Chapter 3 – Quantization of Charge, Light, and Energy

3-1. The radius of curvature is given by Equation 3-2.

$$R = \frac{mu}{qB} = m\left[\frac{2.5\times10^{6}\,m/s}{(1.60\times10^{-19}\,C)(0.40\,T)}\right] = m(3.91\times10^{25}\,m/s\cdot C\cdot T)$$

Substituting particle masses from Appendices A and D:

$$R(proton) = (1.67\times10^{-27}\,kg)(3.91\times10^{25}\,m/s\cdot C\cdot T) = 6.5\times10^{-2}\,m$$

$$R(electron) = (9.11\times10^{-31}\,kg)(3.91\times10^{25}\,m/s\cdot C\cdot T) = 3.6\times10^{-5}\,m$$

$$R(deuteron) = (3.34\times10^{-27}\,kg)(3.91\times10^{25}\,m/s\cdot C\cdot T) = 0.13\,m$$

$$R(H_2) = (3.35\times10^{-27}\,kg)(3.91\times10^{25}\,m/s\cdot C\cdot T) = 0.13\,m$$

$$R(helium) = (6.64\times10^{-27}\,kg)(3.91\times10^{25}\,m/s\cdot C\cdot T) = 0.26\,m$$

3-2. (a) Using ^{198}Au mass number to compute an approximate value for B that will yield R of one meter, Equation 3-2 gives

$$B = \frac{mu}{qR} = m\left[\frac{(198\,u)(1.66\times10^{-27}\,kg/u)(1.5\times10^{5}\,m/s)}{(1.60\times10^{-19}\,C)(1.0\,m)}\right] = 0.31\,T$$

(b) Using Equation 3-2, $R = \frac{mu}{qB} \rightarrow \Delta R = \Delta m\left(\frac{u}{qB}\right)$

$$\Delta R = (m_{Hg} - m_{Au})\left(\frac{u}{qB}\right) = (m_{Hg} - m_{Au})\left[\frac{(1.66\times10^{-27}\,kg/u)(1.5\times10^{5}\,m/s)}{(1.60\times10^{-19}\,C)(0.31\,T)}\right]$$

$$\Delta R = (197.966743\,u - 196.966543\,u)(5.02\times10^{-3}\,m/u) = 0.005\,m = 5\,mm$$

(c) For doubly ionized atoms $q = 2\times1.60\times10^{-19}\,C$, so $B = 0.31\,T/2 = 0.16\,T$ and ΔR is unchanged because

$$qB = (2)(1.6\times10^{-19}\,C)(0.31\,T/2) = (1.60\times10^{-19}\,C)(0.31\,T) \text{ as before.}$$

3-3. $$B = \frac{\mathscr{E}}{u}\,,\ \frac{u}{c} = \frac{pc}{E},\ and\ pc = \sqrt{E^2 - (mc^2)^2}$$

$$pc = \sqrt{(0.561\,MeV)^2 - (0.511\,MeV)^2} = 0.2315\,MeV$$

$$\frac{u}{c} = \frac{0.2315\,MeV}{0.561\,MeV} = 0.41$$

$$\therefore\ B = \frac{2.0\times10^5\,V/m}{0.41\,c} = 1.63\times10^{-3}\,T = 16.3\,G$$

3-4. $F = quB$ and $F_G = m_p g$

$$\frac{F_B}{F_G} = \frac{quB}{m_p g} = \frac{(1.60\times10^{-19}C)(3.0\times10^6 m/s)(3.5\times10^{-5}T)}{(1.67\times10^{-27}kg)(9.80\,m/s^2)} = 1.03\times10^9$$

3-5. (a) $$R = \frac{mu}{qB} = \frac{[(2E_k/e)(e/m)]^{1/2}}{(e/m)(B)}$$

$$= \frac{1}{B}\sqrt{\frac{2E_k/e}{e/m}} = \frac{1}{0.325\,T}\left[\frac{(2)(4.5\times10^4\,eV/e)}{1.76\times10^{11}\,kg}\right]^{1/2} = 2.2\times10^{-3}\,m = 2.2\,mm$$

(b) $$frequency\ \ f = \frac{u}{2\pi R} = \frac{\sqrt{(2E_k/e)(e/m)}}{2\pi R}$$

$$= \frac{[(2)(4.5\times10^4\,eV/e)(1.76\times10^{11}\,C/kg)]^{1/2}}{2\pi(2.2\times10^{-3}\,m)} = 9.1\times10^9\,Hz$$

$$period\ \ T = 1/f = 1.1\times10^{-10}s$$

3-6. (a) $1/2\,mu^2 = E_k,\ so\ u = \sqrt{(2E_k/e)(e/m)}$

$$\therefore\ u = [(2)(2000\,eV/e)(1.76\times10^{11}\,C/kg)]^{1/2} = 2.65\times10^7\,m/s$$

(Problem 3-6 continued)

(b) $\Delta t_1 = \frac{x_1}{u} = \frac{0.05\,m}{2.65\times10^{7}\,m./s} = 1.89\times10^{-9}s = 1.89\,ns$

(c) $mu_y = F\Delta t_1 = e\mathscr{E}\Delta t_1$

$\therefore\; u_y = (e/m)\mathscr{E}\Delta t_1 = (1.76\times10^{11}\,C/kg)(3.33\times10^{3}\,V/m)(1.89\times10^{-9}\,s) = 1.11\times10^{6}\,m/s$

3-7. $NE_k = \Delta W = C_V\Delta T \;\therefore$

$$N = \frac{C_V\Delta T}{E_k} = \frac{(5\times10^{-3}\,cal/°C)(2\,°C)}{2000\,eV}\times\frac{(4.186\,J/cal)}{(1.60\times10^{-19}\,J/eV)} = 1.31\times10^{14}$$

3-8.
$$Q_1 - Q_2 = (25.41 - 20.64)\times10^{-19}C = 4.47\times10^{-19}C = (n_1 - n_2)e$$
$$Q_2 - Q_3 = (20.64 - 17.47)\times10^{-19}C = 3.17\times10^{-19}C = (n_2 - n_3)e$$
$$Q_4 - Q_3 = (19.06 - 17.47)\times10^{-19}C = 1.59\times10^{-19}C = (n_4 - n_3)e$$
$$Q_4 - Q_5 = (19.06 - 12.70)\times10^{-19}C = 6.36\times10^{-19}C = (n_4 - n_5)e$$
$$Q_6 - Q_5 = (14.29 - 12.70)\times10^{-19}C = 1.59\times10^{-19}C = (n_6 - n_5)e$$

where the n_i are integers. Assuming the smallest $\Delta n = 1$, then $\Delta n_{12} = 3.0$, $\Delta n_{23} = 2.0$, $\Delta n_{43} = 1.0$, $\Delta n_{45} = 4.0$, and $\Delta n_{65} = 1.0$. The assumption is valid and the fundamental charge implied is $1.59\times10^{-19}C$.

3-9. For the rise time to equal the field-free fall time, the net upward force must equal the weight. $q\mathscr{E} - mg = mg \;\therefore\; \mathscr{E} = 2mg/q$.

3-10. (See Millikan's Oil Drop Experiment on the home page at www.whfreeman.com/modphysics4e.) The net force in the y-direction is $mg - bv_y = ma_y$. The net force in the x-direction is $q\mathscr{E} - bv_x = ma_x$. At terminal speed $a_x = a_y = 0$ and $v_x / v_f^{\prime\prime} = \sin\theta$.

(Problem 3-10 continued)

$$\sin\theta = \frac{v_x}{v_f'} = \frac{(q\mathscr{E}/b)}{v_f'} = \frac{q\mathscr{E}}{b\,v_f'}$$

3-11. (See Millikan's Oil Drop Experiment on the home page at www.whfreeman.com/physics.)

(a) At terminal speed $mg = bv_f$ where $m = \frac{4}{3}\pi a^3 \rho_{oil}$ and $b = 6\pi\eta a$. Substituting gives

$$a^2 = \frac{18}{4}\left(\frac{\eta v_f}{\rho_{oil} g}\right) \therefore a = \left[\frac{(18)(1.80\times10^{-5}N\cdot s/m^2)(5.0\times10^{-3}m/20s)}{(4)(0.75)(1000\,kg/m^3)(9.8\,m/s^2)}\right]^{1/2}$$

$$= 1.66\times10^{-6}m = 1.66\times10^{-3}\,mm$$

$$m = 4\pi(1.66\times10^{-6}m)^3(750\,kg/m^3)/3 = 1.44\times10^{-14}kg$$

(b) $F_E = q\mathscr{E}$ *and* $F_G = mg \therefore \frac{F_E}{F_G} = \frac{(2)(1.60\times10^{-19}C)(2.5\times10^5 V/m)}{(1.44\times10^{-14}kg)(9.8\,m/s^2)} = 0.57$

3-12. $\lambda_m T = 2.898\times10^{-3}m\cdot K.$ (a) $\lambda_m = \frac{2.898\times10^{-3}m\cdot K}{3K} = 9.66\times10^{-4}m = 0.966\,mm$

(b) $\lambda_m = \frac{2.898\times10^{-3}m\cdot K}{300K} = 9.66\times10^{-6}m = 9.66\,\mu m$

(c) $\lambda_m = \frac{2.898\times10^{-3}m\cdot K}{3000K} = 9.66\times10^{-7}m = 966\,nm$

3-13. Equation 3-10: $R = \sigma T^4.$ Equation 3-12: $R = \frac{1}{4}cU.$ From Example 3-5:

$U = (8\pi^5 k^4 T^4)/(15h^3c^2)$

(Problem 3-13 continued)

$$\sigma = \frac{R}{T^4} = \frac{(1/4)cU}{T^4} = \frac{1}{4}c(8\pi^5 k^4 T^4)/(15h^3c^2T^4)$$

$$= \frac{2\pi^5(1.38\times10^{-23}J/K)^4}{15(6.63\times10^{-34}J\cdot s)^3(3.00\times10^8 m/s)^2} = 5.67\times10^{-8}W/m^2K^4$$

3-14. Equation 3-24: $u(\lambda) = \dfrac{8\pi hc\lambda^{-5}}{e^{hc/\lambda kT}-1}$,

$$u(f)df = u(\lambda)d\lambda \therefore u(f) = u(\lambda)\frac{d\lambda}{df} \quad \text{Because } c = f\lambda,\ \left|\frac{d\lambda}{df}\right| = c/f^2$$

$$u(f) = \frac{8\pi hc(f/c)^5}{e^{hf/kT}-1}\left(\frac{c}{f^2}\right) = \frac{8\pi f^2}{c^3}\frac{hf}{e^{hf/kT}-1}$$

3-15. (a) $\lambda_m T = 2.898\times10^{-3}m\cdot K \quad \therefore \lambda_m = \dfrac{2.898\times10^{-3}m\cdot K}{2.7K} = 1.07\times10^{-3}m = 1.07\,mm$

(b) $c = f\lambda \therefore f = \dfrac{c}{\lambda_m} = \dfrac{3.00\times10^8 m/s}{1.07\times10^{-3}m} = 2.80\times10^{11}\,Hz$

(c) Equation 3-12: $R = \dfrac{1}{4}cU = \dfrac{c}{4}(8\pi^5k^4T^4/15h^3c^3)$

$$= \frac{(3.00\times10^8 m/s)(8\pi^5)(1.38\times10^{-23}J/K)^4(2.7K)^4}{(4)(15)(6.63\times10^{-34}J\cdot s)^3(3.00\times10^8 m/s)^3} = 3.01\times10^{-6}\,W/m^2$$

Area of Earth $A = 4\pi r_E^2 = 4\pi(6.38\times10^6 m)^2$.

Total power = $RA = (3.01\times10^{-6}W/m^2)(4\pi)(6.38\times10^6m)^2 = 1.54\times10^9\,W$

3-16. $\lambda_m T = 2.898\times10^{-3}m\cdot K$ (a) $T = \dfrac{2.898\times10^{-3}m\cdot K}{700\times10^{-9}m} = 4140\,K$

(b) $T = \dfrac{2.898\times10^{-3}m\cdot K}{3\times10^{-2}m} = 9.66\times10^{-2}\,K$ (c) $T = \dfrac{2.898\times10^{-3}m\cdot K}{3m} = 9.66\times10^{-4}\,K$

3-17. Equation 3-10: $R_1 = \sigma T_1^4 \quad R_2 = \sigma T_2^4 = \sigma(2T_1)^4 = 16\sigma T_1^4 = 16R_1$

3-18. (a) Equation 3-23: $\overline{E} = \dfrac{hc/\lambda}{e^{hc/\lambda kT}-1} = \dfrac{hc/(10hc/kT)}{e^{(hc/kT)/(10hc/kT)}-1} = \dfrac{0.1kT}{e^{0.1}-1} = 0.951\,kT$

(b) $\overline{E} = \dfrac{hc/\lambda}{e^{hc/\lambda kT}-1} = \dfrac{hc/(0.1hc/kT)}{e^{(hc/kT)/(0.1hc/kT)}-1} = \dfrac{10kT}{e^{10}-1} = 4.59\times10^{-4}\,kT.$

Equipartition theorem predicts $\overline{E} = kT.$ The long wavelength value is very close to kT, but the short wavelength value is much smaller than the classical prediction.

3-19. (a) $\lambda_m T = 2.898\times10^{-3}m\cdot K \;\therefore\; T_1 = \dfrac{2.898\times10^{-3}m\cdot K}{27.0\times10^{-6}m} = 107\,K$

$$R_1 = \sigma T_1^4 \text{ and } R_2 = \sigma T_2^4 = 2R_1 = 2\sigma T_1^4$$

$$\therefore\; T_2^4 = 2T_1^4 \text{ or } T_2 = 2^{1/4}T_1 = (2^{1/4})(107\,K) = 128\,K$$

(b) $\lambda_m = \dfrac{2.898\times10^{-3}m\cdot K}{128K} = 23\times10^{-6}\,m$

3-20. $\lambda_m = 2.898\times10^{-3}\,m\cdot K$ (Equation 3-20)

$$\lambda_m = \frac{2.898\times10^{-3}m\cdot K}{2\times10^4\,K} = 1.45\times10^{-7}m = 145\,nm$$

3-21. Equation 3-10:

$$R = \sigma T^4$$

$P_{abs} = (1.36\times10^3\,W/m^2)(\pi R_E^2\,m^2)$ where R_E = radius of Earth

$$p_{emit} = (R\,W/m^2)(4\pi R_E^2) = (1.36\times10^3\,W/m^2)(\pi R_E^2\,m^2)$$

$$R = (1.36\times10^3\,W/m^2)\left(\frac{\pi R_E^2}{4\pi R_E^2}\right) = \frac{1.36\times10^3}{4}\,\frac{W}{m^2} = \sigma T^4$$

$$T^4 = \frac{1.36\times10^3\,W/m^2}{4(5.67\times10^{-8}\,W/m^2\cdot K^4)} \quad \therefore\ T = 278.3\,K = 5.3\,C$$

3-22. (a) $\lambda_m T = 2.898\times10^{-3}\,m\cdot K \ \therefore\ \lambda_m = \dfrac{2.898\times10^{-3}\,m\cdot K}{3300\,K} = 8.78\times10^{-7}\,m = 878\,nm$

$$f_m = c/\lambda_m = \frac{3.00\times10^8\,m/s}{8.78\times10^{-7}\,m} = 3.42\times10^{14}\,Hz$$

(b) Each photon has average energy $E = hf$ and $NE = 40\ J/s$.

$$N = \frac{40\,J/s}{hf_m} = \frac{40\,J/s}{(6.63\times10^{-34}\,J\cdot s)(3.42\times10^{14}\,Hz)} = 1.77\times10^{20}\,photons/s$$

(c) At 5m from the lamp N photons are distributed uniformly over an area $A = 4\pi r^2 = 100\pi m^2$. The density of photons on that sphere is $(N/A)/s\cdot m^2$. The area of the pupil of the eye is $\pi(2.5\times10^{-3}\,m)^2$, so the number n of photons entering the eye per second is

$$n = (N/A)(\pi)(2.5\times10^{-3}\,m)^2 = \frac{(1.77\times10^{20}/s)(\pi)(2.5\times10^{-3}\,m)^2}{100\pi m^2}$$

$$= (1.77\times10^{18}/s)(2.5\times10^{-3})^2 = 1.10\times10^{13}\,photons/s$$

3-23. Equation 3-24: $u(\lambda) = \dfrac{8\pi hc\lambda^{-5}}{e^{hc/\lambda kT} - 1}$ Letting $A = \pi hc$, $B = hc/kT$, $U(\lambda) = \dfrac{A\lambda^{-5}}{e^{B/\lambda} - 1}$

$$\frac{du}{d\lambda} = \frac{d}{d\lambda}\left[\frac{A\lambda^{-5}}{e^{B/\lambda} - 1}\right] = A\left[\frac{\lambda^{-5}(-1)e^{B/\lambda}(-B\lambda^{-2})}{(e^{B/\lambda} - 1)^2} - \frac{5\lambda^{-6}}{e^{B/\lambda} - 1}\right]$$

$$= \frac{A\lambda^{-6}}{(e^{B/\lambda} - 1)^2}\left[\frac{B}{\lambda}e^{B/\lambda} - 5(e^{B/\lambda} - 1)\right] = \frac{A\lambda^{-6}e^{B/\lambda}}{(e^{B/\lambda} - 1)^2}\left[\frac{B}{\lambda} - 5(1 - e^{-B/\lambda})\right] = 0$$

The maximum corresponds to the vanishing of the quantity in brackets. Thus, $5\lambda(1 - e^{-B/\lambda}) = B$. This equation is most efficiently solved by iteration; i.e., guess at a value for B/λ in the expression $5\lambda(1 - e^{-B/\lambda})$, solve for a better value of B/λ; substitute the new value to get an even better value, and so on. Repeat the process until the calculated value no longer changes. One succession of values is: 5, 4.966310, 4.965156, 4.965116, 4.965114, 4.965114. Further iterations repeat the same value (to seven digits), so we have

$$\frac{B}{\lambda_m} = 4.965114 = \frac{hc}{\lambda_m kT} \therefore \lambda_m T = \frac{hc}{(4.965114)k} = \frac{(6.63\times10^{-34}J\cdot s)(3.00\times10^{8}m/s)}{(4.965114)(1.38\times10^{-23}J/K)}$$

$$\lambda_m T = 2.898\times10^{-3}\,m\cdot K \qquad (\text{Equation } 3-11)$$

3-24. $hf = hc/\lambda = 0.68\,eV.$ $\lambda_{\max} = \dfrac{hc}{0.68\,eV}$

$$= \frac{(4.14\times10^{-15}eV\cdot s)(3.00\times10^{8}m/s)}{0.68\,eV} = 1.83\times10^{-6}m = 1830\,nm\,(infrared)$$

3-25. (a)

$$hf = hc/\lambda = 0.47\,eV.$$

$$\lambda_{\max} = \frac{hc}{4.87\,eV} = \frac{(4.14\times10^{-15}eV\cdot s)(3.00\times10^{8}m/s)}{4.87\,eV} = 2.55\times10^{-7}m = 255\,nm$$

(b) It is the fraction of the total solar power with wavelengths less than 255 nm, i.e., the area under the Planck curve (Figure 3-7) up to 255 nm divided by the total area. The latter

(Problem 3-25 continued)

is: $R = \sigma T^4 = (5.67\times10^{-8}\,W/m^2\cdot K^4)(5800\,K)^4 = 6.42\times10^7\,W/m^2$. Approximating the former with $u(\lambda)\Delta\lambda$ with $\lambda = 127\,nm \text{ and } \Delta\lambda = 255\,nm$:

$$[u(127\,nm)](255\,nm) = \left[\frac{8\pi hc(127\times10^{-9}\,m)^{-5}}{e^{hc/kT(127\times10^{-9})} - 1}\right](255\times10^{-9}\,m) = 1.23\times10^{-4}\,J/m^3$$

$$R(0-255\,nm) = \frac{c}{4}(1.23\times10^{-4}\,J/m^3) \rightarrow \frac{R(0-255\,nm)}{R}$$

$$= \frac{(3.00\times10^8\,m/s)(1.23\times10^{-4}\,J/m^3)}{(4)(6.42\times10^7\,W/m^2)} \quad fraction = 1.4\times10^{-4}$$

3-26. (a) $\lambda_t = \frac{hc}{\phi} = \frac{1240\,eV\cdot nm}{1.9\,eV} = 653\,nm, \quad f_t = \frac{\phi}{h} = \frac{1.9\,eV}{4.136\times10^{-15}\,eV\cdot s} = 4.59\times10^{14}\,Hz$

(b) $V_0 = \frac{1}{e}\left(\frac{hc}{\lambda} - \phi\right) = \frac{1}{e}\left(\frac{1240\,eV\cdot nm}{300\,nm} - 1.9\,eV\right) = 2.23\,V$

(c) $V_0 = \frac{1}{e}\left(\frac{hc}{\lambda} - \phi\right) = \frac{1}{e}\left(\frac{1240\,eV\cdot nm}{400\,nm} - 1.9\,eV\right) = 1.20\,V$

3-27. (a) Choose λ = 550 nm for visible light. $nhf = E \rightarrow \frac{dn}{dt}hf = \frac{dE}{dt} = P$

$$\frac{dn}{dt} = \frac{P}{hf} = \frac{P\lambda}{hc} = \frac{(0.05\times100\,W)(550\times10^{-9}\,m)}{(6.63\times10^{-34}\,J\cdot s)(3.00\times10^8\,m/s)} = 1.38\times10^{19}/s$$

(b) $flux = \frac{number\ radiated\ /\ unit\ time}{area\ of\ the\ sphere} = \frac{1.38\times10^{19}/s}{4\pi(2\,m)^2} = 2.75\times10^{17}/m^2\cdot s$

3-28. (a) $hf = \phi \;\therefore\; f_t = \dfrac{\phi}{h} = \dfrac{4.22\,eV}{4.14\times10^{-15}\,eV\cdot s} = 1.02\times10^{15}\,Hz$

(b) $f = c/\lambda = \dfrac{3.00\times10^{8}\,m/s}{560\times10^{-9}\,m} = 5.36\times10^{14}\,Hz \qquad No.$

Available energy/photon $hf = (4.14\times10^{-15}\,ev\cdot s)(5.36\times10^{14}\,Hz) = 2.22\,eV$

This is less than ϕ.

3-29. (a) $E = \dfrac{hc}{\lambda} = \dfrac{1240\,eV\cdot nm}{0.1\,nm} = 1.24\times10^{4}\,eV$

(b) $E = \dfrac{hc}{\lambda} = \dfrac{1240\,eV\cdot nm}{1\,fm}\times\dfrac{10^{6}\,fm}{1\,nm} = 1.24\times10^{9}\,eV = 1.24\,GeV$

(c) $E = hf = (4.14\times10^{-15}\,eV\cdot s)(90.7\times10^{6}\,Hz) = 3.75\times10^{-7}\,eV$

3-30. Using Equation 3-36,

(1) $0.95 = \dfrac{h}{e}\left(\dfrac{c}{435.8\times10^{-9}\,m}\right) - \dfrac{\phi}{e}$

(2) $0.38 = \dfrac{h}{e}\left(\dfrac{c}{546.1\times10^{-9}\,m}\right) - \dfrac{\phi}{e}$

Subtracting (2) from (1), $0.57 = \dfrac{hc}{e\,10^{-9}}\left(\dfrac{1}{435.8} - \dfrac{1}{546.1}\right)$

Solving for h yields: $h = 6.56\times10^{-34}$ J·s. Substituting h into either (1) or (2) and solving for ϕ/e yields: ϕ/e = 1.87 eV. Threshold frequency is given by $hf/e = \phi/e$ or

$$f = \left(\frac{\phi}{e}\right)\left(\frac{e}{h}\right) = \frac{(1.87\,eV)(1.60\times10^{-19}\,C)}{6.56\times10^{-34}\,J\cdot s} = 4.57\times10^{14}\,Hz$$

3-31. $E = n\dfrac{hc}{\lambda} = \dfrac{(60)(6.63\times10^{-34}J\cdot s)(3.00\times10^{8}m/s)}{550\times10^{-9}m} = 2.17\times10^{-17}J$

3-32. (a) $\phi = \dfrac{hc}{\lambda} = \dfrac{1240\ eV\cdot nm}{653\,nm} = 1.90\ eV$ (b) $E_k = \dfrac{hc}{\lambda} - \phi = \dfrac{1240\ eV\cdot nm}{300\,nm} - 1.90\,eV = 2.23\ eV$

3-33. Equation 3-31: $\lambda_2 - \lambda_1 = \Delta\lambda = \dfrac{h}{mc}(1 - \cos\theta)$

$$\Delta\lambda = \frac{(6.63\times10^{-34}J\cdot s)(1 - \cos 135°)}{(9.11\times10^{-31}kg)(3.00\times10^{8}m/s)} = 4.14\times10^{-12}m = 4.14\times10^{-3}\ nm$$

$$\frac{\Delta\lambda}{\lambda_1}\times100 = \frac{4.14\times10^{-3}\ nm}{0.0711\ nm}\times100 = 5.8\%$$

3-34. Equation 3-30: $\lambda_m = \dfrac{1.24\times10^{3}}{V}\,nm = \dfrac{1.24\times10^{3}}{80\times10^{3}\ V} = 0.016\,nm$

3-35. $p = \dfrac{h}{\lambda} = \dfrac{hc}{\lambda c}$ (a) $p = \dfrac{1240\,eV\cdot nm}{c(400\,nm)} = 3.10\,eV/c = \dfrac{6.63\times10^{-34}J\cdot s}{400\times10^{-9}m} = 1.66\times10^{-27}kg\cdot m/s$

(b) $p = \dfrac{1240\,eV\cdot nm}{c(0.1\,nm)} = 1.24\times10^{4}eV/c = \dfrac{6.63\times10^{-34}J\cdot s}{0.1\times10^{-9}m} = 6.63\times10^{-24}\ kg\cdot m/s$

(c) $p = \dfrac{1240\,eV\cdot nm}{c(3\times10^{7}nm)} = 4.14\times10^{-5}eV/c = \dfrac{6.63\times10^{-34}J\cdot s}{3\times10^{-2}m} = 2.21\times10^{-32}\ kg\cdot m/s$

(d) $p = \dfrac{1240\,eV\cdot nm}{c(2\,nm)} = 620\,eV/c = \dfrac{6.63\times10^{-34}J\cdot s}{2\times10^{-9}m} = 3.32\times10^{-25}\ kg\cdot m/s$

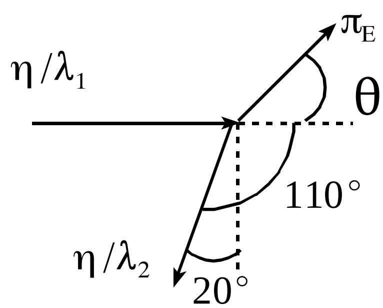
3-36. $\lambda_2 - \lambda_1 = \frac{h}{mc}(1 - \cos\theta) = \frac{(6.63\times10^{-34}J\cdot s)(1-\cos 110°)}{(9.11\times10^{-31}kg)(3.00\times10^{8}m/s)} = 3.26\times10^{-12}\,m$

$$\lambda_1 = \frac{hc}{E_1} = \frac{(6.63\times10^{-34}J\cdot s)(3\times10^{8}m/s)}{(0.511\times10^{6}eV)(1.60\times10^{-19}J/eV)} = 2.43\times10^{-12}\,m$$

$$\lambda_2 = \lambda_1 + 3.26\times10^{-12}m = (2.43 + 3.26)\times10^{-12}m = 5.69\times10^{-12}m$$

$$E_2 = \frac{hc}{\lambda_2} = \frac{1240\,eV\cdot nm}{5.69\times10^{-3}nm} = 2.18\times10^{5}\,eV = 0.218\,MeV$$

Electron recoil energy $E_e = E_1 - E_2$ (Conservation of energy)
$E_e = 0.511\,MeV - 0.218\,MeV = 0.293\,MeV$. The recoil electron momentum makes an angle θ with the direction of the initial photon.



$$\frac{h}{\lambda_2}\cos 20° = p_e \sin\theta = (1/c)\sqrt{E^2 - (mc^2)^2}\,\sin\theta \quad \text{(Conservation of momentum)}$$

$$\sin\theta = \frac{(3.00\times10^{8}m/s)(6.63\times10^{-34}J\cdot s)\cos 20°}{(5.69\times10^{-12}m)[(0.804\,MeV)^2 - (0.511\,MeV)^2]^{1/2}(1.60\times10^{-13}J/MeV)}$$

$$= 0.330 \;\; or \;\; \theta = 19.3°$$

3-37. $\Delta\lambda = \lambda_2 - \lambda_1 = \Delta\lambda = \frac{h}{mc}(1 - \cos\theta) = 0.01\,\lambda_1$ *Equation* 3 – 31

$$\lambda_1 = (100)\frac{h}{mc}(1 - \cos\theta) = (100)(0.00243\,nm)(1 - \cos 90°) = 0.243\,nm$$

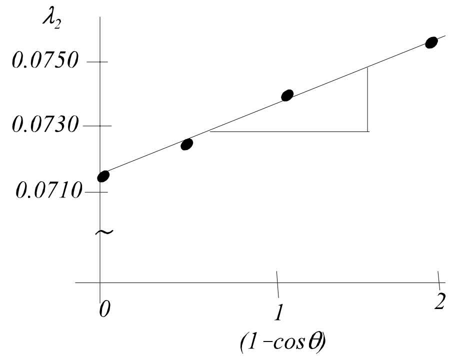
3-38. (a) $E_1 = \dfrac{hc}{\lambda_1} = \dfrac{1240\,eV\cdot nm}{0.0711\,nm} = 1.747\times 10^4\,eV$

(b) $\lambda_2 = \lambda_1 + \dfrac{h}{mc}(1-\cos\theta) = 0.0711 + (0.00243\,nm)(1-\cos 180°) = 0.0760\,nm$

(c) $E_2 = \dfrac{hc}{\lambda_2} = \dfrac{1240\,eV\cdot nm}{0.0760\,nm} = 1.634\times 10^4\,eV$ (d) $E_e = E_1 - E_2 = 1.128\times 10^3\,eV$

3-39. $\Delta\lambda = \dfrac{h}{mc}(1-\cos\theta) \quad \therefore \cos\theta = 1 - \dfrac{mc}{h}\Delta\lambda = 1 - \dfrac{0.29\times 10^{-3}\,nm}{0.00243\,nm} = 0.881 \quad \therefore \theta = 28.3°$

3-40. $\lambda_2 = \lambda_1 + \dfrac{h}{mc}(1-\cos\theta) = 0.0711\,nm + (0.00243\,nm)(1-\cos\theta)$



θ	$1-\cos\theta$	$\lambda_2\,(nm)$
0	0	0.0711
45°	0.293	0.0718
90°	1	0.0735
135°	1.707	0.0752

$$\text{Slope} = \frac{(0.0745 - 0.0720)\,nm}{(1.50 - 0.45)} = 2.381\times 10^{-3}$$

$$= \frac{h}{mc} \Rightarrow h = (2.381\times 10^{-3}\,nm)(9.11\times 10^{-31}\,kg)(3.00\times 10^{8}\,m/s) = 6.51\times 10^{-34}\,J\cdot s$$

3-41. (a) Compton wavelength = $\dfrac{h}{mc}$

electron: $\dfrac{h}{mc} = \dfrac{6.63\times10^{-34}J\cdot s}{(9.11\times10^{-31}kg)(3.00\times10^{8}m/s)} = 2.43\times10^{-12}m = 0.00243\,nm$

proton: $\dfrac{h}{mc} = \dfrac{6.63\times10^{-34}J\cdot s}{(1.67\times10^{-27}kg)(3.00\times10^{8}m/s)} = 1.32\times10^{-15}m = 1.32\,fm$

(b) $E = \dfrac{hc}{\lambda}$ (i) electron: $E = \dfrac{1240\,eV\cdot nm}{0.00243\,nm} = 5.10\times10^{5}eV = 0.510\,MeV$

(ii) proton: $E = \dfrac{1240\,eV\cdot nm}{1.32\times10^{-6}nm} = 9.39\times10^{8}eV = 939\,MeV$

3-42. (a) $eV_0 = hf - \phi = hc/\lambda - \phi$

$e(0.52\,V) = (hc/450\,nm) - \phi \qquad (i)$

$e(1.90\,V) = (hc/300\,nm) - \phi \qquad (ii)$

Multiplying (i) by 450 nm/e and (ii) by 300 nm/e, then subtracting (ii) from (i) and re-arranging gives

$$\frac{\phi}{e} = \frac{[(300\,nm)(1.90\,V) - (450\,nm)(0.52\,V)]}{150\,nm} = 2.24\,eV$$

(b) $\dfrac{hc}{e(300\,nm)} = 1.90 + 2.24 \rightarrow h = \dfrac{e(300\times10^{-9}m)(4.14\,V)}{(3.00\times10^{8}m/s)} = 6.63\times10^{-34}J\cdot s$

3-43. $F_y = e\mathscr{E} = ma_y \rightarrow a_y = e\mathscr{E}/m$ (constant) $\quad t_1 = x_1/u_x \quad u_y = a_y t_1 = e\mathscr{E}x_1/mu_x$

For small θ, $\theta \approx \tan\theta = \dfrac{u_y}{u_x}$ Substituting for u_y, $\theta = \dfrac{e\mathscr{E}x_1}{mx_1^2}$

3-44. Including Earth's magnetic field in computing y_2, Equation 3-6 becomes

$$y_2 = \frac{e}{m}\left[\frac{B^2 x_1 x_2}{\mathscr{E}} + \frac{1}{2}\frac{B_E B x_2^2}{\mathscr{E}}\right]$$

where the second term in the brackets comes from $F_y = euB_E = ma_y$ and

$y = \frac{1}{2}a_y t^2$. Thus, $1 = \frac{e}{m}\left[\frac{B^2 x_1 x_2}{\mathscr{E} y_2} + \frac{1}{2}\frac{B_E B x_2^2}{\mathscr{E} y_2}\right]$ The first term inside the brackets is the

reciprocal of $0.7\times 10^{11} C/kg$, Thomson's value for e/m. Using Thomson's data (B = $5.5\text{x}10^{-4}$ T, $\mathscr{E}$ = $1.5\text{x}10^4$ V/m, x_1 = 5 cm, y_2/x_2 = 8/110) and the modern value for e/m = $1.76\text{x}10^{11}$ C/kg and solving for B_E:

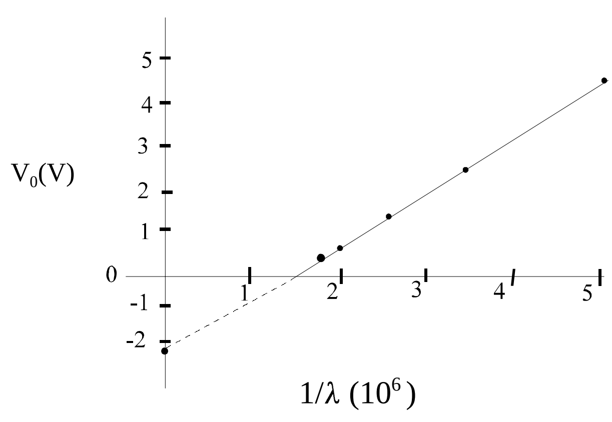
$\frac{1}{2}\frac{B_E B x_2^2}{E y_2} = -8.20\times 10^{-12}$. The minus sign means that B and B_E are in opposite directions,

which is why Thomson's value underestimated the actual value.

$$B_E = \frac{-(8.20\times 10^{-12})(2)(1.5\times 10^4 V/m)(8/110)^2}{(5.5\times 10^{-4} T)(8\times 10^{-2} m)} = -3.1\times 10^{-5} T = -31\,\mu T$$

3-45. Calculate $1/\lambda$ to be used in the graph.

$1/\lambda$ (10^6/m)	5.0	3.3	2.5	2.0	1.7
V_0 (V)	4.20	2.06	1.05	0.41	0.03



(Problem 3-45 continued)

(a) The intercept on the vertical axis is the work function ϕ. $\phi = 2.08$ eV

(b) The intercept on the horizontal axis corresponds to the threshold frequency.

$$\frac{1}{\lambda_t} = 1.65\times 10^{6}\,/m$$

$$f_t = \frac{c}{\lambda_t} = (3.00\times 10^{8}\,m/s)(1.65\times 10^{6}/m) = 4.95\times 10^{14}\,Hz$$

(c) The slope of the graph is h/e. Using the vertical intercept and the largest experimental point,

$$\frac{h}{e} = \frac{1}{c}\frac{\Delta V_0}{\Delta(1/\lambda)} = \frac{4.20\,V - (-2.08\,V)}{(3.00\times 10^{8}\,m/s)(5.0\times 10^{6}/m - 0)} = 4.19\times 10^{-15}\,eV/Hz$$

3-46. In the center of momentum reference frame, the photon and the electron have equal and opposite momenta. $p_\gamma = E_\gamma / c = -p_e$.

The total energy is: $E_\gamma + E_e = E_\gamma + (p_e^2 c^2 + m^2 c^4)^{1/2} = E_\gamma + (E_\gamma^2 + m^2 c^4)^{1/2}$.

By conservation of momentum, the final state is an electron at rest, $p_e' = 0$. Conservation of energy requires that the final state energy E' is

$$E' = E_\gamma + E_e \quad \therefore \quad mc^2 = E_\gamma + [p^2c^2 + (mc^2)^2]^{1/2}$$

$\therefore \quad mc^2 - E_\gamma = [p^2c^2 + (mc^2)^2]^{1/2} = [E_\gamma^2 + (mc^2)^2]^{1/2}$ Squaring yields,

$(mc^2)^2 - 2mc^2E_\gamma + E_\gamma^2 = E_\gamma^2 + (mc^2)^2 \quad \therefore \quad mc^2E_\gamma = 0$. This can be true only if E_Υ vanishes identically, i.e., if there is no photon at all.

3-47. Bragg condition: $m\lambda = 2d\sin\theta$. $\lambda = (2)(0.28\,nm)(\sin 20°) = 1.92\times 10^{-10}m = 0.192\,nm$ This is the minimum wavelength λ_m that must be produced by the X ray tube.

$$\lambda_m = \frac{1.24\times 10^3}{V}\,nm \quad or \quad V = \frac{1.24\times 10^3}{0.192} = 6.47\times 10^3\,V = 6.47\,kV$$

3-48. (a) $E = (100\,W)(10^4 s) = (100\,J/s)(10^4 s) = 10^6\,J$

The momentum p absorbed is $p = \dfrac{E}{c} = \dfrac{10^6 J}{(3.00\times 10^8 m/s)} = 3.33\times 10^{-3} J\cdot s/m$

(b) $\Delta p = m(v_f - v_i) = (2\times 10^{-3} kg)(v_f - 0) = 3.3\times 10^{-3} J\cdot s/m$

$$\therefore\quad v_f = \frac{3.33\times 10^{-3} J\cdot s/m}{2\times 10^{-3} kg} = 1.67\,m/s$$

(c) $E_k = \dfrac{1}{2}mv_f^2 = \dfrac{(2\times 10^{-3} kg)(1.67\,m/s)^2}{2} = 2.78\times 10^{-3} J$

The difference in energy has been (i) used to increase the object's temperature and (ii) radiated into space by the blackbody.

3-49. Conservation of energy: $E_1 + mc^2 = E_2 + E_k + mc^2 \quad \therefore \quad E_k = E_1 - E_2 = hf_1 - hf_2$

From Compton's equation, we have: $\lambda_2 - \lambda_1 = \dfrac{h}{mc}(1-\cos\theta)$ thus $\dfrac{1}{f_2} - \dfrac{1}{f_1} = \dfrac{h}{mc^2}(1-\cos\theta)$

$$\frac{1}{f_2} = \frac{1}{f_1} + \frac{h}{mc^2}(1-\cos\theta) \quad \therefore \quad f_2 = \frac{f_1 mc^2}{mc^2 + hf_1(1-\cos\theta)}$$

Substituting this expression for f_2 into the expression for E_k (and dropping the subscript on f_1):

$$E_k = hf - \frac{hfmc^2}{mc^2 + hf(1-\cos\theta)} = \frac{hfmc^2 + (hf)^2(1-\cos\theta) - hfmc^2}{mc^2 + hf(1-\cos\theta)} = \frac{hf}{1 + \dfrac{mc^2}{[hf(1-\cos\theta)]}}$$

(Problem 3-49 continued)

E_k has its maximum value when the photon energy change is maximum, i.e., when $\theta = \pi$ so $\cos\theta = -1$. Then $$E_k = \frac{hf}{1 + \frac{mc^2}{2hf}}$$

3-50. (a) $\lambda_m T = 2.898\times 10^{-3}\, m\cdot K \quad \therefore \quad T = \frac{2.898\times 10^{-3}\, m\cdot K}{82.8\times 10^{-9}\, m} = 3.50\times 10^4 K$

(b) Equation 3-24: $$\frac{u(70\,nm)}{u(82.8\,nm)} = \frac{(70\,nm)^{-5}/(e^{hc/(70\,nm)kT}-1)}{(82.8\,nm)^{-5}/(e^{hc/(82.8)kT}-1)}$$

where $$\frac{hc}{(70\,nm)kT} = \frac{(6.63\times 10^{-34} J\cdot s)(3.00\times 10^8 m/s)}{(70\times 10^{-9} m)(1.38\times 10^{-23} J/K)(3.5\times 10^4 K)} = 5.88$$ and

$$\frac{hc}{(82.8\,nm)kT} = 4.97 \qquad \frac{u(70\,nm)}{u(82.8\,nm)} = \frac{(70\,nm)^{-5}/(e^{5.88}-1)}{(82.8\,nm)^{-5}/(e^{4.97}-1)} = 0.929$$

Similarly, $$\frac{u(100\,nm)}{u(82.8\,nm)} = \frac{(100\,nm)^{-5}/(e^{4.12}-1)}{(82.8\,nm)^{-5}/(e^{4.97}-1)} = 0.924$$

3-51. Fraction of radiated solar energy in the visible region of the spectrum is the area under the Planck curve (Figure 3-7) between 350 nm and 700 nm divided by the total area. The latter is 6.42×10^7 W/m^2 (see solution to Problem 3-25). Evaluating $u(\lambda)\Delta\lambda$ with λ = 525 nm (midpoint of visible) and $\Delta\lambda$ = 700 nm – 350 nm = 350 nm,

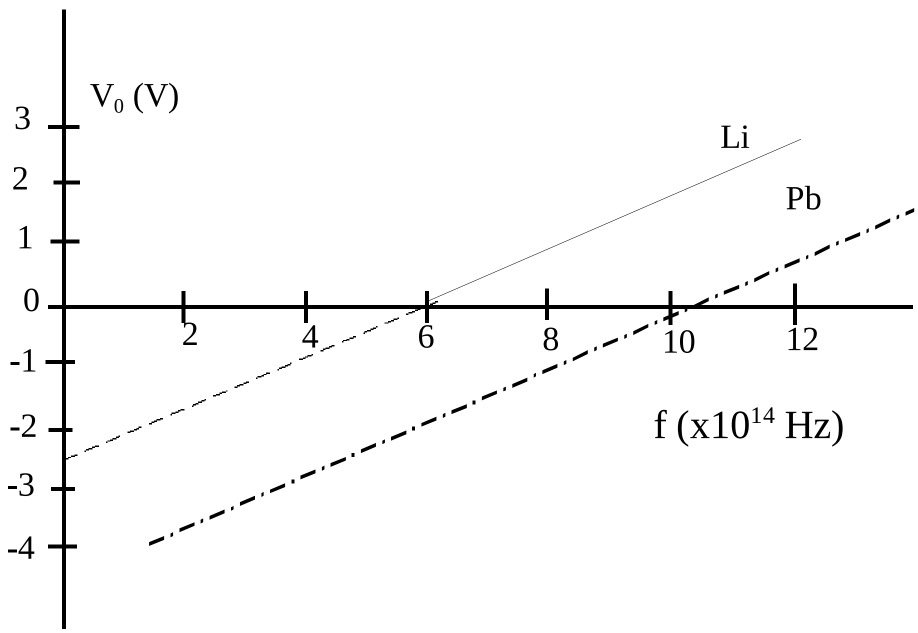
$$u(\lambda)\Delta\lambda = \frac{8\pi(6.63\times 10^{-34} J\cdot s)(3.00\times 10^8 m/s)(525\,nm)^{-5}(350\,nm)}{\exp\left[\frac{(6.63\times 10^{-34} J\cdot s)(3.00\times 10^8 m/s)}{(1.38\times 10^{-23} J/k)(5800 K)(525\,nm)}\right] - 1} = 0.389\, J/m^3$$

$$R(350-700) = \frac{c}{4}u = (3.00\times 10^8 m/s)(0.389\, J/m^3)/4 = 2.92\times 10^7\, W/m^2$$

Fraction in visible = $R(350-700)/R = (2.92\times 10^7 W/m^2)/(6.42\times 10^7 W/m^2) = 0.455$

3-52. (a) Make a table of $f = c/\lambda$ *vs.* V_0.

f (x10^{14} Hz)	11.83	9.6	8.22	7.41	6.91
V_0 (V)	2.57	1.67	1.09	0.73	0.55



The work function for Li (intercept on the vertical axis) is ϕ = 2.40 eV.

(b) The slope of the graph is h/e. Using the largest V_0 and the intercept on the vertical axis,

$$\frac{h}{e} = \frac{2.57\,V-(-2.40\,V)}{11.53\times10^{14}\,Hz-0} \quad or\ h = \frac{(4.97\,V)(1.60\times10^{-19}\,C)}{11.53\times10^{14}\,Hz} = 6.89\times10^{-34}\,J\cdot s$$

(c) The slope is the same for all metals. Draw a line parallel to the Li graph with the work function (vertical intercept) of Pb, ϕ = 4.14 eV. Reading from the graph, the threshold frequency for Pb is 9.8x10^{14} Hz; therefore, no photon wavelengths larger than $\lambda = c/f_t = (3.00\times10^{8}\,m/s)(9.8\times10^{14}\,Hz) = 306\,nm$ will cause emission of photoelectrons from Pb.

3-53. (a) Equation 3-24: $u(\lambda) = \dfrac{8\pi hc\lambda^{-5}}{e^{hc/\lambda kT} - 1}$ Letting $C = 8\pi hc \ and \ a = hc/kT$

gives $u(\lambda) = \dfrac{C\lambda^{-5}}{e^{a/\lambda} - 1}$

(b)
$$\frac{du}{d\lambda} = \frac{d}{d\lambda}\left[\frac{C\lambda_{-5}}{e^{a/\lambda} - 1}\right] = C\left[\frac{\lambda^{-5}(-1)e^{a/\lambda}(-a\lambda^{-2})}{(e^{a/\lambda} - 1)^2} - \frac{5\lambda^{-6}}{e^{a/\lambda} - 1}\right]$$
$$= \frac{C\lambda^{-6}}{(e^{a/\lambda} - 1)^2}\left[\frac{a}{\lambda}e^{a/\lambda} - 5(e^{a/\lambda} - 1)\right] = \frac{C\lambda^{-6}e^{a/\lambda}}{(e^{a/\lambda} - 1)^2}\left[\frac{a}{\lambda} - 5(1 - e^{a/\lambda})\right] = 0$$

The maximum corresponds to the vanishing of the quantity in the brackets. Thus, $5\lambda(1 - e^{-a/\lambda}) = a$.

(c) This equation is most efficiently solved by trial and error; i.e., guess at a value for a/λ in the expression $5\lambda(1 - e^{-a/\lambda}) = a$, solve for a better value of a/λ; substitute the new value to get an even better value, and so on. Repeat the process until the calculated value no longer changes. One succession of values is: 5, 4.966310, 4.965156, 4.965116, 4.965114, 4.965114. Further iterations repeat the same value (to seven digits), so we have $\dfrac{a}{\lambda_m} = 4.965114 = \dfrac{hc}{\lambda_m kT}$

(d) $\lambda_m T = \dfrac{hc}{(4.965114)k} = \dfrac{(6.63\times10^{-34}J\cdot s)(3.00\times10^{8}m/s)}{(4.965114)(1.38\times10^{-23}J/K)}$.

Therefore, $\lambda_m T = 2.898\times10^{-3}m\cdot K$ *Equation* 3 – 11

3-54. (a) $I = \dfrac{P}{4\pi R^2} = \dfrac{1W}{4\pi(1m)^2}\left(\dfrac{1}{1.602\times10^{-19}J/eV}\right) = 4.97\times10^{17}eV/m^2\cdot s$

(b) let the atom occupy an area of $(0.1\,nm)^2$.

$$\frac{dW}{dt} = IA = (4.97\times10^{17}eV/m^2\cdot s)(0.1\,nm)^2(10^{-9}m/nm)^2 = 4.97\times10^{-3}eV/s$$

(c) $t = \dfrac{\phi}{dW/dt} = \dfrac{2eV}{4.97\times10^{-3}eV/s} = 403s = 6.71\min$

3-55. (a) The nonrelativistic expression for the kinetic energy of the recoiling nucleus is

$$E_k = \frac{p^2}{2m} = \frac{(15\,MeV/c)^2}{2\times 12u}\left(\frac{1u}{931.5\,MeV/c^2}\right) = 1.10\times 10^4\,eV$$

Internal energy $U = 15\,MeV - 0.0101\,MeV = 14.9899\,MeV$

(b) the nucleus must recoil with momentum equal to that of the emitted photon, about 14.98 MeV/c.

$$E_k = \frac{p^2}{2m} = \frac{(14.98\,MeV/c)^2}{2\times 12u}\left(\frac{1u}{931.5\,MeV/c^2}\right) = 1.00\times 10^{-2}\,eV$$

$$E_\gamma = U - E_k = 14.9899\,MeV - 0.0100\,MeV = 14.9799\,MeV$$

3-56. Derived in Problem 3-47, the electron's kinetic energy at the Compton edge is

$$E_k = \frac{hf}{1 + mc^2/2hf}$$

$$E_k = 520\,keV = \frac{hf}{1 + (511\,keV)/2hf} \quad \therefore \quad 520\,keV = \frac{2(hf)^2}{2hf + 511\,keV}$$

Thus, $(hf)^2 - 520(hf) - (520)(511)/2 = 0$

Solving with the quadratic formula: $hf = \dfrac{520 \pm [(520)^2 + (2)(520)(511)]^{1/2}}{2} = 708\,keV$ (only the + sign is physically meaningful). Energy of the incident gamma ray hf = 708 keV.

$$\frac{hc}{\lambda} = 708\,keV \quad \sim \quad \lambda = \frac{(6.63\times 10^{-34}\,J\cdot s)(3.00\times 10^8\,m/s)}{(708\,keV)(1.60\times 10^{-16}\,J/keV)} = 1.76\times 10^{-12}\,m = 1.76\,pm$$

3-57. (a) $E_k = 50\,keV \; and \; \lambda_2 = \lambda_1 + 0.095\,nm \quad \dfrac{hc}{\lambda_1} + \dfrac{hc}{\lambda_2} = 5.0\times 10^4\,eV \quad \therefore \quad \dfrac{1}{\lambda_1} + \dfrac{1}{\lambda_1 + 0.095} = \dfrac{5.0\times 10^4\,eV}{hc}$

$$\therefore \quad \frac{2\lambda_1 + 0.095}{\lambda_1^2 + 0.095\,\lambda_1} = \frac{5.0\times 10^4\,eV}{hc}$$

(Problem 3-57 continued)

$$\lambda_1^2 + \left(0.095\,nm - \frac{2hc}{5\times10^4 eV}\right)\lambda_1 - \frac{(0.095\,nm)hc}{5\times10^4 eV} = 0$$

$$\therefore\quad \lambda_1^2 + 0.04532\lambda_1 - 2.35\times10^{-3} = 0$$

Applying the quadratic formula,

$$\lambda_1 = \frac{-0.04532 \pm [(0.04532)^2 + 4(2.3598\times10^{-3})]^{1/2}}{2}$$

$$\lambda_1 = 0.06189\,nm \;\; and \;\; \lambda_2 = 0.08139\,nm$$

(b) $E_1 = \dfrac{hc}{\lambda_1} = \dfrac{1240\,eV\cdot nm}{0.06189\,nm} = 20.04\,keV$

3-58. Let $x = \dfrac{\epsilon}{kT} = \dfrac{hf}{kT}$ in Equation 3-21:

$$\sum_{n=0}^{\infty} f_n = A\sum_{n=0}^{\infty} e^{-nx} = A[e^0 + e^{-x} + (e^{-x})^2 + (e^{-x})^2 + \cdots] = A(1 + y + y^2 + y^3 + \cdots) = 1$$

where y = e^{-x}. This sum is the series expansion of $(1-y)^{-1}$, i.e., $(1-y)^{-1} = 1 + y + y^2 + y^3 + \cdots$

Then $\sum f_n = A(1-y)^{-1} = 1$ gives A = 1–y. Writing Equation 3-22 in terms of x and y:

$$\overline{E} = \sum_{n=0}^{\infty} E_n A e^{-E_n/kT} = A\sum_{n=0}^{\infty} nhfe^{-nhf/kT} = Ahf\sum_{n=0}^{\infty} ne^{-nx}$$

Note that $\sum ne^{-nx} = -(d/dx)\sum e^{-nx}$. But $\sum e^{-nx} = (1-y)^{-1}$, so we have

$$\sum ne^{-nx} = -\frac{d}{dx}\sum e^{-nx} = -\frac{d}{dx}(1-y)^{-1} = (1-y)^{-2}\left(-\frac{dy}{dx}\right) = y(1-y)^{-2}$$

since $\dfrac{dy}{dx} = \dfrac{d(e^{-x})}{dx} = -e^{-x} = -y$. Multiplying this sum by hf and by A = (1–y), the average

(Problem 3-58 continued)

energy is $\overline{E} = hfA\sum_{n=0}^{\infty} ne^{-nx} = hf(1-y)y(1-y)^{-2} = \dfrac{hfy}{1-y} = \dfrac{hfe^{-x}}{1-e^{-x}}$

Multiplying the numerator and the denominator by e^{-x} and substituting for x, we obtain

$\overline{E} = \dfrac{hf}{e^{hf/kT}-1}$, which is Equation 3-23.