Chapter 9

1. a)

$$\overline{v_x^2} = \int_{-\infty}^{\infty} v_x^2 g(v_x)\,dv_x = \left(\frac{\beta m}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} v_x^2 \exp\left(-\frac{1}{2}\beta m v_x^2\right)\,dv_x$$

$$= 2\left(\frac{\beta m}{2\pi}\right)^{1/2} \int_0^{\infty} v_x^2 \exp\left(-\frac{1}{2}\beta m v_x^2\right)\,dv_x = 2\left(\frac{\beta m}{2\pi}\right)^{1/2} \frac{\sqrt{\pi}}{4}\left(\frac{2}{\beta m}\right)^{3/2} = \frac{1}{\beta m}$$

Therefore

$$v_{x\,\mathrm{rms}} = \left(\overline{v_x^2}\right)^{1/2} = \left(\frac{1}{\beta m}\right)^{1/2} = \left(\frac{kT}{m}\right)^{1/2}$$

b)

$$g(v_x) = \left(\frac{\beta m}{2\pi}\right)^{1/2} \exp\left(-\frac{1}{2}\beta m v_x^2\right)$$

and from (a) we see that $(\beta m)^{1/2} = v_{x\,\mathrm{rms}}^{-1}$, so

$$g(v_x)\,dv_x = \frac{1}{\sqrt{2\pi}}\, v_{x\,\mathrm{rms}}^{-1} \exp\left(-\frac{1}{2}\frac{v_x^2}{v_{x\,\mathrm{rms}}^2}\right)\,dv_x$$

2. a) With $v_x = 0.01 v_{x\,\mathrm{rms}}$ we have $\exp\left(-\frac{1}{2}\frac{v_x^2}{v_{x\,\mathrm{rms}}^2}\right) \approx 1$.

$$g(v_x)\,dv_x = \frac{1}{\sqrt{2\pi}} v_{x\,\mathrm{rms}}^{-1} \exp\left(-\frac{1}{2}\frac{v_x^2}{v_{x\,\mathrm{rms}}^2}\right)\,dv_x = \frac{1}{\sqrt{2\pi}} v_{x\,\mathrm{rms}}^{-1}\,(1)\,(0.002 v_{x\,\mathrm{rms}}) = 7.98 \times 10^{-4}$$

This is the probability that a given molecule will be in this range, so in one mole the number is

$$N = \left(7.98 \times 10^{-4}\right) N_A = \left(7.98 \times 10^{-4}\right)\left(6.022 \times 10^{23}\right) = 4.81 \times 10^{20}$$

b) With $v_x = 0.20 v_{x\,\mathrm{rms}}$ we have $\exp\left[-\frac{1}{2}\frac{(0.20 v_{x\,\mathrm{rms}})^2}{(v_{x\,\mathrm{rms}})^2}\right] = 0.980$. Continuing as in (a) we find

$$g(v_x)\,dv_x = \frac{1}{\sqrt{2\pi}} v_{x\,\mathrm{rms}}^{-1}\,(0.98)\,(0.002 v_{x\,\mathrm{rms}}) = 7.82 \times 10^{-4} \text{ and therefore } N = 4.71 \times 10^{20}$$

c) $N = 2.91 \times 10^{20}$

d) $N = 1.79 \times 10^{15}$

e) In this case

$$g(v_x)\,dv_x = \left(7.98 \times 10^{-4}\right) \exp\left(-5 \times 10^{3}\right)$$

which is on the order of 10^{-2175}. Therefore we conclude no molecules travel at that speed.

3. a)

$$\overline{f} = \overline{f_0\left(1 + \frac{v}{c}\right)} = f_0\left(1 + \frac{\overline{v_x}}{c}\right) = f_0\,(1 + 0) = f_0$$

b)

$$\sigma = \left(\overline{(f - f_0)^2}\right)^{1/2} = \left(\overline{\left(\frac{f_0 v_x}{c}\right)^2}\right)^{1/2} = \left(f_0^2 \frac{\overline{v_x^2}}{c^2}\right)^{1/2}$$

But we know that $\overline{v_x^2} = kT/m$, so

$$\sigma = \left(\frac{f_0^2}{c^2}\frac{kT}{m}\right)^{1/2} = \frac{f_0}{c}\sqrt{\frac{kT}{m}}$$

c) From (b) we have $\frac{\sigma}{f_0} = \frac{1}{c}\sqrt{\frac{kT}{m}}$.

$$\text{H}_2 \text{ at } T = 293 \text{ K: } \frac{\sigma}{f_0} = \frac{1}{3.00 \times 10^8 \text{ m/s}}\sqrt{\frac{1.381 \times 10^{-23} \text{ J/K } (293 \text{ K})}{2\,(1.674 \times 10^{-27} \text{ kg})}} = 3.66 \times 10^{-6}$$

$$\text{H at } T = 5500 \text{ K: } \frac{\sigma}{f_0} = \frac{1}{3.00 \times 10^8 \text{ m/s}}\sqrt{\frac{1.381 \times 10^{-23} \text{ J/K } (5500 \text{ K})}{(1.674 \times 10^{-27} \text{ kg})}} = 2.25 \times 10^{-5}$$

This is how we could deduce the surface temperature of a star.

4. a) Letting d be the distance between the two atoms we have

$$\begin{aligned} I_x &= 2\left(mr^2\right) = 2m\left(\frac{d}{2}\right)^2 = \frac{md^2}{2} = \frac{16\,(1.66 \times 10^{-27} \text{ kg})\,(8.5 \times 10^{-10} \text{ m})^2}{2} \\ &= 9.59 \times 10^{-45} \text{ kg} \cdot \text{m}^2 \end{aligned}$$

b)

$$\begin{aligned} I_z &= 2\left(\frac{2}{5}mR^2\right) = \frac{4}{5}mR^2 = 0.8\,(16)\,(1.66 \times 10^{-27} \text{ kg})\,(3.0 \times 10^{-15} \text{ m})^2 \\ &= 1.91 \times 10^{-55} \text{ kg} \cdot \text{m}^2 \end{aligned}$$

c) The rigid rotator is quantized (see Chapter 10) with an energy

$$E = \frac{\hbar^2 l\,(l+1)}{2I} = \frac{\left(1.055 \times 10^{-34} \text{ J} \cdot \text{s}\right)^2 (1)\,(2)}{2\,(9.59 \times 10^{-45} \text{ kg} \cdot \text{m}^2)} = 1.16 \times 10^{-24} \text{ J}$$

d) Rearranging the energy equation in (c) and using the value of I_z to find the ℓ value, we find

$$l(l+1) = \frac{2IE}{\hbar^2} = \frac{2\left(1.91 \times 10^{-55} \text{ kg} \cdot \text{m}^2\right)\left(1.16 \times 10^{-24} \text{ J}\right)}{\left(1.055 \times 10^{-34} \text{ J} \cdot \text{s}\right)^2} = 3.98 \times 10^{-11}$$

This shows that a much larger energy (larger E_{rot}) is required to have $\ell = 1$ for the rotation about the z axis. Therefore the diatomic molecule proceeds as if there were only two degrees of rotational freedom.

* 5. a)

$$\int_c^\infty F(v)\,dv = 4\pi C\int_c^\infty v^2 \exp\left(-\frac{1}{2}\beta mv^2\right)dv$$

with $T = 293$ K and $C = \left(\frac{\beta m}{2\pi}\right)^{3/2}$.

b) For example for H_2 gas at $T = 293$ K we have

$$\frac{1}{2}\beta mc^2 = \frac{(1)\,(2)\,(938 \times 10^6 \text{ eV})}{2\,(8.62 \times 10^{-5} \text{ eV/K})\,(293 \text{ K})} = 3.7 \times 10^{10}$$

The exponential of the negative of this value $\exp\left(-3.7 \times 10^{10}\right)$ is almost zero.

6. Computations depend on the software but should yield numbers very close to zero.

* 7. a)

$$\bar{v} = \frac{4}{\sqrt{2\pi}}\sqrt{\frac{kT}{m}} = \frac{4}{\sqrt{2\pi}}\sqrt{\frac{(1.381 \times 10^{-23}\ \mathrm{J/K})\ \ (300\ \mathrm{K})}{(1.675 \times 10^{-27}\ \mathrm{kg})}} = 2510\ \mathrm{m/s}$$

$$v^* = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2\,(1.381 \times 10^{-23}\ \mathrm{J/K})\ \ (300\ \mathrm{K})}{(1.675 \times 10^{-27}\ \mathrm{kg})}} = 2220\ \mathrm{m/s}$$

b)

$$\bar{v} = \frac{4}{\sqrt{2\pi}}\sqrt{\frac{kT}{m}} = \frac{4}{\sqrt{2\pi}}\sqrt{\frac{(1.381 \times 10^{-23}\ \mathrm{J/K})\ \ (2000\ \mathrm{K})}{(1.675 \times 10^{-27}\ \mathrm{kg})}} = 6480\ \mathrm{m/s}$$

$$v^* = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2\,(1.381 \times 10^{-23}\ \mathrm{J/K})\ \ (2000\ \mathrm{K})}{(1.675 \times 10^{-27}\ \mathrm{kg})}} = 5740\ \mathrm{m/s}$$

8.

$$F(v) = 4\pi C \exp\left(-\frac{1}{2}\beta m v^2\right) v^2$$

In the limit as $v \longrightarrow 0$, the exponential reduces to $e^0 = 1$ and v^2 approaches zero, so clearly

$$\lim_{v \to 0} F(v) = 0$$

The other limit is

$$\lim_{v \to \infty} F(v) = 4\pi C \lim_{v \to \infty} \frac{v^2}{\exp\left(\frac{1}{2}\beta m v^2\right)}$$

Applying L'Hopital's rule,

$$\lim_{v \to \infty} F(v) = \lim_{v \to \infty} \frac{2v}{\beta m v \exp\left(\frac{1}{2}\beta m v^2\right)} = \lim_{v \to \infty} \frac{2}{\beta m} \exp\left(-\frac{1}{2}\beta m v^2\right) = 0$$

9. a)

$$v^* = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2\,(1.381 \times 10^{-23}\ \mathrm{J/K})\ \ (268\ \mathrm{K})}{28\,(1.6605 \times 10^{-27}\ \mathrm{kg})}} = 399\ \mathrm{m/s}$$

b)

$$v^* = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2\,(1.381 \times 10^{-23}\ \mathrm{J/K})\ \ (303\ \mathrm{K})}{28\,(1.6605 \times 10^{-27}\ \mathrm{kg})}} = 424\ \mathrm{m/s}$$

10. The equation to be satisfied is

$$2v^2 \exp\left(-\frac{1}{2}\beta m v^2\right) = v^{*2} \exp\left(-\frac{1}{2}\beta m v^{*2}\right) = \frac{2kT}{m}e^{-1}$$

where we have used the fact that $v^* = \sqrt{\frac{2kT}{m}}$. Thus

$$v^2 \exp\left(-\frac{1}{2}\beta m v^2\right) = \frac{kT}{m}e^{-1} \approx 28000$$

which can be solved graphically to yield $v = 188$ m/s and $v = 639$ m/s. The lower of these is closer to $v^* = 390$ m/s, which follows from the shape of the distribution curve.

11. Various software packages should all give results very close to 1.

12. The calculations start from Equation (9.14) and are of the form:

$$I = 4\pi \left(\frac{\beta m}{2\pi}\right)^{3/2} \int_a^b v^2 \exp\left(-\frac{1}{2}\beta m v^2\right) dv$$

The limits a and b are given in each part. Values are (as a fraction of the total number of molecules):

a) 2×10^{-10} b) 2×10^{-4} c) 0.156 d) 0.494 e) 0.350 f) 0.99987

13. a) From Equation (9.20) we have $v_{\text{rms}} = \sqrt{\frac{3kT}{m}}$. The mass of $H_2 = 2\,(1.008\,\text{u}) = 2.02\,\text{u}$ and the mass of $N_2 = 2\,(14.003\,\text{u}) = 28.0\,\text{u}$.

H_2 $$v_{\text{rms}} = \sqrt{\frac{(3)\,(1.38 \times 10^{-23}\,\text{J/K})\,(293\,\text{K})}{(2.02\,\text{u})\,(1.66 \times 10^{-27}\,\text{kg/u})}} = 1902\,\text{m/s}$$

N_2 $$v_{\text{rms}} = \sqrt{\frac{(3)\,(1.38 \times 10^{-23}\,\text{J/K})\,(293\,\text{K})}{(28.0\,\text{u})\,(1.66 \times 10^{-27}\,\text{kg/u})}} = 511\text{m/s}$$

b) From classical physics we know the escape velocity $v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$. Using the mass and radius of the earth we find $v_{\text{esc}} = 1.12 \times 10^4\,\text{m/s}$ if the object starts at the surface. Neither H_2 nor N_2 has $v_{\text{rms}} > v_{\text{esc}}$; however, a very small percentage of molecules in the exponential tail of the distribution may have speeds greater than v_{esc} and will escape. Since H_2 has a larger v_{rms}, a larger fraction will eventually escape.

* 14. a) Use Equation (9.8) for the translational kinetic energy of one atom and multiply by Avagadro's number for one mole.

$$K = N_A\left(\frac{3}{2}kT\right) = (6.022 \times 10^{23})\left(\frac{3}{2}\,(1.381 \times 10^{-23}\,\text{J/K})\,(273\,\text{K})\right) = 3406\,\text{J}$$

b) Since the translational kinetic energy depends only on temperature and one mole of anything contains the same number of objects, the answer is the same for argon or oxygen.

15. a)

$$\overline{E} = \int_0^\infty E F(E)\, dE = \frac{8\pi C}{\sqrt{2}m^{3/2}} \int_0^\infty E^{3/2} \exp(-\beta E)\, dE = \frac{8\pi C}{\sqrt{2}m^{3/2}} \frac{\Gamma(5/2)}{\beta^{5/2}}$$

Using $\Gamma(5/2) = \frac{3}{2}\Gamma(3/2) = \frac{3\sqrt{\pi}}{4}$ and $C = (\beta m/2\pi)^{3/2}$ we find

$$\overline{E} = \frac{8\pi}{\sqrt{2}m^{3/2}}\left(\frac{\beta m}{2\pi}\right)^{3/2} \frac{3\sqrt{\pi}}{4\beta^{5/2}} = \frac{3}{2\beta} = \frac{3}{2}kT$$

b) As we know from the text $\overline{E} = \frac{1}{2}m\overline{v^2}$ and by Equation (9.17)

$$\frac{1}{2}m\bar{v}^2 = \frac{4}{\pi}kT = 1.27\,kT$$

which is a bit less than $\frac{3}{2}kT$.

* 16. Starting with the distribution

$$F(E) = \frac{8\pi C}{\sqrt{2}m^{3/2}} E^{1/2} \exp(-\beta E)$$

and setting $\frac{dF}{dE} = 0$, we get

$$0 = \frac{d}{dE}\left[E^{1/2}\exp(-\beta E)\right] = \frac{1}{2}E^{-1/2}\exp(-\beta E) - \beta E^{1/2}\exp(-\beta E)$$

Thus $0 = E^{-1/2} - 2\beta E^{1/2}$ which solving for E gives the desired $E^* = \frac{kT}{2}$.

* 17. The ratio of the numbers on the two levels is

$$\frac{n_2(E)}{n_1(E)} = \frac{8\exp(-\beta E_2)}{2\exp(-\beta E_1)} = 4\exp(-\beta(E_2 - E_1)) = 10^{-6}$$

$$\exp(-\beta(E_2 - E_1)) = 2.5 \times 10^{-7}$$

Taking logarithms:

$$-\beta(E_2 - E_1) = -\frac{E_2 - E_1}{kT} = \ln\left(2.5 \times 10^{-7}\right) = -15.20$$

For atomic hydrogen $E_2 - E_1 = \frac{3}{4}E_0 = 10.20$ eV. Finally

$$T = -\frac{E_2 - E_1}{k(-15.20)} = -\frac{10.20 \text{ eV}}{(8.618 \times 10^{-5} \text{ eV/K})(-15.20)} = 7790 \text{ K}$$

18. a) With $E = \frac{p^2}{2m}$ and the mean energy $E = \frac{3}{2}kT$ we obtain

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mK}} = \frac{h}{\sqrt{3mkT}}$$

b) We have $\lambda \ll d$. Using λ from part (a) and $d = (V/N)^{1/3}$ we get

$$\frac{h}{\sqrt{3mkT}} \ll \left(\frac{V}{N}\right)^{1/3}$$

If we cube both sides and rearrange.

$$\frac{N}{V}\frac{h^3}{(3mkT)^{3/2}} \ll 1$$

c) For any ideal gas

$$\frac{N}{V} = \frac{6.022 \times 10^{23}}{22.4 \times 10^{-3} \text{ m}^3} = 2.69 \times 10^{25} \text{ m}^{-3}$$

For argon gas (a monatomic gas) at room temperature

$$\frac{N}{V}\frac{h^3}{(3mkT)^{3/2}} = \left(2.69 \times 10^{25} \text{ m}^{-3}\right)\frac{\left(6.626 \times 10^{-34} \text{ J}\cdot\text{s}\right)^3}{(3(40)(1.66 \times 10^{-27} \text{ kg})(1.38 \times 10^{-23} \text{ J/K})(293 \text{ K}))^{3/2}}$$

$$\approx 3 \times 10^{-7}$$

so Maxwell-Boltzmann statistics are fine. However, for electrons in silver $N/V = 5.86 \times 10^{28}$ $\mathrm{m^{-3}}$ and

$$\begin{aligned} \frac{N}{V}\frac{h^3}{(3mkT)^{3/2}} &= \left(5.86 \times 10^{28}\ \mathrm{m^{-3}}\right)\frac{\left(6.626 \times 10^{-34}\ \mathrm{J\cdot s}\right)^3}{\left(3\left(9.11 \times 10^{-31}\ \mathrm{kg}\right)\left(1.38 \times 10^{-23}\ \mathrm{J/K}\right)(293\ \mathrm{K})\right)^{3/2}} \\ &\approx 1.5 \times 10^4 \end{aligned}$$

and in this case Maxwell-Boltzmann statistics fail.

19.

$$F(v)\,dv = 4\pi C v^2 \exp\left(-\frac{1}{2}\beta m v^2\right) dv = F(E)\,dE$$

With $E = \frac{1}{2}mv^2$ we differentiate to get $dE = mv\,dv$ or $dv = \dfrac{dE}{mv} = \dfrac{dE}{\sqrt{2mE}}$. Then

$$\begin{aligned} F(E)\,dE &= 4\pi C\frac{2E}{m}\exp(-\beta E)\frac{dE}{\sqrt{2mE}} = 8\pi C\frac{E^{1/2}}{\sqrt{2}m^{3/2}}\exp(-\beta E)\,dE \\ &= \frac{8\pi C}{\sqrt{2}m^{3/2}}E^{1/2}\exp(-\beta E)\,dE \end{aligned}$$

20. a) We will assume that the magnetic moment is due to spin alone. In general $n(E) = g(E)F_{MB}$. There is no reason to prefer one spin state or the other, so the two $g(E)$ are the same. Thus the ratio of the numbers in the two spin states is governed by the Maxwell-Boltzmann distribution:

$$\frac{n(E_2)}{n(E_1)} = \frac{F_{MB}(E_2)}{F_{MB}(E_1)} = \frac{\exp(-\beta E_2)}{\exp(-\beta E_1)} = \exp\left(\beta\left(E_1 - E_2\right)\right)$$

The energy of a magnetic moment $\vec{\mu}$ in a magnetic field $\vec{B}$ is $E = -\vec{\mu}\cdot\vec{B}$. We know from Chapter 7 that this works out to be

$$E = \frac{e}{m}\vec{S}\cdot\vec{B} = \frac{e}{m}S_z B = \pm\frac{e\hbar}{2m}B = \pm\mu_B B$$

Then $E_1 = -\mu_B B$ is the energy of an electron aligned with the field, and $E_2 = +\mu_B B$ is the energy of the spin opposed to the field. Therefore

$$\frac{n(E_2)}{n(E_1)} = \exp\left(\beta\left(E_1 - E_2\right)\right) = \exp\left(\frac{-\mu_B B - \mu_B B}{kT}\right) = \exp\left(\frac{-2\mu_B B}{kT}\right)$$

b) At $T = 77$ K

$$\frac{n(E_2)}{n(E_1)} = \exp\left(\frac{-2\mu_B B}{kT}\right) = \exp\left(\frac{-2\left(9.274 \times 10^{-24}\ \mathrm{J/T}\right)(6\ \mathrm{T})}{\left(1.381 \times 10^{-23}\ \mathrm{J/K}\right)(77\ \mathrm{K})}\right) = 0.901$$

At $T = 273$ K

$$\frac{n(E_2)}{n(E_1)} = \exp\left(\frac{-2\mu_B B}{kT}\right) = \exp\left(\frac{-2\left(9.274 \times 10^{-24}\ \mathrm{J/T}\right)(6\ \mathrm{T})}{\left(1.381 \times 10^{-23}\ \mathrm{J/K}\right)(273\ \mathrm{K})}\right) = 0.971$$

At $T = 600$ K

$$\frac{n(E_2)}{n(E_1)} = \exp\left(\frac{-2\mu_B B}{kT}\right) = \exp\left(\frac{-2\,(9.274\times10^{-24}\ \text{J/T})\,(6\ \text{T})}{(1.381\times10^{-23}\ \text{J/K})\,(600\ \text{K})}\right) = 0.987$$

As the temperature is increased, the alignment of the spin with the magnetic field is less probable.

21. Starting with Equation (9.30) and setting $F_{\text{FD}} = 0.5$ when $E = E_{\text{F}}$, we have

$$0.5 = \frac{1}{B_1 \exp(\beta E_{\text{F}}) + 1}$$

Solving for B_1, we find $B_1 \exp(\beta E_{\text{F}}) + 1 = 2$, so $B_1 \exp(\beta E_{\text{F}}) = 1$ and $B_1 = \exp(-\beta E_{\text{F}})$. Therefore in general

$$F_{\text{FD}} = \frac{1}{B_1 \exp(\beta E) + 1} = \frac{1}{\exp(-\beta E_{\text{F}})\exp(\beta E) + 1} = \frac{1}{\exp(\beta(E - E_{\text{F}})) + 1}$$

* 22. At first one may think it should be 0.5. but this is not quite true, due to the asymmetric shape of the distribution. Starting with Equation (9.43) for $g(E)$ and using the fact that $F_{\text{FD}} \approx 1$ in this range, we have

$$N(E < E_{\text{F}}) = \int_0^{\overline{E}} g(E)(1)\,dE = \frac{3}{2} N E_{\text{F}}^{-3/2} \int_0^{\overline{E}} E^{1/2} dE = N E_{\text{F}}^{-3/2} \overline{E}^{3/2}$$

But recalling that $\overline{E} = \frac{3}{5} E_{\text{F}}$, we see that

$$N(E < E_{\text{F}}) = N\left(\frac{3}{5}\right)^{3/2} = 0.465\,N$$

23. a) From dimensional analysis

$$1.05\times10^4\ \text{kg/m}^3 \left(\frac{1\ \text{mol}}{0.10787\ \text{kg}}\right)\left(\frac{6.022\times10^{23}}{\text{mol}}\right) = 5.86\times10^{28}\ \text{m}^{-3}$$

b) For electrons an extra factor of 2 is required due to the Pauli principle:

$$\frac{N}{V} = \frac{2A}{h^3}(2\pi mkT)^{3/2}$$

so

$$T = \frac{\left(\frac{N}{2AV}\right)^{2/3} h^2}{2\pi mk} = \frac{\left(\frac{5.86\times10^{28}\ \text{m}^{-3}}{2(1)}\right)^{2/3} (6.626\times10^{-34}\ \text{J}\cdot\text{s})^2}{2\pi\,(9.109\times10^{-31}\ \text{kg})\,(1.381\times10^{-23}\ \text{J/K})} = 5.28\times10^4\ \text{K}$$

c)

$$T = \frac{\left(\frac{N}{2AV}\right)^{2/3} h^2}{2\pi mk} = \frac{\left(\frac{5.86\times10^{28}\ \text{m}^{-3}}{2(0.001)}\right)^{2/3} (6.626\times10^{-34}\ \text{J}\cdot\text{s})^2}{2\pi\,(9.109\times10^{-31}\ \text{kg})\,(1.381\times10^{-23}\ \text{J/K})} = 5.28\times10^6\ \text{K}$$

24. The number is given by

$$\int_{0.95E_F}^{E_F} g(E)\,dE = \frac{3}{2} N E_F^{-3/2} \int_{0.95E_F}^{E_F} E^{1/2}\,dE = N E_F^{-3/2}\, E^{3/2}\Big|_{0.95E_F}^{E_F}$$
$$= N\left(1.00^{3/2} - 0.95^{3/2}\right) = 0.074\,N$$

We see that about 7.4% of the electrons are in this range, which is about what one would expect from the shape of the distribution.

* 25. a) As in Problem 23, $N/V = 5.86 \times 10^{28}\ \mathrm{m^{-3}}$. Then

$$E_F = \frac{h^2}{8m}\left(\frac{3}{\pi}\frac{N}{V}\right)^{2/3} = \frac{(6.626\times10^{-34}\ \mathrm{J\cdot s})^2}{8\,(9.109\times10^{-31}\ \mathrm{kg})}\left(\frac{3}{\pi}\,(5.86\times10^{28}\ \mathrm{m^{-3}})\right)^{2/3}$$
$$= 8.81\times10^{-19}\ \mathrm{J} = 5.50\ \mathrm{eV}$$

b)

$$u_F = \sqrt{\frac{2E_F}{m}} = \sqrt{\frac{2\,(8.81\times10^{-19}\ \mathrm{J})}{9.109\times10^{-31}\ \mathrm{kg}}} = 1.39\times10^{6}\ \mathrm{m/s}$$

26. a) Note: the term $\alpha\,(kT)^2/E_F$ is a small fraction of one eV and can be ignored. Then

$$\overline{E} = \frac{3}{5}E_F = \frac{3}{5}\,(5.51\ \mathrm{eV}) = 3.31\ \mathrm{eV}$$

b) With $\overline{E} = \frac{3}{2}kT$ we have

$$T = \frac{2\overline{E}}{3k} = \frac{2\,(3.31\ \mathrm{eV})}{3\,(8.617\times10^{-5}\ \mathrm{eV/K})} = 2.56\times10^{4}\ \mathrm{K}$$

c) As discussed in the text, thermal energies are small compared with the Fermi energy, except at high temperatures.

27.

$$8.92\times10^{3}\ \mathrm{kg/m^3}\left(\frac{1\ \mathrm{mol}}{0.063546\ \mathrm{kg}}\right)\left(\frac{6.022\times10^{23}}{\mathrm{mol}}\right) = 8.45\times10^{28}\ \mathrm{m^{-3}}$$

This represents a difference of 0.2% from the experimental value so within rounding errors there is one conduction electron per atom.

28. a)

$$2.70\times10^{3}\ \mathrm{kg/m^3}\left(\frac{1\ \mathrm{mol}}{0.02698\ \mathrm{kg}}\right)\left(\frac{6.022\times10^{23}}{\mathrm{mol}}\right) = 6.03\times10^{28}\ \mathrm{m^{-3}}$$

b)

$$E_F = \frac{h^2}{8m}\left(\frac{3}{\pi}\frac{N}{V}\right)^{2/3}$$

so

$$\frac{N}{V} = \frac{\pi}{3}\left(\frac{8mE_F}{h^2}\right)^{3/2} = \frac{\pi}{3}\left(\frac{8\,(9.109\times10^{-31}\text{ kg})\,(11.63\text{ eV})\,(1.602\times10^{-19}\text{ J/eV})}{(6.626\times10^{-34}\text{ J}\cdot\text{s})^2}\right)^{3/2}$$

$$= 1.80\times10^{29}\text{ m}^{-3}$$

c) Dividing the conduction electron density by the number density we obtain almost exactly 3, from which we conclude that the valence number is three.

* 29. In general $E_F = \frac{1}{2}mu_F^2$, so $u_F = \sqrt{2E_F/m}$.

a)

$$u_F = \sqrt{\frac{2E_F}{m}} = \sqrt{\frac{2\,(3.93\text{ eV})\,(1.602\times10^{-19}\text{ J/eV})}{9.109\times10^{-31}\text{ kg}}} = 1.18\times10^6\text{ m/s}$$

b)

$$u_F = \sqrt{\frac{2E_F}{m}} = \sqrt{\frac{2\,(9.47\text{ eV})\,(1.602\times10^{-19}\text{ J/eV})}{9.109\times10^{-31}\text{ kg}}} = 1.83\times10^6\text{ m/s}$$

30. Beginning with Equation (9.34) consider the following cases as $T \to 0$:

$$E > E_F:\ \frac{E-E_F}{kT} \to \infty \text{ so } F_{FD} \to 0$$

$$E < E_F:\ \frac{E-E_F}{kT} \to -\infty \text{ so } F_{FD} \to 1$$

$$E = E_F:\ \frac{E-E_F}{kT} \to 0 \text{ so } F_{FD} \to \frac{1}{2}$$

31. In general $n(E) = g(E)F_{FD}$. Using Equation (9.43) for $g(E)$ and the result of Problem 21 for F_{FD}, we can substitute to find

$$n(E) = \frac{3N}{2}E_F^{-3/2}\frac{E^{1/2}}{\exp\left(\beta\left(E-E_F\right)\right)+1}$$

32. The graphs will resemble those of Figure 9.11 (b). The $T = 0$ line will match the dashed line and the $T = 300$ K line will match the solid line. The $T = 1500$ K line will deviate a bit more from the dashed line.

33. Numerical integration should yield accurate results with $kT = 0.02586$. However, in Mathcad, for example, the upper limit cannot exceed 20 as the engine calculates e raised to the power before taking the reciprocal and thus rejects the problem. But as you adjust the upper limit above 10, the integral equals 1 within rounding error.

$$1.5\,(7)^{-3/2}\int_0^{\infty}\frac{E^{1/2}}{\exp\left((E-7)\,/\,(0.02586)\right)+1}dE \approx 1$$

34. Setting up the numerical integration in Mathcad we have with $kT = 0.02525$ eV,

$$1.5\,(7)^{-3/2}\int_6^{7}\frac{E^{1/2}}{\exp\left((E-7)\,/\,(0.02525)\right)+1}dE = 0.203$$

So we see that about one-fifth of the electrons are within 1 eV of the Fermi energy, which makes sense given the shape of the distribution.

35. We can use the relationship (9.42)

$$E_{\rm F} = \frac{h^2}{8m}\left(\frac{3N}{\pi L^3}\right)^{2/3}$$

We use the neutron mass and from dimensional analysis

$$\frac{N}{L^3} = \frac{4.50\times 10^{30}\ \text{kg}}{\frac{4}{3}\pi\,(10^4\ \text{m})^3}\frac{1\ (\text{neutron})}{1.675\times 10^{-27}\ \text{kg}} = 6.41\times 10^{44}\ \text{m}^{-3}$$

Then

$$\begin{aligned} E_{\rm F} &= \frac{h^2}{8m}\left(\frac{3N}{\pi L^3}\right)^{2/3} = \frac{(6.626\times 10^{-34}\ \text{J}\cdot\text{s})^2}{8\,(1.675\times 10^{-27}\ \text{kg})}\left(\frac{3}{\pi}\left(6.41\times 10^{44}\ \text{m}^{-3}\right)\right)^{2/3} \\ &= 2.36\times 10^{-11}\ \text{J} = 147\ \text{MeV} \end{aligned}$$

The close packing of the neutrons makes the Fermi energy large compared with Fermi energies in normal matter.

36. The probability that a state will be occupied can be determined from the Fermi-Dirac factor. With $kT = 0.02525\,\text{eV}$:

a)

$$F_{\rm FD} = \frac{1}{\exp\left[\frac{(E-E_{\rm F})}{kT}\right]+1} = \frac{1}{\exp\left[\frac{(-0.1)}{0.02525}\right]+1} = 0.981 = 98.1\,\%$$

b) When $E = E_{\rm F}$, then $F_{FD} = \frac{1}{2} = 50\,\%$.

c)

$$F_{\rm FD} = \frac{1}{\exp\left[\frac{(E-E_{\rm F})}{kT}\right]+1} = \frac{1}{\exp\left[\frac{(0.1)}{0.02525}\right]+1} = 0.0187 = 1.9\,\%$$

Therefore a state with energy less than the Fermi energy is almost certainly occupied and one above the Fermi energy has a very small probability of being occupied.

* 37. We assume that the collection of fermions behaves like an ideal gas. Using Maxwell-Boltzmann statistics, we know that $E = \frac{3}{2}kT$ or $\beta E = 3/2$. We note that $\exp(\beta E) = \exp(3/2) = 4.4817$. Now the MB factor is $\exp(-\beta E)$ and we want this to be within 1 % of the FD factor: $F_{\rm FD} = \frac{1}{\exp\left[\frac{(E-E_{\rm F})}{kT}\right]+1}$. So we want

$$\exp[\beta\,(E-E_{\rm F})]+1 = 4.5265 \qquad \text{or} \qquad \beta\,(E-E_{\rm F}) = \ln(3.5265).$$

At room temperature $(\beta)^{-1} = 2.525\times 10^{-2}\,\text{eV}$ so the expression above can be solved to give $E - E_{\rm F} = 3.2\times 10^{-2}\,\text{eV}$. This small value is possible at room temperature.

* 38. a) To find N/V integrate $n(E)\,dE$ over the whole range of energies:

$$\frac{N}{V} = \left(\frac{1}{L^3}\right)\int_0^\infty n(E)\,dE = \frac{8\pi}{h^3c^3}\int_0^\infty \frac{E^2}{\exp(E/kT)-1}\,dE$$

From integral tables we have the following:

$$\int_0^\infty \frac{x^{n-1}}{e^{mx}-1}dx = m^{-n}\Gamma(n)\zeta(n)$$

For us $m = \frac{1}{kT}$, $\Gamma(3) = 2! = 2$, and from numerical tables $\zeta(3) \approx 1.20$. Thus

$$\frac{N}{V} = \frac{8\pi}{h^3c^3}(kT)^3(2)(1.20) = \frac{8\pi k^3T^3}{h^3c^3}(2.40)$$

b) With $T = 500$ K:

$$\begin{aligned}\frac{N}{V} &= \frac{8\pi k^3T^3}{h^3c^3}(2.40) = 8\pi(2.40)\left(\frac{(1.381\times10^{-23}\ \mathrm{J/K})(500\ \mathrm{K})}{(6.626\times10^{-34}\ \mathrm{J\cdot s})(2.998\times10^{8}\ \mathrm{m/s})}\right)^3\\ &= 2.53\times10^{15}\ \mathrm{m^{-3}}\end{aligned}$$

At $T = 5500$ K:

$$\begin{aligned}\frac{N}{V} &= \frac{8\pi k^3T^3}{h^3c^3}(2.40) = 8\pi(2.40)\left(\frac{(1.381\times10^{-23}\ \mathrm{J/K})(5500\ \mathrm{K})}{(6.626\times10^{-34}\ \mathrm{J\cdot s})(2.998\times10^{8}\ \mathrm{m/s})}\right)^3\\ &= 3.37\times10^{18}\ \mathrm{m^{-3}}\end{aligned}$$

39. Evaluating the following integral in Mathcad we find

$$\int_0^\infty \frac{u^{1/2}}{e^u - 1}du = 2.315$$

40. Expressions for I_1 and I_2 are known from the Appendix.

$$I_3 = -\frac{dI_1}{da} = -\left(-\frac{1}{2a^2}\right) = \frac{1}{2a^2}$$

$$I_4 = -\frac{dI_2}{da} = -\frac{\sqrt{\pi}}{4}\left(-\frac{3}{2}\right)a^{-5/2} = \frac{3\sqrt{\pi}}{8}a^{-5/2}$$

$$I_5 = -\frac{dI_3}{da} = -\frac{1}{2}\left(-2a^{-3}\right) = a^{-3}$$

41.

$$E = K + V = \frac{p^2}{2m} + mgz$$

$$\exp(-\beta E) = \exp\left(-\beta\left(\frac{p^2}{2m} + mgz\right)\right) = \exp\left(-\frac{\beta p^2}{2m}\right)\exp(-\beta mgz)$$

Absorbing the (assumed constant) first exponential factor into the normalization constant C_z,

$$f(z)\,dz = C_z\exp(-\beta mgz)\,dz$$

To find C_z we normalize:

$$\int_0^\infty f(z)\,dz = C_z\int_0^\infty \exp(-\beta mgz)\,dz = C_z(\beta mg)$$

Thus

$$C_z = \frac{1}{\beta mg} = \frac{kT}{mg}$$

42. For air we will us an average $m = 29$ u $= 4.82 \times 10^{-26}$ kg and $T = 273$ K. In general

$$\frac{\rho(h)}{\rho(0)} = \frac{\exp(-\beta mgh)}{\exp(-\beta mg\,(0))} = \exp(-\beta mgh)$$

For Denver:

$$\rho(h) = \exp\left(-\frac{(4.82 \times 10^{-26}\text{ kg})\left(9.80\text{ m/s}^2\right)(1610\text{ m})}{(1.381 \times 10^{-23}\text{ J/K})(273\text{ K})}\right)\rho(0) = 0.817\rho(0)$$

For Mt. Rainier:

$$\rho(h) = \exp\left(-\frac{(4.82 \times 10^{-26}\text{ kg})\left(9.80\text{ m/s}^2\right)(4390\text{ m})}{(1.381 \times 10^{-23}\text{ J/K})(273\text{ K})}\right)\rho(0) = 0.577\rho(0)$$

43. In equilibrium a fluid layer of density ρ, mass M, thickness h, and surface area A has a force $F_2 = P_2A$ acting downward on its upper surface and a force $F_1 = P_1A$ acting upward on its lower surface. The difference between these forces equals the weight of the fluid layer.

$$F_2 - F_1 = (P_1 - P_2)\,A = Mg = \rho gAh$$

With $dP \approx \Delta P = P_2 - P_1$ and $h = \Delta z \approx dz$, we have $dP = -\rho g\,dz$. With N particles of mass m, the mass density is $\rho = Nm/V$. Putting these together:

$$dP = -\rho g\,dz = -\frac{Nmg}{V}\,dz$$

From the ideal gas law, $N/V = P/kT$, so

$$dP = -\frac{mgP}{kT}\,dz$$

Applying separation of variables we can solve this differential equation for P as a function of z:

$$\frac{dP}{P} = -\frac{mg}{kT}\,dz \qquad\qquad \ln P = -\frac{mgz}{kT} + \text{ constant } = -\beta mgz + \text{ constant}$$

$$P = (\text{constant})\exp(-\beta mgz) = P_0\exp(-\beta mgz)$$

44. a)

$$\frac{dN}{dt} = -\frac{n\bar{v}}{4}A = -\frac{N\bar{v}A}{4V}$$

Rearranging this expression we find the following differential equation:

$$\frac{dN}{N} = -\frac{\bar{v}A}{4V}\,dt \qquad\qquad \ln N = -\frac{\bar{v}A}{4V}\,t + \text{ constant}$$

$$N = (\text{constant})\exp\left(-\frac{\bar{v}A}{4V}\,t\right) = N_0\exp\left(-\frac{\bar{v}A}{4V}\,t\right)$$

Setting $\dfrac{N}{N_0} = \dfrac{1}{2}$ at $t = t_{1/2}$, we find

$$\frac{1}{2} = \exp\left(-\frac{\bar{v}A}{4V}\,t_{1/2}\right)$$

$$t_{1/2} = \frac{4V}{\overline{v}A} \ln 2$$

b)

$$V = \frac{\pi D^3}{6} = \frac{\pi\,(0.4\text{ m})^3}{6} = 0.0335\text{ m}^3$$

$$A = \frac{\pi d^2}{4} = \frac{\pi\,(0.001\text{ m})^2}{4} 7.85 \times 10^{-7}\text{ m}^2$$

$$\overline{v} = \frac{4}{\sqrt{2\pi}}\sqrt{\frac{kT}{m}} = \frac{4}{\sqrt{2\pi}}\sqrt{\frac{(1.381 \times 10^{-23}\text{ J/K})\,(293\text{ K})}{29\,(1.66 \times 10^{-27}\text{ kg})}} = 462.6\text{ m/s}$$

$$t_{1/2} = \frac{4V}{\overline{v}A} \ln 2 = \frac{4\,(0.0335\text{ m}^3)}{(462.6\text{ m/s})\,(7.85 \times 10^{-7}\text{ m}^2)} \ln 2 = 256\text{ s}$$

* 45. The number of molecules with speed v that hit the wall per unit time is proportional to v and $F(v)$, so that the distribution $W(v)$ of the escaping molecules is by proportion

$$W(v) \sim vF(v) \sim v^3 \exp\left(-\frac{1}{2}\beta m v^2\right)$$

Let the normalization constant for $W(v)$ be C', so

$$C' \int_0^\infty v^3 \exp\left(-\frac{1}{2}\beta m v^2\right) dv = 1 = C'\left(\frac{1}{2}\right)\left(\frac{\beta m}{2}\right)^{-2}$$

or $C' = \beta^2 m^2/2$. The mean kinetic energy of the escaping molecules is

$$\overline{E} = \frac{1}{2}m\overline{v^2} = \frac{1}{2}mC' \int_0^\infty v^5 \exp\left(-\frac{1}{2}\beta m v^2\right) dv = \frac{1}{2}m\left(\frac{\beta^2 m^2}{2}\right)\left(\frac{\beta m}{2}\right)^{-3} = \frac{2}{\beta} = 2kT$$

46. From Example 9.9 we have $\frac{N}{V} = \frac{A}{h^3}\left[2\pi m k T\right]^{3/2}$:

a) Letting m be the electron mass and inserting a factor of 2 for the Pauli principle,

$$\begin{aligned} \frac{N}{V} &= \frac{2A}{h^3}\left[2\pi m k T\right]^{3/2} \\ &= \frac{2(1)}{(6.626 \times 10^{-34}\text{ J}\cdot\text{s})^3}\left[2\pi\,(9.109 \times 10^{-31}\text{ kg})\,(1.381 \times 10^{-23}\text{ J/K})\,(293\text{ K})\right]^{3/2} \\ &= 2.42 \times 10^{25}\text{ m}^{-3} \end{aligned}$$

This is quite a bit less than the density of conduction electrons in a metal (such as copper). which indicates that Fermi-Dirac statistics should be used.

b)

$$\begin{aligned} \frac{N}{V} &= \frac{2A}{h^3}\left[2\pi m k T\right]^{3/2} \\ &= \frac{2(1)}{(6.626 \times 10^{-34}\text{ J}\cdot\text{s})^3}\left[2\pi\,(1.6749 \times 10^{-27}\text{ kg})\,(1.381 \times 10^{-23}\text{ J/K})\,(293\text{ K})\right]^{3/2} \\ &= 1.91 \times 10^{30}\text{ m}^{-3} \end{aligned}$$

c) For He gas the Pauli principle does not apply, so

$$\begin{aligned}\frac{N}{V} &= \frac{A}{h^3}\left[2\pi mkT\right]^{3/2} \\ &= \frac{1}{(6.626\times10^{-34}\ \mathrm{J\cdot s})^3}\left[2\pi(4)\left(1.66\times10^{-27}\ \mathrm{kg}\right)\left(1.381\times10^{-23}\ \mathrm{J/K}\right)(293\ \mathrm{K})\right]^{3/2} \\ & 7.54\times10^{30}\ \mathrm{m}^{-3}\end{aligned}$$

* 47. For the harmonic oscillator the position and velocity are

$$x = x_0\cos(\omega t) \qquad v = \frac{dx}{dt} = -\omega x_0\sin(\omega t)$$

$$V = \frac{1}{2}kx^2 = \frac{1}{2}kx_0^2\cos^2(\omega t)$$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2x_0^2\sin^2(\omega t) = \frac{1}{2}kx_0^2\sin^2(\omega t)$$

where we have used the fact that $\omega^2 m = k$. Over one cycle the average of the square of the sine or cosine function is one-half. Also the total energy is $E = \frac{1}{2}kx_0^2$. Thus

$$\overline{K} = \overline{V} = \frac{1}{2}kx_0^2\left(\frac{1}{2}\right) = \frac{E}{2}$$

48. The mass of gas molecules would differ slightly depending on the isotope of uranium. $^{235}UF_6$ would have a molecular mass of $235+6(19) = 349$ while $^{238}UF_6$ would have a molecular mass of 352. Find the rms velocity for each molecule.

$$v_{\mathrm{rms}} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3\,(1.3807\times10^{-23}\,\mathrm{J/K})\,(293\,\mathrm{K})}{349\,(1.6605\times10^{-27}\,\mathrm{kg})}} = 1.447\times10^2\,\mathrm{m/s}$$

$$v_{\mathrm{rms}} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3\,(1.3807\times10^{-23}\,\mathrm{J/K})\,(293\,\mathrm{K})}{352\,(1.6605\times10^{-27}\,\mathrm{kg})}} = 1.441\times10^2\,\mathrm{m/s}$$

The difference is about 0.6 m/s which represents a 0.4 % change from one isotope to the other.

* 49. Rearranging equation (9.64)and with $m = 4\,\mathrm{u}$ we have

$$\begin{aligned}\frac{N}{V} &\le \frac{2\pi\,(2.315)}{h^3}\left[2mkT\right]^{3/2} \\ &\le \frac{2\pi\,(2.315)}{(6.626\times10^{-34}\,\mathrm{J\cdot s})^3}\left[2\,(4)\left(1.6605\times10^{-27}\,\mathrm{kg}\right)\left(1.381\times10^{-23}\,\mathrm{J/K}\right)(293\,\mathrm{K})\right]^{3/2} \\ &\le 1.97\times10^{31}\,\mathrm{m}^{-3}\end{aligned}$$

The number density of an ideal gas at STP is $\dfrac{N}{V} = \dfrac{P}{kT} = \dfrac{1.0135\times10^5\,\mathrm{Pa}}{(1.381\times10^{-23}\,\mathrm{J/K})\,(293\,\mathrm{K})}$. Therefore $\dfrac{N}{V} = 2.50\times10^{25}\,\mathrm{m}^{-3}$. As you would expect, the condensate has a number density nearly one million times greater than the ideal gas.

50. Rubidium with atomic number 37 has atomic number of 85.47. The stable isotope has atomic mass of 84.911 u, so we will use 85 for the mass number. Then

$$v_{\text{rms}} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3\,(1.3807 \times 10^{-23}\,\text{J/K})\,(20 \times 10^{-9}\,\text{K})}{85\,(1.6605 \times 10^{-27}\,\text{kg})}} = 2.42 \times 10^{-3}\,\text{m/s}$$

51. a) Beginning with Equation (9.65), and with $\frac{N}{V} = 2.5 \times 10^{28}\,\text{m}^{-3}$ we have

$$\begin{aligned} T &\geq \frac{h^2}{2mk}\left(\frac{1}{2\pi\,(2.315)}\right)^{2/3}\left(\frac{N}{V}\right)^{2/3} \\ &\geq \frac{(6.626 \times 10^{-34}\,\text{J/K})^2}{2\,(40)\,(1.6605 \times 10^{-27}\,\text{kg})\,(1.381 \times 10^{-23}\,\text{J/K})}\left(\frac{1}{2\pi)\,(2.315)}\right)^{2/3}(2.5 \times 10^{28}\,\text{m}^{-3})^{2/3} \\ &\geq 3.43 \times 10^{-1}\,\text{K} = 0.343\,\text{K} \end{aligned}$$

b) The temperature found in a) is far below the freezing point of 84 K so no condensate is possible.

52. Beginning with Equation (9.65), we have

$$\begin{aligned} T &\geq \frac{h^2}{2mk}\left(\frac{N}{2\pi V\,(2.315)}\right)^{2/3} \\ &\geq \frac{(6.626 \times 10^{-34}\,\text{J/K})^2}{2\,(87)\,(1.6605 \times 10^{-27}\,\text{kg})\,(1.381 \times 10^{-23}\,\text{J/K})}\left(\frac{2000}{2\pi\,(1 \times 10^{-15}\,\text{m}^3)\,(2.315)}\right)^{2/3} \\ &\geq 2.93 \times 10^{-8}\,\text{K} = 29.3\,\text{nK} \end{aligned}$$