Chapter 7

* 1. Starting with Equation (7.7), let the electron move in a circle of radius a in the xy-plane, so $\sin\theta = 1$. With both r and θ constant, R and f are also constant. Let $R = f = 1$. Then $g = \psi$ and the derivatives of R and f are zero. With this Equation (7.7) reduces to

$$-\frac{2\mu}{\hbar^2}a^2(E-V) = \frac{1}{\psi}\frac{d^2\psi}{d\phi^2}$$

In uniform circular motion with an inverse-square force, we know from the planetary model that $E = V/2$, and

$$E - V = \frac{V}{2} - V = -\frac{V}{2} = |E|$$

Thus

$$-\frac{2\mu}{\hbar^2}a^2|E| = \frac{1}{\psi}\frac{d^2\psi}{d\phi^2}$$

$$\frac{1}{a^2}\frac{d^2\psi}{d\phi^2} + \frac{2\mu}{\hbar^2}|E| = 0$$

2. This is a simple harmonic oscillator equation. Assume a standard trial solution $\psi = A\exp(iB\phi)$. With this trial solution $d^2\psi/d\phi^2 = -B^2\psi$. Substituting this into the equation from the previous problem

$$\frac{1}{a^2}(-B^2)\psi + \frac{2\mu}{\hbar^2}|E| = 0$$

Solving for B,

$$B = \frac{\sqrt{2\mu|E|}\,a}{\hbar}$$

To find A, normalize

$$\int_0^{2\pi}\psi^*\psi\,d\phi = 1 = A^2\int_0^{2\pi}d\phi = 2\pi A^2$$

so $A = \sqrt{1/2\pi}$. Note that B must be an integer (let $B = n$) so that ψ will be single-valued $[\psi(0) = \psi(2\pi)]$. With $B = n$ we have

$$n^2 = \frac{2\mu}{\hbar^2}|E|a^2 \qquad\qquad |E| = \frac{n^2\hbar^2}{2\mu a^2}$$

For circular motion $|E| = \dfrac{L^2}{2I}$ where rotational inertia $I = \mu a^2$ for a particle of mass μ. Thus

$$L^2 = 2I|E| = 2\mu a^2\frac{n^2\hbar^2}{2\mu a^2} = n^2\hbar^2$$

or $L = n\hbar$, which is the Bohr condition.

3. Assuming a trial solution $g = Ae^{ik\phi}$ (which is easily verified by direct substitution), and using the boundary condition $g(0) = g(2\pi)$, we find

$$Ae^0 = Ae^{2\pi ik}$$

which is only true if k is an integer.

4. Using the transformations it can be shown that for any vector $\vec{A}$

$$\vec{\nabla}\psi = \hat{r}\frac{\partial \psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial \psi}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Because $\nabla^2\psi = \vec{\nabla}\cdot\vec{\nabla}\psi$ we can combine our results to find

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2}$$

and from this a simple rearrangement gives the desired result.

* 5. Letting the constants in the front of R be called A we have

$$R = A\left(2 - \frac{r}{a_0}\right)e^{-r/2a_0}$$

$$\frac{dR}{dr} = A\left(-\frac{2}{a_0} + \frac{r}{2a_0^2}\right)e^{-r/2a_0}$$

$$\frac{d^2R}{dr^2} = A\left(\frac{3}{2a_0^2} - \frac{1}{4a_0^3}\right)e^{-r/2a_0}$$

Substituting these into Equation (7.13) we have

$$\left(-\frac{1}{4a_0^3} - \frac{2\mu E}{a_0\hbar^2}\right)r + \left(\frac{5}{2a_0^2} + \frac{4\mu E}{\hbar^2} - \frac{2\mu e^2}{4\pi\epsilon_0 a_0\hbar^2}\right) + \left(-\frac{4}{a_0} + \frac{4\mu e^2}{4\pi\epsilon_0\hbar^2}\right)\frac{1}{r} = 0$$

To satisfy the equation, each of the expressions in parentheses must equal zero. From the $1/r$ term we find

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}$$

which is correct. From the r term we get

$$E = -\frac{\hbar^2}{8\mu a_0^2} = -\frac{E_0}{4}$$

which is consistent with the Bohr result. The other expression in parentheses also leads directly to $E = -E_0/4$. so the solution is verified.

6. As in the previous problem

$$R = Are^{-r/2a_0}$$

$$\frac{dR}{dr} = A\left(1 - \frac{r}{2a_0}\right)e^{-r/2a_0}$$

$$r^2\frac{dR}{dr} = A\left(r^2 - \frac{r^3}{2a_0}\right)e^{-r/2a_0}$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = A\left(2r - \frac{2r^2}{a_0} + \frac{r^3}{4a_0^2}\right)e^{-r/2a_0}$$

Substituting these into Equation (7.10) with $\ell = 1$ and after substituting the Coulomb potential, we have

$$\left(\frac{1}{4a_0^2} + \frac{2\mu E}{\hbar^2}\right) r + \left(-\frac{1}{2a_0} + \frac{2\mu e^2}{4\pi\epsilon_0\hbar^2}\right) + (2-2)\frac{1}{r} = 0$$

The $1/r$ term vanishes, and the middle expression (without r) reduces to

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}$$

which is correct. From the r term we get

$$E = -\frac{\hbar^2}{8\mu a_0^2} = -\frac{E_0}{4}$$

which is consistent with the Bohr result.

7.

$$R = \frac{e^{-r/2a_0}}{\sqrt{3}\,(2a_0)^{3/2}}\frac{r}{a_0} \qquad R^*R = \frac{e^{-r/a_0}}{3\,(2a_0)^3}\left(\frac{r}{a_0}\right)^2$$

To normalize, we integrate over all space:

$$\int_0^\infty r^2 R^* R\, dr = \frac{1}{24a_0^5}\int_0^\infty r^4 e^{-r/a_0}\, dr = \frac{1}{24a_0^5}\frac{4!}{(1/a_0)^5} = 1$$

so the wave function R_{21} was normalized.

* 8. Do the triple integral over all space

$$\iiint \psi^*\psi\, dV = \frac{1}{\pi a_0^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^2 \sin\theta\, e^{-2r/a_0}\, dr\, d\theta\, d\phi$$

The ϕ integral yields 2π, and the θ integral yields 2. This leaves

$$\iiint \psi^*\psi\, dV = \frac{4\pi}{\pi a_0^3}\int_0^\infty r^2\, e^{-2r/a_0}\, dr = \frac{4}{a_0^3}\frac{2}{(2/a_0)^3} = 1$$

as required.

9. It is required that $\ell < 6$ and $|m_\ell| \le \ell$.

$\ell = 5$: $m_\ell = 0, \pm1, \pm2, \pm3, \pm4, \pm5$ $\qquad$ $\ell = 4$: $m_\ell = 0, \pm1, \pm2, \pm3, \pm4$

$\ell = 3$: $m_\ell = 0, \pm1, \pm2, \pm3$ $\qquad$ $\ell = 2$: $m_\ell = 0, \pm1, \pm2$ $\qquad$ $\ell = 1$: $m_\ell = 0, \pm1$ $\qquad$ $\ell = 0$: $m_\ell = 0$

* 10. $n = 3$ and $\ell = 1$, so $m_\ell = 0$ or ±1. Thus $L_z = 0$ or $\pm\hbar$

$$L = \sqrt{\ell(\ell+1))}\hbar = \sqrt{2}\,\hbar$$

L_y and L_x are unrestricted except for the constraint $L_x^2 + L_y^2 = L^2 - L_z^2$.

11.

$$\psi_{310} = R_{31}Y_{10} = \frac{1}{81}\sqrt{\frac{2}{\pi}}\, a_0^{-3/2}\left(6 - \frac{r}{a_0}\right)\left(\frac{r}{a_0}\right) e^{-r/3a_0}\cos\theta$$

$$\psi_{31\pm1} = R_{31}Y_{1\pm1} = \frac{1}{81\sqrt{\pi}}\, a_0^{-3/2}\left(6 - \frac{r}{a_0}\right)\left(\frac{r}{a_0}\right) e^{-r/3a_0}\sin\theta\, e^{\pm i\phi}$$

12. The sum is of the form

$$\sum_{y=-x}^{x} y^2$$

which by symmetry is equivalent to

$$2\sum_{y=1}^{x} y^2$$

Let us first consider (as a lemma) the sum

$$\begin{aligned}\sum_{y=1}^{x}\left[(y+1)^3 - y^3\right] &= \sum_{y=1}^{x}\left[3y^2 + 3y + 1\right] \\ &= \left(2^3 - 1^3\right) + \left(3^3 - 2^3\right) + \ldots + (x+1)^3 - x^3 \\ &= (x+1)^3 - 1^3 = x^3 + 3x^2 + 3x\end{aligned}$$

Now let us write

$$3\sum_{y=1}^{x} y^2 = \sum_{y=1}^{x}\left[(1+y)^3 - y^3\right] - 3\sum_{y=1}^{x} y - \sum_{y=1}^{x} 1$$

The first of these sums is given by our lemma above. The others are

$$\sum_{y=1}^{x} y = \frac{1}{2}x\,(x+1) \qquad \sum_{y=1}^{x} 1 = x$$

Combining these results

$$3\sum_{y=1}^{x} y^2 = x^3 + 3x^2 + 3x - \frac{3}{2}x\,(x+1) - x = \frac{1}{2}x\,(2x+1)\,(x+1)$$

Therefore

$$\sum_{y=1}^{x} y^2 = \frac{1}{6}x\,(2x+1)\,(x+1)$$

and

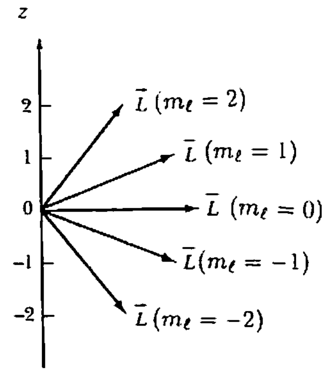
$$\sum_{y=-x}^{x} y^2 = \frac{1}{3}x\,(2x+1)\,(x+1)$$

Then

$$\left\langle L^2\right\rangle = 3\left\langle L_z^2\right\rangle = \frac{3}{2\ell+1}\sum_{m_\ell=-l}^{\ell} m_\ell^2\hbar^2 = \ell\,(\ell+1)\,\hbar^2$$

13. As in Example 7.4 the degeneracy is $n^2 = 36$.

14. There are five possible orientation, corresponding to the five different values of $m_\ell = 0, \pm 1, \pm 2$.



For the $m_\ell = -1$ component we have (with $\ell = 2$)

$$L = \sqrt{\ell(\ell+1)}\,\hbar = \sqrt{6}\,\hbar \qquad L_z = m_\ell\hbar = -\hbar$$

$$L_x^2 + L_y^2 = L^2 - L_z^2 = 6\hbar^2 - \hbar^2 = 5\hbar^2$$

* 15.

$$\cos\theta = \frac{L_z}{L} = \frac{m_\ell}{\sqrt{\ell(\ell+1)}}$$

For this extreme case we could have $\ell = m_\ell$ so

$$\cos(3^\circ) = \frac{\ell}{\sqrt{\ell(\ell+1)}} \qquad \cos^2(3^\circ) = \frac{\ell^2}{\ell(\ell+1)} = \frac{\ell^2}{\ell^2+\ell}$$

Rearranging we find

$$\ell = \left(\frac{1}{\cos^2(3^\circ)} - 1\right)^{-1} = 364.1$$

and we have to round up in order to get within 3°, so $\ell = 365$.

16. There is one possible m_ℓ value for $\ell = 0$, three values of m_ℓ for $\ell = 1$, five values of m_ℓ for $\ell = 2$, and so on, so that the degeneracy of the nth level is

$$1 + 3 + 5 + \ldots = n^2$$

* 17.

$$\psi_{21-1} = R_{21}Y_{1-1} = \frac{1}{8\sqrt{\pi}\,a_0^{3/2}}\left(\frac{r}{a_0}\right)e^{-r/2a_0}\sin\theta e^{-i\phi}$$

$$\psi_{210} = R_{21}Y_{10} = \frac{1}{4\sqrt{2\pi}\,a_0^{3/2}}\left(\frac{r}{a_0}\right)e^{-r/2a_0}\cos\theta$$

$$\psi_{32-1} = R_{32}Y_{2-1} = \frac{1}{81\sqrt{\pi}\,a_0^{3/2}}\left(\frac{r^2}{a_0^2}\right)e^{-r/3a_0}\sin\theta\cos\theta e^{-i\phi}$$

18. We must calculate the triple integral over all space. The definite integrals can be evaluated from a handbook or from Appendix 3.

$$\iiint \psi_{200}^*\psi_{200}\,dV = \frac{1}{32\pi a_0^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left(2 - \frac{r}{a_0}\right)^2 e^{-r/a_0}r^2\sin\theta\,dr\,d\theta\,d\phi$$

The ϕ integral yields 2π, and the θ integral yields 2. This leaves

$$\begin{aligned}\iiint \psi^* \psi \, dV &= \frac{1}{8a_0^3} \int_0^\infty \left(2 - \frac{r}{a_0}\right)^2 r^2 \, e^{-r/a_0} \, dr = \frac{1}{8a_0^3} \int_0^\infty \left(4r^2 - \frac{4r^3}{a_0} + \frac{r^4}{a_0^2}\right) e^{-r/a_0} \, dr \\ &= \frac{1}{8a_0^3} \left[4\,(2!\,a_0^3) - \frac{4}{a_0}\,(3!\,a_0^4) + \frac{1}{a_0^2}\,(4!\,a_0^5)\right] \\ &= 1\end{aligned}$$

as required.

Repeat the process with the second wave function:

$$\iiint \psi_{21-1}^* \psi_{21-1} \, dV = \frac{1}{64\pi a_0^3} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \left(\frac{r}{a_0}\right)^2 r^2 \sin^3\theta \, e^{-r/a_0} \, dr \, d\theta \, d\phi$$

The ϕ integral yields 2π as before. The θ integral can be found in integral tables and equals 4/3. This leaves

$$\iiint \psi^* \psi \, dV = \frac{1}{24a_0^3} \int_0^\infty \left(\frac{r}{a_0}\right)^2 r^2 \, e^{-r/a_0} \, dr = \frac{1}{24a_0^3} \left[\frac{1}{a_0^2}\,(4!\,a_0^5)\right] = 1$$

as required.

19. With $\ell = 1$ we have $m_\ell = 0, \pm 1$ and $L_z = m_\ell \hbar = 0, \pm\hbar$.

20. The maximum difference is between the $m_\ell = -2$ and $m_\ell = +2$ levels, so $\Delta m_\ell = 4$. Then

$$\Delta V = \mu_B \,(\Delta m_\ell)\, B = (5.788 \times 10^{-5} \text{ eV/T})\,(4)\,(2.5 \text{ T}) = 5.79 \times 10^{-4} \text{ eV}$$

* 21. Differentiating $E = \dfrac{hc}{\lambda}$ we find

$$dE = -\frac{hc}{\lambda^2} d\lambda \qquad \text{or} \qquad |\Delta E| = \frac{hc}{\lambda^2}\,|\Delta\lambda|$$

In the Zeeman effect between adjacent m_ℓ states $|\Delta E| = \mu_B B$ so $\mu_B B = \left(\dfrac{hc}{\lambda_0^2}\right) |\Delta\lambda|$ or

$$\Delta\lambda = \frac{\lambda_0^2 \mu_B B}{hc}$$

22. See the solution to problem 14 for the sketch. To compute the angles with $\ell = 2$

$$\cos\theta = \frac{L_z}{L} = \frac{m_\ell}{\sqrt{\ell\,(\ell+1)}} = \frac{m_\ell}{\sqrt{6}}$$

There are five different values of θ, corresponding to the different m_ℓ values $0, \pm 1, \pm 2$:

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) = 35.3^\circ \qquad \theta = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) = 65.9^\circ \qquad \theta = \cos^{-1}(0) = 90^\circ$$

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{6}}\right) = 114.1^\circ \qquad \theta = \cos^{-1}\left(\frac{-2}{\sqrt{6}}\right) = 144.7^\circ$$

23. With $\ell = 3$ we have (as in the previous problem)

$$\cos\theta = \frac{L_z}{L} = \frac{m_\ell}{\sqrt{\ell(\ell+1)}} = \frac{m_\ell}{\sqrt{12}}$$

For the minimum angle $m_\ell = \ell = 3$ and

$$\theta = \cos^{-1}\left(\frac{3}{\sqrt{12}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30°$$

* 24. From Problem 21

$$\Delta\lambda = \frac{\lambda_0^2 \mu_B B}{hc}$$

so the magnetic field is

$$B = \frac{hc\,\Delta\lambda}{\lambda_0^2 \mu_B} = \frac{(1240\ \text{eV}\cdot\text{nm})\,(0.04\ \text{nm})}{(656.5\ \text{nm})^2\,(5.788\times 10^{-5}\ \text{eV/T})} = 1.99\ \text{T}$$

25. There are seven different states, corresponding to $m_\ell = 0, \pm1, \pm2, \pm3$. In the absence of a magnetic field

$$E = -\frac{E_0}{n^2} = -\frac{13.606\ \text{eV}}{25} = -0.544\ \text{eV}$$

The Zeeman splitting is given by

$$\Delta E = \mu_B B\,(\Delta m_\ell) = \left(5.788\times 10^{-5}\ \text{eV/T}\right)(3\ \text{T})\, m_\ell = \left(1.7364\times 10^{-4}\ \text{eV}\right) m_\ell$$

For $m_\ell = 0$ we have $\Delta E = 0$. For the other m_ℓ states

$$m_\ell = \pm1:\ \Delta E = \left(1.7364\times 10^{-4}\ \text{eV}\right)(\pm1) = \pm1.74\times 10^{-4}\ \text{eV}$$

$$m_\ell = \pm2:\ \Delta E = \left(1.7364\times 10^{-4}\ \text{eV}\right)(\pm2) = \pm3.47\times 10^{-4}\ \text{eV}$$

$$m_\ell = \pm3:\ \Delta E = \left(1.7364\times 10^{-4}\ \text{eV}\right)(\pm3) = \pm5.21\times 10^{-4}\ \text{eV}$$

26. From the text the magnitude of the spin magnetic moment is

$$\mu_s = \frac{2\mu_B \left\|\vec{S}\right\|}{\hbar} = \frac{2\mu_B}{\hbar}\frac{\sqrt{3}\hbar}{2} = \sqrt{3}\mu_B$$

The the z-component of the magnetic moment is (see Figure 7.9)

$$\mu_z = \mu_s \cos\theta = \mu_s \frac{1/2}{\sqrt{3}/2} = \frac{\mu_s}{\sqrt{3}} = \mu_B$$

The potential energy is $V = -\vec{\mu}\cdot\vec{B} = -\mu_z B_z$ and so the vertical component of force is $F_z = -dV/dz = \mu_z\,(dB_z/dz)$. From mechanics the acceleration is

$$a_z = \frac{F_z}{m} = \frac{\mu_z}{m}\frac{dB_z}{dz}$$

and with constant acceleration the vertical deflection of each beam is $z = \frac{1}{2}a_z t^2$. With the time equal to the horizontal distance divided by incoming speed, or $t = x/v_x$, we have

$$\begin{aligned} z &= \frac{1}{2}a_z t^2 = \frac{1}{2}\left(\frac{\mu_z}{m}\frac{dB_z}{dz}\right)\left(\frac{x}{v_x}\right)^2 = \frac{1}{2}\left(\frac{9.27 \times 10^{-24}\ \mathrm{J/T}}{1.8 \times 10^{-25}\ \mathrm{kg}}\right)(2000\ \mathrm{T/m})\left(\frac{0.071\ \mathrm{m}}{925\ \mathrm{m/s}}\right)^2 \\ &= 3.034 \times 10^{-4}\ \mathrm{m} \end{aligned}$$

The separation between the two silver beams is twice this amount, or 6.07×10^{-4} m.

27. The kinetic energy of the atoms is

$$K = \frac{3}{2}kT = \frac{3}{2}\left(1.38 \times 10^{-23}\ \mathrm{J/K}\right)(1273\ \mathrm{K}) = 2.64 \times 10^{-20}\ \mathrm{J}$$

From the previous problem, we see that the separation of the beams is (remember $\mu_z = \mu_B$)

$$s = 2z = \left(\frac{\mu_B}{m}\frac{dB_z}{dz}\right)\left(\frac{x}{v_x}\right)^2$$

Rearranging we see that

$$x^2\frac{dB_z}{dz} = \frac{smv^2}{\mu_B} = \frac{2sK}{\mu_B} = \frac{2\,(0.01\ \mathrm{m})\left(2.64 \times 10^{-20}\ \mathrm{J}\right)}{9.27 \times 10^{-24}\ \mathrm{J/T}} = 57.0\ \mathrm{T \cdot m}$$

The magnet should be designed so that the product of its length squared and its vertical magnetic field gradient is 57 T·m.

* 28. As shown in Figure 7.9 the electron spin vector cannot point in the direction of $\vec{B}$, because its magnitude is $S = \sqrt{s(s+1)} = \sqrt{3/4}\,\hbar$ and its z-component is $S_z = m_s\hbar = \hbar/2$. If the z-component of a vector is less than the vector's magnitude, the vector does not lie along the z-axis.

29. For the $5f$ state $n = 5$ and $\ell = 3$. The possible m_ℓ values are $0, \pm1, \pm2$, and ±3 with $m_s = \pm1/2$ for each possible m_ℓ value. The degeneracy of the $5f$ state is then (with 2 spin states per m_l) equal to $2(7) = 14$.

30. For the $6d$ state $n = 6$ and $\ell = 2$. The possible m_ℓ values are $0, \pm1$, and ±2, with $m_s = \pm1/2$ for each possible m_ℓ value. The degeneracy of the $6d$ state is then (with 2 spin states per m_l) equal to $2(5) = 10$.

31. If we determine the thermal energy that equals the energy required for the spin-flip transition, we have

$$5.9 \times 10^{-6}\,\mathrm{eV} = \frac{3}{2}kT = \frac{3}{2}\left(8.617 \times 10^{-5}\,\mathrm{eV/K}\right)T$$

This gives $T = 0.0456\,\mathrm{K}$.

32. The spin degeneracy is 2 and the n^2 is shown in Problem 16.

33. The selection rule $\Delta m_\ell = 0, \pm1$ gives three lines in each case.

34. a) $\Delta\ell = 0$ is forbidden

b) allowed but with $\Delta n = 0$ there is no energy difference unless an external magnetic field is present

c) $\Delta\ell = -2$ is forbidden

d) allowed with absorbed photon of energy

$$\Delta E = E_0\left(\frac{1}{2^2} - \frac{1}{4^2}\right) = 2.55 \text{ eV}$$

* 35. We must find the maxima and minima of the following function.

$$P(r) = r^2 |R(r)|^2 = A^2 e^{-r/a_0}\left(2 - \frac{r}{a_0}\right)^2 r^2 = A^2\left(4r^2 - \frac{4r^3}{a_0} + \frac{r^4}{a_0^2}\right)e^{-r/a_0}$$

To find the extrema set $\frac{dP}{dr} = 0$:

$$0 = -\frac{1}{a_0}\left(4r^2 - \frac{4r^3}{a_0} + \frac{r^4}{a_0^2}\right)e^{-r/a_0} + \left(8r - \frac{12r^2}{a_0} + \frac{4r^3}{a_0^2}\right)e^{-r/a_0}$$

$$0 = -\frac{r^3}{a_0^3} + \frac{8r^2}{a_0^2} - \frac{16r}{a_0} + 8$$

Letting $x = \frac{r}{a_0}$ the equation above can be factored into $(x-2)\left(x^2 - 6x + 4\right) = 0$. From the first factor we get $x = 2$ (or $r = 2a_0$), which from Figure 7.12 we can see is a minimum. The second parenthesis gives a quadratic equation with solutions $x = 3 \pm \sqrt{5}$, so $r = \left(3 \pm \sqrt{5}\right) a_0$. These are both maxima.

36. In the previous problem we found that the two maxima are at $r = \left(3 \pm \sqrt{5}\right) a_0$. From Figure 7.12 it is clear that the peak at $r = \left(3 + \sqrt{5}\right) a_0$ is higher. This can be verified by substituting the two values for r and computing $P(r)$. The most probable location is therefore at $r = \left(3 + \sqrt{5}\right) a_0 = 5.24\, a_0$, which is significantly further from the nucleus than the $2p$ peak at $r = 4a_0$.

37. From the solution to Problem 35 we see that $P(r) = 0$ at $r = 2a_0$. Note that $P(0) = 0$ also. A sketch is found in Figure 7.12.

38. The radial probability distribution for the ground state is

$$P(r) = r^2 |R(r)|^2 = \frac{4}{a_0^3} r^2 e^{-2r/a_0}$$

With $r \ll a_0$ throughout this interval we can say $e^{-2r/a_0} \approx 1$. Therefore the probability of being inside a radius 10^{-15} m is

$$\int_0^{10^{-15}} P(r)\, dr \approx \frac{4}{a_0^3}\int_0^{10^{-15}} r^2\, dr = \left.\frac{4r^3}{3a_0^3}\right|_0^{10^{-15}} = 9.0 \times 10^{-15}$$

39.
$$P(r) = r^2 \left|R(r)\right|^2 = \frac{4}{a_0^3} r^2 e^{-2r/a_0}$$

To find the desired probability, integrate $P(r)$ over the appropriate limits:

$$\int_{0.95a_0}^{1.05a_0} P(r)\,dr = \frac{4}{a_0^3} \int_{0.95a_0}^{1.05a_0} r^2 e^{-2r/a_0}\,dr$$

Letting $x = r/a_0$
$$\int_{0.95a_0}^{1.05a_0} P(r)\,dr = 4 \int_{0.95}^{1.05} x^2 e^{-2x}\,dx = 0.056$$

where the definite integral was evaluated using Mathcad.

40. In general
$$\langle r \rangle = \int_0^\infty r\,P(r)\,dr = \int_0^\infty r^3 \left|R(r)\right|^2 dr$$

For the 2s state
$$\langle r \rangle = \frac{1}{8a_0^3} \int_0^\infty r^3 \left(4 - \frac{4r}{a_0} + \frac{r^2}{a_0^2}\right) e^{-r/a_0} dr$$

Using integral tables
$$\int_0^\infty r^n e^{-r/a_0} dr = n!\,(a_0)^{n+1}$$

$$\langle r \rangle = \frac{1}{8a_0^3}\left[4\,(3!)\,a_0^4 - \frac{4}{a_0}\,(4!)\,(a_0^5) + \frac{1}{a_0^2}\,(5!)\,a_0^6\right] = \frac{a_0}{8}\,(24 - 96 + 120) = 6a_0$$

For the 2p state

$$\begin{aligned}\langle r \rangle &= \frac{1}{24a_0^3} \int_0^\infty r^3 \left(\frac{r}{a_0}\right)^2 e^{-r/a_0} dr = \frac{1}{24a_0^5} \int_0^\infty r^5 e^{-r/a_0} dr \\ &= \frac{1}{24a_0^5}\,(5!)\,(a_0^6) = \frac{120a_0}{24} = 5a_0\end{aligned}$$

41. For the 2s state:
$$P(r) = r^2 \left|R(r)\right|^2 = \frac{1}{8a_0^3} r^2 \left(2 - \frac{r}{a_0}\right)^2 e^{-r/a_0}$$

As in Problem 38 for $r \ll a_0$ we can say $e^{-r/a_0} \approx 1$, so the probability is given by the integral

$$\begin{aligned}\int_0^{10^{-15}} P(r)\,dr &\approx \frac{1}{8a_0^3} \int_0^{10^{-15}} r^2 \left(2 - \frac{r}{a_0}\right)^2 dr = \frac{1}{8a_0^3} \int_0^{10^{-15}} \left(4r^2 - \frac{4r^3}{a_0} + \frac{r^4}{a_0^2}\right) dr \\ &= \frac{1}{8a_0^3} \left(\frac{4}{3} r^3 - \frac{r^4}{a_0} + \frac{r^5}{5a_0^2}\right)\Bigg|_0^{10^{-15}} = 1.1 \times 10^{-15}\end{aligned}$$

Similarly for the 2p state:

$$P(r) = r^2 \left|R(r)\right|^2 = \frac{1}{24a_0^5} r^4 e^{-r/a_0}$$

$$\int_0^{10^{-15}} P(r)\,dr \approx \frac{1}{24a_0^5} \int_0^{10^{-15}} r^4 dr = \frac{r^5}{120a_0^5}\Bigg|_0^{10^{-15}} = 2.01 \times 10^{-26}$$

42. To find the most probable radial position in the $3d$ state:

$$P(r) = r^2 |R(r)|^2 = A^2 e^{-2r/3a_0} \left(\frac{r^2}{a_0^2}\right)^2 r^2 = A^2 \left(\frac{r^6}{a_0^4}\right) e^{-2r/3a_0}$$

To find the extrema set $\frac{dP}{dr} = 0$:

$$0 = \frac{1}{a_0}\left(\frac{6r^5}{a_0^3} - \frac{2r^6}{3a_0^4}\right) e^{-2r/3a_0}$$

$$0 = \frac{6r^5}{a_0^3} - \frac{2r^6}{3a_0^4} \qquad \text{so} \qquad r = 9a_0 \text{ and } r = 0$$

are the two solutions. From Figure (7.12) you can see that $r = 0$ is a minimum where $P(r) = 0$ and $r = 9a_0$ is a maximum.

43. For the $3d$ state:

$$P(r) = r^2 |R(r)|^2 = r^2 \left(\frac{1}{a_0}\right)^3 \left(\frac{16}{(81)^2 (30)}\right) \left(\frac{r^2}{a_0^2}\right)^2 e^{-2r/3a_0} = \left(\frac{8}{(81)^2 (15)}\right) \left(\frac{r^6}{a_0^7}\right) e^{-2r/3a_0}$$

To find that probability that the electron in the $3d$ state is location at a position greater than a_0, we must evaluate

$$\int_{a_0}^{\infty} P(r)\, dr = \int_{a_0}^{\infty} \left(\frac{8}{(81)^2 (15)}\right) \left(\frac{r^6}{a_0^7}\right) e^{-2r/3a_0}\, dr$$

(Alternatively, we could evaluate the integral from 0 to a_0 and subtract that answer from 1 since we know that the wave functions are normalized.) From integral tables, we find integrals of the form

$$\int x^m e^{bx} dx = e^{bx} \sum_{i=0}^{m} (-1)^i \frac{m!\, x^{m-i}}{(m-i)!\, b^{i+1}} \qquad \text{where } m = 6 \text{ and } b = \frac{-2}{3a_0}$$

Evaluation of the summation gives a probability of 0.9999935 that the $3d$ electron will be at a position greater than a_0. Well that means it is almost certain!

* 44.

$$R = \frac{e^2}{4\pi\epsilon_0 mc^2} = \frac{1.44 \times 10^{-9} \text{ eV} \cdot \text{m}}{511 \times 10^3 \text{ eV}} = 2.82 \times 10^{-15} \text{ m}$$

From the angular momentum equation

$$v = \frac{3\hbar}{4mR} = \frac{3\hbar c}{4mc^2 R} c = \frac{3\,(197.33 \text{ eV} \cdot \text{nm})}{4\,(511 \times 10^3 \text{ eV})\,(2.82 \times 10^{-6} \text{ nm})} c = 103c$$

A speed of $103c$ is prohibited by the rules of relativity.

45. The electron radius would be $\frac{\lambda_c}{2} = 1.21 \times 10^{-12}$ m. As in the previous problem

$$v = \frac{3\hbar}{4mR} = \frac{3\hbar c}{4mc^2 R} c = \frac{3\,(197.33 \text{ eV} \cdot \text{nm})}{4\,(511 \times 10^3 \text{ eV})\,(1.21 \times 10^{-3} \text{ nm})} c = 0.24c$$

This result is allowed by relativity. However, in order to obtain this allowed result, we had to assume an unreasonably large size for the electron (one thousand times larger in radius than a proton!).

46. a) The only change in Equation (7.3) is in the potential energy, with

$$V = -\frac{Ze^2}{4\pi\epsilon_0 r}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + \frac{2\mu}{\hbar^2}\left(E + \frac{Ze^2}{4\pi\epsilon_0 r}\right)\psi = 0$$

b) Because V occurs only in the radial part, there is no change in the separation of variables.

c) Yes, from Equation (7.10) is it clear that the radial wave functions will change.

d) No, there is no change in the θ or ϕ dependence.

47. Carrying Z through the derivation in the text [Equations (7.12) through (7.14)] we find

$$\psi_{100} = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$$

48. Use the wave function

$$R_{31} = Ar\left(1 - \frac{r}{6a_0}\right)e^{-r/3a_0}$$

where A is constant. Then

$$P(r) = r^2|R(r)|^2 = A^2\left(r^2 - \frac{r^3}{3a_0} + \frac{r^4}{36a_0^2}\right)e^{-2r/3a_0}$$

To find the extrema set $\frac{dP}{dr} = 0$. Doing so and factoring out $A^2 r e^{-2r/3a_0}$ gives

$$2 - \frac{5r}{3a_0} + \frac{r^2}{3a_0^2} - \frac{r^3}{54a_0^3} = 0$$

Letting $x = \frac{r}{a_0}$ and multiplying both sides by - 54 we get

$$x^3 - 18x^2 + 90x - 108 = 0 = (x-6)\left(x^2 - 12x + 18\right)$$

a) The minimum is at $x = 6$, or $r = 6a_0$, and we find $P(6a_0) = 0$. Clearly $P(0) = 0$ also.

b) The two maxima come from the quadratic equation in parentheses, with $x = 6 \pm 3\sqrt{2}$ or $r = \left(6 \pm 3\sqrt{2}\right)a_0$.

c)

$$Y_{1\pm1} = (\text{constant})\sin\theta\, e^{\pm i\phi}$$

Then Y^*Y is proportional to $\sin^2\theta$, and the probability is zero at $\theta = 0$ and $\theta = 180°$.

* 49. The ground state energy can be obtained using the standard Rydberg formula with the reduced mass μ of the muonic atom

$$E_0 = \frac{e^2}{8\pi\epsilon_0 a_0} = \frac{\mu e^4}{2\left(4\pi\epsilon_0\right)^2\hbar^2}$$

Computing the reduced mass:

$$\mu = \frac{m_p m_\mu}{m_p + m_\mu} = \frac{1}{c^2}\frac{(938.27\text{ MeV})\,(105.66\text{ MeV})}{(938.27\text{ MeV } + 105.66\text{ MeV})} = 94.966\text{ MeV}/c^2$$

Thus

$$E_0 = \frac{\mu e^4}{2\,(4\pi\epsilon_0)^2\,\hbar^2} = \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{\mu c^2}{2\,(\hbar^2 c^2)} = \frac{(1.44\text{ eV}\cdot\text{nm})^2\,(94.966\times 10^6\text{ eV})}{2\,(197.33\text{ eV}\cdot\text{nm})^2} = 2.53\text{ keV}$$

* 50. The interaction between the magnetic moment of the proton and magnetic moment of the electron causes hyperfine splitting. The transition between the two states causes emission of a photon with energy of 5.9×10^{-6} eV. From the uncertainty principle, we know $\Delta E \Delta t \geq \frac{\hbar}{2}$. With a lifetime of $\Delta t = 1\times 10^7$ y then

$$\Delta E \geq \frac{6.5821\times 10^{-16}\,\text{eV}}{2\,(1\times 10^7\,\text{y})\,(3.16\times 107\,\text{s/y})} = 1.041\times 10^{-30}\,\text{eV}$$