Chapter 6 – The Schrödinger Equation

6-1. $\dfrac{d\Psi}{dx} = kAe^{kx-\omega t} = k\Psi$ and $\dfrac{d^2\Psi}{dx^2} = k^2\Psi$

Also, $\dfrac{d\Psi}{dt} = -\omega\Psi$. The Schrödinger equation is then, with these substitutions,

$-\hbar^2 k^2\Psi/2m + V\Psi = -i\hbar\omega\Psi$. Because the left side is real and the right side is a pure imaginary number, the proposed Ψ does not satisfy Schrödinger's equation.

6-2. For the Schrödinger equation:

$\dfrac{d\Psi}{dx} = ik\Psi$ and $\dfrac{d^2\Psi}{dx^2} = -k^2\Psi$. Also, $\dfrac{d\Psi}{dt} = -i\omega\Psi$

Substituting these into the Schrödinger equation yields:

$\hbar^2 k^2\Psi/2m + V\Psi = \hbar\omega\Psi$, which is true, provided $\hbar\omega = \hbar^2 k^2/2m + V$, i.e., if $E = E_k + V$.

For the classical wave equation: (from Equation 6-1)

From above: $\dfrac{d^2\Psi}{dx^2} = -k^2\Psi$ and also $\dfrac{d^2\Psi}{dt} = -\omega^2\Psi$. Substituting into Equation 6-1

(with Ψ replacing $\mathscr{E}$ and v replacing c) $-k^2\Psi = (1/v^2)(-\omega^2\Psi)$, which is true for $v = \omega/k$.

6-3. (a) $\dfrac{d\psi}{dx} = -(x/L^2)\psi$ and $\dfrac{d^2\psi}{dx^2} = \left[\left(-\dfrac{x}{L^2}\right)\left(-\dfrac{x}{L^2}\right) - \dfrac{1}{L^2}\right]\psi = \dfrac{x^2}{L^4}\psi - \dfrac{1}{L}\psi$

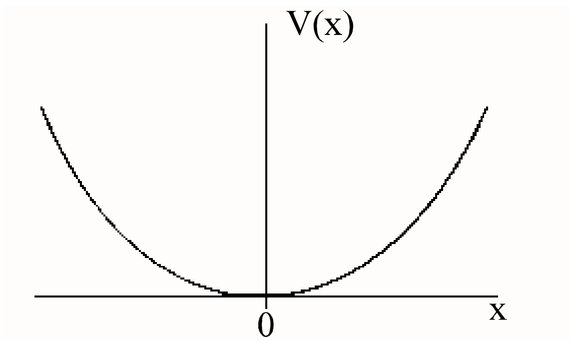
Substituting into the time-independent Schrödinger equation,

$$\left(-\frac{\hbar^2 x^2}{2mL^4} + \frac{\hbar^2}{2mL^2}\right)\psi + V(x) = E\psi = \frac{\hbar^2}{2mL^2}\psi$$

Solving for *V(x)*, $V(x) = \dfrac{\hbar^2}{2mL^2} - \left(-\dfrac{\hbar^2 x^2}{2mL^4} + \dfrac{\hbar^2}{2mL^2}\right) = \dfrac{\hbar^2 x^2}{2mL^4} = \dfrac{1}{2}kx^2$

(Problem 6-3 continued)

where $k = \hbar^2/mL^4$. This is the equation of a parabola centered at $x = 0$.



(b) The classical system with this dependence is the harmonic oscillator.

6-4. (a) $E_k(x) = E - V(x) = \hbar^2/2mL^2 - \hbar^2 x^2/2mL^4 = (\hbar^2/2mL^2)(1 - x^2/L^2)$

(b) The classical turning points are the points where $E = V(x)$ or $E_k(x) = 0$. That occurs when $x^2/L^2 = 1$, or when $x = \pm L$.

(c) For a harmonic oscillator $V(x) = m\omega^2 x^2/2$, so

$$\frac{\hbar^2 x^2}{2mL^4} = \omega^2 x^2/2 \rightarrow \omega^2 = \hbar^2/m^2L^4 \rightarrow \omega = \hbar/mL^2$$

Thus, $E = \dfrac{\hbar^2}{2mL^2} = \left(\dfrac{\hbar}{mL^2}\right)\dfrac{\hbar}{2} = \dfrac{1}{2}\hbar\omega$

6-5. (a)

$$\Psi(x,t) = A\sin(kx - \omega t)$$

$$\frac{\partial \Psi}{\partial t} = -\omega A\cos(kx - \omega t)$$

$$i\hbar\frac{\partial \Psi}{\partial t} = -i\hbar\omega A\cos(kx - \omega t)$$

(Problem 6-5 continued)

$$\frac{\partial^2 \Psi}{\partial x^2} = -k^2 A \sin(kx - \omega t)$$

$$\frac{-\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} = \frac{-\hbar k^2 A}{2m}\sin(kx - \omega t) \neq i\hbar\frac{\partial \Psi}{\partial t}$$

(b) $$\Psi(x,t) = A\cos(kx - \omega t) + iA\sin(kx - \omega t)$$

$$i\hbar\frac{\partial \Psi}{\partial t} = i\hbar\omega A(kx - \omega t) - i^2\hbar\omega A\cos(kx - \omega t)$$

$$= \hbar\omega A\cos(kx - \omega t) + i\hbar\omega A\sin(kx - \omega t)$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} = \frac{\hbar^2 k^2 A}{2m}\cos(kx - \omega t) + \frac{\hbar^2 i k^2 A}{2m}\sin(kx - \omega t)$$

$$= \frac{\hbar^2 k^2}{2m}[A\cos(kx - \omega t) + iA\sin(kx - \omega t)]$$

$$= i\hbar\frac{\partial \Psi}{\partial t} \quad \textit{if } \frac{\hbar^2 k^2}{2m} = \hbar\omega \text{ it does. (Equation 6–5 with V = 0)}$$

6-6. (a) For a free electron *V(x)* = 0, so

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \;\rightarrow\; \frac{d^2\psi}{dx^2} = -(2.5\times 10^{10})^2\,\psi$$

Substituting into the Schrödinger equation gives:

$(2.5\times 10^{10})^2(\hbar^2/2m)\psi = E\psi$ and, since $E = E_k = p^2/2m$ for a free particle,

$p^2 = 2m(2.5\times 10^{10})^2(\hbar^2/2m)$ and $p = (2.5\times 10^{10})\hbar = 2.64\times 10^{-24}\,kg\cdot m/s$

(b) $E = p^2/2m = (2.64\times 10^{-24}\,kg\cdot m/s)^2/(2)(9.11\times 10^{-31}\,kg) = 3.82\times 10^{-18}\,J$

$= (3.82\times 10^{-18}\,J)(1/1.60\times 10^{-19}\,J/eV) = 23.9\,eV$

(c) $\lambda = h/p = 6.63\times 10^{-34}\,J\cdot s/2.64\times 10^{-24}\,kg\cdot m/s = 2.5\times 10^{-10}\,m = 0.251\,nm$

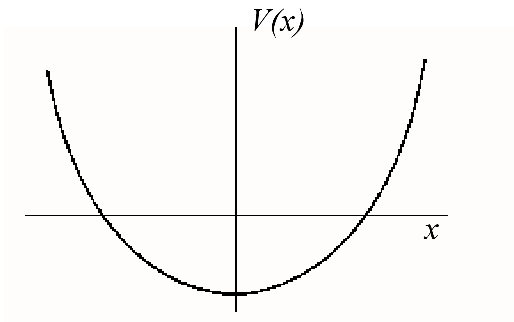
6-7. $\psi(x) = Ce^{-x^2/L^2}$ and $E = 0$

(a) $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + V(x)\psi = 0$

$$\frac{d\psi}{dx} = -\frac{2x}{L^2}\psi(x) \quad \text{and} \quad \frac{d^2\psi}{dx^2} = \left(\frac{4x^2}{L^4} - \frac{2}{L^2}\right)\psi$$

And $-\dfrac{\hbar^2}{2m}\left(\dfrac{4x^2}{L^4} - \dfrac{2}{L^2}\right)\psi + V(x)\psi = 0$ so $V(x) = \dfrac{\hbar^2}{mL^2}\left(\dfrac{2x^2}{L^2} - 1\right)$

(b)



6-8.
$$\int_{-a}^{+a} \psi^* \psi\, dx = A^2 \int_{-a}^{+a} e^{-i(kx - wt)} \times e^{i(kx - wt)}\, dx = 1$$

$$= A^2 \int_{-a}^{+a} dx = A^2 x\Big|_{-a}^{+a} = A^2(2a) = 1$$

$$\therefore A = \frac{1}{(2a)^{1/2}}$$

Normalization between $-\infty$ and $+\infty$ is not possible because the value of the integral is infinite.

6-9. (a) The ground state of an infinite well is $E_1 = h^2/8mL^2 = (hc)^2/8mc^2L^2$

For $m = m_p$, $L = 0.1\ nm$: $E_1 = \dfrac{(1240\,MeV{\cdot}fm)^2}{8(938.3\times10^6\,eV)(0.1\,nm)^2} = 0.021\,eV$

(b) For $m = m_p$, $L = 1\ fm$: $E_1 = \dfrac{(1240\,MeV{\cdot}fm)^2}{8(938.3\times10^6\,eV)(1\,fm)^2} = 205\,MeV$

6-10. The ground state wave function is (n = 1) $\psi_1(x) = \sqrt{2/L}\ \sin(\pi x/L)$ (Equation 6-32)

The probability of finding the particle in Δx is approximately:

$$P(x)\,\Delta x = \frac{2}{L}\sin^2\left(\frac{\pi x}{L}\right)\Delta x = \frac{2\,\Delta x}{L}\sin^2\left(\frac{\pi x}{L}\right)$$

(a) for $x = \dfrac{L}{2}$ *and* $\Delta x = 0.002L$, $P(x)\,\Delta x = \dfrac{2(0.002L)}{L}\sin^2\left(\dfrac{\pi L}{2L}\right) = 0.004\sin\dfrac{\pi}{2} = 0.004$

(b) $x = \dfrac{2L}{3}$, $P(x)\,\Delta x = \dfrac{2(0.002L)}{L}\sin^2\left(\dfrac{2\pi L}{3L}\right) = 0.004\sin\dfrac{2\pi}{3} = 0.0035$

(c) for x = L, $P(x)\,\Delta x = 0.004\sin^2\pi = 0$

6-11. The second excited state wave function is (n = 3) $\psi_3(x) = \sqrt{2/L}\ \sin(3\pi x/L)$

(Equation 6-32). The probability of finding the particle in Δx is approximately:

$$P(x)\,\Delta x = \frac{2}{L}\sin^2\left(\frac{3\pi x}{L}\right)\Delta x$$

(a) $x = \dfrac{L}{2}$ *and* $\Delta x = 0.002L$, $P(x)\,\Delta x = \dfrac{2(0.002L)}{L}\sin^2\left(\dfrac{3\pi L}{2L}\right) = 0.004\sin\dfrac{3\pi}{2} = 0.004$

(b) $x = \dfrac{2L}{3}$, $P(x)\,\Delta x = 0.004\sin^2\left(\dfrac{6\pi L}{3L}\right) = 0.004\sin^2 2\pi = 0$

(c) for x = L, $P(x)\,\Delta x = 0.004\sin^2\left(\dfrac{3\pi L}{L}\right) = 0.004\sin^2 3\pi = 0$

6-12. $E = \frac{1}{2}mv^2 = \frac{n^2\pi^2\hbar^2}{2mL^2}$ (Equation 6-24) $n^2 = \left(\frac{1}{2}mv^2\right)\left(\frac{2mL^2}{\pi^2\hbar^2}\right) = \left(\frac{mvL}{\pi\hbar}\right)^2$

$$n = \frac{mvL}{\pi\hbar} = \frac{(10^{-9}kg)(10^{-3}m/s)(10^{-2}m)}{\pi(1.055\times10^{-34}J{\cdot}s)} = 3\times10^{19}$$

6-13. (a) $\Delta x = 0.0001L = (0.0001)(10^{-2}m) = 10^{-6}m$

$$\Delta p = 0.0001p = (0.0001)(10^{-9}kg)(10^{-3}m/s) = 10^{-16}\,kg{\cdot}m/s$$

(b) $\frac{\Delta x\Delta p}{\hbar} = \frac{(10^{-6}m)(10^{-16}kg{\cdot}m/s)}{1.055\times10^{-34}J{\cdot}s} = 9\times10^{11}$

6-14. (a) This is an infinite square well of width L. $V(x) = 0$ and $E = E_k = p^2/2m$. From uncertainty principle: $E_{k_{\min}} \rightarrow p_{\min} \approx \Delta p = \hbar/\Delta x = \hbar/L$ and

$E_{\min} = p_{\min}^2/2m \approx \hbar^2/2mL^2 = h^2/8\pi^2mL^2$

(b) The solution to the Schrödinger equation for the ground state is:

$\psi_1(x) = (2/L)^{1/2}\sin(\pi x/L)$

and $\frac{d^2\psi_1}{dx^2} = -\left(\frac{\pi}{L}\right)^2\left(\frac{2}{L}\right)^{1/2}\sin(\pi x/L) = -\left(\frac{\pi}{L}\right)^2\psi_1$

So, $\frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2\psi_1 = E\psi_1$ or $E_1 = \frac{h^2}{8mL^2}$

The result in (a) is about 1/10 of the computed value, but has the correct dependence on *h*, *m*, and *L*.

6-15. (a) For the ground state, $L = \lambda/2$, so $\lambda = 2L$.

(b) Recall that state *n* has *n* half-wavelengths between $x = 0$ and $x = L$, so for $n = 3$, $L = 3\lambda/2$ or $\lambda = 2L/3$.

(Problem 6-15 continued)

(c) $p = h/\lambda = h/2L$ in the ground state.

(d) $p^2/2m = (h^2/4L^2)/2m = h^2/8mL^2$, which is the ground state energy.

6-16. $E_n = \dfrac{h^2 n^2}{8mL^2}$ and $\Delta E_n = E_{n+1} - E_n = \dfrac{h^2}{8mL^2}(n^2 + 2n + 1)$

or, $\Delta E_n = (2n+1)\dfrac{h^2}{8mL^2} = \dfrac{hc}{\lambda}$

so, $L = \left(\dfrac{\lambda h}{8mc}\right)^{1/2} = \left(\dfrac{\lambda hc}{8mc^2}\right)^{1/2} = \left(\dfrac{(694.3\,nm)(1240\,eV\cdot nm)}{8(0.511\times 10^6\,eV)}\right)^{1/2} = 0.459\,nm$

6-17. This is an infinite square well with L= 10 *cm*.

$$E_n = \frac{h^2 n^2}{8mL^2} = \frac{1}{2}mv^2 = \frac{(2.0\times 10^{-3}\,kg)(20\,nm)^2}{2(3.16\times 10^7\,s)^2}$$

$$n^2 = \frac{8(2.0\times 10^{-3}\,kg)^2(20\times 10^{-9}\,m)^2(0.1\,m)^2}{2(3.16\times 10^7\,s)^2(6.63\times 10^{-34}\,J\cdot s)^2}$$

$$n = \frac{2(2.0\times 10^{-3}\,kg)(20\times 10^{-9}\,m)(0.1\,m)}{3.16\times 10^7\,s\,(6.63\times 10^{-34}\,J\cdot s)} = 3.8\times 10^{14}$$

6-18. (a) $\psi_5(x) = (2/L)^{1/2}\sin(5\pi x/L)dx$

$$P = \int_{0.2L}^{0.4L}(2/L)\sin^2(5\pi x/L)dx$$

Letting $5\pi x/L = u$, then $5\pi dx/L = du$ and $x = 0.2L \rightarrow u = \pi$ and $x = 0.4L \rightarrow u = 2\pi$, so

$$P = \left(\frac{2}{L}\right)\left(\frac{L}{5\pi}\right)\int_{\pi}^{2\pi}\sin^2 u\,du = \left(\frac{2}{L}\right)\left(\frac{L}{5\pi}\right)\left(\frac{\frac{x}{2} - \sin 2x}{4}\right)\Bigg|_{\pi}^{2\pi} = \frac{1}{5}$$

(Problem 6-18 continued)

(b) $P = (2/L)\sin^2 \dfrac{5\pi(L/2)}{L}(0.01L) = 0.02$ where $0.01L = \Delta x$

6-19. $E_1 = \dfrac{h^2}{8mL^2} = \dfrac{(hc)^2}{8(mc^2)L^2}$

(a) For an electron: $E_1 = \dfrac{(1240\,MeV\cdot fm)^2}{8(0.511\,MeV)(10fm)^2} = 3.76\times 10^3\,MeV$

(b) For a proton: $E_1 = \dfrac{(1240\,MeV\cdot fm)^2}{8(938.3\,MeV)(10fm)^2} = 2.05\,MeV$

(c) $\Delta E_{21} = 3E_1$ (See Problem 6-16)

For the electron: $\Delta E_{21} = 3E_1 = 1.13\times 10^4\,MeV$

For the proton: $\Delta E_{21} = 3E_1 = 6.15\,MeV$

6-20. $F = -dE/dL$ comes from the impulse-momentum theorem $F\Delta t = 2mv$ where $\Delta t \approx L/v$

So, $F \sim mv^2/L \sim E/L$. Because $E_1 = h^2/8mL^2$, $dE/dL = h^2/4mL^3$ where the minus sign means "on the wall". So $F = h^2/4mL^3 = \dfrac{(6.63\times 10^{-34}J\cdot s)^2}{4(9.11\times 10^{-31}kg)(10^{-10}m)^3} = 1.21\times 10^{-7}N$

The weight of an electron is $mg = 9.11\times 10^{-31}kg\,(9.8\,m/s^2) = 8.9\times 10^{-30}N$ which is minuscule by comparison.

6-21. $\psi_n(x) = \sqrt{\dfrac{2}{L}}\,\sin\dfrac{n\pi x}{L}$

To show that

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = 0$$

Using the trig identity 2 *sin*A *sin*B= *cos*(A–B) - *cos*(A+B), the integrand becomes

(Problem 6-21 continued)

$$\frac{1}{2}\{\cos[(n-m)\pi x/L] - \cos[(n+m)\pi x/L]\}$$

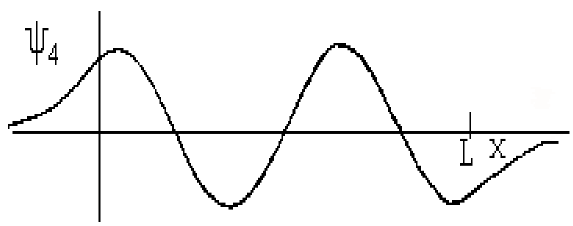
The integral of the first term is

$\frac{L}{\pi}\frac{\sin(n-m)\pi x/L}{(n-m)}$ and similarly for the second term with $(n + m)$ replacing $(n - m)$.

Since n and m are integers and $n \neq m$, the sines both vanish at the limits $x = 0$ and $x = L$.

$$\therefore \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 0 \qquad \text{for } n \neq m.$$

6-22. (a)



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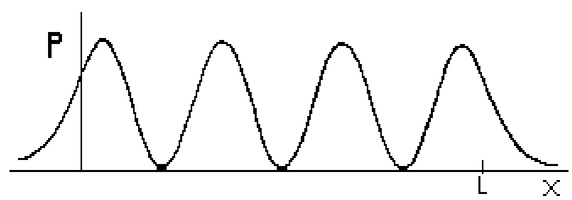
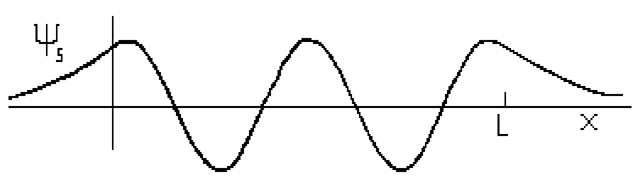


6-23. (a)



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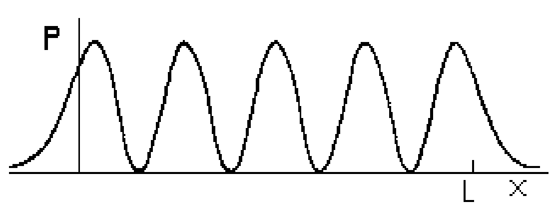
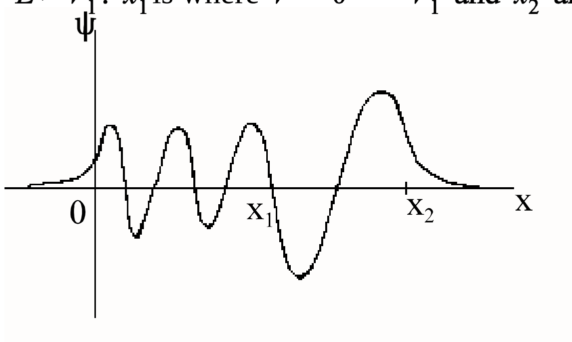


6-24. Because $E_1 = 0.5\,eV$ and for a finite well also $E_n \approx n^2 E_1$, then $n = 4$ is at about 8 eV, i.e., near the top of the well. Referring to Figure 6-14, $ka \approx 2\pi$.

$$ka = \frac{\sqrt{2mE}}{\hbar} \times \frac{L}{2} = (7.24 \times 10^9)L = 2\pi$$

$$L = 2\pi/7.24 \times 10^9 = 8.7 \times 10^{-10}m = 0.87\,nm$$

6-25. For $V_2 > E > V_1$: x_1 is where $V = 0 \rightarrow V_1$ and x_2 and x_2 is where $V_1 \rightarrow V_2$

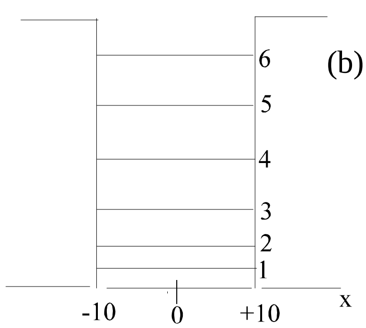


From $-\infty$ to 0 and x_2 to $+\infty$: ψ is exponential

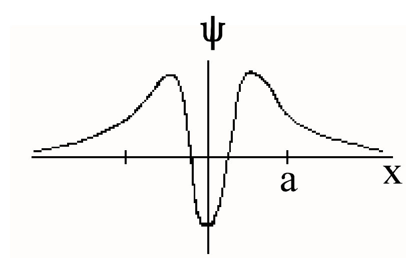
0 to x_1 : ψ is oscillatory; E_k is large so p is large and λ is small; amplitude is small because E_k, hence v is large.

x_1 to x_2: ψ is oscillatory; E_k is small so p is small and λ is large; amplitude is large because E_k, hence v is small.

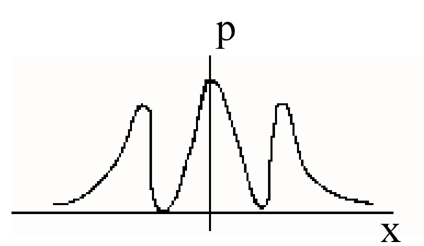
6-26. (a)



(b)



(c)



6-27. Referring to Figure 6-14, there will be two levels in the well when $ka = \pi/2$ (or larger) where

$$ka = \frac{a\sqrt{2mE_2}}{\hbar} = \frac{\pi}{2}$$

Squaring and rearranging,

$$E_2 = \frac{\hbar^2\pi^2}{8ma^2} = \frac{(hc)^2}{32(mc^2)a^2}$$

$$E_2 = \frac{(1240\,eV\cdot nm)^2}{32(1.88\times10^9\,eV)(10^{-6}\,nm)^2} = 25.6\times10^6\,eV$$

$$E_2 = 25.6\,MeV$$

The well must be at least this deep.

6-28. For $n = 3$, $\psi_3 = (2/L)^{1/2}\sin(3\pi x/L)$

(a) $\langle x\rangle = \int_0^L x(2/L)\sin^2(3\pi x/L)\,dx$

substituting $u = 3\pi x/L$, then $x = Lu/3\pi$ and $dx = (L/3\pi)du$. The limits become: $x = 0 \rightarrow u = 0$ and $x = L \rightarrow u = 3\pi$

$$\langle x\rangle = (2/L)(L/3\pi)(l/3\pi)\int_0^{3\pi} u\sin^2 u\,du$$

$$= (2/L)(L/3\pi)^2\left[\frac{u^2}{4} - \frac{u\sin 2u}{4} - \frac{\cos 2u}{8}\right]_0^{3\pi}$$

$$= (2/L)(l/3\pi)^2(3\pi)^2/4 = L/2$$

(b) $\langle x^2\rangle = \int_0^L x^2(2/L)\sin^2(3\pi x/L)\,dx$

Changing the variable exactly as in (a) and noting that:

$$\int_0^{3\pi} u^2\sin^2 u\,du = \left[\frac{u^3}{6} - \left(\frac{u^2}{4} - \frac{1}{8}\right)\sin 2u - \frac{u\cos 2u}{4}\right]_0^{3\pi}$$

(Problem 6-28 continued)

We obtain $\langle x^2 \rangle = \left(1/3 - \frac{1}{8}\pi^2 \right) L^2 = 0.320 L^2$

6-29. (a) Classically, the particle is equally likely to be found anywhere in the box, so *P(x) = constant*.

In addition, $\int_0^L P(x)\,dx = 1$ so $P(x) = 1/L$.

(b) $\langle x \rangle = \int_0^L (x/L)\,dx = L/2$ and $\langle x^2 \rangle = \int_0^L (x^2/L)\,dx = L^2/3$

6-30. $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x)$ (Equation 6-18)

$$\frac{1}{2m}\left(\frac{\hbar}{i}\frac{d}{dx}\right)\left(\frac{\hbar}{i}\frac{d}{dx}\right)\psi(x) = [E - V(x)]\psi(x)$$

$$\frac{1}{2m} p_{op}\, p_{op}\, \psi = [E - V(x)]\,\psi$$

Multiplying by ψ^* and integrating over the range of x,

$$\int_{-\infty}^{\infty} \psi^* \frac{p_{op}^2}{2m}\psi\,dx = \int_{-\infty}^{+\infty} \psi^* [E - V(x)]\psi\,dx$$

$$\left\langle \frac{p^2}{2m} \right\rangle = \langle [E - V(x)] \rangle \quad or \quad \langle p^2 \rangle = \langle 2m[E - V(x)] \rangle$$

For the infinite square well V(x) = 0 wherever ψ(x) does not vanish and vice versa. Thus,

$$\langle V(x) \rangle = 0 \text{ and } \langle p^2 \rangle = \langle 2mE \rangle = \left\langle 2m\frac{n^2\pi^2\hbar^2}{2mL^2} \right\rangle = \frac{\pi^2\hbar^2}{L^2} \ for\ n = 1$$

6-31. $\langle x^2 \rangle = \dfrac{L^2}{3} - \dfrac{L^2}{2\pi^2}$ (See Problem 6-28.) And $\langle x \rangle = \dfrac{L}{2}$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \left[\frac{L^2}{3} - \frac{L^2}{2\pi^2} - \frac{L^2}{4}\right]^{1/2} = L\left[\frac{1}{12} - \frac{1}{2\pi^2}\right]^{1/2} = 0.181\,L$$

$\langle p_x^2 \rangle = \dfrac{\pi^2 \hbar^2}{L^2}$ and $\langle p \rangle = 0$ (See Problem 6-30.)

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \left[\frac{\pi^2\hbar^2}{L^2} - 0\right]^{1/2} = \frac{\pi\hbar}{L}.$$ And $\sigma_x \sigma = (0.181L)(\pi\hbar/L) = 0.568\hbar$

6-32. $\psi_0(x) = A_0 e^{-m\omega x^2/2\hbar}$ where $A_0 = (m\omega/\hbar\pi)^{1/4}$

$\langle x \rangle = \int_{-\infty}^{+\infty} A_0^2 x e^{-m\omega x^2/\hbar}\,dx$ Letting $u^2 = m\omega x^2/\hbar$ and $x = (\hbar/m\omega)^{1/2} u$

$2u\,du = (m\omega/\hbar)(2x\,dx)$. And thus, $(m\omega/\hbar)^{-1} u\,du = x\,dx$; limits are unchanged.

$\langle x \rangle = A_0^2(\hbar/m\omega)\int_{-\infty}^{+\infty} u e^{-u^2}\,du = 0$ (Note that the symmetry of *V(x)* would also tell us that $\langle x \rangle = 0$.)

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} A_0^2 x^2 e^{-m\omega x^2/\hbar}\,dx$$

$$= A_0^2(\hbar/m\omega)^{3/2}\int_{-\infty}^{+\infty} u^2 e^{-u^2}\,du = 2A_0^2(\hbar/m\omega)^{3/2}\int_0^{\infty} u^2 e^{-u^2}\,du$$

$$= 2A_0^2(\hbar/m\omega)^{3/2}\sqrt{\pi}/4 = (m\omega/\hbar\pi)^{1/2}(\hbar/m\omega)^{3/2}\sqrt{\pi}/2 = \hbar/(2m\omega)$$

6-33. $\dfrac{p^2}{2m} + \dfrac{1}{2}m\omega^2 x^2 = (n + 1/2)\hbar\omega$. For the ground state (n = 0),

$\langle x^2 \rangle = \dfrac{2}{m\omega^2}(\hbar\omega/2 - p^2/2m)$ and $\langle x \rangle^2 = \left\langle \dfrac{\hbar}{m\omega} - \dfrac{p^2}{m^2\omega^2} \right\rangle = \hbar/2m\omega$ (See Problem 6-32)

(Problem 6-33 continued)

$$\frac{\hbar}{m\omega}\left\langle 1-\frac{p^2}{m\hbar\omega}\right\rangle = \frac{\hbar}{2m\omega} \quad \text{or} \quad \left\langle 1-\frac{p^2}{m\hbar\omega}\right\rangle = \frac{1}{2} \quad \rightarrow \quad \langle p^2\rangle = \frac{1}{2}m\hbar\omega$$

6-34. (a) $\Psi_0(xt) = (m\omega/\hbar\pi)^{1/4} e^{-m\omega x^2/2\hbar} e^{-i\omega t/2}$

(b) $p_{x\,op} = \frac{\hbar}{i}\frac{\partial}{\partial x}$ $\quad \langle p^2\rangle = \int_{-\infty}^{+\infty} \Psi_0^*(x,t)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \Psi_0(x,t)\,dx$

$$\frac{\partial \Psi_0}{\partial x} = A_0(m\omega x/\hbar\pi)^{1/4} e^{-m\omega x^2/2\hbar} e^{-i\omega t/2}$$

$$\frac{\partial^2 \Psi_0}{\partial x^2} = A_0[(-m\omega x/\hbar)(-m\omega x/\hbar) - m\omega/\hbar] e^{-m\omega x^2/2\hbar} e^{-i\omega t/2}$$

$$\langle p^2\rangle = -\hbar^2 A_0^2 (m\omega/\hbar)\int_{-\infty}^{+\infty}(m\omega x^2/\hbar - 1)e^{-m\omega x^2/\hbar}\,dx$$

$$= -\hbar^2 A_0^2 (m\omega/\hbar)\left[\int_{-\infty}^{+\infty}(m\omega x^2/\hbar)e^{-m\omega x^2/\hbar}\,dx - \int_{-\infty}^{+\infty} e^{-m\omega x^2/\hbar}\,dx\right]$$

Letting $u = (m\omega/\hbar)^{1/2}x$, then

$$\langle p^2\rangle = -\hbar^2 A_0^2 (m\omega/\hbar)(m\omega\hbar)^{-1/2}\left[\int_{-\infty}^{+\infty} u^2 e^{-u^2}\,du - \int_{-\infty}^{+\infty} e^{-u^2}\,du\right]$$

$$= -\hbar^2 A_0^2 (m\omega/\hbar)^{1/2}\, 2\left[\int_0^{\infty} u^2 e^{-u^2}\,du - \int_0^{\infty} e^{-u^2}\,du\right]$$

$$= -\hbar^2 (m\omega/\hbar\pi)^{1/2}(m\omega/\hbar)^{1/2}\, 2\left(\frac{\sqrt{\pi}}{4} - \frac{\sqrt{\pi}}{2}\right)$$

$$= \hbar^2 (m\omega/\hbar)(1/2) = m\hbar\omega/2$$

6-35. $\psi_0(x) = C_0 e^{-m\omega x^2/2\hbar}$ (Equation 6–58)

(a) $\int_{-\infty}^{+\infty} |\psi_0(x)|^2 dx = 1 = \int_{-\infty}^{+\infty} |C_0|^2 e^{-m\omega x^2/\hbar} dx$

$$= |C_0|^2 \times 2I_0 = |C_0|^2 \times 2 \times \frac{1}{2}\sqrt{\frac{\pi}{x}} \text{ with } \lambda = m\omega/\hbar$$

$$= |C_0|^2 \sqrt{\frac{\pi\hbar}{m\omega}}$$

$$C_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

(b) $\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 |\psi_0|^2 dx = \int_{-\infty}^{+\infty} x^2 \sqrt{\frac{m\omega}{\pi\hbar}} e^{-m\omega x^2/\hbar} dx$

$$= \sqrt{\frac{m\omega}{\pi\hbar}} \times 2I_2 = \sqrt{\frac{m\omega}{\pi\hbar}} \times 2 \times \frac{1}{4}\sqrt{\frac{\pi}{\lambda^3}} \text{ with } \lambda = m\omega/\hbar$$

$$= \frac{1}{2}\sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{\pi\hbar^3}{m^3\omega^3}} = \frac{1}{2}\frac{\hbar}{m\omega}$$

(c) $\langle V(x) \rangle = \langle \frac{1}{2} m\omega^2 x^2 \rangle = \frac{1}{2} m\omega^2 \langle x^2 \rangle = \frac{1}{2} m\omega \times \frac{1}{2}\frac{\hbar}{m\omega} = \frac{1}{4}\hbar\omega$

6-36. $\psi_1(x) = C_1 x e^{-m\omega x^2/2\hbar}$ (Equation 6–58)

$\int_{-\infty}^{+\infty} |\psi_1(x)|^2 dx = 1 = \int_{-\infty}^{+\infty} |C_1|^2 x^2 e^{-m\omega x^2/\hbar} dx = |C_1|^2 \times 2I_2$

(a)

$$= |C_1|^2 \times 2 \times \frac{1}{4}\sqrt{\frac{\pi}{\lambda_3}} \text{ with } \lambda = m\omega/\hbar$$

$$= |C_1|^2 \times \frac{1}{2}\sqrt{\frac{\pi\hbar^3}{m^3\omega^3}}$$

$$C_1 = \left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}$$

(Problem 6-36 continued)

(b) $$\langle x\rangle = \int_{-\infty}^{+\infty} x|\psi_1|^2\,dx = \int_{-\infty}^{+\infty} x^3\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/2} e^{-m\omega x^2/\hbar}\,dx = 0$$

(c) $$\langle x^2\rangle = \int_{-\infty}^{+\infty} x^2|\psi_1|^2\,dx = \int_{-\infty}^{+\infty} x^2\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/2} e^{-m\omega x^2/\hbar}x^2\,dx$$

$$= \left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/2}\times 2I_4 = \left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)\times 2\times\frac{3}{8}\sqrt{\frac{\pi}{\lambda^5}} \text{ where } \lambda = m\omega/\hbar$$

$$= \frac{3}{2}\sqrt{\frac{m^3\omega^3}{\pi\hbar^3}}\sqrt{\frac{\pi\hbar^5}{m^5\omega^5}} = \frac{3}{2}\frac{\hbar}{m\omega}$$

(d) $$\langle V(x)\rangle = \langle\frac{1}{2}m\omega^2x^2\rangle = \frac{1}{2}m\omega^2\langle x^2\rangle = \frac{1}{2}m\omega^2\times\frac{3}{2}\frac{\hbar}{m\omega} = \frac{3}{4}\hbar\omega$$

6-37. (a) $\Delta x\Delta p \approx \hbar \;\rightarrow\; \Delta p \approx \hbar/\Delta x \approx \hbar/2A$

(b) $E_k = p^2/2m \approx (\hbar/2A)^2/2m \approx \hbar^2/8mA^2$

(1) $$E_0 = \frac{1}{2}m\omega^2A^2\left(\frac{\hbar^2}{\hbar^2}\right)\left(\frac{2}{2}\right) = \frac{E_0^2\,2mA^2}{\hbar^2}\left(\frac{4}{4}\right) = \frac{E_0^2}{4E_k}$$

Because $E_0 = \hbar\omega/2$ also $E_0 = 4E_k$

(2) $\partial^2\Psi_0/\partial x^2$ is computed in Problem 6-34(b). Using that quantity,

$$\langle E_k\rangle = -\frac{\hbar^2}{2m}\left(-\frac{m\omega}{2\hbar}\right) = \hbar\omega/4 = 2E_k$$

6-38. $E_n = (n+1/2)\hbar\omega \qquad E_{n+1} = (n+3/2)\hbar\omega$

$E_{n+1} - E_n = \Delta E_n (n + 3/2 - n - 1)\hbar\omega = \hbar\omega$

$\Delta E_n/E_n = \hbar\omega(n+1/2)\hbar\omega = 1/(n+1/2)$

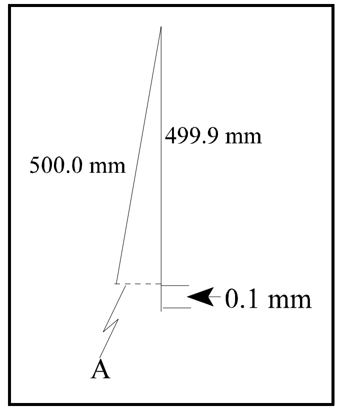
$\lim_{n\to\infty}\dfrac{\Delta E_n}{E_n} = \lim_{n\to\infty}\left(\dfrac{1}{n+1/2}\right) = 0.$ In agreement with the correspondence principle.

6-39. (a) $\omega = 2\pi f = 2\pi/T = 2\pi/1.42s = 4.42\ rad/s$

$$E_0 = \frac{1}{2}\hbar\omega = 1.055\times10^{-34}J\cdot s\,(4.42\ rad/s)/2 = 2.33\times10^{-34}J$$

(b) $A = \sqrt{(500.0)^2 - (499.9)^2} = 10\ mm$

$E = (n + 1/2)\hbar\omega = 1/2 m\omega^2 A^2$



$$n + 1/2 = 1/2(0.010kg)(4.42\ rad/s)(10^{-2}m)^2/1.055\times10^{-34}J\cdot s$$

$$= 2.1\times10^{28} \text{ or } n = 2.1\times10^{28}$$

(c) $f = \omega/2\pi = 0.70\ Hz$

6-40. $\psi_0(x) = A_0 e^{-\omega x^2/2\hbar}$

$$\psi_1(x) = A_1\sqrt{\frac{m\omega}{\hbar}}\,x e^{-m\omega x^2/2\hbar}$$

From Equation 6-58.

Note that ψ_0 is an even function of *x* and ψ_1 is an odd function of *x*. It follows that

$$\int_{-\infty}^{+\infty}\psi_0\psi_1 dx = 0$$

6-41. (a) For $x>0$, $\hbar^2k_2^2/2m + V_0 = E = \hbar^2k_1^2/2m = 2V_0$

So, $k_2 = (2mV_0)^{1/2}/\hbar$. Because $k_1 = (4mV_0)^{1/2}/\hbar$, then $k_2 = k_1/\sqrt{2}$.

(b) $R = (k_1 - k_2)^2/(k_1 + k_2)^2$ (Equation 6-68)

$= (1 - 1/\sqrt{2})^2/(1 + 1/\sqrt{2})^2 = 0.0294$, or 2.94% of the incident particles are reflected.

(Problem 6-41 continued)

(c) $T = 1 - R = 1 - 0.0294 = 0.971$

(d) 97.1% of the particles, or $0.971 \times 10^6 = 9.71 \times 10^5$, continue past the step in the $+x$ direction. Classically, 100% would continue on.

6-42. (a) For $x > 0$, $\hbar^2 k_2^2/2m - V_0 = E = \hbar^2 k_1^2/2m = 2V_0$

So, $k_2 = (6mV_0)^{1/2}/\hbar$. Because $k_1 = (4mV_0)^{1/2}/\hbar$, then $k_2 = \sqrt{3/2}\,k_1$.

(b) $R = (k_1 - k_2)^2/(k_1 + k_2)^2 = (1 - \sqrt{3/2})^2/(1 + \sqrt{3/2})^2 = 0.0102$. Or 1.02% are reflected at $x = 0$.

(c) $T = 1 - R = 0.99$

(d) 99% of the particles, or $0.99 \times 10^6 = 9.9 \times 10^5$, continue in the $+x$ direction. Classically, 100% would continue on.

6-43. $R = \dfrac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$ (Equation 6-68) $\qquad T = \dfrac{4k_1k_2}{(k_1 + k_2)^2}$ (Equation 6-69)

$$T + R = \frac{4k_1 + k_2}{(k_1 + k_2)^2} + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} = \frac{4k_1 + k_1^2 - 2k_1k_2 + k_2^2}{(k_1 + k_2)} = \frac{k_1^2 + 2k_1k_2 + k_2^2}{(k_1 + k_2)^2}$$

$$T + R = \frac{(k_1 + k_2)^2}{(k_1 + k_2)^2} = 1 \quad \text{(Equation 6-70)}$$

6-44. $B = \dfrac{E^{1/2} - (E - V_0)^{1/2}}{E^{1/2} + (E - V_0)^{1/2}} A$ For $E < V_0$, $(E - V_0)^{1/2}$ is imaginary and the numerator and denominator are complex conjugates. Thus, $|B|^2 = |A|^2$ and therefore $R = |B|^2/|A|^2 = 1$, hence $T = 1 - R = 0$

6-45. $A + B = C$ and $k_1 A - k_1 B = k_2 C$ (Equation 6 – 65 a & b)

Substituting for C, $k_1 A - k_1 B = k_2(A + B) = k_2 A + k_2 B$ and solving for B,

$B = \dfrac{k_1 - k_2}{k_1 + k_2} A$, which is Equation 6-66. Substituting this value of B into Equation 6-65(a),

$$A + \frac{k_1 - k_2}{k_1 + k_2} A = C = A\left[\frac{k_1 + k_2 + k_1 - k_2}{k_1 + k_2}\right] \quad \text{or} \quad C = \frac{2k_1}{k_1 + k_2}$$, which is Equation 6-67.

6-46. Using Equation 6-76, $T \approx 16\dfrac{E}{V_0}\left(1 - \dfrac{E}{V_0}\right)e^{-2\alpha a}$ where $E = 2.0 eV$, $V_0 = 6.5\ eV$, and

$a = 0.5$ nm. $T \approx 16\left(\dfrac{2.0}{6.5}\right)\left(1 - \dfrac{2.0}{6.5}\right)e^{-2(10.87)(0.5)} \approx 6.5 \times 10^{-5}$

(Equation 6-75 yields $T = 6.6 \times 10^{-5}$.)

6-47. $R = \dfrac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$ and $T = 1 - R$ (Equations 6 – 68 and 6 – 70)

(a) For protons:

$$k_1 = \sqrt{2mc^2 E}/\hbar c = \sqrt{2(938\,MeV)(40\,MeV)}/197.3\ MeV{\cdot}fm = 1.388$$

$$k_2 = \sqrt{2mc^2(E - V_0)}/\hbar c = \sqrt{2(938\,MeV)(10\,MeV)}/197.3\ MeV{\cdot}fm = 0.694$$

$$R = \left(\frac{1.388 - 0.694}{1.388 + 0.694}\right)^2 = \left(\frac{0.694}{2.082}\right)^2 = 0.111 \quad \text{And } T = 1 - R = 0.889$$

(b) For electrons:

$$k_1 = 1.388\left(\frac{0.511}{938}\right)^{1/2} = 0.0324 \qquad k_2 = 0.694\left(\frac{0.511}{938}\right)^{1/2} = 0.0162$$

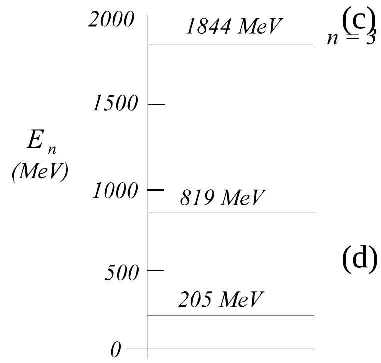
$$R = \left(\frac{0.0324 - 0.0162}{0.0324 + 0.0162}\right)^2 = 0.111 \qquad \text{And } T = 1 - R = 0.889$$

No, the mass of the particle is not a factor. (We might have noticed that $\sqrt{m}$ could be canceled from each term.)

6-48. (a) $E_n = \dfrac{n^2h^2}{8mL^2}$ The ground state is n = 1, so

$$E_1 = \frac{(hc)^2}{8(mc^2)L^2} = \frac{(1240\,MeV\cdot fm)^2}{8(938.3\,MeV)(1fm)^2} = 204.8\,MeV$$

(b)



(c) $\Delta E_{21} = hc/\lambda_{21}$

$$\lambda_{21} = \frac{1240\,MeV\cdot fm}{(819-205)\,MeV} = 2.02\,fm$$

(d) $$\lambda_{32} = \frac{1240\,MeV\cdot fm}{(1844-819)\,MeV} = 1.21\,fm$$

(e) $$\lambda_{31} = \frac{1240\,MeV\cdot fm}{(1844-205)\,MeV} = 0.73\,fm$$

6-49. (a) The probability density for the ground state is $P(x) = \psi^2(x) = (2/L)\sin^2\pi x/L$. The probability of finding the particle in the range $0 < x < L/2$ is :

$$P = \int_0^{L/2} P(x)dx = \frac{2}{L}\frac{L}{\pi}\int_0^{\pi/2}\sin^2 u\,du = \frac{2}{\pi}\left(\frac{\pi}{4} - 0\right) = \frac{1}{2} \quad \text{where } u = \pi x/L$$

(b) $$P = \int_0^{L/3} P(x)dx = \frac{2}{L}\frac{L}{\pi}\int_0^{\pi/3}\sin^2 u\,du = \frac{2}{\pi}\left(\frac{\pi}{6} - \frac{\sin 2\pi/3}{4}\right) = \frac{1}{3} - \frac{\sqrt{3}}{4\pi} = 0.195$$

(Note 1/3 is the classical result.)

(c) $$P = \int_0^{3L/4} P(x)dx = \frac{2}{L}\frac{L}{\pi}\int_0^{3\pi/4}\sin^2 u\,du = \frac{2}{\pi}\left(\frac{3\pi}{8} - \frac{\sin 3\pi/2}{4}\right) = \frac{3}{4} + \frac{1}{2\pi} = 0.909$$

(Note 3/4 is the classical result.)

6-50. (a) $E_n = \frac{n^2h^2}{8mL^2}$ and $E_{n+1} = \frac{(n+1)^2h^2}{8mL^2}$

So, $\frac{E_{n+1}-E_n}{E_n} = \frac{n^2+2n-1-n^2}{n^2} = \frac{2n-1}{n^2} = \frac{2-1/n}{n}$ For large n, $1/n << 2$ and

$$\frac{E_{n+1}-E_n}{E_n} \approx \frac{2}{n}$$

(b) For n = 1000 the fractional energy difference is $\frac{2}{1000} = 0.002 = 0.2\%$

(c) It means that the energy difference between adjacent levels per unit energy for large n is getting smaller, as the correspondence principle requires.

6-51. (a) $\Psi(x,t) = \psi(x)f(t)$

$$\frac{\partial^2\Psi}{\partial x^2} = \frac{d^2\psi(x)}{dx^2}f(t) = \psi''(x)f(t) \quad \frac{\partial\Psi}{\partial t} = \psi(x)\frac{df(t)}{dt} = \psi(x)f'(t)$$

Substituting above in Equation 6-6,

$$-\frac{\hbar^2}{2m}\psi''(x)f(t) + V(x)\psi(x)f(t) = i\hbar\frac{f'(t)}{f(t)}$$

Dividing by $\psi(x)f(t)$, $-\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} + V(x) = i\hbar\frac{f'(t)}{f(t)}$

(b) Set both sides equal to C.

$$i\hbar\frac{f'(t)}{f(t)} = C \quad \rightarrow \quad \frac{df(t)}{f(t)} = \frac{C\,dt}{i\hbar} = d(\ln f)$$

$$\ln f - \ln f_0 = \frac{Ct}{i\hbar} = -\frac{iCt}{\hbar} \quad \therefore \quad f = f_0e^{-iCt/\hbar} = f_0e^{-i\omega t} \text{ with } \omega = C/\hbar$$

and $C = \hbar\omega = E$

(c) $-\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} + V(x) = C = E \qquad -\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\,\psi(x) = E\psi(x)$

6-52. (a) The ball's minimum speed occurs in the ground state where $\lambda = 2L$ and $p = h/\lambda = mv$.

$$v = \frac{h}{2mL} = \frac{6.63\times10^{-24}\,J\cdot s}{2(2\times10^{-3}\,kg)(0.001\,cm)(10^{-2}\,m/cm)} = 1.66\times10^{-26}\,m/s$$

(b) The period T, the time required for the ball to make one round trip across the box is:

$T = 2(0.001\,cm\times10^{-2}\,m/cm)/1.66\times10^{-26}\,m/s = 1.20\times10^{21}\,s$

(This is about 1000 times the age of the universe.)

6-53. (a) The requirement is that $\psi^2(x) = \psi^2(-x) = \psi(-x)\psi(-x)$. This can be true only if:

$\psi(-x) = \psi(x)$ or $\psi(-x) = -\psi(x)$.

(b) Writing the Schrödinger equation in the form $\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$, the general solutions of this 2nd order differential equation are:

$\psi(x) = A\sin kx$ and $\psi(x) = A\cos kx$ where $k = \sqrt{2mE}/\hbar$. Because the boundaries of the box are at $x = \pm L/2$, both solutions are allowed (unlike the treatment in the text where one boundary was at $x = 0$). Still, the solutions are all zero at $x = \pm L/2$ provided that an integral number of half wavelengths fit between $x = -L/2$ and $x = +L/2$. This will occur for: $\psi_n(x) = (2/L)^{1/2}\cos n\pi x/L$ when $n = 1, 3, 5, \cdots$

And for $\psi_n(x) = (2/L)^{1/2}\sin n\pi x/L$ when $n = 2, 4, 6, \cdots$.

The solutions are alternately even and odd.

(c) The allowed energies are: $E = \hbar^2k^2/2m = \hbar^2(n\pi L)^2/2m = n^2h^2/8mL^2$.

6-54. $\psi_0(x) = Ae^{-x^2/2L^2}$

(a) $\frac{d\psi_0}{dx} = (-x/L^2)Ae^{-x^2/2L^2}$ and $\psi_1 = L\frac{d\psi_0}{dx} = (-x/L^2)Ae^{-x^2/2L^2} = (-x/L)\psi_0$

So, $\frac{d\psi_1}{dx} = -(1/L)\psi_0 - (x/L)d\psi_0/dx$

And
$$\frac{d^2\psi_1}{dx^2} = -(1/L)d\psi_0/dx - (1/L)d\psi_0/dx - (x/L)d^2\psi_0/dx^2$$

$$= (2x/L^3)\psi_0 + (x/L^3)\psi_0 + (x^3/L^5)\psi_0$$

(Problem 6-54 continued)

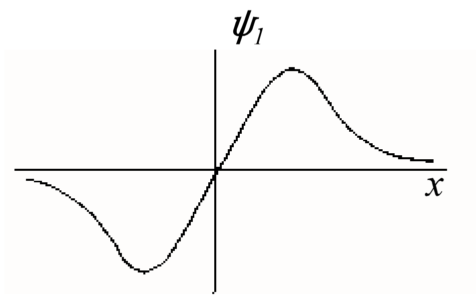
Recalling from Problem 6-3 that $V(x) = \hbar^2 x^2/2mL^4$, the Schrödinger equation becomes

$(-\hbar^2/2m)(3x/L^3 + x^3/L^5)\psi_0 + (\hbar^2 x^3/2mL^5)\psi_0 = E(-x/L)\psi_0$

or, simplifying: $(-3\hbar^2 x/2mL^3)\psi_0 = E(-x/L)\psi_0$. Thus, choosing E appropriately will make ψ_1 a solution.

(b) We see from (a) that $E = 3\hbar^2/2mL^2$, or three times the ground state energy.

(c) ψ_1 plotted looks as below. The single node indicates that ψ_1 is the first excited state. (The energy value in [b] would also tell us that.)



6-55. $\langle x^2 \rangle = \int_0^L \frac{2}{L} x^2 \sin^2 n\pi x/L\, dx$ Letting $u = n\pi x/L$, $du = (n\pi/L)dx$

$$\langle x^2 \rangle = \frac{2}{L}\left(\frac{L}{n\pi}\right)^2\left(\frac{L}{n\pi}\right)\int_0^{n\pi} u^2 \sin^2 u\, du$$

$$= \frac{2}{L}\left(\frac{L}{n\pi}\right)^3\left[\frac{u^3}{6} - \left(\frac{u^2}{4} - \frac{1}{8}\right)\sin 2u - \frac{u\cos 2u}{4}\right]_0^{n\pi}$$

$$= \frac{2}{L}\left(\frac{L}{n\pi}\right)^3\left[\frac{(n\pi)^3}{6} - 0 - \frac{n\pi}{4} - 0\right] = \frac{L^2}{3} - \frac{L^2}{2n^2\pi^2}$$

6-56. $T \approx 16\dfrac{E}{V_0}\left(1-\dfrac{E}{V_0}\right)e^{-\alpha a}$ where $E = 10\,eV$, $V_0 = 25\,eV$, and $a = 1\,nm$.

(a) $\alpha = \sqrt{2m(v_0 - E)}/\hbar = \sqrt{2(m_e c^2)(V_0 - E)}/(\hbar c)$

$= \sqrt{2(0.511\times 10^6 eV)(15\,eV)}/197.3\; eV\cdot nm = 19.84\; nm^{-1}$

And $\alpha a = (19.84\,nm^{-1})(1\,nm) = 19.84$

$T \approx 16\left(\dfrac{10}{25}\right)\left(1-\dfrac{10}{25}\right)e^{-19.84} \approx 9.2\times 10^{-9}$

(b) For a = 0.1 nm: $\alpha a = (19.84\,nm^{-1})(0.1\,nm) = 1.984$

$T \approx 16\left(\dfrac{10}{25}\right)\left(1-\dfrac{10}{25}\right)e^{-1.984} \approx 0.528$

6-57. (a) For $\Psi(x,t) = A\sin(kx-\omega t)$

$\dfrac{d^2\Psi}{dx^2} = -k^2\Psi$ and $\dfrac{\partial\Psi}{\partial t} = -\omega A\cos(kx-\omega t)$ so the Schrödinger equation becomes:

$$-\frac{\hbar^2 k^2}{2m}A\sin(kx-\omega t) + V(x)A\sin(kx-\omega t) = -i\hbar\omega\cos(kx-\omega t)$$

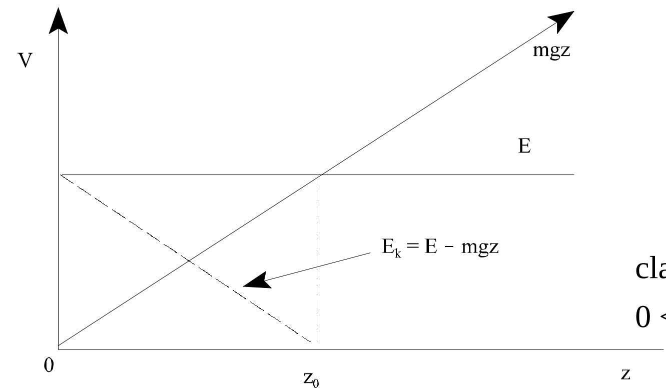
Because the *sin* and *cos* are not proportional, this Ψ cannot be a solution. Similarly, for $\Psi(x,t) = A\cos(kx-\omega t)$, there are no solutions.

(b) For $\Psi(x,t) = A[\cos(kx-\omega t) + i\sin(kx-\omega t)] = Ae^{i(kx-\omega t}$, we have that

$\dfrac{d^2\Psi}{dx^2} = -k^2\Psi$ and $\dfrac{\partial\Psi}{\partial t} = -i\omega\Psi$. And the Schrödinger equation becomes:

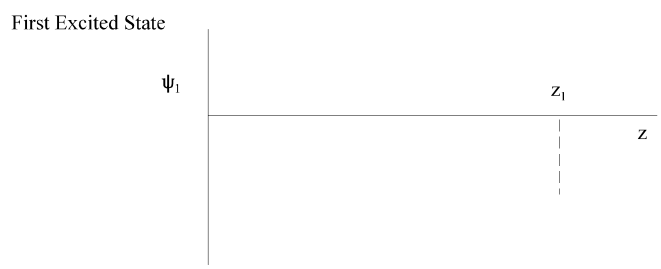
$$-\frac{\hbar^2 k^2}{2m}\Psi + V(x)\Psi = -\hbar\omega\Psi \text{ for } \hbar\omega = \hbar^2 k^2/2m + V.$$

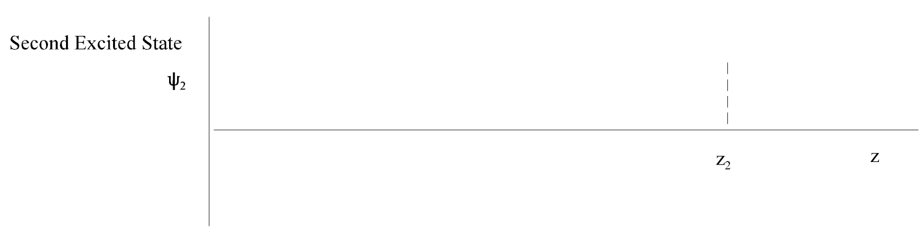
6-58.



classically allowed:
$0 < z < z_0$

The wave function will be concaved toward the *z* axis in the classically allowed region and away from the *z* axis elsewhere. Each wave function vanishes at z = 0 and as z → 0. The smaller amplitude in the regions where the kinetic energy is larger.





6-59. Writing the Schrödinger equation as: $E_k\psi(x) + V(x)\psi(x) = E\psi(x)$ from which we have:

$E_k\psi(x) = [E - V(x)]\psi(x) = (-\hbar^2/2m)(d^2\psi/dx^2)$. The expectation value of E_k is

$\langle E_k\rangle = \int_{-\infty}^{+\infty} E_k\psi(x)\psi(x)dx$ Substituting $E_k\psi(x)$ from above and reordering multiplied quantities

gives: $\langle E_k\rangle = \int_{-\infty}^{+\infty} \psi(x)\left(-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}\right)\psi(x)dx$

6-60. (a) $\Delta p\Delta x \approx \hbar \rightarrow m\Delta v\Delta x \approx \hbar$

$\Delta v \approx \hbar/m\Delta a = (1.055\times10^{-34}J\cdot s)/(9.11\times10^{-31})(10^{-12}m)$

$\Delta v \approx 1.16\times10^{8}m/s = 0.39c$

(b) The width of the well L is still an integer number of half wavelengths, $L = n(\lambda/2)$, and deBroglie's relations still gives: $L = nh/2p$. However, p is *not* given by: $p = \sqrt{2mE_k}$, but by the relativistic expression: $p = [E^2 - (mc^2)^2]^{1/2}/c$. Substituting this yields:

$$L = \frac{nhc}{2[E^2 - (mc^2)^2]^{1/2}} \rightarrow E^2 - (mc^2)^2 = (nhc/2L)^2$$

$$E_n = \left[\left(\frac{(hc)^2}{2L}\right)^2 + (mc^2)^2\right]^{1/2}$$

(c) $E_1 = \left[\left(\frac{(hc)^2}{4L^2}\right)^2 + (mc^2)^2\right]^{1/2} = \left[\frac{(1240eV\cdot nm)^2}{4(10^{-3}nm)^2} + (0.511\times10^6 eV)^2\right]^{1/2} = 8.03\times10^5 eV$

(d) Nonrelativistic: $E_1 = \frac{h^2}{8mL^2} = \frac{(hc)^2}{8(mc^2)L^2} = \frac{(1240eV\cdot nm)^2}{8(0.511\times10^6 eV)(10^{-3}nm)^2} = 3.76\times10^5 eV$

E_1 computed in (c) is 2.14 times the nonrelativistic value.

6-61. (a) Applying the boundary conditions of continuity to ψ and $d\psi/dx$ at $x = 0$ and $x = a$, where the various wave functions are given by Equations 6-74, results in the two pairs of equations below:

(Problem 6-61 continued)

At $x = 0$: $A + B = C + D$ and $ikA - ikB = -\alpha C + \alpha D$

At $x = a$: $Fe^{ika} = Ce^{-\alpha a} + De^{\alpha a}$ and $ikFe^{ika} = -\alpha Ce^{-\alpha a} + \alpha De^{\alpha a}$

Eliminating the coefficients C and D from these four equations, a straightforward but lengthy task, yields:

* $4ik\alpha A = [(\alpha + ik)^2 e^{-\alpha a} - (\alpha - ik)^2 e^{\alpha a}] Fe^{ika}$

The transmission coefficient T is then:

$$T = \frac{|F|^2}{|A|^2} = \left\{ \frac{4ik\alpha}{e^{ika}[(\alpha + ik)^2 e^{-\alpha a} - (\alpha - ik)^2 e^{\alpha a}]} \right\}^2$$

Recalling that $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$ and noting that $(\alpha + ik)$ and $(\alpha - ik)$ are complex conjugates, substituting $k = \sqrt{2mE}/\hbar$ and $\alpha = \sqrt{2m(V_0 - E)}/\hbar$, T then can be written as

$$T = \left[1 + \frac{\sinh^2 \alpha a}{4\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)} \right]^{-1}$$

(b) If $\alpha a >> 1$, then the first term in the bracket on the right side of the * equation in part (a) is much smaller than the second and we can write:

$$\frac{F}{A} \approx \frac{4ik\alpha e^{-(\alpha + ik)a}}{(\alpha - ik)^2} \qquad \text{And } T = \left|\frac{F}{A}\right|^2 \approx \frac{16\alpha^2 k^2 e^{-2\alpha a}}{(\alpha^2 + k^2)^2}$$

Or $T \approx 16\left(\frac{E}{V_0}\right)\left(1 - \frac{E}{V_0}\right) e^{-2\alpha a}$

6-62. $|\psi_{II}|^2 = |C|^2 e^{-2\alpha a}$ (Equation 6-72)

Where $|C|^2 = \left|\frac{2E^{1/2}}{E^{1/2} + (E - V_0)^{1/2}}\right|^2 |A|^2 = \left|\frac{2(0.5V_0)^{1/2}}{(0.5V_0)^{1/2} + (-0.5V_0)^{1/2}}\right|^2 = 2.000$

$$\alpha = \sqrt{2m(v_0 - E)}/\hbar = \sqrt{2(m_p c^2)(20MeV)}/\hbar c$$

$$= \sqrt{2(938.3MeV)(20MeV)}/197.3Mev\cdot fm = 0.982fm^{-1}$$

(Problem 6-62 continued)

x (fm)	$e^{-2\alpha x}$	$\|\psi_{II}\|^2 = \|C\|^2 e^{-2\alpha x}$
1	0.1403	0.5612
2	0.0197	0.0788
3	2.76×10^{-3}	1.10×10^{-2}
4	3.87×10^{-4}	1.55×10^{-3}
5	5.4×10^{-5}	2.2×10^{-4}