Chapter 15

* 1. From Newton's second law we have for a pendulum of length L

$$F = m_G g \sin\theta = m_I a = m_I L \frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = \frac{m_G g}{m_I L}\sin\theta \approx \frac{m_G g}{m_I L}\theta$$

where we have made the small angle approximation $\sin\theta \approx \theta$. This is a simple harmonic oscillator equation with solution $\theta = \theta_0 \cos(\omega t)$ where θ_0 is the amplitude and the angular frequency is

$$\omega = \sqrt{\frac{m_G g}{m_I L}}$$

The period of oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m_I L}{m_G g}}$$

Therefore two masses with different ratios m_I/m_G will have different small-amplitude periods.

2.

$$\Delta f = \frac{gHf}{c^2} = \frac{\left(9.80\ \mathrm{m/s^2}\right)\left(4\times10^5\ \mathrm{m}\right)\left(10^8\ \mathrm{s^{-1}}\right)}{\left(2.998\times10^8\ \mathrm{m/s}\right)^2} = 4.36\times10^{-3}\,\mathrm{Hz}$$

3.

$$\frac{\Delta f}{f} = -\frac{GM}{c^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) = -\frac{GM}{r_1 r_2 c^2}\left(r_2 - r_1\right) = \frac{GM}{r_1 r_2 c^2}\left(r_1 - r_2\right)$$

Use $r_1 - r_2 = H$ and let $r_1 \approx r_2 = r$. From classical mechanics $g = GM/r^2$, so

$$\frac{\Delta f}{f} = \frac{gH}{c^2}$$

4.

$$\frac{\Delta T}{T} = -\frac{GM}{c^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

We use $r_2 = 6378$ km and $r_1 = 6378$ km $+10$ km $= 6388$ km.

$$\begin{aligned}\frac{\Delta T}{T} &= -\frac{\left(6.673\times10^{-11}\ \mathrm{m^3\cdot kg^{-1}\cdot s^{-2}}\right)\left(5.98\times10^{24}\ \mathrm{kg}\right)}{\left(2.998\times10^8\ \mathrm{m/s}\right)^2}\left(\frac{1}{6388\times10^3\ \mathrm{m}}-\frac{1}{6378\times10^3\ \mathrm{m}}\right)\\ &= 1.09\times10^{-12}\end{aligned}$$

which is the same as in the example. to three significant digits.

5. The distance d is the sum of the radii of the earth's orbit and Venus's orbit (assuming circular orbits).

$$d = 149.6\times10^9\ \mathrm{m} + 108.2\times10^9\ \mathrm{m} = 257.8\times10^9\,\mathrm{m}$$

The round-trip time is

$$t = \frac{2d}{c} = \frac{2\left(257.8\times10^9\,\mathrm{m}\right)}{2.998\times10^8\ \mathrm{m/s}} = 1719.8\,\mathrm{s}$$

Using Shapiro's measurement of 200 μs the percent change is therefore

$$\frac{200 \times 10^{-6}\,\text{s}}{1719.8\,\text{s}}(100\,\%) = 1.16 \times 10^{-5}\,\%$$

* 6. Using the mass of the neutron star and the radius given

$$\frac{\Delta f}{f} = \frac{GM}{rc^2} = \frac{\left(6.673 \times 10^{-11}\,\text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}\right)\left(5 \times 10^{30}\,\text{kg}\right)}{\left(1.0 \times 10^4\ \text{m}\right)\left(2.998 \times 10^8\,\text{m/s}\right)^2} = 3.712 \times 10^{-1}$$

The wavelength is affected by the same factor, so the redshift at the given wavelength is

$$\Delta\lambda = \left(3.712 \times 10^{-1}\right)(550\ \text{nm}) = 204\,\text{nm}$$

7. Using the mass and radius of the sun

$$\frac{\Delta f}{f} = \frac{GM}{rc^2} = \frac{\left(6.673 \times 10^{-11}\ \text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}\right)\left(1.99 \times 10^{30}\ \text{kg}\right)}{\left(6.96 \times 10^8\ \text{m}\right)\left(2.998 \times 10^8\ \text{m/s}\right)^2} = 2.123 \times 10^{-6}$$

The redshift of the wavelength is

$$\Delta\lambda = \left(2.123 \times 10^{-6}\right)(550\ \text{nm}) = 1.17 \times 10^{-3}\ \text{nm}$$

8. We can assume that g is constant over this distance. We find the frequency of the gamma ray:

$$f = \frac{E}{h} = \frac{14.4 \times 10^3\,\text{eV}}{4.136 \times 10^{-15}\,\text{eV} \cdot \text{s}} = 3.48 \times 10^{18}\ \text{Hz}$$

We can use Equation (15.3) to find:

$$\frac{\Delta f}{f} = \frac{GM}{rc^2} = \frac{gH}{c^2} = \frac{\left(9.80\,\text{m/s}^2\right)(381\,\text{m})}{\left(2.998 \times 10^8\,\text{m/s}\right)^2} = 4.15 \times 10^{-14}\ \text{which equals a}\ 4.15 \times 10^{-12}\%\ \text{change.}$$

The change in frequency is $\Delta f = 4.15 \times 10^{-14}\left(3.48 \times 10^{18}\,\text{Hz}\right) = 145\,\text{kHz}$

9. Let us assume that g is constant over this distance. Using $E = hf$ we find

$$\Delta f = \frac{gHf}{c^2} = \frac{gHE}{c^2 h} = \frac{\left(9.80\ \text{m/s}^2\right)(22.5\ \text{m})\left(14.4 \times 10^3\ \text{eV}\right)}{\left(2.998 \times 10^8\ \text{m/s}\right)^2\left(4.136 \times 10^{-15}\ \text{eV} \cdot \text{s}\right)} = 8541\ \text{Hz}$$

From the previous problem we know the frequency of the gamma ray is 3.48×10^{18} Hz so the percentage change is

$$\frac{8541\ \text{Hz}}{3.48 \times 10^{18}\,\text{Hz}}(100\ \%) = 2.45 \times 10^{-13}\ \%$$

10.

$$r_s = \frac{2GM}{c^2} = \frac{2\left(6.673 \times 10^{-11}\ \text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}\right)\left(7.35 \times 10^{22}\ \text{kg}\right)}{\left(2.998 \times 10^8\ \text{m/s}\right)^2} = 1.09 \times 10^{-4}\ \text{m}$$

* 11.

$$r_s = \frac{2GM}{c^2} = \frac{2\left(6.673 \times 10^{-11}\ \text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}\right)\left(1.90 \times 10^{27}\ \text{kg}\right)}{\left(2.998 \times 10^8\ \text{m/s}\right)^2} = 2.82\ \text{m}$$

* 12. From Equation (15.7) we have

$$T = \frac{\hbar c^3}{8\pi kGM} = \frac{(1.0546\times10^{-34}\ \mathrm{J\cdot s})\left(2.998\times10^8\ \mathrm{m/s}\right)^3}{8\pi\,(1.381\times10^{-23}\ \mathrm{J/K})\left(6.673\times10^{-11}\ \mathrm{m^3\cdot kg^{-1}\cdot s^{-2}}\right)(1.99\times10^{30}\ \mathrm{kg})}$$
$$= 6.17\times10^{-8}\,\mathrm{K}$$

13. Rearranging Equation (15.7), we have

$$M = \frac{\hbar c^3}{8\pi kGT} = \frac{(1.0546\times10^{-34}\ \mathrm{J\cdot s})\left(2.998\times10^8\ \mathrm{m/s}\right)^3}{8\pi\,(1.381\times10^{-23}\ \mathrm{J/K})\left(6.673\times10^{-11}\ \mathrm{m^3\cdot kg^{-1}\cdot s^{-2}}\right)(293\ \mathrm{K})} = 4.19\times10^{20}\ \mathrm{kg}$$

which is about

$$\frac{4.19\times10^{20}\ \mathrm{kg}}{1.99\times10^{30}\ \mathrm{kg}} = 2.11\times10^{-10}$$

solar masses.

$$r_s = \frac{2GM}{c^2} = \frac{2\left(6.673\times10^{-11}\ \mathrm{m^3\cdot kg^{-1}\cdot s^{-2}}\right)\left(4.19\times10^{20}\ \mathrm{kg}\right)}{\left(2.998\times10^8\ \mathrm{m/s}\right)^2} = 6.22\times10^{-7}\ \mathrm{m}$$

14. a) The mass is $1\times10^9\,(M_{\mathrm{Sun}}) = 1\times10^9\left(1.99\times10^{30}\,\mathrm{kg}\right) = 1.99\times10^{39}\,\mathrm{kg}$. We use Equation (15.5) to determine the Schwarzschild radius.

$$r_s = \frac{2GM}{c^2} = \frac{2\left(6.673\times10^{-11}\ \mathrm{m^3\cdot kg^{-1}\cdot s^{-2}}\right)\left(1.99\times10^{30}\,\mathrm{kg}\right)}{\left(2.998\times10^8\ \mathrm{m/s}\right)^2} = 2.95\times10^{12}\,\mathrm{m}$$

Pluto's orbit is extremely eccentric, ranging from about 4.34×10^{12} m to 7.4×10^{12} m, so this black hole would be smaller than our solar system.

b) We use the result of Example 15.3, with the value of α from the example (or from problem 16 below)

$$t = \frac{M_0^3}{3\alpha} = \frac{\left(1.99\times10^{39}\right)^3}{3\left(3.965\times10^{15}\ \mathrm{kg^3/s}\right)} = 6.63\times10^{101}\,\mathrm{s}$$

This is much, much greater than the age of our universe!

* 15. a) We use the results of Example 15.3 that give the time in terms of the mass. Rearranging the equation, using the age of the universe as 13.7 billion years, and using the value of α from problem 16, we have

$$M_0 = (3\alpha t)^{1/3} = \left[3\left(3.965\times10^{15}\,\mathrm{kg^3/s}\right)\left(13.7\times10^9\,\mathrm{y}\frac{3.156\times10^7\,\mathrm{s}}{1\,\mathrm{y}}\right)\right]^{1/3} = 1.73\times10^{11}\,\mathrm{kg}$$

b) Current evidence for the smallest black holes require a mass of about 5 to 20 times the mass of the Sun. The mass from part a) is only $M_0 \approx 10^{-19}\,M_{\mathrm{Sun}}$ which is too small to form a black hole.

16. From Equation (15.10) we see

$$\alpha = \frac{2\sigma\hbar^4c^6}{(8\pi)^3k^4G^2} = \frac{2\left(5.6705\times10^{-8}\,\mathrm{W\cdot m^{-2}\cdot K^{-4}}\right)\left(1.0546\times10^{-34}\,\mathrm{J\cdot s}\right)^4\left(2.9979\times10^8\,\mathrm{m/s}\right)^6}{(8\pi)^3\left(1.3807\times10^{-23}\,\mathrm{J/K}\right)^4\left(6.6726\times10^{-11}\,\mathrm{N\cdot m^2\cdot kg^{-2}}\right)^2}$$
$$= 3.965\times10^{15}\,\mathrm{kg^3/s}$$

17. We use Equation (15.4) and find

$$\begin{aligned}
\frac{\Delta f}{f} &= -\frac{GM}{c^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\\
&= -\frac{(6.6726\times10^{-11}\,\mathrm{N\cdot m^2\cdot kg^{-2}})\,(5.976\times10^{24}\,\mathrm{kg})}{(2.998\times10^{8}\,\mathrm{m/s})^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\\
&= -\left(4.439\times10^{-3}\,\mathrm{m}\right)\left[\frac{1}{(2.02\times10^{7}+6.37\times10^{6})\,\mathrm{m}}-\frac{1}{6.37\times10^{6}\,\mathrm{m}}\right]\\
&= 5.30\times10^{-10}
\end{aligned}$$

The frequency change is then $\Delta f = (5.30\times10^{-10})\,(1575.42\,\mathrm{MHz}) = 0.834\,\mathrm{Hz}$. This is a small, but significant change with the precision required of the GPS system.

* 18. Set the change in the photon's energy equal to the change in gravitational potential energy:

$$\Delta E = h\,\Delta f = -\frac{GMm}{r_1} - \left(-\frac{GMm}{r_2}\right) = -GMm\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

where M is the mass of the earth and m is the equivalent mass of the photon. Now $m = E/c^2 = hf/c^2$, so

$$h\,\Delta f = -\frac{GMhf}{c^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

$$\frac{\Delta f}{f} = -\frac{GM}{c^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

19. The problem advises us to assume that g is constant. Therefore we can use Equation (15.3) to determine the change in frequency. We have $\frac{\Delta f}{f} = \frac{gH}{c^2}$ so

$$\Delta f = \left(\frac{gH}{c^2}\right) f = \left(\frac{(9.80\,\mathrm{m/s})\,(3.0\times10^{5}\,\mathrm{m})}{(2.998\times10^{8}\,\mathrm{m/s})^2}\right)(294\times10^{6}\,\mathrm{Hz}) = 9.61\times10^{-3}\,\mathrm{Hz}$$

We can obtain a more precise answer using the formula from the previous problem and recalling that the earth's radius is 6378 km.

$$\begin{aligned}
\frac{\Delta f}{f} &= -\frac{GM}{c^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\\
&= -\frac{(6.673\times10^{-11}\,\mathrm{m^3\cdot kg^{-1}\cdot s^{-2}})\,(5.976\times10^{24}\,\mathrm{kg})}{(2.998\times10^{8}\,\mathrm{m/s})^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\\
&= -\left(4.439\times10^{3}\,\mathrm{m}\right)\left(\frac{1}{6.678\times10^{6}\,\mathrm{m}}-\frac{1}{6.378\times10^{6}\,\mathrm{m}}\right)\\
&= 3.127\times10^{-11}
\end{aligned}$$

Therefore

$$\Delta f = (3.127\times10^{-11})\,(294\times10^{6}\,\mathrm{Hz}) = 9.19\times10^{-3}\,\mathrm{Hz}$$

which is about a 5 % difference.

20. $g = GM/r^2$ which can be differentiated to give

$$dg = -\frac{2GM}{r^3}dr$$

For a small change let $dg \approx |dg|$ and $dr \approx \Delta r = 3$ m. Also notice that the distance from the center of the earth is 6.378×10^6 m $+3 \times 10^5$ m $= 6.678 \times 10^6$ m. Then

$$\Delta g \approx \frac{2\left(6.673 \times 10^{-11}\ \mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}}\right)\left(5.98 \times 10^{24}\ \mathrm{kg}\right)}{\left(6.678 \times 10^6\ \mathrm{m}\right)^3}(3\ \mathrm{m}) = 8.04 \times 10^{-6}\ \mathrm{m/s^2}$$

This is about $10^{-6}g$, so it is a very small effect.

21. If we use $\lambda = h/mc$ for a relativistic particle of mass m , we have

$$\lambda = \frac{h}{mc} = \pi r_s = \frac{2\pi Gm}{c^2}$$

Solving for m we have

$$m = \sqrt{\frac{hc}{2\pi G}} = \sqrt{\frac{(6.626 \times 10^{-34}\ \mathrm{J \cdot s})(2.998 \times 10^8\ \mathrm{m/s})}{2\pi\left(6.673 \times 10^{-11}\ \mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}}\right)}} = 2.18 \times 10^{-8}\ \mathrm{kg}$$

The Planck energy is

$$E_{\mathrm{Pl}} = mc^2 = \left(2.18 \times 10^{-8}\ \mathrm{kg}\right)\left(2.998 \times 10^8\ \mathrm{m/s}\right)^2 = 1.96 \times 10^9\ \mathrm{J} = 1.22 \times 10^{28}\ \mathrm{eV}$$

22. a) The combination of G, h, and c that has the right units is

$$\lambda_{\mathrm{Pl}} = \sqrt{\frac{Gh}{c^3}} = \sqrt{\frac{\left(6.673 \times 10^{-11}\ \mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}}\right)(6.626 \times 10^{-34}\ \mathrm{J \cdot s})}{(2.998 \times 10^8\ \mathrm{m/s})^3}} = 4.05 \times 10^{-35}\ \mathrm{m}$$

b)

$$\lambda = \frac{h}{mc} = \frac{hc}{mc^2} = \frac{1240 \times 10^{-9}\ \mathrm{eV \cdot m}}{1.22 \times 10^{28}\ \mathrm{eV}} = 1.02 \times 10^{-34}\ \mathrm{m}$$

which is the same order of magnitude as (a).

23. The combination of constants that gives time is

$$t_{\mathrm{Pl}} = \sqrt{\frac{Gh}{c^5}} = \sqrt{\frac{\left(6.673 \times 10^{-11}\ \mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}}\right)(6.626 \times 10^{-34}\ \mathrm{J \cdot s})}{(2.998 \times 10^8\ \mathrm{m/s})^5}} = 1.35 \times 10^{-43}\ \mathrm{s}$$

The time for light to travel the Planck length is

$$t = \frac{\lambda_{\mathrm{Pl}}}{c} = \sqrt{\frac{Gh}{c^5}} = 1.35 \times 10^{-43}\ \mathrm{s}$$

as we found in this problem.

24. As in Problem 19

$$\begin{aligned}
\frac{\Delta f}{f} &= -\frac{GM}{c^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) \\
&= -\frac{\left(6.673\times 10^{-11}\ \mathrm{m}^3\cdot\mathrm{kg}^{-1}\cdot\mathrm{s}^{-2}\right)\left(5.976\times 10^{24}\ \mathrm{kg}\right)}{\left(2.998\times 10^{8}\ \mathrm{m/s}\right)^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) \\
&= -\left(4.439\times 10^{-3}\,\mathrm{m}\right)\left(\frac{1}{3.587\times 10^{7}\ \mathrm{m}+6.378\times 10^{6}\ \mathrm{m}}-\frac{1}{6.378\times 10^{6}\ \mathrm{m}}\right) \\
&= 5.91\times 10^{-10}
\end{aligned}$$

Therefore

$$\Delta f = \left(5.91\times 10^{-10}\right)\left(2\times 10^{9}\ \mathrm{Hz}\right) = 1.18\,\mathrm{Hz}$$