21. (a) The object oscillates about its equilibrium point, where the downward force of gravity is balanced by the upward force of the spring. If ℓ is the elongation of the spring at equilibrium, then $k\ell = mg$, where k is the spring constant and m is the mass of the object. Thus $k/m = g/\ell$ and $f = \omega/2\pi = (1/2\pi)\sqrt{k/m} = (1/2\pi)\sqrt{g/\ell}$. Now the equilibrium point is halfway between the points where the object is momentarily at rest. One of these points is where the spring is unstretched and the other is the lowest point, 10 cm below. Thus $\ell = 5.0\,\text{cm} = 0.050\,\text{m}$ and

$$f = \frac{1}{2\pi}\sqrt{\frac{9.8\,\text{m/s}^2}{0.050\,\text{m}}} = 2.23\,\text{Hz} .$$

(b) Use conservation of energy. We take the zero of gravitational potential energy to be at the initial position of the object, where the spring is unstretched. Then both the initial potential and kinetic energies are zero. We take the y axis to be positive in the downward direction and let $y = 0.080\,\text{m}$. The potential energy when the object is at this point is $U = \frac{1}{2}ky^2 - mgy$. The energy equation becomes $0 = \frac{1}{2}ky^2 - mgy + \frac{1}{2}mv^2$. We solve for the speed.

$$\begin{aligned} v &= \sqrt{2gy - \frac{k}{m}y^2} = \sqrt{2gy - \frac{g}{\ell}y^2} \\ &= \sqrt{2(9.8\,\text{m/s}^2)(0.080\,\text{m}) - \left(\frac{9.8\,\text{m/s}^2}{0.050\,\text{m}}\right)(0.080\,\text{m})^2} = 0.56\ \text{m/s} \end{aligned}$$

(c) Let m be the original mass and Δm be the additional mass. The new angular frequency is $\omega' = \sqrt{k/(m+\Delta m)}$. This should be half the original angular frequency, or $\frac{1}{2}\sqrt{k/m}$. We solve $\sqrt{k/(m+\Delta m)} = \frac{1}{2}\sqrt{k/m}$ for m. Square both sides of the equation, then take the reciprocal to obtain $m + \Delta m = 4m$. This gives $m = \Delta m/3 = (300\,\text{g})/3 = 100\,\text{g}$.

(d) The equilibrium position is determined by the balancing of the gravitational and spring forces: $ky = (m+\Delta m)g$. Thus $y = (m+\Delta m)g/k$. We will need to find the value of the spring constant k. Use $k = m\omega^2 = m(2\pi f)^2$. Then

$$y = \frac{(m+\Delta m)g}{m(2\pi f)^2} = \frac{(0.10\,\text{kg} + 0.30\,\text{kg})\left(9.8\,\text{m/s}^2\right)}{(0.10\,\text{kg})(2\pi \times 2.24\,\text{Hz})^2} = 0.20\,\text{m} .$$

This is measured from the initial position.