51. The proposed wave function is

$$\psi = \frac{1}{\sqrt{\pi} a^{3/2}} e^{-r/a}$$

where a is the Bohr radius. Substituting this into the right side of Schrödinger's equation, our goal is to show that the result is zero. The derivative is

$$\frac{d\psi}{dr} = -\frac{1}{\sqrt{\pi} a^{5/2}} e^{-r/a}$$

so

$$r^2 \frac{d\psi}{dr} = -\frac{r^2}{\sqrt{\pi} a^{5/2}} e^{-r/a}$$

and

$$\frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d\psi}{dr}\right) = \frac{1}{\sqrt{\pi} a^{5/2}} \left[-\frac{2}{r} + \frac{1}{a}\right] e^{-r/a} = \frac{1}{a}\left[-\frac{2}{r} + \frac{1}{a}\right] \psi \; .$$

The energy of the ground state is given by $E = -me^4/8\varepsilon_0^2 h^2$, and the Bohr radius is given by $a = h^2\varepsilon_0/\pi m e^2$, so $E = -e^2/8\pi\varepsilon_0 a$. The potential energy is given by $U = -e^2/4\pi\varepsilon_0 r$, so

$$\begin{aligned} \frac{8\pi^2 m}{h^2}\left[E - U\right]\psi &= \frac{8\pi^2 m}{h^2}\left[-\frac{e^2}{8\pi\varepsilon_0 a} + \frac{e^2}{4\pi\varepsilon_0 r}\right]\psi = \frac{8\pi^2 m}{h^2}\frac{e^2}{8\pi\varepsilon_0}\left[-\frac{1}{a} + \frac{2}{r}\right]\psi \\ &= \frac{\pi m e^2}{h^2\varepsilon_0}\left[-\frac{1}{a} + \frac{2}{r}\right]\psi = \frac{1}{a}\left[-\frac{1}{a} + \frac{2}{r}\right]\psi \; . \end{aligned}$$

The two terms in Schrödinger's equation cancel, and the proposed function ψ satisfies that equation.