Chapter 6

1. The function's output does not approach zero in the limits $-\infty$ and $+\infty$, so it cannot be normalized over these limits.

2. a) Given the placement of the $-$ sign, it moves in the $+x$-direction.

   b) By the same reasoning as in (a), it moves in the $-x$-direction.

   c) It is a complex number.

   d) It moves in the $+x$-direction. Looking at a particular phase $kx - \omega t$, x must increase as t increases in order to keep the phase constant.

3. The derivatives are $\partial\Psi/\partial t = -i\omega\Psi$ and $\partial^2\Psi/\partial x^2 = -k^2\Psi$. Substiruting in the time-dependent Schrödinger equation

$$i\hbar\left(-i\omega\Psi\right) = -\frac{\hbar^2}{2m}\left(-k^2\Psi\right) + V\Psi$$

The term on the left reduces to $\hbar\omega\Psi = E\Psi$ and the first term on the right is

$$\frac{\hbar^2 k^2}{2m}\Psi = \frac{p^2}{2m}\Psi = K\Psi$$

where K is the kinetic energy. The result is $E = K + V$. which is a statement of conservation of mechanical energy.

* 4.

$$\Psi^*\Psi = A^2 \exp\left[-i(kx - \omega t) + i(kx - \omega t)\right] = A^2$$

$$\int_0^a \Psi^*\Psi dx = A^2 \int_0^a dx = A^2 a = 1 \qquad \text{so} \qquad A = \frac{1}{\sqrt{a}} \quad \text{and}$$

$$\Psi = \frac{1}{\sqrt{a}} \exp\left[i(kx - \omega t)\right]$$

* 5.

$$\Psi^*\Psi = A^2 r^2 \exp\left(\frac{-2r}{\alpha}\right)$$

$$\int_0^\infty \Psi^*\Psi dr = A^2 \int_0^\infty r^2 \exp\left(\frac{-2r}{\alpha}\right) dr = A^2\left[\frac{2}{(2/\alpha)^3}\right] = \frac{A^2\alpha^3}{4} = 1$$

Therefore

$$A = \sqrt{\frac{4}{\alpha^3}} = 2\alpha^{-3/2}$$

6. In order for a particle to have a greater probability of being at a given point than at an adjacent point, it would need to have infinite speed. This is seen as $p = i\hbar\left(\frac{\partial\Psi}{\partial x}\right) \to \infty$ at a discontinuity. Another problem is that the second derivative must exist in order to satisfy the Schrödinger equation.

7. a) The wave function does not satisfy condition 3. The derivative of the wave function is not continuous at $x = 0$.

   b) Based on (a) the wave function cannot be realized physically.

   c) Very close to $x = 0$ we could modify the function so that its derivative is continuous. If we do so just in the neighborhood of $x = 0$, we need not change the function elsewhere.

8.

$$\begin{aligned}\bar{x} &= \frac{3.4 + 3.9 + 5.2 + 4.7 + 4.1 + 3.8 + 3.9 + 4.7 + 4.1 + 4.5 + 3.8 + 4.5 + 4.8 + 3.9 + 4.4}{15} \\ &= 4.247\end{aligned}$$

$\langle x^2 \rangle =$

$$\frac{3.4^2 + 3.9^2 + 5.2^2 + 4.7^2 + 4.1^2 + 3.8^2 + 3.9^2 + 4.7^2 + 4.1^2 + 4.5^2 + 3.8^2 + 4.5^2 + 4.8^2 + 3.9^2 + 4.4^2}{15}$$

$$\langle x^2 \rangle = 18.254$$

The standard deviation is

$$\sigma = \sqrt{\frac{\sum (x_i - \bar{x})^2}{N}} = \sqrt{\frac{\sum \left(x_i^2 - 2x_i\bar{x} + \bar{x}^2\right)}{N}} = \sqrt{\frac{\sum x_i^2}{N} - 2\bar{x}\frac{\sum x_i}{N} + \frac{\sum \bar{x}^2}{N}}$$

Look at the three terms in the sum. The first is just $\langle x^2 \rangle$. The second is $-2\bar{x}(\bar{x}) = -2\bar{x}^2$. The third term is

$$\frac{\sum \bar{x}^2}{N} = \frac{N\bar{x}^2}{N} = \bar{x}^2$$

Putting the results together

$$\sigma = \sqrt{\langle x^2 \rangle - 2\bar{x}^2 + \bar{x}^2} = \sqrt{\langle x^2 \rangle - \bar{x}^2}$$

For the data given we have

$$\sigma = \sqrt{\langle x^2 \rangle - \bar{x}^2} = \sqrt{18.254 - (4.247)^2} = 0.466$$

9. If V is independent of time, then we can use the time-independent Schrödinger equation. Then by Equation (6.15)

$$\Psi^*\Psi = \psi^*(x)\psi(x)e^{-i\omega t}e^{i\omega t} = \psi^*(x)\psi(x)$$

Then

$$\int \Psi^*\Psi x \, dx = \int \psi^*(x)\psi(x)\, x \, dx$$

which is independent of time.

* 10. Using the Euler relations between exponential and trig functions

$$\psi = A\left(e^{ix} + e^{-ix}\right) = 2A\cos(x)$$

Normalization:

$$\int_{-\pi}^{\pi} \psi^* \psi \, dx = 4A^2 \int_{-\pi}^{\pi} \cos^2(x) \, dx = 4A^2\pi = 1$$

Thus $A = \frac{1}{2\sqrt{\pi}}$ and the probability of being in the interval $[0, \pi/8]$ is

$$\begin{aligned} P &= \int_0^{\pi/8} \psi^* \psi \, dx = \frac{1}{\pi} \int_0^{\pi/8} \cos^2(x) \, dx = \frac{1}{\pi} \left(\frac{x}{2} + \frac{1}{4}\sin(2x) \right) \Big|_0^{\pi/8} \\ &= \frac{1}{16} + \frac{1}{4\pi\sqrt{2}} = 0.119 \end{aligned}$$

11.

$$\int_0^{\pi} \psi^* \psi \, dx = A^2 \int_0^{\pi} \sin^2(x) \, dx = A^2 \frac{\pi}{2} = 1$$

so $A = \sqrt{\frac{2}{\pi}}$ and the probability of being in the interval $[0, \pi/4]$ is

$$\begin{aligned} P &= \int_0^{\pi/4} \psi^* \psi \, dx = \frac{2}{\pi} \int_0^{\pi/4} \sin^2(x) \, dx = \frac{2}{\pi} \left(\frac{x}{2} - \frac{1}{4}\sin(2x) \right) \Big|_0^{\pi/4} \\ &= \frac{2}{\pi}\left(\frac{\pi}{8} - \frac{1}{4} \right) = \frac{1}{4} - \frac{1}{2\pi} = 0.091 \end{aligned}$$

12.

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \qquad E_{n+1} = \frac{(n+1)^2\pi^2\hbar^2}{2mL^2}$$

$$\Delta E_n = E_{n+1} - E_n = \frac{\pi^2\hbar^2}{2mL^2}\left[(n+1)^2 - n^2\right] = \frac{\pi^2\hbar^2}{2mL^2}(2n+1)$$

Computing specific values

$$\Delta E_1 = \frac{\pi^2\hbar^2}{2mL^2}(3)$$

$$\Delta E_8 = \frac{\pi^2\hbar^2}{2mL^2}(17)$$

$$\Delta E_{800} = \frac{\pi^2\hbar^2}{2mL^2}(1601)$$

* 13. The wave function for the nth level is $\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$ so the average value of the square of the wave function is

$$\begin{aligned} \langle \psi_n^2(x) \rangle &= \frac{\int_0^L \psi^* \psi \, dx}{\int_0^L dx} = \frac{1}{L} \int_0^L \psi^* \psi \, dx = \frac{2}{L^2} \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2}{L^2}\frac{L}{2} = \frac{1}{L} \end{aligned}$$

This result for the average value of the wave function is independent of n and is the same as the classical probability. The classical probability is uniform throughout the box, but this is not so in the quantum mechanical case which is $\frac{2}{L}\sin^2(k_n x)$.

14. a) We know the energy values from Equation (6.35). The energy value E_n is proportional to n^2 where n is the quantum number. If the ground state energy is 4.3 eV, then the next three levels correspond to: $4E_1 = 17.2$ eV for $n = 2$; $9E_1 = 38.7$ eV for $n = 3$; and $16E_1 = 68.8$ eV for $n = 4$.

b) The wave functions and energy levels will be like those shown in Figure 6.3.

* 15. a) Starting with Equation (6.35) and using the electron mass and the length given, we have

$$\begin{aligned} E_n &= n^2 \frac{\pi^2 \hbar^2}{2mL^2} = n^2 \frac{\pi^2 (\hbar c)^2}{2(mc^2) L^2} \\ &= n^2 \frac{\pi^2 (197.3\,\text{eV}\cdot\text{nm})^2}{2(5.11\times 10^5\,\text{eV})(2000\,\text{nm})^2} = n^2 (9.40\times 10^{-8}\,\text{eV}) \end{aligned}$$

Then the three lowest energy levels are: $E_1 = 9.40 \times 10^{-8}$ eV; $E_2 = 1.88 \times 10^{-7}$ eV; and $E_1 = 2.82 \times 10^{-7}$ eV;

b) Average kinetic energy equals $\frac{3}{2}kT = \frac{3}{2}\left[(1.381\times 10^{-23}\,\text{J/K})\left(\frac{1\,\text{eV}}{1.602\times 10^{-19}\,\text{J}}\right)\right] 13\,\text{K}$ which equals 1.68×10^{-3} eV. Substitute this value into the equation above as E_n and solve for n. We find $n = 134$.

16. For this non-relativistic speed we have $E = \frac{1}{2}mv^2 = \frac{1}{2}(mc^2)\beta^2 = 0.002555$ eV. Using $E = \frac{n^2h^2}{8mL^2}$ we find

$$n^2 = \frac{8mL^2E}{h^2} = \frac{8mc^2L^2E}{h^2c^2} = \frac{8(511\times 10^3\,\text{eV})(48.5\,\text{nm})^2(0.002555\,\text{eV})}{(1240\,\text{eV}\cdot\text{nm})^2} = 16.0$$

and therefore $n = 4$.

17. The ground-state wave function is $\psi_1 = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)$.

$$\begin{aligned} P_1 &= \int_0^{L/3} \psi_1^2 dx = \frac{2}{L}\int_0^{L/3} \sin^2\left(\frac{\pi x}{L}\right) dx = \frac{2}{L}\left(\frac{x}{2} - \frac{L}{4\pi}\sin\left(\frac{2\pi x}{L}\right)\right)\Bigg|_0^{L/3} \\ &= 2\left(\frac{1}{6} - \frac{\sqrt{3}}{8\pi}\right) = 0.1955 \end{aligned}$$

$$\begin{aligned} P_2 &= \int_{L/3}^{2L/3} \psi_1^2 dx = \frac{2}{L}\left(\frac{x}{2} - \frac{L}{4\pi}\sin\left(\frac{2\pi x}{L}\right)\right)\Bigg|_{L/3}^{2L/3} \\ &= 2\left(\frac{1}{6} + \frac{\sqrt{3}}{4\pi}\right) = 0.6090 \end{aligned}$$

$$\begin{aligned} P_3 &= \int_{2L/3}^{L} \psi_1^2 dx = \frac{2}{L}\left(\frac{x}{2} - \frac{L}{4\pi}\sin\left(\frac{2\pi x}{L}\right)\right)\Bigg|_{2L/3}^{L} \\ &= 2\left(\frac{1}{6} - \frac{\sqrt{3}}{8\pi}\right) = 0.1955 \end{aligned}$$

Notice that $P_1 + P_2 + P_3 = 1$ as required.

18. The first excited state has a wave function $\psi_2 = \sqrt{\frac{2}{L}} \sin(kx)$ with $k = \frac{2\pi}{L}$.

$$P_1 = \int_0^{L/3} \psi_2^2 dx = \frac{2}{L}\left(\frac{x}{2} - \frac{1}{4k}\sin(2kx)\right)\Big|_0^{L/3}$$
$$= 2\left(\frac{1}{6} + \frac{\sqrt{3}}{16\pi}\right) = 0.402$$

Similarly

$$P_2 = 2\left(\frac{1}{6} - \frac{\sqrt{3}}{8\pi}\right) = 0.196$$

and $P_3 = P_1 = 0.402$. Notice that $P_1 + P_2 + P_3 = 1$.

* 19.

$$E_1 = \frac{h^2}{8mL^2} = \frac{h^2c^2}{8mc^2L^2} = \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8(511 \times 10^3 \text{ eV})(10^{-5} \text{ nm})^2} = 3.76 \text{ GeV}$$

Then $E_2 = 4E_1 = 15.05$ GeV and $\Delta E = E_2 - E_1 = 11.3$ GeV.

20. a) As in the previous problem

$$E_1 = \frac{h^2c^2}{8mc^2L^2} = \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8(938.27 \times 10^6 \text{ eV})(1.4 \times 10^{-5} \text{ nm})^2} = 1.05 \text{ MeV}$$

b)

$$E_1 = \frac{h^2c^2}{8mc^2L^2} = \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8(3727 \times 10^6 \text{ eV})(1.4 \times 10^{-5} \text{ nm})^2} = 263 \text{ keV}$$

21. As in previous problems the ground state energy is

$$E_1 = \frac{h^2}{8mL^2} = \frac{h^2c^2}{8mc^2L^2} = \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8(511 \times 10^3 \text{ eV})(0.5 \text{ nm})^2} = 1.5045 \text{ eV}$$

The other energy levels are $E_n = n^2E_1$:

$$E_2 = 4E_1 = 6.02 \text{ eV} \qquad E_3 = 9E_1 = 13.54 \text{ eV} \qquad E_4 = 16E_1 = 24.07 \text{ eV}$$

The allowed photon energies are

$$\begin{array}{lll} E_4 - E_3 = 10.5 \text{ eV} & E_4 - E_2 = 18.1 \text{ eV} & E_4 - E_1 = 22.6 \text{ eV} \\ E_3 - E_2 = 7.5 \text{ eV} & E_3 - E_1 = 12.0 \text{ eV} & E_2 - E_1 = 4.52 \text{ eV} \end{array}$$

22. Lacking an explicit equation for finite square well energies, we will approximate using the infinite square well formula. In order to contain three energy levels the depth of the well must be at least

$$E = \frac{n^2h^2}{8mL^2} = \frac{9h^2}{8mL^2}$$

Evaluating numerically with the given mass

$$E = \frac{9h^2}{8mL^2} = \frac{9h^2c^2}{8mc^2L^2} = \frac{9(1240 \text{ eV} \cdot \text{nm})^2}{8(2 \times 10^9 \text{ eV})(3 \times 10^{-6} \text{ nm})^2} = 96.1 \text{ MeV}$$

23. a) The wavelengths are longer for the finite well, because the wave functions can leak outside the box.

b) Generally shorter wavelengths correspond to higher energies, so we expect energies to be lower for the finite well.

c) Generally the number of bound states is limited by the depth of the well. We expect no bound states for $E > V_0$.

* 24. From the boundary condition $\psi_1(x=0) = \psi_2(x=0)$ we have $Ae^0 = Ce^0 + De^0$ or $A = C + D$. From the condition $\psi_1'(x=0) = \psi_2'(x=0)$ we have $\alpha A = ikC - ikD$. Solving this last expression for A and combining with the first boundary condition gives

$$C + D = \frac{ik}{\alpha}C - \frac{ik}{\alpha}D$$

or after rearranging

$$\frac{C}{D} = \frac{ik+\alpha}{ik-\alpha}$$

25. As in the previous problem matching the wave function at the boundary gives

$$Ce^{ikL} + De^{-ikL} = Be^{-\alpha L}$$

and matching the first derivative gives

$$ikCe^{ikL} - ikDe^{-ikL} = -\alpha Be^{-\alpha L}$$

Eliminating B from these two equations gives

$$Ce^{ikL} + De^{-ikL} = \frac{1}{\alpha}\left(ikDe^{-ikL} - ikCe^{ikL}\right)$$

Thus

$$\frac{C}{D} = \frac{ik-\alpha}{ik+\alpha}e^{-2ikL}$$

26.

$$E = \frac{\pi^2\hbar^2}{2mL^2}\left(n_1^2 + n_2^2 + n_3^2\right) = E_0\left(n_1^2 + n_2^2 + n_3^2\right)$$

where

$$E_0 = \frac{\pi^2\hbar^2}{2mL^2}$$

Then the second, third, fourth, and fifth levels are

$$E_2 = \left(2^2 + 1^2 + 1^2\right)E_0 = 6E_0 \qquad \text{(degenerate)}$$

$$E_3 = \left(2^2 + 2^2 + 1^2\right)E_0 = 9E_0 \qquad \text{(degenerate)}$$

$$E_4 = \left(3^2 + 1^2 + 1^2\right)E_0 = 11E_0 \qquad \text{(degenerate)}$$

$$E_5 = \left(2^2 + 2^2 + 2^2\right)E_0 = 12E_0 \qquad \text{(not degenerate)}$$

27. In general we have

$$\psi(x) = A\sin\left(\frac{n_1\pi x}{L}\right)\sin\left(\frac{n_2\pi y}{L}\right)\sin\left(\frac{n_3\pi z}{L}\right)$$

For $\psi_2(x)$ we can have $(n_1, n_2, n_3) = (1,1,2)$ or $(1,2,1)$ or $(2,1,1)$

For $\psi_3(x)$ we can have $(n_1, n_2, n_3) = (1,2,2)$ or $(2,2,1)$ or $(2,1,2)$

For $\psi_4(x)$ we can have $(n_1, n_2, n_3) = (1,1,3)$ or $(1,3,1)$ or $(3,1,1)$

For $\psi_5(x)$ we can have $(n_1, n_2, n_3) = (2,2,2)$

* 28. We must normalize by evaluating the triple integral of $\psi^*\psi$:

$$\iiint \psi^*\psi\, dx dy dz = 1$$

with $\psi(x, y, z)$ given by Equation (6.47) in the text. We can evaluate the iterated triple integral

$$A^2\int_0^L \sin^2\left(\frac{\pi x}{L}\right)dx \int_0^L \sin^2\left(\frac{\pi y}{L}\right)dy \int_0^L \sin^2\left(\frac{\pi z}{L}\right)dz = A^2\left(\frac{L}{2}\right)^3 = 1$$

Solving for A we find

$$A = \left(\frac{2}{L}\right)^{3/2}$$

29. Taking the derivatives we find $\nabla^2\psi = -\left(k_1^2 + k_2^2 + k_3^2\right)\psi$ so the Schrödinger equation becomes

$$-\frac{\hbar^2}{2m}\nabla^2\psi = \frac{\hbar^2}{2m}\left(k_1^2 + k_2^2 + k_3^2\right)\psi = E\psi$$

From the boundary conditions $k_i = \dfrac{n_i\pi}{L_i}$ so

$$E = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2}\right)$$

The three quantum numbers come directly from the three boundary conditions.

30. $\psi_0(x)$ has these features: it is symmetric about $x = 0$; it has a maximum at $x = 0$ because the wave function must tend toward zero for $x \to \pm\infty$; there is no node in the ground state; the wave function decreases exponentially where $V > E$.

* 31.

$$\Delta E_n = E_{n+1} - E_n = \left(n + 1 + \frac{1}{2}\right)\hbar\omega - \left(n + \frac{1}{2}\right)\hbar\omega = \hbar\omega \qquad \text{for all } n$$

This is true for all n, and there is no restriction on the number of levels.

32. Normalization

$$1 = \int_{-\infty}^{\infty}\psi^*\psi\,dx = A^2\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}dx = 2A^2\int_0^{\infty} x^2 e^{-\alpha x^2}dx = 2A^2\frac{1}{4\alpha}\sqrt{\frac{\pi}{\alpha}}$$

Solving for A we find $A = \sqrt{2}\pi^{-1/4}\alpha^{3/4}$

$$\langle x \rangle = A^2 \int_{-\infty}^{\infty} x^3 e^{-\alpha x^2}\, dx = 0$$

because the integrand is odd over symmetric limits.

$$\langle x^2 \rangle = A^2 \int_{-\infty}^{\infty} x^4 e^{-\alpha x^2}\, dx = \frac{3}{4} A^2 \pi^{1/2} \alpha^{-5/2} = \frac{3}{2\alpha}$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{3}{2\alpha}}$$

33.

$$\begin{aligned} E &= \left(n + \frac{1}{2}\right) \hbar\omega = \left(n + \frac{1}{2}\right) hf = \left(4.136 \times 10^{-15}\ \mathrm{eV \cdot s}\right)\left(10^{13}\ \mathrm{s}^{-1}\right)\left(n + \frac{1}{2}\right) \\ &= \left(4.136 \times 10^{-2}\ \mathrm{eV}\right)\left(n + \frac{1}{2}\right) \end{aligned}$$

For the harmonic oscillator $\omega^2 = \dfrac{k}{m}$ so

$$k = \omega^2 m = 4\pi^2 f^2 m = 4\pi^2 \left(10^{13}\ \mathrm{s}^{-1}\right)^2 \left(3.32 \times 10^{-26}\ \mathrm{kg}\right) = 131\ \mathrm{N/m}$$

34. Taking the second derivative of ψ for the Schrödinger equation:

$$\frac{d\psi}{dx} = 5A\alpha x e^{-\alpha x^2/2} - 2\alpha^2 A x^3 e^{-\alpha x^2/2}$$

$$\frac{d^2\psi}{dx^2} = \left(5\alpha - 5\alpha^2 x^2 - 6\alpha^2 x^2 + 2\alpha^3 x^4\right) A e^{-\alpha x^2/2}$$

Starting with Equation (6.56), and with the wave function given in the problem, we have

$$\begin{aligned} \frac{d^2\psi}{dx^2} &= \left(\alpha^2 x^2 - \beta\right)\psi = \left(\alpha^2 x^2 - \beta\right) A \left(2\alpha x^2 - 1\right) e^{-\alpha x^2/2} \\ &= \left(2\alpha^3 x^4 + x^2\left(-2\alpha\beta - \alpha^2\right) + \beta\right) A e^{-\alpha x^2/2} \end{aligned}$$

Matching our two values of $d^2\psi/dx^2$ we see that the Schrödinger equation can only be satisfied if $\beta = 5\alpha$. Then

$$\frac{2mE}{\hbar^2} = 5\sqrt{\frac{mk}{\hbar^2}}$$

or $E = \frac{5}{2}\hbar\omega$. This is the expected result, because the wave function contains a second-order polynomial (in x), and with $n = 2$ we expect $E = \left(n + \frac{1}{2}\right)\hbar\omega = \frac{5}{2}\hbar\omega$.

35. By symmetry $\langle p \rangle = 0$. Setting the ground state energy equal to $\dfrac{\langle p^2 \rangle}{2m}$ we find

$$\frac{\langle p^2 \rangle}{2m} = \frac{1}{2}\hbar\omega \qquad\qquad \langle p^2 \rangle = \hbar\omega m$$

Note, however, that a detailed calculation gives $\langle p^2 \rangle = \frac{1}{2}\hbar\omega m$. The factor of one-half is evidently the difference between the kinetic energy and the total energy, which if taken into account does give the correct result.

* 36. Taking the derivatives for the Schrödinger equation

$$\frac{d\psi}{dx} = Ae^{-\alpha x^2/2} - A\alpha x^2 e^{-\alpha x^2/2}$$

$$\frac{d^2\psi}{dx^2} = -3A\alpha x e^{-\alpha x^2/2} + A\alpha^2 x^3 e^{-\alpha x^2/2} = \left(\alpha^2 x^2 - 3\alpha\right)\psi$$

Combining Equation (6.56) with this, we see

$$\frac{d^2\psi}{dx^2} = \left(\alpha^2 x^2 - \beta\right)\psi = \left(\alpha^2 x^2 - 3\alpha\right)\psi$$

Thus we see that $\beta = 3\alpha$ or

$$\frac{2mE}{\hbar^2} = 3\sqrt{\frac{mk}{\hbar^2}}$$

$$E = \frac{3}{2}\hbar\sqrt{\frac{k}{m}} = \frac{3}{2}\hbar\omega$$

37. The classical frequency for a two-particle oscillator is (see Chapter 10, Equation (10.4)) $\omega = \sqrt{k/\mu} = \sqrt{k\left(m_1 + m_2\right)/m_1 m_2} = \sqrt{2k/m}$ since the masses are equal in this case. The energies of the ground state (E_0) and the first three excited states are given by $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$ so the possible transitions (from E_3 to E_2, E_3 to E_1, etc. are $\Delta E = \hbar\omega$, $2\hbar\omega$, and $3\hbar\omega$, or

$$\hbar\omega = \hbar\sqrt{\frac{2k}{m}} = \left(6.582 \times 10^{-16}\ \text{eV}\cdot\text{s}\right)\sqrt{\frac{2\left(1.1 \times 10^3\ \text{N/m}\right)}{1.673 \times 10^{-27}\ \text{kg}}} = 0.755\ \text{eV}$$

$$\lambda = \frac{hc}{E} = \frac{1240\,\text{eV}\cdot\text{nm}}{0.755\,\text{eV}} = 1640\,\text{nm}$$

$$2\hbar\omega = 2\left(6.582 \times 10^{-16}\ \text{eV}\cdot\text{s}\right)\sqrt{\frac{2\left(1.1 \times 10^3\ \text{N/m}\right)}{1.673 \times 10^{-27}\ \text{kg}}} = 1.51\ \text{eV}$$

$$\lambda = \frac{hc}{E} = \frac{1240\,\text{eV}\cdot\text{nm}}{1.51\,\text{eV}} = 821\,\text{nm}$$

$$3\hbar\omega = 3\left(6.582 \times 10^{-16}\ \text{eV}\cdot\text{s}\right)\sqrt{\frac{2\left(1.1 \times 10^3\ \text{N/m}\right)}{1.673 \times 10^{-27}\ \text{kg}}} = 2.26\ \text{eV}$$

$$\lambda = \frac{hc}{E} = \frac{1240\,\text{eV}\cdot\text{nm}}{2.26\,\text{eV}} = 549\,\text{nm}$$

38. The kinetic energy is (see Chapter 9)

$$K = \frac{3}{2}kT = \frac{3}{2}\left(8.617 \times 10^{-5}\ \text{eV/K}\right)\left(12000\ \text{K}\right) = 1.551\ \text{eV}$$

Assume a square-top potential of height

$$V_0 = \frac{q_1 q_2}{4\pi\varepsilon_0 r} = \frac{6e^2}{4\pi\varepsilon_0 r} = \frac{6\left(1.440\ \text{eV}\cdot\text{nm}\right)}{\left(1.2 \times 10^{-6}\ \text{nm}\right)\left(12\right)^{1/3}} = 3.145\ \text{MeV}$$

where we have used the fact that the radius of a nucleus is approximately $1.2A^{1/3}$ fm (see Chapter 12). For the width of the potential barrier use twice the radius or

$$L = 2\,(1.2 \times 10^{-6}\ \text{nm})\,(12)^{1/3} = 5.49 \times 10^{-6}\ \text{nm}$$

Then

$$\begin{aligned}\kappa &= \frac{\sqrt{2mc^2\,(V_0 - E)}}{\hbar c} = \frac{\sqrt{2\,(938.27 \times 10^6\ \text{eV})\,[3.145 \times 10^6\ \text{eV} - 1.551\ \text{eV}]}}{197.3\ \text{eV}\cdot\text{nm}}\\ &= 3.89 \times 10^5\ \text{nm}^{-1}\end{aligned}$$

$$\kappa L = (3.89 \times 10^5\ \text{nm}^{-1})\,(5.49 \times 10^{-6}\ \text{nm}) = 2.135$$

$$T = \left(1 + \frac{(3.145 \times 10^6\ \text{eV})^2 \sinh^2(2.135)}{4\,(1.551\ \text{eV})\,(3.145 \times 10^6\ \text{eV} - 1.551\ \text{eV})}\right)^{-1} = 1.14 \times 10^{-7}$$

39. a)

$$p = \sqrt{2m\,(E - V_0)}$$

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m\,(E - V_0)}} \qquad K = E - V_0$$

b)

$$p = \sqrt{2m\,(E + V_0)}$$

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m\,(E + V_0)}} \qquad K = E + V_0$$

* 40. In each case $\kappa L \gg 1$ so we can use

$$T = 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2\kappa L}$$

where

$$\kappa = \frac{\sqrt{2mc^2\,(V_0 - E)}}{\hbar c} = \frac{(2\,(3727 \times 10^6\ \text{eV})\,(10 \times 10^6\ \text{eV}))^{1/2}}{197.4\ \text{eV}\cdot\text{nm}} = 1.38 \times 10^{15}\ \text{m}^{-1}$$

a) With $L = 1.3 \times 10^{-14}$ m

$$T_a = 16\frac{5\ \text{MeV}}{15\ \text{MeV}}\left(1 - \frac{5\ \text{MeV}}{15\ \text{MeV}}\right)e^{-2(1.38 \times 10^{15}\ \text{m}^{-1})(1.3 \times 10^{-14}\ \text{m})} = 9.3 \times 10^{-16}$$

b) With $V_0 = 30$ MeV

$$\kappa = \frac{\sqrt{2mc^2\,(V_0 - E)}}{\hbar c} = \frac{(2\,(3727 \times 10^6\ \text{eV})\,(25 \times 10^6\ \text{eV}))^{1/2}}{197.4\ \text{eV}\cdot\text{nm}} = 2.19 \times 10^{15}\ \text{m}^{-1}$$

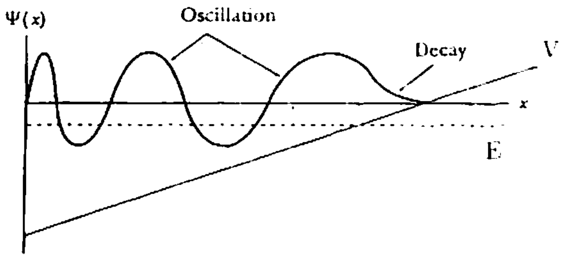
$$T_b = 16\frac{5\ \text{MeV}}{30\ \text{MeV}}\left(1 - \frac{5\ \text{MeV}}{30\ \text{MeV}}\right)e^{-2(2.19 \times 10^{15}\ \text{m}^{-1})(1.3 \times 10^{-14}\ \text{m})} = 4.2 \times 10^{-25}$$

c) With $V_0 = 15$ MeV we return to the original value of κ, but now $L = 2.6 \times 10^{-14}$ m and

$$T_c = 16\frac{5\ \text{MeV}}{15\ \text{MeV}}\left(1 - \frac{5\ \text{MeV}}{15\ \text{MeV}}\right)e^{-2(1.38 \times 10^{15}\ \text{m}^{-1})(2.6 \times 10^{-14}\ \text{m})} = 2.4 \times 10^{-31}$$

By comparison $T_a > T_b > T_c$.

41. When $E > V$ the wave function is oscillating, with a longer wavelength as $E - V$ decreases. Then when $E < V$ the wave function decays.



42. In general for $E > V_0$

$$R = 1 - T = 1 - \left[1 + \frac{V_0^2 \sin^2(k_2 L)}{4E(E - V_0)}\right]^{-1}$$

If $E \gg V_0$ then $4E(E - V_0) \approx 4E^2$. From the binomial theorem $(1 + x)^{-1} \approx 1 - x$ for small x and

$$R \approx 1 - \left[1 - \frac{V_0^2 \sin^2(k_2 L)}{4E^2}\right] = \frac{V_0^2 \sin^2(k_2 L)}{4E^2}$$

$$R \approx \left(\frac{V_0 \sin(k_2 L)}{2E}\right)^2$$

43.

$$T = \left[1 + \frac{V_0^2 \sin^2(k_2 L)}{4E(E - V_0)}\right]^{-1}$$

a) To obtain $T = 1$ we require $\sin(k_2 L) = 0$. Except for the trivial solution $L = 0$, this occurs whenever $k_2 L = n\pi$ with n an integer. Letting $n = 1$ we find

$$L = \frac{\pi}{k_2} = \frac{\pi\hbar}{\sqrt{2m(E - V_0)}} = \frac{1}{2}\frac{hc}{\sqrt{2mc^2(E - V_0)}} = \frac{1}{2}\frac{1240\ \text{eV}\cdot\text{nm}}{\sqrt{2(511\times10^3\ \text{eV})(7.2\ \text{eV})}} = 0.229\ \text{nm}$$

Any integer multiple of this value will work.

b) For maximum reflection $\sin^2(k_2 L) = 1$ or $L = \dfrac{n\pi}{2k_2}$ for any odd integer n. From the result of (a) we see that the first maximum is with L equal to half the value of L for the first minimum, or $L = 0.114$ nm.

44.

$$\kappa = \frac{\sqrt{2mc^2(V_0 - E)}}{\hbar c} = \frac{[2(511\times10^3\ \text{eV})(1.5\ \text{eV})]^{1/2}}{197.4\ \text{eV}\cdot\text{nm}} = 6.27\ \text{nm}^{-1}$$

With a probability of 10^{-4} we know $\kappa L \gg 1$ and we can use

$$T = 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2\kappa L} = 16\frac{1}{2.5}\left(1 - \frac{1}{2.5}\right)e^{-2\kappa L} = 3.84e^{-2\kappa L} = 10^{-4}$$

Solving for L:

$$L = \frac{\ln\left(3.84 \times 10^4\right)}{2\left(6.27 \times 10^9 \text{ m}^{-1}\right)} = 8.42 \times 10^{-10}\,\text{m}.$$

Now using the proton mass

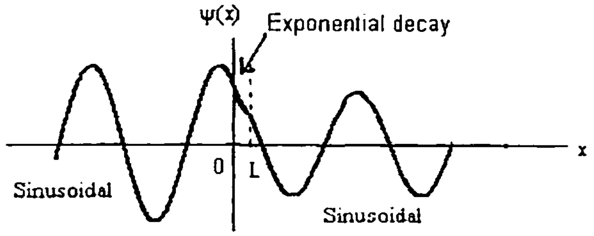
$$\kappa = \frac{\sqrt{2mc^2\left(V_0 - E\right)}}{\hbar c} = \frac{\left[2\left(938.27 \times 10^6 \text{ eV}\right)\left(1.5 \text{ eV}\right)\right]^{1/2}}{197.4 \text{ eV} \cdot \text{nm}} = 268.8 \text{ nm}^{-1}$$

$$T = 3.84e^{-2\kappa L} = 3.84e^{-2\left(268.8\times10^9 \text{ m}^{-1}\right)\left(8.42\times10^{-10} \text{ m}\right)} = 9.9 \times 10^{-197}$$

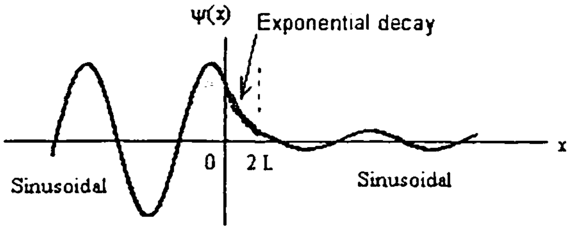
The proton's probability is much lower!

45. The sketch of the potential will be like Figure 6.12. Since the question mentions tunneling, the assumption is that the total energy is less than the potential. Therefore the wave function will be sinusoidal on either side of the barrier with an exponential decay in the barrier region. Each sketch will be similar to Figure 6.15. From Equation (6.70) we can see that in part b) with a barrier twice as wide, the exponential factor will be markedly smaller (a ratio of $e^{-2} = 0.135$ while doubling the barrier height in c) will reduce the transmission coefficient as compared to a) by less than 1/2.

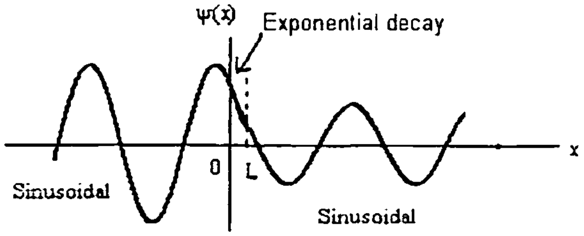
a)



b)



c)



46. Starting with $T \approx e^{-2\kappa L}$ and $\kappa = \dfrac{\sqrt{2mc(V_0 - E)}}{\hbar} = \dfrac{\sqrt{2mc^2(V_0 - E)}}{\hbar c}$, we find:

a) $\kappa = \dfrac{\sqrt{2\,(511 \times 10^3\,\text{eV})\,(6.4\,\text{eV} - 1.4\,\text{eV})}}{197.33\,\text{eV}} = 11.5\,\text{nm}^{-1}$

Then $T \approx e^{-2\kappa L} = 2e^{-2\left(11.5\,\text{nm}^{-1}(2.8\,\text{nm})\right)} = 2e^{-64.4} = 2.1 \times 10^{-28}$

b) $\kappa = \dfrac{\sqrt{2\,(3727 \times 10^6\,\text{eV})\,(19.2 \times 10^6\,\text{eV} - 4.4 \times 10^6\,\text{eV})}}{197.33\,\text{eV}} = 1.683 \times 10^6\,\text{nm}^{-1}$

Then $T \approx e^{-2\kappa L} = 2e^{-2\left(1.683\times 10^{15}\,\text{m}^{-1}(6.7\times 10^{-15}\,\text{m})\right)} = 2e^{-22.6} = 3.1 \times 10^{-10}$

For value of $\kappa L \gg 1$, Equation (6.70) is a good approximation for Equation (6.67) and Equation (6.73) gives at least an estimate of the transmission that is to within an order of magnitude.

* 47. As in the text we find

$$E = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2}\right)$$

and substituting the given values of L we find

$$E = \frac{\hbar^2\pi^2}{2mL^2}\left(n_1^2 + 2n_2^2 + \frac{n_3^2}{4}\right)$$

Letting $E_0 = \hbar^2\pi^2/2mL^2$ we have

$$E_1 = E_0\left(1 + 2 + \frac{1}{4}\right) = \frac{13}{4}E_0$$

$$E_2 = E_0\left(1 + 2 + \frac{2^2}{4}\right) = 4E_0$$

$$E_3 = E_0\left(1 + 2 + \frac{3^2}{4}\right) = \frac{21}{4}E_0$$

$$E_4 = E_0\left(2^2 + 2 + \frac{1}{4}\right) = \frac{25}{4}E_0$$

$$E_5 = E_0\left(1 + 2 + \frac{4^2}{4}\right) = E_0\left(2^2 + 2 + \frac{2^2}{4}\right) = 7E_0$$

Of those listed, only E_5 is degenerate.

48. Recognizing this as the infinite square well wave function we see that $k = \dfrac{3\pi}{\alpha}$ and

$$E = \frac{p^2}{2m} = \frac{\hbar^2k^2}{2m} = \frac{9\hbar^2\pi^2}{2m\alpha^2}$$

* 49. a) In general inside the box we have a superposition of sine and cosine functions, but only the sine function satisfies the boundary condition $\psi(0) = 0$, and thus $\psi = A\sin(kx)$. With $V = 0$ inside the well $E = \dfrac{p^2}{2m} = \dfrac{\hbar^2k^2}{2m}$ or $k = \dfrac{\sqrt{2mE}}{\hbar}$. Outside the well the decaying exponential

is required as explained in section 6.4 of the text, with $E = \frac{\hbar^2 k^2}{2m} + V_0$ which reduces to $\kappa = ik = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$.

b) Equating the wavefunctions and first derivatives at $x = L$:

$$A\sin(kL) = Be^{-\kappa L}$$
$$kA\cos(kL) = -\kappa Be^{-\kappa L}$$

Dividing these two equations

$$\frac{\tan(kL)}{k} = -\frac{1}{\kappa}$$
$$\kappa\tan(kL) = -k$$

50. From the previous problem $\kappa\tan(kL) = -k$ or $\kappa L\tan(kL) = -kL$. Let $\alpha = kL$ and $\beta = \kappa L$ so that

$$\beta\tan\alpha = -\alpha \qquad (1)$$

Now from the value of V_0 given in the problem and the definitions of α and β we have

$$1 = \frac{2mV_0L^2}{\hbar^2} = \frac{2mEL^2}{\hbar^2} + \frac{2m(V_0 - E)L^2}{\hbar^2} = k^2L^2 + \kappa^2L^2 = \alpha^2 + \beta^2 \qquad (2)$$

Solving Equations (1) and (2) numerically we find $\alpha = 0.20$ and $\beta = -0.98$. Then E is given by

$$E = \frac{\hbar^2k^2}{2m} = \hbar^2\frac{(\alpha/L)^2}{2m} = \frac{0.04\,\hbar^2}{2mL^2} = \frac{0.004\,h^2}{8mL^2}$$

51. Referring to the solution to the previous problem, we see that only a finite number of solutions to Equation (1) exist up to any particular (finite) value of V_0. Therefore for any finite V_0 only a finite number of combinations of α and β will satisfy both equations, and the number of bound states is finite.

52. Using the nomenclature of Problem 49

$$\kappa L = \frac{\sqrt{2m(V_0 - E)}\,L}{\hbar} = \frac{\sqrt{2(939 \times 10^6\text{ eV})(2.2 \times 10^6\text{ eV})}\,(3.5 \times 10^{-6}\text{ nm})}{197.4\text{ eV}\cdot\text{nm}} = 1.14$$

where we have used the mass of one nucleon, because one nucleon is "bound" by the other. Now $\kappa L = \beta = -\alpha/\tan\alpha$ so $\alpha \approx 2.07 = kL$. Then

$$E = \frac{\hbar^2k^2}{2m} = \hbar^2\frac{(\alpha/L)^2}{2m} = \hbar^2\frac{(2.07/L)^2}{2m} = \frac{(2.07)^2(197.4\text{ eV}\cdot\text{nm})^2}{2(939 \times 10^6\text{ eV})(3.5 \times 10^{-6}\text{ nm})^2} = 7.26\text{ MeV}$$

This means that $V_0 = 2.2$ MeV $+E = 9.46$ MeV. The next solution of the equation $\beta = -\alpha/\tan\alpha$ is at $\alpha \approx 4.94$, a value that will put $E > V_0$. Therefore there are no excited states.

53. a) This was calculated in Problem 34.

b)

$$\langle x\rangle = \int_{-\infty}^{\infty} x\,\psi^*\psi\,dx = 0$$

because the integrand is odd over symmetric limits. To find $\langle x^2 \rangle$ we first need to normalize:

$$\begin{aligned} A^2 \int_{-\infty}^{\infty} \left(1 - 2\alpha x^2\right)^2 e^{-\alpha x^2} dx &= 2A^2 \int_0^{\infty} \left(1 - 2\alpha x^2\right)^2 e^{-\alpha x^2} dx = 1 \\ &= 2A^2 \int_0^{\infty} \left(1 - 4\alpha x^2 + 4\alpha^2 x^4\right)^2 e^{-\alpha x^2} dx = 1 \\ &= 2A^2 \sqrt{\frac{\pi}{\alpha}} \left(\frac{1}{2} - 1 + \frac{3}{2}\right) = 1 \end{aligned}$$

Thus $A^2 = \frac{1}{2}\sqrt{\frac{\alpha}{\pi}}$ and

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \psi^* \psi \, dx = 2A^2 \int_0^{\infty} x^2 \left(1 - 4\alpha x^2 + 4\alpha^2 x^4\right)^2 e^{-\alpha x^2} dx = \frac{1}{\alpha}\left(\frac{1}{4} - \frac{3}{2} + \frac{15}{4}\right) = \frac{5}{2\alpha}$$

54. Using the known values of ψ_1 and ψ_2 we see

$$\begin{aligned} \psi &= \frac{1}{2}\psi_1 + \frac{\sqrt{3}}{2}\psi_2 = \frac{1}{2}\sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) + \frac{\sqrt{3}}{2}\sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right) \\ \psi &= \sqrt{\frac{1}{2L}} \sin\left(\frac{\pi x}{L}\right) + \sqrt{\frac{3}{2L}} \sin\left(\frac{2\pi x}{L}\right) \end{aligned}$$

For normalization

$$\int_0^L \psi^* \psi dx = \int_0^L \left(\frac{1}{2L}\sin^2\left(\frac{\pi x}{L}\right) + \frac{3}{2L}\sin^2\left(\frac{2\pi x}{L}\right) + \frac{\sqrt{3}}{2L}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\right) dx$$

The third term vanishes because of the orthogonality of the trig functions, leaving

$$\begin{aligned} \int_0^L \psi^* \psi dx &= \int_0^L \left(\frac{1}{2L}\sin^2\left(\frac{\pi x}{L}\right) + \frac{3}{2L}\sin^2\left(\frac{2\pi x}{L}\right)\right) dx \\ &= \frac{1}{2L}\int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx + \frac{3}{2L}\int_0^L \sin^2\left(\frac{2\pi x}{L}\right) dx \\ &= \frac{1}{2L}\left(\frac{L}{2}\right) + \frac{3}{2L}\left(\frac{L}{2}\right) = 1 \text{ as required} \end{aligned}$$

55. Using the Taylor approximation for the exponential $e^x \approx 1 + x$ for small x, we have

$$V(r) = D\left(1 - e^{-a(r-r_e)}\right)^2 \approx D\left(1 - (1 - a(r - r_e))\right)^2 = D\left(a(r - r_e)\right)^2 = Da^2 (r - r_e)^2$$

56. We will solve for numerical values of the factors in front of the quantum numbers:

$$\hbar\omega = \hbar a\sqrt{\frac{2D}{\mu}} = \hbar a\sqrt{\frac{2D\,(m_1 + m_2)}{m_1 m_2}}$$

$$\hbar\omega = \left(6.582 \times 10^{-16} \text{ eV} \cdot \text{s}\right)\left(7.8 \times 10^9 \text{ m}^{-1}\right)$$

$$\times\sqrt{\frac{2\,(4.42\text{ eV})\,(39.10\text{ u }+35.45\text{ u})}{(39.10\text{ u})\,(35.45\text{ u})}\,\frac{1\text{ u}}{931.5\times10^{6}\text{ eV}/c^{2}}}\,\frac{2.998\times10^{8}\text{ m/s}}{c}$$

$$\hbar\omega=0.03477\text{ eV}$$

$$\frac{\hbar^2\omega^2}{4D}=\frac{(0.03477\text{ eV})^2}{4\,(4.42\text{ eV})}=6.838\times10^{-5}\text{ eV}$$

Evaluating for specific energy levels:

$$E_0=\left(0+\frac{1}{2}\right)(0.03477\text{ eV})-\left(0+\frac{1}{2}\right)^2\left(6.838\times10^{-5}\text{ eV}\right)=0.017\text{ eV}$$

$$E_1=\left(1+\frac{1}{2}\right)(0.03477\text{ eV})-\left(1+\frac{1}{2}\right)^2\left(6.838\times10^{-5}\text{ eV}\right)=0.052\text{ eV}$$

$$E_2=\left(2+\frac{1}{2}\right)(0.03477\text{ eV})-\left(2+\frac{1}{2}\right)^2\left(6.838\times10^{-5}\text{ eV}\right)=0.086\text{ eV}$$

$$E_3=\left(3+\frac{1}{2}\right)(0.03477\text{ eV})-\left(3+\frac{1}{2}\right)^2\left(6.838\times10^{-5}\text{ eV}\right)=0.121\text{ eV}$$

Note that for these low quantum numbers the second-order correction is small.

* 57. The solution is identical to the presentation in the text for the three-dimensional box but without the z dimension. Briefly, we assume a trial function for the form

$$\psi(x,y)=A\sin(k_1x)\sin(k_2y)$$

Assuming that one corner is at the origin, applying the boundary conditions leads to

$$k_1=\frac{n_x\pi}{L}\qquad\qquad k_2=\frac{n_y\pi}{L}$$

and substituting into the Schrödinger equation leads to

$$E=\frac{\pi^2\hbar^2}{2mL^2}\left(n_x^2+n_y^2\right)$$

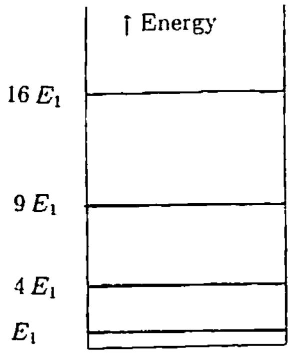
To normalize do the iterated double integral

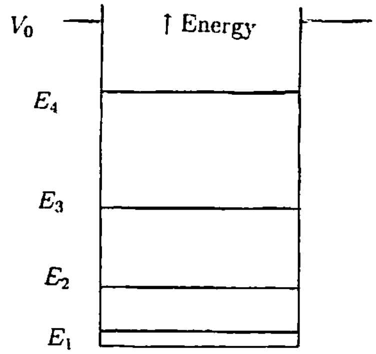
$$\begin{aligned}\int_0^L\int_0^L\psi^*\psi\,dxdy &= A^2\int_0^L\int_0^L\sin^2\left(\frac{n_x\pi x}{L}\right)\sin^2\left(\frac{n_y\pi y}{L}\right)dxdy\\ &= A^2\left(\frac{L}{2}\right)\left(\frac{L}{2}\right)=1\end{aligned}$$

so $A=\dfrac{2}{L}$. Now to find the energy levels use the energy equation with different values of the quantum numbers. Letting $E_0=\dfrac{\pi^2\hbar^2}{2mL^2}$ we have

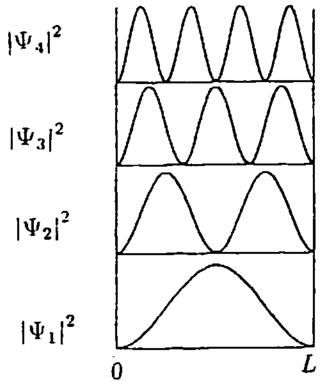
$E_1=E_0\left(1^2+1^2\right)=2E_0$	$n_1=1,\,n_2=1$
$E_2=E_0\left(2^2+1^2\right)=5E_0$	$n_1=2,\,n_2=1$ or vice versa
$E_3=E_0\left(2^2+2^2\right)=8E_0$	$n_1=2,\,n_2=2$
$E_4=E_0\left(3^2+1^2\right)=10E_0$	$n_1=3,\,n_2=1$ or vice versa
$E_5=E_0\left(3^2+2^2\right)=13E_0$	$n_1=3,\,n_2=2$ or vice versa
$E_6=E_0\left(4^2+1^2\right)=17E_0$	$n_1=4,\,n_2=1$ or vice versa

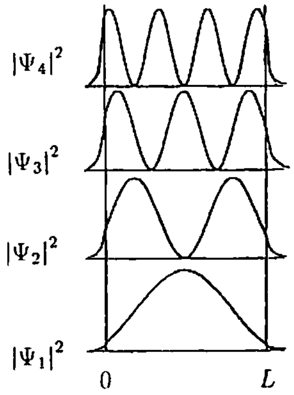
58. a) For the infinite square well, the energies are given by Equation (6.35) and increase as n^2.





b) For the infinite square well, the wave functions are given by Equation (6.34) and the $|\psi|^2$ can be determined easily. Note that for the finite well, the wave functions extend beyond the boundaries.





c) For the infinite square well, the wave functions equal zero exactly at the boundaries. This is not true for the finite square well. The determination of the exact energies is rather difficult and is often treated in Quantum Mechanics texts. The number of available energies depends on the value of the potential. We assume that four states exist in this problem.

59. This tunneling problem is similar to Example 6.14. Because of the information given in the problem we will use Equation (6.73) to approximate the transmission probability. We know $\kappa = \dfrac{\sqrt{2m\,(V_0 - E)}}{\hbar} = \dfrac{\sqrt{2mc^2\,(V_0 - E)}}{\hbar c}$ so

$$\kappa = \frac{\sqrt{2\,(5.11 \times 10^5)\,(0.9\,\text{eV})}}{197.33\,\text{eV} \cdot \text{nm}} = 4.86\,\text{nm}^{-1}\,.$$

Then from Equation (6.73)

$$T \approx 2e^{-2\kappa L} = 2e^{-2\left(4.86\,\text{nm}^{-1}\right)(1.3\,\text{nm})} = 6.51 \times 10^{-6}$$

* 60. a) Note that this is an approximate procedure for one-dimensional problems with a gradually varying potential, $V(x)$. We begin with Equation (6.62b) which was derived for a scenario where $E > V$ but with V constant. We found $k = \dfrac{\sqrt{2m\,(E - V_0)}}{\hbar}$ for a constant V_0. Since our potential varies slowly, we approximate the wave number by $k = \dfrac{\sqrt{2m\,[E - V(x)]}}{\hbar}$. We also know that $p = \hbar k$ and $\lambda = \dfrac{h}{p}$. Combining all of the above, we find a position-dependent wavelength

$$\lambda(x) = \frac{h}{\sqrt{2m\,[E - V(x)]}}$$

b) If we neglect barrier penetration, then the wave function must be zero at the turning points. From the particle in a box example, we know that the number of wavelengths that fit between the turning points is $\frac{1}{2}$, or 1, or $\frac{3}{2}$, etc., which equals the distance divided by the wavelength. By analogy, the number of wavelengths that can fit inside our potential well with a slowly varying wavelength is

$$\int \frac{dx}{\lambda(x)} = \frac{n}{2}\,;\ \text{where } n \text{ is an integer.}$$

Substituting from above and rearranging, we have

$$2\int \sqrt{2m\,[E - V(x)]}dx = nh\,;\ \text{where } n \text{ is an integer.}$$

61. a) Strictly speaking this approach is valid only when the potential varies slowly. From problem 60, we know

$$2\int \sqrt{2m\,[E - V(x)]}dx = nh\,;\ \text{where } n \text{ is an integer.}$$

For the potential energy $V(x) = \infty$ for $x \le 0$ and $V(x) = Ax$ for $x > 0$, we know the classical turning points occur at $x = 0$ and where $E - V(x) = 0 = E - Ax$. Let us call this point b; then $E - Ab = 0$ so $b = E/A$. So we must evaluate

$$2\int_0^b \sqrt{2m\,[E - Ax)]}dx = nh$$

with $b = E/A$. We know that $\int (a - qx)^{1/2}\,dx = \dfrac{-2}{3q}\sqrt{(a - qx)^3}$ so

$$2\sqrt{2m}\int_0^b \sqrt{[E - Ax)]}dx = 2\sqrt{2m}\left[\left(\frac{-2}{3A}\right)(E - Ax)^{3/2}\Big|_0^b\right] = nh$$

Simplifying, we find $\dfrac{-4\sqrt{2m}}{3A}\left[-E^{3/2}\right] = nh$ so $E^{3/2} = \dfrac{3Anh}{4\sqrt{2m}}$ or $E = \dfrac{(3Anh)^{2/3}}{2^{5/3}m^{1/3}}$.

b) A sketch is shown below. The wave functions are oscillating with the lowest state being one-half a cycle, the second state a full cycle, etc. As mentioned in the answer for Problem 60 part b), these sketches ignore barrier penetration in the region where the potential is finite. No barrier penetration occurs where $x = 0$ where the potential is infinite.