Chapter 9

1. a)

$$\begin{aligned}\overline{v_x^2} &= \int_{-\infty}^{\infty} v_x^2 g(v_x)\,dv_x = C'\int_{-\infty}^{\infty} v_x^2 \exp\left(-\frac{1}{2}\beta m v_x^2\right)\,dv_x \\ &= 2C'\int_0^{\infty} v_x^2 \exp\left(-\frac{1}{2}\beta m v_x^2\right)\,dv_x = 2\left(\frac{\beta m}{2\pi}\right)^{1/2}\frac{\sqrt{\pi}}{4}\left(\frac{2}{\beta m}\right)^{3/2} = \frac{1}{\beta m}\end{aligned}$$

Therefore

$$v_{x\text{rms}} = \left(\overline{v_x^2}\right)^{1/2} = \left(\frac{1}{\beta m}\right)^{1/2} = \left(\frac{kT}{m}\right)^{1/2}$$

b)

$$g(v_x) = \left(\frac{\beta m}{2\pi}\right)^{1/2}\exp\left(-\frac{1}{2}\beta m v_x^2\right)$$

and from (a) we see that $(\beta m)^{1/2} = v_{x\text{rms}}^{-1}$, so

$$g(v_x)\,dv_x = \frac{1}{\sqrt{2\pi}}v_{x\text{rms}}^{-1}\exp\left(-\frac{1}{2}\frac{v_x^2}{v_{x\text{rms}}^2}\right)\,dv_x$$

2. a) With $v_x = 0.01 v_{x\text{rms}}$ we have $\exp\left(-\frac{1}{2}\frac{v_x^2}{v_{x\text{rms}}^2}\right) \cong 1$.

$$g(v_x)\,dv_x = \frac{1}{\sqrt{2\pi}}v_{x\text{rms}}^{-1}\exp\left(-\frac{1}{2}\frac{v_x^2}{v_{x\text{rms}}^2}\right)\,dv_x = \frac{1}{\sqrt{2\pi}}v_{x\text{rms}}^{-1}\,(1)\,(0.002 v_{x\text{rms}}) = 7.98\times 10^{-4}$$

This is the probability that a given molecule will be in this range, so in one mole the number is

$$N = \left(7.98\times 10^{-4}\right)N_A = \left(7.98\times 10^{-4}\right)\left(6.022\times 10^{23}\right) = 4.81\times 10^{20}$$

b) As in (a) we find $N = 4.71\times 10^{20}$.

c) $N = 2.91\times 10^{20}$

d) $N = 1.79\times 10^{15}$

e) In this case

$$g(v_x)\,dv_x = \left(7.98\times 10^{-4}\right)\exp\left(-5\times 10^{3}\right)$$

which is on the order of 10^{-2175}. Therefore we conclude no molecules travel at that speed.

3. a)

$$\overline{\nu} = \overline{\nu_0\left(1+\frac{v}{c}\right)} = \nu_0\left(1+\frac{\overline{v_x}}{c}\right) = \nu_0\,(1+0) = \nu_0$$

b)

$$\sigma = \left(\overline{(\nu - \nu_0)^2}\right)^{1/2} = \left(\overline{\left(\frac{\nu_0 v_x}{c}\right)^2}\right)^{1/2} = \nu_0^2 \frac{\overline{v_x^2}}{c^2}$$

But we know that $\overline{v_x^2} = kT/m$, so

$$\sigma = \left(\frac{\nu_0^2}{c^2}\frac{kT}{m}\right)^{1/2} = \frac{\nu_0}{c}\sqrt{\frac{kT}{m}}$$

c) From (b) we have $\sigma/\nu_0 = \frac{1}{c}\sqrt{kT/m}$.

$$\text{H}_2 \text{ at } T = 293 \text{ K: } \frac{\sigma}{\nu_0} = \frac{1}{3.00 \times 10^8 \text{ m/s}}\sqrt{\frac{1.381 \times 10^{-23} \text{ J/K } (293 \text{ K})}{2\,(1.674 \times 10^{-27} \text{ kg})}} = 3.66 \times 10^{-6}$$

$$\text{H at } T = 5500 \text{ K: } \frac{\sigma}{\nu_0} = \frac{1}{3.00 \times 10^8 \text{ m/s}}\sqrt{\frac{1.381 \times 10^{-23} \text{ J/K } (5500 \text{ K})}{(1.674 \times 10^{-27} \text{ kg})}} = 2.25 \times 10^{-5}$$

This is how we could deduce the surface temperature of a star.

4. a) Letting d be the distance between the two atoms we have

$$\begin{aligned} I_x &= 2\left(mr^2\right) = 2m\left(\frac{d}{2}\right)^2 = \frac{md^2}{2} = \frac{16\,(1.66 \times 10^{-27} \text{ kg})\,(8.5 \times 10^{-10} \text{ m})^2}{2} \\ &= 9.59 \times 10^{-45} \text{ kg} \cdot \text{m}^2 \end{aligned}$$

b)

$$\begin{aligned} I_z &= 2\left(\frac{2}{5}mR^2\right) = \frac{4}{5}mR^2 = 0.8\,(16)\left(1.66 \times 10^{-27} \text{ kg}\right)\left(3.0 \times 10^{-15} \text{ m}\right)^2 \\ &= 1.91 \times 10^{-55} \text{ kg} \cdot \text{m}^2 \end{aligned}$$

c) The rigid rotator is quantized (see Chapter 10) with an energy

$$E = \frac{\hbar^2 l\,(l+1)}{2I} = \frac{(1.055 \times 10^{-34} \text{ J} \cdot \text{s})^2\,(1)\,(2)}{2\,(9.59 \times 10^{-45} \text{ kg} \cdot \text{m}^2)} = 1.16 \times 10^{-24} \text{ J}$$

d) Rearranging the energy equation in (c) we find

$$l(l+1) = \frac{2IE}{\hbar^2} = \frac{2\,(1.20 \times 10^{-56} \text{ kg} \cdot \text{m}^2)\,(1.16 \times 10^{-24} \text{ J})}{(1.055 \times 10^{-34} \text{ J} \cdot \text{s})^2} = 2.5 \times 10^{-12}$$

Equipartition requires that the available energy be shared equally among accessible degrees of freedom. We have shown that the rotation about the z-axis is not accessible. Therefore, the rotation of the diatomic molecule proceeds as if there were only two degrees of rotational freedom.

*5. a)

$$\int_c^\infty F(v)\,dv = 4\pi C \int_c^\infty v^2 \exp\left(-\frac{1}{2}\beta m v^2\right)\,dv$$

with $T = 293$ K and $C = (\beta m/2\pi)^{3/2}$.

b) For example for H_2 gas at $T = 293$ K we have

$$\frac{1}{2}\beta m c^2 = \frac{(1)\,(2)\,(938\times 10^6\ \text{eV})}{2\,(8.62\times 10^{-5}\ \text{eV/K})\,(293\ \text{K})} = 3.7\times 10^{10}$$

The exponential of the negative of this value is $\exp\left(-3.7\times 10^{10}\right)$ which is almost zero.

6. Computations depend on the software but should yield numbers very close to zero.

*7. a)

$$\overline{v} = \frac{4}{\sqrt{2\pi}}\sqrt{\frac{kT}{m}} = \frac{4}{\sqrt{2\pi}}\sqrt{\frac{(1.381\times 10^{-23}\ \text{J/K})\ \ (500\ \text{K})}{(1.675\times 10^{-27}\ \text{kg})}} = 3240\ \text{m/s}$$

$$v^* = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2\,(1.381\times 10^{-23}\ \text{J/K})\ \ (500\ \text{K})}{(1.675\times 10^{-27}\ \text{kg})}} = 2870\ \text{m/s}$$

b)

$$\overline{v} = \frac{4}{\sqrt{2\pi}}\sqrt{\frac{kT}{m}} = \frac{4}{\sqrt{2\pi}}\sqrt{\frac{(1.381\times 10^{-23}\ \text{J/K})\ \ (2500\ \text{K})}{(1.675\times 10^{-27}\ \text{kg})}} = 7240\ \text{m/s}$$

$$v^* = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2\,(1.381\times 10^{-23}\ \text{J/K})\ \ (2500\ \text{K})}{(1.675\times 10^{-27}\ \text{kg})}} = 6420\ \text{m/s}$$

8.

$$F(v) = 4\pi C \exp\left(-\frac{1}{2}\beta m v^2\right) v^2$$

In the limit as $v \to 0$, the exponential reduces to $e^0 = 1$ and v^2 approaches zero, so clearly

$$\lim_{v\to 0} F(v) = 0$$

The other limit is

$$\lim_{v\to\infty} F(v) = 4\pi C \lim_{v\to\infty} \frac{v^2}{\exp\left(\frac{1}{2}\beta m v^2\right)}$$

Applying L'Hopital's rule,

$$\lim_{v\to\infty} F(v) = \lim_{v\to\infty} \frac{2v}{\beta m v \exp\left(\frac{1}{2}\beta m v^2\right)} = \lim_{v\to\infty} \frac{2}{\beta m} \exp\left(-\frac{1}{2}\beta m v^2\right) = 0$$

9. a)

$$v^* = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2\,(1.381 \times 10^{-23}\ \text{J/K})\ \ (263\ \text{K})}{28\,(1.6605 \times 10^{-27}\ \text{kg})}} = 395\ \text{m/s}$$

b)

$$v^* = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2\,(1.381 \times 10^{-23}\ \text{J/K})\ \ (308\ \text{K})}{28\,(1.6605 \times 10^{-27}\ \text{kg})}} = 428\ \text{m/s}$$

10. The equation to be satisfied is

$$2v^2 \exp\left(-\frac{1}{2}\beta m v^2\right) = v^{*2} \exp\left(-\frac{1}{2}\beta m v^{*2}\right) = \frac{2kT}{m}e^{-1}$$

where we have used the fact that $v^* = \sqrt{2kT/m}$. Thus

$$v^2 \exp\left(-\frac{1}{2}\beta m v^2\right) = \frac{kT}{m}e^{-1} \cong 28000$$

which can be solved graphically to yield $v = 188$ m/s and $v = 639$ m/s. The lower of these is closer to $v^* = 390$ m/s, which follows from the shape of the distribution curve.

11. Various software packages should all give results very close to 1.

12. Typical values are (as a fraction of the total number of molecules):

a) 2×10^{-10} b) 2×10^{-4} c) 0.157 d) 0.496 e) 0.347 f) 0.99987

13. a)

$$\overline{E} = \int_0^\infty E\,F(E)\,dE = \frac{8\pi C}{\sqrt{2}m^{3/2}} \int_0^\infty E^{3/2} \exp\left(-\beta E\right)\,dE = \frac{8\pi C}{\sqrt{2}m^{3/2}} \frac{\Gamma(5/2)}{\beta^{5/2}}$$

Using $\Gamma(5/2) = \frac{3}{2}\Gamma(3/2) = 3\sqrt{\pi}/4$ and $C = (\beta m/2\pi)^{3/2}$ we find

$$\overline{E} = \frac{8\pi}{\sqrt{2}m^{3/2}} \left(\frac{\beta m}{2\pi}\right)^{3/2} \frac{3\sqrt{\pi}}{4\beta^{5/2}} = \frac{3}{2\beta} = \frac{3}{2}kT$$

b) As we know from the text $\overline{E} = \frac{1}{2}m\overline{v^2}$ and by Equation (9.17)

$$\frac{1}{2}m\overline{v}^2 = \frac{4}{\pi}kT \cong 1.27kT$$

which is a bit less than $\frac{3}{2}kT$.

*14. Starting with the distribution

$$F(E) = \frac{8\pi C}{\sqrt{2}m^{3/2}} E^{1/2} \exp(-\beta E)$$

and setting $dF/dE = 0$, we get

$$0 = \frac{d}{dE}\left[E^{1/2}\exp(-\beta E)\right] = \frac{1}{2}E^{-1/2}\exp(-\beta E) - \beta E^{1/2}\exp(-\beta E)$$

Thus $0 = E^{-1/2} + 2\beta E^{1/2}$ which solving for E gives the desired $E^* = kT/2$.

*15. The ratio of the numbers on the two levels is

$$\frac{n_2(E)}{n_1(E)} = \frac{8\exp(-\beta E_2)}{2\exp(-\beta E_1)} = 4\exp(-\beta(E_2 - E_1)) = 10^{-6}$$

$$\exp(-\beta(E_2 - E_1)) = 2.5 \times 10^{-7}$$

Taking logarithms:

$$-\beta(E_2 - E_1) = -\frac{E_2 - E_1}{kT} = \ln\left(2.5 \times 10^{-7}\right) = -15.20$$

For atomic hydrogen $E_2 - E_1 = \frac{3}{4}E_0 = 10.20$ eV. Finally

$$T = -\frac{E_2 - E_1}{k(-15.20)} = -\frac{10.20 \text{ eV}}{(8.617 \times 10^{-5} \text{ eV/K})(-15.20)} = 7790 \text{ K}$$

16. a) With $E = p^2/2m$ and the mean energy $E = \frac{3}{2}kT$ we get

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mK}} = \frac{h}{\sqrt{3mkT}}$$

b) We have $\lambda \ll d$. Using λ from part (a) and $d = (V/N)^{1/3}$ we get

$$\frac{h}{\sqrt{3mkT}} \ll \left(\frac{V}{N}\right)^{1/3}$$

If we cube both sides and rearrange,

$$\frac{N}{V}\frac{h^3}{(3mkT)^{3/2}} \ll 1$$

c) For any ideal gas

$$\frac{N}{V} = \frac{6.022 \times 10^{23}}{22.4 \times 10^{-3} \text{ m}^3} = 2.69 \times 10^{25} \text{ m}^{-3}$$

For argon gas (a monatomic gas) at room temperature

$$\begin{aligned}\frac{N}{V}\frac{h^3}{(3mkT)^{3/2}} &= \left(2.69\times10^{25}\ \mathrm{m}^{-3}\right)\frac{\left(6.626\times10^{-34}\ \mathrm{J\cdot s}\right)^3}{\left(3\,(40)\,(1.66\times10^{-27}\ \mathrm{kg})\,(1.38\times10^{-23}\ \mathrm{J/K})\,(293\ \mathrm{K})\right)^{3/2}} \\ &\cong 3\times10^{-7}\end{aligned}$$

so Maxwell-Boltzmann statistics are fine. However, for electrons in silver $N/V = 8.47\times10^{28}$ m^{-3} and

$$\begin{aligned}\frac{N}{V}\frac{h^3}{(3mkT)^{3/2}} &= \left(8.47\times10^{28}\ \mathrm{m}^{-3}\right)\frac{\left(6.626\times10^{-34}\ \mathrm{J\cdot s}\right)^3}{\left(3\,(9.11\times10^{-31}\ \mathrm{kg})\,(1.38\times10^{-23}\ \mathrm{J/K})\,(293\ \mathrm{K})\right)^{3/2}} \\ &\cong 2\times10^{4}\end{aligned}$$

and in this case Maxwell-Boltzmann statistics fail.

17.

$$F(v)\,dv = 4\pi C v^2 \exp\left(-\frac{1}{2}\beta m v^2\right)\,dv = F(E)\,dE$$

With $E=\frac{1}{2}mv^2$ we differentiate to get $dE = mv\,dv$ or $dv = dE/mv = dE/\sqrt{2mE}$. Then

$$\begin{aligned}F(E)\,dE &= 4\pi C\frac{2E}{m}\exp\left(-\beta E\right)\frac{dE}{\sqrt{2mE}} = 8\pi C\frac{E^{1/2}}{\sqrt{2}m^{3/2}}\exp\left(-\beta E\right)\,dE \\ &= \frac{8\pi C}{\sqrt{2}m^{3/2}}E^{1/2}\exp\left(-\beta E\right)\,dE\end{aligned}$$

18. a) We will assume that the magnetic moment is due to spin alone. In general $n(E) = g(E)F_{MB}$. There is no reason to prefer one spin state or the other, so the two $g(E)$ are the same. Thus the ratio of the numbers in the two spin states is governed by the Maxwell-Boltzmann distribution:

$$\frac{n(E_2)}{n(E_1)} = \frac{F_{MB}(E_2)}{F_{MB}(E_1)} = \frac{\exp\left(-\beta E_2\right)}{\exp\left(-\beta E_1\right)} = \exp\left(\beta\left(E_1 - E_2\right)\right)$$

The energy of a magnetic moment $\vec{\mu}$ in a magnetic field $\vec{B}$ is $E = -\vec{\mu}\cdot\vec{B}$. We know from Chapter 8 that this works out to be

$$E = \frac{e}{m}\vec{S}\cdot\vec{B} = \frac{e}{m}S_zB = \pm\frac{e\hbar}{2m}B = \pm\mu_B B$$

Then $E_1 = -\mu_B B$ is the energy of an electron aligned with the field, and $E_2 = +\mu_B B$ is the energy of the spin opposed to the field. Therefore

$$\frac{n(E_2)}{n(E_1)} = \exp\left(\beta\left(E_1 - E_2\right)\right) = \exp\left(\frac{-\mu_B B - \mu_B B}{kT}\right) = \exp\left(\frac{-2\mu_B B}{kT}\right)$$

b) At $T = 77$ K

$$\frac{n(E_2)}{n(E_1)} = \exp\left(\frac{-2\mu_B B}{kT}\right) = \exp\left(\frac{-2\,(9.274\times10^{-24}\ \mathrm{J/T})\,(8\ \mathrm{T})}{(1.381\times10^{-23}\ \mathrm{J/K})\,(77\ \mathrm{K})}\right) = 0.870$$

At $T = 273$ K

$$\frac{n(E_2)}{n(E_1)} = \exp\left(\frac{-2\mu_B B}{kT}\right) = \exp\left(\frac{-2\,(9.274\times10^{-24}\ \mathrm{J/T})\,(8\ \mathrm{T})}{(1.381\times10^{-23}\ \mathrm{J/K})\,(273\ \mathrm{K})}\right) = 0.961$$

At $T = 800$ K

$$\frac{n(E_2)}{n(E_1)} = \exp\left(\frac{-2\mu_B B}{kT}\right) = \exp\left(\frac{-2\,(9.274\times10^{-24}\ \mathrm{J/T})\,(8\ \mathrm{T})}{(1.381\times10^{-23}\ \mathrm{J/K})\,(800\ \mathrm{K})}\right) = 0.987$$

As the temperature is increased, the alignment of the spin with the magnetic field is less probable.

19. Setting $F_{FD} = 0.5$ when $E = E_F$, we have

$$0.5 = \frac{1}{B_1 \exp(\beta E_F) + 1}$$

Solving for B_1, we find $B_1 \exp(\beta E_F) + 1 = 2$, so $B_1 \exp(\beta E_F) = 1$ and $B_1 = \exp(-\beta E_F)$. Therefore in general

$$F_{FD} = \frac{1}{B_1 \exp(\beta E) + 1} = \frac{1}{\exp(-\beta E_F)\exp(\beta E) + 1} = \frac{1}{\exp(\beta(E - E_F)) + 1}$$

*20. At first one may think it should be 0.5, but this is not quite true, due to the asymmetric shape of the distribution. Starting with Equation (9.43) for $g(E)$ and using the fact that $F_{FD} \cong 1$ in this range, we have

$$N\left(E < \overline{E}\right) = \int_0^{\overline{E}} g(E)(1)\,dE = \frac{3}{2} N E_F^{-3/2} \int_0^{\overline{E}} E^{1/2} dE = N E_F^{-3/2} \overline{E}^{3/2}$$

But recalling that $\overline{E} = \frac{3}{5} E_F$, we see that

$$N\left(E < \overline{E}\right) = N\left(\frac{3}{5}\right)^{3/2} = 0.465N$$

21. a) From dimensional analysis

$$1.05\times10^4\ \mathrm{kg/m^3}\left(\frac{1\ \mathrm{mol}}{0.10787\ \mathrm{kg}}\right)\left(\frac{6.022\times10^{23}}{\mathrm{mol}}\right) = 5.86\times10^{28}\ \mathrm{m^{-3}}$$

b) For electrons an extra factor of 2 is required due to the Pauli principle:

$$\frac{N}{V} = \frac{2A}{h^3}\left(2\pi mkT\right)^{3/2}$$

so

$$T = \frac{\left(\frac{N}{2AV}\right)^{2/3} h^2}{2\pi mk} = \frac{\left(\frac{5.86\times 10^{28}\ \mathrm{m^{-3}}}{2(1)}\right)^{2/3}\left(6.626\times 10^{-34}\ \mathrm{J\cdot s}\right)^2}{2\pi\left(9.109\times 10^{-31}\ \mathrm{kg}\right)\left(1.38\times 10^{-23}\ \mathrm{J/K}\right)} = 5.28\times 10^4\ \mathrm{K}$$

c)

$$T = \frac{\left(\frac{N}{2AV}\right)^{2/3} h^2}{2\pi mk} = \frac{\left(\frac{5.86\times 10^{28}\ \mathrm{m^{-3}}}{2(0.001)}\right)^{2/3}\left(6.626\times 10^{-34}\ \mathrm{J\cdot s}\right)^2}{2\pi\left(9.109\times 10^{-31}\ \mathrm{kg}\right)\left(1.38\times 10^{-23}\ \mathrm{J/K}\right)} = 5.28\times 10^6\ \mathrm{K}$$

22. At $T = 0$, $F_{FD} = 1$ and so $n(E) = g(E)\ (1) = g(E)$. The number of electrons in this range is given by

$$\begin{aligned}\int_{0.90E_F}^{E_F} g(E)\,dE &= \frac{3}{2}NE_F^{-3/2}\int_{0.90E_F}^{E_F} E^{1/2}\,dE = NE_F^{-3/2}\ E^{3/2}\Big|_{0.90E_F}^{E_F} \\ &= N\left(1.00^{3/2} - 0.90^{3/2}\right) \cong 0.146N\end{aligned}$$

We see that about 14.6% of the electrons are in this range, which is about what one would expect from the shape of the distribution.

*23. a) As in Problem 21, $N/V = 5.86\times 10^{28}\ \mathrm{m^{-3}}$. Then

$$\begin{aligned}E_F &= \frac{h^2}{8m}\left(\frac{3}{\pi}\frac{N}{V}\right)^{2/3} = \frac{\left(6.626\times 10^{-34}\ \mathrm{J\cdot s}\right)^2}{8\left(9.109\times 10^{-31}\ \mathrm{kg}\right)}\left(\frac{3}{\pi}\left(5.86\times 10^{28}\ \mathrm{m^{-3}}\right)\right)^{2/3} \\ &= 8.81\times 10^{-19}\ \mathrm{J}\ = 5.50\ \mathrm{eV}\end{aligned}$$

b)

$$u_F = \sqrt{\frac{2E_F}{m}} = \sqrt{\frac{2\left(8.81\times 10^{-19}\ \mathrm{J}\right)}{9.109\times 10^{-31}\ \mathrm{kg}}} = 1.39\times 10^6\ \mathrm{m/s}$$

24. a) Note: the term $\alpha\,(kT)^2/E_F$ is a small fraction of one eV and can be ignored. Then

$$\overline{E} = \frac{3}{5}E_F = \frac{3}{5}\left(5.51\ \mathrm{eV}\right) = 3.31\ \mathrm{eV}$$

b) With $\overline{E} = \frac{3}{2}kT$ we have

$$T = \frac{2\overline{E}}{3k} = \frac{2\left(3.31\ \mathrm{eV}\right)}{3\left(8.617\times 10^{-5}\ \mathrm{eV/K}\right)} = 2.56\times 10^4\ \mathrm{K}$$

c) As discussed in the text, thermal energies are small compared with the Fermi energy, except at high temperatures.

25.

$$8.92 \times 10^3 \text{ kg/m}^3 \left(\frac{1 \text{ mol}}{0.063546 \text{ kg}}\right)\left(\frac{6.022 \times 10^{23}}{\text{mol}}\right) = 8.45 \times 10^{28} \text{ m}^{-3}$$

The difference is 0.2%. Within rounding errors there is one conduction electron per atom.

26. a)

$$2.70 \times 10^3 \text{ kg/m}^3 \left(\frac{1 \text{ mol}}{0.02698 \text{ kg}}\right)\left(\frac{6.022 \times 10^{23}}{\text{mol}}\right) = 6.03 \times 10^{28} \text{ m}^{-3}$$

b)

$$E_F = \frac{h^2}{8m}\left(\frac{3}{\pi}\frac{N}{V}\right)^{2/3}$$

so

$$\begin{aligned}\frac{N}{V} &= \frac{\pi}{3}\left(\frac{8mE_F}{h^2}\right)^{3/2} = \frac{\pi}{3}\left(\frac{8\,(9.109 \times 10^{-31} \text{ kg})\,(11.63 \text{ eV})\,(1.602 \times 10^{-19} \text{ J/eV})}{(6.626 \times 10^{-34} \text{ J}\cdot\text{s})^2}\right)^{3/2} \\ &= 1.81 \times 10^{29} \text{ m}^{-3}\end{aligned}$$

c) Dividing he conduction electron density by the number density we get almost exactly 3, from which we conclude that the valence number is three.

*27. In general $E_F = \frac{1}{2}mu_F^2$, so $u_F = \sqrt{2E_F/m}$.

a)

$$u_F = \sqrt{\frac{2E_F}{m}} = \sqrt{\frac{2\,(3.93 \text{ eV})\,(1.602 \times 10^{-19} \text{ J/eV})}{9.109 \times 10^{-31} \text{ kg}}} = 1.18 \times 10^6 \text{ m/s}$$

b)

$$u_F = \sqrt{\frac{2E_F}{m}} = \sqrt{\frac{2\,(9.47 \text{ eV})\,(1.602 \times 10^{-19} \text{ J/eV})}{9.109 \times 10^{-31} \text{ kg}}} = 1.83 \times 10^6 \text{ m/s}$$

28.

$$E > E_F\text{:}\ \frac{E - E_F}{kT} \to \infty \text{ so } F_{FD} \to 0$$

$$E < E_F\text{:}\ \frac{E - E_F}{kT} \to -\infty \text{ so } F_{FD} \to 1$$

$$E = E_F\text{:}\ \frac{E - E_F}{kT} \to 0 \text{ so } F_{FD} \to \frac{1}{2}$$

29. In general $n(E) = g(E)F_{FD}$. Using Equation (9.43) for $g(E)$ and the result of Problem 19 for F_{FD}, we can substitute to find

$$n(E) = \frac{3N}{2} E_F^{-3/2} \frac{E^{1/2}}{\exp\left(\beta\left(E - E_F\right)\right) + 1}$$

30. Graphs will resemble those in Figure 9.10 (b). The $T = 0$ line matches the dashed line shown, and at $T = 293$ K we get the solid line. At the higher temperature (1800 K) the graph deviates a bit more from the dashed line.

31. Numerical integration should yield accurate results.

$$1.5\,(7)^{-3/2} \int_0^\infty \frac{E^{1/2}}{\exp\left(\left(E - 7\right) / \left(0.02525\right)\right) + 1} dE \cong 1$$

32. Setting up the numerical integration in Maple we have with $kT = 0.02525$ eV,

$$1.5\,(7)^{-3/2} \int_6^7 \frac{E^{1/2}}{\exp\left(\left(E - 7\right) / \left(0.02525\right)\right) + 1} dE \cong 0.203$$

So we see that about one-fifth of the electrons are within 1 eV of the Fermi energy, which makes sense given the shape of the distribution.

33. We can use the relationship (9.42)

$$E_F = \frac{h^2}{8m} \left(\frac{3N}{\pi L^3}\right)^{2/3}$$

We use the neutron mass and from dimensional analysis

$$\frac{N}{L^3} = \frac{4.50 \times 10^{30}\ \text{kg}}{\frac{4}{3}\pi\left(10^4\ \text{m}\right)^3} \frac{1\ (\text{neutron})}{1.675 \times 10^{-27}\ \text{kg}} = 6.41 \times 10^{44}\ \text{m}^{-3}$$

Then

$$\begin{aligned} E_F &= \frac{h^2}{8m} \left(\frac{3N}{\pi L^3}\right)^{2/3} = \frac{\left(6.626 \times 10^{-34}\ \text{J} \cdot \text{s}\right)^2}{2\left(1.675 \times 10^{-27}\ \text{kg}\right)} \left(\frac{3}{\pi}\left(6.41 \times 10^{44}\ \text{m}^{-3}\right)\right)^{2/3} = 9.45 \times 10^{-11}\ \text{J} \\ &= 590\ \text{MeV} \end{aligned}$$

The close packing of the neutrons makes the Fermi energy large compared with Fermi energies in normal matter.

*34. a) To find N/V integrate $n(E)\,dE$ over the whole range of energies:

$$\frac{N}{V} = \int_0^\infty n(E)\,dE = \frac{8\pi}{h^3c^3}\int_0^\infty \frac{E^2}{\exp(E/kT)-1}\,dE$$

From integral tables we have the following:

$$\int_0^\infty \frac{x^{n-1}}{e^{mx}-1}dx = m^{-n}\Gamma(n)\zeta(n)$$

For us $m = 1/kT$, $\Gamma(3) = 2! = 2$, and from numerical tables $\zeta(3) \cong 1.20$. Thus

$$\frac{N}{V} = \frac{8\pi}{h^3c^3}(kT)^3(2)(1.20) = \frac{8\pi k^3T^3}{h^3c^3}(2.40)$$

b) With $T = 400$ K:

$$\frac{N}{V} = \frac{8\pi k^3T^3}{h^3c^3}(2.40) = 8\pi(2.40)\left(\frac{(1.381\times10^{-23}\ \mathrm{J/K})(400\ \mathrm{K})}{(6.626\times10^{-34}\ \mathrm{J\cdot s})(2.998\times10^8\ \mathrm{m/s})}\right)^3 = 1.30\times10^{15}\ \mathrm{m^{-3}}$$

At $T = 5500$ K:

$$\frac{N}{V} = \frac{8\pi k^3T^3}{h^3c^3}(2.40) = 8\pi(2.40)\left(\frac{(1.381\times10^{-23}\ \mathrm{J/K})(5500\ \mathrm{K})}{(6.626\times10^{-34}\ \mathrm{J\cdot s})(2.998\times10^8\ \mathrm{m/s})}\right)^3 = 3.37\times10^{18}\ \mathrm{m^{-3}}$$

35. Evaluating by computer we find

$$\int_0^\infty \frac{u^{1/2}}{e^u-1}du \cong 2.315$$

36.

$$I_3 = -\frac{dI_1}{da} = -\left(-\frac{1}{2a^2}\right) = \frac{1}{2a^2}$$

$$I_4 = -\frac{dI_2}{da} = -\frac{\sqrt{\pi}}{4}\left(-\frac{3}{2}\right)a^{-5/2} = \frac{3\sqrt{\pi}}{8}a^{-5/2}$$

$$I_5 = -\frac{dI_3}{da} = -\frac{1}{2}\left(-2a^{-3}\right) = a^{-3}$$

37.

$$E = K + V = \frac{p^2}{2m} + mgz$$

$$\exp(-\beta E) = \exp\left(-\beta\left(\frac{p^2}{2m}+mgz\right)\right) = \exp\left(-\frac{\beta p^2}{2m}\right)\exp\left(-\beta mgz\right)$$

Absorbing the (assumed constant) first exponential factor into the normalization constant C_z,

$$f(z)\,dz = C_z \exp\left(-\beta m g z\right)\,dz$$

To find C_z we normalize:

$$\int_0^\infty f(z)\,dz = C_z \int_0^\infty \exp\left(-\beta m g z\right)\,dz = C_z\left(\beta m g\right)$$

Thus

$$C_z = \frac{1}{\beta m g} = \frac{kT}{mg}$$

38. For air we will use an average $m = 29$ u $= 4.82 \times 10^{-26}$ kg and $T = 273$ K. In general

$$\frac{\rho(h)}{\rho(0)} = \frac{\exp\left(-\beta m g h\right)}{\exp\left(-\beta m g 0\right)} = \exp\left(-\beta m g h\right)$$

For Denver:

$$\rho(h) = \exp\left(-\frac{(4.82 \times 10^{-26}\ \text{kg})\,(9.80\ \text{m/s}^2)\,(1610\ \text{m})}{(1.381 \times 10^{-23}\ \text{J/K})\,(273\ \text{K})}\right)\rho(0) = 0.817\rho(0)$$

For Mt. Rainier:

$$\rho(h) = \exp\left(-\frac{(4.82 \times 10^{-26}\ \text{kg})\,(9.80\ \text{m/s}^2)\,(4390\ \text{m})}{(1.381 \times 10^{-23}\ \text{J/K})\,(273\ \text{K})}\right)\rho(0) = 0.577\rho(0)$$

39. In equilibrium a fluid layer of density ρ, mass M, thickness h, and surface area A has a force $F_2 = P_2 A$ acting downward on its upper surface and a force $F_1 = P_1 A$ acting upward on its lower surface. The difference between these forces equals the weight of the fluid layer.

$$F_2 - F_1 = \left(P_1 - P_2\right)A = Mg = \rho g A h$$

Let $dP \cong \Delta P = P_2 - P_1$ and $h = \Delta z \cong dz$, we have $dP = -\rho g\,dz$. With N particles of mass m, the mass density is $\rho = Nm/V$. Putting these together:

$$dP = -\rho g\,dz = -\frac{Nmg}{V}\,dz$$

From the ideal gas law, $N/V = P/kT$, so

$$dP = -\frac{mgP}{kT}\,dz$$

Applying separation of variables we can solve this differential equation for P as a function of z:

$$\frac{dP}{P} = -\frac{mg}{kT}\,dz \qquad\qquad \ln P = -\frac{mgz}{kT} + \text{ constant } = -\beta m g z + \text{ constant}$$

$$P = (\text{constant}) \exp(-\beta mgz) = P_0 \exp(-\beta mgz)$$

40. a)

$$\frac{dN}{dt} = -\frac{n\overline{v}}{4} A = -\frac{N\overline{v}A}{4V}$$

Solving this differential equation:

$$\frac{dN}{N} = -\frac{\overline{v}A}{4V}\, dt \qquad\qquad \ln N = -\frac{\overline{v}A}{4V}\, t + \text{ constant}$$

$$N = (\text{constant}) \exp\left(-\frac{\overline{v}A}{4V}\, t\right) = N_0 \exp\left(-\frac{\overline{v}A}{4V}\, t\right)$$

Setting $N/N_0 = 1/2$ at $t = t_{1/2}$, we find

$$\frac{1}{2} = \exp\left(-\frac{\overline{v}A}{4V}\, t_{1/2}\right)$$

$$t_{1/2} = \frac{4V}{\overline{v}A} \ln 2$$

b)

$$V = \frac{\pi D^3}{6} = \frac{\pi\,(0.4 \text{ m})^3}{6} = 0.0335 \text{ m}^3$$

$$A = \frac{\pi d^2}{4} = \frac{\pi\,(0.001 \text{ m})^2}{4} = 7.85 \times 10^{-7} \text{ m}^2$$

$$\overline{v} = \frac{4}{\sqrt{2\pi}}\sqrt{\frac{kT}{m}} = \frac{4}{\sqrt{2\pi}}\sqrt{\frac{(1.381 \times 10^{-23} \text{ J/K})\,(293 \text{ K})}{29\,(1.66 \times 10^{-27} \text{ kg})}} = 462.6 \text{ m/s}$$

$$t_{1/2} = \frac{4V}{\overline{v}A} \ln 2 = \frac{4\,(0.0335 \text{ m}^3)}{(462.6 \text{ m/s})\,(7.85 \times 10^{-7} \text{ m}^2)} \ln 2 = 256 \text{ s}$$

*41. The number of molecules with speed v that hit the wall per unit time is proportional to v and $F(v)$, so that the distribution $W(v)$ of the escaping molecules is by proportion

$$W(v) \sim vF(v) \sim v^3 \exp\left(-\frac{1}{2}\beta m v^2\right)$$

Let the normalization constant for $W(v)$ be C', so

$$C' \int_0^\infty v^3 \exp\left(-\frac{1}{2}\beta m v^2\right) dv = 1 = C' \left(\frac{1}{2}\right)\left(\frac{\beta m}{2}\right)^{-2}$$

or $C' = \beta^2 m^2/2$. The mean kinetic energy of the escaping molecules is

$$\overline{E} = \frac{1}{2} m\overline{v^2} = \frac{1}{2} mC' \int_0^\infty v^5 \exp\left(-\frac{1}{2}\beta m v^2\right) dv = \frac{1}{2} m \left(\frac{\beta^2 m^2}{2}\right)\left(\frac{\beta m}{2}\right)^{-3} = \frac{2}{\beta} = 2kT$$

42. From Example 9.5

$$\frac{N}{V} = \frac{A}{h^3}(2\pi mkT)^{3/2}$$

a) Letting m be the electron mass and inserting a factor of 2 for the Pauli principle,

$$\begin{aligned}\frac{N}{V} &= \frac{2A}{h^3}(2\pi mkT)^{3/2} \\ &= \frac{2(1)}{(6.626\times10^{-34}\ \mathrm{J\cdot s})^3}\left(2\pi\left(9.109\times10^{-31}\ \mathrm{kg}\right)\left(1.381\times10^{-23}\ \mathrm{J/K}\right)(293\ \mathrm{K})\right)^{3/2} \\ &= 2.42\times10^{25}\ \mathrm{m}^{-3}\end{aligned}$$

This is quite a bit less than the density of conduction electrons in a metal (such as copper), which indicates that Fermi-Dirac statistics should be used.

b)

$$\begin{aligned}\frac{N}{V} &= \frac{2A}{h^3}(2\pi mkT)^{3/2} \\ &= \frac{2(1)}{(6.626\times10^{-34}\ \mathrm{J\cdot s})^3}\left(2\pi\left(1.6749\times10^{-27}\ \mathrm{kg}\right)\left(1.381\times10^{-23}\ \mathrm{J/K}\right)(293\ \mathrm{K})\right)^{3/2} \\ &= 1.91\times10^{30}\ \mathrm{m}^{-3}\end{aligned}$$

c) For He gas the Pauli principle does not apply, so

$$\begin{aligned}\frac{N}{V} &= \frac{A}{h^3}(2\pi mkT)^{3/2} \\ &= \frac{1}{(6.626\times10^{-34}\ \mathrm{J\cdot s})^3}\left(2\pi(4)\left(1.66\times10^{-27}\ \mathrm{kg}\right)\left(1.381\times10^{-23}\ \mathrm{J/K}\right)(293\ \mathrm{K})\right)^{3/2} \\ &= 7.54\times10^{30}\ \mathrm{m}^{-3}\end{aligned}$$

*43. For the harmonic oscillator the position and velocity are

$$x = x_0\cos(\omega t) \qquad v = \frac{dx}{dt} = -\omega x_0\sin(\omega t)$$

$$V = \frac{1}{2}kx^2 = \frac{1}{2}kx_0^2\cos^2(\omega t)$$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2x_0^2\sin^2(\omega t) = \frac{1}{2}kx_0^2\sin^2(\omega t)$$

where we have used the fact that $\omega^2 m = k$. Over one cycle the average of the square of the sine or cosine function is one-half. Also the total energy is $E = \frac{1}{2}kx_0^2$. Thus

$$\overline{K} = \overline{V} = \frac{1}{2}kx_0^2\left(\frac{1}{2}\right) = \frac{E}{2}$$