12
Introduction to the Laplace Transform

Assessment Problems

AP 12.1 **[a]** $\cosh\beta t = \dfrac{e^{\beta t}+e^{-\beta t}}{2}$

Therefore,

$$\mathcal{L}\{\cosh\beta t\} = \frac{1}{2}\int_{0^-}^{\infty}[e^{-(s-\beta)t}+e^{-(s+\beta)t}]dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-\beta)t}}{-(s-\beta)}\Big|_{0^-}^{\infty} + \frac{e^{-(s+\beta)t}}{-(s+\beta)}\Big|_{0^-}^{\infty}\right]$$

$$= \frac{1}{2}\left(\frac{1}{s-\beta}+\frac{1}{s+\beta}\right) = \frac{s}{s^2-\beta^2}$$

[b] $\sinh\beta t = \dfrac{e^{\beta t}-e^{-\beta t}}{2}$

Therefore,

$$\mathcal{L}\{\sinh\beta t\} = \frac{1}{2}\int_{0^-}^{\infty}\left[e^{-(s-\beta)t}-e^{-(s+\beta)t}\right]\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-\beta)t}}{-(s-\beta)}\right]_{0^-}^{\infty} - \frac{1}{2}\left[\frac{e^{-(s+\beta)t}}{-(s+\beta)}\right]_{0^-}^{\infty}$$

$$= \frac{1}{2}\left(\frac{1}{s-\beta}-\frac{1}{s+\beta}\right) = \frac{\beta}{(s^2-\beta^2)}$$

AP 12.2 **[a]** Let $f(t) = te^{-at}$:

$$F(s) = \mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$$

Now, $\quad \mathcal{L}\{tf(t)\} = -\dfrac{dF(s)}{ds}$

So, $\mathcal{L}\{t \cdot te^{-at}\} = -\frac{d}{ds}\left[\frac{1}{(s+a)^2}\right] = \frac{2}{(s+a)^2}$

[b] Let $f(t) = e^{-at}\sinh\beta t$, then

$$\mathcal{L}\{f(t)\} = F(s) = \frac{\beta}{(s+a)^2 - \beta^2}$$

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-) = \frac{s(\beta)}{(s+a)^2 - \beta^2} - 0 = \frac{\beta s}{(s+a)^2 - \beta^2}$$

[c] Let $f(t) = \cos\omega t$. Then

$$F(s) = \frac{s}{(s^2+\omega^2)} \quad \text{and} \quad \frac{dF(s)}{ds} = \frac{-(s^2-\omega^2)}{(s^2+\omega^2)^2}$$

Therefore $\mathcal{L}\{t\cos\omega t\} = -\frac{dF(s)}{ds} = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$

AP 12.3 $F(s) = \frac{6s^2+26s+26}{(s+1)(s+2)(s+3)} = \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+3}$

$$K_1 = \frac{6-26+26}{(1)(2)} = 3; \qquad K_2 = \frac{24-52+26}{(-1)(1)} = 2$$

$$K_3 = \frac{54-78+26}{(-2)(-1)} = 1$$

Therefore $f(t) = [3e^{-t} + 2e^{-2t} + e^{-3t}]\,u(t)$

AP 12.4 $F(s) = \frac{7s^2+63s+134}{(s+3)(s+4)(s+5)} = \frac{K_1}{s+3} + \frac{K_2}{s+4} + \frac{K_3}{s+5}$

$$K_1 = \frac{63-189+134}{1(2)} = 4; \qquad K_2 = \frac{112-252+134}{(-1)(1)} = 6$$

$$K_3 = \frac{175-315+134}{(-2)(-1)} = -3$$

$$f(t) = [4e^{-3t} + 6e^{-4t} - 3e^{-5t}]u(t)$$

AP 12.5 $F(s) = \frac{10(s^2+119)}{(s+5)(s^2+10s+169)}$

$$s_{1,2} = -5 + \sqrt{25-169} = -5 + j12$$

$$F(s) = \frac{K_1}{s+5} + \frac{K_2}{s+5-j12} + \frac{K_2^*}{s+5+j12}$$

$$K_1 = \frac{10(25+119)}{25-50+169} = 10$$

$$K_2 = \frac{10[(-5+j12)^2+119]}{(j12)(j24)} = j4.167 = 4.167\underline{/90^\circ}$$

Therefore

$$\begin{aligned} f(t) &= [10e^{-5t} + 8.33e^{-5t}\cos(12t+90^\circ)]\,u(t) \\ &= [10e^{-5t} - 8.33e^{-5t}\sin 12t]\,u(t) \end{aligned}$$

AP 12.6 $F(s) = \dfrac{4s^2+7s+1}{s(s+1)^2} = \dfrac{K_0}{s} + \dfrac{K_1}{(s+1)^2} + \dfrac{K_2}{s+1}$

$$K_0 = \frac{1}{(1)^2} = 1; \qquad K_1 = \frac{4-7+1}{-1} = 2$$

$$\begin{aligned} K_2 &= \frac{d}{ds}\left[\frac{4s^2+7s+1}{s}\right]_{s=-1} = \left.\frac{s(8s+7)-(4s^2+7s+1)}{s^2}\right|_{s=-1} \\ &= \frac{1+2}{1} = 3 \end{aligned}$$

Therefore $\quad f(t) = [1 + 2te^{-t} + 3e^{-t}]\,u(t)$

AP 12.7 $\begin{aligned} F(s) &= \frac{40}{(s^2+4s+5)^2} = \frac{40}{(s+2-j1)^2(s+2+j1)^2} \\ &= \frac{K_1}{(s+2-j1)^2} + \frac{K_2}{(s+2-j1)} + \frac{K_1^*}{(s+2+j1)^2} \\ &\quad + \frac{K_2^*}{(s+2+j1)} \end{aligned}$

$$K_1 = \frac{40}{(j2)^2} = -10 = 10\underline{/180^\circ} \quad \text{and} \quad K_1^* = -10$$

$$K_2 = \frac{d}{ds}\left[\frac{40}{(s+2+j1)^2}\right]_{s=-2+j1} = \frac{-80}{(j2)^3} = -j10 = 10\underline{/-90^\circ}$$

$$K_2^* = j10$$

Therefore

$$\begin{aligned} f(t) &= [20te^{-2t}\cos(t+180^\circ) + 20e^{-2t}\cos(t-90^\circ)]\,u(t) \\ &= 20e^{-2t}[\sin t - t\cos t]\,u(t) \end{aligned}$$

AP 12.8 $F(s) = \dfrac{5s^2+29s+32}{(s+2)(s+4)} = \dfrac{5s^2+29s+32}{s^2+6s+8} = 5 - \dfrac{s+8}{(s+2)(s+4)}$

$$\frac{s+8}{(s+2)(s+4)} = \frac{K_1}{s+2} + \frac{K_2}{s+4}$$

$$K_1 = \frac{-2+8}{2} = 3; \qquad K_2 = \frac{-4+8}{-2} = -2$$

Therefore,

$$F(s) = 5 - \frac{3}{s+2} + \frac{2}{s+4}$$

$$f(t) = 5\delta(t) + [-3e^{-2t} + 2e^{-4t}]u(t)$$

AP 12.9 $F(s) = \dfrac{2s^3+8s^2+2s-4}{s^2+5s+4} = 2s - 2 + \dfrac{4(s+1)}{(s+1)(s+4)} = 2s - 2 + \dfrac{4}{s+4}$

$$f(t) = 2\frac{d\delta(t)}{dt} - 2\delta(t) + 4e^{-4t}\,u(t)$$

AP 12.10

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{7s^3[1+(9/s)+(134/7s^2)]}{s^3[1+(3/s)][1+(4/s)][1+(5/s)]}\right] = 7$$

$$\therefore\quad f(0^+) = 7$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{7s^3+63s^2+134s}{(s+3)(s+4)(s+5)}\right] = 0$$

$$\therefore\quad f(\infty) = 0$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{s^3[4+(7/s)+(1/s^2)]}{s^3[1+(1/s)]^2}\right] = 4$$

$$\therefore\quad f(0^+) = 4$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{4s^2+7s+1}{(s+1)^2}\right] = 1$$

$$\therefore \quad f(\infty) = 1$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{40s}{s^4[1+(4/s)+(5/s^2)]^2}\right] = 0$$

$$\therefore \quad f(0^+) = 0$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{40s}{(s^2+4s+5)^2}\right] = 0$$

$$\therefore \quad f(\infty) = 0$$

Problems

P 12.1 **[a]** $f(t) = 5t[u(t) - u(t-2)] + 10[u(t-2) - u(t-6)]+$
$$(-5t + 40)[u(t-6) - u(t-8)]$$

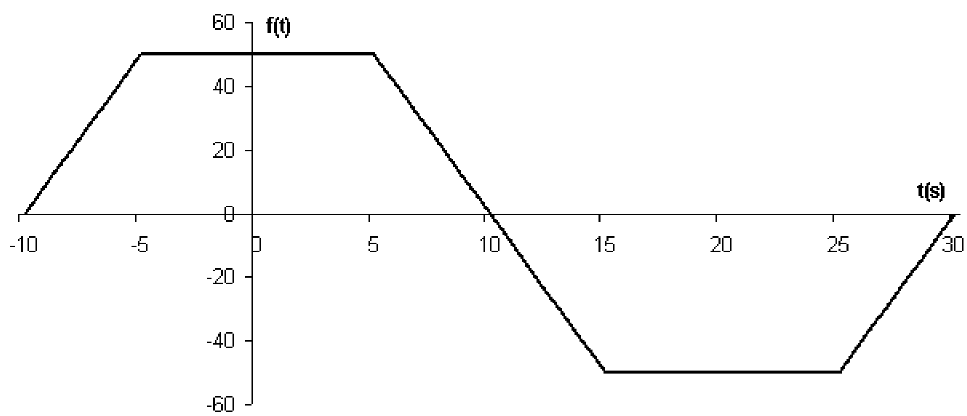
[b] $f(t) = (10\sin \pi t)[u(t) - u(t-2)]$

[c] $f(t) = 4t[u(t) - u(t-5)]$

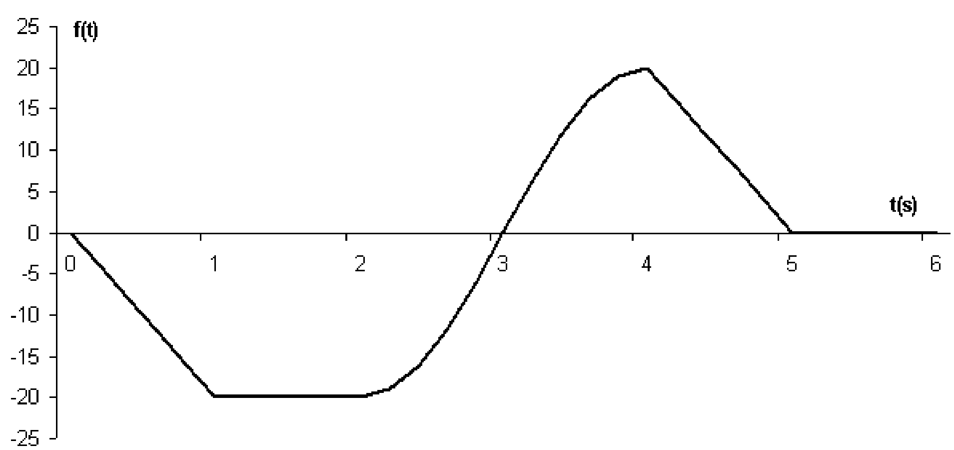
P 12.2 **[a]** $(10 + t)[u(t+10) - u(t)] + (10 - t)[u(t) - u(t-10)]$
$$= (t + 10)u(t + 10) - 2tu(t) + (t - 10)u(t - 10)$$

[b] $(-24 - 8t)[u(t+3) - u(t+2)] - 8[u(t+2) - u(t+1)] + 8t[u(t+1) - u(t-1)]$
$$+8[u(t-1) - u(t-2)] + (24 - 8t)[u(t-2) - u(t-3)]$$
$$= -8(t+3)u(t+3) + 8(t+2)u(t+2) + 8(t+1)u(t+1) - 8(t-1)u(t-1)$$
$$-8(t-2)u(t-2) + 8(t-3)u(t-3)$$

P 12.3



P 12.4 **[a]**



[b] $f(t) = -20t[u(t) - u(t-1)] - 20[u(t-1) - u(t-2)]$

$+20\cos(\frac{\pi}{2}t)[u(t-2) - u(t-4)]$

$+(100 - 20t)[u(t-4) - u(t-5)]$

P 12.5 **[a]** $A = \left(\frac{1}{2}\right) bh = \left(\frac{1}{2}\right)(2\varepsilon)\left(\frac{1}{\varepsilon}\right) = 1$

[b] 0; **[c]** ∞

P 12.6 **[a]** $I = \int_{-1}^{3}(t^3+2)\delta(t)\,dt + \int_{-1}^{3} 8(t^3+2)\delta(t-1)\,dt$

$$= (0^3+2) + 8(1^3+2) = 2 + 8(3) = 26$$

[b] $I = \int_{-2}^{2} t^2\delta(t)\,dt + \int_{-2}^{2} t^2\delta(t+1.5)\,dt + \int_{-2}^{2} t^2\delta(t-3)\,dt$

$$= 0^2 + (-1.5)^2 + 0 = 2.25$$

P 12.7 $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(4+j\omega)}{(9+j\omega)}\cdot\pi\delta(\omega)\cdot e^{jt\omega}\,d\omega = \left(\frac{1}{2\pi}\right)\left(\frac{4+j0}{9+j0}\pi e^{jt0}\right) = \frac{2}{9}$

P 12.8 As $\varepsilon \to 0$ the amplitude $\to \infty$; the duration $\to 0$; and the area is independent of ε, i.e.,

$$A = \int_{-\infty}^{\infty}\frac{\varepsilon}{\pi}\frac{1}{\varepsilon^2+t^2}\,dt = 1$$

P 12.9 $F(s) = \int_{-\varepsilon}^{\varepsilon}\frac{1}{2\varepsilon}e^{-st}\,dt = \frac{e^{s\varepsilon} - e^{-s\varepsilon}}{2\varepsilon s}$

$$F(s) = \frac{1}{2s}\lim_{\varepsilon\to 0}\left[\frac{se^{s\varepsilon} + se^{-s\varepsilon}}{1}\right] = \frac{1}{2s}\cdot\frac{2s}{1} = 1$$

P 12.10 **[a]** Let $dv = \delta'(t-a)\,dt, \qquad v = \delta(t-a)$

$u = f(t), \qquad du = f'(t)\,dt$

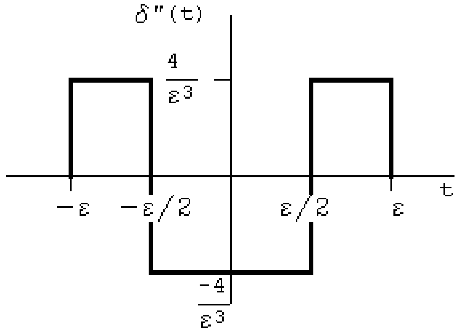
Therefore

$$\int_{-\infty}^{\infty} f(t)\delta'(t-a)\,dt = f(t)\delta(t-a)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\delta(t-a)f'(t)\,dt$$

$$= 0 - f'(a)$$

[b] $\mathcal{L}\{\delta'(t)\} = \int_{0^-}^{\infty}\delta'(t)e^{-st}\,dt = -\left[\frac{d(e^{-st})}{dt}\right]_{t=0} = -\left[-se^{-st}\right]_{t=0} = s$

P 12.11



$$F(s) = \int_{-\varepsilon}^{-\varepsilon/2} \frac{4}{\varepsilon^3} e^{-st}\,dt + \int_{-\varepsilon/2}^{\varepsilon/2} \left(\frac{-4}{\varepsilon^3}\right) e^{-st}\,dt + \int_{\varepsilon/2}^{\varepsilon} \frac{4}{\varepsilon^3} e^{-st}\,dt$$

Therefore $\quad F(s) = \dfrac{4}{s\varepsilon^3}[e^{s\varepsilon} - 2e^{s\varepsilon/2} + 2e^{-s\varepsilon/2} - e^{-s\varepsilon}]$

$$\mathcal{L}\{\delta''(t)\} = \lim_{\varepsilon\to 0} F(s)$$

After applying L'Hopital's rule three times, we have

$$\lim_{\varepsilon\to 0} \frac{2s}{3}\left[se^{s\varepsilon} - \frac{s}{4}e^{s\varepsilon/2} - \frac{s}{4}e^{-s\varepsilon/2} + se^{-s\varepsilon}\right] = \frac{2s}{3}\left(\frac{3s}{2}\right)$$

Therefore $\quad \mathcal{L}\{\delta''(t)\} = s^2$

P 12.12 $\mathcal{L}\left\{\dfrac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots,$

Therefore

$$\mathcal{L}\{\delta^{(n)}(t)\} = s^n(1) - s^{n-1}\delta(0^-) - s^{n-2}\delta'(0^-) - \cdots = s^n$$

P 12.13 **[a]** $\mathcal{L}\{t\} = \dfrac{1}{s^2}$; $\qquad$ therefore $\quad \mathcal{L}\{te^{-at}\} = \dfrac{1}{(s+a)^2}$

[b] $\sin\omega t = \dfrac{e^{j\omega t} - e^{-j\omega t}}{j2}$

Therefore

$$\mathcal{L}\{\sin\omega t\} = \left(\frac{1}{j2}\right)\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) = \left(\frac{1}{j2}\right)\left(\frac{2j\omega}{s^2+\omega^2}\right)$$

$$= \frac{\omega}{s^2+\omega^2}$$

[c] $\sin(\omega t + \theta) = (\sin\omega t\cos\theta + \cos\omega t\sin\theta)$

Therefore

$$\mathcal{L}\{\sin(\omega t+\theta)\} = \cos\theta\mathcal{L}\{\sin\omega t\} + \sin\theta\mathcal{L}\{\cos\omega t\}$$
$$= \frac{\omega\cos\theta + s\sin\theta}{s^2+\omega^2}$$

[d] $\mathcal{L}\{t\} = \int_0^\infty te^{-st}\,dt = \frac{e^{-st}}{s^2}(-st-1)\Big|_0^\infty = 0 - \frac{1}{s^2}(0-1) = \frac{1}{s^2}$

[e] $f(t) = \cosh t\cosh\theta + \sinh t\sinh\theta$

From Assessment Problem 12.1(a)

$$\mathcal{L}\{\cosh t\} = \frac{s}{s^2-1}$$

From Assessment Problem 12.1(b)

$$\mathcal{L}\{\sinh t\} = \frac{1}{s^2-1}$$

$$\therefore\quad \mathcal{L}\{\cosh(t+\theta)\} = \cosh\theta\left[\frac{s}{s^2-1}\right] + \sinh\theta\left[\frac{1}{s^2-1}\right]$$
$$= \frac{\sinh\theta + s[\cosh\theta]}{s^2-1}$$

P 12.14 **[a]** $\mathcal{L}\{te^{-at}\} = \int_{0^-}^\infty te^{-(s+a)t}\,dt$

$$= \frac{e^{-(s+a)t}}{(s+a)^2}\Big[-(s+a)t-1\Big]_{0^-}^\infty$$
$$= 0 + \frac{1}{(s+a)^2}$$
$$\therefore\quad \mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$$

[b]

$$\mathcal{L}\left\{\frac{d}{dt}(te^{-at})\right\} = \frac{s}{(s+a)^2} - 0$$
$$= \frac{s}{(s+a)^2}$$

[c] $\frac{d}{dt}(te^{-at}) = -ate^{-at} + e^{-at}$

$$\mathcal{L}\{-ate^{-at} + e^{-at}\} = \frac{-a}{(s+a)^2} + \frac{1}{(s+a)} = \frac{-a}{(s+a)^2} + \frac{s+a}{(s+a)^2}$$

$$\therefore\quad \mathcal{L}\left\{\frac{d}{dt}(te^{-at})\right\} = \frac{s}{(s+a)^2} \qquad \text{CHECKS}$$

P 12.15 **[a]** $\mathcal{L}\{f'(t)\} = \int_{-\varepsilon}^{\varepsilon} \frac{e^{-st}}{2\varepsilon}\,dt + \int_{\varepsilon}^{\infty} -\,ae^{-a(t-\varepsilon)}e^{-st}\,dt$

$$= \frac{1}{2s\varepsilon}(e^{s\varepsilon} - e^{-s\varepsilon}) - \left(\frac{a}{s+a}\right)e^{-s\varepsilon} = F(s)$$

$$\lim_{\varepsilon\to 0} F(s) = 1 - \frac{a}{s+a} = \frac{s}{s+a}$$

[b] $\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$

Therefore $\quad \mathcal{L}\{f'(t)\} = sF(s) - f(0^-) = \frac{s}{s+a} - 0 = \frac{s}{s+a}$

P 12.16 $\mathcal{L}\{e^{-at}f(t)\} = \int_{0^-}^{\infty} [e^{-at}f(t)]e^{-st}\,dt = \int_{0^-}^{\infty} f(t)e^{-(s+a)t}\,dt = F(s+a)$

P 12.17 **[a]** $\mathcal{L}\left\{\int_{0^-}^{t} e^{-ax}\,dx\right\} = \frac{F(s)}{s} = \frac{1}{s(s+a)}$

[b] $\mathcal{L}\left\{\int_{0^-}^{t} y\,dy\right\} = \frac{1}{s}\left(\frac{1}{s^2}\right) = \frac{1}{s^3}$

[c] $\int_{0^-}^{t} e^{-ax}\,dx = \frac{1}{a} - \frac{e^{-at}}{a}$

$$\mathcal{L}\left\{\frac{1}{a} - \frac{e^{-at}}{a}\right\} = \frac{1}{a}\left[\frac{1}{s} - \frac{1}{s+a}\right] = \frac{1}{s(s+a)}$$

$$\int_{0^-}^{t} y\,dy = \frac{t^2}{2}; \qquad \mathcal{L}\left\{\frac{t^2}{2}\right\} = \frac{1}{2}\cdot\frac{2}{s^3} = \frac{1}{s^3}$$

P 12.18 **[a]** $\mathcal{L}\left\{\frac{d\sin\omega t}{dt}\right\} = \frac{s\omega}{s^2+\omega^2} - 0$

[b] $\mathcal{L}\left\{\frac{d\cos\omega t}{dt}\right\} = \frac{s^2}{s^2+\omega^2} - 0$

[c] $\mathcal{L}\left\{\frac{d^3(t^2)}{dt^3}\right\} = s^3\left(\frac{2}{s^3}\right) - s^2(0) - s(0) - 2(0) = 2$

[d] $\frac{d\sin\omega t}{dt} = (\cos\omega t)\cdot\omega, \qquad \mathcal{L}\{\omega\cos\omega t\} = \frac{\omega s}{s^2+\omega^2}$

$$\frac{d\cos\omega t}{dt} = -\omega\sin\omega t + \delta(t)$$

$$\mathcal{L}\{-\omega\sin\omega t + \delta(t)\} = -\frac{\omega^2}{s^2+\omega^2} + 1 = \frac{s^2}{s^2+\omega^2}$$

$$\frac{d^2(t^2)}{dt^2} = 2u(t); \qquad \frac{d^3(t^2)}{dt^3} = 2\delta(t); \qquad \mathcal{L}\{2\delta(t)\} = 2$$

P 12.19 **[a]** $f(t) = 5t[u(t) - u(t-2)]$

$$+(20-5t)[u(t-2) - u(t-6)]$$

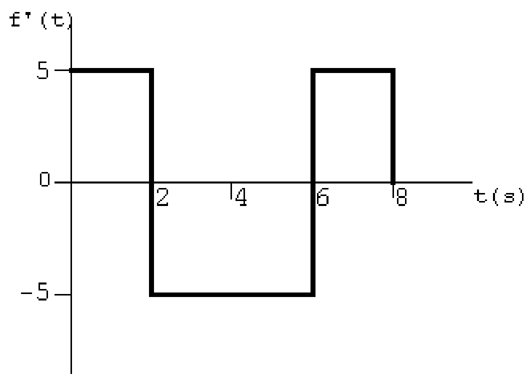
$$+(5t-40)[u(t-6) - u(t-8)]$$

$$= 5tu(t) - 10(t-2)u(t-2)$$

$$+10(t-6)u(t-6) - 5(t-8)u(t-8)$$

$$\therefore \quad F(s) = \frac{5[1 - 2e^{-2s} + 2e^{-6s} - e^{-8s}]}{s^2}$$

[b]



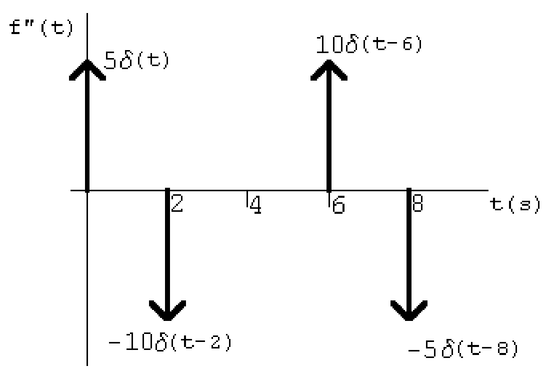
$$f'(t) = 5[u(t) - u(t-2)] - 5[u(t-2) - u(t-6)]$$

$$+5[u(t-6) - u(t-8)]$$

$$= 5u(t) - 10u(t-2) + 10u(t-6) - 5u(t-8)$$

$$\mathcal{L}\{f'(t)\} = \frac{5[1 - 2e^{-2s} + 2e^{-6s} - e^{-8s}]}{s}$$

[c]



$$f''(t) = 5\delta(t) - 10\delta(t-2) + 10\delta(t-6) - 5\delta(t-8)$$

$$\mathcal{L}\{f''(t)\} = 5[1 - 2e^{-2s} + 2e^{-6s} - e^{-8s}]$$

P 12.20 **[a]** $\int_{0^-}^{t} x\,dx = \dfrac{t^2}{2}$

$$\mathcal{L}\left\{\frac{t^2}{2}\right\} = \frac{1}{2}\int_{0^-}^{\infty} t^2 e^{-st}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-st}}{-s^3}(s^2t^2 + 2st + 2)\Big|_{0^-}^{\infty}\right]$$

$$= \frac{1}{2s^3}(2) = \frac{1}{s^3}$$

$$\therefore \quad \mathcal{L}\left\{\int_{0^-}^{t} x\,dx\right\} = \frac{1}{s^3}$$

[b] $\mathcal{L}\left\{\int_{0^-}^{t} x\,dx\right\} = \dfrac{\mathcal{L}\{t\}}{s} = \dfrac{1/s^2}{s} = \dfrac{1}{s^3}$

$\therefore \quad \mathcal{L}\left\{\int_{0^-}^{t} x\,dx\right\} = \dfrac{1}{s^3}$ CHECKS

P 12.21 **[a]** $\mathcal{L}\{40e^{-8(t-3)}u(t-3)\} = \dfrac{40e^{-3s}}{(s+8)}$

[b] First rewrite $f(t)$ as

$$f(t) = (5t-10)u(t-2) + (40-10t)u(t-4)$$
$$+(10t-80)u(t-8) + (50-5t)u(t-10)$$
$$= 5(t-2)u(t-2) - 10(t-4)u(t-4)$$
$$+10(t-8)u(t-8) - 5(t-10)u(t-10)$$

$$\therefore \quad F(s) = \frac{5[e^{-2s} - 2e^{-4s} + 2e^{-8s} - e^{-10s}]}{s^2}$$

P 12.22 $\mathcal{L}\{f(at)\} = \int_{0^-}^{\infty} f(at)e^{-st}\,dt$

Let $u = at$, $du = a\,dt$, $u = 0^-$ when $t = 0^-$

and $u = \infty$ when $t = \infty$

Therefore $\mathcal{L}\{f(at)\} = \int_{0^-}^{\infty} f(u)e^{-(u/a)s}\dfrac{du}{a} = \dfrac{1}{a}F(s/a)$

P 12.23 **[a]** $f_1(t) = e^{-at}\sin\omega t;$ $F_1(s) = \dfrac{\omega}{(s+a)^2 + \omega^2}$

$$F(s) = sF_1(s) - f_1(0^-) = \frac{s\omega}{(s+a)^2 + \omega^2} - 0$$

[b] $f_1(t) = e^{-at}\cos\omega t; \qquad F_1(s) = \dfrac{s+a}{(s+a)^2+\omega^2}$

$$F(s) = \frac{F_1(s)}{s} = \frac{s+a}{s[(s+a)^2+\omega^2]}$$

[c] $\dfrac{d}{dt}[e^{-at}\sin\omega t] = \omega e^{-at}\cos\omega t - ae^{-at}\sin\omega t$

Therefore $\quad F(s) = \dfrac{\omega(s+a)-\omega a}{(s+a)^2+\omega^2} = \dfrac{\omega s}{(s+a)^2+\omega^2}$

$$\int_{0^-}^{t} e^{-ax}\cos\omega x\,dx = \frac{-ae^{-at}\cos\omega t + \omega e^{-at}\sin\omega t + a}{a^2+\omega^2}$$

Therefore

$$F(s) = \frac{1}{a^2+\omega^2}\left[\frac{-a(s+a)}{(s+a)^2+\omega^2} + \frac{\omega^2}{(s+a)^2+\omega^2} + \frac{a}{s}\right]$$

$$= \frac{s+a}{s[(s+a)^2+\omega^2]}$$

P 12.24 **[a]** $\dfrac{dF(s)}{ds} = \dfrac{d}{ds}\left[\int_{0^-}^{\infty} f(t)e^{-st}\,dt\right] = -\int_{0^-}^{\infty} tf(t)e^{-st}\,dt$

Therefore $\quad \mathcal{L}\{tf(t)\} = -\dfrac{dF(s)}{ds}$

[b] $\dfrac{d^2F(s)}{ds^2} = \int_{0^-}^{\infty} t^2f(t)e^{-st}\,dt; \qquad \dfrac{d^3F(s)}{ds^3} = \int_{0^-}^{\infty} -t^3f(t)e^{-st}\,dt$

Therefore $\quad \dfrac{d^nF(s)}{ds^n} = (-1)^n\int_{0^-}^{\infty} t^nf(t)e^{-st}\,dt = (-1)^n\mathcal{L}\{t^nf(t)\}$

[c] $\mathcal{L}\{t^5\} = \mathcal{L}\{t^4t\} = (-1)^4\dfrac{d^4}{ds^4}\left(\dfrac{1}{s^2}\right) = \dfrac{120}{s^6}$

$$\mathcal{L}\{t\sin\beta t\} = (-1)^1\frac{d}{ds}\left(\frac{\beta}{s^2+\beta^2}\right) = \frac{2\beta s}{(s^2+\beta^2)^2}$$

$\mathcal{L}\{te^{-t}\cosh t\}$:

From Assessment Problem 12.1(a),

$$F(s) = \mathcal{L}\{\cosh t\} = \frac{s}{s^2-1}$$

$$\frac{dF}{ds} = \frac{(s^2-1)1 - s(2s)}{(s^2-1)^2} = -\frac{s^2+1}{(s^2-1)^2}$$

Therefore $\quad -\dfrac{dF}{ds} = \dfrac{s^2+1}{(s^2-1)^2}$

Thus

$$\mathcal{L}\{t\cosh t\} = \frac{s^2+1}{(s^2-1)^2}$$

$$\mathcal{L}\{e^{-t}t\cosh t\} = \frac{(s+1)^2+1}{[(s+1)^2-1]^2} = \frac{s^2+2s+2}{s^2(s+2)^2}$$

P 12.25 **[a]** $\int_s^\infty F(u)du = \int_s^\infty \left[\int_{0^-}^\infty f(t)e^{-ut}\,dt\right]du = \int_{0^-}^\infty \left[\int_s^\infty f(t)e^{-ut}\,du\right]dt$

$$= \int_{0^-}^\infty f(t)\int_s^\infty e^{-ut}\,du\,dt = \int_{0^-}^\infty f(t)\left[\frac{e^{-tu}}{-t}\Big|_s^\infty\right]dt$$

$$= \int_{0^-}^\infty f(t)\left[\frac{-e^{-st}}{-t}\right]dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

[b] $\mathcal{L}\{t\sin\beta t\} = \dfrac{2\beta s}{(s^2+\beta^2)^2}$

therefore $\quad \mathcal{L}\left\{\dfrac{t\sin\beta t}{t}\right\} = \displaystyle\int_s^\infty \left[\frac{2\beta u}{(u^2+\beta^2)^2}\right]du$

Let $\omega = u^2+\beta^2$, then $\omega = s^2+\beta^2$ when $u = s$, and $\omega = \infty$ when $u = \infty$; also $d\omega = 2u\,du$, thus

$$\mathcal{L}\left\{\frac{t\sin\beta t}{t}\right\} = \beta\int_{s^2+\beta^2}^\infty \left[\frac{d\omega}{\omega^2}\right] = \beta\left(\frac{-1}{\omega}\right)\Big|_{s^2+\beta^2}^\infty = \frac{\beta}{s^2+\beta^2}$$

P 12.26 $I_g(s) = \dfrac{1.2s}{s^2+1}; \qquad \dfrac{1}{RC} = 1.6; \qquad \dfrac{1}{LC} = 1; \qquad \dfrac{1}{C} = 1.6$

$$\frac{V(s)}{R} + \frac{1}{L}\frac{V(s)}{s} + C[sV(s) - v(0^-)] = I_g(s)$$

$$V(s)\left[\frac{1}{R} + \frac{1}{Ls} + sC\right] = I_g(s)$$

$$V(s) = \frac{I_g(s)}{\frac{1}{R}+\frac{1}{Ls}+sC} = \frac{LsI_g(s)}{\frac{L}{R}s+1+s^2LC} = \frac{\frac{1}{C}sI_g(s)}{s^2+\frac{1}{RC}s+\frac{1}{LC}}$$

$$= \frac{(1.6)(1.2)s^2}{(s^2+1.6s+1)(s^2+1)} = \frac{1.92s^2}{(s^2+1.6s+1)(s^2+1)}$$

P 12.27 **[a]** $\dfrac{v_o - V_{\rm dc}}{R} + \dfrac{1}{L}\displaystyle\int_0^t v_o\,dx + C\frac{dv_o}{dt} = 0$

$$\therefore \qquad v_o + \frac{R}{L}\int_0^t v_o\,dx + RC\frac{dv_o}{dt} = V_{\rm dc}$$

[b] $V_o + \frac{R}{L}\frac{V_o}{s} + RCsV_o = \frac{V_{\rm dc}}{s}$

$\therefore \quad sLV_o + RV_o + RCLs^2V_o = LV_{\rm dc}$

$\therefore \quad V_o(s) = \frac{(1/RC)V_{\rm dc}}{s^2 + (1/RC)s + (1/LC)}$

[c] $i_o = \frac{1}{L}\int_0^t v_o\,dx$

$$I_o(s) = \frac{V_o}{sL} = \frac{(1/RCL)V_{\rm dc}}{s[s^2 + (1/RC)s + (1/LC)]}$$

P 12.28 **[a]** $\frac{1}{LC} = \frac{1}{(200\times10^{-3})(100\times10^{-9})} = 50\times10^6$

$$\frac{1}{RC} = \frac{1}{(5000)(100\times10^{-9})} = 2000$$

$$V_o(s) = \frac{70{,}000}{s^2 + 2000s + 50\times10^6}$$

$s_{1,2} = -1000 \pm j7000$ rad/s

$$V_o(s) = \frac{70{,}000}{(s+1000-j7000)(s+1000+j7000)}$$

$$= \frac{K_1}{s+1000-j7000} + \frac{K_1^*}{s+1000+j7000}$$

$$K_1 = \frac{70{,}000}{j14{,}000} = 5\underline{/-90^\circ}$$

$$v_o(t) = 10e^{-1000t}\cos(7000t - 90^\circ)u(t)\text{ V}$$

$$= 10e^{-1000t}\sin(7000t)u(t)\text{ V}$$

[b] $I_o(s) = \frac{35(10{,}000)}{s(s+1000-j7000)(s+1000+j7000)}$

$$= \frac{K_1}{s} + \frac{K_2}{s+1000-j7000} + \frac{K_2^*}{s+1000+j7000}$$

$$K_1 = \frac{35(10{,}000)}{50\times10^6} = 7\text{ mA}$$

$$K_2 = \frac{35(10{,}000)}{(-1000+j7000)(j14{,}000)} = 3.54\underline{/171.87^\circ}\text{ mA}$$

$$i_o(t) = [7 + 7.07e^{-1000t}\cos(7000t + 171.87^\circ)]u(t)\text{ mA}$$

P 12.29 **[a]** $I_{\rm dc} = \dfrac{1}{L}\displaystyle\int_0^t v_o\,dx + \dfrac{v_o}{R} + C\dfrac{dv_o}{dt}$

[b] $\dfrac{I_{\rm dc}}{s} = \dfrac{V_o(s)}{sL} + \dfrac{V_o(s)}{R} + sCV_o(s)$

$$\therefore\ \ V_o(s) = \frac{I_{\rm dc}/C}{s^2 + (1/RC)s + (1/LC)}$$

[c] $i_o = C\dfrac{dv_o}{dt}$

$$\therefore\ \ I_o(s) = sCV_o(s) = \frac{sI_{\rm dc}}{s^2 + (1/RC)s + (1/LC)}$$

P 12.30 **[a]** $\dfrac{1}{RC} = \dfrac{1}{(1\times 10^3)(2\times 10^{-6})} = 500$

$$\frac{1}{LC} = \frac{1}{(12.5)(2\times 10^{-6})} = 40{,}000$$

$$V_o(s) = \frac{500{,}000I_{\rm dc}}{s + 500s + 40{,}000}$$

$$= \frac{500{,}000I_{\rm dc}}{(s+100)(s+400)}$$

$$= \frac{15{,}000}{(s+100)(s+400)}$$

$$= \frac{K_1}{s+100} + \frac{K_2}{s+400}$$

$$K_1 = \frac{15{,}000}{300} = 50; \qquad K_2 = \frac{15{,}000}{-300} = -50$$

$$V_o(s) = \frac{50}{s+100} - \frac{50}{s+400}$$

$$v_o(t) = [50e^{-100t} - 50e^{-400t}]u(t)\,\mathrm{V}$$

[b] $I_o(s) = \dfrac{0.03s}{(s+100)(s+400)}$

$$= \frac{K_1}{s+100} + \frac{K_2}{s+400}$$

$$K_1 = \frac{0.03(-100)}{300} = -0.01$$

$$K_2 = \frac{0.03(-400)}{-300} = 0.04$$

$$I_o(s) = \frac{-0.01}{s+100} + \frac{0.04}{s+400}$$

$$i_o(t) = (40e^{-400t} - 10e^{-100t})u(t)\,\text{mA}$$

[c] $i_o(0) = 40 - 10 = 30\,\text{mA}$

Yes. The initial inductor current is zero by hypothesis, the initial resistor current is zero because the initial capacitor voltage is zero by hypothesis. Thus at $t = 0$ the source current appears in the capacitor.

P 12.31 **[a]** $C\dfrac{dv_1}{dt} + \dfrac{v_1 - v_2}{R} = i_g$

$$\frac{1}{L}\int_0^t v_2\,d\tau + \frac{v_2 - v_1}{R} = 0$$

or

$$C\frac{dv_1}{dt} + \frac{v_1}{R} - \frac{v_2}{R} = i_g$$

$$-\frac{v_1}{R} + \frac{v_2}{R} + \frac{1}{L}\int_0^t v_2\,d\tau = 0$$

[b] $CsV_1(s) + \dfrac{V_1(s)}{R} - \dfrac{V_2(s)}{R} = I_g(s)$

$$-\frac{V_1(s)}{R} + \frac{V_2(s)}{R} + \frac{V_2(s)}{sL} = 0$$

or

$$(RCs+1)V_1(s) - V_2(s) = RI_g(s)$$

$$-sLV_1(s) + (R + sL)V_2(s) = 0$$

Solving,

$$V_2(s) = \frac{sI_g(s)}{C[s^2 + (R/L)s + (1/LC)]}$$

P 12.32 $\dfrac{1}{C} = 5 \times 10^6; \qquad \dfrac{1}{LC} = 25 \times 10^6; \qquad \dfrac{R}{L} = 8000$

$$V_2(s) = \frac{(6 \times 10^{-3})(5 \times 10^6)}{s^2 + 8000s + 25 \times 10^6}$$

$$s_{1,2} = -4000 \pm j3000$$

$$V_2(s) = \frac{30{,}000}{(s + 4000 - j3000)(s + 4000 + j3000)}$$

$$= \frac{K_1}{s + 4000 - j3000} + \frac{K_1^*}{s + 4000 + j3000}$$

$$K_1 = \frac{30{,}000}{j6000} = -j5 = 5\underline{/-90^\circ}$$

$$v_2(t) = 10e^{-4000t}\cos(3000t - 90^\circ)$$
$$= [10e^{-4000t}\sin 3000t]u(t)\,\mathrm{V}$$

P 12.33 **[a]** For $t \geq 0^+$:

$$\frac{v_o}{R} + C\frac{dv_o}{dt} + i_o = 0$$

$$v_o = L\frac{di_o}{dt}; \qquad \frac{dv_o}{dt} = L\frac{d^2i_o}{dt^2}$$

$$\therefore \qquad \frac{L}{R}\frac{di_o}{dt} + LC\frac{d^2i_o}{dt^2}$$

or $$\frac{d^2i_o}{dt^2} + \frac{1}{RC}\frac{di_o}{dt} + \frac{1}{LC}i_o = 0$$

[b] $$s^2I_o(s) - sI_{\mathrm{dc}} - 0 + \frac{1}{RC}[sI_o(s) - I_{\mathrm{dc}}] + \frac{1}{LC}I_o(s) = 0$$

$$I_o(s)\left[s^2 + \frac{1}{RC}s + \frac{1}{LC}\right] = I_{\mathrm{dc}}(s + 1/RC)$$

$$I_o(s) = \frac{I_{\mathrm{dc}}[s + (1/RC)]}{[s^2 + (1/RC)s + (1/LC)]}$$

P 12.34 $$\frac{1}{RC} = 8000; \qquad \frac{1}{LC} = 16\times 10^6$$

$$I_o(s) = \frac{0.005(s + 8000)}{s^2 + 8000s + 16\times 10^6}$$

$$s_{1,2} = -4000$$

$$I_o(s) = \frac{0.005(s + 8000)}{(s + 4000)^2} = \frac{K_1}{(s + 4000)^2} + \frac{K_2}{s + 4000}$$

$$K_1 = 0.005(s + 8000)\bigg|_{s=-4000} = 20$$

$$K_2 = \frac{d}{ds}\left[0.005(s + 8000)\right]_{s=-4000} = 0.005$$

$$I_o(s) = \frac{20}{(s + 4000)^2} + \frac{0.005}{s + 4000}$$

$$i_o(t) = [20te^{-4000t} + 0.005e^{-4000t}]u(t)\,\mathrm{V}$$

P 12.35 **[a]** $300 = 60i_1 + 25\frac{di_1}{dt} + 10\frac{d}{dt}(i_2 - i_1) + 5\frac{d}{dt}(i_1 - i_2) - 10\frac{di_1}{dt}$

$$0 = 5\frac{d}{dt}(i_2 - i_1) + 10\frac{di_1}{dt} + 40i_2$$

Simplifying the above equations gives:

$$300 = 60i_1 + 10\frac{di_1}{dt} + 5\frac{di_2}{dt}$$

$$0 = 40i_2 + 5\frac{di_1}{dt} + 5\frac{di_2}{dt}$$

[b] $\frac{300}{s} = (10s + 60)I_1(s) + 5sI_2(s)$

$$0 = 5sI_1(s) + (5s + 40)I_2(s)$$

[c] Solving the equations in (b),

$$I_1(s) = \frac{60(s+8)}{s(s+4)(s+24)}$$

$$I_2(s) = \frac{-60}{(s+4)(s+24)}$$

[d] $I_1(s) = \frac{K_1}{s} + \frac{K_2}{s+4} + \frac{K_3}{s+24}$

$$K_1 = \frac{(60)(8)}{(4)(24)} = 5; \qquad K_2 = \frac{(60)(4)}{(-4)(20)} = -3$$

$$K_3 = \frac{(60)(-16)}{(-24)(-20)} = -2$$

$$I_1(s) = \left(\frac{5}{s} - \frac{3}{s+4} - \frac{2}{s+24}\right)$$

$$i_1(t) = (5 - 3e^{-4t} - 2e^{-24t})u(t)\,\text{A}$$

$$I_2(s) = \frac{K_1}{s+4} + \frac{K_2}{s+24}$$

$$K_1 = \frac{-60}{20} = -3; \qquad K_2 = \frac{-60}{-20} = 3$$

$$I_2(s) = \left(\frac{-3}{s+4} + \frac{3}{s+24}\right)$$

$$i_2(t) = (3e^{-24t} - 3e^{-4t})u(t)\,\text{A}$$

[e] $i_1(\infty) = 5\,\text{A}; \qquad i_2(\infty) = 0\,\text{A}$

[f] Yes, at $t = \infty$

$$i_1 = \frac{300}{60} = 5\,\mathrm{A}$$

Since i_1 is a dc current at $t = \infty$ there is no voltage induced in the 10 H inductor; hence, $i_2 = 0$. Also note that $i_1(0) = 0$ and $i_2(0) = 0$. Thus our solutions satisfy the condition of no initial energy stored in the circuit.

P 12.36 From Problem 12.26:

$$V(s) = \frac{1.92s^2}{(s^2 + 1.6s + 1)(s^2 + 1)}$$

$$s^2 + 1.6s + 1 = (s + 0.8 + j0.6)(s + 0.8 - j0.6); \qquad s^2 + 1 = (s - j1)(s + j1)$$

Therefore

$$V(s) = \frac{1.92s^2}{(s + 0.8 + j0.6)(s + 0.8 - j0.6)(s - j1)(s + j1)}$$

$$= \frac{K_1}{s + 0.8 - j0.6} + \frac{K_1^*}{s + 0.8 + j0.6} + \frac{K_2}{s - j1} + \frac{K_2^*}{s + j1}$$

$$K_1 = \frac{1.92s^2}{(s + 0.8 + j0.6)(s^2 + 1)}\bigg|_{s=-0.8+j0.6} = 1\underline{/-126.87^\circ}$$

$$K_2 = \frac{1.92s^2}{(s + j1)(s^2 + 1.6s + 1)}\bigg|_{s=j1} = 0.6\underline{/0^\circ}$$

Therefore

$$v(t) = [2e^{-0.8t}\cos(0.6t - 126.87^\circ) + 1.2\cos(t)]u(t)\,\mathrm{V}$$

P 12.37 **[a]** $F(s) = \dfrac{K_1}{s+1} + \dfrac{K_2}{s+2} + \dfrac{K_3}{s+4}$

$$K_1 = \frac{8s^2 + 37s + 32}{(s+2)(s+4)}\bigg|_{s=-1} = 1$$

$$K_2 = \frac{8s^2 + 37s + 32}{(s+1)(s+4)}\bigg|_{s=-2} = 5$$

$$K_3 = \frac{8s^2 + 37s + 32}{(s+1)(s+2)}\bigg|_{s=-4} = 2$$

$$f(t) = [e^{-t} + 5e^{-2t} + 2e^{-4t}]u(t)$$

[b] $F(s) = \dfrac{K_1}{s} + \dfrac{K_2}{s+2} + \dfrac{K_3}{s+3} + \dfrac{K_4}{s+5}$

$$K_1 = \frac{8s^3 + 89s^2 + 311s + 300}{(s+2)(s+3)(s+5)}\bigg|_{s=0} = 10$$

$$K_2 = \frac{8s^3 + 89s^2 + 311s + 300}{s(s+3)(s+5)}\bigg|_{s=-2} = 5$$

$$K_3 = \frac{8s^3 + 89s^2 + 311s + 300}{s(s+2)(s+5)}\bigg|_{s=-3} = -8$$

$$K_4 = \frac{8s^3 + 89s^2 + 311s + 300}{s(s+2)(s+3)}\bigg|_{s=-5} = 1$$

$$f(t) = [10 + 5e^{-2t} - 8e^{-3t} + e^{-5t}]u(t)$$

[c] $F(s) = \dfrac{K_1}{s+1} + \dfrac{K_2}{s+2-j} + \dfrac{K_2^*}{s+2+j}$

$$K_1 = \frac{22s^2 + 60s + 58}{s^2 + 4s + 5}\bigg|_{s=-1} = 10$$

$$K_2 = \frac{22s^2 + 60s + 58}{(s+1)(s+2+j)}\bigg|_{s=-2+j} = 6 + j8 = 10\underline{/53.13^\circ}$$

$$f(t) = [10e^{-t} + 20e^{-2t}\cos(t + 53.13^\circ)]u(t)$$

[d] $F(s) = \dfrac{K_1}{s} + \dfrac{K_2}{s+7-j} + \dfrac{K_2^*}{s+7+j}$

$$K_1 = \frac{250(s+7)(s+14)}{s^2 + 14s + 50}\bigg|_{s=0} = 490$$

$$K_2 = \frac{250(s+7)(s+14)}{s(s+7+j)}\bigg|_{s=-7+j} = 125\underline{/-163.74^\circ}$$

$$f(t) = [490 + 250e^{-7t}\cos(t - 163.74^\circ)]u(t)$$

P 12.38 **[a]** $F(s) = \dfrac{K_1}{s^2} + \dfrac{K_2}{s} + \dfrac{K_3}{s+5}$

$$K_1 = \frac{100}{s+5}\bigg|_{s=0} = 20$$

$$K_2 = \frac{d}{ds}\left[\frac{100}{s+5}\right] = \frac{-100}{(s+5)^2}\bigg|_{s=0} = -4$$

$$K_3 = \frac{100}{s^2}\bigg|_{s=-5} = 4$$

$$f(t) = [20t - 4 + 4e^{-5t}]u(t)$$

[b] $F(s) = \frac{K_1}{s} + \frac{K_2}{(s+1)^2} + \frac{K_3}{s+1}$

$$K_1 = \frac{50(s+5)}{(s+1)^2}\bigg|_{s=0} = 250$$

$$K_2 = \frac{50(s+5)}{s}\bigg|_{s=-1} = -200$$

$$K_3 = \frac{d}{ds}\left[\frac{50(s+5)}{s}\right] = \left[\frac{50}{s} - \frac{50(s+5)}{s^2}\right]_{s=-1} = -250$$

$$f(t) = [250 - 200te^{-t} - 250e^{-t}]u(t)$$

[c] $F(s) = \frac{K_1}{s^2} + \frac{K_2}{s} + \frac{K_3}{s+3-j} + \frac{K_3^*}{s+3+j}$

$$K_1 = \frac{100(s+3)}{s^2+6s+10}\bigg|_{s=0} = 30$$

$$K_2 = \frac{d}{ds}\left[\frac{100(s+3)}{s^2+6s+10}\right]$$

$$= \left[\frac{100}{s^2+6s+10} - \frac{100(s+3)(2s+6)}{(s^2+6s+10)^2}\right]_{s=0} = 10 - 18 = -8$$

$$K_3 = \frac{100(s+3)}{s^2(s+3+j)}\bigg|_{s=-3+j} = 4 + j3 = 5\underline{/36.87^\circ}$$

$$f(t) = [30t - 8 + 10e^{-3t}\cos(t + 36.87^\circ)]u(t)$$

[d] $F(s) = \frac{K_1}{s} + \frac{K_2}{(s+1)^3} + \frac{K_3}{(s+1)^2} + \frac{K_4}{s+1}$

$$K_1 = \frac{5(s+2)^2}{(s+1)^3}\bigg|_{s=0} = 20$$

$$K_2 = \frac{5(s+2)^2}{s}\bigg|_{s=-1} = -5$$

$$K_3 = \frac{d}{ds}\left[\frac{5(s+2)^2}{s}\right] = \left[\frac{10(s+2)}{s} - \frac{5(s+2)^2}{s^2}\right]_{s=-1}$$

$$= -10 - 5 = -15$$

$$K_4 = \frac{1}{2}\frac{d}{ds}\left[\frac{10(s+2)}{s} - \frac{5(s+2)^2}{s^2}\right]$$

$$= \frac{1}{2}\left[\frac{10}{s} - \frac{10(s+2)}{s^2} - \frac{10(s+2)}{s^2} + \frac{10(s+2)^2}{s^3}\right]_{s=-1}$$

$$= \frac{1}{2}(-10 - 10 - 10 - 10) = -20$$

$$f(t) = [20 - 2.5t^2e^{-t} - 15te^{-t} - 20e^{-t}]u(t)$$

[e] $$F(s) = \frac{K_1}{s} + \frac{K_2}{(s+2-j)^2} + \frac{K_2^*}{(s+2+j)^2} + \frac{K_3}{s+2-j} + \frac{K_3^*}{s+2-j}$$

$$K_1 = \left.\frac{400}{(s^2+4s+5)^2}\right|_{s=0} = 16$$

$$K_2 = \left.\frac{400}{s(s+2+j)^2}\right|_{s=-2+j} = 44.72\underline{/26.57^\circ}$$

$$K_3 = \frac{d}{ds}\left[\frac{400}{s(s+2+j)^2}\right] = \left[\frac{-400}{s^2(s+2+j)^2} + \frac{-800}{s(s+2+j)^3}\right]_{s=-2+j}$$

$$= 12 + j16 - 20 + j40 = -8 + j56 = 56.57\underline{/98.13^\circ}$$

$$f(t) = [16 + 89.44te^{-2t}\cos(t + 26.57^\circ) + 113.14e^{-2t}\cos(t + 98.13^\circ)]u(t)$$

P 12.39 **[a]**

$$F(s) = \begin{array}{r|l} & 5 \\ \hline s^2+6s+8 & 5s^2 + 38s + 80 \\ & 5s^2 + 30s + 40 \\ \hline & 8s + 40 \end{array}$$

$$F(s) = 5 + \frac{8s+40}{s^2+6s+8} = 5 + \frac{K_1}{s+2} + \frac{K_2}{s+4}$$

$$K_1 = \left.\frac{8s+40}{s+4}\right|_{s=-2} = 12$$

$$K_2 = \left.\frac{8s+40}{s+2}\right|_{s=-4} = -4$$

$$f(t) = 5\delta(t) + [12e^{-2t} - 4e^{-4t}]u(t)$$

[b]

$$F(s) = \begin{array}{r|l} & 10 \\ \hline s^2+48s+625 & 10s^2 + 512s + 7186 \\ & 10s^2 + 480s + 6250 \\ \hline & 32s + 936 \end{array}$$

$$F(s) = 10 + \frac{32s+936}{s^2+48s+625} = 10 + \frac{K_1}{s+24-j7} + \frac{K_2^*}{s+24+j7}$$

$$K_1 = \left.\frac{32s+936}{s+24+j7}\right|_{s=-24+j7} = 16 - j12 = 20\underline{/-36.87^\circ}$$

$$f(t) = 10\delta(t) + [40e^{-24t}\cos(7t - 36.87^\circ)]u(t)$$

[c]

$$F(s) = \begin{array}{r|l} & \quad\quad\quad\quad\quad\quad s-10 \\ \hline s^2+15s+50 & s^3+5s^2-50s-100 \\ & s^3+15s^2+50s \\ \cline{2-2} & -10s^2-100s-100 \\ & -10s^2-150s-500 \\ \cline{2-2} & \quad\quad\quad\quad 50s+400 \end{array}$$

$$F(s) = s - 10 + \frac{K_1}{s+5} + \frac{K_2}{s+10}$$

$$K_1 = \left.\frac{50s+400}{s+10}\right|_{s=-5} = 30$$

$$K_2 = \left.\frac{50s+400}{s+5}\right|_{s=-10} = 20$$

$$f(t) = \delta'(t) - 10\delta(t) + [30e^{-5t} + 20e^{-10t}]u(t)$$

P 12.40 **[a]** $F(s) = \dfrac{K_1}{s^2} + \dfrac{K_2}{s} + \dfrac{K_3}{s+1-j2} + \dfrac{K_3^*}{s+1+j2}$

$$K_1 = \left.\frac{100(s+1)}{s^2+2s+5}\right|_{s=0} = 20$$

$$K_2 = \frac{d}{ds}\left[\frac{100(s+1)}{s^2+2s+5}\right] = \left[\frac{100}{s^2+2s+5} - \frac{100(s+1)(2s+2)}{(s^2+2s+5)^2}\right]_{s=0}$$

$$= 20 - 8 = 12$$

$$K_3 = \left.\frac{100(s+1)}{s^2(s+1+j2)}\right|_{s=-1+j2} = -6 + j8 = 10\underline{/126.87^\circ}$$

$$f(t) = [20t + 12 + 20e^{-t}\cos(2t + 126.87^\circ)]u(t)$$

[b] $F(s) = \dfrac{K_1}{s} + \dfrac{K_2}{(s+5)^3} + \dfrac{K_3}{(s+5)^2} + \dfrac{K_4}{s+5}$

$$K_1 = \left.\frac{500}{(s+5)^3}\right|_{s=0} = 4$$

$$K_2 = \left.\frac{500}{s}\right|_{s=-5} = -100$$

$$K_3 = \frac{d}{ds}\left[\frac{500}{s}\right] = \left.\frac{-500}{s^2}\right|_{s=-5} = -20$$

$$K_4 = \frac{1}{2}\frac{d}{ds}\left[\frac{-500}{s^2}\right] = \left.\frac{1}{2}\frac{1000}{(s^3)}\right|_{s=-5} = -4$$

$$f(t) = [4 - 50t^2e^{-5t} - 20te^{-5t} - 4e^{-5t}]u(t)$$

[c] $F(s) = \frac{K_1}{s} + \frac{K_2}{(s+1)^3} + \frac{K_3}{(s+1)^2} + \frac{K_4}{s+1}$

$$K_1 = \frac{40(s+2)}{(s+1)^3}\Big|_{s=0} = 80$$

$$K_2 = \frac{40(s+2)}{s}\Big|_{s=-1} = -40$$

$$K_3 = \frac{d}{ds}\left[\frac{40(s+2)}{s}\right] = \left[\frac{40}{s} - \frac{40(s+2)}{s^2}\right]_{s=-1} = -40 - 40 = -80$$

$$K_4 = \frac{1}{2}\frac{d}{ds}\left[\frac{40}{s} - \frac{40(s+2)}{s^2}\right]$$

$$= \frac{1}{2}\left[\frac{-40}{s^2} - \frac{40}{s^2} + \frac{80(s+2)}{s^3}\right]_{s=-1} = \frac{1}{2}(-40-40-80) = -80$$

$$f(t) = [80 - 20t^2e^{-t} - 80te^{-t} - 80e^{-t}]u(t)$$

[d] $F(s) = \frac{K_1}{s} + \frac{K_2}{(s+1)^4} + \frac{K_3}{(s+1)^3} + \frac{K_4}{(s+1)^2} + \frac{K_5}{s+1}$

$$K_1 = \frac{(s+5)^2}{(s+1)^4}\Big|_{s=0} = 25$$

$$K_2 = \frac{(s+5)^2}{s}\Big|_{s=-1} = -16$$

$$K_3 = \frac{d}{ds}\left[\frac{(s+5)^2}{s}\right] = \left[\frac{2(s+5)}{s} - \frac{(s+5)^2}{s^2}\right]_{s=-1}$$

$$= -8 - 16 = -24$$

$$K_4 = \frac{1}{2}\frac{d}{ds}\left[\frac{2(s+5)}{s} - \frac{(s+5)^2}{s^2}\right]$$

$$= \frac{1}{2}\left[\frac{2}{s} - \frac{2(s+5)}{s^2} - \frac{2(s+5)}{s^2} + \frac{2(s+5)^2}{s^3}\right]_{s=-1}$$

$$= \frac{1}{2}(-2-8-8-32) = -25$$

$$K_5 = \frac{1}{6}\frac{d}{ds}\left[\frac{2}{s} - \frac{2(s+5)}{s^2} - \frac{2(s+5)}{s^2} + \frac{2(s+5)^2}{s^3}\right]$$

$$= \frac{1}{6}\left[\frac{-2}{s^2} - \frac{2}{s^2} + \frac{4(s+5)}{s^3} - \frac{2}{s^2} + \frac{4(s+5)}{s^3} + \frac{4(s+5)}{s^3} - \frac{6(s+5)^2}{s^4}\right]_{s=-1}$$

$$= \frac{1}{6}(-2-2-16-2-16-16-96) = -25$$

$$f(t) = [25 - (8/3)t^3e^{-t} - 12t^2e^{-t} - 25te^{-t} - 25e^{-t}]u(t)$$

P 12.41 $$f(t) = \mathcal{L}^{-1}\left\{\frac{K}{s+\alpha-j\beta} + \frac{K^*}{s+\alpha+j\beta}\right\}$$

$$= Ke^{-\alpha t}e^{j\beta t} + K^* e^{-\alpha t}e^{-j\beta t}$$

$$= |K|e^{-\alpha t}[e^{j\theta}e^{j\beta t} + e^{-j\theta}e^{-j\beta t}]$$

$$= |K|e^{-\alpha t}[e^{j(\beta t+\theta)} + e^{-j(\beta t+\theta)}]$$

$$= 2|K|e^{-\alpha t}\cos(\beta t+\theta)$$

P 12.42 **[a]** $$\mathcal{L}\{t^n f(t)\} = (-1)^n\left[\frac{d^n F(s)}{ds^n}\right]$$

Let $f(t) = 1$, then $F(s) = \dfrac{1}{s}$, thus $\dfrac{d^n F(s)}{ds^n} = \dfrac{(-1)^n n!}{s^{(n+1)}}$

Therefore $\mathcal{L}\{t^n\} = (-1)^n\left[\dfrac{(-1)^n n!}{s^{(n+1)}}\right] = \dfrac{n!}{s^{(n+1)}}$

It follows that $\mathcal{L}\{t^{(r-1)}\} = \dfrac{(r-1)!}{s^r}$

and $\mathcal{L}\{t^{(r-1)}e^{-at}\} = \dfrac{(r-1)!}{(s+a)^r}$

Therefore $\dfrac{K}{(r-1)!}\mathcal{L}\{t^{r-1}e^{-at}\} = \dfrac{K}{(s+a)^r} = \mathcal{L}\left\{\dfrac{Kt^{r-1}e^{-at}}{(r-1)!}\right\}$

[b] $$f(t) = \mathcal{L}^{-1}\left\{\frac{K}{(s+\alpha-j\beta)^r} + \frac{K^*}{(s+\alpha+j\beta)^r}\right\}$$

Therefore

$$f(t) = \frac{Kt^{r-1}}{(r-1)!}e^{-(\alpha-j\beta)t} + \frac{K^*t^{r-1}}{(r-1)!}e^{-(\alpha+j\beta)t}$$

$$= \frac{|K|t^{r-1}e^{-\alpha t}}{(r-1)!}\left[e^{j\theta}e^{j\beta t} + e^{-j\theta}e^{-j\beta t}\right]$$

$$= \left[\frac{2|K|t^{r-1}e^{-\alpha t}}{(r-1)!}\right]\cos(\beta t+\theta)$$

P 12.43 **[a]** $$\lim_{s\to\infty} sV(s) = \lim_{s\to\infty}\left[\frac{1.92s^3}{s^4[1+(1.6/s)+(1/s^2)][1+(1/s^2)]}\right] = 0$$

Therefore $v(0^+) = 0$

[b] No, V has a pair of poles on the imaginary axis.

P 12.44 **[a]** $sF(s) = \dfrac{8s^3 + 37s^2 + 32s}{(s+1)(s+2)(s+4)}$

$$\lim_{s\to 0} sF(s) = 0, \qquad \therefore \quad f(\infty) = 0$$

$$\lim_{s\to \infty} sF(s) = 8, \qquad \therefore \quad f(0^+) = 8$$

[b] $sF(s) = \dfrac{8s^3 + 89s^2 + 311s + 300}{(s+2)(s^2+8s+15)}$

$$\lim_{s\to 0} sF(s) = 10; \qquad \therefore \quad f(\infty) = 10$$

$$\lim_{s\to \infty} sF(s) = 8, \qquad \therefore \quad f(0^+) = 8$$

[c] $sF(s) = \dfrac{22s^3 + 60s^2 + 58s}{(s+1)(s^2+4s+5)}$

$$\lim_{s\to 0} sF(s) = 0, \qquad \therefore \quad f(\infty) = 0$$

$$\lim_{s\to \infty} sF(s) = 22, \qquad \therefore \quad f(0^+) = 22$$

[d] $sF(s) = \dfrac{250(s+7)(s+14)}{(s^2+14s+50)}$

$$\lim_{s\to 0} sF(s) = \frac{250(7)(14)}{50} = 490, \qquad \therefore \quad f(\infty) = 490$$

$$\lim_{s\to \infty} sF(s) = 250, \qquad \therefore \quad f(0^+) = 250$$

P 12.45 **[a]** $sF(s) = \dfrac{100}{s(s+5)}$

$F(s)$ has a second-order pole at the origin so we cannot use the final value theorem.

$$\lim_{s\to \infty} sF(s) = 0, \qquad \therefore \quad f(0^+) = 0$$

[b] $sF(s) = \dfrac{50(s+5)}{(s+1)^2}$

$$\lim_{s\to 0} sF(s) = 250, \qquad \therefore \quad f(\infty) = 250$$

$$\lim_{s\to \infty} sF(s) = 0, \qquad \therefore \quad f(0^+) = 0$$

[c] $sF(s) = \dfrac{100(s+3)}{s(s^2+6s+10)}$

$F(s)$ has a second-order pole at the origin so we cannot use the final value theorem.

$$\lim_{s\to \infty} sF(s) = 0, \qquad \therefore \quad f(0^+) = 0$$

[d] $sF(s) = \dfrac{5(s+2)^2}{(s+1)^3}$

$$\lim_{s\to 0} sF(s) = 20, \qquad \therefore \quad f(\infty) = 20$$

$$\lim_{s\to \infty} sF(s) = 0, \qquad \therefore \quad f(0^+) = 0$$

[e] $sF(s) = \dfrac{400}{(s^2+4s+5)^2}$

$$\lim_{s\to 0} sF(s) = 16, \qquad \therefore \quad f(\infty) = 16$$

$$\lim_{s\to \infty} sF(s) = 0, \qquad \therefore \quad f(0^+) = 0$$

P 12.46 All of the $F(s)$ functions referenced in this problem are improper rational functions, and thus the corresponding $f(t)$ functions contain impulses ($\delta(t)$). Thus, neither the initial value theorem nor the final value theorem may be applied to these $F(s)$ functions!

P 12.47 $sV_o(s) = \dfrac{sV_{\text{dc}}/RC}{s^2 + (1/RC)s + (1/LC)}$

$$\lim_{s\to 0} sV_o(s) = 0, \qquad \therefore \quad v_o(\infty) = 0$$

$$\lim_{s\to \infty} sV_o(s) = 0, \qquad \therefore \quad v_o(0^+) = 0$$

$$sI_o(s) = \frac{V_{\text{dc}}/RCL)}{s^2 + (1/RC)s + (1/LC)}$$

$$\lim_{s\to 0} sI_o(s) = \frac{V_{\text{dc}}/RLC}{1/LC} = \frac{V_{\text{dc}}}{R}, \qquad \therefore \quad i_o(\infty) = \frac{V_{\text{dc}}}{R}$$

$$\lim_{s\to \infty} sI_o(s) = 0, \qquad \therefore \quad i_o(0^+) = 0$$

P 12.48 $sV_o(s) = \dfrac{(I_{\text{dc}}/C)s}{s^2 + (1/RC)s + (1/LC)}$

$$\lim_{s\to 0} sV_o(s) = 0, \qquad \therefore \quad v_o(\infty) = 0$$

$$\lim_{s\to \infty} sV_o(s) = 0, \qquad \therefore \quad v_o(0^+) = 0$$

$$sI_o(s) = \frac{s^2 I_{\text{dc}}}{s^2 + (1/RC)s + (1/LC)}$$

$$\lim_{s\to 0} sI_o(s) = 0, \qquad \therefore \quad i_o(\infty) = 0$$

$$\lim_{s\to \infty} sI_o(s) = I_{\text{dc}}, \qquad \therefore \quad v_o(0^+) = I_{\text{dc}}$$

P 12.49 **[a]** $sF(s) = \dfrac{100(s+1)}{s(s^2+2s+5)}$

$F(s)$ has a second-order pole at the origin, so we cannot use the final value theorem here.

$$\lim_{s\to\infty} sF(s) = 0, \qquad \therefore \quad f(0^+) = 0$$

[b] $sF(s) = \dfrac{500}{(s+5)^3}$

$$\lim_{s\to 0} sF(s) = 4, \qquad \therefore \quad f(\infty) = 4$$

$$\lim_{s\to\infty} sF(s) = 0, \qquad \therefore \quad f(0^+) = 0$$

[c] $sF(s) = \dfrac{40(s+2)}{(s+1)^3}$

$$\lim_{s\to 0} sF(s) = 80, \qquad \therefore \quad f(\infty) = 80$$

$$\lim_{s\to\infty} sF(s) = 0, \qquad \therefore \quad f(0^+) = 0$$

[d] $sF(s) = \dfrac{(s+5)^2}{(s+1)^4}$

$$\lim_{s\to 0} sF(s) = 25, \qquad \therefore \quad f(\infty) = 25$$

$$\lim_{s\to\infty} sF(s) = 0, \qquad \therefore \quad f(0^+) = 0$$

P 12.50 $sI_o(s) = \dfrac{I_{\rm dc}s[s+(1/RC)]}{s^2+(1/RC)s+(1/LC)}$

$$\lim_{s\to 0} sI_o(s) = 0, \qquad \therefore \quad i_o(\infty) = 0$$

$$\lim_{s\to\infty} sI_o(s) = I_{\rm dc}, \qquad \therefore \quad i_o(0^+) = I_{\rm dc}$$