49. We consider an infinitesimal segment of a string oscillating in a standing wave pattern. Its length is $\mathrm{d}x$ and its mass is $\mathrm{d}m = \mu\,\mathrm{d}x$, where μ is its linear mass density. If it is moving with speed u its kinetic energy is $\mathrm{d}K = \frac{1}{2}u^2\,\mathrm{d}m = \frac{1}{2}\mu u^2\,\mathrm{d}x$. If the segment is located at x its displacement at time t is $y = 2y_m \sin(kx)\cos(\omega t)$ and its velocity is $u = \partial y/\partial t = -2\omega y_m \sin(kx)\sin(\omega t)$, so its kinetic energy is

$$\mathrm{d}K = \left(\frac{1}{2}\right)\left(4\mu\omega^2 y_m^2\right)\sin^2(kx)\sin^2(\omega t) = 2\mu\omega^2 y_m^2 \sin^2(kx)\sin^2(\omega t)\ .$$

Here y_m is the amplitude of each of the traveling waves that combine to form the standing wave. The infinitesimal segment has maximum kinetic energy when $\sin^2(\omega t) = 1$ and the maximum kinetic energy is given by the differential amount

$$dK_m = 2\mu\omega^2 y_m^2 \sin^2(kx)\ .$$

Note that every portion of the string has its maximum kinetic energy at the same time although the values of these maxima are different for different parts of the string. If the string is oscillating with n loops, the length of string in any one loop is L/n and the kinetic energy of the loop is given by the integral

$$K_m = 2\mu\omega^2 y_m^2 \int_0^{L/n} \sin^2(kx)\,dx\ .$$

We use the trigonometric identity $\sin^2(kx) = \frac{1}{2}\left[1 + 2\cos(2kx)\right]$ to obtain

$$K_m = \mu\omega^2 y_m^2 \int_0^{L/n} \left[1 + 2\cos(2kx)\right]\mathrm{d}x = \mu\omega^2 y_m^2 \left[\frac{L}{n} + \frac{1}{k}\sin\frac{2kL}{n}\right]\ .$$

For a standing wave of n loops the wavelength is $\lambda = 2L/n$ and the angular wave number is $k = 2\pi/\lambda = n\pi/L$, so $2kL/n = 2\pi$ and $\sin(2kL/n) = 0$, no matter what the value of n. Thus,

$$K_m = \frac{\mu\omega^2 y_m^2 L}{n}\ .$$

To obtain the expression given in the problem statement, we first make the substitutions $\omega = 2\pi f$ and $L/n = \lambda/2$, where f is the frequency and λ is the wavelength. This produces $K_m = 2\pi^2\mu y_m^2 f^2\lambda$. We now substitute the wave speed v for $f\lambda$ and obtain $K_m = 2\pi^2\mu y_m^2 f v$.