Chapter 15

*1. From Newton's second law we have for a pendulum of length L

$$F = m_G g \sin\theta = m_I a = m_I L \frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = \frac{m_G g}{m_I L} \sin\theta \cong \frac{m_G g}{m_I L}\theta$$

where we have made the small angle approximation $\sin\theta \cong \theta$. This is a simple harmonic oscillator equation with solution $\theta = \theta_0 \cos(\omega t)$ where θ_0 is the amplitude and the angular frequency is

$$\omega = \sqrt{\frac{m_G g}{m_I L}}$$

The period of oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m_I L}{m_G g}}$$

Therefore two masses with different ratios m_I/m_G will have different small-amplitude periods.

2.

$$\Delta\nu = \frac{gH\nu}{c^2} = \frac{(9.80\ \text{m/s}^2)\,(4\times 10^5\ \text{m})\,(10^8\ \text{s}^{-1})}{(2.998\times 10^8\ \text{m/s})^2} = 4.36\times 10^{-3}\ \text{Hz}$$

3.

$$\frac{\Delta\nu}{\nu} = -\frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = -\frac{GM}{r_1 r_2 c^2}\,(r_2 - r_1) = \frac{GM}{r_1 r_2 c^2}\,(r_1 - r_2)$$

Use $r_1 - r_2 = H$ and let $r_1 \cong r_2 = r$. From classical mechanics $g = GM/r^2$, so

$$\Delta\nu = \frac{gH\nu}{c^2}$$

4.

$$\frac{\Delta T}{T} = -\frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

We use $r_2 = 6378$ km and $r_1 = 6378$ km $+10$ km $= 6388$ km.

$$\begin{aligned}\frac{\Delta T}{T} &= -\frac{(6.673\times 10^{-11}\ \text{m}^3\cdot\text{kg}^{-1}\cdot\text{s}^{-2})\,(5.98\times 10^{24}\ \text{kg})}{(2.998\times 10^8\ \text{m/s})^2}\left(\frac{1}{6388\times 10^3\ \text{m}} - \frac{1}{6378\times 10^3\ \text{m}}\right)\\ &= 1.09\times 10^{-12}\end{aligned}$$

which is the same as in the example, to three significant digits.

5. The distance d is the sum of the radii of the earth's orbit and Venus's orbit (assuming circular orbits).

$$d = 149.6 \times 10^9 \text{ m } + 108.2 \times 10^9 \text{ m } = 258.8 \times 10^9 \text{ m}$$

The round-trip time is

$$t = \frac{2d}{c} = \frac{2\,(258.8 \times 10^9 \text{ m})}{2.998 \times 10^8 \text{ m/s}} = 1726.5 \text{ s}$$

The percent change is therefore

$$\frac{200 \times 10^{-6} \text{ s}}{1726.5 \text{ s}}\,(100\ \%) = 1.16 \times 10^{-5}\ \%$$

*6. If the photon "falls" at a rate g then during the time taken to travel a distance $x = 40,000$ km it has fallen a distance d :

$$d = \frac{1}{2}gt^2 = \frac{1}{2}\left(9.80 \text{ m/s}^2\right)\left(\frac{4 \times 10^7 \text{ m}}{2.998 \times 10^8 \text{ m/s}}\right)^2 = 8.72 \text{ cm}$$

7. Using the mass and radius of the sun

$$\frac{\Delta\nu}{\nu} = \frac{GM}{rc^2} = \frac{(6.673 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2})\,(1.99 \times 10^{30} \text{ kg})}{(6.96 \times 10^8 \text{ m})\,(2.998 \times 10^8 \text{ m/s})^2} = 2.123 \times 10^{-6}$$

The wavelength is affected by the same factor, so the redshift of the given wavelength is

$$\Delta\lambda = \left(2.123 \times 10^{-6}\right)(550 \text{ nm}) = 1.17 \times 10^{-3} \text{ nm}$$

8. As in the previous problem

$$\frac{\Delta\nu}{\nu} = \frac{GM}{rc^2} = \frac{(6.673 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2})\,(5 \times 10^{30} \text{ kg})}{(10^4 \text{ m})\,(2.998 \times 10^8 \text{ m/s})^2} = 0.371$$

$$\Delta\lambda = (0.371)\,(550 \text{ nm}) = 204 \text{ nm}$$

9. Let us assume that g is constant over this short distance. Using $E = h\nu$ we find

$$\Delta\nu = \frac{gH\nu}{c^2} = \frac{gHE}{c^2 h} = \frac{(9.80 \text{ m/s}^2)\,(22.5 \text{ m})\,(14.4 \times 10^3 \text{ eV})}{(2.998 \times 10^8 \text{ m/s})^2\,(4.136 \times 10^{-15} \text{ eV} \cdot \text{s})} = 8541 \text{ Hz}$$

The percentage change is

$$\frac{8541 \text{ Hz}}{(14.4 \times 10^3 \text{ eV})\,/\,(4.136 \times 10^{-15} \text{ eV} \cdot \text{s})}\,(100\ \%) = 2.45 \times 10^{-13}\ \%$$

10.

$$r_s = \frac{2GM}{c^2} = \frac{2\,(6.673 \times 10^{-11}\ \mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}})\,(7.35 \times 10^{22}\ \mathrm{kg})}{(2.998 \times 10^8\ \mathrm{m/s})^2} = 1.09 \times 10^{-4}\ \mathrm{m}$$

*11.

$$r_s = \frac{2GM}{c^2} = \frac{2\,(6.673 \times 10^{-11}\ \mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}})\,(1.90 \times 10^{27}\ \mathrm{kg})}{(2.998 \times 10^8\ \mathrm{m/s})^2} = 2.82\ \mathrm{m}$$

*12.

$$\begin{aligned} T &= \frac{hc^3}{8\pi kGM} = \frac{(6.626 \times 10^{-34}\ \mathrm{J \cdot s})\,(2.998 \times 10^8\ \mathrm{m/s})^3}{8\pi\,(1.381 \times 10^{-23}\ \mathrm{J/K})\,(6.673 \times 10^{-11}\ \mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}})\,(1.99 \times 10^{30}\ \mathrm{kg})} \\ &= 3.87 \times 10^{-7}\ \mathrm{K} \end{aligned}$$

13. Rearranging the formula given in the previous problem,

$$M = \frac{hc^3}{8\pi kGT} = \frac{(6.626 \times 10^{-34}\ \mathrm{J \cdot s})\,(2.998 \times 10^8\ \mathrm{m/s})^3}{8\pi\,(1.381 \times 10^{-23}\ \mathrm{J/K})\,(6.673 \times 10^{-11}\ \mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}})\,(293\ \mathrm{K})} = 2.63 \times 10^{21}\ \mathrm{kg}$$

which is about

$$\frac{2.63 \times 10^{21}\ \mathrm{kg}}{1.99 \times 10^{30}\ \mathrm{kg}} = 1.32 \times 10^{-9}$$

solar masses.

$$r_s = \frac{2GM}{c^2} = \frac{2\,(6.673 \times 10^{-11}\ \mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}})\,(2.63 \times 10^{21}\ \mathrm{kg})}{(2.998 \times 10^8\ \mathrm{m/s})^2} = 3.91 \times 10^{-6}\ \mathrm{m}$$

*14. Set the change in the photon's energy equal to the change in gravitational potential energy:

$$\Delta E = h\,\Delta\nu = -\frac{GMm}{r_1} - \left(-\frac{GMm}{r_2}\right) = -GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

where M is the mass of the earth and m is the equivalent mass of the photon. Now $m = E/c^2 = h\nu/c^2$, so

$$h\,\Delta\nu = -\frac{GMh\nu}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$\frac{\Delta\nu}{\nu} = -\frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

15. Using the formula from the previous problem and recalling that the earth's radius is 6378 km,

$$\begin{aligned}\frac{\Delta\nu}{\nu} &= -\frac{GM}{c^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\\ &= -\frac{\left(6.673\times10^{-11}\ \mathrm{m^3\cdot kg^{-1}\cdot s^{-2}}\right)\left(5.98\times10^{24}\ \mathrm{kg}\right)}{\left(2.998\times10^{8}\ \mathrm{m/s}\right)^2}\left(\frac{1}{6.678\times10^{6}\ \mathrm{m}}-\frac{1}{6.378\times10^{6}\ \mathrm{m}}\right)\\ &= 3.127\times10^{-11}\end{aligned}$$

Therefore

$$\Delta\nu = \left(3.127\times10^{-11}\right)\left(294\times10^{6}\ \mathrm{Hz}\right) = 9.19\times10^{-3}\ \mathrm{Hz}$$

16. $g = GM/r^2$ which can be differentiated to give

$$dg = -\frac{2GM}{r^3}dr$$

For a small change let $dg \cong |dg|$ and $dr \cong \Delta r = 3$ m. Also notice that the distance from the center of the earth is 6.378×10^{6} m $+3\times10^{5}$ m $= 6.678\times10^{6}$ m. Then

$$\Delta g \cong \frac{2\left(6.673\times10^{-11}\ \mathrm{m^3\cdot kg^{-1}\cdot s^{-2}}\right)\left(5.98\times10^{24}\ \mathrm{kg}\right)}{\left(6.678\times10^{6}\ \mathrm{m}\right)^3}(3\ \mathrm{m}) = 8.04\times10^{-6}\ \mathrm{m/s^2}$$

This is about $10^{-7}g$, so it is a very small effect.

17. If we use $\lambda = h/mc$ for a relativistic particle of mass m , we have

$$\lambda = \frac{h}{mc} = \pi r_s = \frac{2\pi G m}{c^2}$$

Solving for m we have

$$m = \sqrt{\frac{hc}{2\pi G}} = \sqrt{\frac{\left(6.626\times10^{-34}\ \mathrm{J\cdot s}\right)\left(2.998\times10^{8}\ \mathrm{m/s}\right)}{2\pi\left(6.673\times10^{-11}\ \mathrm{m^3\cdot kg^{-1}\cdot s^{-2}}\right)}} = 2.18\times10^{-8}\ \mathrm{kg}$$

The Planck energy is

$$E_{\mathrm{Pl}} = mc^2 = \left(2.18\times10^{-8}\ \mathrm{kg}\right)\left(2.998\times10^{8}\ \mathrm{m/s}\right)^2 = 1.96\times10^{9}\ \mathrm{J} = 1.22\times10^{28}\ \mathrm{eV}$$

18. a) The combination of G, h, and c that has the right units is

$$\lambda_{\mathrm{Pl}} = \sqrt{\frac{Gh}{c^3}} = \sqrt{\frac{\left(6.673\times10^{-11}\ \mathrm{m^3\cdot kg^{-1}\cdot s^{-2}}\right)\left(6.626\times10^{-34}\ \mathrm{J\cdot s}\right)}{\left(2.998\times10^{8}\ \mathrm{m/s}\right)^3}} = 4.05\times10^{-35}\ \mathrm{m}$$

b)

$$\lambda = \frac{h}{mc} = \frac{hc}{mc^2} = \frac{1240\times10^{-9}\ \mathrm{eV\cdot m}}{1.22\times10^{28}\ \mathrm{eV}} = 1.02\times10^{-34}\ \mathrm{m}$$

which is the same order of magnitude as (a).

19. The combination of constants that gives time is

$$t_{\mathrm{Pl}} = \sqrt{\frac{Gh}{c^5}} = \sqrt{\frac{\left(6.673\times10^{-11}\ \mathrm{m^3\cdot kg^{-1}\cdot s^{-2}}\right)\left(6.626\times10^{-34}\ \mathrm{J\cdot s}\right)}{\left(2.998\times10^{8}\ \mathrm{m/s}\right)^5}} = 1.35\times10^{-43}\ \mathrm{s}$$

The time for light to travel the Planck length is

$$t = \frac{\lambda_{\mathrm{Pl}}}{c} = \sqrt{\frac{Gh}{c^5}} = 1.35\times10^{-43}\ \mathrm{s}$$

as we found in this problem.

20. As in Problem 15

$$\begin{aligned}\frac{\Delta\nu}{\nu} &= -\frac{GM}{c^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\\ &= -\frac{\left(6.673\times10^{-11}\ \mathrm{m^3\cdot kg^{-1}\cdot s^{-2}}\right)\left(5.98\times10^{24}\ \mathrm{kg}\right)}{\left(2.998\times10^{8}\ \mathrm{m/s}\right)^2}\\ &\quad\times\left(\frac{1}{3.587\times10^{7}\ \mathrm{m}+6.378\times10^{6}\ \mathrm{m}}-\frac{1}{6.378\times10^{6}\ \mathrm{m}}\right)\\ &= 5.91\times10^{-10}\end{aligned}$$

Therefore

$$\Delta\nu = \left(5.91\times10^{-10}\right)\left(2\times10^{9}\ \mathrm{Hz}\right) = 1.18\ \mathrm{Hz}$$