epsilon_0 - 1)l(l+1)}{(\epsilon/\epsilon_0 + 1)l + 1}. \tag{8}$$

(d) Find the leading term in the force when $d \gg a$. Provide a simple explanation of your result [you should be able to deduce the leading term from simple considerations even if you couldn't solve part (c)].

Solution. The leading term is the $l = 1$ term,

$$F_z = -\frac{q^2}{4\pi\epsilon_0} \frac{2(\epsilon/\epsilon_0 - 1)}{\epsilon/\epsilon_0 + 2} \frac{a^3}{d^5}. \tag{9}$$

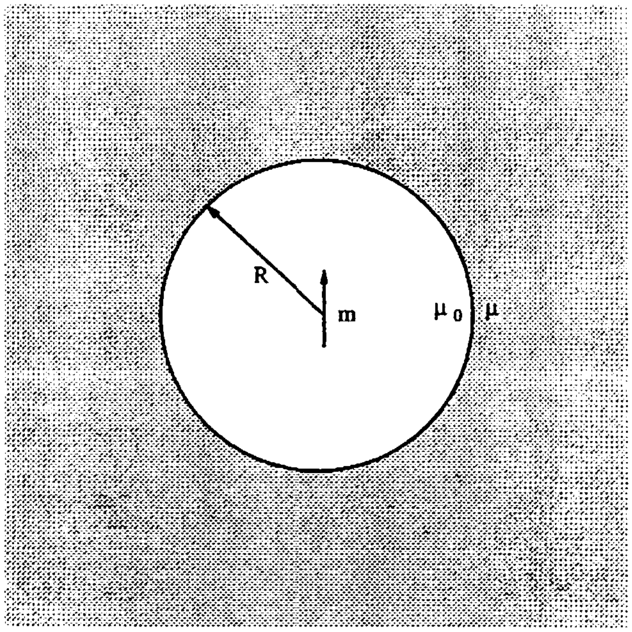
This is the interaction between the dipole moment induced in the sphere (proportional to $1/d^2$) and the point charge.

2. **Spherical void in a permeable material** (25 points). A spherical void of radius R is in an otherwise homogeneous magnetic material of permeability μ. At the center of the void is a point dipole $\mathbf{m}$.

(a) Suppose that $R \to \infty$. What are $\mathbf{B}$ and $\mathbf{H}$ for the dipole in this case?

Solution. The magnetic flux density for a point dipole is

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{3(\mathbf{m} \cdot \mathbf{n})\mathbf{n} - \mathbf{m}}{r^3}. \tag{10}$$



With $\mathbf{n} = \mathbf{e}_r$ and $\mathbf{m} = m\mathbf{e}_z$, in component form this becomes

$$B_r = \frac{\mu_0}{4\pi}\frac{2m\cos\theta}{r^3}, \quad B_\theta = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^3}. \tag{11}$$

The magnetic field $\mathbf{H} = \mathbf{B}/\mu_0$. If we were to derive $\mathbf{H}$ from a potential, we would have $\mathbf{H} = -\nabla\Phi_M$, with

$$\Phi_M = \frac{1}{4\pi}\frac{m\cos\theta}{r^2}. \tag{12}$$

(b) Write down the fundamental equations for $\mathbf{B}$ and $\mathbf{H}$. What are the boundary conditions on these fields at the surface of the void?

Solution. With $\mathbf{J} = 0$, the equations of magnetostatics are

$$\nabla\cdot\mathbf{B} = 0, \quad \nabla\times\mathbf{H} = 0. \tag{13}$$

The boundary conditions are that the normal component of $\mathbf{B}$ is continuous and the tangential component of $\mathbf{H}$ is continuous.

(c) Now solve the equations and apply the boundary conditions to find $\mathbf{B}$ and $\mathbf{H}$ both inside and outside the void. [Hint: you might want to introduce a magnetic scalar potential Φ_M, such that $\mathbf{H} = -\nabla\Phi_M$.]

Solution. Since $\nabla\times\mathbf{H} = 0$, we can write $\mathbf{H} = -\nabla\Phi_M$. In spherical coordinates, the boundary conditions on Φ_M are

$$-\mu_0\left(\frac{\partial\Phi_M}{\partial r}\right)_{r=a^-} = -\mu\left(\frac{\partial\Phi_M}{\partial r}\right)_{r=a^+}, \tag{14}$$

$$-\frac{1}{a}\left(\frac{\partial\Phi_M}{\partial\theta}\right)_{r=a^-} = -\frac{1}{a}\left(\frac{\partial\Phi_M}{\partial\theta}\right)_{r=a^+}. \quad (15)$$

This problem has azimuthal symmetry, so the magnetic scalar potential may be expanded in Legendre polynomials $P_l(\cos\theta)$; since the dipole itself corresponds to $l = 1$, and the response in the void and in the dielectric is induced, we know on physical grounds that we only need to keep the $l = 1$ terms. Therefore, inside the void $(r < a)$ we have

$$\Phi_M = \frac{1}{4\pi}\frac{m\cos\theta}{r^2} + Ar\cos\theta, \quad (16)$$

while inside the void $(r > a)$ we have

$$\Phi_M = \frac{1}{4\pi}\frac{m'\cos\theta}{r^2}, \quad (17)$$

with m' and A constants which are to be determined from the boundary conditions. Applying the two boundary conditions, we obtain

$$\frac{2m}{4\pi a^3} - A = \frac{2\mu_r m'}{4\pi a^3}, \quad (18)$$

$$\frac{m}{4\pi a^3} + A = \frac{m'}{4\pi a^3}, \quad (19)$$

where $\mu_r = \mu/\mu_0$. Solving, we obtain

$$m' = \frac{3}{2\mu_r + 1}m, \quad A = -\frac{\mu_r - 1}{2\mu_r + 1}\frac{m}{2\pi a^3}. \quad (20)$$

Therefore, for $r < a$ the magnetic field is that of a dipole plus a uniform field,

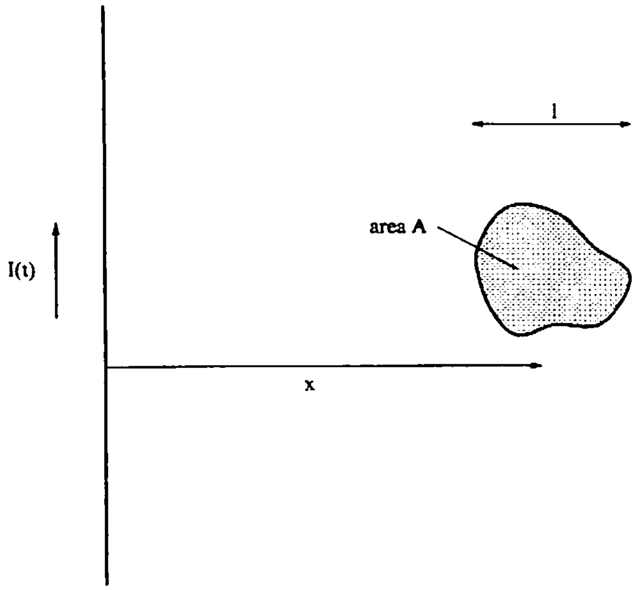
$$\mathbf{H}_{\text{in}} = \frac{1}{4\pi}\frac{3(\mathbf{m}\cdot\mathbf{n})\mathbf{n} - \mathbf{m}}{r^3} + \frac{\mu_r - 1}{2\mu_r + 1}\frac{m}{2\pi a^3}\mathbf{e}_z, \quad (21)$$

and the magnetic flux density is $\mathbf{B}_{\text{in}} = \mu_0\mathbf{H}_{\text{in}}$. For $r > a$ the magnetic field is a dipole field.

$$\mathbf{H}_{\text{out}} = \frac{3}{2\mu_r + 1}\frac{1}{4\pi}\frac{3(\mathbf{m}\cdot\mathbf{n})\mathbf{n} - \mathbf{m}}{r^3}, \quad (22)$$

and $\mathbf{B}_{\text{out}} = \mu\mathbf{H}_{\text{out}}$.

3. **Induction** (25 points). A planar wire loop of *arbitrary* shape is coplanar with a long, straight wire which carries a current $I(t)$. The loop has a resistance R, encloses an area A, and is a fixed distance x away from the straight wire. **Assume that x is much larger than the characteristic size l of the loop, and assume that the self-inductance of the loop is negligible.**



(a) Determine the sense of the current induced in the loop (clockwise or counterclockwise) and the direction of the force on the loop (left or right) when (i) $I > 0$ and $\dot{I} > 0$, (ii) $I > 0$ and $\dot{I} < 0$, (iii) $I < 0$ and $\dot{I} > 0$, (iv) $I < 0$ and $\dot{I} < 0$ (here $\dot{I} \equiv dI/dt$).

Solution.

i. $I > 0$ and $\dot{I} > 0$: ccw, right (repel).
ii. $I > 0$ and $\dot{I} < 0$: cw, left (attract).
iii. $I < 0$ and $\dot{I} > 0$: cw, right (repel).
iv. $I < 0$ and $\dot{I} < 0$: ccw, left (attract).

(b) Find the flux through the loop due to the field produced by the straight wire, and therefore find the mutual inductance M.

Solution. The magnitude of the magnetic field is

$$B = \frac{\mu_0 I}{2\pi x}. \tag{23}$$

In the approximation that $x \gg l$, we can take B to be constant over the loop, so the flux is just the field times the area A:

$$\Phi = \frac{\mu_0 A}{2\pi x} I, \tag{24}$$

so we see that the mutual inductance is

$$M = \frac{\mu_0 A}{2\pi x}. \tag{25}$$

(c) What is the current I_{loop} induced in the loop?

Solution. Since we are neglecting the self-inductance, we have

$$\begin{aligned} \mathcal{E} &= I_{\text{loop}} R \\ &= -\frac{d\Phi}{dt} \\ &= -\frac{\mu_0 A}{2\pi x}\frac{dI}{dt}. \end{aligned} \tag{26}$$

Therefore,

$$I_{\text{loop}} = -\frac{\mu_0 A}{2\pi x R}\frac{dI}{dt}. \tag{27}$$

(d) What is the magnetic dipole moment of the loop?

Solution. For a planar loop the dipole moment is simply the current in the loop times its area:

$$m = |\mathbf{m}| = |I_{\text{loop}}| A = \frac{\mu_0 A^2}{2\pi x R}\left|\frac{dI}{dt}\right|. \tag{28}$$

The direction of $\mathbf{m}$ is determined by the sense of I_{loop}; it is out of the page when I_{loop} is counterclockwise and into the page when I_{loop} is clockwise.

(e) Assuming that $I(t) = I_0 \cos\omega t$, find the force on the loop as a function of x, A, I_0, ω, R, and any physical constants. Find the time average of this force.

Solution. The force is $\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$. We have

$$\begin{aligned} \mathbf{m}\cdot\mathbf{B} &= -\frac{\mu_0^2 A^2}{4\pi^2 R x^2} I\dot{I} \\ &= \frac{\mu_0^2 A^2 \omega I_0^2}{4\pi^2 R x^2}\sin\omega t\cos\omega t. \end{aligned} \tag{29}$$

Therefore, we have

$$F_x = -\frac{\mu_0^2 A^2 \omega I_0^2}{2\pi^2 R x^3}\sin\omega t\cos\omega t. \tag{30}$$

Taking the time average, we find that $\langle F_x\rangle = 0$; the force oscillates between being to the left and to the right.

(f) Finally, suppose that the self-inductance L of the loop is *not* negligible. Calculate the time-averaged force acting on the ring. [Hint: Write down a differential equation for the current I_{loop} which includes both the self-inductance and the resistance. For the case in which $I(t) = I_0\cos\omega t$, show that the steady-state solution of this equation is $I_{\text{loop}}(t) = \tilde{I}\sin(\omega t - \phi)$, and find $\tilde{I}$ and ϕ.]

Solution. The induced current satisfies

$$\begin{aligned} \frac{L}{R}\frac{dI_{\text{loop}}}{dt} + I_{\text{loop}} &= -\frac{M}{R}\frac{dI}{dt} \\ &= \frac{M I_0 \omega}{R}\sin\omega t. \end{aligned} \tag{31}$$

(c) Let's assume that magnetic charge is conserved, just like electric charge. If the current density for magnetic charge is $\mathbf{J}_M$, what is the equation relating $\mathbf{J}_M$ and ρ_M?

Solution. The magnetic charge density and magnetic current density will satisfy the equation of continuity,

$$\frac{\partial \rho_M}{\partial t} + \nabla \cdot \mathbf{J}_M = 0. \tag{38}$$

(d) Show that Faraday's law is inconsistent in the presence of a magnetic charge density which changes with time. Modify Faraday's law so that it is consistent with conservation of magnetic charge. [Hint: think about the way that Maxwell "fixed" Ampére's law.]

Solution. Faraday's law is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{39}$$

Take the divergence of both sides:

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{E}) &= 0 \\ &= -\frac{\partial \nabla \cdot \mathbf{B}}{\partial t} \\ &= -\frac{\partial \rho_M}{\partial t}, \end{aligned} \tag{40}$$

so the equation of continuity is *not* satisfied. To fix this, add the magnetic current density $\mathbf{J}_M$ to the right hand side:

$$\nabla \times \mathbf{E} = -\mathbf{J}_M - \frac{\partial \mathbf{B}}{\partial t}. \tag{41}$$

Taking the divergence of both sides of this equation, we see that the equation of continuity is now satisfied.

(e) Finally, to have a complete description of electromagnetic phenomena we need to know how the fields act upon the magnetic charges. It is possible to show that a point particle of magnetic charge g and electrical charge q moving with a velocity $\mathbf{v}$ experiences a force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + g(\mathbf{H} - \mathbf{v} \times \mathbf{D}). \tag{42}$$

Use this force law to prove *Poynting's theorem*:

$$\frac{d}{dt}(E_{\text{mech}} + E_{\text{field}}) = -\oint_S \mathbf{S} \cdot \mathbf{n}\, da, \tag{43}$$

where

$$E_{\text{field}} = \int_V u\, d^3x = \frac{1}{2}\int_V (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})\, d^3x. \tag{44}$$

$$\frac{dE_{\text{mech}}}{dt} = \int_V (\mathbf{J}\cdot\mathbf{E} + \mathbf{J}_M\cdot\mathbf{H})\, d^3x, \tag{45}$$

and

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}. \tag{46}$$

Assume that the medium is linear, with negligible dispersion or losses.

Solution. For a single particle, the time rate of change of the mechanical work done is $\mathbf{v}\cdot\mathbf{F} = q\mathbf{v}\cdot\mathbf{E} + g\mathbf{v}\cdot\mathbf{H}$. For a collection of particles we replace $q\mathbf{v}$ by $\mathbf{J}$ and $g\mathbf{v}$ by $\mathbf{J}_M$, and integrate over the volume, so that

$$\frac{dE_{\text{mech}}}{dt} = \int_V (\mathbf{J}\cdot\mathbf{E} + \mathbf{J}_M\cdot\mathbf{H})\, d^3x. \tag{47}$$

We now eliminate the sources in favor of the fields, using the fourth Maxwell equation and our modified Faraday's law:

$$\frac{dE_{\text{mech}}}{dt} = \int_V \left[\left(\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}\right)\cdot\mathbf{E} + \left(-\nabla\times\mathbf{E} - \frac{\partial\mathbf{B}}{\partial t}\right)\cdot\mathbf{H}\right] d^3x. \tag{48}$$

Using

$$-\nabla\cdot(\mathbf{E}\times\mathbf{H}) = \mathbf{E}\cdot\nabla\times\mathbf{H} - \mathbf{H}\cdot\nabla\times\mathbf{E}, \tag{49}$$

and the fact that the medium is linear so that

$$\frac{\partial\mathbf{D}}{\partial t}\cdot\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{H} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{E}\cdot\mathbf{D} + \frac{1}{2}\mathbf{B}\cdot\mathbf{H}\right), \tag{50}$$

we obtain

$$\frac{dE_{\text{mech}}}{dt} = -\frac{d}{dt}\int_V \left(\frac{1}{2}\mathbf{E}\cdot\mathbf{D} + \frac{1}{2}\mathbf{B}\cdot\mathbf{H}\right) d^3x - \int_V \nabla\cdot\mathbf{S}\, d^3x. \tag{51}$$

The first term on the right hand side is $-dE_{\text{field}}/dt$, and the second term can be converted into a surface integral using the divergence theorem, so that we obtain Eq. (43).