

Chapter 6

1. The function's output does not approach zero in the limits $-\infty$ and $+\infty$, so it cannot be normalized over these limits.
2. a) Given the placement of the $-$ sign, it moves in the $+x$ -direction.
 b) By the same reasoning as in (a), it moves in the $-x$ -direction.
 c) It is a complex number.
 d) It moves in the $+x$ -direction. Looking at a particular phase $kx - \omega t$, x must increase as t increases in order to keep the phase constant.
3. The derivatives are $\partial\Psi/\partial t = -i\omega\Psi$ and $\partial^2\Psi/\partial x^2 = -k^2\Psi$. Plugging into the time-dependent Schrödinger equation

$$i\hbar(-i\omega\Psi) = -\frac{\hbar^2}{2m}(-k^2\Psi) + V\Psi$$

The term on the left reduces to $\hbar\omega\Psi = E\Psi$ and the first term on the right is

$$\frac{\hbar^2 k^2}{2m}\Psi = \frac{p^2}{2m}\Psi = K\Psi$$

where K is the kinetic energy. The result is $E = K + V$, which is a statement of conservation of mechanical energy.

*4.

$$\begin{aligned}\Psi^*\Psi &= A^2 \exp[-i(kx - \omega t) + i(kx - \omega t)] = A^2 \\ \int_0^a \Psi^*\Psi dx &= A^2 \int_0^a dx = A^2 a = 1\end{aligned}$$

so $A = 1/\sqrt{a}$ and

$$\Psi = \frac{1}{\sqrt{a}} \exp[i(kx - \omega t)]$$

*5.

$$\begin{aligned}\Psi^*\Psi &= A^2 r^2 \exp\left(\frac{-2r}{\alpha}\right) \\ \int_0^\infty \Psi^*\Psi dr &= A^2 \int_0^\infty r^2 \exp\left(\frac{-2r}{\alpha}\right) dr = A^2 \left[\frac{2}{(2/\alpha)^3} \right] = \frac{A^2 \alpha^3}{4} = 1\end{aligned}$$

Therefore

$$A = \sqrt{\frac{4}{\alpha^3}} = 2\alpha^{-3/2}$$

6. In order for a particle to have a greater probability of being at given point than at an adjacent point, it would need to have infinite speed. This is seen as $p = i\hbar(\partial\Psi/\partial x) \rightarrow \infty$ at a discontinuity. Another problem is that the second derivative must exist in order to satisfy the Schrödinger equation.
7. a) The wave function does not satisfy condition 3. The derivative of the wave function is not continuous at $x = 0$.
 b) Based on (a) the wave function cannot be realized physically.
 c) Very close to $x = 0$ we could modify the function so that its derivative is continuous. If we do so just in the neighborhood of $x = 0$, we need not change the function elsewhere.

8.

$$\begin{aligned}\bar{x} &= \frac{3.4 + 3.9 + 5.2 + 4.7 + 4.1 + 3.8 + 3.9 + 4.7 + 4.1 + 4.5 + 3.8 + 4.5 + 4.8 + 3.9 + 4.4}{15} \\ &= 4.247\end{aligned}$$

$$\begin{aligned}\langle x^2 \rangle &= \frac{3.4^2 + 3.9^2 + 5.2^2 + 4.7^2 + 4.1^2 + 3.8^2 + 3.9^2 + 4.7^2 + 4.1^2 + 4.5^2 + 3.8^2 + 4.5^2 + 4.8^2 + 3.9^2 + 4.4^2}{15} \\ \langle x^2 \rangle &= 18.254\end{aligned}$$

The standard deviation is

$$\sigma = \sqrt{\frac{\sum (x_i - \bar{x})^2}{N}} = \sqrt{\frac{\sum (x_i^2 - 2x_i\bar{x} + \bar{x}^2)}{N}} = \sqrt{\frac{\sum x_i^2}{N} - 2\bar{x}\frac{\sum x_i}{N} + \frac{\sum \bar{x}^2}{N}}$$

Look at the three terms in the sum. The first is just $\langle x^2 \rangle$. The second is $-2\bar{x}(\bar{x}) = -2\bar{x}^2$. The third term is

$$\frac{\sum \bar{x}^2}{N} = \frac{N\bar{x}^2}{N} = \bar{x}^2$$

Putting the results together

$$\sigma = \sqrt{\langle x^2 \rangle - 2\bar{x}^2 + \bar{x}^2} = \sqrt{\langle x^2 \rangle - \bar{x}^2}$$

For the data given we have

$$\sigma = \sqrt{\langle x^2 \rangle - \bar{x}^2} = \sqrt{18.254 - (4.247)^2} = 0.466$$

9. If V is independent of time, then we can use the time-independent Schrödinger equation. Then by Equation (6.15)

$$\Psi^* \Psi = \psi^*(x) \psi(x) e^{-i\omega t} e^{i\omega t} = \psi^*(x) \psi(x)$$

Then

$$\int \Psi^* \Psi x dx = \int \psi^*(x) \psi(x) x dx$$

which is independent of time.

- *10. Using the Euler relations between exponential and trig functions

$$\psi = A (e^{ix} + e^{-ix}) = 2A \cos(x)$$

Normalization:

$$\int_{-\pi}^{\pi} \psi^* \psi dx = 4A^2 \int_{-\pi}^{\pi} \cos^2(x) dx = 4A^2 \pi = 1$$

Thus $A = 1/2\sqrt{\pi}$ and the probability of being in the interval $[0, \pi/8]$ is

$$\begin{aligned} P &= \int_0^{\pi/8} \psi^* \psi dx = \frac{1}{\pi} \int_0^{\pi/8} \cos^2(x) dx = \frac{1}{\pi} \left(\frac{x}{2} + \frac{1}{4} \sin(2x) \right) \Big|_0^{\pi/8} \\ &= \frac{1}{16} + \frac{1}{4\pi\sqrt{2}} \cong 0.119 \end{aligned}$$

11.

$$\int_0^{\pi} \psi^* \psi dx = A^2 \int_0^{\pi} \sin^2(x) dx = A^2 \frac{\pi}{2} = 1$$

so $A = \sqrt{2/\pi}$ and the probability of being in the interval $[0, \pi/4]$ is

$$\begin{aligned} P &= \int_0^{\pi/4} \psi^* \psi dx = \frac{2}{\pi} \int_0^{\pi/4} \sin^2(x) dx = \frac{2}{\pi} \left(\frac{x}{2} - \frac{1}{4} \sin(2x) \right) \Big|_0^{\pi/4} \\ &= \frac{2}{\pi} \left(\frac{\pi}{8} - \frac{1}{4} \right) = \frac{1}{4} - \frac{1}{2\pi} \cong 0.091 \end{aligned}$$

12.

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2} \quad E_{n+1} = \frac{(n+1)^2 \pi^2 \hbar^2}{2mL^2}$$

$$\Delta E_n = E_{n+1} - E_n = \frac{\pi^2 \hbar^2}{2mL^2} [(n+1)^2 - n^2] = \frac{\pi^2 \hbar^2}{2mL^2} (2n+1)$$

Computing specific values

$$\Delta E_1 = \frac{\pi^2 \hbar^2}{2mL^2} (3)$$

$$\Delta E_8 = \frac{\pi^2 \hbar^2}{2mL^2} (17)$$

$$\Delta E_{800} = \frac{\pi^2 \hbar^2}{2mL^2} (1601)$$

- *13. The wave function for the n th level is $\psi_n(x) = \sqrt{2/L} \sin(n\pi x/L)$ so the average value of the square of the wave function is

$$\begin{aligned}\langle \psi_n^2(x) \rangle &= \frac{\int_0^L \psi_n^* \psi_n dx}{\int_0^L dx} = \frac{1}{L} \int_0^L \psi_n^* \psi_n dx = \frac{2}{L^2} \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2}{L^2} \frac{L}{2} = \frac{1}{L}\end{aligned}$$

This result is independent of n and is the same as the classical probability. The classical probability is uniform throughout the box, but this is not so in the quantum mechanical case.

14. For this non-relativistic speed we have $E = \frac{1}{2}mv^2 = \frac{1}{2}(mc^2)\beta^2 = 0.002555 \text{ eV}$. Using $E = n^2 h^2 / 8mL^2$ we find

$$n^2 = \frac{8mL^2 E}{h^2} = \frac{8mc^2 L^2 E}{h^2 c^2} = \frac{8(511 \times 10^3 \text{ eV})(48.5 \text{ nm})^2 (0.002555 \text{ eV})}{(1240 \text{ eV} \cdot \text{nm})^2} = 16.0$$

and therefore $n = 4$.

15. The ground-state wave function is $\psi_1 = \sqrt{2/L} \sin(\pi x/L)$.

$$\begin{aligned}P_1 &= \int_0^{L/3} \psi_1^2 dx = \frac{2}{L} \int_0^{L/3} \sin^2\left(\frac{\pi x}{L}\right) dx = \frac{2}{L} \left(\frac{x}{2} - \frac{L}{4\pi} \sin\left(\frac{2\pi x}{L}\right) \right) \Big|_0^{L/3} \\ &= 2 \left(\frac{1}{6} - \frac{\sqrt{3}}{8\pi} \right) \cong 0.1955\end{aligned}$$

$$\begin{aligned}P_2 &= \int_{L/3}^{2L/3} \psi_1^2 dx = \frac{2}{L} \left(\frac{x}{2} - \frac{L}{4\pi} \sin\left(\frac{2\pi x}{L}\right) \right) \Big|_{L/3}^{2L/3} \\ &= 2 \left(\frac{1}{6} + \frac{\sqrt{3}}{4\pi} \right) \cong 0.6090\end{aligned}$$

$$\begin{aligned}P_3 &= \int_{2L/3}^L \psi_1^2 dx = \frac{2}{L} \left(\frac{x}{2} - \frac{L}{4\pi} \sin\left(\frac{2\pi x}{L}\right) \right) \Big|_{2L/3}^L \\ &= 2 \left(\frac{1}{6} - \frac{\sqrt{3}}{8\pi} \right) \cong 0.1955\end{aligned}$$

Notice that $P_1 + P_2 + P_3 = 1$ as required.

16. The first excited state has a wave function $\psi_2 = \sqrt{2/L} \sin(kx)$ with $k = 2\pi/L$.

$$\begin{aligned} P_1 &= \int_0^{L/3} \psi_2^2 dx = \frac{2}{L} \left(\frac{x}{2} - \frac{1}{4k} \sin(2kx) \right) \Big|_0^{L/3} \\ &= 2 \left(\frac{1}{6} + \frac{\sqrt{3}}{16\pi} \right) \cong 0.402 \end{aligned}$$

Similarly

$$P_2 = 2 \left(\frac{1}{6} - \frac{\sqrt{3}}{8\pi} \right) \cong 0.196$$

and $P_3 = P_1 \cong 0.402$. Notice that $P_1 + P_2 + P_3 = 1$.

*17.

$$E_1 = \frac{h^2}{8mL^2} = \frac{h^2 c^2}{8mc^2 L^2} = \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8(511 \times 10^3 \text{ eV})(10^{-5} \text{ nm})^2} = 3.76 \text{ GeV}$$

Then $E_2 = 4E_1 = 15.05 \text{ GeV}$ and $\Delta E = E_2 - E_1 = 11.3 \text{ GeV}$. As noted in the text, this is unreasonably high, implying that electrons are not bound to the nucleus.

18. a) As in the previous problem

$$E_1 = \frac{h^2 c^2}{8mc^2 L^2} = \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8(938.27 \times 10^6 \text{ eV})(1.4 \times 10^{-5} \text{ nm})^2} = 1.05 \text{ MeV}$$

b)

$$E_1 = \frac{h^2 c^2}{8mc^2 L^2} = \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8(3727 \times 10^6 \text{ eV})(1.4 \times 10^{-5} \text{ nm})^2} = 263 \text{ keV}$$

19. As in previous problems the ground state energy is

$$E_1 = \frac{h^2}{8mL^2} = \frac{h^2 c^2}{8mc^2 L^2} = \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8(511 \times 10^3 \text{ eV})(0.5 \text{ nm})^2} = 1.5045 \text{ eV}$$

The other energy levels are $E_n = n^2 E_1$:

$$E_2 = 4E_1 = 6.02 \text{ eV} \quad E_3 = 9E_1 = 13.54 \text{ eV} \quad E_4 = 16E_1 = 24.07 \text{ eV}$$

The allowed jumps are

$$\begin{array}{lll} E_4 - E_3 & = & 10.5 \text{ eV} \quad E_4 - E_2 = 18.1 \text{ eV} \quad E_4 - E_1 = 22.6 \text{ eV} \\ E_3 - E_2 & = & 7.5 \text{ eV} \quad E_3 - E_1 = 12.0 \text{ eV} \quad E_2 - E_1 = 4.52 \text{ eV} \end{array}$$

20. Lacking an explicit equation for finite square well energies, we will approximate using the infinite square well formula. In order to contain three energy levels the depth of the well should be at least

$$E = \frac{n^2 h^2}{8mL^2} = \frac{9h^2}{8mL^2}$$

Evaluating numerically with the given mass

$$E = \frac{9h^2}{8mL^2} = \frac{9h^2 c^2}{8mc^2 L^2} = \frac{9(1240 \text{ eV} \cdot \text{nm})^2}{8(2 \times 10^9 \text{ eV})(3 \times 10^{-6} \text{ nm})^2} = 96.1 \text{ MeV}$$

21. a) The wavelengths are longer for the finite well, because the wave functions can leak outside the box.
 b) Generally shorter wavelengths correspond to higher energies, so we expect energies to be lower for the finite well.
 c) Generally the number of bound states is limited by the depth of the well. We expect no bound states for $E > V_0$.

- *22. From the boundary condition $\psi_1(x=0) = \psi_2(x=0)$ we have $Ae^0 = Ce^0 + De^0$ or $A = C + D$. From the condition $\psi'_1(x=0) = \psi'_2(x=0)$ we have $\alpha A = ikC - ikD$. Solving for A and combining with the first boundary condition gives

$$C + D = \frac{ik}{\alpha}C - \frac{ik}{\alpha}D$$

or after rearranging

$$\frac{C}{D} = \frac{ik + \alpha}{ik - \alpha}$$

23. As in the previous problem matching the wavefunction at the boundary gives

$$Ce^{ikL} + De^{ikL} = Be^{-\alpha L}$$

and matching the first derivative gives

$$ikCe^{ikL} - ikDe^{-ikL} = -\alpha Be^{-\alpha L}$$

Eliminating B from these two equations gives

$$Ce^{ikL} + De^{ikL} = \frac{1}{\alpha} (ikDe^{-ikL} - ikCe^{ikL})$$

Thus

$$\frac{C}{D} = \frac{ik - 1}{ik + 1} \frac{e^{2ikL}}{\alpha}$$

24.

$$E = \frac{\pi^2 \hbar^2}{2mL^2} (n_1^2 + n_2^2 + n_3^2) = E_0 (n_1^2 + n_2^2 + n_3^2)$$

where

$$E_0 = \frac{\pi^2 \hbar^2}{2mL^2}$$

Then the second, third, fourth, and fifth levels are

$$\begin{aligned} E_2 &= (2^2 + 1^2 + 1^2) E_0 = 6E_0 && \text{(degenerate)} \\ E_3 &= (2^2 + 2^2 + 1^2) E_0 = 9E_0 && \text{(degenerate)} \\ E_4 &= (3^2 + 1^2 + 1^2) E_0 = 11E_0 && \text{(degenerate)} \\ E_5 &= (2^2 + 2^2 + 2^2) E_0 = 12E_0 && \text{(not degenerate)} \end{aligned}$$

25. In general we have

$$\psi(x) = A \sin\left(\frac{n_1 \pi x}{L}\right) \sin\left(\frac{n_2 \pi y}{L}\right) \sin\left(\frac{n_3 \pi z}{L}\right)$$

For $\psi_2(x)$ we can have $(n_1, n_2, n_3) = (1, 1, 2)$ or $(1, 2, 1)$ or $(2, 1, 1)$

For $\psi_3(x)$ we can have $(n_1, n_2, n_3) = (1, 2, 2)$ or $(2, 2, 1)$ or $(2, 1, 2)$

For $\psi_4(x)$ we can have $(n_1, n_2, n_3) = (1, 1, 3)$ or $(1, 3, 1)$ or $(3, 1, 1)$

For $\psi_5(x)$ we can have $(n_1, n_2, n_3) = (2, 2, 2)$

26. We must normalize by doing the triple integral of $\psi^ \psi$:

$$\iiint \psi^* \psi \, dx \, dy \, dz = 1$$

with $\psi(x, y, z)$ given in the text. We can evaluate the iterated triple integral

$$A^2 \int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx \int_0^L \sin^2\left(\frac{\pi y}{L}\right) dy \int_0^L \sin^2\left(\frac{\pi z}{L}\right) dz = A^2 \left(\frac{L}{2}\right)^3 = 1$$

Solving for A we find

$$A = \left(\frac{2}{L}\right)^{3/2}$$

27. Taking the derivatives we find

$$\nabla^2 \psi = -(k_1^2 + k_2^2 + k_3^2) \psi$$

so the Schrödinger equation becomes

$$-\frac{\hbar^2}{2m} \nabla^2 \psi = \frac{\hbar^2}{2m} (k_1^2 + k_2^2 + k_3^2) \psi = E \psi$$

From the boundary conditions $k_i = n_i \pi / L_i$ so

$$E = \frac{\hbar^2 \pi^2}{2m} \left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2} \right)$$

The three quantum numbers come directly from the three boundary conditions.

28. $\psi_0(x)$ has these features: it is symmetric about $x = 0$; it has a maximum at $x = 0$ because the wavefunction must tend toward zero for $x \rightarrow \pm\infty$; there is no node in the ground state; the wavefunction decreases exponentially where $V > E$.

*29.

$$\Delta E_n = E_{n+1} - E_n = \left(n + 1 + \frac{1}{2}\right) \hbar\omega - \left(n + \frac{1}{2}\right) \hbar\omega = \hbar\omega \quad \text{for all } n$$

This is true for all n , and there is no restriction on the number of levels.

30. Normalization

$$1 = \int_{-\infty}^{\infty} \psi^* \psi dx = A^2 \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = 2A^2 \int_0^{\infty} x^2 e^{-\alpha x^2} dx = 2A^2 \frac{1}{4\alpha} \sqrt{\frac{\pi}{\alpha}}$$

Solving for A we find $A = \sqrt{2\pi}^{-1/4} \alpha^{3/4}$

$$\langle x \rangle = A^2 \int_{-\infty}^{\infty} x^3 e^{-\alpha x^2} dx = 0$$

because the integrand is odd over symmetric limits.

$$\langle x^2 \rangle = A^2 \int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} dx = \frac{3}{4} A^2 \pi^{1/2} \alpha^{-5/2} = \frac{3}{2\alpha}$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{3}{2\alpha}}$$

31.

$$\begin{aligned} E &= \left(n + \frac{1}{2}\right) \hbar\omega = \left(n + \frac{1}{2}\right) h\nu = (4.136 \times 10^{-15} \text{ eV} \cdot \text{s}) (10^{13} \text{ s}^{-1}) \left(n + \frac{1}{2}\right) \\ &= (4.136 \times 10^{-2} \text{ eV}) \left(n + \frac{1}{2}\right) \end{aligned}$$

For the harmonic oscillator $\omega^2 = k/m$ so

$$k = \omega^2 m = 4\pi^2 \nu^2 m = 4\pi^2 (10^{13} \text{ s}^{-1})^2 (3.32 \times 10^{-26} \text{ kg}) = 131 \text{ N/m}$$

32. Taking the second derivative of ψ for the Schrödinger equation:

$$\frac{d\psi}{dx} = 5A\alpha x e^{-\alpha x^2/2} - 2\alpha^2 A x^3 e^{-\alpha x^2/2}$$

$$\frac{d^2\psi}{dx^2} = (5A\alpha - 5A\alpha^2 x^2 - 6A\alpha^2 x^2 + 2A\alpha^3 x^4) e^{-\alpha x^2/2}$$

The Schrödinger equation says that

$$\begin{aligned} \frac{d^2\psi}{dx^2} &= (\alpha^2 x^2 - \beta) \psi = (\alpha^2 x^2 - \beta) A (2\alpha x^2 - 1) e^{-\alpha x^2/2} \\ &= (2\alpha^2 x^4 + x^2 (-2\alpha\beta - \alpha^2) + \beta) A e^{-\alpha x^2/2} \end{aligned}$$

Matching our two values of $d^2\psi/dx^2$ we see that the Schrödinger equation can only be satisfied if $\beta = 5\alpha$. Then

$$\frac{2mE}{\hbar^2} = 5\sqrt{\frac{mk}{\hbar^2}}$$

or $E = \frac{5}{2}\hbar\omega$. This is the expected result, because the wave function contains a second-order polynomial (in x), and with $n = 2$ we expect $E = (n + \frac{1}{2})\hbar\omega = \frac{5}{2}\hbar\omega$.

33. By symmetry $\langle p \rangle = 0$. Setting the ground state energy equal to $\langle p^2 \rangle / 2m$ we find

$$\frac{\langle p^2 \rangle}{2m} = \frac{1}{2}\hbar\omega \qquad \langle p^2 \rangle = \hbar\omega m$$

Note, however, that a detailed calculation gives $\langle p^2 \rangle = \frac{1}{2}\hbar\omega m$. The factor of one-half is evidently the difference between the kinetic energy and the total energy, which if taken into account does give the correct result.

*34. Taking the derivatives for the Schrödinger equation

$$\frac{d\psi}{dx} = Ae^{-\alpha x^2/2} - A\alpha x^2 e^{-\alpha x^2/2}$$

$$\frac{d^2\psi}{dx^2} = -3A\alpha x e^{-\alpha x^2/2} + A\alpha^2 x^3 e^{-\alpha x^2/2} = (\alpha^2 x^2 - 3\alpha)\psi$$

Thus

$$\frac{d^2\psi}{dx^2} = (\alpha^2 x^2 - \beta)\psi = (\alpha^2 x^2 - 3\alpha)\psi$$

Thus we see that $\alpha = 3\beta$ or

$$\begin{aligned} \frac{2mE}{\hbar^2} &= 3\sqrt{\frac{mk}{\hbar^2}} \\ E &= \frac{3}{2}\hbar\sqrt{\frac{k}{m}} = \frac{3}{2}\hbar\omega \end{aligned}$$

35. The classical frequency is (see Chapter 10) $\omega = \sqrt{k/\mu} = \sqrt{k(m_1 + m_2)/m_1 m_2} = \sqrt{2k/m}$ since the masses are equal in this case. The energies of the ground state (E_0) and the first three excited states are given by $E_n = (n + \frac{1}{2})\hbar\omega$ so the possible transitions (from E_3 to E_2 , E_3 to E_1 , etc. are $\Delta E = \hbar\omega$, $2\hbar\omega$, and $3\hbar\omega$, or

$$\hbar\omega = \hbar\sqrt{\frac{2k}{m}} = (6.582 \times 10^{-16} \text{ eV} \cdot \text{s}) \sqrt{\frac{2(1.1 \times 10^3 \text{ N/m})}{1.673 \times 10^{-27} \text{ kg}}} = 0.755 \text{ eV}$$

$$\lambda = \frac{hc}{E} = \frac{1240 \text{ eV} \cdot \text{nm}}{0.755 \text{ eV}} = 1640 \text{ nm}$$

$$2\hbar\omega = 2(6.582 \times 10^{-16} \text{ eV} \cdot \text{s}) \sqrt{\frac{2(1.1 \times 10^3 \text{ N/m})}{1.673 \times 10^{-27} \text{ kg}}} = 1.51 \text{ eV}$$

$$\lambda = \frac{hc}{E} = \frac{1240 \text{ eV} \cdot \text{nm}}{1.51 \text{ eV}} = 822 \text{ nm}$$

$$3\hbar\omega = 3 \left(6.582 \times 10^{-16} \text{ eV} \cdot \text{s} \right) \sqrt{\frac{2 (1.1 \times 10^3 \text{ N/m})}{1.673 \times 10^{-27} \text{ kg}}} = 2.26 \text{ eV}$$

$$\lambda = \frac{hc}{E} = \frac{1240 \text{ eV} \cdot \text{nm}}{2.26 \text{ eV}} = 549 \text{ nm}$$

36. The kinetic energy is (see Chapter 9)

$$K = \frac{3}{2}kT = \frac{3}{2} (8.617 \times 10^{-5} \text{ eV/K}) (12000 \text{ K}) = 1.551 \text{ eV}$$

Assume a square-top potential of height

$$V_0 = \frac{q_1 q_2}{4\pi\epsilon_0 r} = \frac{6e^2}{4\pi\epsilon_0 r} = \frac{6 (1.440 \text{ eV} \cdot \text{nm})}{(1.2 \times 10^{-6} \text{ nm}) (12)^{1/3}} = 3.145 \text{ MeV}$$

where we have used the fact that the radius of a nucleus is approximately $1.2A^{1/3} \text{ fm}$ (see Chapter 12). For the width of the potential barrier use twice the radius or

$$L = 2 (1.2 \times 10^{-6} \text{ nm}) (12)^{1/3} = 5.49 \times 10^{-6} \text{ nm}$$

Then

$$\kappa = \frac{\sqrt{2mc^2 (V_0 - E)}}{\hbar c} = \frac{(2 (938.27 \times 10^6 \text{ eV}) (3.145 \times 10^6 \text{ eV} - 1.551 \text{ eV}))^{1/2}}{197.3 \text{ eV} \cdot \text{nm}} = 3.89 \times 10^5 \text{ nm}^{-1}$$

$$\kappa L = (3.89 \times 10^5 \text{ nm}^{-1}) (5.49 \times 10^{-6} \text{ nm}) = 2.135$$

$$T = \left(1 + \frac{(3.145 \times 10^6 \text{ eV})^2 \sinh^2 (2.135)}{4 (1.551 \text{ eV}) (3.145 \times 10^6 \text{ eV} - 1.551 \text{ eV})} \right)^{-1} = 1.14 \times 10^{-7}$$

37. a)

$$p = \sqrt{2m(E - V_0)}$$

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m(E - V_0)}} \quad K = E - V_0$$

b)

$$p = \sqrt{2m(E + V_0)}$$

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m(E + V_0)}} \quad K = E + V_0$$

*38. In each case $\kappa L \gg 1$ so we can use

$$T = 16 \frac{E}{V_0} \left(1 - \frac{E}{V_0} \right) e^{-2\kappa L}$$

where

$$\kappa = \frac{\sqrt{2mc^2(V_0 - E)}}{\hbar c} = \frac{(2(3727 \times 10^6 \text{ eV})(10 \times 10^6 \text{ eV}))^{1/2}}{197.3 \text{ eV} \cdot \text{nm}} = 1.38 \times 10^{15} \text{ nm}^{-1}$$

a) With $L = 1.3 \times 10^{-14} \text{ m}$

$$T_a = 16 \frac{5 \text{ MeV}}{15 \text{ MeV}} \left(1 - \frac{5 \text{ MeV}}{15 \text{ MeV}}\right) e^{-2(1.38 \times 10^{15} \text{ m}^{-1})(1.3 \times 10^{-14} \text{ m})} = 9.3 \times 10^{-16}$$

b) With $V_0 = 30 \text{ MeV}$

$$\kappa = \frac{\sqrt{2mc^2(V_0 - E)}}{\hbar c} = \frac{(2(3727 \times 10^6 \text{ eV})(25 \times 10^6 \text{ eV}))^{1/2}}{197.3 \text{ eV} \cdot \text{nm}} = 2.19 \times 10^{15} \text{ m}^{-1}$$

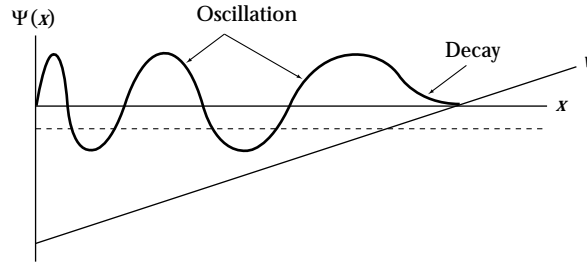
$$T_b = 16 \frac{5 \text{ MeV}}{30 \text{ MeV}} \left(1 - \frac{5 \text{ MeV}}{30 \text{ MeV}}\right) e^{-2(2.19 \times 10^{15} \text{ m}^{-1})(1.3 \times 10^{-14} \text{ m})} = 4.2 \times 10^{-25}$$

c) With $V_0 = 15 \text{ MeV}$ we return to the original value of κ , but now $L = 2.6 \times 10^{-14} \text{ m}$ and

$$T_c = 16 \frac{5 \text{ MeV}}{15 \text{ MeV}} \left(1 - \frac{5 \text{ MeV}}{15 \text{ MeV}}\right) e^{-2(1.38 \times 10^{15} \text{ m}^{-1})(2.6 \times 10^{-14} \text{ m})} = 2.4 \times 10^{-31}$$

By comparison $T_a > T_b > T_c$.

39. When $E > V$ the wavefunction is oscillating, with a longer wavelength as $E - V$ decreases. Then when $E < V$ the wavefunction decays.



40. In general for $E > V_0$

$$R = 1 - T = 1 - \left[1 + \frac{V_0^2 \sin^2(k_2 L)}{4E(E - V_0)}\right]^{-1}$$

If $E \gg V_0$ then $4E(E - V_0) \cong 4E^2$. From the binomial theorem $(1 + x)^{-1} \cong 1 - x$ for small x and

$$R \cong 1 - \left[1 - \frac{V_0^2 \sin^2(k_2 L)}{4E^2}\right] = \frac{V_0^2 \sin^2(k_2 L)}{4E^2}$$

$$R \cong \left(\frac{V_0 \sin(k_2 L)}{2E}\right)^2$$

41.

$$T = \left[1 + \frac{V_0^2 \sin^2(k_2 L)}{4E(E - V_0)} \right]^{-1}$$

a) To get $T = 1$ we require $\sin(k_2 L) = 0$. Except for the trivial solution $L = 0$, this occurs whenever $k_2 L = n\pi$ with n an integer. Letting $n = 1$ we find

$$L = \frac{\pi}{k_2} = \frac{\pi \hbar}{\sqrt{2m(E - V_0)}} = \frac{1}{2} \frac{hc}{\sqrt{2mc^2(E - V_0)}} = \frac{1}{2} \frac{1240 \text{ eV} \cdot \text{nm}}{\sqrt{2(511 \times 10^3 \text{ eV})(7.2 \text{ eV})}} = 0.229 \text{ nm}$$

Any integer multiple of this value will work.

b) For maximum reflection $\sin^2(k_2 L) = 1$ or $L = n\pi/2k_2$ for any odd integer n . From the result of (a) we see that the first maximum is with L equal to half the value of L for the first minimum, or $L = 0.114 \text{ nm}$.

42.

$$\kappa = \frac{\sqrt{2mc^2(V_0 - E)}}{\hbar c} = \frac{(2(511 \times 10^3 \text{ eV})(1.5 \text{ eV}))^{1/2}}{197.4 \text{ eV} \cdot \text{nm}} = 6.27 \text{ nm}^{-1}$$

With a probability of 10^{-4} we know $\kappa L \gg 1$ and we can use

$$T = 16 \frac{E}{V_0} \left(1 - \frac{E}{V_0} \right) e^{-2\kappa L} = 16 \frac{1}{2.5} \left(1 - \frac{1}{2.5} \right) e^{-2\kappa L} = 3.84 e^{-2\kappa L} = 10^{-4}$$

Solving for L :

$$L = \frac{\ln(3.84 \times 10^4)}{2(6.27 \times 10^9 \text{ m}^{-1})} = 8.42 \times 10^{-10} \text{ m}.$$

Now using the proton mass

$$\kappa = \frac{\sqrt{2mc^2(V_0 - E)}}{\hbar c} = \frac{(2(938.27 \times 10^6 \text{ eV})(1.5 \text{ eV}))^{1/2}}{197.4 \text{ eV} \cdot \text{nm}} = 268.8 \text{ nm}^{-1}$$

$$T = 3.84 e^{-2\kappa L} = 3.84 e^{-2(268.8 \times 10^9 \text{ m}^{-1})(8.42 \times 10^{-10} \text{ m})} = 9.9 \times 10^{-197}$$

The proton's probability is much lower!

*43. As in the text we find

$$E = \frac{\hbar^2 \pi^2}{2m} \left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2} \right)$$

and substituting the given values of L we find

$$E = \frac{\hbar^2 \pi^2}{2mL^2} \left(n_1^2 + 2n_2^2 + \frac{n_3^2}{4} \right)$$

Letting $E_0 = \hbar^2 \pi^2 / 2mL^2$ we have

$$E_1 = E_0 \left(1 + 2 + \frac{1}{4} \right) = \frac{13}{4} E_0$$

$$E_2 = E_0 \left(1 + 2 + \frac{2^2}{4} \right) = 4E_0$$

$$E_3 = E_0 \left(1 + 2 + \frac{3^2}{4} \right) = \frac{21}{4}E_0$$

$$E_4 = E_0 \left(2^2 + 2 + \frac{1}{4} \right) = \frac{25}{4}E_0$$

$$E_5 = E_0 \left(1 + 2 + \frac{4^2}{4} \right) = E_0 \left(2^2 + 2 + \frac{2^2}{4} \right) = 7E_0$$

Of those listed, only E_5 is degenerate.

44. Recognizing this as the infinite square well wave function we see that $k = 3\pi/\alpha$ and

$$E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} = \frac{9\hbar^2 \pi^2}{2m\alpha^2}$$

- *45. a) In general inside the box we have a superposition of sine and cosine functions, but the boundary condition $\psi(0) = 0$ allows only the sine function to survive, and thus $\psi = A \sin(kx)$. With $V = 0$ inside the well $E = p^2/2m = \hbar^2 k^2/2m$ or $k = \sqrt{2mE}/\hbar$. Outside the well the decaying exponential is required as in the tunneling example in the text, with $E = \hbar^2 k^2/2m + V_0$ which reduces to $\kappa = ik = \sqrt{2m(V_0 - E)}/\hbar$.

b) Equating the wavefunctions and first derivatives at $x = L$:

$$A \sin(kL) = B e^{-\kappa L}$$

$$kA \cos(kL) = -\kappa B e^{-\kappa L}$$

Dividing these two equations

$$\frac{\tan(kL)}{k} = -\frac{1}{\kappa}$$

$$\kappa \tan(kL) = -k$$

46. From the previous problem $\kappa \tan(kL) = -k$ or $\kappa L \tan(kL) = -kL$. Let $\alpha = kL$ and $\beta = \kappa L$ so that

$$\beta \tan \alpha = -\alpha \tag{1}$$

Now from their definitions

$$\alpha^2 + \beta^2 = k^2 L^2 + \kappa^2 L^2 = \frac{2mEL^2}{\hbar^2} + \frac{2m(V_0 - E)L^2}{\hbar^2} = \frac{2mV_0 L^2}{\hbar^2} = 1 \tag{2}$$

Solving Equations (1) and (2) numerically we find $\alpha = 0.20$ and $\beta = -0.98$. Then E is given by

$$E = \frac{\hbar^2 k^2}{2m} = \hbar^2 \frac{(\alpha/L)^2}{2m} = \frac{0.04 \hbar^2}{2mL^2}$$

47. Referring to the solution to the previous problem, we see that only a finite number of solutions to Equation (1) exist up to any particular (finite) value of V_0 . Therefore for any finite V_0 only a finite number of combinations of α and β will satisfy both equations, and the number of bound states is finite.

48. Using the nomenclature of Problem 46

$$\kappa L = \frac{\sqrt{2m(V_0 - E)} L}{\hbar} = \frac{\sqrt{2(939 \times 10^6 \text{ eV})(2.2 \times 10^6 \text{ eV})(3.5 \times 10^{-6} \text{ nm})}}{197.4 \text{ eV} \cdot \text{nm}} = 1.14$$

where we have used the mass of one nucleon, because one nucleon is “bound” by the other. Now $\kappa L = \beta = -\alpha/\tan \alpha$ so $\alpha \cong 2.07 = kL$. Then

$$E = \frac{\hbar^2 k^2}{2m} = \hbar^2 \frac{(\alpha/L)^2}{2m} = \hbar^2 \frac{(2.07/L)^2}{2m} = \frac{2.07^2 (197.4 \text{ eV} \cdot \text{nm})^2}{2(939 \times 10^6 \text{ eV})(3.5 \times 10^{-6} \text{ nm})^2} = 7.26 \text{ MeV}$$

This means that $V_0 = 2.2 \text{ MeV} + E = 9.46 \text{ MeV}$. The next solution of the equation $\beta = -\alpha/\tan \alpha$ is at $\alpha \cong 4.94$, a value that will put $E > V_0$. Therefore there are no excited states.

49. a) This was done in Problem 32.

b)

$$\langle x \rangle = \int_{-\infty}^{\infty} x \psi^* \psi dx = 0$$

because the integrand is odd over symmetric limits. To find $\langle x^2 \rangle$ we first need to normalize:

$$\begin{aligned} A^2 \int_{-\infty}^{\infty} (1 - 2\alpha x^2)^2 e^{-\alpha x^2} dx &= 2A^2 \int_0^{\infty} (1 - 2\alpha x^2)^2 e^{-\alpha x^2} dx \\ &= 2A^2 \int_0^{\infty} (1 - 4\alpha x^2 + 4\alpha^2 x^4) e^{-\alpha x^2} dx = 1 \\ &= 2A^2 \sqrt{\frac{\pi}{\alpha}} \left(\frac{1}{2} - 1 + \frac{3}{2} \right) = 1 \end{aligned}$$

Thus $A^2 = \frac{1}{2} \sqrt{\frac{\alpha}{\pi}}$ and

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \psi^* \psi dx = 2A^2 \int_0^{\infty} x^2 (1 - 4\alpha x^2 + 4\alpha^2 x^4) e^{-\alpha x^2} dx = \frac{1}{\alpha} \left(\frac{1}{4} - \frac{3}{2} + \frac{15}{4} \right) = \frac{5}{2\alpha}$$

50. Using the known values of ψ_1 and ψ_2 we see

$$\begin{aligned} \psi &= \frac{1}{2}\psi_1 + \frac{\sqrt{3}}{2}\psi_2 = \frac{1}{2}\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right) + \frac{\sqrt{3}}{2}\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right) \\ \psi &= \sqrt{\frac{1}{2L}}\sin\left(\frac{\pi x}{L}\right) + \sqrt{\frac{3}{2L}}\sin\left(\frac{2\pi x}{L}\right) \end{aligned}$$

For normalization

$$\int_0^L \psi^* \psi dx = \int_0^L \left(\frac{1}{2L} \sin^2\left(\frac{\pi x}{L}\right) + \frac{3}{2L} \sin^2\left(\frac{2\pi x}{L}\right) + \frac{\sqrt{3}}{2L} \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi x}{L}\right) \right) dx$$

The third term vanishes because of the orthogonality of the trig functions, leaving

$$\begin{aligned} \int_0^L \psi^* \psi dx &= \int_0^L \left(\frac{1}{2L} \sin^2\left(\frac{\pi x}{L}\right) + \frac{3}{2L} \sin^2\left(\frac{2\pi x}{L}\right) \right) dx \\ &= \frac{1}{2L} \int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx + \frac{3}{2L} \int_0^L \sin^2\left(\frac{2\pi x}{L}\right) dx \\ &= \frac{1}{2L} \left(\frac{L}{2} \right) + \frac{3}{2L} \left(\frac{L}{2} \right) = 1 \text{ as required} \end{aligned}$$

51. Using the Taylor approximation for the exponential $e^x \cong 1 + x$ for small x , we have

$$V(r) = D (1 - e^{-a(r-r_e)})^2 \cong D (1 - (1 - a(r - r_e)))^2 = D (a(r - r_e))^2 = Da^2 (r - r_e)^2$$

52. We will solve for numerical values of the factors in front of the quantum numbers:

$$\begin{aligned} \hbar\omega &= \hbar a \sqrt{\frac{2D}{\mu}} = \hbar a \sqrt{\frac{2D(m_1 + m_2)}{m_1 m_2}} \\ &= (6.582 \times 10^{-16} \text{ eV} \cdot \text{s}) (7.8 \times 10^9 \text{ m}^{-1}) \\ &\quad \times \sqrt{\frac{2(4.42 \text{ eV})(39.10 \text{ u} + 35.45 \text{ u})}{(39.10 \text{ u})(35.45 \text{ u})} \frac{1 \text{ u}}{931.5 \times 10^6 \text{ eV}/c^2} \left(\frac{2.998 \times 10^8 \text{ m/s}}{c} \right)} \\ &= 0.03477 \text{ eV} \\ \frac{\hbar^2 \omega^2}{4D} &= \frac{(0.03477 \text{ eV})^2}{4(4.42 \text{ eV})} = 6.838 \times 10^{-5} \text{ eV} \end{aligned}$$

Evaluating for specific energy levels:

$$E_0 = \left(0 + \frac{1}{2}\right) (0.03477 \text{ eV}) - \left(0 + \frac{1}{2}\right)^2 (6.838 \times 10^{-5} \text{ eV}) = 0.017 \text{ eV}$$

$$E_1 = \left(1 + \frac{1}{2}\right) (0.03477 \text{ eV}) - \left(1 + \frac{1}{2}\right)^2 (6.838 \times 10^{-5} \text{ eV}) = 0.052 \text{ eV}$$

$$E_2 = \left(2 + \frac{1}{2}\right) (0.03477 \text{ eV}) - \left(2 + \frac{1}{2}\right)^2 (6.838 \times 10^{-5} \text{ eV}) = 0.086 \text{ eV}$$

$$E_3 = \left(3 + \frac{1}{2}\right) (0.03477 \text{ eV}) - \left(3 + \frac{1}{2}\right)^2 (6.838 \times 10^{-5} \text{ eV}) = 0.121 \text{ eV}$$

Note that for these low quantum numbers the second-order correction is small.

*53. The solution is identical to the presentation in the text for the three-dimensional box but without the z dimension. Briefly, assuming a trial function for the form

$$\psi(x, y) = A \sin(k_1 x) \sin(k_2 y)$$

Assuming that one corner is at the origin, applying the boundary conditions leads to

$$k_1 = \frac{n_x \pi}{L} \qquad k_2 = \frac{n_y \pi}{L}$$

and substituting into the Schrödinger equation leads to

$$E = \frac{\pi^2 \hbar^2}{2mL^2} (n_x^2 + n_y^2)$$

To normalize do the iterated double integral

$$\begin{aligned}\int_0^L \int_0^L \psi^* \psi dx dy &= A^2 \int_0^L \int_0^L \sin^2 \left(\frac{n_x \pi x}{L} \right) \sin^2 \left(\frac{n_y \pi y}{L} \right) dx dy \\ &= A^2 \left(\frac{L}{2} \right) \left(\frac{L}{2} \right) = 1\end{aligned}$$

so $A = 2/L$. Now to find the energy levels use the energy equation with different values of the quantum numbers. Letting $E_0 = \pi^2 \hbar^2 / 2mL^2$ we have

$$E_1 = E_0 (1^2 + 1^2) = 2E_0 \qquad n_1 = 1, n_2 = 1$$

$$E_2 = E_0 (2^2 + 1^2) = 5E_0 \qquad n_1 = 2, n_2 = 1 \text{ or vice versa}$$

$$E_3 = E_0 (2^2 + 2^2) = 8E_0 \qquad n_1 = 2, n_2 = 2$$

$$E_4 = E_0 (3^2 + 1^2) = 10E_0 \qquad n_1 = 3, n_2 = 1 \text{ or vice versa}$$

$$E_5 = E_0 (3^2 + 2^2) = 13E_0 \qquad n_1 = 3, n_2 = 2 \text{ or vice versa}$$

$$E_6 = E_0 (4^2 + 1^2) = 17E_0 \qquad n_1 = 4, n_2 = 1 \text{ or vice versa}$$