Chapter 5

1. $\sin\theta_1 = \lambda/2d = 0.259$

Second order:

$$\sin\theta_2 = \frac{2\lambda}{2d} = 2\sin\theta_1 \qquad \theta_2 = \sin^{-1}(2\sin\theta_1) = 31.2°$$

$$\theta_3 = \sin^{-1}(3\sin\theta_1) = 50.9°$$

*2. Use $\lambda = 0.16$ nm, and we know from the text that $d = 0.282$ nm for NaCl.

$$n = 1: \qquad \sin\theta = \frac{n\lambda}{2d} = \frac{\lambda}{2d} = 0.284 \qquad \theta = 16.5°$$

$$n = 2: \qquad \sin\theta = \frac{n\lambda}{2d} = \frac{\lambda}{d} = 0.567 \qquad \theta = 34.6°$$
$$\Delta\theta = 34.6° - 16.5° = 18.1°$$

3. For $n = 1$ we have $\lambda = 2d\sin\theta = 2(0.314 \text{ nm})(\sin 14°) = 0.152$ nm

$$E = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{0.152 \text{ nm}} = 8.16 \text{ keV}$$

The largest order n is the largest integer for which $n\lambda/2d < 1$

$$n < \frac{2d}{\lambda} = 4.13$$

so we can observe up through $n = 4$.

4. As in Davisson-Germer scattering $n\lambda = D\sin\phi$

$$\phi = \sin^{-1}\left(\frac{\lambda}{D}\right) = \sin^{-1}\left(\frac{hc}{ED}\right) = \sin^{-1}\left(\frac{1240 \text{ eV} \cdot \text{nm}}{(10^5 \text{ eV})(0.24 \text{ nm})}\right) = 3.0°$$

5.

$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34} \text{ J} \cdot \text{s}}{(3.0 \text{ kg})(6.0 \text{ m/s})} = 3.68 \times 10^{-35} \text{ m}$$

No, the wavelength of the water waves depends on the medium; they are strictly mechanical waves.

6. Using the mean speed from kinetic theory (Chapter 9)

$$\overline{v} = \frac{4}{\sqrt{2\pi}}\sqrt{\frac{kT}{m}} = \frac{4}{\sqrt{2\pi}}\sqrt{\frac{(1.38 \times 10^{-23}\ \text{J/K})\,(308.15\ \text{K})}{28\,(1.66 \times 10^{-27}\ \text{kg})}} = 482.7\ \text{m/s}$$

$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34}\ \text{J}\cdot\text{s}}{28\,(1.66 \times 10^{-27}\ \text{kg})\,(482.7\ \text{m/s})} = 2.95 \times 10^{-11}\ \text{m}$$

or roughly 3% of the size of the molecule.

7. $(50\ \text{eV})\,(1.602 \times 10^{-19}\ \text{J/eV}) = 8.01 \times 10^{-18}\ \text{J}$

$$\lambda = \frac{h}{\sqrt{2mK}} = \frac{6.626 \times 10^{-34}\ \text{J}\cdot\text{s}}{\sqrt{2\,(9.109 \times 10^{-31}\ \text{kg})\,(8.01 \times 10^{-18}\ \text{J})}} = 1.73 \times 10^{-10}\ \text{m}$$

This is the same as in the textbook's example, to two places.

8. When $E \gg E_0$ then $E \cong pc$ for the particle, just as for a photon. Therefore the electron's energy is approximately equal to the photon energy.
   If $E = 2E_0$ then we cannot use $E \cong pc$. The exact expression is

$$p = \frac{\sqrt{E^2 - E_0^2}}{c} = \frac{\sqrt{3}E_0}{c}$$

for the electron's momentum. Then the de Broglie wavelength is

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{3}E_0}$$

If the photon has the same wavelength its energy is

$$E = \frac{hc}{\lambda} = \sqrt{3}E_0$$

*9. a) Relativistically

$$p = \frac{\sqrt{E^2 - E_0^2}}{c} = \frac{\sqrt{(K + mc^2)^2 - (mc^2)^2}}{c} = \frac{\sqrt{K^2 + 2Kmc^2}}{c}$$

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{K^2 + 2Kmc^2}}$$

b) Non-relativistically, as in the text

$$\lambda = \frac{h}{\sqrt{2mK}}$$

*10.

$$p = \frac{\sqrt{E^2 - E_0^2}}{c} = 50 GeV/c$$

$$\lambda = \frac{h}{p} = \frac{1240 \text{ eV} \cdot \text{nm}}{50 \text{ GeV}} = 2.48 \times 10^{-17} \text{ m}$$

$$\text{fraction} = \frac{2.48 \times 10^{-17} \text{ m}}{2 \times 10^{-15} \text{ m}} = 0.012$$

11. a) For photons kinetic energy equals total energy and de Broglie wavelength is wavelength

$$K = E = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{0.15 \text{ nm}} = 8.27 \text{ keV}$$

b) The energy is low enough that we can use the non-relativistic formula:

$$K = \frac{h^2}{2m\lambda^2} = \frac{(hc)^2}{2mc^2\lambda^2} = \frac{(1240 \text{ eV} \cdot \text{nm})^2}{2 (511 \times 10^3 \text{ eV}) (0.15 \text{ nm})^2} = 66.9 \text{ eV}$$

c)

$$K = \frac{h^2}{2m\lambda^2} = \frac{(hc)^2}{2mc^2\lambda^2} = \frac{(1240 \text{ eV} \cdot \text{nm})^2}{2 (939 \times 10^6 \text{ eV}) (0.15 \text{ nm})^2} = 0.036 \text{ eV}$$

d)

$$K = \frac{h^2}{2m\lambda^2} = \frac{(hc)^2}{2mc^2\lambda^2} = \frac{(1240 \text{ eV} \cdot \text{nm})^2}{2 (3727 \times 10^6 \text{ eV}) (0.15 \text{ nm})^2} = 9.17 \times 10^{-3} \text{ eV}$$

12. a) As in Problem 6 we use the mean speed formula from kinetic theory:

$$\overline{v} = \frac{4}{\sqrt{2\pi}} \sqrt{\frac{kT}{m}} = \frac{4}{\sqrt{2\pi}} \sqrt{\frac{(1.38 \times 10^{-23} \text{ J/K}) (10 \text{ K})}{(1.675 \times 10^{-27} \text{ kg})}} = 458.0 \text{ m/s}$$

$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34} \text{ J} \cdot \text{s}}{(1.675 \times 10^{-27} \text{ kg}) (458.0 \text{ m/s})} = 0.86 \text{ nm}$$

b)

$$\overline{v} = \frac{4}{\sqrt{2\pi}} \sqrt{\frac{kT}{m}} = \frac{4}{\sqrt{2\pi}} \sqrt{\frac{(1.38 \times 10^{-23} \text{ J/K}) (0.1 \text{ K})}{(1.675 \times 10^{-27} \text{ kg})}} = 45.8 \text{ m/s}$$

$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34} \text{ J} \cdot \text{s}}{(1.675 \times 10^{-27} \text{ kg}) (45.8 \text{ m/s})} = 8.6 \text{ nm}$$

*13. From the accelerating potential we know $K = eV = 3$ keV.

$$E = K + E_0 = 514 \text{ keV}$$

$$p = \frac{\sqrt{E^2 - E_0^2}}{c} = \frac{\sqrt{(514 \text{ keV})^2 - (511 \text{ keV})^2}}{c} = 55.4 \text{ keV}/c$$

$$\lambda = \frac{h}{p} = \frac{hc}{pc} = \frac{1240 \text{ eV} \cdot \text{nm}}{55.4 \times 10^3 \text{ eV}} = 22.4 \text{ pm}$$

14. We use the relativistic formula derived in Problem 9:

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{K^2 + 2Kmc^2}}$$

a) $\lambda = 0.194$ nm   b) $\lambda = 6.13 \times 10^{-2}$ nm   c) $\lambda = 1.94 \times 10^{-2}$ nm
d) $\lambda = 6.02 \times 10^{-3}$ nm   e) $\lambda = 1.64 \times 10^{-3}$ nm   f) $\lambda = 2.77 \times 10^{-4}$ nm

15. a)
$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34} \text{ J} \cdot \text{s}}{32 \left(1.661 \times 10^{-27} \text{ kg}\right) (480 \text{ m/s})} = 2.60 \times 10^{-11} \text{ m}$$

b)
$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34} \text{ J} \cdot \text{s}}{\left(1.5 \times 10^{-15} \text{ kg}\right) \left(10^{-6} \text{ m/s}\right)} = 4.42 \times 10^{-13} \text{ m}$$

16. Using the relativistic formula from Problem 9

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{K^2 + 2Kmc^2}} = \frac{1240 \text{ eV} \cdot \text{nm}}{\sqrt{\left(10^{12} \text{ eV}\right)^2 + 2\left(10^{12} \text{ eV}\right)\left(938 \times 10^6 \text{ eV}\right)}} = 1.24 \times 10^{-18} \text{ m}$$

17.

$$d = D\sin(\phi/2) = (0.23 \text{ nm})\sin 16° = 0.063 \text{ nm}$$

$$\lambda = D\sin\phi = (0.23 \text{ nm})\sin 32° = 0.122 \text{ nm}$$

$$p = \frac{h}{\lambda} = \frac{hc}{\lambda c} = \frac{1240 \text{ eV} \cdot \text{nm}}{(0.122 \text{ nm})\, c} = 10.2 \text{ keV}/c$$

$$E = \sqrt{p^2c^2 + E_0^2} = \sqrt{(10.2 \text{ keV})^2 + (511 \text{ keV})^2} = 511.102 \text{ keV}$$

$$K = E - E_0 = 102 \text{ eV}$$

18. At 48 eV

$$\lambda = \frac{1.226 \text{ nm} \cdot V^{1/2}}{\sqrt{48 \text{ V}}} = 0.177 \text{ nm}$$

$$\phi = \sin^{-1}\left(\frac{d}{\lambda}\right) = \sin^{-1}\left(\frac{0.177 \text{ nm}}{0.215 \text{ nm}}\right) = 55.4°$$

At 64 eV

$$\lambda = \frac{1.226 \text{ nm} \cdot V^{1/2}}{\sqrt{64 \text{ V}}} = 0.153 \text{ nm}$$

$$\phi = \sin^{-1}\left(\frac{d}{\lambda}\right) = \sin^{-1}\left(\frac{0.153 \text{ nm}}{0.215 \text{ nm}}\right) = 45.4°$$

19. First we compute the wavelength of the electrons:

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{E^2 - E_0^2}} = \frac{1240 \text{ eV} \cdot \text{nm}}{\sqrt{(513 \text{ keV})^2 - (511 \text{ keV})^2}} = 2.74 \times 10^{-2} \text{ nm}$$

From Figure 5.6(a) we see that $2\theta_1 = \tan^{-1}(2.1 \text{ cm}/35 \text{ cm}) = 3.434°$ or $\theta_1 = 1.717°$. Now since $\lambda = 2d\sin\theta$ we have

$$d_1 = \frac{\lambda}{2\sin\theta_1} = \frac{2.74 \times 10^{-2} \text{ nm}}{2\sin(1.717°)} = 0.457 \text{ nm}$$

$$\theta_2 = \frac{1}{2}\tan^{-1}(2.3 \text{ cm}/35 \text{ cm}) = 1.880°$$

$$d_2 = \frac{\lambda}{2\sin\theta_2} = \frac{2.74 \times 10^{-2} \text{ nm}}{2\sin(1.880°)} = 0.412 \text{ nm}$$

$$\theta_3 = \frac{1}{2}\tan^{-1}(3.2 \text{ cm}/35 \text{ cm}) = 2.612°$$

$$d_3 = \frac{\lambda}{2\sin\theta_3} = \frac{2.74 \times 10^{-2} \text{ nm}}{2\sin(2.612°)} = 0.301 \text{ nm}$$

*20.

$$\lambda = \frac{h}{\sqrt{2mK}} = \frac{hc}{\sqrt{2mc^2K}} = \frac{1240 \text{ eV} \cdot \text{nm}}{\sqrt{2(939 \times 10^6 \text{ eV})(0.025 \text{ eV})}} = 0.181 \text{ nm}$$

$$\lambda = D\sin\phi \qquad \phi = \sin^{-1}\left(\frac{\lambda}{D}\right) = \sin^{-1}\left(\frac{0.181 \text{ nm}}{0.45 \text{ nm}}\right) = 23.7°$$

21. a) $\nu = v/\lambda = 0.571$ Hz

b) From the initial conditions given, we should use a cosine function.

$$\Psi = A\cos\left(\left(\frac{2\pi}{\lambda}\right)(x - vt)\right) = (3.0 \text{ cm})\cos\left(\left(\frac{2\pi}{7 \text{ cm}}\right)(10 \text{ cm} - (4 \text{ cm/s})(13 \text{ s}))\right) = 3.0 \text{ cm}$$

22. a) $\nu = v/\lambda = 1$ Hz   b) $T = 1/\nu = 1$ s

c) $k = 2\pi/\lambda = \pi/2$ cm^{-1} d) $\omega = 2\pi/T = 2\pi$ rad/s

23. a)
$$\Psi = \Psi_1 + \Psi_2 = 0.003\left[\sin(6.0x - 300t) + \sin(7.0x - 250t)\right]$$

We can use a trig identity $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$$\Psi = 0.006\sin(6.5x - 275t)\cos(-0.5x - 25t)$$

or because cosine is an even function

$$\Psi = 0.006\sin(6.5x - 275t)\cos(0.5x + 25t)$$

b)
$$v_{ph} = \frac{\omega_1 + \omega_2}{k_1 + k_2} = \frac{550 \text{ rad/s}}{13 \text{ m}^{-1}} = 42.3 \text{ m/s}$$

$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{50 \text{ rad/s}}{1 \text{ m}^{-1}} = 50 \text{ m/s}$$

c) As in Equation (5.22) $\Delta x = 2\pi/\Delta k = 2\pi$ m and the separation between zeroes is half of this or π meters.

d) $\Delta k \Delta x = (1 \text{ m}^{-1})(2\pi \text{ m}) = 2\pi$

24.
$$u_{gr} = \frac{d\omega}{dk} = \frac{dE}{dp} = \frac{d}{dp}\left(p^2c^2 + E_0^2\right)^{1/2} = \frac{pc^2}{\sqrt{p^2c^2 + E_0^2}} = \frac{pc^2}{E} = \beta c$$

$$v_{ph} = \lambda\nu = \frac{h}{p}\frac{\omega}{2\pi} = \frac{E}{p} = \frac{pc^2/v}{p} = \frac{c^2}{v} = \frac{c}{\beta}$$

The particle and its "signal" are associated with the group velocity, not the phase velocity.

*25. As in Example 5.4 $u_{gr} = c\lambda^n - cn\lambda^n$. Setting $u_{gr} = v_{ph} = c\lambda^n$, we find $c\lambda^n = c\lambda^n - cn\lambda^n$. This can be satisfied only if $n = 0$, so v_{ph} is independent of λ. This is consistent with the idea that when a medium is non-dispersive, the phase and group velocities are equal and the speed independent of wavelength.

26. Protons:

$$\gamma = \frac{K + E_0}{E_0} = \frac{946.27 \text{ MeV}}{938.27 \text{ MeV}} = 1.00853 \qquad \beta = \sqrt{1 - \frac{1}{1.00853^2}} = 0.130$$

$$u_{gr} = \beta c = 0.130c \qquad v_{ph} = \frac{c}{\beta} = 7.7c$$
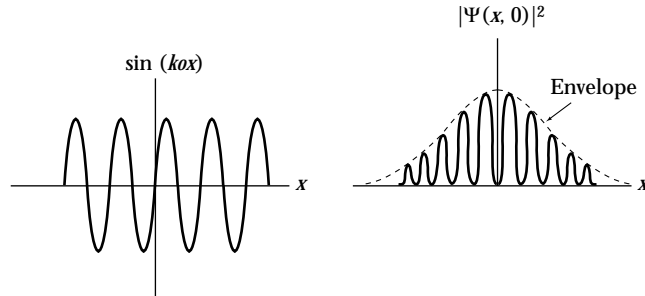
Electrons:

$$\gamma = \frac{K + E_0}{E_0} = \frac{8.511 \text{ MeV}}{0.511 \text{ MeV}} = 16.66 \qquad \beta = \sqrt{1 - \frac{1}{16.66^2}} = 0.9982$$

$$u_{gr} = \beta c = 0.9982c \qquad v_{ph} = \frac{c}{\beta} = 1.002c$$

27.

$$\begin{aligned}
\Psi(x, 0) &= \int \tilde{A}(k) \cos(kx)\, dk = A_0 \int_{k_0 - \Delta k/2}^{k_0 + \Delta k/2} \cos(kx)\, dk \\
&= A_0 \left. \frac{\sin(kx)}{x} \right|_{k_0 - \Delta k/2}^{k_0 + \Delta k/2} = \frac{A_0}{x} \left(\sin\left(k_0 + \Delta k/2\right) x - \sin\left(k_0 - \Delta k/2\right) x \right) \\
&= -\frac{2A_0}{x} \sin\left(\frac{\Delta k x}{2}\right) \sin\left(k_0 x\right)
\end{aligned}$$

See the diagrams below. At the half-width of $|\Psi(x, 0)|^2$, we have $\sin^2(\Delta k x/2) = 1/2$, so $\Delta k x/2 = \pi/4$ and $x = \pi/(2\Delta k)$. Then $\Delta x = 2x = \pi/\Delta k$, and $\Delta k \Delta x = \pi$.

*28. Relativistically

$$u = \frac{dE}{dp} = \frac{d}{dp}\left(p^2 c^2 + E_0^2\right)^{1/2} = \frac{pc^2}{\sqrt{p^2 c^2 + E_0^2}} = \frac{pc^2}{E}$$

Classically

$$u = \frac{dE}{dp} = \frac{d}{dp}\left(\frac{p^2}{2m}\right) = \frac{p}{m}$$

29. For a double slit, the amplitude of E is doubled, and hence the intensity (proportional to E^2) is higher by a factor of four for the double slit.

30.

$$d = \frac{\lambda}{\sin\theta} = \frac{h/p}{\sin\theta} = \frac{hc}{\sqrt{E^2 - E_0^2}\,\sin\theta} = \frac{1240 \text{ eV} \cdot \text{nm}}{\sqrt{(512 \text{ keV})^2 - (511 \text{ keV})^2}\,\sin 1°} = 2.22 \text{ nm}$$

*31.

$$\sin\theta \cong \tan\theta = \frac{0.3 \text{ mm}}{0.8 \text{ m}} = 3.75 \times 10^{-4}$$

$$\lambda = d\sin\theta \cong d\theta = (2000 \text{ nm})\left(3.75 \times 10^{-4}\right) = 0.75 \text{ nm}$$

$$p = \frac{h}{\lambda} = \frac{hc}{\lambda c} = \frac{1240 \text{ eV} \cdot \text{nm}}{(0.75 \text{ nm})\,c} = 1.653 \text{ keV}/c$$

$$K = E - E_0 = \sqrt{p^2c^2 + E_0^2} - E_0 = \sqrt{(1.653 \text{ keV})^2 + (511 \text{ keV})^2} - 511 \text{ keV} = 2.67 \text{ eV}$$

Such low energies will present problems, because low-energy electrons are more easily deflected by stray electric fields.

32. Normalization:

$$\int_0^L A^2 \sin^2\left(\frac{\pi x}{L}\right) dx = 1 = A^2 \frac{L}{2} \qquad\qquad \text{or} \qquad\qquad A = \sqrt{\frac{2}{L}}$$

33. As before $\lambda = 2L/n$ so $p_n = nh/2L$. At high energies we must use relativity, so

$$E = \sqrt{p^2c^2 + E_0^2} = E_0\sqrt{\frac{p^2c^2}{E_0^2} + 1}$$

$$\frac{E_2}{E_1} = \left[\frac{1 + h^2c^2/L^2E_0^2}{1 + h^2c^2/4L^2E_0^2}\right]^{1/2}$$

$$\frac{E_3}{E_1} = \left[\frac{1 + 9h^2c^2/4L^2E_0^2}{1 + h^2c^2/4L^2E_0^2}\right]^{1/2}$$

$$\frac{E_4}{E_1} = \left[\frac{1 + 4h^2c^2/L^2E_0^2}{1 + h^2c^2/4L^2E_0^2}\right]^{1/2}$$

These are quite different from the non-relativistic results, as one might expect. They do reduce to the non-relativistic results in the low-energy limit.

34.

$$\psi_1 = A \sin\left(\frac{\pi x}{L}\right) \qquad\qquad \psi_2 = A \sin\left(\frac{2\pi x}{L}\right) \qquad\qquad \psi_3 = A \sin\left(\frac{3\pi x}{L}\right)$$

where $A = \sqrt{2/L}$

35. For the $n = 1$ energy level

$$E = \frac{h^2}{8md^2} = \frac{h^2 c^2}{8mc^2 d^2} = \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8\,(939.57 \times 10^6 \text{ eV})\,(2 \times 10^{-6} \text{ nm})^2} = 51.1 \text{ MeV}$$

By the uncertainty principle $\Delta p \Delta x = \hbar/2$ at minimum. Non-relativistically (with $\Delta x = d$)

$$E_{\min} = \frac{(\Delta p)^2}{2m} = \frac{\hbar^2}{8md^2} = 1.30 \text{ MeV}$$

*36. The uncertainty ratio is the same for any mass.

$$\Delta p \Delta x = m\Delta v \Delta x \geq \frac{\hbar}{2} \qquad\qquad \text{or} \qquad\qquad \Delta v \geq \frac{\hbar}{2m\Delta x} = \frac{\hbar}{2mL}$$

$$E = \frac{1}{2}mv^2 = \frac{h^2}{8mL^2} \qquad\qquad \text{or} \qquad\qquad v = \sqrt{\frac{h^2}{4m^2 L^2}} = \frac{h}{2mL}$$

$$\frac{\Delta v}{v} = \frac{\hbar/2mL}{h/2mL} = \frac{1}{2\pi}$$

37. For circular motion $L = rp$ and so $\Delta L = r\Delta p$. Along the circle $x = r\theta$ and $\Delta x = r\Delta\theta$. Thus

$$\Delta p \Delta x = \frac{\Delta L}{r}\,(r\,\Delta\theta) = \Delta L \Delta\theta \geq \frac{\hbar}{2}$$

For complete uncertainty $\Delta\theta = 2\pi$ and

$$\Delta L = \frac{\hbar/2}{2\pi} = \frac{\hbar}{4\pi}$$

38. $\Delta E \Delta t \geq \hbar/2$ so

$$\Delta E \geq \frac{\hbar}{2\Delta t} = \frac{1.054 \times 10^{-34} \text{ J} \cdot \text{s}}{(2 \times 10^{36} \text{ y})\,(3.16 \times 10^7 \text{ s/y})} = 1.67 \times 10^{-78} \text{ J}$$

39. If we use $\Delta\omega\Delta t = 1/2$ we find

$$\Delta\omega = \frac{1}{2\Delta t} = \frac{1}{2\,(2\ \mu s)} = 2.5 \times 10^5 \text{ rad/s}$$

40. $\Delta p\Delta x = m\Delta v\Delta x \geq \hbar/2$ so at minimum uncertainty

$$\Delta v = \frac{\hbar}{2m\Delta x} = \frac{1.054 \times 10^{-34} \text{ J} \cdot \text{s}}{2\,(3 \times 10^{-15} \text{ kg})\,(10^{-6} \text{ m})} = 1.76 \times 10^{-14} \text{ m/s}$$

*41. a) $\Delta E\Delta t \geq \hbar/2$ so

$$\Delta E \geq \frac{\hbar}{2\Delta t} = \frac{6.582 \times 10^{-16} \text{ eV} \cdot \text{s}}{2 \times 10^{-13} \text{ s}} = 3.29 \times 10^{-3} \text{ eV}$$

b) Using the photon relation $E = hc/\lambda$ and taking a derivative

$$dE = -\frac{hc}{\lambda^2}d\lambda = -\frac{E^2}{hc}d\lambda$$

Then letting $\Delta\lambda = d\lambda$ and $\Delta E = dE$ we have

$$|\Delta\lambda| = hc\frac{\Delta E}{E^2} = (1240 \text{ eV} \cdot \text{nm})\frac{3.29 \times 10^{-3} \text{ eV}}{(4.7 \text{ eV})^2} = 0.18 \text{ nm}$$

*42. The wavelength of the electrons should be 0.14 nm or less. For this wavelength

$$p = \frac{h}{\lambda} = \frac{hc}{\lambda c} = \frac{1240 \text{ eV} \cdot \text{nm}}{(0.14 \text{ nm})\ c} = 8.86 \text{ keV}/c$$

$$K = E - E_0 = \sqrt{p^2c^2 + E_0^2} - E_0 = \sqrt{(8.86 \text{ keV})^2 + (511 \text{ keV})^2} - 511 \text{ keV} = 77 \text{ eV}$$

43. For the angle θ_R we find $\theta_R \cong \tan\theta_R = 4000 \text{ nm}/20 \text{ cm} = 2 \times 10^{-5}$. The wavelength is

$$\lambda = \frac{d\theta_R}{1.22} = \frac{(0.05 \text{ m})\,(2 \times 10^{-5})}{1.22} = 820 \text{ nm}$$

a)
$$E = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{820 \text{ nm}} = 1.51 \text{ eV}$$

b) For non-relativistic electrons

$$K = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} = \frac{h^2c^2}{2mc^2\lambda^2} = \frac{(1240 \text{ eV} \cdot \text{nm})^2}{2\,(511 \times 10^3 \text{ eV})\,(820 \text{ nm})^2} = 2.24 \times 10^{-6} \text{ eV}$$

Check with the uncertainty principle:

$$\Delta p \geq \frac{\hbar}{2\Delta x} = \frac{1.055 \times 10^{-34} \text{ J} \cdot \text{s}}{2\,(4000 \text{ nm})} = 1.32 \times 10^{-29} \text{ kg} \cdot \text{m/s}$$

The actual momentum is

$$p = \frac{h}{\lambda} = \frac{6.626 \times 10^{-34} \text{ J} \cdot \text{s}}{820 \text{ nm}} = 8.08 \times 10^{-28} \text{ kg} \cdot \text{m/s}$$

This is allowed because $p > \Delta p$.

44.

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mK}} = \frac{1240 \text{ eV} \cdot \text{nm}}{\sqrt{2\,(3727 \text{ MeV})\,(5.5 \text{ MeV})}} = 6.12 \text{ fm}$$

The minimum kinetic energy according to the uncertainty principle is

$$K = \frac{(\Delta p)^2}{2m} = \frac{\hbar^2}{8m\,(\Delta x)^2} = \frac{(1240 \text{ eV} \cdot \text{nm})^2 / 4\pi^2}{8\,(3727 \text{ MeV})\,(16 \text{ fm})^2} = 5.10 \text{ keV}$$

Since the kinetic energy exceeds the minimum, it is allowed.

*45. The proof is done in Example 6.9. With $\omega = \sqrt{k/m}$ we have a minimum energy

$$E = \frac{\hbar\omega}{2} = \frac{\hbar}{2}\sqrt{\frac{k}{m}} = \frac{1.055 \times 10^{-34} \text{ J} \cdot \text{s}}{2}\sqrt{\frac{8 \text{ N/m}}{0.002 \text{ kg}}} = 3.34 \times 10^{-33} \text{ J}$$

46. At time $t = 0$ the velocity is uncertain by at least $\Delta v_0 = \Delta p/m = \hbar/2m\Delta x$. After a time $t = m\,(\Delta x)^2/\hbar$ we have

$$\Delta x' = (\Delta v_0)\,t = \frac{\hbar}{2m\Delta x}\frac{m\,(\Delta x)^2}{\hbar} = \frac{\Delta x}{2}$$

or half the distance of travel.

47. Both the spatial distribution $\psi(x)$ and the wavenumber distribution $\phi(k)$ should have the same Gaussian form:

$$\psi(x) \cong \exp\left[-\frac{x^2}{(2\Delta x)^2}\right] \qquad \phi(k) \cong \exp\left[-\frac{k^2}{(2\Delta k)^2}\right]$$

For conjugate variables (x, k) it is possible to obtain one distribution by taking a Fourier transform of the other.

Letting A be a normalization constant for $\phi(k)$ we have

$$\psi(x) = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \exp\left[-\frac{k^2}{(2\Delta k)^2}\right] \exp\left(ikx\right)$$

The integral is done by completing the square:

$$\begin{aligned}
\psi(x) &= \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \exp\left[-\frac{k^2}{4\Delta k^2} + ikx - x^2\Delta k^2 + x^2\Delta k^2\right] \\
&= \frac{A}{\sqrt{2\pi}} \exp\left(-x^2\Delta k^2\right) \int_{-\infty}^{\infty} dk \, \exp\left[\left(-\frac{1}{4\Delta k^2}\right)\left(k - 2ix\Delta k^2\right)^2\right]
\end{aligned}$$

Letting $u = \left(k - 2ix\Delta k^2\right)/2\Delta k$ we have

$$\psi(x) = A\sqrt{\frac{2}{\pi}} \Delta k \exp\left(-x^2\Delta k^2\right) \int_{-\infty}^{\infty} \exp\left(-u^2\right) \, du$$

The integral has a value $\sqrt{\pi}$ so

$$\psi(x) = \sqrt{2}A\Delta k \exp\left(-x^2\Delta k^2\right)$$

Now comparing with the Gaussian form

$$\psi(x) \cong \exp\left[-\frac{x^2}{(2\Delta x)^2}\right]$$

we see that

$$(\Delta k)^2 = \frac{1}{(2\Delta x)^2} \qquad \text{or} \qquad \Delta k \Delta x = \frac{1}{2}$$

48. $\Delta E \Delta t \geq \hbar/2$

$$\Delta E \geq \frac{\hbar}{2\Delta t} = \frac{6.582 \times 10^{-16} \text{ eV} \cdot \text{s}}{2 \times 10^{-16} \text{ s}} = 3.29 \text{ eV}$$

*49.

$$\Delta t = \frac{d}{c} = \frac{1.2 \times 10^{-15} \text{ m}}{3.0 \times 10^{8} \text{ m/s}} = 4.0 \times 10^{-24} \text{ s}$$

$$\Delta E \geq \frac{\hbar}{2\Delta t} = \frac{6.582 \times 10^{-16} \text{ eV} \cdot \text{s}}{2\left(4.0 \times 10^{-24} \text{ s}\right)} = 82 \text{ MeV}$$

This "lower bound" estimate of the rest energy is within a factor of two of the rest energy.

50. a) In general $n\lambda = 2d\sin\theta$ so

$$\sin\theta = \frac{n\lambda}{2d} = n\frac{0.5 \text{ nm}}{2\,(0.8 \text{ nm})} = 0.3125n$$

It is required that $\sin\theta \le 1$ so the allowed values of n are $1, 2, 3$. Plugging in we find $\theta = 18.2°$ for $n = 1$, $\theta = 38.7°$ for $n = 2$, and $\theta = 69.6°$ for $n = 3$.

b) For electrons

$$p = \frac{h}{\lambda} = \frac{hc}{\lambda c} = \frac{1240 \text{ eV} \cdot \text{nm}}{(0.5 \text{ nm})\,c} = 2480 \text{ eV}/c$$

$$K = E - E_0 = \sqrt{p^2c^2 + E_0^2} - E_0 = \sqrt{(2.480 \text{ keV})^2 + (511 \text{ keV})^2} - 511 \text{ keV} = 6.0 \text{ eV}$$

51. The uncertainty of the strike zone width is $\Delta x < 0.38$ m. There is a second uncertainty, in the x-component of the ball's velocity, given by

$$\Delta p_x = m\Delta v_x \ge \frac{\hbar}{2\,\Delta x}$$

Due to the uncertainty in velocity the ball's x position will be uncertain by the time it reaches home plate in an amount

$$\Delta v_x t = \frac{\hbar t}{2m\,\Delta x} \le 0.38 \text{ m}$$

where we have added the inequality because this is the condition for a strike. Since this inequality must be satisfied simultaneously with the first one $\Delta x < 0.38$ m, we multiply the two inequalities together to find

$$\frac{\hbar t}{2m} \le (0.38 \text{ m})^2 = 0.144 \text{ m}^2$$

Rearranging we find

$$\hbar \le \frac{2m\,(0.144 \text{ m}^2)}{t} = \frac{2\,(0.145 \text{ kg})\,(0.144 \text{ m}^2)}{(18 \text{ m})\,/\,(35 \text{ m/s})} = 0.081 \text{ J} \cdot \text{s}$$