Chapter 10

*1. a) For each state the energy is given by $E_{\text{rot}} = \hbar^2 l(l+1)/2I$, so the transition energy is

$$\Delta E = \frac{\hbar^2}{2I}\left(2(3) - 1(2)\right) = \frac{2\hbar^2}{I} = \frac{2\left(1.055 \times 10^{-34}\ \text{J}\cdot\text{s}\right)^2}{10^{-46}\ \text{kg}\cdot\text{m}^2} = 2.2 \times 10^{-22}\ \text{J}\ = 1.4 \times 10^{-3}\ \text{eV}$$

b) As in part (a)

$$\Delta E = \frac{\hbar^2}{2I}\left(20(21) - 19(20)\right) = \frac{20\hbar^2}{I} = 2.2 \times 10^{-21}\ \text{J}\ = 1.4 \times 10^{-2}\ \text{eV}$$

This is still in the infrared part of the spectrum.

2. From Table 10.1 $\kappa = 1860$ N/m and $\nu = 6.42 \times 10^{13}$ Hz.

a)

$$\Delta E = h\nu = \left(4.136 \times 10^{-15}\ \text{eV}\cdot\text{s}\right)\left(6.42 \times 10^{13}\ \text{Hz}\right) = 0.266\ \text{eV}$$

b) Set $\Delta E = kT$ with two degrees of freedom in the vibrational mode. Then

$$T = \frac{\Delta E}{k} = \frac{0.266\ \text{eV}}{8.617 \times 10^{-5}\ \text{eV/K}} = 3090\ \text{K}$$

3. In the ground state $E = \left(n + \frac{1}{2}\right)\hbar\omega = \frac{1}{2}\hbar\omega = \frac{1}{2}kA^2 = \frac{1}{2}\mu\omega^2 A^2$. Solving for A:

$$A = \sqrt{\frac{\hbar}{\mu\omega}}$$

Using the ^{35}Cl isotope

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{35}{36}\ \text{u}\ = 1.614 \times 10^{-27}\ \text{kg}$$

From Table 10.1 $\nu = 8.66 \times 10^{13}$ Hz. With $\omega = 2\pi v$ we have

$$A = \sqrt{\frac{\hbar}{\mu\omega}} = \sqrt{\frac{1.055 \times 10^{-34}\ \text{J}\cdot\text{s}}{\left(1.614 \times 10^{-27}\ \text{kg}\right)(2\pi)\left(8.66 \times 10^{13}\ \text{s}^{-1}\right)}} = 1.10 \times 10^{-11}\ \text{m}$$

4. a) Using the result of Problem 8 for the rotational inertia we have

$$I = \mu R^2 = \left(1.614 \times 10^{-27}\ \text{kg}\right)\left(1.28 \times 10^{-10}\ \text{m}\right)^2 = 2.64 \times 10^{-47}\ \text{kg}\cdot\text{m}^2$$

For $l = 1$ we have

$$E_{\text{rot}} = \frac{l(l+1)\hbar^2}{2I} = \frac{\hbar^2}{I} = \frac{1}{2}I\omega^2$$

Therefore

$$\omega = \sqrt{\frac{2\hbar^2}{I^2}} = \sqrt{\frac{2\left(1.055 \times 10^{-34}\ \mathrm{J \cdot s}\right)^2}{\left(2.64 \times 10^{-47}\ \mathrm{kg \cdot m^2}\right)^2}} = 5.65 \times 10^{12}\ \mathrm{rad/s}$$

For $l = 10$ we have

$$E_{\mathrm{rot}} = \frac{l(l+1)\hbar^2}{2I} = \frac{55\hbar^2}{I} = \frac{1}{2}I\omega^2$$

Therefore

$$\omega = \sqrt{\frac{110\hbar^2}{I^2}} = \sqrt{\frac{110\left(1.055 \times 10^{-34}\ \mathrm{J \cdot s}\right)^2}{\left(2.64 \times 10^{-47}\ \mathrm{kg \cdot m^2}\right)^2}} = 4.19 \times 10^{13}\ \mathrm{rad/s}$$

b) The distance of the center of mass from the H atom is

$$x_{\mathrm{cm}} = \frac{m_{\mathrm{Cl}}x_{\mathrm{Cl}}}{M} = \frac{35}{36}R = 1.24 \times 10^{-10}\ \mathrm{m}$$

Then from rotational kinematics we have for $l = 1$

$$v = \omega x_{\mathrm{cm}} = \left(5.65 \times 10^{12}\ \mathrm{rad/s}\right)\left(1.24 \times 10^{-10}\ \mathrm{m}\right) = 700\ \mathrm{m/s}$$

and for $l = 10$:

$$v = \omega x_{\mathrm{cm}} = \left(4.19 \times 10^{13}\ \mathrm{rad/s}\right)\left(1.24 \times 10^{-10}\ \mathrm{m}\right) = 5200\ \mathrm{m/s}$$

c)

$$\omega = \frac{v}{x_{\mathrm{cm}}} = \frac{0.1c}{x_{\mathrm{cm}}} = \frac{2.998 \times 10^{7}\ \mathrm{m/s}}{1.24 \times 10^{-10}\ \mathrm{m}} = 2.42 \times 10^{17}\ \mathrm{rad/s}$$

Then

$$\omega = \sqrt{\frac{l(l+1)\hbar^2}{I^2}}$$

$$l(l+1) = \frac{\omega^2 I^2}{\hbar^2} = \left(\frac{\left(2.42 \times 10^{17}\ \mathrm{rad/s}\right)\left(2.64 \times 10^{-47}\ \mathrm{kg \cdot m^2}\right)}{1.055 \times 10^{-34}\ \mathrm{J \cdot s}}\right)^2 = 3.67 \times 10^{9}$$

from which it follows that $l \cong 6.1 \times 10^4$.

d) $E_{\mathrm{rot}} = kT = \hbar^2 l(l+1)/2I$. Thus

$$T = \frac{\hbar^2 l\,(l+1)}{2Ik} = \frac{\left(1.055 \times 10^{-34}\ \mathrm{J \cdot s}\right)^2\left(3.67 \times 10^{9}\right)}{2\left(2.64 \times 10^{-47}\ \mathrm{kg \cdot m^2}\right)\left(1.38 \times 10^{-23}\ \mathrm{J/K}\right)} = 5.6 \times 10^{10}\ \mathrm{K}$$

*5. With Bohr's condition $L = n\hbar$ we find

$$E_{\mathrm{rot}} = \frac{L^2}{2I} = \frac{n^2\hbar^2}{2I}$$

The Bohr version and the correct version become similar for large values of quantum number n or l, but they are quite different for small l.

*6. $\Delta E = E_1 - E_0 = E_1 = \hbar^2/I = hc/\lambda$.

$$I = \frac{\hbar^2 \lambda}{hc} = \frac{\hbar\lambda}{2\pi c} = \frac{(1.055 \times 10^{-34}\ \mathrm{J\cdot s})(1.3 \times 10^{-3}\ \mathrm{m})}{2\pi(2.998 \times 10^{8}\ \mathrm{m/s})} = 7.28 \times 10^{-47}\ \mathrm{kg\cdot m^2}$$

b) The minimum energy in a vibrational transition is $\Delta E = h\nu$. From Table 10.1 $\nu = 6.42 \times 10^{13}$ Hz, which corresponds to a photon of wavelength $\lambda = c/\nu = 4.67\mu$m. A photon of this wavelength or less is required to excite the vibrational mode, so the 1.30 mm photon is too weak.

7.

$$\Delta E = h\nu = \left(6.626 \times 10^{-34}\ \mathrm{J\cdot s}\right)\left(6.42 \times 10^{13}\ \mathrm{Hz}\right) = 4.25 \times 10^{-20}\ \mathrm{J}$$

Using this energy for a rotational transition from the ground state to the lth state we have

$$\Delta E = \frac{\hbar^2 l\,(l+1)}{2I}$$

so

$$l\,(l+1) = \frac{2I\Delta E}{\hbar^2} = \frac{2\,(7.28 \times 10^{-47}\ \mathrm{kg\cdot m^2})(4.25 \times 10^{-20}\ \mathrm{J})}{(1.055 \times 10^{-34}\ \mathrm{J\cdot s})^2} = 556$$

so $l \cong 24$. This is prohibited by the $\Delta l = \pm 1$ selection rule.

8. Combining $R = r_1 + r_2$ with $m_1 r_1 = m_2 r_2$ we find that

$$r_1 = \frac{m_2}{m_1 + m_2} R$$

and

$$r_2 = \frac{m_1}{m_1 + m_2} R$$

Therefore

$$\begin{aligned} I &= m_1 r_1^2 + m_2 r_2^2 = m_1 \left(\frac{m_2}{m_1+m_2} R\right)^2 + m_2 \left(\frac{m_1}{m_1+m_2} R\right)^2 \\ &= \frac{m_1 m_2 (m_1 + m_2)}{(m_1+m_2)^2} R^2 = \mu R^2 \end{aligned}$$

9. a)

$$\Delta E = \frac{\hbar^2}{2I}\left(3\,(4) - 2\,(3)\right) = \frac{3\hbar^2}{I}$$

$$I = \frac{3\hbar^2}{\Delta E} = \frac{3\,(1.055 \times 10^{-34}\ \mathrm{J\cdot s})^2}{(1.43 \times 10^{-3}\ \mathrm{eV})(1.602 \times 10^{-19}\ \mathrm{J/eV})} = 1.46 \times 10^{-46}\ \mathrm{kg\cdot m^2}$$

b)

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{(12)(16)}{28}\mathrm{u} = 1.139 \times 10^{-26}\ \mathrm{kg}$$

Then $I = \mu R^2$, so

$$R = \sqrt{\frac{I}{\mu}} = \sqrt{\frac{1.46 \times 10^{-46}\ \text{kg} \cdot \text{m}^2}{1.139 \times 10^{-26}\ \text{kg}}} = 1.13 \times 10^{-10}\ \text{m}$$

which is a reasonable answer.

*10. a) The distance of each H atom from the line is $d = (0.0958\ \text{nm})\,(\sin 52.5°) = 7.60 \times 10^{-2}$ nm. Then

$$I = 2m_H d^2 = 2\left(1.67 \times 10^{-27}\ \text{kg}\right)\left(7.60 \times 10^{-11}\ \text{m}\right)^2 = 1.93 \times 10^{-47}\ \text{kg} \cdot \text{m}^2$$

b)

$$E_1 = \frac{\hbar^2}{I} = \frac{(1.055 \times 10^{-34}\ \text{J} \cdot \text{s})^2}{1.93 \times 10^{-47}\ \text{kg} \cdot \text{m}^2} = 5.77 \times 10^{-22}\ \text{J}\ = 3.61\ \text{meV}$$

$$E_2 = \frac{3\hbar^2}{I} = \frac{3\,(1.055 \times 10^{-34}\ \text{J} \cdot \text{s})^2}{1.93 \times 10^{-47}\ \text{kg} \cdot \text{m}^2} = 1.73 \times 10^{-21}\ \text{J}\ = 10.81\ \text{meV}$$

c)

$$\lambda = \frac{hc}{E_1} = \frac{(6.626 \times 10^{-34}\ \text{J} \cdot \text{s})\,(2.998 \times 10^8\ \text{m/s})}{5.77 \times 10^{-22}\ \text{J}} = 344\,\mu\text{m}$$

11. First we need to compute the rotational inertia. Including both the nucleus and electrons we have

$$\begin{aligned} I &= \frac{2}{5} m_\alpha r_{\text{nuc}}^2 + \frac{2}{5}\,(2m_e)\,a_0^2 \\ &= \frac{2}{5}\left(\left(6.64 \times 10^{-27}\ \text{kg}\right)\left(1.9 \times 10^{-15}\ \text{m}\right)^2 + 2\left(9.109 \times 10^{-31}\ \text{kg}\right)\left(5.29 \times 10^{-11}\ \text{m}\right)^2\right) \\ &= 2.04 \times 10^{-51}\ \text{kg} \cdot \text{m}^2 \end{aligned}$$

Notice that the nuclear contribution was negligible.

a)

$$E_{\text{rot}} = \frac{\hbar^2}{I} = \frac{(1.055 \times 10^{-34}\ \text{J} \cdot \text{s})^2}{2.04 \times 10^{-51}\ \text{kg} \cdot \text{m}^2} = 5.46 \times 10^{-18}\ \text{J}\ = 34.1\ \text{eV}$$

b) This is greater than the ionization energy for helium, and therefore it is not likely to be observed.

12.

$$\nu' = \nu \pm \frac{\hbar}{2\pi I}\,(2l + 3)$$

with $\nu = c/\lambda = 1.00 \times 10^{13}$ Hz. Thus

$$\nu' = 1.00 \times 10^{13}\ \text{Hz}\ \pm \frac{1.055 \times 10^{-34}\ \text{J} \cdot \text{s}}{2\pi\,(1.46 \times 10^{-45}\ \text{kg} \cdot \text{m}^2)}\,(2l + 3)$$

or

$$\nu' = 1.00 \times 10^{13}\ \text{Hz}\ \pm \left(\left(2.30 \times 10^{10}\ \text{Hz}\right) l + 3.45 \times 10^{10}\ \text{Hz}\right)$$

where l is an integer and then as usual $\lambda = c/\nu'$.

13.

$$\Delta E = h\nu = \frac{\hbar^2}{2I}\left[(l+1)(l+2) - l(l+1)\right]$$

Therefore we can say that for a particular transition $\nu = C/\mu$ where C is a constant, because $I = \mu R^2$. Then $d\nu/d\mu = -C/\mu^2$, or, taking absolute values, $d\nu/d\mu = C/\mu^2 = \nu/\mu$. Thus $\Delta\nu/\nu = \Delta\mu/\mu$ as required.

14. a) The Maxwell-Boltzmann factor is $F_{MB} = A\exp(-E/kT)$. The energies of the three rotational levels are

$$E_0 = 0 \qquad E_1 = \frac{\hbar^2}{I} \qquad E_2 = \frac{3\hbar^2}{I}$$

Thus the Maxwell-Boltzmann factors are

$$l = 0: \qquad F_{MB} = A$$

$l = 1$:

$$F_{MB} = A\exp\left(-\frac{\hbar^2}{IkT}\right) = A\exp\left(-\frac{(1.055\times10^{-34}\ \text{J}\cdot\text{s})^2}{(10^{-46}\ \text{kg}\cdot\text{m}^2)(1.381\times10^{-23}\ \text{J/K})(293\ \text{K})}\right) = 0.973A$$

$l = 2$:

$$F_{MB} = A\exp\left(-\frac{3\hbar^2}{IkT}\right) = A\exp\left(-\frac{3(1.055\times10^{-34}\ \text{J}\cdot\text{s})^2}{(10^{-46}\ \text{kg}\cdot\text{m}^2)(1.381\times10^{-23}\ \text{J/K})(293\ \text{K})}\right) = 0.921A$$

b) The degeneracy factor $g(E)$ is 1 for $l = 0$, 3 for $l = 1$, and 5 for $l = 2$. Therefore the level populations $n(E)$ are

$$l = 0: \qquad n(E) = g(E)F_{MB} = A$$
$$l = 1: \qquad n(E) = g(E)F_{MB} = 3(0.973A) = 2.92A$$
$$l = 2: \qquad n(E) = g(E)F_{MB} = 5(0.921A) = 4.61A$$

c) For the lower rotational states the degeneracy factor causes the state populations to increase with increasing l. However, as l increases and the rotational energy increases, the exponential factor begins to take over and decrease the state populations.

*15. The gap between adjacent lines is $h(\Delta\nu) = \hbar^2/I$.

a)

$$I = \frac{\hbar^2}{h(\Delta\nu)} = \frac{(1.055\times10^{-34}\ \text{J}\cdot\text{s})^2}{(6.626\times10^{-34}\ \text{J}\cdot\text{s})(7\times10^{11}\ \text{Hz})} = 2.4\times10^{-47}\ \text{kg}\cdot\text{m}^2$$

b) $\omega = 2\pi\nu = \sqrt{\kappa/\mu}$ with $\nu = 8.65\times10^{13}$ Hz. The reduced mass is (using the ^{35}Cl isotope)

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{35}{36}\ \text{u} = 1.614\times10^{-27}\ \text{kg}$$

Solving for κ we find

$$\kappa = 4\pi^2\nu^2\mu = 4\pi^2\left(8.65\times10^{13}\ \text{Hz}\right)^2\left(1.614\times10^{-27}\ \text{kg}\right) = 478\ \text{N/m}$$

in good agreement with Table 10.1.

16. a)

$$P = qx = \left(1.602 \times 10^{-19}\ \mathrm{C}\right)\left(2.67 \times 10^{-10}\ \mathrm{m}\right) = 4.28 \times 10^{-29}\ \mathrm{C \cdot m}$$

b) The fractional ionic character is the ratio

$$\frac{4.28 \times 10^{-29}\ \mathrm{C \cdot m}}{5.41 \times 10^{-29}\ \mathrm{C \cdot m}} = 0.79$$

17. $\Delta E = h\nu = hc/\lambda$, so the wavelength is

$$\lambda = \frac{c}{\nu} = \frac{2.998 \times 10^{8}\ \mathrm{m/s}}{8.72 \times 10^{13}\ \mathrm{Hz}} = 3.44\,\mu\mathrm{m}$$

which is in the infrared part of the spectrum.

*18. a) Using dimensional analysis and the fact that the energy of each photon is $hc/\lambda = 3.14 \times 10^{-19}$ J,

$$\frac{N}{t} = \frac{2.5 \times 10^{-3}\ \mathrm{J/s}}{3.14 \times 10^{-19}\ \mathrm{J}} = 7.96 \times 10^{15}\ \mathrm{s}^{-1}$$

b) 0.02 mole is equal to $0.02N_A = 1.20 \times 10^{22}$ atoms. Then the fraction participating is

$$\frac{7.96 \times 10^{15}}{1.20 \times 10^{22}} = 6.63 \times 10^{-7}$$

c) The transitions involved have a fairly low probability, even with stimulated radiation. We are saved by the large number of atoms available.

19. a) From Chapter 9 we have

$$\Delta\nu = \frac{\nu_0}{c}\sqrt{\frac{kT}{m}}$$

We also have in general $\nu = c/\lambda$ so $\Delta\nu = \left(c/\lambda^2\right)\Delta\lambda$. Therefore

$$\Delta\lambda = \frac{\lambda^2 \Delta\nu}{c} = \frac{\lambda^2 \nu_0}{c^2}\sqrt{\frac{kT}{m}} = \frac{\lambda}{c}\sqrt{\frac{kT}{m}}$$

Using the neon mass 3.32×10^{-26} kg we get

$$\Delta\lambda = \frac{6.328 \times 10^{-7}\ \mathrm{m}}{2.998 \times 10^{8}\ \mathrm{m/s}}\sqrt{\frac{(1.381 \times 10^{-23}\ \mathrm{J/K})\,(293\ \mathrm{K})}{3.32 \times 10^{-26}\ \mathrm{kg}}} = 7.37 \times 10^{-13}\ \mathrm{m}$$

b)

$$\Delta E = \frac{hc}{\lambda^2}\Delta\lambda = \frac{\hbar}{2\tau}$$

$$\Delta\lambda = \frac{\hbar\lambda^2}{2hc\tau} = \frac{\lambda^2}{4\pi c\tau} = \frac{\left(6.328 \times 10^{-7}\ \mathrm{m}\right)^2}{4\pi\,(2.998 \times 10^{8}\ \mathrm{m/s})\,(10^{-3}\ \mathrm{s})} = 1.06 \times 10^{-19}\ \mathrm{m}$$

The Doppler broadening is much more significant than the Heisenberg broadening.

20. a)

$$\Delta t = \frac{2\,(1\ \mathrm{m})}{2.998 \times 10^8\ \mathrm{m/s}} = 6.67 \times 10^{-9}\ \mathrm{s}$$

b) For the 16 km round trip

$$\Delta t = \frac{16 \times 10^3\ \mathrm{m}}{2.998 \times 10^8\ \mathrm{m/s}} - \frac{16 \times 10^3\ \mathrm{m}}{(1 - 3 \times 10^{-4})\, 2.998 \times 10^8\ \mathrm{m/s}} = -1.6 \times 10^{-8}\ \mathrm{s}$$

Because this result is larger than the desired uncertainty in timing, it is important to take atmospheric effects into account.

21. In a three-level system the population of the upper level must exceed the population of the ground state. This is not necessary in a four-level system.

22. Because the 3s state has two possible configurations and the $n = 2$ level has eight, let us assign a density of states $g = 2$ to the excited state and $g = 8$ to the ground state.

a)

$$\begin{aligned}\frac{f_3}{f_2} &= \frac{2\exp(-\beta E_3)}{8\exp(-\beta E_2)} = 0.25\exp\left(-\frac{E_3 - E_2}{kT}\right) \\ &= 0.25\exp\left(-\frac{16.6\ \mathrm{eV}}{(8.617 \times 10^{-5}\ \mathrm{eV/K})\,(293\ \mathrm{K})}\right) = 7 \times 10^{-287}\end{aligned}$$

b)

$$\frac{f_3}{f_2} = 0.25\exp\left(-\frac{16.6\ \mathrm{eV}}{(8.617 \times 10^{-5}\ \mathrm{eV/K})\,(200\ \mathrm{K})}\right) = 1.2 \times 10^{-419}$$

c)

$$\frac{f_3}{f_2} = 0.25\exp\left(-\frac{16.6\ \mathrm{eV}}{(8.617 \times 10^{-5}\ \mathrm{eV/K})\,(500\ \mathrm{K})}\right) = 1.2 \times 10^{-168}$$

d) Thermal excitations can be neglected.

*23. Using dimensional analysis the number density is

$$1980\ \mathrm{kg/m^3}\frac{1\ \mathrm{mol}}{0.07455\ \mathrm{kg}}\frac{2\,(6.022 \times 10^{23})}{\mathrm{mol}} = 3.20 \times 10^{28}\ \mathrm{m^{-3}}$$

Therefore the distance is

$$d = \left(3.20 \times 10^{28}\ \mathrm{m^{-3}}\right)^{-1/3} = 3.15 \times 10^{-10}\ \mathrm{m} = 0.315\ \mathrm{nm}$$

*24. Each charge has two unlike charges a distance r away, two like charges a distance $2r$ away, and so on:

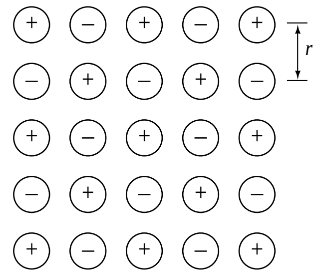
$$V = -\frac{2e^2}{4\pi\epsilon_0 r}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...\right)$$

The bracketed expression is the Taylor series expansion for $\ln 2$, so

$$V = -\frac{2e^2}{4\pi\epsilon_0 r}\ln 2 = -\frac{\alpha e^2}{4\pi\epsilon_0 r}$$

and we see that $\alpha = 2\ln 2$.

25. Using the positive central charge as a guide, we find



$$V = \frac{e^2}{4\pi\epsilon_0 r}\left(-\frac{4}{r} + \frac{4}{\sqrt{2}r} + \frac{4}{2r} - \frac{8}{\sqrt{5}r} + \frac{4}{\sqrt{8}r} - ...\right)$$

so by definition of α we have

$$\alpha = 4 - \frac{4}{\sqrt{2}} - 2 + \frac{8}{\sqrt{5}} - \frac{4}{\sqrt{8}} + ...$$

26.

$$F = -\frac{dV}{dr} = -\frac{\alpha e^2}{4\pi\epsilon_0 r^2} + \frac{\lambda}{\rho}e^{-r/\rho}$$

From Equation (10.20a) we have

$$1 = e^{r_0/\rho}\frac{\rho\alpha e^2}{4\pi\epsilon_0 r_0^2\lambda}$$

Multiplying this factor of 1 by the last term in the force equation, we get

$$\begin{aligned} F &= -\frac{\alpha e^2}{4\pi\epsilon_0 r^2} + \frac{\lambda}{\rho}e^{-r/\rho}e^{r_0/\rho}\frac{\rho\alpha e^2}{4\pi\epsilon_0 r_0^2\lambda} \\ &= \frac{\alpha e^2}{4\pi\epsilon_0 r_0^2}\left(-\frac{r_0^2}{r^2} + e^{-(r-r_0)/\rho}\right) \end{aligned}$$

27. Inserting $r = r_0 + \delta r$ into the force equation from Problem 26

$$F = \frac{\alpha e^2}{4\pi\epsilon_0 r_0^2}\left(-\frac{r_0^2}{(r_0+\delta r)^2} + e^{-\delta r/\rho}\right)$$

Factoring the first term in the parentheses leaves

$$F = \frac{\alpha e^2}{4\pi\epsilon_0 r_0^2}\left(-\frac{1}{\left(1+\frac{\delta r}{r_0}\right)^2} + e^{-\delta r/\rho}\right)$$

Applying the binomial theorem

$$\left(1+\frac{\delta r}{r_0}\right)^{-2} = 1 - \frac{2}{r_0}\delta r + \frac{3}{r_0^2}(\delta r)^2 - ...$$

and we end the series at that point for small δr. The Taylor series for the exponential is

$$e^{-\delta r/\rho} = 1 - \frac{\delta r}{\rho} + \frac{(\delta r)^2}{2\rho^2} - ...$$

Putting these two series approximations together:

$$F \cong \frac{\alpha e^2}{4\pi\epsilon_0 r_0^2}\left(-1 + \frac{2}{r_0}\delta r - \frac{3}{r_0^2}(\delta r)^2 + 1 - \frac{\delta r}{\rho} + \frac{(\delta r)^2}{2\rho^2}\right)$$

Collecting terms:

$$\begin{aligned} F &\cong \frac{\alpha e^2}{4\pi\epsilon_0 r_0^2}\left(\left(\frac{2}{r_0} - \frac{1}{\rho}\right)(\delta r) + \left(-\frac{3}{r_0^2} + \frac{1}{2\rho^2}\right)(\delta r)^2\right) \\ &\cong K_1(\delta r) + K_2(\delta r)^2 \end{aligned}$$

where

$$K_1 = \frac{\alpha e^2}{4\pi\epsilon_0 r_0^2}\left(\frac{2}{r_0} - \frac{1}{\rho}\right) \quad \text{and} \quad K_2 = \frac{\alpha e^2}{4\pi\epsilon_0 r_0^2}\left(-\frac{3}{r_0^2} + \frac{1}{2\rho^2}\right)$$

28. a) Looking at the result of the previous problem we see the spring constant is $\kappa = -K_1$, and we know that for the harmonic oscillator $\omega = \sqrt{\kappa/\mu}$. For NaCl

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = 2.32 \times 10^{-26} \text{ kg}$$

Recall that for NaCl we know $\alpha = 1.7476$, $r_0 = 0.282$ nm, and $\rho = 0.0316$ nm. These values give

$$-K_1 = \kappa = -\frac{\alpha e^2}{4\pi\epsilon_0 r_0^2}\left(\frac{2}{r_0} - \frac{1}{\rho}\right) = 124.7 \text{ N/m}$$

Then the oscillation frequency is

$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{\kappa}{\mu}} = \frac{1}{2\pi}\sqrt{\frac{124.7 \text{ N/m}}{2.32 \times 10^{-26} \text{ kg}}} = 1.17 \times 10^{13} \text{ Hz}$$

b)

$$\lambda = \frac{c}{\nu} = \frac{2.998 \times 10^8 \text{ m/s}}{1.17 \times 10^{13} \text{ Hz}} = 25.6\,\mu\text{m}$$

which is about half the observed value.

29. a)

$$\overline{F} = 0 = K_1 \overline{\delta r} + K_2 \overline{(\delta r)^2}$$

Therefore

$$\overline{\delta r} = -\frac{K_2}{K_1} \overline{(\delta r)^2}$$

b) From the equipartition theorem

$$K_1 \overline{(\delta r)^2} = kT$$

so from (a)

$$\overline{\delta r} = \frac{K_2}{K_1^2} kT$$

The coefficient of thermal expansion α comes from $\Delta L = L\alpha\, \Delta T$, or

$$\alpha = \frac{1}{L} \frac{\Delta L}{\Delta T}$$

Therefore in our nomenclature

$$\alpha = \frac{1}{r_0} \frac{d\left(\overline{\delta r}\right)}{dT} = \frac{1}{r_0} \frac{K_2}{K_1^2} k$$

Evaluation with $K_1 = -124.7$ N/m and $K_2 = 2.35 \times 10^{12}$ N/m^2, we find $\alpha = 7.4 \times 10^{-6}$ K^{-1}, which is on the right order of magnitude.

30. a)

$$\frac{\frac{3\sqrt{\pi}}{4} b a^{-5/2} \beta^{-3/2}}{\pi^{1/2} a^{-1/2} \beta^{-1/2}} = \frac{3}{4} b a^{-2} \beta^{-1} = \frac{3bkT}{4a^2}$$

b) Let $3bk/4a^2 = C_0$. Then $\langle x \rangle = C_0 T$.

$$C_0 = \frac{\Delta \langle x \rangle}{\Delta T} = \alpha x = \left(1.67 \times 10^{-5} \text{ K}^{-1}\right) \left(8.47 \times 10^{28} \text{ m}^{-3}\right)^{-1/3} = 3.80 \times 10^{-15} \text{ m/K}$$

where we used the number density of copper from Chapter 9.

31. b) From Table 10.1 we see that typically $\kappa \cong 10^3$ N/m. Using this value for a along with the definition of C_0 from Problem 30 we find

$$b = \frac{4a^2 C_0}{3k} = \frac{4\,(10^3 \text{ N/m})^2\,(3.80 \times 10^{-15} \text{ m/K})}{3\,(1.381 \times 10^{-23} \text{ J/K})} = 4 \times 10^{14} \text{ N/m}^2$$

32. a) $\overline{E} = \frac{3}{2}kT$ so $T = 2\overline{E}/3k$ and the ideal gas law becomes

$$PV = NkT = Nk\frac{2\overline{E}}{3k} = \frac{2N\overline{E}}{3}$$

b) From Chapter 9, $N/V = 8.47 \times 10^{28}\ \mathrm{m}^{-3}$ and we know $\overline{E} = \frac{3}{5}E_F = 6.76 \times 10^{-19}$ J. Thus in SI units we have

$$P = \frac{2N\overline{E}}{3V} = \frac{2}{3}\left(8.47 \times 10^{28}\ \mathrm{m}^{-3}\right)\left(6.76 \times 10^{-19}\ \mathrm{J}\right) = 3.8 \times 10^{10}\ \mathrm{N/m}^2$$

which is quite high. The ideal gas law may not be the best assumption for conduction electrons.

33. From the previous problem

$$P = \frac{2N\overline{E}}{3V} = \frac{2N}{3V}\frac{3}{5}E_F = \frac{2NE_F}{5V}$$

We must be careful, because E_F depends on the volume:

$$E_F = \frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{2/3}$$

$$P = \frac{2N}{5V}\frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{2/3} = \left(\frac{3}{\pi}\right)^{2/3}\frac{N^{5/3}h^2}{20m}V^{-5/3}$$

The bulk modulus is

$$\begin{aligned} B &= -V\frac{\partial P}{dV} = -V\left(\frac{3}{\pi}\right)^{2/3}\frac{N^{5/3}h^2}{20m}\left(-\frac{5}{3}\right)V^{-8/3} \\ &= \frac{5}{3}\left(\frac{3}{\pi}\right)^{2/3}\frac{N^{5/3}h^2}{20m}V^{-5/3} = \frac{5P}{3} \end{aligned}$$

Using the fact from above that $P = 2NE_F/5V$ we find

$$B = \frac{5}{3}\frac{2NE_F}{5V} = \frac{2NE_F}{3V}$$

34. For silver $N/V = 5.86 \times 10^{28}\ \mathrm{m}^{-3}$ and $E_F = 5.49$ eV.

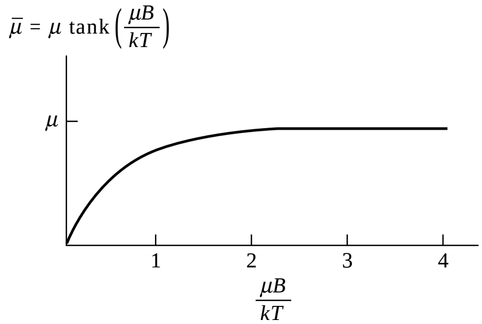
a)

$$B = \frac{2NE_F}{3V} = \frac{2}{3}\left(5.86 \times 10^{28}\ \mathrm{m}^{-3}\right)(5.49\ \mathrm{eV})\left(1.602 \times 10^{-19}\ \mathrm{J/eV}\right) = 3.44 \times 10^{10}\ \mathrm{N/m}^2$$

b) The computed result is about one-third of the measured value.

35. This is the same as the high-field limit. With $\mu B/kT \gg 1$ we have $\tanh(\mu B/kT) \cong 1$ so $\overline{\mu} \cong \mu$.

36. a) See graph.



b)

$$\overline{\mu} = \mu \tanh(5) = 0.99991\mu$$

and the approximate result of Problem 35 is off by only 0.009%.

c)

$$\overline{\mu} = \mu \tanh(0.10) = 0.0997\mu$$

The approximate result is off by just 0.3%.

*37. Magnetic dipole moment has units A·m^2, so M has units A·m^2/m^3 = A/m. μ_0 has units T·m/A and B has units T, so $\chi = \mu_0 M/B$ has units $(\mathrm{T \cdot m/A})\,(\mathrm{A/m})\,/\mathrm{T}$ which reduces to no units.

38. a) If we assume that every atom's magnetic moment is a Bohr magneton aligned in the same direction, $M = n\mu_B$ where n is the number density.

$$n = \frac{7.92 \ \times 10^3 \text{ kg}}{\text{m}^3} \frac{1 \text{ mol}}{0.05585 \text{ kg}} \left(\frac{6.022 \times 10^{23}}{1 \text{ mol}}\right) = 8.54 \times 10^{28} \text{ m}^{-3}$$

Thus

$$M = n\mu_B = \left(8.54 \times 10^{28} \text{ m}^{-3}\right)\left(9.274 \times 10^{-24} \text{ J/T}\right) = 7.92 \times 10^5 \text{ A/m}$$

b) The computed value is almost exactly one-half the measured value.

c) This implies that there are two unpaired spins per atom.

*39.

$$B_c = B_c(0)\left(1 - \left(\frac{T}{T_c}\right)^2\right) = 0.1B_c(0)$$

Thus $(T/T_c)^2 = 0.9$ and $T = \sqrt{0.9}T_c \cong 0.95T_c$. Similarly for a ratio of 0.5 we find $T = \sqrt{0.5}T_c \cong 0.71T_c$, and for a ratio of 0.9 we find $T = \sqrt{0.1}T_c \cong 0.32T_c$.

40. The energy gap at $T = 2$ K is

$$E_g(2\text{ K}) = \frac{hc}{\lambda} = \frac{1240\text{ eV}\cdot\text{nm}}{5.68\times 10^5\text{ nm}} = 2.18\times 10^{-3}\text{ eV}$$

Inserting Equation (10.45) into Equation (10.45a) gives

$$E_g(T) = 1.74\,(3.54kT_c)\left(1-\frac{T}{T_c}\right)^{1/2}$$

Using $E_g(2\text{ K}) = 2.18\times 10^{-3}$ eV and $k = 8.62\times 10^{-5}$ eV/K we get

$$4.11\text{ K} = T_c\left(1-\frac{T}{T_c}\right)^{1/2}$$

Solving by calculator we get $T_c = 5.3$ K which is closest to vanadium.

*41. Using the value given in the text, $T_c = 4.146$ K for a mass of 203.4 u, we get

$$M^{0.5}T_c = \text{ constant } = 5.91296\text{ u}^{0.5}\cdot\text{K}$$

and so for a mass of 201 u we find $T_c = 4.171$ K and for a mass of 204 u we find $T_c = 4.140$ K.

42. With ^{16}O the molar mass in grams is

$$88.906 + 2(137.33) + 3(63.546) + 7(16.00) = 666.204$$

Replacing all the ^{16}O atoms with ^{18}O adds 14 grams per mole, changing the mass to 680.204. Using the BCS formula for the isotope effect

$$M_1^{0.5}T_{c1} = M_2^{0.5}T_{c2}$$

and assuming $T_c = 93$ K (exactly) for the first sample

$$T_{c2} = \left(\frac{M_1}{M_2}\right)^{1/2}T_{c1} = \left(\frac{666.204}{680.204}\right)^{1/2}(93\text{ K}) = 92.0\text{ K}$$

a change of 1.0 kelvin.

43. Extrapolating on the graph it could be at about 130 K.

*44.

$$B = \mu_0 In = \left(4\pi\times 10^{-7}\text{ N/A}^2\right)(5.0\text{ A})\left(3000\text{ m}^{-1}\right) = 18.85\text{ mT}$$

$$\Phi = BA = B\pi d^2/4 = \left(18.85\times 10^{-3}\text{ T}\right)\frac{\pi\,(0.025\text{ m})^2}{4} = 9.25\times 10^{-6}\text{ T}\cdot\text{m}^2$$

$$\frac{\Phi}{\Phi_0} = \frac{9.25\times 10^{-6}\text{ T}\cdot\text{m}^2}{2.068\times 10^{-15}\text{ T}\cdot\text{m}^2} = 4.5\times 10^9\text{ flux quanta}$$

This large number shows how small the flux quantum is.

45. We know that for niobium $B_c = 0.206$ T. Then the diameter (twice the radius) is

$$D = 2R = \frac{\mu_0 I}{\pi B} = \frac{(4\pi \times 10^{-7}\ \mathrm{N/A^2})\,(5.5\ \mathrm{A})}{\pi\,(0.206\ \mathrm{T})} = 1.07 \times 10^{-5}\ \mathrm{m}$$

which is quite small.

46.

$$P = I^2 R = \frac{I^2 \rho L}{A}$$

where ρ is resistivity, L is length, and A is area. Now $A = \pi r^2$, so

$$P = \frac{I^2 \rho L}{\pi r^2}$$

The surface area is $2\pi r L$ so the power per unit area is

$$\frac{P}{\mathrm{area}} = \frac{I^2 \rho L}{\pi r^2\,(2\pi r L)} = \frac{I^2 \rho}{2\pi^2 r^3} = 100\ \mathrm{W/m^2}$$

Using $r = 3.75 \times 10^{-4}$ m we find

$$I^2 = \frac{2\pi^2\,(3.75 \times 10^{-4}\ \mathrm{m})^3\,(100\ \mathrm{W/m^2})}{1.72 \times 10^{-8}\ \Omega \cdot \mathrm{m}} = 6.052\ \mathrm{A^2}$$

or $I = 2.46$ A.

47. a) From the BCS theory we have

$$B_c(4.2\ \mathrm{K}) = B_c(0)\left(1 - \left(\frac{T}{T_c}\right)^2\right) = (0.206\ \mathrm{T})\left(1 - \left(\frac{4.2}{9.25}\right)^2\right) = 0.1635\ \mathrm{T}$$

From the result of Problem 45 we know that

$$I = \frac{\pi B_c D}{\mu_0} = \frac{\pi\,(0.1635\ \mathrm{T})\,(7.5 \times 10^{-4}\ \mathrm{m})}{4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}} = 307\ \mathrm{A}$$

b) This is a lot more current than the copper can carry (more than 100 times).

48. Because ν_j is directly proportional to V, it is known to within one part in 10^{10}, or

$$\left(10^{-10}\right)\left(483.6 \times 10^9\ \mathrm{Hz}\right) = 48.36\ \mathrm{Hz}$$

which is pretty fine tuning.

49. By conversion 550 km/h = 152.78 m/s. Then from kinematics $v^2 = 2ax$, so

$$a = \frac{v^2}{2x} = \frac{(152.78 \text{ m/s})^2}{2\,(3500 \text{ m})} = 3.33 \text{ m/s}^2$$

This is about $g/3$, which would certainly be noticeable.

50. a) To compute escape speed use conservation of energy, with $\frac{1}{2}mv_{\text{esc}}^2 = GMm/R_e$:

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R_e}} = \sqrt{\frac{2\,(6.673 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-1})\,(5.98 \times 10^{24} \text{ kg})}{6.378 \times 10^6 \text{ m}}} = 11.1 \text{ km/s}$$

b) From Chapter 9

$$\overline{v} = \frac{4}{\sqrt{2\pi}}\sqrt{\frac{kT}{m}} = \frac{4}{\sqrt{2\pi}}\sqrt{\frac{(1.381 \times 10^{-23} \text{ J/K})\,(293 \text{ K})}{4\,(1.661 \times 10^{-27} \text{ kg})}} = 1245 \text{ m/s}$$

c) There are always enough helium atoms on the (high-speed) tail of the Maxwell-Boltzmann distribution that a significant number can escape, given enough time.

51. Equating the centripetal force with the Lorentz (magnetic) force we get

$$\frac{mv^2}{R} = qvB$$

or $mv = p = qBR$. The formula $p = qBR$ is also correct relativistically, and note that for these extremely high energies $E \cong pc = qBRc$. Therefore the energy is

$$\begin{aligned} E &\cong qBRc = \left(1.602 \times 10^{-19} \text{ C}\right)(13.5 \text{ T})\left(\frac{27000 \text{ m}}{2\pi}\right)\left(2.998 \times 10^8 \text{ m/s}\right) \\ &= 2.786 \times 10^{-6} \text{ J } = 17.4 \text{ TeV} \end{aligned}$$

*52. a) In a RL circuit the current is

$$I = I_0 e^{-Rt/L}$$

For small values of R let us approximate the exponential with the Taylor expansion $1 - Rt/L$. Then

$$10^{-9} = 1 - \frac{I}{I_0} = 1 - e^{-Rt/L} \cong \frac{Rt}{L}$$

$$R \le 10^{-9}\frac{L}{t} = 10^{-9}\left(\frac{3.14 \times 10^{-8} \text{ H}}{2.5 \text{ y } \ (3.16 \times 10^7 \text{ s/y})}\right) = 4.0 \times 10^{-25}\ \Omega$$

b) For a 10% loss

$$t = \frac{0.1}{10^{-9}}\,(2.5 \text{ y}) = 2.5 \times 10^8 \text{ y}$$

53. From the BCS theory

$$B = B_c(0)\left(1 - \left(\frac{T}{T_c}\right)^2\right)$$

Then

$$\frac{\Delta S}{V} = -\frac{\partial}{\partial T}\left(\frac{B^2}{2\mu_0}\right) = \frac{2B_c^2(0)}{\mu_0 T_c}\left(\frac{T}{T_c} - \left(\frac{T}{T_c}\right)^3\right)$$

For numerical values use $T = 6$ K, $T_c = 9.25$ K, and $B_c(0) = 0.206$ T.

$$\frac{\Delta S}{V} = \frac{2\,(0.206\ \mathrm{T})^2}{(4\pi \times 10^{-7}\ \mathrm{T\cdot m/A})\,(9.25\ \mathrm{K})}\left(\frac{6}{9.25} - \left(\frac{6}{9.25}\right)^3\right) = 2743\ \mathrm{J\cdot m^{-3}\cdot K^{-1}}$$

The volume of one mole of niobium is

$$V = \frac{92.91\ \mathrm{g}}{8.57\ \mathrm{g/cm^3}} = 10.84\ \mathrm{cm^3} = 1.084 \times 10^{-5}\ \mathrm{m^3}$$

Thus

$$\Delta S = \left(2743\ \mathrm{J\cdot m^{-3}\cdot K^{-1}}\right)\left(1.084 \times 10^{-5}\ \mathrm{m^3}\right) = 2.97 \times 10^{-2}\ \mathrm{J/K}$$

for one mole of niobium. The superconducting state has a lower entropy than the normal state.