

33. (a) Letting $v(r) = Hr \leq v_e = \sqrt{2GM/r}$, we get $M/r^3 \geq H^2/2G$. Thus,

$$\rho = \frac{M}{4\pi r^3/3} = \frac{3}{4\pi} \frac{M}{r^3} \geq \frac{3H^2}{8\pi G} .$$

(b) The density being expressed in H-atoms/m³ is equivalent to expressing it in terms of $\rho_0 = m_{\text{H}}/\text{m}^3 = 1.67 \times 10^{-27} \text{ kg/m}^3$. Thus,

$$\begin{aligned} \rho &= \frac{3H^2}{8\pi G\rho_0} \left(\text{H atoms/m}^3 \right) = \frac{3(0.0193 \text{ m/s}\cdot\text{ly})^2 (1.00 \text{ ly}/9.460 \times 10^{15} \text{ m})^2 (\text{H atoms/m}^3)}{8\pi (6.67 \times 10^{-11} \text{ m}^3/\text{kg}\cdot\text{s}^2) (1.67 \times 10^{-27} \text{ kg/m}^3)} \\ &= 4.5 \text{ H atoms/m}^3 . \end{aligned}$$