Part IV: A Deeper Look at Nature

Chapter S2. Space and Time

General Notes on Part IV (Chapters S2–S4)

The chapters of Part IV (Chapters S2–S4) are labeled "supplementary" because coverage of them is optional. Covering these chapters will give your students a deeper understanding of the topics that follow, on stars, galaxies, and cosmology, but the later chapters are self-contained and may be covered without having covered Chapters S2–S4 at all. We have therefore designed these chapters so that they can be used in any of three ways:

1. You may skip Part IV entirely. We have written the chapters of Parts V and VI so that they are *not* dependent on the Part IV chapters.
2. If you want to present a brief overview of relativity and quantum mechanics, you can cover only the first section in each of Chapters S2–S4. Each first section presents an overview of the key ideas. Covering only the first section will provide about the amount of background found in most other introductory astronomy texts, in which relativity and quantum mechanics are given just a few pages when relevant topics come up.
3. You can cover Chapters S2–S4 in depth. Our experience shows that each of these three chapters requires a minimum of about a week in class (e.g., 3 hours) to do it justice; it is even better if you can spread the three chapters over four weeks. Note that, while this coverage takes significant class time, it should enable you to move more quickly through later chapters of the book, especially Chapter 18 on degenerate objects and black holes and the chapters with significant cosmology (especially Chapters 20–23).

The placement of these three supplementary chapters is dictated by the fact that, *if* you cover them, they will enhance students' understanding of the chapters in Parts V and VI (even though they are not prerequisites to those chapters).

Incidentally, if you have not previously covered relativity in your astronomy classes, we hope you will at least consider incorporating it in the future. Based on having taught it for many years in the way presented in these chapters, we've found not only that students are able to understand the topic, but also that most of our students name relativity as their favorite part of their course in astronomy.

General Notes on Chapter S2

This chapter provides students with a short but solid introduction to the special theory of relativity. The first section provides a concise overview of special relativity, and the remaining sections explain the logic and evidence behind this overview. Thus, it is possible to teach only the first section if you want to give students an overview of special relativity but do not have time to go into any details of the theory.

Because relativity forces students to think in a different way than they are accustomed to, we recommend that you cover the key elements and thought experiments of this chapter essentially verbatim from the text. We have found that repetition of explanations in class and in the text helps solidify student understanding of unfamiliar topics such as relativity.

As always, when you prepare to teach this chapter, be sure you are familiar with the relevant media resources (see the complete, section-by-section resource grid in Appendix 3 of this Instructor's Guide) and the on-line quizzes and other study resources available on the Astronomy Place website.

Teaching Notes (By Section)

Section S2.1 Einstein's Revolution

This first section introduces the idea of relativity and summarizes the key findings of relativity that are discussed in the remainder of the chapter.

- The bullet list in this section lists what we refer to as five key ideas of special relativity: (1) inability to exceed c, (2) time dilation, (3) relativity of simultaneity, (4) length contraction, and (5) mass increase and $E = mc^2$. The rest of the chapter essentially explains these five ideas in detail; if you cover only this first section, use this list as the main element in your overview of relativity.
- We recommend that you bring a large globe (a large, inflatable Earth globe works great) when explaining the example of the supersonic airplane flight from Nairobi to Quito that appears in Figure S2.1. We have found this example to be particularly effective in helping students understand that motion really is relative.
- Please note that, although many people often state that relativity defies common sense, we have found that such statements only serve to make students resistant to learning relativity. Thus, we instead choose the strategy of arguing that relativity does *not* violate common sense because we have no common experience with speeds at which the effects of relativity become noticeable.

Section S2.2 Relative Motion

This section introduces thought experiments to help students understand the idea of relative motion and then explains why the absoluteness of the speed of light is such a surprising fact—and why this fact implies that we cannot reach or exceed the speed of light.

- The idea of relative motion is counterintuitive for many students even in its low-speed, everyday applications. Thus, we begin with three thought experiments involving ordinary speeds before moving to thought experiments with relative motion at speeds approaching the speed of light. We have found that it helps to "act out" these low-speed thought experiments in class. For example, you might recruit two volunteers—one to play the role of Jackie and one to hold the baseball. You and your volunteers can then demonstrate the motion involved in the thought experiments, which help students see why the different observers see things in different ways. If your students are having difficulty with this concept, you may wish to do several additional thought experiments of a similar nature.
- The key thought experiment of this section is Thought Experiment 5, because this is the one that shows students the counterintuitive nature of the idea that the speed of light is absolute. Unless students understand what is "strange" about the absoluteness of the speed of light, they will not be able to follow the remaining thought experiments.
- Thought Experiment 6 then uses the absoluteness of the speed of light to "prove" that no material object can reach or exceed the speed of light; note that this is the first of the five bullets from the bullet list in Section S2.1. Students will invariably look for loopholes in the logic establishing the absoluteness of the speed of light, so be prepared for many such questions. (You may wish to refer the students to the Thinking About box titled "What If Light Can't Catch You?") In addition, students will question the postulate that the speed of light is absolute. For this latter objection, we recommend emphasizing that the absoluteness of the speed of light is an experimentally verified fact—and reminding the students that this point is discussed further in Section S2.4.

- Note that we have chosen to use the term *free-float frame* in place of the more common *inertial reference frame* because we feel it is much more intuitive. The term *free-float frame* was coined by E. F. Taylor and J. A. Wheeler in their book *Spacetime Physics,* 2d ed. (W. H. Freeman and Company, New York, 1992).
- FYI: We are often asked why we chose a woman named Jackie to be the participant in our thought experiments. The answer is that we wrote the first draft of this chapter during the 1996 Olympics in Atlanta—and decided that if anyone was capable of moving at speeds close to the speed of light, it would probably be Jackie Joyner Kersee. Thus, the Jackie in our thought experiments is named in her honor.

Section S2.3 The Reality of Space and Time

By this point, students should understand what we mean by the absoluteness of the speed of light. This section therefore uses this idea in a series of thought experiments to establish the ideas listed in the remaining four bullets of the bullet list in Section S2.1.

- Under the subheading "Time Differs in Different Reference Frames," you may be wondering why we begin with the example of the ball being tossed in the train rather than going directly to the light paths in Thought Experiment 7. We've found that, if you ask students what a vertical light path would look like to an observer moving past them, most students initially think the observer would see the light path slanting *backward*, rather than forward with your perceived motion. By beginning with the example of the ball in the train, we usually can clear up this confusion. In addition, the comparison of the two situations helps students understand why the absoluteness of the speed of light implies that time varies in different reference frames, whereas there is no such obvious implication with the ball in the train (since different observers can disagree on the ball's speed).
- There is a subtlety in the relativity of simultaneity that may bother some thoughtful students. At the instant of the flashes, both you and Jackie agree that she is midway between them (i.e., she is in the middle of the train, and flashes occur on its front and rear ends). You also both agree that the flashes travel at the speed of light. Thus, from *your* point of view, the green flash reaches Jackie first because its speed *relative* to her is greater than the speed of light: It is the light's speed *plus* Jackie's speed toward the position where you saw the light emitted. Similarly, you see the red flash moving relative to her at less than the speed of light. There are no violations of relativity: Everyone still agrees on the speed of the light, and no one sees any individual object moving at greater than the speed of light.

  A good way to explain this point in class is with the following thought experiment: Suppose that you see spaceship A going at $0.99c$ to your left and spaceship B going at $0.99c$ to your right. According to you, they are separating at $1.98c$ and therefore will be 1.98 light-years apart after 1 year. But either of the two spaceships will see the other going at less than c and therefore will measure the other to travel less than 1 light-year in 1 year. Of course, the three observers (A, B, and you) will disagree about both the distances and the times between events.
- Note that we have deliberately chosen not to address the fact that Jackie sees your time dilated just as you see hers dilated; this issue is covered in Section S2.5.
- FYI: Note that, while time dilation applies to any reference frame moving relative to you, what you actually *see* may be more complex. If the moving frame has a radial component of motion toward or away from you, what you see is complicated by kinematic effects arising from the fact that the reference frame is moving at a speed close to the speed at which its light moves out ahead of or behind it. In fact, if the reference frame is moving toward you, these kinematic effects will cause it to appear to be moving faster than the speed of light—which is the origin of the apparent superluminal motion often seen in quasars. This issue is discussed at a level

accessible to many students (if their math skills are reasonably strong) in Taylor and Wheeler's *Spacetime Physics*, 2d ed.

- FYI: It is also difficult to *see* length contraction, because motion near the speed of light introduces apparent rotations of objects. Again, this point is discussed at a level accessible to many students in Taylor and Wheeler's *Spacetime Physics*, 2d ed.

Section S2.4 Is It True?

Having established all the key ideas of special relativity as laid out in the bullet list of Section S2.1, we now turn to the evidence supporting these ideas.

- Note that we have chosen not to go into detail concerning the Michelson–Morley experiment or to discuss the historical issue of the ether. While both topics are interesting and relevant, they are not necessary to making the case for special relativity. If you have time in your class, you may wish to discuss these topics to give historical context to the subject of relativity.

Section S2.5 Toward a New Common Sense

In this section we point out the fact that Jackie must claim the same things about you (e.g., time dilation, length contraction, and mass increase) that you claim about her, as long as you are moving at a constant relative velocity. We then use this idea to help students begin to establish what we refer to as a "new common sense"—a common sense that can incorporate the ideas of relativity. Note that we do *not* expect students to develop such a new common sense easily; at this point, we want them only to recognize that it is *possible* to do so. They can move further toward developing this new common sense when they study Chapter S3, but they will need to study relativity further if they wish to truly get a handle on this new common sense. If they have enjoyed learning about relativity, we hope they will be encouraged to read more about it in the future.

- As discussed above (fourth bullet under Section S2.3), Thought Experiment 12 works as written only if Jackie's motion is transverse to yours. We have chosen not to mention this subtlety in the text.

Section S2.6 Ticket to the Stars

This final section points out that special relativity offers a way to make interstellar trips in times that are reasonably short for the traveler, if not for those who stay behind on Earth.

- Note that we do not discuss the twin paradox at this point, except for in a footnote. As described in the footnote, it is possible to resolve the twin paradox with special relativity. (Again, a good discussion of this issue can be found in Taylor and Wheeler's *Spacetime Physics*, 2d ed.) However, we have found that it is easier for students at this level to understand the twin paradox in the context of general relativity, so we discuss it in a Thinking About box in Chapter S3.
- The actual distance to Vega is close to 26 light-years; we use "about 25 light-years" to make the mathematics easier for students to follow.

Answers/Discussion Points for Think About It Questions

The Think About It questions are not numbered in the book, so we list them in the order in which they appear, keyed by section number.

Section S2.1

- On the treadmill, your speed is measured relative to the moving tread. Your speed relative to the ground is essentially zero. An observer on the Moon would see you moving with the rotation of the Earth (and also with the motion of Earth relative to the Moon). Other view-points on your speed might include an observer on the Sun, who would see you moving with the Earth in its orbit at some 100,000 km/hr, or an observer in a distant galaxy, who might see you moving away with the expansion of the universe at a substantial fraction of the speed of light.

- This question is designed to help students recognize that maps and globes are made with the Northern Hemisphere bias of most map makers. (Note: If you can get hold of one, you should show students a map in which the Southern Hemisphere is placed "up"—ask an Australian friend to buy one for you.)

Section S2.2

- If you throw the ball in Jackie's direction at 80 km/hr while Jackie is moving away from you at 90 km/hr (as in Thought Experiment 3), Jackie will see the ball moving away from her at 10 km/hr. If you throw it toward her at 80 km/hr while she is moving toward you at 90 km/hr, she will see the ball coming toward her at 170 km/hr.
- If Jackie is moving away from you at a speed short of the speed of light by only 3 km/s, you'll see a light beam traveling faster than Jackie by only 3 km/s. She'll measure the same light beam to be traveling past her at the full speed of light.

Section S2.4

- The point of this question is to make students think about the fact that you can't simply discount an idea without considering the consequences. In this case, many students would like to make the strange ideas of relativity "go away" by saying that the speed of light is not truly absolute. Here, they must consider the fact that a nonabsolute speed of light would introduce other consequences in terms of how people perceive events.

Solutions to End-of-Chapter Problems (Chapter S2)

1. Einstein proved that everything is relative. *This is a false statement that reflects a common misconception. He showed that motion is relative, while other things—such as the laws of nature and the speed of light—are absolute.*
2. An object moving by you at very high speed will appear to have a higher density than it has at rest. (*Hint:* Think about the effects on both length and mass.) *This is true, because its mass will be increased while its length will be decreased. A greater mass in a smaller volume means a higher density.*
3. Suppose you and a friend are standing at opposite sides of a room, and you each pop a peanut into your mouth at precisely the same instant. According to the theory of relativity, it is possible for a person moving past you at high speed to observe that you ate your peanut before your friend ate hers. *This is true. If two events appear simultaneous in one reference frame, they may not appear simultaneous in a different reference frame.*
4. Suppose you and a friend are standing at opposite sides of a room, and you each pop a peanut into your mouth at precisely the same instant. According to the theory of relativity, it is possible for a person moving past you at high speed to observe that you ate cashews rather than peanuts. *This is false. While people in different reference frames may disagree about the timing of events, they cannot disagree on the specifics of a particular event.*
5. Because we can't build spaceships that travel at nearly the speed of light, we have never been able to test whether time dilation really occurs. *This statement is false. We have tested time dilation even at slow speeds, and we have tested it at high speeds for subatomic particles in particle accelerators.*
6. The detonation of a nuclear bomb is a test of the special theory of relativity. *This is true, because the bomb releases energy from its mass, thereby verifying a prediction of Einstein's special theory of relativity.*

7. If a person is moving past you at a speed close to the speed of light, you will see that person's time running slow, while he or she will see your time running fast. *This statement is partly true and partly false. It's true that you will see the moving person's time running slow. However, that person will also see your time moving slow—not fast.*
8. If you had a sufficiently fast spaceship, you could leave today, make a round-trip to a star 500 light-years away, and return home to Earth in the year 2020. *This is false. As the traveler on the fast ship, you could potentially make the 1,000 light-year roundtrip in just a few years of ship time. However, more than 1,000 years would have to pass on Earth, so you would return in the distant future.*
9. On a stationary bike, the speed is supposed to represent how fast the bike would be moving if it had wheels in contact with the ground. Clearly, it is a good example of the idea of relative motion, since the actual speed of the bike through the room is zero.
10. a. If Bob is coming toward you at 75 km/hr and you throw a baseball in his direction at 75 km/hr, he'll see the ball coming at him at 75 + 75 = 150 km/hr.
    b. If Shawn is traveling away from you at 120 km/hr when he throws a baseball in your direction at 100 km/hr, you'll see the ball traveling away from you at 20 km/hr.
    c. If Carol is traveling away from you at 75 km/hr and Sam is going away from you in the opposite direction at 90 km/hr, Carol will see Sam going away from her at 165 km/hr.
    d. If you now throw a ball toward Sam at 120 km/hr, he'll see the ball coming at him at 120 – 75 = 45 km/hr. Carol will see the ball going away from her at 75 + 120 = 195 km/hr.
11. a. As you observe a spaceship moving past, you will see their clocks running slow. That is, everything on the spaceship would appear to be taking place in slow motion.
    b. The spaceship would be shortened in the direction of its motion. Its height and width would be unchanged.
    c. The mass of the spaceship would be increased compared to its rest mass.
    d. A passenger on the spaceship would say that your clocks are slow, your length is contracted, and your mass is increased. Because all inertial frames are equivalent, the situations seen by you and by passengers on the spaceship must be symmetric.
12. a. All observers, including Bob, will agree that Jackie is illuminated first by the green flash and then by the red flash.
    b. Because Bob is traveling in the opposite direction of Jackie, he will be illuminated first by the red flash and then by the green flash. Again, all observers will agree on this point.
    c. Bob will conclude that the red flash occurred before the green flash—the opposite of the order seen by Jackie, and not simultaneously as seen by you.
13. We will let t' represent the time for the traveler, and t is the time according to those who stay at "rest" on the Earth. The traveler's time will be slowed by the time dilation factor so that:

$$t' = t\sqrt{1-\left(\frac{v}{c}\right)^2}$$

In this problem $v = 89$ km/hr, which we must convert to a fraction of the speed of light:

$$\frac{v}{c} = \frac{89\,\frac{\text{km}}{\text{hr}} \times \frac{1\ \text{hr}}{3600\ \text{s}}}{3\times10^5\,\frac{\text{km}}{\text{s}}} = 8.2\times10^{-8} \quad \Rightarrow \quad \left(\frac{v}{c}\right)^2 = 6.8\times10^{-15}$$

In this case, $(v/c)^2$ is so small that, when you subtract it from 1, your calculator still gives an answer of 1. Thus, to the accuracy of your calculator, the time that has passed for the traveling student is the same as that for the people at "rest" on the Earth, or 70 years.

We can get a more accurate answer by using the *binomial expansion*, which has the general form:

$$\left(a+x\right)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots$$

In this case, we set $a = 1$ and $x = -(v/c)^2$ to find:

$$\sqrt{1-\left(\frac{v}{c}\right)^2} = \underbrace{\left(1+\underbrace{-\left(\frac{v}{c}\right)^2}_{x}\right)^{\frac{1}{2}}}_{a=1,\ x=-\left(\frac{v}{c}\right)^2,\ n=\frac{1}{2}} = 1-\tfrac{1}{2}\left(\frac{v}{c}\right)^2+\tfrac{3}{8}\left(\frac{v}{c}\right)^4 -+ \cdots$$

For $(v/c)^2 < 1$, the series converges rapidly and we may neglect all but the first two terms:

$$\begin{aligned}\sqrt{1-\left(\frac{v}{c}\right)^2} &\approx 1-\tfrac{1}{2}\left(\frac{v}{c}\right)^2\\ &= 1-\tfrac{1}{2}\left(6.9\times 10^{-15}\right)\\ &= 1-\left(3.5\times 10^{-15}\right)\end{aligned}$$

The time for the traveling student can then be computed as:

$$t' = (70\ \text{yr})\left(1-\left(3.5\times 10^{-15}\right)\right) = 70\ \text{yr} - \left(2.5\times 10^{-13}\ \text{yr}\right) = 70\ \text{yr} - \left(7.7\times 10^{-6}\,\text{s}\right)$$

That is, the time for the traveler at 55 mph is less than the time for those at "rest" on Earth by only 7.7 microseconds!

14. We use the same formula as in the previous problem to figure out the time passage for our second student. Because she is traveling at 95% of the speed of light, we have $(v/c) = 0.95$ or $(v/c)^2 = 0.9025$. Plugging this value into the time dilation formula, we find:

$$t' = t\sqrt{1-\left(\frac{v}{c}\right)^2} = \left(70\ \text{yr}\right)\sqrt{1-\left(0.9025\right)} = \left(70\ \text{yr}\right)\sqrt{0.0975} = \left(70\ \text{yr}\right)(0.312) \approx 21.9\ \text{yr}$$

For the traveler moving at 95% of the speed of light, only a little less than 22 years passes while 70 years pass for those of us left behind on Earth.

15. a. The π^+ meson will always "think" it is at rest and therefore decay after about 18 billionths of a second (1.8×10^{-8} s) in its *own* reference frame. However, from the point of view of the scientists conducting the experiment, the π^+ meson represents the *moving* reference frame. Hence, the π^+ meson's lifetime of 1.8×10^{-8} s represents t' in the time dilation formula, or $t' = 1.8 \times 10^{-8}$ s. To solve for the lifetime observed by the scientists, t, divide both sides of the time dilation formula by the square root term:

$$t' = t\sqrt{1-\left(\frac{v}{c}\right)^2} \quad\Rightarrow\quad t = \frac{t'}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{1.8\times 10^{-8}\,\text{s}}{\sqrt{1-\left(0.998\right)^2}} = \frac{1.8\times 10^{-8}\,\text{s}}{0.0632} = 2.8\times 10^{-7}\,\text{s}$$

When produced at $0.998c$, the π^+ meson is expected to last about 280 billionths of a second, rather than its "normal" lifetime of 18 billionths of a second. That is, its moving lifetime is more than 15 times as long as its lifetime at rest.

b. Actual experiments in particle accelerators allow scientists to measure the lifetime of π^+ mesons (and other particles) produced at high speeds. The formulas of special relativity allow us to calculate the expected lifetimes. The fact that predicted lifetimes match observed experimental results provides strong evidence in support of the special theory of relativity.

16. Your sister's speed is $v = 0.99c$, or $v/c = 0.99$. The time that passes for you is $t = 20$ years (you age from 25 to 45). The time that passes for your sister is:

$$t' = t\sqrt{1-\left(\frac{v}{c}\right)^2} = (20\text{ yr})\sqrt{1-(0.99)^2} = (20\text{ yr})\sqrt{1-0.9801} = (20\text{ yr})\sqrt{0.0199} = 2.8\text{ yr}$$

While 20 years pass for you, only a little less than 3 years passes for your traveling sister. This is a real difference: Although 20 years will have passed on Earth, only 3 years will have passed for your sister—she will return only 3 years older, at age 28. Although you were the same age before she left, you now are 17 years older than she.

17. This problem is simply velocity addition. Let u be the speed of the baseball according to those on the Earth. Let u' be the speed of the baseball according to Nomo ($u' = 0.8c$), and let v be the speed of the train carrying Nomo ($v = 0.9c$). To then find the velocity, we measure u by use of the addition formula:

$$u = \frac{u'+v}{1+\frac{u'v}{c^2}} = \frac{0.8c+0.95c}{1+\frac{(0.8c)(0.95c)}{c^2}} = \frac{1.75c}{1+0.76} = \frac{1.75}{1.76}c = 0.994c$$

We will see the baseball traveling at 99.4% of the speed of light—less than the speed of light, but faster than either the train or the baseball independently.

18. The light beam travels at the speed of light and therefore covers a distance of 100 meters in a time of:

$$\text{time} = \frac{\text{distance}}{\text{speed}} = \frac{100\text{ m}}{3\times10^8\text{ m/s}} = 3.3\times10^{-7}\text{ s}$$

The light beam covers the 100-meter course in less than a millionth of a second, much faster than the sprinter's 8.7 seconds. Thus, the light beam wins the race easily.

19. a. As seen from the stands, Jo is going at a speed of $0.999c$ while the light beam's speed is c. Thus, the light beam is only $0.001c$ faster than Jo and wins a very close race.

b. As seen by Jo, the light beam goes the full speed of light ahead of him because everyone always measures the same absolute value for c.

c. From Jo's point of view, the light is going faster than him by the same 300,000 km/s as before; thus, he has gained nothing in terms of catching up with light. In contrast, the spectators see Jo staying much closer to the light beam in the second race than in the first.

d. Jo sees a shorter course, because of length contraction due to his high speed relative to the Earth. We simply plug values into the length contraction formula:

$$\begin{aligned}\text{length according to Jo} &= (\text{rest length})\sqrt{1-\left(\frac{v}{c}\right)^2}\\ &= (100\text{ m})\sqrt{1-(0.999)^2}\\ &= (100\text{ m})(0.0447)\\ &\approx 4.5\text{ m}\end{aligned}$$

Jo finds the course to be only about 4.5 meters long, instead of the usual 100 meters.

Chapter S3. Spacetime and Gravity

Please see the general notes about Chapters S2–S4 that appear on page 126.

This chapter provides students with a short but solid introduction to the general theory of relativity. As with Chapter S2, this chapter begins with a section that provides a concise overview of the topic; the remaining sections explain the logic and evidence behind this overview. Thus, it is possible to teach only the first section if you want to give students an overview of general relativity but do not have time to go into any details of the theory.

As always, when you prepare to teach this chapter, be sure you are familiar with the relevant media resources (see the complete, section-by-section resource grid in Appendix 3 of this Instructor's Guide) and the on-line quizzes and other study resources available on the Astronomy Place website.

Teaching Notes (By Section)

Section S3.1 Einstein's Second Revolution

This first section introduces the ideas of general relativity, including a bullet list that highlights the ideas that we will encounter in the rest of the book and that are discussed in further detail in the rest of this chapter.

- Historical note: In the box called "Thinking About Einstein's Leap," we state that someone was bound to come up with special relativity around the time that Einstein did. In fact, French mathematician Jules Henri Poincaré gave a speech in 1904 in which he predicted the development of "an entirely new mechanics" in which the speed of light became an absolute limit.

Section S3.2 The Equivalence Principle

This section introduces the equivalence principle and discusses how it plays a role in general relativity similar to the role played by the absoluteness of the speed of light in special relativity.

- Note that, in stating the equivalence principle in the main body of the text, we do not explicitly state that it holds true only on a highly localized scale—although we do mention this fact in footnote 1. We have found that, unless you have a great deal of time to discuss this point, it tends to confuse all but the best students. Thus, we have chosen to be technically correct by addressing it in the footnote, but otherwise to treat it as a subtlety that is beyond the scope of this book.

Section S3.3 Understanding Spacetime

This section discusses the concept of spacetime, including the construction of spacetime diagrams and a discussion of what it means for spacetime to be curved.

- Note that we do not make much use of spacetime diagrams elsewhere in the chapter; we introduce them because we have found that they help students understand the idea of spacetime.
- Confronted with the idea of history as being "viewable" in spacetime, many students will wonder whether the future is similarly "viewable" in spacetime and whether this issue has any implications for the idea of fate or a predetermined future. This issue is the topic of Discussion Question 1 at the end of the chapter and can make for a great in-class discussion if time permits. You may wish to tell students about the "many worlds" hypothesis, in which each possible future has an independent reality, as one of many possible ways that spacetime might have a fixed structure and yet not imply a predetermined future.

Section S3.4 A New View of Gravity

This section builds on the previous sections to explain why general relativity provides a new view of gravity in which gravity arises from the curvature of spacetime. It also explains gravitational time dilation and discusses the possibilities of an overall geometry of the universe.

- We suggest emphasizing the bullet list of caveats about rubber-sheet analogies; without these caveats, these analogies often lead to misconceptions instead of helping students understand spacetime curvature.
- Note that we discuss black holes only very briefly in this section; more detailed discussion of black holes appears in Chapter 18.

Section S3.5 Is It True?

By this point we have discussed all the key ideas of general relativity laid out in the bullet list of Section S3.1 except gravitational waves. In this section, we turn to the evidence supporting these ideas and also introduce gravitational waves in the context of discussing evidence of their existence.

- Note that we keep the discussion of gravitational lensing brief here; it is discussed further in Chapter 22.

Section S3.6 Hyperspace, Wormholes, and Warp Drive

This short section discusses the science fiction devices of hyperspace, wormholes, and warp drive, explaining how these ideas arise from speculation about the implications of relativity.

- It is important to emphasize the speculative nature of these ideas so that students do not confuse them with established science. Nevertheless, students usually enjoy this discussion.

Section S3.7 The Last Word

This very short section essentially consists of a quote from Einstein that summarizes his thoughts about the meaning of relativity. It is a great quote with which to end a series of classes on relativity and can spawn great discussions.

Answers/Discussion Points for Think About It Questions

The Think About It questions are not numbered in the book, so we list them in the order in which they appear, keyed by section number.

Section S3.2

- After 1 second of 1 *g* acceleration, Jackie will be traveling relative to you at about 10 m/s. After 10 seconds she will be going about 100 m/s. After a minute, she will be going about 600 m/s.

Section S3.3

- The idea in this question is that your body would look like its normal 3-dimensional self, but with an added dimension stretching through time. Bumping into someone on a bus would be represented by a short stretch along the time axis at which your 4-dimensional self would be in contact with his or hers.
- On a spacetime diagram in which the speed of light is represented by a 45° line, ordinary speeds are so slow that their angles are nearly indistinguishable from the vertical.
- This question asks students to examine great circle routes for themselves with the aid of a globe. Airplanes try to follow great circle routes because they are the shortest distance between points on Earth.
- When you are standing on a scale—or for that matter standing anywhere on Earth—your worldline is *not* following the straightest possible path through spacetime because you are feeling weight.

Section S3.4

- You would age more slowly on the Earth than on the Moon because of the Earth's stronger gravitational field. However, the Earth's gravitational field is still so weak that the effects of gravitational time dilation are barely measurable, and the difference between your aging rate on the Earth and your aging rate on the Moon would be virtually unnoticeable.

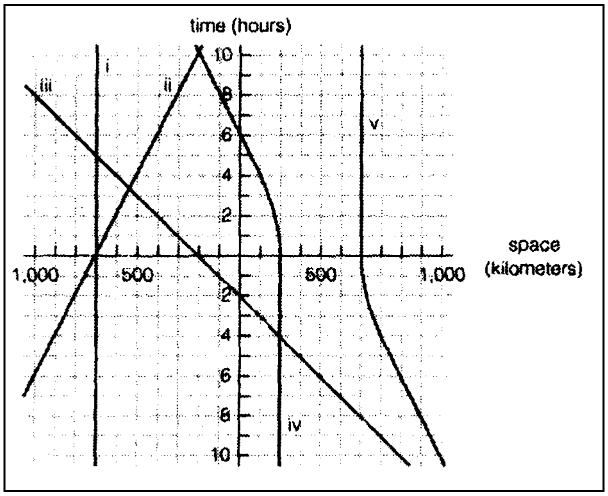
Section S3.5

- If Mercury's perihelion were closer to the Sun, it would be in a region of spacetime with greater curvature and hence greater gravitational time dilation. Hence, the discrepancy between the actual orbit and the orbit predicted by Newton's laws would be greater than it actually is.

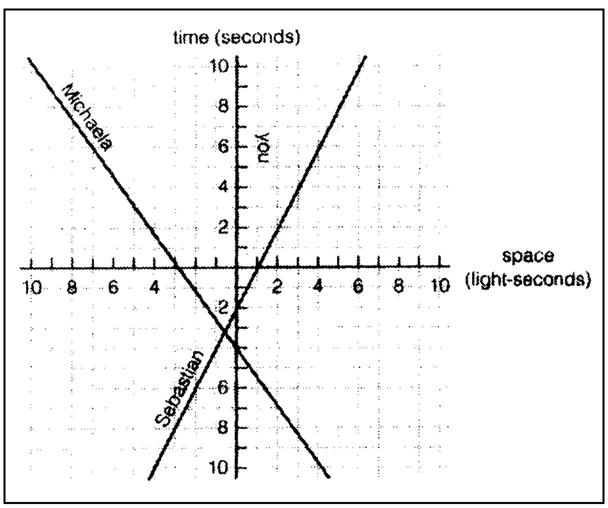
Solutions to End-of-Chapter Problems (Chapter S3)

1. Imagine that you are sitting in an enclosed room with no windows. According to the principle of equivalence, there are no experiments you could perform to tell you whether you were still on Earth or in a spaceship accelerating through space at 1 *g*. *This is true, and basically defines the equivalence principle.* (*Technical note:* In reality, you could measure a slight tidal effect that occurs on Earth but not in the spaceship. That is why the equivalence principle is technically valid only in small regions where tidal effects are negligible.)
2. According to the principle of equivalence, if you feel weight, you must be standing on a planetary surface. *This is false. You could also be in an accelerating reference frame.*
3. A person moving by you at high speed will measure time and space differently from you, but you will both agree that there is just a single spacetime reality. *This is true: spacetime is the same for everyone.*
4. The term *black hole* is a misnomer, because a black hole is neither black nor a hole of any kind. *This is false. A black hole is black because it emits no light, and it really is a hole in the universe.*
5. Time runs slightly slower on the surface of the Sun than it does here on Earth. *This is true: time runs slower in stronger gravitational fields, and gravity is stronger on the Sun's surface than it is here on Earth.*
6. Telescopes sometimes see multiple images of a single object, just as we should expect from the general theory of relativity. *This is true, and represents the phenomenon of gravitational lensing.*
7. The general theory of relativity offers nice explanations for a number of paradoxes that Einstein thought about, but no real evidence supports this theory. *This is false. There is a great deal of observational evidence in support of the general theory of relativity, some of which is discussed in the text.*

8. According to the general theory of relativity, it is impossible to travel through hyperspace or to use anything like Star Trek's "warp drive." *This is false. General relativity allows such travel in principle, though other laws of physics (not yet known) may prohibit it in reality.*
9. This question asks students to briefly restate and explain ideas taken directly from the reading. The key in grading it is to make sure that students demonstrate that they *understand* the concepts about which they are writing.
10. The following diagram shows the worldlines for the five situations described:



11. The following diagram shows the worldlines for the three cases described:



12. Essay question; answers will vary, but the key point should be that, from the point of view of general relativity, masses move in response to the curvature of spacetime. Because the curvature is well-defined by any large mass, small masses will all follow the same trajectories through spacetime regardless of differences in mass between them.
13. If we have a constant acceleration through space of 1 *g*, we will feel gravity in the same way we feel it on Earth. Thus, the trip will be very comfortable. Moreover, we will not notice anything unusual about our own length, mass, or time. Thus, from our point of view, we will not notice anything different from what we would notice in a closed room on Earth (aside from the view out the windows).

14. a. As long as time dilation is not a major factor, we can determine the velocity after some time with a given acceleration from the formula:

$$\text{velocity} = \text{acceleration} \times \text{time}$$

Although time dilation is noticeable at half the speed of light (1.5×10^8 m/s), it is still a relatively small factor until much higher speeds are reached. Thus, we can get a reasonable approximation of how long it would take the ship to reach half the speed of light at a constant acceleration of 1 *g* by neglecting time dilation. We therefore just solve the above formula for the time:

$$\text{time} = \frac{\text{velocity}}{\text{acceleration}} = \frac{1.5 \times 10^8 \ \text{m/s}}{9.8 \ \text{m/s}^2} = 1.5 \times 10^7 \ \text{s}$$

Converting to days gives a time of:

$$1.5 \times 10^7 \text{s} \times \frac{1 \ \text{hr}}{3600 \ \text{s}} \times \frac{1 \ \text{day}}{24 \ \text{hr}} \approx 174 \ \text{day}$$

At a constant acceleration of 1 *g*, the ship will reach a speed of about half the speed of light in about 174 days, or almost 6 months.

b. As the ship continues to accelerate away from Earth at 1 *g*, you (on Earth) will *not* continue to see its speed increase by 9.8 m/s with each passing second. From your point of view on Earth, time dilation will begin to noticeably affect the rate at which time is passing on the accelerating ship. Thus, each second on the ship will become much longer than a second on Earth, which means the ship's speed will increase by much less than 9.8 m/s during an Earth second. This effect will become more and more pronounced as the ship's speed approaches the speed of light away from Earth.

c. Because the ship will reach half the speed of light in just a few months, we can conclude that on a long journey it will be traveling *very* close to the speed of light for most of its journey. Thus, the ship will take only slightly longer to make a long trip than it would take a light beam from the point of view of observers on Earth. Thus, from your point of view on Earth, it will take the ship only a little more than 500 years to reach a star 500 light-years away.

15. Answers will vary; this is a fun question in which students evaluate the science in a science fiction movie that involves interstellar travel.

16. Answers will vary; this question requires research to learn about the Eötvös experiment.

17. Answers will vary; this question requires students to read a book or article about wormholes.

18. As mentioned in the problem, these solutions require putting D in *meters*, with $g = 9.8$ m/s^2 and $c = 3 \times 10^8$ m/s, to get an answer in units of *seconds*.

a. A spaceship is traveling to a star at a distance of 500 light-years, with a constant acceleration of 1 *g*. To find the amount of time that passes on the ship, we set:

$$D = 500 \ \text{light-years} \times \frac{9.5 \times 10^{15} \text{m}}{\text{light-year}} = 4.75 \times 10^{18} \text{m}$$

We now use this value for D in the formula to find the ship time:

$$\begin{aligned}
T_{\text{ship}} &= \frac{2c}{g}\ln\left(\frac{g\times D}{c^2}\right) \\
&= \frac{2\times 3\times 10^8\ \text{m/s}}{9.8\ \text{m/s}^2}\times\ln\left(\frac{9.8\ \text{m/s}^2\times 4.75\times 10^{18}\ \text{m}}{\left[3\times 10^8\ \text{m/s}\right]^2}\right) \\
&= \left(6.1\times 10^7\ \text{s}\right)\times\ln(517.2) \\
&= \left(6.1\times 10^7\ \text{s}\right)\times 6.25 = 3.8\times 10^8\ \text{s}
\end{aligned}$$

Finally, we convert this answer to years:

$$3.8\times 10^8\ \text{s}\times\frac{1\ \text{hr}}{3600\ \text{s}}\times\frac{1\ \text{day}}{24\ \text{hr}}\times\frac{1\ \text{yr}}{365\ \text{day}} = 12\ \text{yr}$$

The ship accelerating at 1 g will reach the star at a distance of 500 light-years in a time of about 12 years, ship time. Meanwhile, on Earth, about 500 years will go by: The ship will be traveling *at very* close to the speed of light for most of its journey, so we can assume that it takes the ship only slightly longer than it would take a light beam from the point of view of observers on Earth.

b. This time your spaceship travels 28,000 light-years to the center of the Milky Way Galaxy. Thus, we set:

$$D = 28{,}000\ \text{light-years}\times\frac{9.5\times 10^{15}\ \text{m}}{\text{light-year}}\approx 2.7\times 10^{20}\ \text{m}$$

We now use this value for D in the formula to find the ship time:

$$\begin{aligned}
T_{\text{ship}} &= \frac{2c}{g}\ln\left(\frac{g\times D}{c^2}\right) \\
&= \frac{2\times 3\times 10^8\ \text{m/s}}{9.8\ \text{m/s}^2}\times\ln\left(\frac{9.8\ \text{m/s}^2\times 2.7\times 10^{20}\ \text{m}}{\left[3\times 10^8\ \text{m/s}\right]^2}\right) \\
&= \left(6.1\times 10^7\ \text{s}\right)\times\ln(29{,}400) \\
&= \left(6.1\times 10^7\ \text{s}\right)\times 10.3 = 6.3\times 10^8\ \text{s}
\end{aligned}$$

Finally, we convert this answer to years:

$$6.3\times 10^8\ \text{s}\times\frac{1\ \text{hr}}{3600\ \text{s}}\times\frac{1\ \text{day}}{24\ \text{hr}}\times\frac{1\ \text{yr}}{365\ \text{day}} = 20\ \text{yr}$$

The ship accelerating at 1 g will reach the center of the galaxy in a time of about 20 years, ship time. Note that, even though this distance is more than 50 times as far as the 500-light-year distance from part (a), the travel time is less than twice as long—this is because the continuing acceleration makes the average speed much faster for the longer trip. During this trip of 20 years, ship time, about 28,000 years will pass on Earth because the ship's average speed will be *very* close to the speed of light, and thus the travel time will be only slightly longer than the time required for a light beam to make the trip.

c. The Andromeda Galaxy is about 2.5 million light-years away, or:

$$D = 2.5\times10^{6} \text{ light-years} \times \frac{9.5\times10^{15}\,\text{m}}{\text{light-year}} = 2.4\times10^{22}\,\text{m}$$

If you traveled in a spaceship accelerating at 1 *g*, the time it would take for the trip is given by the formula in the text, from which we find:

$$\begin{aligned}
T_{\text{ship}} &= \frac{2c}{g}\ln\left(\frac{g\times D}{c^2}\right)\\
&= \frac{2\times3\times10^{8}\,\text{m/s}}{9.8\,\text{m/s}^2}\times\ln\left(\frac{9.8\,\text{m/s}^2\times2.4\times10^{22}\,\text{m}}{\left[3\times10^{8}\,\text{m/s}\right]^2}\right)\\
&= \left(6.1\times10^{7}\,\text{s}\right)\times\ln(2,613,333)\\
&= \left(6.1\times10^{7}\,\text{s}\right)\times14.8 = 9.0\times10^{8}\,\text{s}
\end{aligned}$$

Converting this answer to years, we find:

$$9.0\times10^{8}\,\text{s}\times\frac{1\text{ hr}}{3600\text{ s}}\times\frac{1\text{ day}}{24\text{ hr}}\times\frac{1\text{ yr}}{365\text{ day}} = 28.5\text{ yr}$$

The round-trip travel time is twice as long, or 57 years. In other words, the ship time for the round-trip to Andromeda would be about 57 years, so you could easily make this trip in your lifetime. However, when you returned, you would find that 5 million years would have passed on Earth!

Chapter S4. Building Blocks of the Universe

Please see the general notes about Chapters S2–S4 that appear on page 126.

This chapter provides students with a brief overview of the standard model of physics and of the two key principles of quantum mechanics (uncertainty and exclusion). Like the chapters on relativity, it is designed to enhance student appreciation of the remaining chapters in the book, but it is not prerequisite to those chapters. Also like the relativity chapters, its first section provides a concise overview of the chapter ideas, while the remaining sections discuss these ideas in greater detail. Thus, it is possible to teach only the first section if you want to give your students an overview of modern particle physics without the details.

As always, when you prepare to teach this chapter, be sure you are familiar with the relevant media resources (see the complete, section-by-section resource grid in Appendix 3 of this Instructor's Guide) and the on-line quizzes and other study resources available on the Astronomy Place website.

Teaching Notes (By Section)

Section S4.1 The Quantum Revolution

This first section introduces the history of the quantum revolution and provides a bullet list of key quantum and particle physics ideas that are important in astronomy.

Section S4.2 Fundamental Particles and Forces

This section essentially provides a summary of the standard model of physics, explaining what we mean by fundamental particles and forces.

- We have included the terminology of the standard model (e.g., fermions and bosons, names of the quarks and leptons) because students are likely to encounter it in articles they read about cosmology. However, only the terms *quarks* and *leptons* arise again in this text, primarily in Chapter 23 on cosmology.

Section S4.3 The Uncertainty Principle

This section serves to introduce and explain the meaning of the uncertainty principle.

- Because the uncertainty principle is often misused by those who don't really understand it (and especially in the pseudosciences), we recommend focusing not only on what the uncertainty principle says, but also on what it doesn't say; e.g., it does not say that uncertainty is inevitable on macroscopic scales. (And, contrary to many news articles at the time, it did not play a role in the uncertainty surrounding the 2000 presidential election results in Florida.)

Section S4.4 The Exclusion Principle

This section serves to introduce and explain the meaning of the exclusion principle.

- As physicists, we use the term *state* so often that its meaning seems obvious. It is important to remember that this meaning of *state* in physics is not obvious to most students. Hopefully, the discussion of state in this section will clarify the point for students, but you may need to use additional examples of your own.

Section S4.5 Key Quantum Effects in Astronomy

Having established the key ideas of quantum mechanics, in this section we describe several important quantum effects that arise in astronomical contexts: degeneracy pressure, quantum tunneling, and virtual particles and Hawking radiation.

- The effects described here will come up again in context in later chapters. You should suggest that students return to this section to review the physics behind the effects when they arise later in context.

Answers/Discussion Points for Think About It Questions

The Think About It questions are not numbered in the book, so we list them in the order in which they appear, keyed by section number.

Section S4.2

- Quarks and leptons are more fundamental than protons, neutrons, and electrons, because protons and neutrons are made of quarks, while electrons represent just one type of lepton.

Section S4.3

- This question is meant to encourage thought about how a precise quantum statement such as the uncertainty principle relates to the way it is often stated colloquially. You can use this question to get into a discussion of how the uncertainty principle is often misused.
- The quantum uncertainties in the position and momentum of a baseball are so small in comparison to the baseball's size and momentum that they are unnoticeable. (In class, it can be fun to discuss what the world would be like if Planck's constant were much larger.)

Section S4.4

- The velocity of 5 km/hr is consistent with walking or swimming, but not with driving or riding a bike down a hill. The high heart rate of 160 and high metabolic rate of 1,200 calories per hour suggest that the person is swimming vigorously, not walking slowly.

Section S4.5

- This question is designed to point out to students that the concept of something being "virtual" is not as unfamiliar as it may at first seem.

Solutions to End-of-Chapter Problems (Chapter S4)

1. Although there are six known types of quark, ordinary atoms contain only two of these types. *This is true. Ordinary atomic nuclei are made only from up and down quarks.*
2. Ordinary atoms contain two of the six known types of leptons. *This is false. The only leptons in ordinary atoms are electrons, which represent just one of the six lepton types.*
3. There's no such thing as antimatter, except in science fiction. *This is false. Antimatter is routinely produced in laboratories and in the cosmos.*
4. According to the uncertainty principle, we can never be certain whether one theory is really better than another. *This is false. The uncertainty principle concerns only particular measurements that cannot be made to perfect precision. It does not apply to scientific theories or to anything else.*
5. The exclusion principle applies to protons and neutrons as well as to electrons. *This is true. The exclusion principle applies to all fermions; protons, neutrons, and electrons are all fermions.*
6. No known astronomical objects exhibit any type of degeneracy pressure. *This is false. Degeneracy pressure is important in many astronomical objects, including brown dwarfs, white dwarfs, and neutron stars.*
7. Although we speak of four fundamental forces—gravity, electromagnetic, strong, and weak—it is likely that these forces are different manifestations of a smaller number of truly fundamental forces. *This is true. A well-established theory already links the electromagnetic and weak forces (into the electroweak force), and physicists suspect that the other forces are also linked.*
8. As bizarre as the effects of quantum mechanics may seem, their reality is supported by a huge amount of observational and experimental evidence. *This is true. Every tested prediction of quantum theory has been verified.*
9. An atomic nucleus is made of protons and neutrons. The neutrons have no charge, but protons are positively charged. Thus, if it were just up to the electromagnetic force, a nucleus would fall apart due to the repulsion between the positive protons. Because the strong force is holding the nucleus together despite the electromagnetic repulsion, it must be the stronger (per particle) force *within* the nucleus. Note that this strength holds only over distances roughly the size of an atomic nucleus. Over larger distances, the strong force cannot be felt at all.
10. Gravity is far too weak to play a role in creating and breaking bonds between atoms or molecules; in fact, its only role in life is keeping us "stuck" to the ground. The nuclear forces are of such short range that they have no effects outside the nucleus itself. The only force that remains is the electromagnetic force, which influences interactions between the charged electrons and nuclei. Thus, all events in our ordinary lives—all chemistry and biology—are dominated by the electromagnetic force.
11. Despite its far greater strength per particle, the electromagnetic force is unable to attain very large values because it is impossible to accumulate a very large charge. This is because large objects tend to have *equal* amounts of positive (protons) and negative (electrons) charge, and

the electromagnetic force thus "cancels" itself out. Gravity, on the other hand, always attracts: As objects get more and more massive, gravity continues to gain strength. Thus, for very large objects, there will be a great deal of gravitational attraction but virtually no electromagnetic force, because the overall object is neutral.

12. Short essay question; answers will vary, but the key point should be that quantum tunneling plays a major role in nuclear fusion in stars and thus is crucial to the creation of the elements by stars. *Note:* In fact, quantum tunneling is most important in low-mass stars, not the high-mass stars that make most heavy elements. However, low-mass carbon stars are probably the source of most carbon in the universe.

13. a. We calculate the gravitational force, in *newtons*, simply by plugging the given numbers into the formula:

$$F_g = G\frac{M_1 M_2}{d^2} = \left(6.67\times10^{-11}\,\frac{\text{N}\times\text{m}^2}{\text{kg}^2}\right)\frac{\left(9.1\times10^{-31}\,\text{kg}\right)\times\left(9.1\times10^{-31}\,\text{kg}\right)}{\left(10^{-10}\,\text{m}\right)^2} = 5.52\times10^{-51}\,\text{N}$$

The gravitational force of attraction between the two electrons is 5.52×10^{-51} newton.

b. We calculate the electromagnetic force by plugging the charges of the two electrons into the formula for the electromagnetic force:

$$F_{EM} = k\frac{q_1 q_2}{d^2} = \left(9.0\times10^{9}\,\frac{\text{N}\times\text{m}^2}{\text{Coul}^2}\right)\frac{\left(-1.6\times10^{-19}\,\text{Coul}\right)\times\left(-1.6\times10^{-19}\,\text{Coul}\right)}{\left(10^{-10}\,\text{m}\right)^2} = 2.3\times10^{-8}\,\text{N}$$

The electromagnetic force of repulsion between the two electrons is 2.3×10^{-8} newton.

c. The ratio of the two forces is:

$$\frac{F_{EM}}{F_g} = \frac{2.3\times10^{-8}\,\text{N}}{5.52\times10^{-51}\,\text{N}} = 4.2\times10^{42}$$

The electromagnetic repulsion between the electrons is stronger than their gravitational attraction by a factor of over 10^{42}! Clearly, gravity will not play an important role until collections of very large amounts of mass are put together.

14. a. Note that the lifetime of a black hole depends on the *cube* of its mass. Thus, lower-mass black holes have much shorter lifetimes than more massive ones. For example, the lifetime of a 2-solar-mass black hole is $2^3 = 8$ times longer than that of a 1-solar-mass black hole. This also means that the evaporation process accelerates as the black hole loses mass. For example, suppose you calculate the lifetime of some black hole with a mass *M*. Some time in the future, the mass of the black hole will have decreased by a factor of 2, to 0.5*M*. Its remaining lifetime at that point will be $0.5^3 = 1/8$ of its original lifetime. Therefore, each successive "half-life" for a black hole requires only one-eighth of the time it previously took to reduce the mass in half. The evaporation process thus begins slowly, while the final evaporation is a runaway process emitting a violent burst of energy. If such bursts occur, they might be observable with gamma-ray detectors.

b. To calculate the lifetime of a black hole with the mass of the Sun (2.0×10^{30} kg), we simply plug values into the formula given:

$$t = 10240\,\pi^2 \frac{G^2 M^3}{hc^4} = 10240\,\pi^2 \frac{\left(6.67\times10^{-11}\,\frac{\text{m}^3}{\text{kg}\times\text{s}^2}\right)^2\left(2\times10^{30}\,\text{kg}\right)^3}{\left(6.63\times10^{-34}\,\frac{\text{kg}\times\text{m}^2}{\text{s}}\right)\left(3\times10^{8}\,\frac{\text{m}}{\text{s}}\right)^4} = 6.7\times10^{74}\,\text{s}$$

Converting this answer to years, we find:

$$t = 6.7\times10^{74}\,\text{s}\times\frac{1\text{ hr}}{3600\text{ s}}\times\frac{1\text{ day}}{24\text{ hr}}\times\frac{1\text{ yr}}{365\text{ day}} \approx 2\times10^{67}\,\text{yr}$$

The lifetime of a black hole with the mass of the Sun is some 10^{67} years—which is some 10^{57} times the current age of the universe (which is about 10^{10} years).

c. One trillion (10^{12}) solar masses is about 2.0×10^{42} kg. To calculate the lifetime of a black hole with this mass, we simply plug values into the formula given:

$$t = 10240\,\pi^2 \frac{G^2 M^3}{hc^4} = 10240\,\pi^2 \frac{\left(6.67\times10^{-11}\,\frac{\text{m}^3}{\text{kg}\times\text{s}^2}\right)^2\left(2\times10^{42}\,\text{kg}\right)^3}{\left(6.63\times10^{-34}\,\frac{\text{kg}\times\text{m}^2}{\text{s}}\right)\left(3\times10^{8}\,\frac{\text{m}}{\text{s}}\right)^4}$$

Because most calculators do not register numbers over 10^{100}, it is necessary to rearrange the calculation so that you can do the powers of 10 in your head:

$$\begin{aligned}
t &= 10240\,\pi^2 \frac{\left(6.67\times10^{-11}\,\frac{\text{m}^3}{\text{kg}\times\text{s}^2}\right)^2\left(2\times10^{42}\,\text{kg}\right)^3}{\left(6.63\times10^{-34}\,\frac{\text{kg}\times\text{m}^2}{\text{s}}\right)\left(3\times10^{8}\,\frac{\text{m}}{\text{s}}\right)^4} \\
&= 10240\,\pi^2 \frac{\left(6.67^2\times10^{-22}\,\frac{\text{m}^6}{\text{kg}^2\times\text{s}^4}\right)\left(8\times10^{126}\,\text{kg}^3\right)}{\left(6.63\times10^{-34}\,\frac{\text{kg}\times\text{m}^2}{\text{s}}\right)\left(81\times10^{32}\,\frac{\text{m}}{\text{s}}\right)} \\
&= 10240\,\pi^2 \frac{6.67^2\times8}{6.63\times81}\times\left(10^{-22+126-(-34)-32}\right)\left(\frac{\text{m}^6\times\text{kg}^3}{\text{kg}^2\times\text{s}^4}\times\frac{\text{s}^2}{\text{kg}\times\text{m}^3}\right) \\
&= 66979\times10^{106} \\
&\approx 7\times10^{110}
\end{aligned}$$

Thus, the lifetime of the trillion-solar-mass black hole is about 7×10^{110} seconds, which is equivalent to about 2×10^{103} years—a long time, but still a lot shorter than infinite time!

15. a. To calculate the lifetime of a black hole with the mass of the Earth (6×10^{24} kg), we plug values into the formula given:

$$t = 10240\,\pi^2 \frac{G^2 M^3}{hc^4} = 10240\,\pi^2 \frac{\left(6.67\times10^{-11}\,\frac{\text{m}^3}{\text{kg}\times\text{s}^2}\right)^2\left(6\times10^{24}\,\text{kg}\right)^3}{\left(6.63\times10^{-34}\,\frac{\text{kg}\times\text{m}^2}{\text{s}}\right)\left(3\times10^{8}\,\frac{\text{m}}{\text{s}}\right)^4} = 1.8\times10^{58}\,\text{s}$$

Converting this answer to years, we find:

$$t = 1.8 \times 10^{58}\,\text{s} \times \frac{1\ \text{hr}}{3600\ \text{s}} \times \frac{1\ \text{day}}{24\ \text{hr}} \times \frac{1\ \text{yr}}{365\ \text{day}} \approx 6 \times 10^{50}\,\text{yr}$$

The lifetime of a black hole with the mass of the Earth is some 10^{50} years—which is some 10^{40} times the current age of the universe (which is about 10^{10} years).

b. To solve the lifetime formula for the mass, we first divide both sides by all of the constants to isolate the mass term, then take the cube root of both sides. We find:

$$t = 10240\,\pi^2 \frac{G^2 M^3}{hc^4} \quad \Rightarrow \quad M = \sqrt[3]{\frac{t \times hc^4}{10240\,\pi^2 \times G^2}}$$

Before plugging in the lifetime of 12 billion years, we must convert it to units of seconds; you should find that 12 billion years is equivalent to about 3.8×10^{17} seconds. Plugging in all the values, we find that the mass of the black hole is:

$$M = \sqrt[3]{\frac{\left(3.8 \times 10^{17}\,\text{s}\right) \times \left(6.63 \times 10^{-34}\,\frac{\text{kg} \times \text{m}^2}{\text{s}}\right)\left(3 \times 10^{8}\,\frac{\text{m}}{\text{s}}\right)^4}{10240\,\pi^2 \times \left(6.67 \times 10^{-11}\,\frac{\text{m}^3}{\text{kg} \times \text{s}^2}\right)^2}} = 1.7 \times 10^{11}\,\text{kg}$$

Dividing by 1,000 kg per metric ton, we see that a mini–black hole with a mass of about 170 million tons would be evaporating during the present epoch.