Chapter 24
Electrostatic Energy and Capacitance

Conceptual Problems

***1** •

Determine the Concept The capacitance of a parallel-plate capacitor is a function of the surface area of its plates, the separation of these plates, and the electrical properties of the matter between them. The capacitance is, therefore, independent of the voltage across the capacitor. $\boxed{(c)\text{ is correct.}}$

2 •

Determine the Concept The capacitance of a parallel-plate capacitor is a function of the surface area of its plates, the separation of these plates, and the electrical properties of the matter between them. The capacitance is, therefore, independent of the charge of the capacitor. $\boxed{(c)\text{ is correct.}}$

3 •

Determine the Concept True. The energy density of an electrostatic field is given by $u_e = \frac{1}{2}\epsilon_0 E^2$.

4 •

Picture the Problem The energy stored in the electric field of a parallel-plate capacitor is related to the potential difference across the capacitor by $U = \frac{1}{2}QV$.

Relate the potential energy stored in the electric field of the capacitor to the potential difference across the capacitor:

$$U = \tfrac{1}{2}QV$$

$$\boxed{\text{With } Q \text{ constant, } U \text{ is directly proportional to } V\text{. Hence, doubling } V \text{ doubles } U.}$$

***5** ••

Picture the Problem The energy stored in a capacitor is given by $U = \frac{1}{2}QV$ and the capacitance of a parallel-plate capacitor by $C = \epsilon_0 A/d$. We can combine these relationships, using the definition of capacitance and the condition that the potential difference across the capacitor is constant, to express U as a function of *d*.

Express the energy stored in the capacitor:

$$U = \tfrac{1}{2}QV$$

Use the definition of capacitance to express the charge of the capacitor:

$$Q = CV$$

Substitute to obtain:

$$U = \tfrac{1}{2}CV^2$$

Express the capacitance of a parallel-plate capacitor in terms of the separation d of its plates:

$$C = \frac{\epsilon_0 A}{d}$$

where A is the area of one plate.

Substitute to obtain:

$$U = \frac{\epsilon_0 AV^2}{2d}$$

Because $U \propto \dfrac{1}{d}$, doubling the separation of the plates will reduce the energy stored in the capacitor to 1/2 its previous value:

$$\boxed{(d)\text{ is correct.}}$$

6 ••

Picture the Problem Let V represent the initial potential difference between the plates, U the energy stored in the capacitor initially, d the initial separation of the plates, and $V\,'$, $U\,'$, and $d\,'$ these physical quantities when the plate separation has been doubled. We can use $U = \tfrac{1}{2}QV$ to relate the energy stored in the capacitor to the potential difference across it and $V = Ed$ to relate the potential difference to the separation of the plates.

Express the energy stored in the capacitor before the doubling of the separation of the plates:

$$U = \tfrac{1}{2}QV$$

Express the energy stored in the capacitor after the doubling of the separation of the plates:

$$U' = \tfrac{1}{2}QV'$$

because the charge on the plates does not change.

Express the ratio of U' to U:

$$\frac{U'}{U} = \frac{V'}{V}$$

Express the potential differences across the capacitor plates before and after the plate separation in terms of the electric field E between the plates:

$$V = Ed$$

and

$$V' = Ed'$$

because E depends solely on the charge on the plates and, as observed above, the charge does not change during the separation process.

Substitute to obtain: $$\frac{U'}{U}=\frac{Ed'}{Ed}=\frac{d'}{d}$$

For $d' = 2d$: $$\frac{U'}{U}=\frac{2d}{d}=2 \text{ and } \boxed{(b) \text{ is correct}}$$

7 •

Determine the Concept Both statements are true. The total charge stored by two capacitors in parallel is the sum of the charges on the capacitors and the equivalent capacitance is the sum of the individual capacitances. Two capacitors in series have the same charge and their equivalent capacitance is found by taking the reciprocal of the sum of the reciprocals of the individual capacitances.

8 ••

(*a*) False. Capacitors connected in series carry the same charge.

(*b*) False. The voltage across the capacitor whose capacitance is C_0 is Q/C_0 and that across the second capacitor is $Q/2C_0$.

(*c*) False. The energy stored by the capacitor whose capacitance is C_0 is $Q^2/2C_0$ and the energy stored by the second capacitor is $Q^2/4C_0$.

(*d*) $\boxed{\text{True}}$

9 •

Determine the Concept True. The capacitance of a parallel-plate capacitor filled with a dielectric of constant κ is given by $C=\dfrac{\kappa\in_0 A}{d}$ or $C \propto \kappa$.

***10** ••

Picture the Problem We can treat the configuration in (*a*) as two capacitors in parallel and the configuration in (*b*) as two capacitors in series. Finding the equivalent capacitance of each configuration and examining their ratio will allow us to decide whether (*a*) or (*b*) has the greater capacitance. In both cases, we'll let C_1 be the capacitance of the dielectric-filled capacitor and C_2 be the capacitance of the air capacitor.

In configuration (*a*) we have: $$C_a=C_1+C_2$$

Express C_1 and C_2:

$$C_1 = \frac{\kappa \in_0 A_1}{d_1} = \frac{\kappa \in_0 \frac{1}{2}A}{d} = \frac{\kappa \in_0 A}{2d}$$

and

$$C_2 = \frac{\in_0 A_2}{d_2} = \frac{\in_0 \frac{1}{2}A}{d} = \frac{\in_0 A}{2d}$$

Substitute for C_1 and C_2 and simplify to obtain:

$$C_a = \frac{\kappa \in_0 A}{2d} + \frac{\in_0 A}{2d} = \frac{\in_0 A}{2d}(\kappa + 1)$$

In configuration (b) we have:

$$\frac{1}{C_b} = \frac{1}{C_1} + \frac{1}{C_2} \Rightarrow C_b = \frac{C_1 C_2}{C_1 + C_2}$$

Express C_1 and C_2:

$$C_1 = \frac{\in_0 A_1}{d_1} = \frac{\in_0 A}{\frac{1}{2}d} = \frac{2 \in_0 A}{d}$$

and

$$C_2 = \frac{\kappa \in_0 A_2}{d_2} = \frac{\kappa \in_0 A}{\frac{1}{2}d} = \frac{2\kappa \in_0 A}{d}$$

Substitute for C_1 and C_2 and simplify to obtain:

$$C_b = \frac{\left(\frac{2 \in_0 A}{d}\right)\left(\frac{2\kappa \in_0 A}{d}\right)}{\frac{2 \in_0 A}{d} + \frac{2\kappa \in_0 A}{d}} = \frac{\left(\frac{2 \in_0 A}{d}\right)\left(\frac{2\kappa \in_0 A}{d}\right)}{\frac{2 \in_0 A}{d}(\kappa + 1)} = \frac{2 \in_0 A}{d}\left(\frac{\kappa}{\kappa + 1}\right)$$

Divide C_b by C_a:

$$\frac{C_b}{C_a} = \frac{\frac{2 \in_0 A}{d}\left(\frac{\kappa}{\kappa + 1}\right)}{\frac{\in_0 A}{2d}(\kappa + 1)} = \frac{4\kappa}{(\kappa + 1)^2}$$

Because $\frac{4\kappa}{(\kappa + 1)^2} < 1$ for $\kappa > 1$:

$$\boxed{C_a > C_b}$$

11 •

(*a*) False. The capacitance of a parallel-plate capacitor is defined to be the ratio of the charge on the capacitor to the potential difference across it.

(*b*) False. The capacitance of a parallel-plate capacitor depends on the area of its plates *A*, their separation *d*, and the dielectric constant κ of the material between the plates according to $C = \kappa \in_0 A/d$.

(*c*) False. As in part (*b*), the capacitance of a parallel-plate capacitor depends on the area of its plates *A*, their separation *d*, and the dielectric constant κ of the material between the plates according to $C = \kappa \in_0 A/d$.

12 ••

Picture the Problem We can use the expression $U = \frac{1}{2}CV^2$ to express the ratio of the energy stored in the single capacitor and in the identical-capacitors-in-series combination.

Express the energy stored in capacitors when they are connected to the 100-V battery:

$$U = \tfrac{1}{2}C_{\text{eq}}V^2$$

Express the equivalent capacitance of the two identical capacitors connected in series:

$$C_{\text{eq}} = \frac{C^2}{2C} = \tfrac{1}{2}C$$

Substitute to obtain:

$$U = \tfrac{1}{2}\left(\tfrac{1}{2}C\right)V^2 = \tfrac{1}{4}CV^2$$

Express the energy stored in one capacitor when it is connected to the 100-V battery:

$$U_0 = \tfrac{1}{2}CV^2$$

Express the ratio of U to U_0:

$$\frac{U}{U_0} = \frac{\frac{1}{4}CV^2}{\frac{1}{2}CV^2} = \frac{1}{2}$$

or

$$U = \tfrac{1}{2}U_0 \text{ and } \boxed{(d)\text{ is correct}}$$

Estimation and Approximation

13 ••

Picture the Problem The outer diameter of a "typical" coaxial cable is about 5 mm, while the inner diameter is about 1 mm. From Table 24-1 we see that a reasonable range of values for κ is 3-5. We can use the expression for the capacitance of a cylindrical capacitor to estimate the capacitance per unit length of a coaxial cable.

The capacitance of a cylindrical dielectric-filled capacitor is given by:

$$C=\frac{2\pi\kappa\in_0 L}{\ln\left(\frac{R_2}{R_1}\right)}$$

where L is the length of the capacitor, R_1 is the radius of the inner conductor, and R_2 is the radius of the second (outer) conductor.

Divide both sides by L to obtain an expression for the capacitance per unit length of the cable:

$$\frac{C}{L}=\frac{2\pi\kappa\in_0}{\ln\left(\frac{R_2}{R_1}\right)}=\frac{\kappa}{2k\ln\left(\frac{R_2}{R_1}\right)}$$

If $\kappa = 3$:

$$\frac{\boldsymbol{C}}{\boldsymbol{L}}=\frac{3}{2\left(8.99\times10^9\ \mathrm{N\cdot m^2/C^2}\right)\ln\left(\frac{2.5\,\mathrm{mm}}{0.5\,\mathrm{mm}}\right)}=0.104\,\mathrm{nF/m}$$

If $\kappa = 5$:

$$\frac{\boldsymbol{C}}{\boldsymbol{L}}=\frac{5}{2\left(8.99\times10^9\ \mathrm{N\cdot m^2/C^2}\right)\ln\left(\frac{2.5\,\mathrm{mm}}{0.5\,\mathrm{mm}}\right)}=0.173\,\mathrm{nF/m}$$

A reasonable range of values for C/L, corresponding to $3 \le \kappa \le 5$, is:

$$\boxed{0.104\,\mathrm{nF/m}\le\frac{\boldsymbol{C}}{\boldsymbol{L}}\le0.173\,\mathrm{nF/m}}$$

***14 ••**

Picture the Problem The energy stored in a capacitor is given by $U=\frac{1}{2}CV^2$.

Relate the energy stored in a capacitor to its capacitance and the potential difference across it:

$$U=\tfrac{1}{2}CV^2$$

Solve for C:

$$C=\frac{2U}{V^2}$$

The potential difference across the spark gap is related to the width of the gap d and the electric field E in the gap:

$$V=Ed$$

Substitute for V in the expression for C to obtain:

$$C = \frac{2U}{E^2 d^2}$$

Substitute numerical values and evaluate C:

$$C = \frac{2(100\,\text{J})}{(3\times10^6\,\text{V/m})^2(0.001\,\text{m})^2} = \boxed{22.2\,\mu\text{F}}$$

15 ••

Picture the Problem Because $\Delta R << R_E$ we can treat the atmosphere as a flat slab with an area equal to the surface area of the earth. Then the energy stored in the atmosphere can be estimated from $U = uV$, where u is the energy density of the atmosphere and V is its volume.

Express the electric energy stored in the atmosphere in terms of its energy density and volume:

$$U = uV$$

Because $\Delta R << R_E$ = 6370 km, we can consider the volume:

$$V = A_{\text{surface area of the earth}}\Delta R = 4\pi R_E^2 \Delta R$$

Express the energy density of the Earth's atmosphere in terms of the average magnitude of its electric field:

$$u = \tfrac{1}{2}\epsilon_0 E^2$$

Substitute for V and u to obtain:

$$U = \left(\tfrac{1}{2}\epsilon_0 E^2\right)\left(4\pi R_E^2 \Delta R\right) = \frac{R_E^2 E^2 \Delta R}{2k}$$

Substitute numerical values and evaluate U:

$$U = \frac{(6370\,\text{km})^2(200\,\text{V/m})^2(1\,\text{km})}{2(8.99\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2)} = \boxed{9.03\times10^{10}\,\text{J}}$$

16 ••

Picture the Problem We'll approximate the balloon by a sphere of radius R = 3 m and use the expression for the capacitance of an isolated spherical conductor.

Relate the capacitance of an isolated spherical conductor to its radius:

$$C = 4\pi\epsilon_0 R = \frac{R}{k}$$

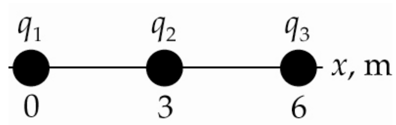
Substitute numerical values and evaluate C:

$$C = \frac{3\,\text{m}}{8.99\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2} = \boxed{0.334\,\text{nF}}$$

Electrostatic Potential Energy

17 •

The electrostatic potential energy of this system of three point charges is the work needed to bring the charges from an infinite separation to the final positions shown in the diagram.



Express the work required to assemble this system of charges:

$$U = \frac{kq_1q_2}{r_{1,2}} + \frac{kq_1q_3}{r_{1,3}} + \frac{kq_2q_3}{r_{2,3}} = k\left(\frac{q_1q_2}{r_{1,2}} + \frac{q_1q_3}{r_{1,3}} + \frac{q_2q_3}{r_{2,3}}\right)$$

Find the distances $r_{1,2}$, $r_{1,3}$, and $r_{2,3}$:

$$r_{1,2} = 3\,\text{m}, r_{2,3} = 3\,\text{m}, \text{and}\, r_{1,3} = 6\,\text{m}$$

(a) Evaluate U for $q_1 = q_2 = q_3 = 2\ \mu\text{C}$:

$$U = \left(8.99\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left[\frac{(2\,\mu\text{C})(2\,\mu\text{C})}{3\,\text{m}} + \frac{(2\,\mu\text{C})(2\,\mu\text{C})}{6\,\text{m}} + \frac{(2\,\mu\text{C})(2\,\mu\text{C})}{3\,\text{m}}\right] = \boxed{30.0\,\text{mJ}}$$

(b) Evaluate U for $q_1 = q_2 = 2\ \mu\text{C}$ and $q_3 = -2\ \mu\text{C}$:

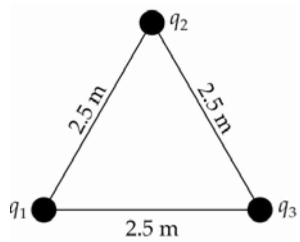
$$U = \left(8.99\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left[\frac{(2\,\mu\text{C})(2\,\mu\text{C})}{3\,\text{m}} + \frac{(2\,\mu\text{C})(-2\,\mu\text{C})}{6\,\text{m}} + \frac{(2\,\mu\text{C})(-2\,\mu\text{C})}{3\,\text{m}}\right] = \boxed{-5.99\,\text{mJ}}$$

(c) Evaluate U for $q_1 = q_3 = 2\ \mu\text{C}$ and $q_2 = -2\ \mu\text{C}$:

$$U = \left(8.99\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left[\frac{(2\,\mu\text{C})(-2\,\mu\text{C})}{3\,\text{m}} + \frac{(2\,\mu\text{C})(2\,\mu\text{C})}{6\,\text{m}} + \frac{(-2\,\mu\text{C})(2\,\mu\text{C})}{3\,\text{m}}\right] = \boxed{-18.0\,\text{mJ}}$$

18 •

Picture the Problem The electrostatic potential energy of this system of three point charges is the work needed to bring the charges from an infinite separation to the final positions shown in the diagram.



Express the work required to assemble this system of charges:

$$U = \frac{kq_1q_2}{r_{1,2}} + \frac{kq_1q_3}{r_{1,3}} + \frac{kq_2q_3}{r_{2,3}}$$
$$= k\left(\frac{q_1q_2}{r_{1,2}} + \frac{q_1q_3}{r_{1,3}} + \frac{q_2q_3}{r_{2,3}}\right)$$

Find the distances $r_{1,2}$, $r_{1,3}$, and $r_{2,3}$:

$$r_{1,2} = r_{2,3} = r_{1,3} = 2.5\,\text{m}$$

(*a*) Evaluate U for $q_1 = q_2 = q_3 = 4.2\ \mu\text{C}$:

$$U = (8.99\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\left[\frac{(4.2\,\mu\text{C})(4.2\,\mu\text{C})}{2.5\,\text{m}} + \frac{(4.2\,\mu\text{C})(4.2\,\mu\text{C})}{2.5\,\text{m}} + \frac{(4.2\,\mu\text{C})(4.2\,\mu\text{C})}{2.5\,\text{m}}\right]$$
$$= \boxed{0.190\,\text{J}}$$

(*b*) Evaluate U for $q_1 = q_2 = 4.2\ \mu\text{C}$ and $q_3 = -4.2\ \mu\text{C}$:

$$U = (8.988\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\left[\frac{(4.2\,\mu\text{C})(4.2\,\mu\text{C})}{2.5\,\text{m}} + \frac{(4.2\,\mu\text{C})(-4.2\,\mu\text{C})}{2.5\,\text{m}} + \frac{(4.2\,\mu\text{C})(-4.2\,\mu\text{C})}{2.5\,\text{m}}\right]$$
$$= \boxed{-63.4\,\text{mJ}}$$

(*c*) Evaluate U for $q_1 = q_2 = -4.2\ \mu\text{C}$ and $q_3 = +4.2\ \mu\text{C}$:

$$U=\left(8.99\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left[\frac{(-4.2\,\mu\text{C})(-4.2\,\mu\text{C})}{2.5\,\text{m}}+\frac{(-4.2\,\mu\text{C})(4.2\,\mu\text{C})}{2.5\,\text{m}}\right.$$
$$\left.+\frac{(-4.2\,\mu\text{C})(4.2\,\mu\text{C})}{2.5\,\text{m}}\right]$$
$$=\boxed{-63.4\,\text{mJ}}$$

***19 •**

Picture the Problem The potential of an isolated spherical conductor is given by $V=kQ/r$,where Q is its charge and r its radius, and its electrostatic potential energy by $U=\frac{1}{2}QV$. We can combine these relationships to find the sphere's electrostatic potential energy.

Express the electrostatic potential energy of the isolated spherical conductor as a function of its charge Q and potential V:

$$U=\tfrac{1}{2}QV$$

Express the potential of the spherical conductor:

$$V=\frac{kQ}{r}$$

Solve for Q to obtain:

$$Q=\frac{rV}{k}$$

Substitute to obtain:

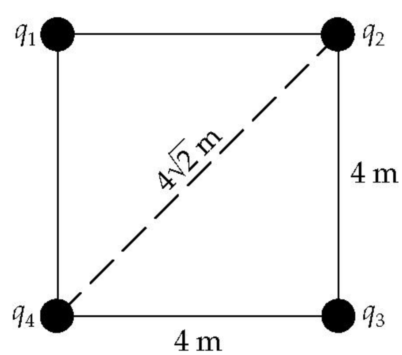
$$U=\tfrac{1}{2}\left(\frac{rV}{k}\right)V=\frac{rV^2}{2k}$$

Substitute numerical values and evaluate U:

$$U=\frac{(0.1\,\text{m})(2\,\text{kV})^2}{2\left(8.99\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)}=\boxed{22.2\,\mu\text{J}}$$

20 ••

Picture the Problem The electrostatic potential energy of this system of four point charges is the work needed to bring the charges from an infinite separation to the final positions shown in the diagram. In part (c), depending on the configuration of the positive and negative charges, two energies are possible.



Express the work required to assemble this system of charges:

$$U=\frac{kq_1q_2}{r_{1,2}}+\frac{kq_1q_3}{r_{1,3}}+\frac{kq_1q_4}{r_{1,4}}+\frac{kq_2q_3}{r_{2,3}}+\frac{kq_2q_4}{r_{2,4}}+\frac{kq_3q_4}{r_{3,4}}$$
$$=k\left(\frac{q_1q_2}{r_{1,2}}+\frac{q_1q_3}{r_{1,3}}+\frac{q_1q_4}{r_{1,4}}+\frac{q_2q_3}{r_{2,3}}+\frac{q_2q_4}{r_{2,4}}+\frac{q_3q_4}{r_{3,4}}\right)$$

Find the distances $r_{1,2}$, $r_{1,3}$, $r_{1,4}$, $r_{2,3}$, $r_{2,4}$, and $r_{3,4}$,:

$$r_{1,2}=r_{2,3}=r_{3,4}=r_{1,4}=4\,\text{m}$$

and

$$r_{1,3}=r_{2,4}=4\sqrt{2}\,\text{m}$$

(*a*) Evaluate U for $q_1 = q_2 = q_3 = q_4 = -2\ \mu\text{C}$:

$$U=\left(8.99\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left[\frac{(-2\,\mu\text{C})(-2\,\mu\text{C})}{4\,\text{m}}+\frac{(-2\,\mu\text{C})(-2\,\mu\text{C})}{4\sqrt{2}\,\text{m}}+\frac{(-2\,\mu\text{C})(-2\,\mu\text{C})}{4\,\text{m}}\right.$$
$$\left.+\frac{(-2\,\mu\text{C})(-2\,\mu\text{C})}{4\,\text{m}}+\frac{(-2\,\mu\text{C})(-2\,\mu\text{C})}{4\sqrt{2}\,\text{m}}+\frac{(-2\,\mu\text{C})(-2\,\mu\text{C})}{4\,\text{m}}\right]$$
$$=\boxed{48.7\,\text{mJ}}$$

(*b*) Evaluate U for $q_1 = q_2 = q_3 = 2\ \mu\text{C}$ and $q_4 = -2\ \mu\text{C}$:

$$U=\left(8.99\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left[\frac{(2\,\mu\text{C})(2\,\mu\text{C})}{4\,\text{m}}+\frac{(2\,\mu\text{C})(2\,\mu\text{C})}{4\sqrt{2}\,\text{m}}+\frac{(2\,\mu\text{C})(-2\,\mu\text{C})}{4\,\text{m}}\right.$$
$$\left.+\frac{(2\,\mu\text{C})(2\,\mu\text{C})}{4\,\text{m}}+\frac{(2\,\mu\text{C})(-2\,\mu\text{C})}{4\sqrt{2}\,\text{m}}+\frac{(2\,\mu\text{C})(-2\,\mu\text{C})}{4\,\text{m}}\right]$$
$$=\boxed{0}$$

(*c*) Let $q_1 = q_2 = 2\ \mu\text{C}$ and $q_3 = q_4 = -2\ \mu\text{C}$:

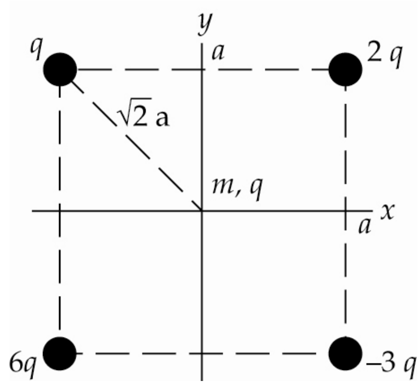
$$U=\left(8.99\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left[\frac{(2\,\mu\text{C})(2\,\mu\text{C})}{4\,\text{m}}+\frac{(2\,\mu\text{C})(-2\,\mu\text{C})}{4\sqrt{2}\,\text{m}}+\frac{(2\,\mu\text{C})(-2\,\mu\text{C})}{4\,\text{m}}\right.$$
$$\left.+\frac{(2\,\mu\text{C})(-2\,\mu\text{C})}{4\,\text{m}}+\frac{(2\,\mu\text{C})(-2\,\mu\text{C})}{4\sqrt{2}\,\text{m}}+\frac{(-2\,\mu\text{C})(-2\,\mu\text{C})}{4\,\text{m}}\right]$$
$$=\boxed{-12.7\,\text{mJ}}$$

Let $q_1 = q_3 = 2\ \mu\text{C}$ and $q_2 = q_4 = -2\ \mu\text{C}$:

$$U=\left(8.99\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left[\frac{(2\,\mu\text{C})(-2\,\mu\text{C})}{4\,\text{m}}+\frac{(2\,\mu\text{C})(2\,\mu\text{C})}{4\sqrt{2}\,\text{m}}+\frac{(2\,\mu\text{C})(-2\,\mu\text{C})}{4\,\text{m}}\right.$$
$$\left.+\frac{(-2\,\mu\text{C})(2\,\mu\text{C})}{4\,\text{m}}+\frac{(-2\,\mu\text{C})(-2\,\mu\text{C})}{4\sqrt{2}\,\text{m}}+\frac{(2\,\mu\text{C})(-2\,\mu\text{C})}{4\,\text{m}}\right]$$
$$=\boxed{-23.2\,\text{mJ}}$$

21 ••

Picture the Problem The diagram shows the four charges fixed at the corners of the square and the fifth charge that is released from rest at the origin. We can use conservation of energy to relate the initial potential energy of the fifth particle to its kinetic energy when it is at a great distance from the origin and the electrostatic potential at the origin to express U_i.



Use conservation of energy to relate the initial potential energy of the particle to its kinetic energy when it is at a great distance from the origin:

$$\Delta K+\Delta U=0$$

or, because $K_i = U_f = 0$,

$$K_f-U_i=0$$

Express the initial potential energy of the particle to its charge and the electrostatic potential at the origin:

$$U_i=qV(0)$$

Substitute for K_f and U_i to obtain:

$$\tfrac{1}{2}mv^2-qV(0)=0$$

Solve for v:

$$v=\sqrt{\frac{2qV(0)}{m}}$$

Express the electrostatic potential at the origin:

$$V(0)=\frac{kq}{\sqrt{2}a}+\frac{2kq}{\sqrt{2}a}+\frac{-3kq}{\sqrt{2}a}+\frac{6kq}{\sqrt{2}a}=\frac{6kq}{\sqrt{2}a}$$

Substitute and simplify to obtain:

$$v=\sqrt{\frac{2q}{m}\left(\frac{6kq}{\sqrt{2}a}\right)}=\boxed{q\sqrt{\frac{6\sqrt{2}k}{ma}}}$$

Capacitance

***22 •**

Picture the Problem The charge on the spherical conductor is related to its radius and potential according to $V = kQ/r$ and we can use the definition of capacitance to find the capacitance of the sphere.

(*a*) Relate the potential *V* of the spherical conductor to the charge on it and to its radius:

$$V = \frac{kQ}{r}$$

Solve for and evaluate *Q*:

$$Q = \frac{rV}{k} = \frac{(0.1\,\text{m})(2\,\text{kV})}{8.99\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2} = \boxed{22.2\,\text{nC}}$$

(*b*) Use the definition of capacitance to relate the capacitance of the sphere to its charge and potential:

$$C = \frac{Q}{V} = \frac{22.2\,\text{nC}}{2\,\text{kV}} = \boxed{11.1\,\text{pF}}$$

(*c*) $\boxed{\text{It doesn't. The capacitance of a sphere is a function of its radius.}}$

23 •

Picture the Problem We can use its definition to find the capacitance of this capacitor.

Use the definition of capacitance to obtain:

$$C = \frac{Q}{V} = \frac{30\,\mu\text{C}}{400\,\text{V}} = \boxed{75.0\,\text{nF}}$$

24 ••

Picture the Problem Let the separation of the spheres be *d* and their radii be *R*. Outside the two spheres the electric field is approximately the field due to point charges of $+Q$ and $-Q$, each located at the centers of spheres, separated by distance *d*. We can derive an expression for the potential at the surface of each sphere and then use the potential difference between the spheres and the definition of capacitance and to find the capacitance of the two-sphere system.

The capacitance of the two-sphere system is given by:

$$C = \frac{Q}{\Delta V}$$

where ΔV is the potential difference between the spheres.

The potential at any point outside the two spheres is:

$$V = \frac{k(+Q)}{r_1} + \frac{k(-Q)}{r_2}$$

where r_1 and r_2 are the distances from the given point to the centers of the spheres.

For a point on the surface of the sphere with charge $+Q$:

$r_1 = R$ and $r_2 = d + \delta$
where $|\delta| < R$

Substitute to obtain:

$$V_{+Q} = \frac{k(+Q)}{R} + \frac{k(-Q)}{d+\delta}$$

For $\delta << d$:

$$V_{+Q} = \frac{kQ}{R} - \frac{kQ}{d}$$

and

$$V_{-Q} = -\frac{kQ}{R} + \frac{kQ}{d}$$

The potential difference between the spheres is:

$$\begin{aligned}\Delta V &= V_Q - V_{-Q} \\ &= \frac{kQ}{R} - \frac{kQ}{d} - \left(\frac{-kQ}{R} + \frac{kQ}{d}\right) \\ &= 2kQ\left(\frac{1}{R} - \frac{1}{d}\right)\end{aligned}$$

Substitute for ΔV in the expression for C to obtain:

$$\begin{aligned}C &= \frac{Q}{2kQ\left(\frac{1}{R} - \frac{1}{d}\right)} = \frac{2\pi \in_0}{\left(\frac{1}{R} - \frac{1}{d}\right)} \\ &= \frac{2\pi \in_0 R}{1 - \frac{R}{d}}\end{aligned}$$

For d very large:

$$C = \boxed{2\pi \in_0 R}$$

The Storage of Electrical Energy

25 •

Picture the Problem Of the three equivalent expressions for the energy stored in a charged capacitor, the one that relates U to C and V is $U = \frac{1}{2}CV^2$.

(a) Express the energy stored in the capacitor as a function of C and V:

$$U = \tfrac{1}{2}CV^2$$

Substitute numerical values and evaluate U:

$$U = \tfrac{1}{2}(3\,\mu\text{F})(100\,\text{V})^2 = \boxed{15.0\,\text{mJ}}$$

(*b*) Express the additional energy required as the difference between the energy stored in the capacitor at 200 V and the energy stored at 100 V:

$$\begin{aligned}\Delta U &= U(200\,\text{V}) - U(100\,\text{V})\\ &= \tfrac{1}{2}(3\,\mu\text{F})(200\,\text{V})^2 - 15.0\,\text{mJ}\\ &= \boxed{45.0\,\text{mJ}}\end{aligned}$$

26 •

Picture the Problem Of the three equivalent expressions for the energy stored in a charged capacitor, the one that relates U to Q and C is $U = \dfrac{1}{2}\dfrac{Q^2}{C}$.

(*a*) Express the energy stored in the capacitor as a function of C and Q:

$$U = \frac{1}{2}\frac{Q^2}{C}$$

Substitute numerical values and evaluate U:

$$U = \frac{1}{2}\frac{(4\,\mu\text{C})^2}{10\,\mu\text{F}} = \boxed{0.800\,\mu\text{J}}$$

(*b*) Express the energy remaining when half the charge is removed:

$$U\left(\tfrac{1}{2}Q\right) = \frac{1}{2}\frac{(2\,\mu\text{C})^2}{10\,\mu\text{F}} = \boxed{0.200\,\mu\text{J}}$$

27 •

Picture the Problem Of the three equivalent expressions for the energy stored in a charged capacitor, the one that relates U to Q and C is $U = \dfrac{1}{2}\dfrac{Q^2}{C}$.

(*a*) Express the energy stored in the capacitor as a function of C and Q:

$$U = \frac{1}{2}\frac{Q^2}{C}$$

Substitute numerical values and evaluate U:

$$U(5\,\mu\text{C}) = \frac{1}{2}\frac{(5\,\mu\text{C})^2}{20\,\text{pF}} = \boxed{0.625\,\text{J}}$$

(*b*) Express the additional energy required as the difference between the energy stored in the capacitor when its charge is 5 μC and when its charge is 10 μC:

$$\begin{aligned}\Delta U &= U(10\,\mu\text{C}) - U(5\,\mu\text{C})\\ &= \frac{1}{2}\frac{(10\,\mu\text{C})^2}{20\,\text{pF}} - 0.625\,\text{J}\\ &= 2.50\,\text{J} - 0.625\,\text{J}\\ &= \boxed{1.88\,\text{J}}\end{aligned}$$

***28 •**

Picture the Problem The energy per unit volume in an electric field varies with the square of the electric field according to $u = \epsilon_0 E^2/2$.

Express the energy per unit volume in an electric field:

$$u = \tfrac{1}{2}\epsilon_0 E^2$$

Substitute numerical values and evaluate u:

$$u = \tfrac{1}{2}\left(8.85\times10^{-12}\ \mathrm{C^2/N\cdot m^2}\right)\left(3\ \mathrm{MV/m}\right)^2 = \boxed{39.8\ \mathrm{J/m^3}}$$

29 •

Picture the Problem Knowing the potential difference between the plates, we can use $E = V/d$ to find the electric field between them. The energy per unit volume is given by $u = \tfrac{1}{2}\epsilon_0 E^2$ and we can find the capacitance of the parallel-plate capacitor using $C = \epsilon_0 A/d$.

(*a*) Express the electric field between the plates in terms of their separation and the potential difference between them:

$$E = \frac{V}{d} = \frac{100\ \mathrm{V}}{1\ \mathrm{mm}} = \boxed{100\ \mathrm{kV/m}}$$

(*b*) Express the energy per unit volume in an electric field:

$$u = \tfrac{1}{2}\epsilon_0 E^2$$

Substitute numerical values and evaluate u:

$$u = \tfrac{1}{2}\left(8.85\times10^{-12}\ \mathrm{C^2/N\cdot m^2}\right)\left(100\ \mathrm{kV/m}\right)^2 = \boxed{44.3\ \mathrm{mJ/m^3}}$$

(*c*) The total energy is given by:

$$U = uV = uAd = \left(44.3\ \mathrm{mJ/m^3}\right)\left(2\ \mathrm{m^2}\right)\left(1\ \mathrm{mm}\right) = \boxed{88.6\ \mu J}$$

(*d*) The capacitance of a parallel-plate capacitor is given by:

$$C = \frac{\epsilon_0 A}{d} = \frac{\left(8.85\times10^{-12}\ \mathrm{C^2/N\cdot m^2}\right)\left(2\ \mathrm{m^2}\right)}{1\ \mathrm{mm}} = \boxed{17.7\ \mathrm{nF}}$$

(*e*) The total energy is given by:

$$U = \tfrac{1}{2}CV^2$$

Substitute numerical values and evaluate *U*:

$$U = \tfrac{1}{2}(17.7\,\text{nF})(100\,\text{V})^2 = \boxed{88.5\,\mu\text{J, in agreement with } (c).}$$

30 ••

Picture the Problem The total energy stored in the electric field is the product of the energy density in the space between the spheres and the volume of this space.

(*a*) The total energy *U* stored in the electric field is given by:

$$U = uV$$

where *u* is the energy density and *V* is the volume between the spheres.

The energy density of the field is:

$$u = \frac{1}{2} \epsilon_0 E^2$$

where *E* is the field between the spheres.

The volume between the spheres is approximately:

$$V \approx 4\pi r_1^2 (r_2 - r_1)$$

Substitute for *u* and *V* to obtain:

$$U = 2\pi \epsilon_0 E^2 r_1^2 (r_2 - r_1) \qquad (1)$$

The magnitude of the electric field between the concentric spheres is the sum of the electric fields due to each charge distribution:

$$E = E_Q + E_{-Q}$$

Because the two surfaces are so close together, the electric field between them is approximately the sum of the fields due to two plane charge distributions:

$$E = \frac{\sigma_Q}{2\epsilon_0} + \frac{\sigma_{-Q}}{2\epsilon_0} = \frac{\sigma_Q}{\epsilon_0}$$

Substitute for σ_Q to obtain:

$$E \approx \frac{Q}{4\pi r_1^2 \epsilon_0}$$

Substitute for *E* in equation (1) and simplify:

$$U = 2\pi \epsilon_0 \left(\frac{Q}{4\pi r_1^2 \epsilon_0}\right)^2 r_1^2 (r_2 - r_1) = \frac{Q^2}{8\pi \epsilon_0} \frac{r_2 - r_1}{r_1^2}$$

Substitute numerical values and evaluate U:

$$U=\frac{(5\,\text{nC})^2(10.5\,\text{cm}-10.0\,\text{cm})}{8\pi\left(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)(10.0\,\text{cm})^2}=\boxed{56.0\,\text{nJ}}$$

(b) The capacitance of the two-sphere system is given by:

$$C=\frac{Q}{\Delta V}$$

where ΔV is the potential difference between the two spheres.

The electric potentials at the surfaces of the spheres are:

$$V_1=\frac{Q}{4\pi\in_0 r_1}\text{ and }V_2=\frac{Q}{4\pi\in_0 r_2}$$

Substitute for ΔV and simplify to obtain:

$$C=\frac{Q}{\dfrac{Q}{4\pi\in_0 r_1}-\dfrac{Q}{4\pi\in_0 r_2}}=4\pi\in_0\frac{r_1r_2}{r_2-r_1}$$

Substitute numerical values and evaluate C:

$$C=4\pi\left(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)\frac{(10.0\,\text{cm})(10.5\,\text{cm})}{10.5\,\text{cm}-10.0\,\text{cm}}=\boxed{0.234\,\text{nF}}$$

Use ½ Q^2/C to find the total energy stored in the electric field between the spheres:

$$U=\frac{1}{2}\left[\frac{(5\,\text{nC})^2}{0.234\,\text{nF}}\right]=\boxed{53.4\,\text{nJ}}$$

Note that our approximate result in (a) is within 5% of our exact result obtained in (b).

***31** ••

Picture the Problem We can relate the charge Q on the positive plate of the capacitor to the charge density of the plate σ using its definition. The charge density, in turn, is related to the electric field between the plates according to $\sigma=\in_0 E$ and the electric field can be found from $E = \Delta V/\Delta d$. We can use $\Delta U=\frac{1}{2}Q\Delta V$ in part (b) to find the increase in the energy stored due to the movement of the plates.

(a) Express the charge Q on the positive plate of the capacitor in terms of the plate's charge density σ and surface area A:

$$Q=\sigma A$$

Relate σ to the electric field E between the plates of the capacitor:

$$\sigma = \epsilon_0 E$$

Express E in terms of the change in V as the plates are separated a distance Δd:

$$E = \frac{\Delta V}{\Delta d}$$

Substitute for σ and E to obtain:

$$Q = \epsilon_0 EA = \epsilon_0 A\frac{\Delta V}{\Delta d}$$

Substitute numerical values and evaluate Q:

$$Q = \left(8.85\times10^{-12}\,\mathrm{C^2/N\cdot m^2}\right)\left(500\,\mathrm{cm}^2\right)\frac{100\,\mathrm{V}}{0.4\,\mathrm{cm}} = \boxed{11.1\,\mathrm{nC}}$$

(b) Express the change in the electrostatic energy in terms of the change in the potential difference:

$$\Delta U = \tfrac{1}{2}Q\Delta V$$

Substitute numerical values and evaluate ΔU:

$$\Delta U = \tfrac{1}{2}\left(11.1\,\mathrm{nC}\right)\left(100\,\mathrm{V}\right) = \boxed{0.553\,\mu\mathrm{J}}$$

32 •••

Picture the Problem By symmetry, the electric field must be radial. In part (a) we can find E_r both inside and outside the ball by choosing a spherical Gaussian surface first inside and then outside the surface of the ball and applying Gauss's law.

(a) Relate the electrostatic energy density at a distance r from the center of the ball to the electric field due to the uniformly distributed charge Q:

$$u_e = \tfrac{1}{2}\epsilon_0 E^2 \qquad (1)$$

Relate the flux through the Gaussian surface to the electric field E_r on the Gaussian surface at $r < R$:

$$E_r\left(4\pi r^2\right) = \frac{Q_{\mathrm{inside}}}{\epsilon_0} \qquad (2)$$

Using the fact that the charge is uniformly distributed, express the ratio of the charge enclosed by the Gaussian surface to the total charge of the sphere:

$$\frac{Q_{\mathrm{inside}}}{Q} = \frac{\rho V_{\mathrm{Gaussian\ surface}}}{\rho V_{\mathrm{ball}}} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} = \frac{r^3}{R^3}$$

Solve for Q_{inside} to obtain:

$$Q_{\text{inside}} = Q\frac{r^3}{R^3}$$

Substitute in equation (2):

$$E_r\left(4\pi r^2\right) = \frac{Qr^3}{\epsilon_0 R^3}$$

Solve for $E_{r<R}$:

$$E_{r<R} = \frac{Qr}{4\pi \epsilon_0 R^3} = \frac{kQ}{R^3}r$$

Substitute in equation (1) to obtain:

$$u_{\text{e}}(r<R) = \tfrac{1}{2}\epsilon_0\left(\frac{kQ}{R^3}r\right)^2 = \boxed{\frac{\epsilon_0 k^2Q^2}{2R^6}r^2}$$

Relate the flux through the Gaussian surface to the electric field E_r on the Gaussian surface at $r > R$:

$$E_r\left(4\pi r^2\right) = \frac{Q_{\text{inside}}}{\epsilon_0} = \frac{Q}{\epsilon_0}$$

Solve for $E_{r>R}$:

$$E_{r>R} = \frac{Q}{4\pi r^2 \epsilon_0} = kQr^{-2}$$

Substitute in equation (1) to obtain:

$$u_{\text{e}}(r>R) = \tfrac{1}{2}\epsilon_0\left(kQr^{-2}\right)^2 = \boxed{\tfrac{1}{2}\epsilon_0 k^2Q^2r^{-4}}$$

(*b*) Express the energy dU in a spherical shell of thickness dr and surface area $4\pi r^2$:

$$dU_{\text{shell}} = 4\pi r^2 u(r)dr$$

For $r < R$:

$$dU_{\text{shell}}(r<R) = 4\pi r^2\left(\frac{\epsilon_0 k^2Q^2}{2R^6}r^2\right)dr = \boxed{\frac{kQ^2}{2R^6}r^4dr}$$

For $r > R$:

$$dU_{\text{shell}}(r>R) = 4\pi r^2\left(\tfrac{1}{2}\epsilon_0 k^2Q^2r^{-4}\right)dr = \boxed{\tfrac{1}{2}kQ^2r^{-2}dr}$$

(*c*) Express the total electrostatic energy:

$$U = U(r<R) + U(r>R) \qquad (3)$$

Integrate $U_{\text{shell}}(r < R)$ from 0 to R:

$$U_{\text{shell}}(r < R) = \frac{kQ^2}{2R^6}\int_0^R r^4 dr = \frac{kQ^2}{10R}$$

Integrate $U_{\text{shell}}(r > R)$ from R to ∞:

$$U_{\text{shell}}(r > R) = \tfrac{1}{2}kQ^2\int_R^\infty r^{-2} dr = \frac{kQ^2}{2R}$$

Substitute in equation (3) to obtain:

$$U = \frac{kQ^2}{10R} + \frac{kQ^2}{2R} = \boxed{\frac{3kQ^2}{5R}}$$

The field inside the shell is zero, so the first integral vanishes. The result is greater for the sphere because it includes the field energy within the sphere.

Combinations of Capacitors

33 •

Picture the Problem We can apply the properties of capacitors connected in parallel to determine the number of 1.0-μF capacitors connected in parallel it would take to store a total charge of 1 mC with a potential difference of 10 V across each capacitor. Knowing that the capacitors are connected in parallel (parts (*a*) and (*b*)) we determine the potential difference across the combination. In part (*c*) we can use our knowledge of how potential differences add in a series circuit to find the potential difference across the combination and the definition of capacitance to find the charge on each capacitor.

(*a*) Express the number of capacitors n in terms of the charge q on each and the total charge Q:

$$n = \frac{Q}{q}$$

Relate the charge q on one capacitor to its capacitance C and the potential difference across it:

$$q = CV$$

Substitute to obtain:

$$n = \frac{Q}{CV}$$

Substitute numerical values and evaluate n:

$$n = \frac{1\,\text{mC}}{(1\,\mu\text{F})(10\,\text{V})} = \boxed{100}$$

(*b*) Because the capacitors are connected in parallel the potential difference across the combination is the same as the potential difference across each of them:

$$V_{\text{parallel combination}} = V = \boxed{10\,\text{V}}$$

(*c*) With the capacitors connected in series, the potential difference across the combination will be the sum of the potential differences across the 100 capacitors:

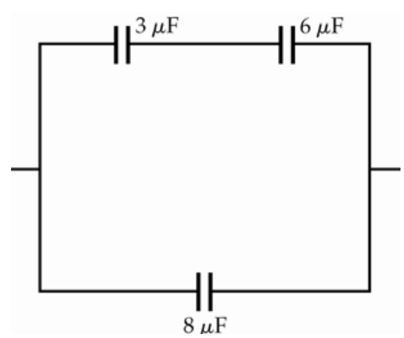
$$\begin{aligned} V_{\text{series combination}} &= 100V \\ &= 100(10\,\text{V}) \\ &= \boxed{1.00\,\text{kV}} \end{aligned}$$

Use the definition of capacitance to find the charge on each capacitor:

$$q = CV = (1\,\mu\text{F})(10\,\text{V}) = \boxed{10.0\,\mu\text{C}}$$

34 •

Picture the Problem The capacitor array is shown in the diagram. We can find the equivalent capacitance of this combination by first finding the equivalent capacitance of the 3.0-μF and 6.0-μF capacitors in series and then the equivalent capacitance of this capacitor with the 8.0-μF capacitor in parallel.



Express the equivalent capacitance for the 3.0-μF and 6.0-μF capacitors in series:

$$\frac{1}{C_{3+6}} = \frac{1}{3\,\mu\text{F}} + \frac{1}{6\,\mu\text{F}}$$

Solve for C_{3+6}:

$$C_{3+6} = 2\,\mu\text{F}$$

Find the equivalent capacitance of a 2-μF capacitor in parallel with an 8-μF capacitor:

$$C_{2+8} = 2\,\mu\text{F} + 8\,\mu\text{F} = \boxed{10\,\mu\text{F}}$$

***35** •

Picture the Problem Because we're interested in the equivalent capacitance across terminals *a* and *c*, we need to recognize that capacitors C_1 and C_3 are in series with each other and in parallel with capacitor C_2.

Find the equivalent capacitance of C_1 and C_3 in series:

$$\frac{1}{C_{1+3}} = \frac{1}{C_1} + \frac{1}{C_3}$$

Solve for C_{1+3}:

$$C_{1+3}=\frac{C_1C_3}{C_1+C_3}$$

Find the equivalent capacitance of C_{1+3} and C_2 in parallel:

$$C_{eq}=C_2+C_{1+3}=\boxed{C_2+\frac{C_1C_3}{C_1+C_3}}$$

36 •

Picture the Problem Because the capacitors are connected in parallel we can add their capacitances to find the equivalent capacitance of the combination. Also, because they are in parallel, they have a common potential difference across them. We can use the definition of capacitance to find the charge on each capacitor.

(*a*) Find the equivalent capacitance of the two capacitors in parallel:

$$C_{eq}=10.0\,\mu\text{F}+20\,\mu\text{F}=\boxed{30\,\mu\text{F}}$$

(*b*) Because capacitors in parallel have a common potential difference across them:

$$V=V_{10}+V_{20}=\boxed{6.00\,\text{V}}$$

(*c*) Use the definition of capacitance to find the charge on each capacitor:

$$Q_{10}=C_{10}V=(10\,\mu\text{F})(6\,\text{V})=\boxed{60.0\,\mu\text{C}}$$

and

$$Q_{20}=C_{20}V=(20\,\mu\text{F})(6\,\text{V})=\boxed{120\,\mu\text{C}}$$

37 ••

Picture the Problem We can use the properties of capacitors in series to find the equivalent capacitance and the charge on each capacitor. We can then apply the definition of capacitance to find the potential difference across each capacitor.

(*a*) Because the capacitors are connected in series they have equal charges:

$$Q_{10}=Q_{20}=C_{eq}V$$

Express the equivalent capacitance of the two capacitors in series:

$$\frac{1}{C_{eq}}=\frac{1}{10\,\mu\text{F}}+\frac{1}{20\,\mu\text{F}}$$

Solve for C_{eq} to obtain:

$$C_{eq}=\frac{(10\,\mu\text{F})(20\,\mu\text{F})}{10\,\mu\text{F}+20\,\mu\text{F}}=6.67\,\mu\text{F}$$

Substitute to obtain:

$$Q_{10}=Q_{20}=(6.67\,\mu\text{F})(6\,\text{V})=\boxed{40.0\,\mu\text{C}}$$

(*b*) Apply the definition of capacitance to find the potential difference across each capacitor:

$$V_{10} = \frac{Q_{10}}{C_{10}} = \frac{40.0\,\mu\text{C}}{10\,\mu\text{F}} = \boxed{4.00\,\text{V}}$$

and

$$V_{20} = \frac{Q_{20}}{C_{20}} = \frac{40.0\,\mu\text{C}}{20\,\mu\text{F}} = \boxed{2.00\,\text{V}}$$

***38 ••**

Picture the Problem We can use the properties of capacitors connected in series and in parallel to find the equivalent capacitances for various connection combinations.

(*a*) $\boxed{\text{If their capacitance is to be a maximum, they must be connected in parallel.}}$

Find the capacitance of each capacitor:

$$C_{\text{eq}} = 3C = 15\,\mu\text{F}$$

and

$$C = 5\,\mu\text{F}$$

(*b*) (1) Connect the three capacitors in series:

$$\frac{1}{C_{\text{eq}}} = \frac{3}{5\,\mu\text{F}} \text{ and } C_{\text{eq}} = \boxed{1.67\,\mu\text{F}}$$

(2) Connect two in parallel, with the third in series with that combination:

$$C_{\text{eq, two in parallel}} = 2(5\,\mu\text{F}) = 10\,\mu\text{F}$$

and

$$\frac{1}{C_{\text{eq}}} = \frac{1}{10\,\mu\text{F}} + \frac{1}{5\,\mu\text{F}}$$

Solve for C_{eq}:

$$C_{\text{eq}} = \frac{(10\,\mu\text{F})(5\,\mu\text{F})}{10\,\mu\text{F} + 5\,\mu\text{F}} = \boxed{3.33\,\mu\text{F}}$$

(3) Connect two in series, with the third in parallel with that combination:

$$\frac{1}{C_{\text{eq, two in series}}} = \frac{2}{5\,\mu\text{F}}$$

or

$$C_{\text{eq, two in series}} = 2.5\,\mu\text{F}$$

Find the capacitance equivalent to 2.5 μF and 5 μF in parallel:

$$C_{\text{eq}} = 2.5\,\mu\text{F} + 5\,\mu\text{F} = \boxed{7.50\,\mu\text{F}}$$

39 ••

Picture the Problem We can use the properties of capacitors connected in series and in parallel to find the equivalent capacitance between the terminals and these properties and the definition of capacitance to find the charge on each capacitor.

(*a*) Relate the equivalent capacitance of the two capacitors in series to their individual capacitances:

$$\frac{1}{C_{4+15}} = \frac{1}{4\ \mu\text{F}} + \frac{1}{15\ \mu\text{F}}$$

Solve for C_{4+15}:

$$C_{4+15} = \frac{(4\ \mu\text{F})(15\ \mu\text{F})}{4\ \mu\text{F} + 15\ \mu\text{F}} = 3.16\ \mu\text{F}$$

Find the equivalent capacitance of C_{4+15} in parallel with the 12-μF capacitor:

$$C_{\text{eq}} = 3.16\ \mu\text{F} + 12\ \mu\text{F} = \boxed{15.2\ \mu\text{F}}$$

(*b*) Using the definition of capacitance, express and evaluate the charge stored on the 12-μF capacitor:

$$Q_{12} = C_{12}V_{12} = C_{12}V = (12\ \mu\text{F})(200\ \text{V}) = \boxed{2.40\ \text{mC}}$$

Because the capacitors in series have the same charge:

$$Q_4 = Q_{15} = C_{4+15}V = (3.16\ \mu\text{F})(200\ \text{V}) = \boxed{0.632\ \text{mC}}$$

(*c*) The total energy stored is given by:

$$U_{\text{total}} = \frac{1}{2}C_{\text{eq}}V^2$$

Substitute numerical values and evaluate U_{total}:

$$U_{\text{total}} = \frac{1}{2}(15.2\ \mu\text{F})(200\ \text{V})^2 = \boxed{0.304\ \text{J}}$$

40 ••

Picture the Problem We can use the properties of capacitors in series to establish the results called for in this problem.

(*a*) Express the equivalent capacitance of two capacitors in series:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{C_2 + C_1}{C_1C_2}$$

Solve for C_{eq} by taking the reciprocal of both sides of the equation to obtain:

$$C_{\text{eq}} = \boxed{\frac{C_1C_2}{C_1 + C_2}}$$

(*b*) Divide numerator and denominator of this expression by C_1 to obtain:

$$C_{\text{eq}} = \frac{C_2}{1+\frac{C_2}{C_1}} < \boxed{C_2}$$

because $1+\frac{C_2}{C_1} > 1$.

Divide numerator and denominator of this expression by C_2 to obtain:

$$C_{\text{eq}} = \frac{C_1}{1+\frac{C_1}{C_2}} < \boxed{C_1}$$

because $1+\frac{C_1}{C_2} > 1$.

Using our result from part (*a*) for two of the capacitors, add a third capacitor C_3 in series to obtain:

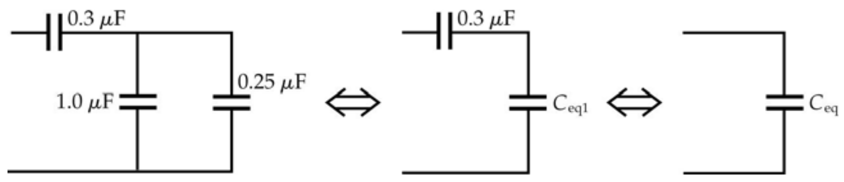
$$\frac{1}{C_{\text{eq}}} = \frac{C_1+C_2}{C_1C_2} + \frac{1}{C_3}$$
$$= \frac{C_1C_3 + C_2C_3 + C_1C_2}{C_1C_2C_3}$$

Take the reciprocal of both sides of the equation to obtain:

$$C_{\text{eq}} = \boxed{\frac{C_1C_2C_3}{C_1C_2 + C_2C_3 + C_1C_3}}$$

41 ••

Picture the Problem Let C_{eq1} represent the equivalent capacitance of the parallel combination and C_{eq} the total equivalent capacitance between the terminals. We can use the equations for capacitors in parallel and then in series to find C_{eq}. Because the charge on C_{eq} is the same as on the 0.3-μF capacitor and C_{eq1}, we'll know the charge on the 0.3-μF capacitor when we have found the total charge Q_{eq} stored by the circuit. We can find the charges on the 1.0-μF and 0.25-μF capacitors by first finding the potential difference across them and then using the definition of capacitance.



(*a*) Find the equivalent capacitance for the parallel combination:

$$C_{\text{eq1}} = 1\,\mu\text{F} + 0.25\,\mu\text{F} = 1.25\,\mu\text{F}$$

The 0.30-μF capacitor is in series with C_{eq1} … find their equivalent capacitance C_{eq}:

$$\frac{1}{C_{eq}} = \frac{1}{0.3\,\mu\text{F}} + \frac{1}{1.25\,\mu\text{F}}$$

and

$$C_{eq} = \boxed{0.242\,\mu\text{F}}$$

(*b*) Express the total charge stored by the circuit Q_{eq}:

$$Q_{eq} = Q_{0.3} = Q_{1.25} = C_{eq}V = (0.242\,\mu\text{F})(10\,\text{V}) = \boxed{2.42\,\mu\text{C}}$$

The 1-μF and 0.25-μF capacitors, being in parallel, have a common potential difference. Express this potential difference in terms of the 10 V across the system and the potential difference across the 0.3-μF capacitor:

$$V_{1.25} = 10\,\text{V} - V_{0.3} = 10\,\text{V} - \frac{Q_{0.3}}{C_{0.3}} = 10\,\text{V} - \frac{2.42\,\mu\text{C}}{0.3\,\mu\text{F}} = 1.93\,\text{V}$$

Using the definition of capacitance, find the charge on the 1-μF and 0.25-μF capacitors:

$$Q_1 = C_1V_1 = (1\,\mu\text{F})(1.93\,\text{V}) = \boxed{1.93\,\mu\text{C}}$$

and

$$Q_{0.25} = C_{0.25}V_{0.25} = (0.25\,\mu\text{F})(1.93\,\text{V}) = \boxed{0.483\,\mu\text{C}}$$

(*c*) The total stored energy is given by:

$$U = \tfrac{1}{2}C_{eq}V^2$$

Substitute numerical values and evaluate U:

$$U = \tfrac{1}{2}(0.242\,\mu\text{F})(10\,\text{V})^2 = \boxed{12.1\,\mu\text{J}}$$

42 ••

Picture the Problem Note that there are three parallel paths between *a* and *b*. We can find the equivalent capacitance of the capacitors connected in series in the upper and lower branches and then find the equivalent capacitance of three capacitors in parallel.

(*a*) Find the equivalent capacitance of the series combination of capacitors in the upper and lower branch:

$$\frac{1}{C_{eq}} = \frac{1}{C_0} + \frac{1}{C_0}$$

or

$$\text{C}_{eq} = \frac{C_0^2}{2C_0} = \tfrac{1}{2}C_0$$

Now we have two capacitors with capacitance $C_0/2$ in parallel with a capacitor whose capacitance is C_0. Find their equivalent capacitance:

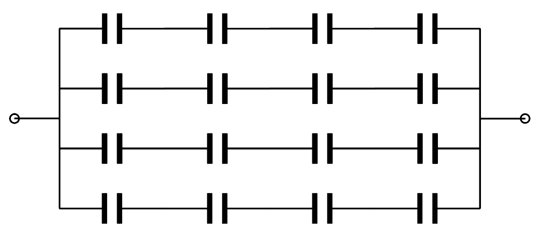
$$C'_{\text{eq}} = \tfrac{1}{2}C_0 + C_0 + \tfrac{1}{2}C_0 = \boxed{2C_0}$$

(*b*) If the central capacitance is $10C_0$, then:

$$C'_{\text{eq}} = \tfrac{1}{2}C_0 + 10C_0 + \tfrac{1}{2}C_0 = \boxed{11C_0}$$

43 ••

Picture the Problem Place four of the capacitors in series. Then the potential across each is 100 V when the potential across the combination is 400 V. The equivalent capacitance of the series is 2/4 μF = 0.5 μF. If we place four such series combinations in parallel, as shown in the circuit diagram, the total capacitance between the terminals is 2 μF.



***44** ••

Picture the Problem We can connect two capacitors in parallel, all three in parallel, two in series, three in series, two in parallel in series with the third, and two in series in parallel with the third.

Connect 2 in parallel to obtain:

$$C_{\text{eq}} = 1\,\mu\text{F} + 2\,\mu\text{F} = \boxed{3\,\mu\text{F}}$$

or

$$C_{\text{eq}} = 1\,\mu\text{F} + 4\,\mu\text{F} = \boxed{5\,\mu\text{F}}$$

or

$$C_{\text{eq}} = 2\,\mu\text{F} + 4\,\mu\text{F} = \boxed{6\,\mu\text{F}}$$

Connect all three in parallel to obtain:

$$C_{\text{eq}} = 1\,\mu\text{F} + 2\,\mu\text{F} + 4\,\mu\text{F} = \boxed{7\,\mu\text{F}}$$

Connect two in series:

$$C_{\text{eq}} = \frac{(1\,\mu\text{F})(2\,\mu\text{F})}{1\,\mu\text{F} + 2\,\mu\text{F}} = \boxed{\frac{2}{3}\,\mu\text{F}}$$

or

$$C_{\text{eq}} = \frac{(1\,\mu\text{F})(4\,\mu\text{F})}{1\,\mu\text{F} + 4\,\mu\text{F}} = \boxed{\frac{4}{5}\,\mu\text{F}}$$

or

$$C_{eq} = \frac{(2\,\mu\text{F})(4\,\mu\text{F})}{2\,\mu\text{F} + 4\,\mu\text{F}} = \boxed{\frac{4}{3}\,\mu\text{F}}$$

Connect all three in series:

$$C_{eq} = \frac{(1\,\mu\text{F})(2\,\mu\text{F})(4\,\mu\text{F})}{(1\,\mu\text{F})(2\,\mu\text{F}) + (2\,\mu\text{F})(4\,\mu\text{F}) + (1\,\mu\text{F})(4\,\mu\text{F})} = \boxed{\frac{4}{7}\,\mu\text{F}}$$

Connect two in parallel, in series with the third:

$$C_{eq} = \frac{(4\,\mu\text{F})(1\,\mu\text{F} + 2\,\mu\text{F})}{1\,\mu\text{F} + 2\,\mu\text{F} + 4\,\mu\text{F}} = \boxed{\frac{12}{7}\,\mu\text{F}}$$

or

$$C_{eq} = \frac{(1\,\mu\text{F})(4\,\mu\text{F} + 2\,\mu\text{F})}{1\,\mu\text{F} + 2\,\mu\text{F} + 4\,\mu\text{F}} = \boxed{\frac{6}{7}\,\mu\text{F}}$$

or

$$C_{eq} = \frac{(2\,\mu\text{F})(4\,\mu\text{F} + 1\,\mu\text{F})}{1\,\mu\text{F} + 2\,\mu\text{F} + 4\,\mu\text{F}} = \boxed{\frac{10}{7}\,\mu\text{F}}$$

Connect two in series, in parallel with the third:

$$C_{eq} = \frac{(1\,\mu\text{F})(2\,\mu\text{F})}{1\,\mu\text{F} + 2\,\mu\text{F}} + 4\,\mu\text{F} = \boxed{\frac{14}{3}\,\mu\text{F}}$$

or

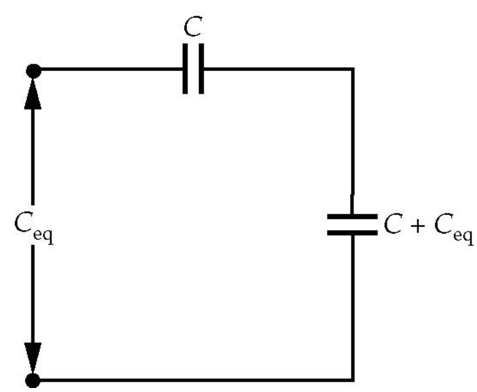
$$C_{eq} = \frac{(4\,\mu\text{F})(2\,\mu\text{F})}{4\,\mu\text{F} + 2\,\mu\text{F}} + 1\,\mu\text{F} = \boxed{\frac{7}{3}\,\mu\text{F}}$$

or

$$C_{eq} = \frac{(1\,\mu\text{F})(4\,\mu\text{F})}{1\,\mu\text{F} + 4\,\mu\text{F}} + 2\,\mu\text{F} = \boxed{\frac{14}{5}\,\mu\text{F}}$$

45 •••

Picture the Problem Let C be the capacitance of each capacitor in the ladder and let C_{eq} be the equivalent capacitance of the infinite ladder less the series capacitor in the first rung. Because the capacitance is finite and non-zero, adding one more stage to the ladder will not change the capacitance of the network. The capacitance of the two capacitor combination shown to the right is the equivalent of the infinite ladder, so it has capacitance C_{eq} also.



(*a*) The equivalent capacitance of the parallel combination of C and C_{eq} is:

$$C + C_{eq}$$

The equivalent capacitance of the series combination of C and $(C + C_{eq})$ is C_{eq}, so:

$$\frac{1}{C_{eq}} = \frac{1}{C} + \frac{1}{C + C_{eq}}$$

Simply this expression to obtain a quadratic equation in C_{eq}:

$$C_{eq}^2 + CC_{eq} - C^2 = 0$$

Solve for the positive value of C_{eq} to obtain:

$$C_{eq} = \left(\frac{\sqrt{5} - 1}{2}\right)C = 0.618C$$

Because $C = 1\ \mu$F:

$$C_{eq} = \boxed{0.618\,\mu\text{F}}$$

(*b*) The capacitance C' required so that the combination has the same capacitance as the infinite ladder is:

$$C' = C + C_{eq}$$

Substitute for C_{eq} and evaluate C':

$$C' = C + 0.618C = 1.618C$$

Because $C = 1\ \mu$F:

$$C' = \boxed{1.618\,\mu\text{F}}$$

Parallel-Plate Capacitors

46 •

Picture the Problem The potential difference V across a parallel-plate capacitor, the electric field E between its plates, and the separation d of the plates are related according to $V = Ed$. We can use this relationship to find V_{max} corresponding to dielectric breakdown and the definition of capacitance to find the maximum charge on the capacitor.

(*a*) Express the potential difference V across the plates of the capacitor in terms of the electric field between the plates E and their separation d:

$$V = Ed$$

V_{max} corresponds to E_{max}:

$$V_{max} = (3\,\text{MV/m})(1.6\,\text{mm}) = \boxed{4.80\,\text{kV}}$$

(*b*) Using the definition of capacitance, find the charge Q

$$Q = CV_{max} = (2.0\,\mu\text{F})(4.80\,\text{kV}) = \boxed{9.60\,\text{mC}}$$

stored at this maximum potential difference:

47 •

Picture the Problem The potential difference V across a parallel-plate capacitor, the electric field E between its plates, and the separation d of the plates are related according to $V = Ed$. In part (b) we can use the definition of capacitance and the expression for the capacitance of a parallel-plate capacitor to find the required plate radius.

(a) Express the potential difference V across the plates of the capacitor in terms of the electric field between the plates E and their separation d:

$$V = Ed$$

Substitute numerical values and evaluate V:

$$V = \left(2\times10^{4}\ \text{V/m}\right)\left(2\ \text{mm}\right) = \boxed{40.0\ \text{V}}$$

(b) Use the definition of capacitance to relate the capacitance of the capacitor to its charge and the potential difference across it:

$$C = \frac{Q}{V}$$

Express the capacitance of a parallel-plate capacitor:

$$C = \frac{\epsilon_0 A}{d} = \frac{\epsilon_0 \pi R^2}{d}$$

where R is the radius of the circular plates.

Equate these two expressions for C:

$$\frac{\epsilon_0 \pi R^2}{d} = \frac{Q}{V}$$

Solve for R to obtain:

$$R = \sqrt{\frac{Qd}{\epsilon_0 \pi V}}$$

Substitute numerical values and evaluate R:

$$R = \sqrt{\frac{\left(10\ \mu\text{C}\right)\left(2\ \text{mm}\right)}{\pi\left(8.85\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)\left(40\ \text{V}\right)}} = \boxed{4.24\ \text{m}}$$

48 ••

Picture the Problem We can use the expression for the capacitance of a parallel-plate capacitor to find the area of each plate and the definition of capacitance to find the potential difference when the capacitor is charged to 3.2 μC. We can find the stored energy using $U = \frac{1}{2}CV^2$ and the definition of capacitance and the relationship between the potential difference across a parallel-plate capacitor and the electric field between its plates to find the charge at which dielectric breakdown occurs. Recall that $E_{\text{max, air}} = 3$ MV/m.

(*a*) Relate the capacitance of a parallel-plate capacitor to the area *A* of its plates and their separation *d*:

$$C = \frac{\epsilon_0 A}{d}$$

Solve for *A*:

$$A = \frac{Cd}{\epsilon_0}$$

Substitute numerical values and evaluate *A*:

$$A = \frac{(0.14\,\mu\text{F})(0.5\,\text{mm})}{8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2} = \boxed{7.91\,\text{m}^2}$$

(*b*) Using the definition of capacitance, express and evaluate the potential difference across the capacitor when it is charged to 3.2 μC:

$$V = \frac{Q}{C} = \frac{3.2\,\mu\text{C}}{0.14\,\mu\text{F}} = \boxed{22.9\,\text{V}}$$

(*c*) Express the stored energy as a function of the capacitor's capacitance and the potential difference across it:

$$U = \tfrac{1}{2}CV^2$$

Substitute numerical values and evaluate *U*:

$$U = \tfrac{1}{2}(0.14\,\mu\text{F})(22.9\,\text{V})^2 = \boxed{36.7\,\mu\text{J}}$$

(*d*) Using the definition of capacitance, relate the charge on the capacitor to breakdown potential difference:

$$Q_{\text{max}} = CV_{\text{max}}$$

Relate the maximum potential difference to the maximum electric field between the plates:

$$V_{\text{max}} = E_{\text{max}}d$$

Substitute to obtain:

$$Q_{\text{max}} = CE_{\text{max}}d$$

Substitute numerical values and evaluate Q_{max}:

$$Q_{\text{max}} = (0.14\,\mu\text{F})(3\,\text{MV/m})(0.5\,\text{mm}) = \boxed{210\,\mu\text{C}}$$

***49** ••

Picture the Problem The potential difference across the capacitor plates V is related to their separation d and the electric field between them according to $V = Ed$. We can use this equation with E_{max} = 3 MV/m to find d_{min}. In part (*b*) we can use the expression for the capacitance of a parallel-plate capacitor to find the required area of the plates.

(*a*) Use the relationship between the potential difference across the plates and the electric field between them to find the minimum separation of the plates:

$$d_{min} = \frac{V}{E_{max}} = \frac{1000\,\text{V}}{3\,\text{MV/m}} = \boxed{0.333\,\text{mm}}$$

(*b*) Use the expression for the capacitance of a parallel-plate capacitor to relate the capacitance to the area of a plate:

$$C = \frac{\epsilon_0 A}{d}$$

Solve for A:

$$A = \frac{Cd}{\epsilon_0}$$

Substitute numerical values and evaluate A:

$$A = \frac{(0.1\,\mu\text{F})(0.333\,\text{mm})}{8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2} = \boxed{3.76\,\text{m}^2}$$

Cylindrical Capacitors

50 •

Picture the Problem The capacitance of a cylindrical capacitor is given by $C = 2\pi\kappa\epsilon_0 L/\ln(r_2/r_1)$ where L is its length and r_1 and r_2 the radii of the inner and outer conductors.

(*a*) Express the capacitance of the coaxial cylindrical shell:

$$C = \frac{2\pi\kappa\epsilon_0 L}{\ln\left(\frac{r}{R}\right)}$$

Substitute numerical values and evaluate C:

$$C = \frac{2\pi(1)(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)(0.12\,\text{m})}{\ln\left(\frac{1.5\,\text{cm}}{0.2\,\text{mm}}\right)}$$

$$= \boxed{1.55\,\text{pF}}$$

(*b*) Use the definition of capacitance to express the charge per unit length:

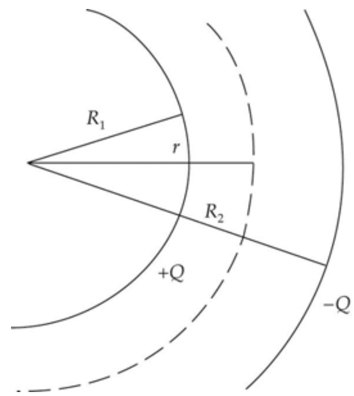
$$\lambda = \frac{Q}{L} = \frac{CV}{L}$$

Substitute numerical values and evaluate λ:

$$\lambda = \frac{(1.55\,\text{pF})(1.2\,\text{kV})}{0.12\,\text{m}} = \boxed{15.5\,\text{nC/m}}$$

51 ••

Picture the Problem The diagram shows a partial cross-sectional view of the inner wire and the outer cylindrical shell. By symmetry, the electric field is radial in the space between the wire and the concentric cylindrical shell. We can apply Gauss's law to cylindrical surfaces of radii $r < R_1$, $R_1 < r < R_2$, and $r > R_2$ to find the electric field and, hence, the energy density in these regions.



(*a*) Apply Gauss's law to a cylindrical surface of radius $r < R_1$ and length L to obtain:

$$E_r(2\pi rL) = \frac{Q_{\text{inside}}}{\epsilon_0} = 0$$

and

$$E_{r<R_1} = \boxed{0}$$

Because $E = 0$ for $r < R_1$:

$$u_{r<R_1} = \boxed{0}$$

Apply Gauss's law to a cylindrical surface of radius $R_1 < r < R_2$ and length L to obtain:

$$E_r(2\pi rL) = \frac{Q_{\text{inside}}}{\epsilon_0} = \frac{\lambda L}{\epsilon_0}$$

where λ is the linear charge density.

Solve for E_r to obtain:

$$E_r = \frac{\lambda L}{2\pi \epsilon_0 r} = \frac{2k\lambda}{r}$$

Express the energy density in the region $R_1 < r < R_2$:

$$u = \tfrac{1}{2}\epsilon_0 E_r^2 = \tfrac{1}{2}\epsilon_0\left(\frac{2k\lambda}{r}\right)^2$$

$$= \tfrac{1}{2}\epsilon_0\left(\frac{2kQ}{rL}\right)^2 = \frac{2k^2\epsilon_0 Q^2}{r^2L^2}$$

Apply Gauss's law to a cylindrical surface of radius $r > R_2$ and length L to obtain:

$$E_r(2\pi rL) = \frac{Q_{\text{inside}}}{\in_0} = 0$$

and

$$E_{r>R_2} = \boxed{0}$$

Because $E = 0$ for $r > R_2$:

$$u_{r>R_2} = \boxed{0}$$

(*b*) Express the energy residing in a cylindrical shell between the conductors of radius r, thickness dr, and volume $2\pi rL\,dr$:

$$\begin{aligned} dU &= 2\pi rLu(r)dr \\ &= 2\pi rL\left(\frac{2k^2 \in_0 Q^2}{r^2L^2}\right)dr = \frac{kQ^2}{rL}dr \end{aligned}$$

(*c*) Integrate dU from $r = R_1$ to R_2 to obtain:

$$U = \frac{kQ^2}{L}\int_{R_1}^{R_2}\frac{dr}{r} = \boxed{\frac{kQ^2}{L}\ln\left(\frac{R_2}{R_1}\right)}$$

Use $U = \frac{1}{2}CV^2$ and the expression for the capacitance of a cylindrical capacitor to obtain:

$$\begin{aligned} U &= \tfrac{1}{2}CV^2 \\ &= \tfrac{1}{2}\frac{Q^2}{C} = \frac{Q^2}{2\left(\dfrac{2\pi \in_0 L}{\ln\dfrac{R_2}{R_1}}\right)} \\ &= \boxed{\frac{kQ^2}{L}\ln\left(\frac{R_2}{R_1}\right)} \end{aligned}$$

in agreement with the result from part (*b*).

52 •••

Picture the Problem Note that with the innermost and outermost cylinders connected together the system corresponds to two cylindrical capacitors connected in parallel. We can use $C = \dfrac{2\pi \in_0 \kappa L}{\ln(R_o/R_i)}$ to express the capacitance per unit length and then calculate and add the capacitances per unit length of each of the cylindrical shell capacitors.

Relate the capacitance of a cylindrical capacitor to its length L and inner and outer radii R_i and R_o:

$$C = \frac{2\pi \in_0 \kappa L}{\ln(R_o/R_i)}$$

Divide both sides of the equation by L to express the capacitance per unit

$$\frac{C}{L} = \frac{2\pi \in_0 \kappa}{\ln(R_o/R_i)}$$

length:

Express the capacitance per unit length of the cylindrical system:

$$\frac{C}{L} = \left(\frac{C}{L}\right)_{\text{outer}} + \left(\frac{C}{L}\right)_{\text{inner}} \qquad (1)$$

Find the capacitance per unit length of the outer cylindrical shell combination:

$$\left(\frac{C}{L}\right)_{\text{outer}} = \frac{2\pi\left(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)(1)}{\ln(0.8\,\text{cm}/0.5\,\text{cm})}$$
$$= 118.3\,\text{pF/m}$$

Find the capacitance per unit length of the inner cylindrical shell combination:

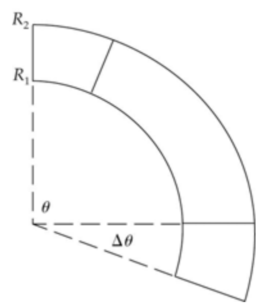
$$\left(\frac{C}{L}\right)_{\text{inner}} = \frac{2\pi\left(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)(1)}{\ln(0.5\,\text{cm}/0.2\,\text{cm})}$$
$$= 60.7\,\text{pF/m}$$

Substitute in equation (1) to obtain:

$$\frac{C}{L} = 118.3\,\text{pF/m} + 60.7\,\text{pF/m}$$
$$= \boxed{179\,\text{pF/m}}$$

***53 ••**

Picture the Problem We can use the expression for the capacitance of a parallel-plate capacitor of variable area and the geometry of the figure to express the capacitance of the goniometer.



The capacitance of the parallel-plate capacitor is given by:

$$C = \frac{\epsilon_0\,(A - \Delta A)}{d}$$

The area of the plates is:

$$A = \pi\left(R_2^2 - R_1^2\right)\frac{\theta}{2\pi} = \left(R_2^2 - R_1^2\right)\frac{\theta}{2}$$

If the top plate rotates through an angle $\Delta\theta$, then the area is reduced by:

$$\Delta A = \pi\left(R_2^2 - R_1^2\right)\frac{\Delta\theta}{2\pi} = \left(R_2^2 - R_1^2\right)\frac{\Delta\theta}{2}$$

Substitute for A and ΔA in the expression for C to obtain:

$$C = \frac{\epsilon_0}{d}\left[\left(R_2^2 - R_1^2\right)\frac{\theta}{2} - \left(R_2^2 - R_1^2\right)\frac{\Delta\theta}{2}\right]$$
$$= \boxed{\frac{\epsilon_0\left(R_2^2 - R_1^2\right)}{2d}(\theta - \Delta\theta)}$$

54 ••

Picture the Problem Let C be the capacitance of the capacitor when the pressure is P and C' be the capacitance when the pressure is $P + \Delta P$. We'll assume that (*a*) the change in the thickness of the plates is small, and (*b*) the total volume of material between the plates is conserved. We can use the expression for the capacitance of a dielectric-filled parallel-plate capacitor and the definition of Young's modulus to express the change in the capacitance ΔC of the given capacitor when the pressure on its plates is increased by ΔP.

Express the change in capacitance resulting from the decrease in separation of the capacitor plates by Δd:

$$\Delta C = C' - C = \frac{\kappa \in_0 A'}{d - \Delta d} - \frac{\kappa \in_0 A}{d}$$

Because the volume is constant:

$$A'd' = Ad$$

or

$$A' = \left(\frac{d}{d'}\right)A = \left(\frac{d}{d - \Delta d}\right)A$$

Substitute for A' in the expression for ΔC and simplify to obtain:

$$\begin{aligned}\Delta C &= \frac{\kappa \in_0 A}{d - \Delta d}\left(\frac{d}{d - \Delta d}\right) - \frac{\kappa \in_0 A}{d} \\ &= \frac{\kappa \in_0 A}{d(d - \Delta d)^2}d^2 - \frac{\kappa \in_0 A}{d} \\ &= \frac{\kappa \in_0 A}{d}\left[\frac{d^2}{(d - \Delta d)^2} - 1\right] \\ &= C\left[\frac{d^2}{(d - \Delta d)^2} - 1\right]\end{aligned}$$

From the definition of Young's modulus:

$$\frac{\Delta d}{d} = -\frac{P}{Y} \Rightarrow \Delta d = -\left(\frac{P}{Y}\right)d$$

Substitute for Δd in the expression for ΔC to obtain:

$$\begin{aligned}\Delta C &= \frac{\kappa \in_0 A}{d}\left[\frac{d^2}{\left\{d + \left(\frac{P}{Y}\right)d\right\}^2} - 1\right] \\ &= C\left[\left\{1 + \left(\frac{P}{Y}\right)\right\}^{-2} - 1\right]\end{aligned}$$

Expand $\left(1 - \frac{P}{Y}\right)^{-2}$ binomially to obtain:

$$\left(1 - \frac{P}{Y}\right)^{-2} = 1 - 2\frac{P}{Y} + 3\left(\frac{P}{Y}\right)^2 + \ldots$$

Provided $P << Y$:

$$\left(1-\frac{P}{Y}\right)^{-2} \approx 1-2\frac{P}{Y}$$

Substitute in the expression for ΔC and simplify to obtain:

$$\Delta C = C\left[1-2\frac{P}{Y}-1\right] = \boxed{-2\frac{P}{Y}C}$$

Spherical Capacitors

***55 ••**

Picture the Problem We can use the definition of capacitance and the expression for the potential difference between charged concentric spherical shells to show that $C = 4\pi \in_0 R_1R_2/(R_2 - R_1)$.

(*a*) Using its definition, relate the capacitance of the concentric spherical shells to their charge Q and the potential difference V between their surfaces:

$$C = \frac{Q}{V}$$

Express the potential difference between the conductors:

$$V = kQ\left(\frac{1}{R_1}-\frac{1}{R_2}\right) = kQ\frac{R_2 - R_1}{R_1R_2}$$

Substitute to obtain:

$$C = \frac{Q}{kQ\frac{R_2 - R_1}{R_1R_2}} = \frac{R_1R_2}{k(R_2 - R_1)}$$

$$= \boxed{\frac{4\pi \in_0 R_1R_2}{R_2 - R_1}}$$

(*b*) Because $R_2 = R_1 + d$:

$$R_1R_2 = R_1(R_1 + d)$$
$$= R_1^2 + R_1 d$$
$$\approx R_1^2 = R^2$$

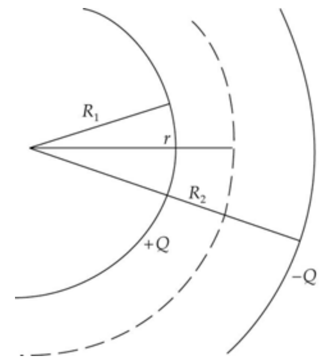
because d is small.

Substitute to obtain:

$$C \approx \frac{4\pi \in_0 R^2}{d} = \boxed{\frac{\in_0 A}{d}}$$

56 ••

Picture the Problem The diagram shows a partial cross-sectional view of the inner and outer spherical shells. By symmetry, the electric field is radial. We can apply Gauss's law to spherical surfaces of radii $r < R_1$, $R_1 < r < R_2$, and $r > R_2$ to find the electric field and, hence, the energy density in these regions.



(*a*) Apply Gauss's law to a spherical surface of radius $r < R_1$ to obtain:

$$E_r\left(4\pi r^2\right) = \frac{Q_{\text{inside}}}{\epsilon_0} = 0$$

and

$$E_{r<R_1} = \boxed{0}$$

Because $E = 0$ for $r < R_1$:

$$u_{r<R_1} = \boxed{0}$$

Apply Gauss's law to a spherical surface of radius $R_1 < r < R_2$ to obtain:

$$E_r\left(4\pi r^2\right) = \frac{Q_{\text{inside}}}{\epsilon_0} = \frac{Q}{\epsilon_0}$$

Solve for E_r to obtain:

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2} = \boxed{\frac{kQ}{r^2}}$$

Express the energy density in the region $R_1 < r < R_2$:

$$u = \tfrac{1}{2}\epsilon_0 E_r^2 = \tfrac{1}{2}\epsilon_0\left(\frac{kQ}{r^2}\right)^2$$

$$= \boxed{\frac{k^2\epsilon_0 Q^2}{2r^4}}$$

Apply Gauss's law to a cylindrical surface of radius $r > R_2$ to obtain:

$$E_r\left(4\pi r^2\right) = \frac{Q_{\text{inside}}}{\epsilon_0} = 0$$

and

$$E_{r>R_2} = \boxed{0}$$

Because $E = 0$ for $r > R_2$:

$$u_{r>R_2} = \boxed{0}$$

(*b*) Express the energy in the electrostatic field in a spherical shell of radius *r*, thickness *dr*, and volume $4\pi r^2 dr$ between the conductors:

$$dU = 4\pi r^2 u(r)dr = 4\pi r^2\left(\frac{k^2 \in_0 Q^2}{2r^4}\right)dr$$
$$= \boxed{\frac{kQ^2}{2r^2}dr}$$

(*c*) Integrate *dU* from $r = R_1$ to R_2 to obtain:

$$U = \frac{kQ^2}{2}\int_{R_1}^{R_2}\frac{dr}{r^2} = \frac{kQ^2(R_2 - R_1)}{2R_1R_2}$$
$$= \boxed{\frac{1}{2}Q^2\left(\frac{R_2 - R_1}{4\pi \in_0 R_1R_2}\right)}$$

Note that the quantity in parentheses is $1/C$, so we have $U = \frac{1}{2}Q^2/C$.

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Picture the Problem We know, from Gauss's law, that the field inside the shell is zero. Applying Gauss's law to a spherical surface of radius $R > r$ will allow us to find the energy density in this region. We can then express the energy in the electrostatic field in a spherical shell of radius *R*, thickness *dR*, and volume $4\pi R^2 dR$ outside the spherical shell and find the total energy in the electric field by integrating from *r* to ∞. If we then integrate the same expression from *r* to *R* we can find the radius *R* of the sphere such that half the total electrostatic field energy of the system is contained within that sphere.

Apply Gauss's law to a spherical shell of radius $R > r$ to obtain:

$$E_r\left(4\pi R^2\right) = \frac{Q_{\text{inside}}}{\in_0} = \frac{Q}{\in_0}$$

Solve for E_r outside the spherical shell:

$$E_r = \frac{kQ}{R^2}$$

Express the energy density in the region $R > r$:

$$u = \tfrac{1}{2}\in_0 E_R^2 = \tfrac{1}{2}\in_0\left(\frac{kQ}{R^2}\right)^2 = \frac{k^2 \in_0 Q^2}{2R^4}$$

Express the energy in the electrostatic field in a spherical shell of radius *R*, thickness *dR*, and volume $4\pi R^2 dR$ outside the spherical shell:

$$dU = 4\pi R^2 u(R)dR$$
$$= 4\pi R^2\left(\frac{k^2 \in_0 Q^2}{2R^4}\right)dR$$
$$= \frac{kQ^2}{2R^2}dR$$

Integrate *dU* from *r* to ∞ to obtain:

$$U_{\text{tot}} = \frac{kQ^2}{2}\int_r^{\infty}\frac{dR}{R^2} = \frac{kQ^2}{2r}$$

Integrate dU from r to R to obtain:

$$U = \frac{kQ^2}{2}\int_r^R \frac{dR'}{R'^2} = \frac{kQ^2}{2}\left(\frac{1}{r} - \frac{1}{R}\right)$$

Set $U = \frac{1}{2}U_{\text{tot}}$ to obtain:

$$\frac{kQ^2}{2}\left(\frac{1}{r} - \frac{1}{R}\right) = \frac{kQ^2}{4r}$$

Solve for R:

$$R = \boxed{2r}$$

Disconnected and Reconnected Capacitors

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Picture the Problem Let C_1 represent the capacitance of the 2.0-μF capacitor and C_2 the capacitance of the 2nd capacitor. Note that when they are connected as described in the problem statement they are in parallel and, hence, share a common potential difference. We can use the equation for the equivalent capacitance of two capacitors in parallel and the definition of capacitance to relate C_2 to C_1 and to the charge stored in and the potential difference across the equivalent capacitor.

Using the definition of capacitance, find the charge on capacitor C_1:

$$Q_1 = C_1V = (2\,\mu\text{F})(12\,\text{V}) = 24\,\mu\text{C}$$

Express the equivalent capacitance of the two-capacitor system and solve for C_2:

$$C_{\text{eq}} = C_1 + C_2$$

and

$$C_2 = C_{\text{eq}} - C_1$$

Using the definition of capacitance, express C_{eq} in terms of Q_2 and V_2:

$$C_{\text{eq}} = \frac{Q_2}{V_2} = \frac{Q_1}{V_2}$$

where V_2 is the common potential difference (they are in parallel) across the two capacitors and Q_1 and Q_2 are the (equal) charges on the two capacitors.

Substitute to obtain:

$$C_2 = \frac{Q_1}{V_2} - C_1$$

Substitute numerical values and evaluate C_2:

$$C_2 = \frac{24\,\mu\text{C}}{4\,\text{V}} - 2\,\mu\text{F} = \boxed{4.00\,\mu\text{F}}$$

59 ••

Picture the Problem Because, when the capacitors are connected as described in the problem statement, they are in parallel, they will have the same potential difference across them. In part (*b*) we can find the energy lost when the connections are made by comparing the energies stored in the capacitors before and after the connections.

(*a*) Because the capacitors are in parallel:

$$V_{100} = V_{400} = \boxed{2.00\,\text{kV}}$$

(*b*) Express the energy lost when the connections are made in terms of the energy stored in the capacitors before and after their connection:

$$\Delta U = U_{\text{before}} - U_{\text{after}}$$

Express and evaluate U_{before}:

$$\begin{aligned} U_{\text{before}} &= U_{100} + U_{400} \\ &= \tfrac{1}{2}C_{100}V_{100}^2 + \tfrac{1}{2}C_{400}V_{400}^2 \\ &= \tfrac{1}{2}V^2\left(C_{100} + C_{400}\right) \\ &= \tfrac{1}{2}(2\,\text{kV})^2(500\,\text{pF}) \\ &= 1.00\,\text{mJ} \end{aligned}$$

Express and evaluate U_{after}:

$$\begin{aligned} U_{\text{after}} &= U_{100} + U_{400} \\ &= \tfrac{1}{2}C_{100}V_{100}^2 + \tfrac{1}{2}C_{400}V_{400}^2 \\ &= \tfrac{1}{2}V^2\left(C_{100} + C_{400}\right) \\ &= \tfrac{1}{2}(2\,\text{kV})^2(500\,\text{pF}) \\ &= 1.00\,\text{mJ} \end{aligned}$$

Substitute to obtain:

$$\Delta U = 1.00\,\text{mJ} - 1.00\,\text{mJ} = \boxed{0}$$

***60** ••

Picture the Problem When the capacitors are reconnected, each will have the charge it acquired while they were connected in series across the 12-V battery and we can use the definition of capacitance and their equivalent capacitance to find the common potential difference across them. In part (*b*) we can use $U = \tfrac{1}{2}CV^2$ to find the initial and final energy stored in the capacitors.

(*a*) Using the definition of capacitance, express the potential difference across each capacitor when they are reconnected:

$$V = \frac{2Q}{C_{\text{eq}}} \qquad (1)$$

where Q is the charge on each capacitor *before* they are disconnected.

Find the equivalent capacitance of the two capacitors after they are connected in parallel:

$$C_{\text{eq}} = C_1 + C_2 = 4\,\mu\text{F} + 12\,\mu\text{F} = 16\,\mu\text{F}$$

Express the charge Q on each capacitor before they are disconnected:

$$Q = C'_{\text{eq}}V$$

Express the equivalent capacitance of the two capacitors connected in series:

$$C'_{\text{eq}} = \frac{C_1C_2}{C_1 + C_2} = \frac{(4\,\mu\text{F})(12\,\mu\text{F})}{4\,\mu\text{F} + 12\,\mu\text{F}} = 3\,\mu\text{F}$$

Substitute to find Q:

$$Q = (3\,\mu\text{F})(12\,\text{V}) = 36\,\mu\text{C}$$

Substitute in equation (1) and evaluate V:

$$V = \frac{2(36\,\mu\text{C})}{16\,\mu\text{F}} = \boxed{4.50\,\text{V}}$$

(*b*) Express and evaluate the energy stored in the capacitors initially:

$$U_{\text{i}} = \tfrac{1}{2}C'_{\text{eq}}V_{\text{i}}^2 = \tfrac{1}{2}(3\,\mu\text{F})(12\,\text{V})^2 = \boxed{216\,\mu\text{J}}$$

Express and evaluate the energy stored in the capacitors when they have been reconnected:

$$U_{\text{f}} = \tfrac{1}{2}C_{\text{eq}}V_{\text{f}}^2 = \tfrac{1}{2}(16\,\mu\text{F})(4.5\,\text{V})^2 = \boxed{162\,\mu\text{J}}$$

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Picture the Problem Let C_1 represent the capacitance of the 1.2-μF capacitor and C_2 the capacitance of the 2nd capacitor. Note that when they are connected as described in the problem statement they are in parallel and, hence, share a common potential difference. We can use the equation for the equivalent capacitance of two capacitors in parallel and the definition of capacitance to relate C_2 to C_1 and to the charge stored in and the potential difference across the equivalent capacitor. In part (*b*) we can use $U = \frac{1}{2}CV^2$ to find the energy before and after the connection was made and, hence, the energy lost when the connection was made.

(*a*) Using the definition of capacitance, find the charge on capacitor C_1:

$$Q_1 = C_1V = (1.2\,\mu\text{F})(30\,\text{V}) = 36\,\mu\text{C}$$

Express the equivalent capacitance of the two-capacitor system and solve for C_2:

$$C_{\text{eq}} = C_1 + C_2$$

and

$$C_2 = C_{eq} - C_1$$

Using the definition of capacitance, express C_{eq} in terms of Q_2 and V_2:

$$C_{eq} = \frac{Q_2}{V_2} = \frac{Q_1}{V_2}$$

where V_2 is the common potential difference (they are in parallel) across the two capacitors.

Substitute to obtain:

$$C_2 = \frac{Q_1}{V_2} - C_1$$

Substitute numerical values and evaluate C_2:

$$C_2 = \frac{36\,\mu\text{C}}{10\,\text{V}} - 1.2\,\mu\text{F} = \boxed{2.40\,\mu\text{F}}$$

(*b*) Express the energy lost when the connections are made in terms of the energy stored in the capacitors before and after their connection:

$$\begin{aligned}\Delta U &= U_{\text{before}} - U_{\text{after}}\\ &= \tfrac{1}{2}C_1V_1^2 - \tfrac{1}{2}C_{eq}V_f^2\\ &= \tfrac{1}{2}\left(C_1V_1^2 - C_{eq}V_f^2\right)\end{aligned}$$

Substitute numerical values and evaluate ΔU:

$$\Delta U = \tfrac{1}{2}\left[(1.2\,\mu\text{F})(30\,\text{V})^2 - (3.6\,\mu\text{F})(10\,\text{V})^2\right] = \boxed{360\,\mu\text{J}}$$

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Picture the Problem Because, when the capacitors are connected as described in the problem statement, they are in parallel, they will have the same potential difference across them. In part (*b*) we can find the energy lost when the connections are made by comparing the energies stored in the capacitors before and after the connections.

(*a*) Using the definition of capacitance, express the charge Q on the capacitors when they have been reconnected:

$$\begin{aligned}Q &= Q_{400} - Q_{100}\\ &= C_{400}V_{400} - C_{100}V_{100}\\ &= (C_{400} - C_{100})V\end{aligned}$$

where V is the common potential difference to which the capacitors have been charged.

Substitute numerical values to obtain:

$$Q = (400\,\text{pF} - 100\,\text{pF})(2\,\text{kV}) = 600\,\text{nC}$$

Using the definition of capacitance, relate the equivalent capacitance, charge, and final potential difference for the parallel connection:

$$Q = (C_1 + C_2)V_f$$

Solve for and evaluate V_f:

$$V_f = \frac{Q}{C_1 + C_2} = \frac{600\,n\text{C}}{100\,\text{pF} + 400\,\text{pF}} = \boxed{1.20\,\text{kV}}$$

across both capacitors.

(*b*) Express the energy lost when the connections are made in terms of the energy stored in the capacitors before and after their connection:

$$\begin{aligned}\Delta U &= U_{\text{before}} - U_{\text{after}} \\ &= \tfrac{1}{2}C_1V_1^2 + \tfrac{1}{2}C_2V_2^2 - \tfrac{1}{2}C_{\text{eq}}V_f^2 \\ &= \tfrac{1}{2}\left(C_1V_1^2 + C_2V_2^2 - C_{\text{eq}}V_f^2\right)\end{aligned}$$

Substitute numerical values and evaluate ΔU:

$$\Delta U = \tfrac{1}{2}\left[(100\,\text{pF})(2\,\text{kV})^2 + (400\,\text{pF})(2\,\text{kV})^2 - (500\,\text{pF})(1.2\,\text{kV})^2\right] = \boxed{0.640\,\text{mJ}}$$

63 ••

Picture the Problem When the capacitors are reconnected, each will have a charge equal to the difference between the charges they acquired while they were connected in parallel across the 12-V battery. We can use the definition of capacitance and their equivalent capacitance to find the common potential difference across them. In part (*b*) we can use $U = \frac{1}{2}CV^2$ to find the initial and final energy stored in the capacitors.

(*a*) Using the definition of capacitance, express the potential difference across the capacitors when they are reconnected:

$$V_f = \frac{Q_f}{C_{\text{eq}}} = \frac{Q_f}{C_1 + C_2} \qquad (1)$$

where Q_f is the common charge on the capacitors *after* they are reconnected.

Express the final charge Q_f on each capacitor:

$$Q_f = Q_2 - Q_1$$

Use the definition of capacitance to substitute for Q_2 and Q_1:

$$Q_f = C_2V - C_1V = (C_2 - C_1)V$$

Substitute in equation (1) to obtain:

$$V_f = \frac{C_2 - C_1}{C_1 + C_2}V$$

Substitute numerical values and evaluate V_f:

$$V_f = \frac{12\,\mu\text{F} - 4\,\mu\text{F}}{12\,\mu\text{F} + 4\,\mu\text{F}}(12\,\text{V}) = \boxed{6.00\,\text{V}}$$

(*b*) Express and evaluate the energy stored in the capacitors initially:

$$\begin{aligned} U_{\mathrm{i}} &= \tfrac{1}{2}C_1V^2 + \tfrac{1}{2}C_2V^2 \\ &= \tfrac{1}{2}V^2\left(C_1 + C_2\right) \\ &= \tfrac{1}{2}\left(12\,\mathrm{V}\right)^2\left(12\,\mu\mathrm{F} + 4\,\mu\mathrm{F}\right) \\ &= \boxed{1.15\,\mathrm{mJ}} \end{aligned}$$

Express and evaluate the energy stored in the capacitors when they have been reconnected:

$$\begin{aligned} U_{\mathrm{f}} &= \tfrac{1}{2}C_1V_{\mathrm{f}}^2 + \tfrac{1}{2}C_2V_{\mathrm{f}}^2 \\ &= \tfrac{1}{2}V_{\mathrm{f}}^2\left(C_1 + C_2\right) \\ &= \tfrac{1}{2}\left(6\,\mathrm{V}\right)^2\left(12\,\mu\mathrm{F} + 4\,\mu\mathrm{F}\right) \\ &= \boxed{0.288\,\mathrm{mJ}} \end{aligned}$$

***64** ••

Picture the Problem Let the numeral 1 refer to the 20-pF capacitor and the numeral 2 to the 50-pF capacitor. We can use conservation of charge and the fact that the connected capacitors will have the same potential difference across them to find the charge on each capacitor. We can decide whether electrostatic potential energy is gained or lost when the two capacitors are connected by calculating the change ΔU in the electrostatic energy during this process.

(*a*) Using the fact that no charge is lost in connecting the capacitors, relate the charge Q initially on the 20-pF capacitor to the charges on the two capacitors when they have been connected:

$$Q = Q_1 + Q_2 \qquad (1)$$

Because the capacitors are in parallel, the potential difference across them is the same:

$$V_1 = V_2 \Rightarrow \frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

Solve for Q_1 to obtain:

$$Q_1 = \frac{C_1}{C_2}Q_2$$

Substitute in equation (1) and solve for Q_2 to obtain:

$$Q_2 = \frac{Q}{1 + C_1/C_2} \qquad (2)$$

Use the definition of capacitance to find the charge Q initially on the 20-pF capacitor:

$$Q = C_1V = \left(20\,\mathrm{pF}\right)\left(3\,\mathrm{kV}\right) = 60\,\mathrm{nC}$$

Substitute in equation (2) and evaluate Q_2:

$$Q_2 = \frac{60\,\text{nC}}{1 + 20\,\text{pF}/50\,\text{pF}} = \boxed{42.9\,\text{nC}}$$

Substitute in equation (1) to obtain:

$$Q_1 = Q - Q_2$$
$$= 60\text{nC} - 42.9\,\text{nC} = \boxed{17.1\text{nC}}$$

(*b*) Express the change in the electrostatic potential energy of the system when the two capacitors are connected:

$$\Delta U = U_\text{f} - U_\text{i}$$
$$= \frac{Q^2}{2C_\text{eq}} - \frac{Q^2}{2C_1}$$
$$= \frac{Q^2}{2}\left(\frac{1}{C_\text{eq}} - \frac{1}{C_1}\right)$$

Substitute numerical values and evaluate ΔU:

$$\Delta U = \frac{(60\,\text{nC})^2}{2}\left(\frac{1}{70\,\text{pF}} - \frac{1}{20\,\text{pF}}\right)$$
$$= -64.3\,\mu\text{J}$$

$\boxed{\text{Because } \Delta U < 0\text{, electrostatic energy is lost when the two capacitors are connected.}}$

65 •••

Picture the Problem Let upper case Qs refer to the charges before S_3 is closed and lower case qs refer to the charges after this switch is closed. We can use conservation of charge to relate the charges on the capacitors before S_3 is closed to their charges when this switch is closed. We also know that the sum of the potential differences around the circuit when S_3 is closed must be zero and can use this to obtain a fourth equation relating the charges on the capacitors after the switch is closed to their capacitances. Solving these equations simultaneously will yield the charges q_1, q_2, and q_3. Knowing these charges, we can use the definition of capacitance to find the potential difference across each of the capacitors.

(*a*) With S_1 and S_2 closed, but S_3 open, the charges on and the potential differences across the capacitors do not change and:

$$V_1 = V_2 = V_3 = \boxed{200\,\text{V}}$$

(*b*) When S_3 is closed, the charges can redistribute; express the conditions on the charges that must be satisfied as a result of this

$$q_2 - q_1 = Q_2 - Q_1,$$
$$q_3 - q_2 = Q_3 - Q_2,$$

and

redistribution: $q_1 - q_3 = Q_1 - Q_3$.

Express the condition on the potential differences that must be satisfied when S_3 is closed:

$$V_1 + V_2 + V_3 = 0$$

where the subscripts refer to the three capacitors.

Use the definition of capacitance to eliminate the potential differences:

$$\frac{q_1}{C_1} + \frac{q_2}{C_2} + \frac{q_3}{C_3} = 0 \qquad (1)$$

Use the definition of capacitance to find the initial charge on each capacitor:

$$Q_1 = C_1 V = (2\,\mu\text{F})(200\,\text{V}) = 400\,\mu\text{C},$$
$$Q_2 = C_2 V = (4\,\mu\text{F})(200\,\text{V}) = 800\,\mu\text{C},$$

and

$$Q_3 = C_3 V = (6\,\mu\text{F})(200\,\text{V}) = 1200\,\mu\text{C}$$

Let $Q = Q_1$. Then: $Q_2 = 2Q$ and $Q_3 = 3Q$

Express q_2 and q_3 in terms of q_1 and Q:

$$q_2 = Q + q_1 \qquad (2)$$

and

$$q_3 = q_1 + 2Q \qquad (3)$$

Substitute in equation (1) to obtain:

$$\frac{q_1}{C_1} + \frac{Q + q_1}{C_2} + \frac{q_1 + 2Q}{C_3} = 0$$

or

$$\frac{q_1}{2\,\mu\text{F}} + \frac{Q + q_1}{4\,\mu\text{F}} + \frac{q_1 + 2Q}{6\,\mu\text{F}} = 0$$

Solve for and evaluate q_1 to obtain:

$$q_1 = -\tfrac{7}{11}Q = -\tfrac{7}{11}(400\,\mu\text{C}) = \boxed{-254\,\mu\text{C}}$$

Substitute in equation (2) to obtain:

$$q_2 = 400\,\mu\text{C} - 254\,\mu\text{C} = \boxed{146\,\mu\text{C}}$$

Substitute in equation (3) to obtain:

$$q_3 = -254\,\mu\text{C} + 2(400\,\mu\text{C}) = \boxed{546\,\mu\text{C}}$$

(*c*) Use the definition of capacitance to find the potential difference across each capacitor with S_3 closed:

$$V_1 = \frac{q_1}{C_1} = \frac{-254\,\mu\text{C}}{2\,\mu\text{F}} = \boxed{-127\,\text{V}},$$

$$V_2 = \frac{q_2}{C_2} = \frac{146\,\mu\text{C}}{4\,\mu\text{F}} = \boxed{36.5\,\text{V}},$$

and

$$V_3 = \frac{q_3}{C_3} = \frac{546\,\mu\text{C}}{6\,\mu\text{F}} = \boxed{91.0\,\text{V}}$$

***66 ••**

Picture the Problem We can use the expression for the energy stored in a capacitor to express the ratio of the energy stored in the system after the discharge of the first capacitor to the energy stored in the system prior to the discharge.

Express the energy U initially stored in the capacitor whose capacitance is C:

$$U = \frac{Q^2}{2C}$$

The energy U' stored in the two capacitors after the first capacitor has discharged is:

$$U' = \frac{\left(\frac{Q}{2}\right)^2}{2C} + \frac{\left(\frac{Q}{2}\right)^2}{2C} = \frac{Q^2}{4C}$$

Express the ratio of U' to U:

$$\frac{U'}{U} = \frac{\frac{Q^2}{4C}}{\frac{Q^2}{2C}} = \frac{1}{2} \Rightarrow U' = \boxed{\tfrac{1}{2}U}$$

Dielectrics

67 •

Picture the Problem The capacitance of a parallel-plate capacitor filled with a dielectric of constant κ is given by $C = \frac{\kappa \in_0 A}{d}$.

Relate the capacitance of the parallel-plate capacitor to the area of its plates, their separation, and the dielectric constant of the material between the plates:

$$C = \frac{\kappa \in_0 A}{d}$$

Substitute numerical values and evaluate C:

$$C = \frac{2.3\left(8.85\times10^{-12}\,\mathrm{C^2/N\cdot m^2}\right)\left(400\,\mathrm{cm^2}\right)}{0.3\,\mathrm{mm}} = \boxed{2.71\,\mathrm{nF}}$$

68 ••

Picture the Problem The capacitance of a cylindrical capacitor is given by $C = 2\pi\kappa \in_0 L/\ln(r_2/r_1)$, where L is its length and r_1 and r_2 the radii of the inner and outer conductors. We can use this expression, in conjunction with the definition of capacitance, to express the potential difference between the wire and the cylindrical shell in the Geiger tube. Because the electric field E in the tube is related to the linear charge density λ on the wire according to $E = 2k\lambda/\kappa r$, we can use this expression to find $2k\lambda/\kappa$ for $E = E_{max}$. In part (b) we'll use this relationship to find the charge per unit length λ on the wire.

(*a*) Use the definition of capacitance and the expression for the capacitance of a cylindrical capacitor to express the potential difference between the wire and the cylindrical shell in the tube:

$$\Delta V=\frac{Q}{C}=\frac{Q}{\dfrac{2\pi\kappa\in_0 L}{\ln(R/r)}}$$
$$=\frac{2\lambda}{4\pi\in_0\kappa}\ln\left(\frac{R}{r}\right)=\frac{2k\lambda}{\kappa}\ln\left(\frac{R}{r}\right)$$

where λ is the linear charge density, κ is the dielectric constant of the gas in the Geiger tube, r is the radius of the wire, and R the radius of the coaxial cylindrical shell of length L.

Express the electric field at a distance r greater than its radius from the center of the wire:

$$E=\frac{2k\lambda}{\kappa r}$$

Solve for $2k\lambda/\kappa$:

$$\frac{2k\lambda}{\kappa}=Er \qquad (1)$$

Noting that E is a maximum at r = 0.2 mm, evaluate $2k\lambda/\kappa$:

$$\frac{2k\lambda}{\kappa}=E_{\text{max}}r=\left(2\times10^{6}\text{ V/m}\right)\left(0.2\,\text{mm}\right)$$
$$=400\,\text{V}$$

Substitute and evaluate ΔV_{max}:

$$\Delta V_{\text{max}}=\left(400\,\text{V}\right)\ln\left(\frac{1.5\,\text{cm}}{0.2\,\text{mm}}\right)=\boxed{1.73\,\text{kV}}$$

(*b*) Solve equation (1) for λ:

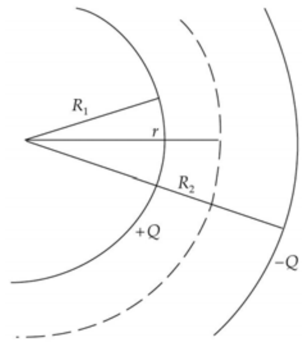
$$\lambda=\frac{E_{\text{max}}\kappa r}{2k}$$

Substitute numerical values and evaluate λ:

$$\lambda=\frac{1.8\left(2\times10^{6}\text{ V/m}\right)\left(0.2\,\text{mm}\right)}{2\left(8.99\times10^{9}\text{ N}\cdot\text{m}^2/\text{C}^2\right)}$$
$$=\boxed{40.0\,\text{nC/m}}$$

69 ••

Picture the Problem The diagram shows a partial cross-sectional view of the inner and outer spherical shells. By symmetry, the electric field is radial. We can apply Gauss's law to spherical surfaces of radii $r < R_1$, $R_1 < r < R_2$, and $r > R_2$ to find the electric field and, hence, the energy density in these regions.



(*a*) Apply Gauss's law to a spherical surface of radius $r < R_1$ to obtain:

$$E_r\left(4\pi r^2\right)=\frac{Q_{\text{inside}}}{\kappa\in_0}=0$$

and

$$E_{r<R_1}=\boxed{0}$$

Because $E = 0$ for $r < R_1$:

$$u_{r<R_1}=\boxed{0}$$

Apply Gauss's law to a spherical surface of radius $R_1 < r < R_2$ to obtain:

$$E_r\left(4\pi r^2\right)=\frac{Q_{\text{inside}}}{\kappa\in_0}=\frac{Q}{\kappa\in_0}$$

Solve for E_r to obtain:

$$E_r=\frac{Q}{4\pi\kappa\in_0 r^2}=\frac{kQ}{\kappa r^2}$$

Express the energy density in the region $R_1 < r < R_2$:

$$u=\tfrac{1}{2}\kappa\in_0 E_r^2=\tfrac{1}{2}\kappa\in_0\left(\frac{kQ}{\kappa r^2}\right)^2$$

$$=\frac{k^2\in_0 Q^2}{2\kappa r^4}$$

Apply Gauss's law to a cylindrical surface of radius $r > R_2$ to obtain:

$$E_r\left(4\pi r^2\right)=\frac{Q_{\text{inside}}}{\kappa\in_0}=0$$

and

$$E_{r>R_2}=\boxed{0}$$

Because $E = 0$ for $r > R_2$:

$$u_{r>R_2}=\boxed{0}$$

(*b*) Express the energy in the electrostatic field in a spherical shell of radius *r*, thickness *dr*, and volume $4\pi r^2 dr$ between the conductors:

$$dU = 4\pi r^2 u(r) dr$$
$$= 4\pi r^2 \left(\frac{k^2 \kappa \in_0 Q^2}{2\kappa^2 r^4} \right) dr$$
$$= \boxed{\frac{kQ^2}{2\kappa r^2} dr}$$

(*c*) Integrate *dU* from $r = R_1$ to R_2 to obtain:

$$U = \frac{kQ^2}{2\kappa} \int_{R_1}^{R_2} \frac{dr}{r^2} =$$
$$= \frac{kQ^2 (R_2 - R_1)}{2\kappa R_1 R_2}$$
$$= \boxed{\frac{1}{2} Q^2 \left(\frac{R_2 - R_1}{4\pi\kappa\varepsilon_0 R_1 R_2} \right)}$$

Note that the quantity in parentheses is $1/C$, so we have $U = \frac{1}{2} Q^2 / C$.

70 ••

Picture the Problem We can use the relationship between the electric field between the plates of a capacitor, their separation, and the potential difference between them to find the minimum plate separation. We can use the expression for the capacitance of a dielectric-filled parallel-plate capacitor to determine the necessary area of the plates.

(*a*) Relate the electric field of the capacitor to the potential difference across its plates:

$$E = \frac{V}{d}$$

where *d* is the plate separation.

Solve for *d*:

$$d = \frac{V}{E}$$

Noting that d_{min} corresponds to E_{max}, evaluate d_{min}:

$$d_{min} = \frac{V}{E_{max}} = \frac{2000\text{ V}}{4 \times 10^7 \text{ V/m}} = \boxed{50.0\ \mu\text{m}}$$

(*b*) Relate the capacitance of a parallel-plate capacitor to the area of its plates:

$$C = \frac{\kappa \in_0 A}{d}$$

Solve for *A*:

$$A = \frac{Cd}{\kappa \in_0}$$

Substitute numerical values and evaluate A:

$$A = \frac{(0.1\,\mu\text{F})(50\,\mu\text{m})}{24\left(8.85\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)} = 2.35\times10^{-2}\ \text{m}^2 = \boxed{235\,\text{cm}^2}$$

71 ••

Picture the Problem We can model this system as two capacitors in series, C_1 of thickness $d/4$ and C_2 of thickness $3d/4$ and use the equation for the equivalent capacitance of two capacitors connected in series.

Express the equivalent capacitance of the two capacitors connected in series:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2}$$

or

$$C_{\text{eq}} = \frac{C_1 C_2}{C_1 + C_2}$$

Relate the capacitance of C_1 to its dielectric constant and thickness:

$$C_1 = \frac{\kappa_1 \in_0 A}{\frac{1}{4}d} = \frac{4\kappa_1 \in_0 A}{d}$$

Relate the capacitance of C_2 to its dielectric constant and thickness:

$$C_2 = \frac{\kappa_2 \in_0 A}{\frac{3}{4}d} = \frac{4\kappa_2 \in_0 A}{3d}$$

Substitute and simplify to obtain:

$$C_{\text{eq}} = \frac{\left(\frac{4\kappa_1 \in_0 A}{d}\right)\left(\frac{4\kappa_2 \in_0 A}{3d}\right)}{\frac{4\kappa_1 \in_0 A}{d} + \frac{4\kappa_2 \in_0 A}{3d}} = \frac{\left(\frac{\kappa_1}{d}\right)\left(\frac{4\kappa_2}{3d}\right)}{\frac{3\kappa_1}{3d} + \frac{\kappa_2}{3d}} \in_0 A = \frac{\frac{4\kappa_1\kappa_2}{d}}{3\kappa_1 + \kappa_2} \in_0 A$$

$$= \frac{4\kappa_1\kappa_2}{3\kappa_1 + \kappa_2}\left(\frac{\in_0 A}{d}\right) = \boxed{\left(\frac{4\kappa_1\kappa_2}{3\kappa_1 + \kappa_2}\right)C_0}$$

***72** ••

Picture the Problem Let the charge on the capacitor with the air gap be Q_1 and the charge on the capacitor with the dielectric gap be Q_2. If the capacitances of the capacitors were initially C, then the capacitance of the capacitor with the dielectric inserted is $C' = \kappa C$. We can use the conservation of charge and the equivalence of the potential difference across the capacitors to obtain two equations that we can solve simultaneously for Q_1 and Q_2.

Apply conservation of charge during

$$Q_1 + Q_2 = 2Q \qquad (1)$$

the insertion of the dielectric to obtain:

Because the capacitors have the same potential difference across them:

$$\frac{Q_1}{C}=\frac{Q_2}{\kappa C} \qquad (2)$$

Solve equations (1) and (2) simultaneously to obtain:

$$Q_1=\boxed{\frac{2Q}{1+\kappa}} \text{ and } Q_2=\boxed{\frac{2Q\kappa}{1+\kappa}}$$

73 ••

Picture the Problem We can model this system as two capacitors in series, C_1 of thickness t and C_2 of thickness $d-t$ and use the equation for the equivalent capacitance of two capacitors connected in series.

Express the equivalent capacitance of the two capacitors connected in series:

$$\frac{1}{C_{\text{eq}}}=\frac{1}{C_1}+\frac{1}{C_2}$$

or

$$C_{\text{eq}}=\frac{C_1C_2}{C_1+C_2}$$

Relate the capacitance of C_1 to its dielectric constant and thickness:

$$C_1=\frac{\kappa\in_0 A}{t}$$

Relate the capacitance of C_2 to its dielectric constant and thickness:

$$C_2=\frac{\in_0 A}{d-t}$$

Substitute and simplify to obtain:

$$C_{\text{eq}}=\frac{\left(\frac{\kappa\in_0 A}{t}\right)\left(\frac{\in_0 A}{d-t}\right)}{\frac{\kappa\in_0 A}{t}+\frac{\in_0 A}{d-t}}=\frac{\left(\frac{\kappa}{t}\right)\left(\frac{1}{d-t}\right)}{\frac{\kappa}{t}+\frac{1}{d-t}}\in_0 A=\frac{\left(\frac{\kappa}{t}\right)\left(\frac{1}{d-t}\right)}{\frac{\kappa}{t}+\frac{1}{d-t}}\in_0 A$$

$$=\frac{\kappa}{\kappa(d-t)+t}\in_0 A=\boxed{\left(\frac{\kappa d}{\kappa(d-t)+t}\right)C_0}$$

74 ••

Picture the Problem Because $d << r$, we can model the membrane as a parallel-plate capacitor. We can use the definition of capacitance to find the charge on each side of the membrane in part (*b*) and the relationship between the potential difference across the membrane, its thickness, and the electric field in it to find the electric field called for in part (*c*).

(*a*) Express the capacitance of a parallel-plate capacitor:

$$C = \frac{\kappa \in_0 A}{d}$$

Substitute for the area of the plates:

$$C = \frac{2\pi\kappa \in_0 rL}{d} = \frac{\kappa rL}{2kd}$$

Substitute numerical values and evaluate C:

$$C = \frac{3\left(10^{-5}\,\text{m}\right)\left(0.1\,\text{m}\right)}{2\left(8.99\times10^{9}\,\text{N}\cdot\text{m}^2/\text{C}^2\right)\left(10^{-8}\,\text{m}\right)} = \boxed{16.7\,\text{nF}}$$

(*b*) Use the definition of capacitance to find the charge on each side of the membrane:

$$Q = CV = \left(16.7\,\text{nF}\right)\left(70\,\text{mV}\right) = \boxed{1.17\,\text{nC}}$$

(*c*) Express the electric field through the membrane as a function of its thickness d and the potential difference V across it:

$$E = \frac{V}{d}$$

Substitute numerical values and evaluate E:

$$E = \frac{70\,\text{mV}}{10^{-8}\,\text{m}} = \boxed{7.00\,\text{MV/m}}$$

***75** ••

Picture the Problem The bound charge density is related to the dielectric constant and the free charge density according to $\sigma_\text{b} = \left(1 - \dfrac{1}{\kappa}\right)\sigma_\text{f}$.

Solve the equation relating σ_b, σ_f, and κ for κ to obtain:

$$\kappa = \frac{1}{1 - \sigma_\text{b}/\sigma_\text{f}}$$

(*a*) Evaluate this expression for $\sigma_\text{b}/\sigma_\text{f} = 0.8$:

$$\kappa = \frac{1}{1-0.8} = \boxed{5.00}$$

(*b*) Evaluate this expression for $\sigma_\text{b}/\sigma_\text{f} = 0.2$:

$$\kappa = \frac{1}{1-0.2} = \boxed{1.25}$$

(*c*) Evaluate this expression for $\sigma_\text{b}/\sigma_\text{f} = 0.98$:

$$\kappa = \frac{1}{1-0.98} = \boxed{50.0}$$

76 ••

Picture the Problem We can use the definition of the dielectric constant to find its value. In part (*b*) we can use the expression for the electric field in the space between the charged capacitor plates to find the area of the plates and in part (*c*) we can relate the surface charge densities to the induced charges on the plates.

(*a*) Using the definition of the dielectric constant, relate the electric field without a dielectric E_0 to the field with a dielectric E:

$$E = \frac{E_0}{\kappa}$$

Solve for and evaluate κ:

$$\kappa = \frac{E_0}{E} = \frac{2.5\times10^5\ \text{V/m}}{1.2\times10^5\ \text{V/m}} = \boxed{2.08}$$

(*b*) Relate the electric field in the region between the plates to the surface charge density of the plates:

$$E_0 = \frac{\sigma}{\epsilon_0} = \frac{Q/A}{\epsilon_0}$$

Solve for A:

$$A = \frac{Q}{E_0\ \epsilon_0}$$

Substitute numerical values and evaluate A:

$$A = \frac{10\,\text{nC}}{\left(2.5\times10^5\ \text{V/m}\right)\left(8.85\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)} = 4.52\times10^{-3}\ \text{m}^2 = \boxed{45.2\,\text{cm}^2}$$

(*c*) Relate the surface charge densities to the induced charges on the plates:

$$\sigma_b = \left(1 - \frac{1}{\kappa}\right)\sigma_f$$

or

$$\frac{\sigma_b}{\sigma_f} = \frac{Q_b}{Q_f} = 1 - \frac{1}{\kappa}$$

Solve for Q_b:

$$Q_b = \left(1 - \frac{1}{\kappa}\right)Q_f$$

Substitute numerical values and evaluate Q_b:

$$Q_b = \left(1 - \frac{1}{2.08}\right)(10\,\text{nC}) = \boxed{5.19\,\text{nC}}$$

***77 ••**

Picture the Problem We can model this parallel-plate capacitor as a combination of two capacitors C_1 and C_2 in series with capacitor C_3 in parallel.

Express the capacitance of two series-connected capacitors in parallel with a third:

$$C = C_3 + C_s \quad (1)$$

where

$$C_s = \frac{C_1C_2}{C_1 + C_2} \quad (2)$$

Express each of the capacitances C_1, C_2, and C_3 in terms of the dielectric constants, plate areas, and plate separations:

$$C_1 = \frac{\kappa_1 \in_0 \left(\frac{1}{2}A\right)}{\frac{1}{2}d} = \frac{\kappa_1 \in_0 A}{d},$$

$$C_2 = \frac{\kappa_2 \in_0 \left(\frac{1}{2}A\right)}{\frac{1}{2}d} = \frac{\kappa_2 \in_0 A}{d},$$

and

$$C_3 = \frac{\kappa_3 \in_0 \left(\frac{1}{2}A\right)}{d} = \frac{\kappa_3 \in_0 A}{2d}$$

Substitute in equation (2) to obtain:

$$C_s = \frac{\left(\frac{\kappa_1 \in_0 A}{d}\right)\left(\frac{\kappa_2 \in_0 A}{d}\right)}{\frac{\kappa_1 \in_0 A}{d} + \frac{\kappa_2 \in_0 A}{d}} = \frac{\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\left(\frac{\in_0 A}{d}\right)$$

Substitute in equation (1) to obtain:

$$C = \frac{\kappa_3 \in_0 A}{2d} + \frac{\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\left(\frac{\in_0 A}{d}\right) = \boxed{\left(\kappa_3 + \frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\right)\left(\frac{\in_0 A}{2d}\right)}$$

78 ••

Picture the Problem The electric field E between the plates of a parallel-plate capacitor is related to the potential difference V between the plates and their separation d according to $V = Ed$ and the electrostatic energy U depends on the electric field according to $U = \frac{1}{2} \in_0 E_0^2 Ad$. We can use these relationships to find E, V, and U with and without the dielectric in place.

(*a*) Relate the electric field E_0 to the potential difference V between the plates and the plate separation d:

$$E_0 = \frac{V}{d} = \frac{100\,\text{V}}{4\,\text{mm}} = \boxed{25.0\,\text{kV/m}}$$

Use the definition of energy density to relate the electrostatic energy U_0 to the volume of the space between the plates:

$$U_0 = u_0 Ad$$

Express the energy density in the electric field:

$$u_0 = \tfrac{1}{2}\in_0 E_0^2$$

Substitute to obtain:

$$U_0 = \tfrac{1}{2}\in_0 E_0^2 Ad$$

Substitute numerical values and evaluate U_0:

$$U_0 = \tfrac{1}{2}\left(8.85\times10^{-12}\ \mathrm{C^2/N\cdot m^2}\right)(25\,\mathrm{kV/m})^2 \times\left(600\,\mathrm{cm^2}\right)(4\,\mathrm{mm}) = \boxed{0.664\,\mu\mathrm{J}}$$

(*b*) With the dielectric in place the electric field becomes:

$$E = \frac{E_0}{\kappa} = \frac{25\,\mathrm{kV/m}}{4} = \boxed{6.25\,\mathrm{kV/m}}$$

(*c*) Relate the potential difference *V* to the electric field *E* and the separation of the plates:

$$V = Ed = (6.25\,\mathrm{kV/m})(4\,\mathrm{mm}) = \boxed{25.0\,\mathrm{V}}$$

(*d*) Relate the new electrostatic energy *U* to the initial electrostatic energy U_0 and the dielectric constant κ:

$$U = \frac{U_0}{\kappa} = \frac{0.664\,\mu\mathrm{J}}{4} = \boxed{0.166\,\mu\mathrm{J}}$$

79 •••

Picture the Problem We can use the definition of capacitance and the relationship between the electric field in the capacitor and the potential difference across its plates to express *C*. In part (*b*) we can use $\sigma_b = (1 - 1/\kappa)\sigma_f$ and $\kappa = 1 + (3/y_0)y$ to express the ratio σ_b/σ_f and evaluate it at $y = 0$ and $y = y_0$. The application of Gauss's law in part (*c*) will yield an expression for $\rho(y)$ within the dielectric that we can integrate in part (*d*) to find the total induced bound charge.

(*a*) Using its definition, express the capacitance of the parallel-plate capacitor:

$$C = \frac{Q}{V} = \frac{\sigma A}{V} \qquad (1)$$

Express the potential difference *V* between the plates in terms of the electric field *E* between the plates:

$$dV = Edy$$

Express the electric field in the region between the plates:

$$E=\frac{E_0}{\kappa(y)}=\frac{\sigma}{\in_0\kappa(y)}$$

Substitute to obtain:

$$dV=\frac{\sigma}{\in_0\kappa(y)}dy=\frac{\sigma}{\in_0\left\{1+\left(\frac{3}{y_0}\right)y\right\}}dy$$

Integrate from $y = 0$ to $y = y_0$:

$$\begin{aligned}V&=\frac{\sigma}{\in_0}\int_0^{y_0}\frac{dy}{1+(3/y_0)y}\\&=\frac{\sigma y_0}{3\in_0}\ln(1+3y/y_0)\Big|_0^{y_0}\\&=\frac{\sigma y_0}{3\in_0}\ln(4)\end{aligned}$$

Substitute in equation (1) and simplify to obtain:

$$C=\frac{\sigma A}{\frac{\sigma y_0}{3\in_0}\ln(4)}=\boxed{\frac{3\in_0 A}{y_0\ln(4)}}$$

(*b*) Relate σ_b to σ_f and κ:

$$\sigma_\mathrm{b}=\left(1-\frac{1}{\kappa}\right)\sigma_\mathrm{f}$$

and

$$\frac{\sigma_\mathrm{b}}{\sigma_\mathrm{f}}=1-\frac{1}{\kappa}$$

Substitute for κ to obtain:

$$\frac{\sigma_\mathrm{b}}{\sigma_\mathrm{f}}=1-\frac{1}{1+(3/y_0)y}$$

Evaluate $\sigma_\mathrm{b}/\sigma_\mathrm{f}$ at $y = 0$:

$$\left.\frac{\sigma_\mathrm{b}}{\sigma_\mathrm{f}}\right|_{y=0}=1-\frac{1}{1+(3/y_0)(0)}=\boxed{0}$$

Evaluate $\sigma_\mathrm{b}/\sigma_\mathrm{f}$ at $y = y_0$:

$$\left.\frac{\sigma_\mathrm{b}}{\sigma_\mathrm{f}}\right|_{y=y_0}=1-\frac{1}{1+(3/y_0)y_0}=\boxed{0.750}$$

(*c*) Consider a Gaussian surface of area *A* and width *dy* and recall that *E* into the surface is taken to be negative. Apply Gauss's law to obtain:

$$\begin{aligned}[E(y)-E(y+dy)]A&=\frac{Q_\mathrm{inside}}{\in_0}\\&=\frac{Ady\rho(y)}{\in_0}\end{aligned}$$

Divide both sides of the equation by dy:

$$\frac{[E(y)-E(y+dy)]}{dy}=\frac{\rho(y)}{\epsilon_0}$$

or

$$-\frac{dE}{dy}=\frac{\rho(y)}{\epsilon_0}$$

Solve for $\rho(y)$ to obtain:

$$\begin{aligned}\rho(y)&=-\epsilon_0\frac{dE}{dy}\\&=-\epsilon_0\frac{d}{dy}\left[\frac{\sigma}{\epsilon_0\,\kappa(y)}\right]\\&=-\sigma\frac{d}{dy}\left[\frac{1}{1+(3/y_0)y}\right]\\&=\boxed{\frac{3\sigma}{\left[y_0(1+3y/y_0)^2\right]}}\end{aligned}$$

(*d*) Integrate $\rho(y)$ from $y = 0$ to $y = y_0$ to obtain:

$$\rho=3\sigma\int_0^{y_0}\frac{dy}{\left[y_0(1+3y/y_0)^2\right]}$$

$=\boxed{-\frac{3}{4}\sigma}$, the charge per unit area in the dielectric, and just cancels out the induced surface charge density.

General Problems

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Picture the Problem We can use the expression $U_0=\frac{1}{2}C_{\text{eq}}V^2$ to express the total energy stored in the combination of four capacitors in terms of their equivalent capacitance C_{eq}.

The energy stored in one capacitor when it is connected to the 100-V battery is:

$$U_0=\tfrac{1}{2}CV^2$$

When the four capacitors are connected to the battery in some combination, the total energy stored in them is:

$$U=\tfrac{1}{2}C_{\text{eq}}V^2$$

Equate U and U_0 and solve for C_{eq} to obtain:

$$C_{\text{eq}}=C$$

The equivalent capacitance C' of two capacitors of capacitance C connected in series is their product divided by their sum:

$$C' = \frac{C^2}{C + C} = \tfrac{1}{2}C$$

If we connect two of the capacitors in series in parallel with the other two capacitors connected in series, their equivalent capacitance will be:

$$C_{\text{eq}} = C' + C' = \tfrac{1}{2}C + \tfrac{1}{2}C = C$$

Thus, a series combination of two of the capacitors in parallel with a series combination of the other two capacitors will result in total energy U_0 stored in all four capacitors.

***81** •

Picture the Problem We can use the equations for the equivalent capacitance of three capacitors connected in parallel and in series to find these equivalent capacitances.

(*a*) Express the equivalent capacitance of three capacitors connected in parallel:

$$C_{\text{eq}} = C_1 + C_2 + C_3$$

Substitute numerical values and evaluate C_{eq}:

$$C_{\text{eq}} = 2.0\,\mu\text{F} + 4.0\,\mu\text{F} + 8.0\,\mu\text{F} = \boxed{14.0\,\mu\text{F}}$$

(*b*) Express the equivalent capacitance of the three capacitors connected in series:

$$C_{\text{eq}} = \frac{C_1C_2C_3}{C_1C_2 + C_2C_3 + C_1C_3}$$

Substitute numerical values and evaluate C_{eq}:

$$C_{\text{eq}} = \frac{(2\,\mu\text{F})(4\,\mu\text{F})(8\,\mu\text{F})}{(2\,\mu\text{F})(4\,\mu\text{F}) + (4\,\mu\text{F})(8\,\mu\text{F}) + (2\,\mu\text{F})(8\,\mu\text{F})} = \boxed{1.14\,\mu\text{F}}$$

82 •

Picture the Problem We can first use the equation for the equivalent capacitance of two capacitors connected in parallel and then the equation for two capacitors connected in series to find the equivalent capacitance.

Find the equivalent capacitance of a 1.0-μF capacitor connected in parallel with a 2.0-μF capacitor:

$$C_{\text{eq},1} = C_1 + C_2 = 1.0\,\mu\text{F} + 2.0\,\mu\text{F} = 3.0\,\mu\text{F}$$

Find the equivalent capacitance of a 3.0-μF capacitor connected in series with a 6.0-μF capacitor:

$$\boldsymbol{C}_{\text{eq},2} = \frac{\boldsymbol{C}_{\text{eq},1}\boldsymbol{C}_6}{\boldsymbol{C}_{\text{eq},1} + \boldsymbol{C}_6} = \frac{(3.0\,\mu\text{F})(6.0\,\mu\text{F})}{3.0\,\mu\text{F} + 6.0\,\mu\text{F}} = \boxed{2.00\,\mu\text{F}}$$

83 •

Picture the Problem The charge Q and the charge density σ are independent of the separation of the plates and do not change during the process described in the problem statement. Because the electric field E depends on σ, it too is constant. We can use $U = \frac{1}{2}CV^2$ and the relationship between V and E, together with the expression for the capacitance of a parallel-plate capacitor, to show that $U \propto d$.

Express the energy stored in the capacitor in terms of its capacitance C and the potential difference across its plates:

$$U = \tfrac{1}{2}CV^2$$

Express V in terms of E:

$$V = Ed$$

where d is the separation of the plates.

Express the capacitance of a parallel-plate capacitor:

$$C = \frac{\kappa \in_0 A}{d}$$

Substitute to obtain:

$$U = \frac{1}{2}\frac{\kappa \in_0 A}{d}(Ed)^2 = \left(\tfrac{1}{2}\kappa \in_0 AE^2\right)d$$

Because $U \propto d$, to double U one must double d. Hence:

$$d_f = 2d = 2(0.5\,\text{mm}) = \boxed{1.00\,\text{mm}}$$

84 ••

Picture the Problem We can use the equations for the equivalent capacitance of capacitors connected in parallel and in series to find the single capacitor that will store the same amount of charge as each of the networks shown above.

(a) Find the capacitance of the two capacitors in parallel:

$$C_{\text{eq},1} = C_0 + C_0 = 2C_0$$

Find the capacitance equivalent to $2C_0$ in series with C_0:

$$C_{\text{eq},2} = \frac{C_{\text{eq},1}C_0}{C_{\text{eq},1} + C_0} = \frac{(2C_0)C_0}{2C_0 + C_0} = \boxed{\tfrac{2}{3}C_0}$$

(*b*) Find the capacitance of two capacitors of capacitance C_0 in parallel:

$$C_{\text{eq},1} = 2C_0$$

Find the capacitance equivalent to $2C_0$ in series with $2C_0$:

$$C_{\text{eq},2} = \frac{C_{\text{eq},1}C_0}{C_{\text{eq},1} + C_0} = \frac{(2C_0)(2C_0)}{2C_0 + 2C_0} = \boxed{C_0}$$

(*c*) Find the equivalent capacitance of three equal capacitors connected in parallel:

$$C_{\text{eq}} = C_0 + C_0 + C_0$$
$$= \boxed{3C_0}$$

***85** ••

Picture the Problem Note that with V applied between a and b, C_1 and C_3 are in series, and so are C_2 and C_4. Because in a series combination the potential differences across the two capacitors are inversely proportional to the capacitances, we can establish proportions involving the capacitances and potential differences for the left- and right-hand side of the network and then use the condition that $V_c = V_d$ to eliminate the potential differences and establish the relationship between the capacitances.

Letting Q represent the charge on capacitors 1 and 2, relate the potential differences across the capacitors to their common charge and capacitances:

$$V_1 = \frac{Q}{C_1}$$

and

$$V_3 = \frac{Q}{C_3}$$

Divide the first of these equations by the second to obtain:

$$\frac{V_1}{V_3} = \frac{C_3}{C_1} \quad (1)$$

Proceed similarly to obtain:

$$\frac{V_2}{V_4} = \frac{C_4}{C_2} \quad (2)$$

Divide equation (1) by equation (2) to obtain:

$$\frac{V_1V_4}{V_3V_2} = \frac{C_3C_2}{C_1C_4} \quad (3)$$

If $V_c = V_d$ then we must have:

$$V_1 = V_2 \text{ and } V_3 = V_4$$

Substitute in equation (3) and

$$\boxed{C_2C_3 = C_1C_4}$$

rearrange to obtain:

86 ••

Picture the Problem Because the spheres are identical, each will have half the charge of the initially charged sphere when they are connected. We can find the fraction of the initial energy that is dissipated by finding the energy stored initially and the energy stored when the two spheres are connected.

Express the fraction of the initial energy that is dissipated when the two spheres are connected:

$$f=\frac{U_{\mathrm{i}}-U_{\mathrm{f}}}{U_{\mathrm{i}}}=1-\frac{U_{\mathrm{f}}}{U_{\mathrm{i}}} \quad (1)$$

Express the initial energy of the sphere whose charge is Q:

$$U_{\mathrm{i}}=\frac{1}{2}\frac{Q^2}{C}$$

Relate the capacitance of a an isolated spherical conductor to its radius:

$$C=4\pi\in_0 R$$

Substitute to obtain:

$$U_{\mathrm{i}}=\frac{1}{2}\frac{Q^2}{4\pi\in_0 R}=\frac{1}{2}\frac{kQ^2}{R}$$

Express the energy of the connected spheres:

$$U_{\mathrm{f}}=\frac{1}{2}\frac{k(Q/2)^2}{R}+\frac{1}{2}\frac{k(Q/2)^2}{R}=\frac{1}{4}\frac{kQ^2}{R}$$

Substitute in equation (1) and simplify:

$$f=1-\frac{\frac{1}{4}\frac{kQ^2}{R}}{\frac{1}{2}\frac{kQ^2}{R}}=1-\frac{1}{2}=\boxed{\frac{1}{2}}$$

87 ••

Picture the Problem We can use the expression for the capacitance of a parallel-plate capacitor as a function of A and d to determine the effect on the capacitance of doubling the plate separation. We can use $V=Ed$ to determine the effect on the potential difference across the capacitor of doubling the plate separation. Finally, we can use $U=CV^2/2$ to determine the effect of doubling the plate separation on the energy stored in the capacitor.

(*a*) Express the capacitance of a capacitor whose plates are separated by a distance 2*d*:

$$C_{\mathrm{new}}=\boxed{\frac{\in_0 A}{2d}}$$

(*b*) Express the potential difference across a parallel-plate capacitor whose plates are separated by a distance *d*:

$$V = Ed$$

where the electric field *E* depends solely on the charge on the capacitor plates.

Express the new potential difference across the plates resulting from the doubling of their separation:

$$V_{\text{new}} = E(2d) = 2(Ed) = \boxed{2V}$$

(*c*) Relate the energy stored in a parallel-plate capacitor to the separation of the plates:

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\epsilon_0 A}{d}V^2$$

When the plate separation is doubled we have:

$$U_{\text{new}} = \frac{1}{2}\frac{\epsilon_0 A}{2d}(2V)^2 = \boxed{\frac{\epsilon_0 AV^2}{d}}$$

(*d*) Relate the work required to change the plate separation from *d* to 2*d* to the change in the electrostatic potential energy of the system:

$$W = \Delta U = U_{\text{new}} - U_{\text{i}}$$
$$= \frac{\epsilon_0 AV^2}{d} - \frac{\epsilon_0 AV^2}{2d}$$
$$= \boxed{\frac{\epsilon_0 AV^2}{2d}}$$

88 ••

Picture the Problem We can use the equation for the equivalent capacitance of two capacitors in series to relate C_0 to C' and the capacitance of the dielectric-filled parallel-plate capacitor and then solve the resulting equation for C'.

Express the equivalent capacitance of the system in terms of C' and C, where C is the dielectric-filled capacitor:

$$C_0 = \frac{C'C}{C'+C}$$

Solve for C' to obtain:

$$C' = \frac{C_0 C}{C - C_0}$$

Express the capacitance of the dielectric-filled capacitor:

$$C = \frac{\kappa \epsilon_0 A}{d} = \kappa C_0$$

Substitute to obtain:

$$C' = \frac{C_0(\kappa C_0)}{\kappa C_0 - C_0} = \boxed{\frac{\kappa}{\kappa - 1}C_0}$$

89 ••

Picture the Problem Modeling the Leyden jar as a parallel-plate capacitor, we can use the equation relating the capacitance of a parallel-plate capacitor to the area A and separation d of its plates to find the jar's capacitance. To find the maximum charge the jar can carry without undergoing dielectric breakdown we can use the definition of capacitance to express Q_{max} in terms of V_{max} … and then relate V_{max} to E_{max} using $V_{max} = E_{max}d$, where d is the thickness of the glass wall of the jar.

(*a*) Treating it as a parallel-plate capacitor, express the capacitance of the Leyden jar

$$C = \frac{\kappa \in_0 A}{d} = \frac{\kappa \in_0 (2\pi Rh)}{d} = \frac{4\pi \in_0 \kappa Rh}{2d} = \frac{\kappa Rh}{2kd}$$

where h is the height of the jar and R is its inside radius.

Substitute numerical values and evaluate C:

$$C = \frac{5(0.04\,\text{m})(0.4\,\text{m})}{2(8.99\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2)(2\times10^{-3}\,\text{m})} = \boxed{2.22\,\text{nF}}$$

(*b*) Using the definition of capacitance, relate the maximum charge of the capacitor to the breakdown voltage of the dielectric:

$$Q_{max} = CV_{max}$$

Express the breakdown voltage in terms of the dielectric strength and thickness of the dielectric:

$$V_{max} = E_{max}d$$

Substitute to obtain:

$$Q_{max} = CE_{max}d$$

Substitute numerical values and evaluate Q_{max}:

$$Q_{max} = (2.22\,\text{nF})(15\,\text{MV/m})(2\times10^{-3}\,\text{m}) = \boxed{66.6\,\mu\text{C}}$$

***90** ••

Picture the Problem The maximum voltage is related to the dielectric strength of the medium according to $V_{max} = E_{max}d$ and we can use the expression for the capacitance of a parallel-plate capacitor to determine the required area of the plates.

(*a*) Relate the maximum voltage that can be applied across this capacitor

$$V_{max} = E_{max}d$$

to the dielectric strength of silicon dioxide:

Substitute numerical values and evaluate V_{max}:

$$V_{max} = (8\times10^{6}\text{ V/m})(5\times10^{-6}\text{ m}) = \boxed{40.0\text{ V}}$$

(*b*) Relate the capacitance of a parallel-plate capacitor to area *A* of its plates:

$$C = \frac{\kappa \in_0 A}{d}$$

Solve for *A* to obtain:

$$A = \frac{Cd}{\kappa \in_0}$$

Substitute numerical values and evaluate *A*:

$$A = \frac{(10\text{ pF})(5\times10^{-6}\text{ m})}{3.8(8.85\times10^{-12}\text{ C}^2/\text{N}\cdot\text{m}^2)} = 1.49\times10^{-6}\text{ m}^2 = \boxed{1.49\text{ mm}^2}$$

(*c*) Express the number of capacitors *n* in terms of the area of a square 1 cm by 1cm and the area required for each capacitor:

$$n = \frac{(1\text{cm})^2}{A} = \frac{100\text{ mm}^2}{1.49\text{ mm}^2} \approx \boxed{67}$$

91 ••

Picture the Problem When the battery is removed, after having initially charged both capacitors, and the separation of one of the capacitors is doubled, the charge is redistributed subject to the condition that the total charge remains constant; i.e., $Q = Q_1 + Q_2$ where Q is the initial charge on both capacitors and Q_2 is the charge on the capacitor whose plate separation has been doubled. We can use the conservation of charge during the plate separation process and the fact that, because the capacitors are in parallel, they share a common potential difference.

Find the equivalent capacitance of the two 2-μF parallel-plate capacitors connected in parallel:

$$C_{eq} = 2\,\mu\text{F} + 2\,\mu\text{F} = 4\,\mu\text{F}$$

Use the definition of capacitance to find the charge on the equivalent capacitor:

$$Q = C_{eq}V = (4\,\mu\text{F})(100\text{ V}) = 400\,\mu\text{C}$$

Relate this total charge to charges distributed on capacitors 1 and 2 when the battery is removed and the separation of the plates of capacitor 2 is doubled:

$$Q = Q_1 + Q_2 \qquad (1)$$

Because the capacitors are in parallel:

$$V_1 = V_2$$

and

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2'} = \frac{Q_2}{\frac{1}{2}C_2} = \frac{2Q_2}{C_2}$$

Solve for Q_1 to obtain:

$$Q_1 = 2\left(\frac{C_1}{C_2}\right)Q_2 \qquad (2)$$

Substitute equation (2) in equation (1) and solve for Q_2 to obtain:

$$Q_2 = \frac{Q}{2(C_1/C_2)+1}$$

Substitute numerical values and evaluate Q_2:

$$Q_2 = \frac{400\ \mu\text{C}}{2(2\ \mu\text{F}/2\ \mu\text{F})+1} = \boxed{133\ \mu\text{C}}$$

Substitute in equation (1) or equation (2) and evaluate Q_1:

$$Q_1 = \boxed{267\ \mu\text{C}}$$

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Picture the Problem We can relate the electric field in the dielectric to the electric field between the capacitor's plates in the absence of a dielectric using $E = E_0/\kappa$. In part (*b*) we can express the potential difference between the plates as the sum of the potential differences across the dielectrics and then express the potential differences in terms of the electric fields in the dielectrics. In part (*c*) we can use our result from (*b*) and the definition of capacitance to express the capacitance of the dielectric-filled capacitor. In part (*d*) we can confirm the result of part (*c*) by using the addition formula for capacitors in series.

(*a*) Express the electric field E in a dielectric of constant κ in terms of the electric field E_0 in the absence of the dielectric:

$$E = \frac{E_0}{\kappa}$$

Express the electric field E_0 in the absence of the dielectrics:

$$E_0 = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

Substitute to obtain:

$$E = \frac{Q}{\kappa \in_0 A}$$

Use this relationship to express the electric fields in dielectrics whose constants are κ_1 and κ_2:

$$E_1 = \boxed{\frac{Q}{\kappa_1 \in_0 A}} \text{ and } E_2 = \boxed{\frac{Q}{\kappa_2 \in_0 A}}$$

(*b*) Express the potential difference between the plates as the sum of the potential differences across the dielectrics:

$$V = V_1 + V_2$$

Relate the potential differences to the electric fields and the thicknesses of the dielectrics:

$$V_1 = E_1 \frac{d}{2} = \frac{Qd}{2\kappa_1 \in_0 A}$$

and

$$V_2 = E_2 \frac{d}{2} = \frac{Qd}{2\kappa_2 \in_0 A}$$

Substitute and simplify to obtain:

$$V = \frac{Qd}{2\kappa_1 \in_0 A} + \frac{Qd}{2\kappa_2 \in_0 A}$$

$$= \boxed{\frac{Qd}{2 \in_0 A}\left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2}\right)}$$

(*c*) Use the definition of capacitance to obtain:

$$C = \frac{Q}{V} = \frac{Q}{\dfrac{Qd}{2 \in_0 A}\left(\dfrac{1}{\kappa_1} + \dfrac{1}{\kappa_2}\right)}$$

$$= \frac{2 \in_0 A}{d\left(\dfrac{\kappa_1 + \kappa_2}{\kappa_1 \kappa_2}\right)} = \frac{2 \in_0 A}{d}\left(\frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}\right)$$

$$= \boxed{2C_0\left(\frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}\right)}$$

where $C_0 = \in_0 A/d$.

(*d*) Express the equivalent capacitance C of capacitors C_1 and C_2 in series:

$$C = \frac{C_1 C_2}{C_1 + C_2}$$

Express C_1:

$$C_1 = \frac{\kappa_1 \in_0 A}{d/2} = \frac{2\kappa_1 \in_0 A}{d} = 2\kappa_1 C_0$$

Express C_2:

$$C_2 = \frac{\kappa_2 \in_0 A}{d/2} = \frac{2\kappa_2 \in_0 A}{d} = 2\kappa_2 C_0$$

Substitute to obtain:

$$C = \frac{(2\kappa_1 C_0)(2\kappa_2 C_0)}{2\kappa_1 C_0 + 2\kappa_2 C_0} = \boxed{2C_0\left(\frac{\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\right)},$$

a result in agreement with part (*c*).

93 ••

Picture the Problem Recall that within a conductor $E = 0$. We can use the definition of capacitance to express C in terms of the charge on the capacitor Q and the potential difference across the plates V. We can then express V in terms of E and the thickness of the air gap between the plates. Finally, we can express the electric field between the plates in terms of the charge on them and their area. Substitution in our expression for C will give us C in terms of $d - t$. In part (*b*) we can use the expression for the equivalent capacitance of two capacitors connected in series to derive the same expression for C.

(*a*) Use its definition to express the capacitance of this parallel-plate capacitor:

$$C = \frac{Q}{V}$$

where Q is the charge on the capacitor.

Relate the electric potential between the plates to the electric field between the plates:

$$V = E(d - t)$$

Express the electric field E between the plates but outside the metal slab:

$$E = \frac{\sigma}{\in_0} = \frac{Q}{\in_0 A}$$

Substitute and simplify to obtain:

$$C = \frac{Q}{E(d-t)} = \frac{Q}{\frac{Q}{\in_0 A}(d-t)} = \boxed{\frac{\in_0 A}{d-t}}$$

(*b*) Express the equivalent capacitance C of two capacitors C_1 and C_2 connected in series:

$$C = \frac{C_1 C_2}{C_1 + C_2}$$

Express the capacitances C_1 and C_2 of the plates separated by a and b, respectively:

$$C_1 = \frac{\in_0 A}{a}$$

and

$$C_2 = \frac{\epsilon_0 A}{b}$$

Substitute and simplify to obtain:

$$C = \frac{\left(\frac{\epsilon_0 A}{a}\right)\left(\frac{\epsilon_0 A}{b}\right)}{\frac{\epsilon_0 A}{a} + \frac{\epsilon_0 A}{b}} = \frac{\epsilon_0 A}{a+b}$$

Solve the constraint that $a + b + t = d$ for $a + b$ to obtain:

$$a + b = d - t$$

Substitute for $a + b$ to obtain:

$$C = \boxed{\frac{\epsilon_0 A}{d-t}}$$

***94** ••

Picture the Problem We can express the ratio of C_{eq} to C_0 to show that the capacitance with the dielectrics in place is $(\kappa_1 + \kappa_2)/2$ times greater than that of the capacitor in the absence of the dielectrics.

(*a*) $\boxed{\text{Because the capacitor plates are conductors, the potentials are the same across the entire upper and lower plates. Hence, the system is equivalent to two capacitors, each of area } A/2\text{, in parallel.}}$

(*b*) Relate the capacitance C_0, in the absence of the dielectrics, to the plate area and separation:

$$C_0 = \frac{\epsilon_0 A}{d}$$

Express the equivalent capacitance of capacitors C_1 and C_2, each with plate area $A/2$, connected in parallel:

$$\begin{aligned} C_{eq} &= C_1 + C_2 \\ &= \frac{\kappa_1 \epsilon_0 \left(\frac{1}{2}A\right)}{d} + \frac{\kappa_1 \epsilon_0 \left(\frac{1}{2}A\right)}{d} \\ &= \frac{\kappa_1 \epsilon_0 A}{2d}(\kappa_1 + \kappa_2) \end{aligned}$$

Express the ratio of C_{eq} to C_0 and simplify to obtain:

$$\frac{C_{eq}}{C_0} = \frac{\frac{\kappa_1 \epsilon_0 A}{2d}(\kappa_1 + \kappa_2)}{\frac{\epsilon_0 A}{d}} = \boxed{\tfrac{1}{2}(\kappa_1 + \kappa_2)}$$

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Picture the Problem We can use $U = Q^2/2C$ and the expression for the capacitance as a function of plate separation to express U as a function of x. Differentiation of this result with respect to x will yield dU. Because the work done in increasing the plate separation a distance dx equals the change in the electrostatic potential energy of the capacitor, we can evaluate F from dU/dx. Finally, we can express F in terms of Q and E by relating E to x using $E = Vx$ and using the definition of capacitance and the expression for the capacitance of a parallel-plate capacitor.

(*a*) Relate the electrostatic energy U stored in the capacitor to its capacitance C:

$$U = \frac{1}{2}\frac{Q^2}{C}$$

Express the capacitance as a function of the plate separation:

$$C = \frac{\epsilon_0 A}{x}$$

Substitute for C to obtain:

$$U = \boxed{\frac{Q^2}{2\epsilon_0 A}x}$$

(*b*) Use the result obtained in (*a*) to evaluate dU:

$$dU = \frac{dU}{dx}dx = \frac{d}{dx}\left[\frac{Q^2}{2\epsilon_0 A}x\right]dx$$

$$= \boxed{\frac{Q^2}{2\epsilon_0 A}dx}$$

(*c*) Relate the work needed to move one plate a distance dx to the change in the electrostatic potential energy of the system:

$$W = dU = Fdx$$

Solve for and evaluate F:

$$F = \frac{dU}{dx} = \frac{d}{dx}\left[\frac{Q^2}{2\epsilon_0 A}x\right] = \boxed{\frac{Q^2}{2\epsilon_0 A}}$$

(*d*) Express the electric field between the plates in terms of their separation and their potential difference:

$$E = \frac{V}{x}$$

Use the definition of capacitance to eliminate V:

$$E = \frac{Q}{Cx}$$

Use the expression for the capacitance of a parallel-plate capacitor to eliminate C:

$$E = \frac{Q}{\frac{\epsilon_0 A}{x}x} = \frac{Q}{\epsilon_0 A}$$

Substitute in our result from part (*c*) to obtain:

$$F = \frac{Q(\epsilon_0 AE)}{2\epsilon_0 A} = \boxed{\tfrac{1}{2}QE}$$

The field E is due to the sum of the fields from charges $+Q$ and $-Q$ on the opposite plates of the capacitor. Each plate produces a field ½ E and the force is the product of charge Q and the field ½ E.

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Picture the Problem We can model this capacitor as the equivalent of two capacitors connected in parallel. Let the numeral 1 denote the capacitor with the dielectric material whose constant is κ and the numeral 2 the air-filled capacitor.

(*a*) Express the equivalent capacitance of the two capacitors in parallel:

$$C(x) = C_1 + C_2 \tag{1}$$

Use the expression for the capacitance of a parallel-plate capacitor to express C_1:

$$C_1 = \frac{\kappa \epsilon_0 A_1}{d} = \frac{\kappa \epsilon_0 bx}{d}$$

Express the capacitance C_0 of the capacitor with the dielectric removed, i.e., $x = 0$:

$$C_0 = \frac{\epsilon_0 ab}{d}$$

Divide C_1 by C_0 to obtain:

$$\frac{C_1}{C_0} = \frac{\frac{\kappa \epsilon_0 bx}{d}}{\frac{\epsilon_0 ab}{d}} = \frac{\kappa x}{a}$$

or

$$C_1 = \frac{\kappa x}{a} C_0$$

Use the expression for the capacitance of a parallel-plate capacitor to express C_2:

$$C_2 = \frac{\epsilon_0 A_2}{d} = \frac{\epsilon_0 b(a-x)}{d}$$

Divide C_2 by C_0 to obtain:

$$\frac{C_2}{C_0} = \frac{\frac{\epsilon_0 b(a-x)}{d}}{\frac{\epsilon_0 ab}{d}} = \frac{a-x}{a}$$

or

$$C_2 = \frac{a-x}{a}C_0$$

Substitute in equation (1) and simplify to obtain:

$$C(x) = \frac{\kappa x}{a}C_0 + \frac{a-x}{a}C_0 = \frac{C_0}{a}[a+(\kappa-1)x] = \boxed{\frac{\in_0 b}{d}[a+(\kappa-1)x]}$$

(*b*) Evaluate C for $x = 0$:

$$C(0) = \frac{\in_0 b}{d}[a] = \frac{\in_0 ab}{d} = \boxed{C_0}$$

as expected.

Evaluate C for $x = a$:

$$C(a) = \frac{\in_0 b}{d}[a+(\kappa-1)a] = \boxed{\frac{\kappa \in_0 ab}{d}} \text{ as expected.}$$

***97 •••**

Picture the Problem We can model this capacitor as the equivalent of two capacitors connected in parallel, one with an air gap and other filled with a dielectric of constant κ. Let the numeral 1 denote the capacitor with the dielectric material whose constant is κ and the numeral 2 the air-filled capacitor.

(*a*) Using the hint, express the energy stored in the capacitor as a function of the equivalent capacitance C_{eq}:

$$U = \frac{1}{2}\frac{Q^2}{C_{eq}}$$

The capacitances of the two capacitors are:

$$C_1 = \frac{\kappa \in_0 ax}{d} \text{ and } C_2 = \frac{\in_0 a(a-x)}{d}$$

Because the capacitors are in parallel, C_{eq} is the sum of C_1 and C_2:

$$C_{eq} = C_1 + C_2 = \frac{\kappa \in_0 ax}{d} + \frac{\in_0 a(a-x)}{d} = \frac{\in_0 a}{d}(\kappa x + a - x) = \frac{\in_0 a}{d}[(\kappa-1)x + a]$$

Substitute for C_{eq} in the expression for U and simplify to obtain:

$$U = \boxed{\frac{Q^2 d}{2 \in_0 a\left[(\kappa-1)x+a\right]}}$$

(*b*) The force exerted by the electric field is given by:

$$\begin{aligned} F &= -\frac{dU}{dx} \\ &= -\frac{d}{dx}\left[\frac{1}{2}\frac{Q^2 d}{\in_0 a\left[(\kappa-1)x+a\right]}\right] \\ &= -\frac{Q^2 d}{2 \in_0 a}\frac{d}{dx}\left\{\left[(\kappa-1)x+a\right]^{-1}\right\} \\ &= \boxed{\frac{(\kappa-1)Q^2 d}{2a \in_0 \left[(\kappa-1)x+a\right]^2}} \end{aligned}$$

(*c*) Rewrite our result in (*b*) to obtain:

$$\begin{aligned} F &= \frac{(\kappa-1)Q^2\left(\dfrac{a \in_0}{d}\right)}{2\left(\dfrac{a \in_0}{d}\right)^2\left[(\kappa-1)x+a\right]^2} \\ &= \frac{(\kappa-1)Q^2\left(\dfrac{a \in_0}{d}\right)}{2C_{\text{eq}}^2} \\ &= \boxed{\frac{(\kappa-1)a \in_0 V^2}{2d}} \end{aligned}$$

Note that this expression is independent of x.

(*d*) $\boxed{\text{This force originates from the fringing fields around the edges of the capacitor. The effect of the force is to pull the dielectric into the space between the capacitor plates.}}$

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Picture the Problem Because capacitors connected in series have a common charge, we can find the charge on each capacitor by finding the charge on the equivalent capacitor. We can also find the total energy stored in the capacitors, with and without the dielectric inserted in one of them, by using $U = \frac{1}{2}C_{\text{eq}}V^2$. In part (*d*) we can use our knowledge of the charge on each capacitor and the definition of capacitance to the potential differences across them.

(*a*) Using the definition of capacitance, relate the charge on each capacitor to the equivalent

$$Q = Q_1 = Q_2 = C_{\text{eq}}V$$

capacitance:

Express the equivalent capacitance of two capacitors in series:

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$$

Substitute numerical values and evaluate C_{eq}:

$$C_{eq} = \frac{(4\,\mu\text{F})(4\,\mu\text{F})}{4\,\mu\text{F} + 4\,\mu\text{F}} = 2\,\mu\text{F}$$

Substitute numerical values and evaluate Q:

$$Q_1 = Q_2 = (2\,\mu\text{F})(24\,\text{V}) = \boxed{48.0\,\mu\text{C}}$$

(*b*) Express the energy U stored in the capacitors as a function of C_{eq} and V:

$$U = \tfrac{1}{2} C_{eq} V^2$$
$$= \tfrac{1}{2}(2\,\mu\text{F})(24\,\text{V})^2 = \boxed{576\,\mu\text{J}}$$

(*c*) Using the definition of capacitance, relate the charge on each capacitor to the new equivalent capacitance C_{eq}':

$$Q' = Q_1' = Q_2' = C_{eq}'V \qquad (1)$$

Express the new equivalent capacitance C_{eq}' when the dielectric of constant κ has been inserted between the plates of one of the capacitors:

$$C_{eq}' = \frac{C_1' C_2'}{C_1' + C_2'}$$

Letting the capacitor with the dielectric between its plates be denoted by the numeral 1, express C_1' and C_2':

$$C_1' = \kappa C_1$$
and
$$C_2' = C_2$$

Substitute to obtain:

$$C_{eq}' = \frac{\kappa C_1 C_2}{\kappa C_1 + C_2}$$

Substitute numerical values and evaluate C_{eq}':

$$C_{eq}' = \frac{4.2(4\,\mu\text{F})(4\,\mu\text{F})}{4.2(4\,\mu\text{F}) + 4\,\mu\text{F}} = 3.23\,\mu\text{F}$$

Substitute in equation (1) to obtain:

$$Q_1' = Q_2' = (3.23\,\mu\text{F})(24\,\text{V}) = \boxed{77.5\,\mu\text{C}}$$

(*d*) Express the potential difference across each capacitor in terms of its

$$V_1' = \frac{Q_1'}{C_1'} = \frac{Q_1'}{\kappa C_1} = \frac{77.5\,\mu\text{C}}{4.2(4\,\mu\text{F})} = \boxed{4.61\,\text{V}}$$

charge and capacitance:

and

$$V_2' = \frac{Q_2'}{C_2'} = \frac{Q_2'}{C_2} = \frac{77.5\,\mu\text{C}}{4\,\mu\text{F}} = \boxed{19.4\,\text{V}}$$

(*e*) Express the total stored energy in terms of the equivalent capacitance:

$$U = \tfrac{1}{2}C_{\text{eq}}'V^2$$
$$= \tfrac{1}{2}(3.23\,\mu\text{F})(24\,\text{V})^2 = \boxed{930\,\mu\text{J}}$$

99 ••

Picture the Problem We can find the work required to pull the glass plate out of the capacitor by finding the change in the electrostatic energy of the system as a consequence of the removal of the dielectric plate.

Express the change in the electrostatic energy of the system resulting from the removal of the glass plate:

$$W = \Delta U = U_{\text{f}} - U_{\text{i}}$$
$$= \frac{1}{2}\frac{Q^2}{C_0} - \frac{1}{2}\frac{Q^2}{C}$$

Express the capacitance C with the dielectric plate in place in terms of the dielectric constant κ and the air-only capacitance C_0:

$$C = \kappa C_0$$

where $C_0 = \dfrac{\epsilon_0 A}{d}$.

Substitute and factor to obtain:

$$W = \frac{1}{2}\frac{Q^2}{C_0} - \frac{1}{2}\frac{Q^2}{\kappa C_0} = \frac{Q^2}{2C_0}\left(1 - \frac{1}{\kappa}\right)$$

Use the definition of capacitance to relate the charge on the capacitor to the potential difference across its plates:

$$Q = CV = \kappa C_0 V$$

Substitute to obtain:

$$W = \frac{\kappa^2 C_0^2 V^2}{2C_0}\left(1 - \frac{1}{\kappa}\right) = \frac{\kappa^2 C_0 V^2}{2}\left(1 - \frac{1}{\kappa}\right)$$
$$= \frac{\kappa^2 \varepsilon_0 A V^2}{2d}\left(1 - \frac{1}{\kappa}\right)$$

Substitute numerical values and evaluate W:

$$W = \frac{(5)^2\left(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)\left(1\text{m}^2\right)(12\,\text{V})^2}{2\left(0.5\times10^{-2}\,\text{m}\right)}\left(1 - \frac{1}{5}\right) = \boxed{2.55\,\mu\text{J}}$$

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Picture the Problem The problem statement provides us with two conditions relating the potential between the plates of the capacitor and the charge on them. We can use the definition of capacitance to obtain simultaneous equations in *Q* and *V* and solve these equations to determine the capacitance of the capacitor and the initial and final voltages.

Using the definition of capacitance, relate the initial potential between the plates of the capacitor to the charge carried by these plates:

$$15\,\mu\text{C} = CV_\text{i}$$

Again using the definition of capacitance, express the relationship between the charge on the capacitor and the increased voltage:

$$18\,\mu\text{C} = CV_\text{f} = C(V_\text{i} + 6\,\text{V})$$

Divide the second of these equations by the first to obtain:

$$\frac{18\,\mu\text{C}}{15\,\mu\text{C}} = \frac{C(V_\text{i} + 6\,\text{V})}{CV_\text{i}}$$

or

$$6V_\text{i} = 5(V_\text{i} + 6\,\text{V})$$

Solve for *V* to obtain:

$$V_\text{i} = \boxed{30.0\,\text{V}}$$

and

$$V_\text{f} = V_\text{i} + 6\,\text{V} = \boxed{36.0\,\text{V}}$$

Substitute in either of the first two equations to obtain:

$$C = \boxed{0.500\,\mu\text{F}}$$

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Picture the Problem Let ℓ be the variable separation of the plates. We can use the definition of the work done in charging the capacitor to relate the force on the upper plate to the energy stored in the capacitor. Solving this expression for the force and substituting for the energy stored in a parallel-plate capacitor will yield an expression that we can use to decide whether the balance is stable. We can use this same expression and a condition for equilibrium to find the voltage required to balance the object whose mass is *M*.

(*a*) Express the work done in charging the capacitor (the energy stored in it) in terms of the force between the plates:

$$dW = dE = -F\,d\ell$$

or

$$F = -\frac{dE}{d\ell}$$

The energy stored in the capacitor is given by:

$$E = \frac{1}{2}CV^2 = \frac{1}{2}\left(\frac{\in_0 A}{\ell}\right)V^2$$

Differentiate E with respect to ℓ to obtain:

$$F = -\frac{d}{d\ell}\left[\frac{1}{2}\left(\frac{\in_0 A}{\ell}\right)V^2\right] = \left(\frac{\in_0 A}{2\ell^2}\right)V^2$$

Because $\boldsymbol{F}$ increases as ℓ decreases, a decrease in plate separation will unbalance the system and the balance is unstable.

(*b*) Apply $\sum F = 0$ to the object whose mass is M to obtain:

$$Mg - \left(\frac{\in_0 A}{2\ell^2}\right)V^2 = 0$$

Solve for V:

$$V = \boxed{\ell\sqrt{\frac{2Mg}{\in_0 A}}}$$

***102 •••**

Picture the Problem Recall that the dielectric strength of air is 3 MV/m. We can express the maximum energy to be stored in terms of the capacitance of the air-gap capacitor and the maximum potential difference between its plates. This maximum potential can, in turn, be expressed in terms of the maximum electric field (dielectric strength) possible in the air gap. We can solve the resulting equation for the volume of the space between the plates. In part (*b*) we can modify the equation we derive in part (*a*) to accommodate a dielectric with a constant other than 1.

(*a*) Express the energy stored in the capacitor in terms of its capacitance and the potential difference across it:

$$U_{\max} = \tfrac{1}{2}CV_{\max}^2$$

Express the capacitance of the air-gap parallel-plate capacitor:

$$C = \frac{\in_0 A}{d}$$

Relate the maximum potential difference across the plates to the maximum electric field between them:

$$V_{\max} = E_{\max}d$$

Substitute to obtain:

$$U_{\max} = \tfrac{1}{2}\left(\frac{\in_0 A}{d}\right)(E_{\max}d)^2 = \tfrac{1}{2}\in_0 (Ad)E_{\max}^2$$

$$= \tfrac{1}{2}\in_0 \upsilon E^2$$

where $\upsilon = Ad$ is the volume between the plates.

Solve for υ:

$$\upsilon = \frac{2U_{max}}{\epsilon_0 E_{max}^2} \qquad (1)$$

Substitute numerical values and evaluate υ:

$$\upsilon = \frac{2(100\,\text{kJ})}{(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)(3\,\text{MV/m})^2} = \boxed{2.51\times10^{3}\,\text{m}^3}$$

(*b*) With the dielectric in place equation (1) becomes:

$$\upsilon = \frac{2U_{max}}{\kappa \epsilon_0 E_{max}^2} \qquad (2)$$

Evaluate equation (2) with $\kappa = 5$ and $E_{max} = 3\times10^8$ V/m:

$$\upsilon = \frac{2(100\,\text{kJ})}{5(8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2)(3\times10^8\,\text{V/m})^2} = \boxed{5.02\times10^{-2}\,\text{m}^3}$$

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Picture the Problem We can use the definition of capacitance to find the charge on each capacitor in part (*a*). In part (*b*) we can express the total energy stored as the sum of the energy stored on the two capacitors … using our result from (*a*) for the charge on each capacitor. When the dielectric is removed in part (*c*) each capacitor will carry half the charge carried by the capacitor system previously and we can proceed as in (*b*). Knowing the total charge stored by the capacitors, we can use the definition of capacitance to find the final voltage across the two capacitors in part (*d*).

(*a*) Use the definition of capacitance to express the charge on each capacitor as a function of its capacitance:

$$Q_1 = C_1V = \boxed{(200\,\text{V})C_1}$$

and

$$Q_2 = C_2V = \kappa C_1V = \boxed{(200\,\text{V})\kappa C_1}$$

(*b*) Express the total stored energy of the capacitors as the sum of stored energy in each capacitor:

$$\begin{aligned} U &= U_1 + U_2 = \tfrac{1}{2}C_1V^2 + \tfrac{1}{2}C_2V^2 \\ &= \tfrac{1}{2}C_1V^2 + \tfrac{1}{2}\kappa C_1V^2 \\ &= \tfrac{1}{2}C_1V^2(1+\kappa) \\ &= \tfrac{1}{2}(200\,\text{V})^2C_1(1+\kappa) \\ &= \boxed{(2\times10^4\,\text{V}^2)(1+\kappa)C_1} \end{aligned}$$

(*c*) With the dielectric removed, each capacitor carries charge $Q/2$. Express the final energy stored by the capacitors under this condition:

$$U_f = \frac{1}{2}\frac{Q_1^2}{C_1} + \frac{1}{2}\frac{Q_2^2}{C_2} = \frac{1}{2}\frac{Q^2}{4C_1} + \frac{1}{2}\frac{Q^2}{4C_1} = \frac{Q^2}{4C_1}$$

Using the definition of capacitance, express the total charge carried by the capacitors with the dielectric in place in C_2:

$$Q = Q_1 + Q_2 = C_1V + C_2V = C_1V + \kappa C_1V = C_1V(1+\kappa) = (200\,\text{V})C_1(1+\kappa)$$

Substitute to obtain:

$$U_f = \frac{[(200\,\text{V})C_1(1+\kappa)]^2}{4C_1} = \boxed{(10^4\,\text{V}^2)C_1(1+\kappa)^2}$$

(*d*) Use the definition of capacitance to express the final voltage across the capacitors:

$$V_f = \frac{Q}{C_{eq}} = \frac{(200\,\text{V})C_1(1+\kappa)}{2C_1} = \boxed{100(1+\kappa)\,\text{V}}$$

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Picture the Problem We can use the definition of capacitance and the expression for the capacitance of a cylindrical capacitor to find the potential difference between the cylinders. In part (*b*) we can apply the definition of surface charge density to find the density of the free charge σ_f on the inner and outer cylindrical surfaces. We can use the fact that that Q and Q_b are proportional to E and E_b to express Q_b at a and b and then apply the definition of surface charge density to express $\sigma_b(a)$ and $\sigma_b(b)$. In part (*d*) we can use $U = \frac{1}{2}QV$ to find the total stored electrostatic energy and in (*e*) find the mechanical work required from the change in electrostatic energy of the system resulting from the removal of the dielectric cylindrical shell.

(*a*) Using the definition of capacitance, relate the potential difference between the cylinders to their charge and capacitance:

$$V = \frac{Q}{C}$$

Express the capacitance of a cylindrical capacitor as a function of its radii a and b and length L:

$$C = \frac{2\pi \in_0 \kappa L}{\ln(b/a)}$$

Substitute to obtain:

$$V = \frac{Q\ln(b/a)}{2\pi \in_0 \kappa L} = \boxed{\frac{2kQ\ln(b/a)}{\kappa L}}$$

(*b*) Apply the definition of surface charge density to obtain:

$$\sigma_f(a) = \boxed{\frac{Q}{2\pi aL}}$$

and

$$\sigma_f(b) = \boxed{\frac{-Q}{2\pi bL}}$$

(*c*) Noting that Q and Q_b are proportional to E and E_b, express Q_b at a and b:

$$Q_b(a) = \frac{-Q(\kappa-1)}{\kappa}$$

and

$$Q_b(b) = \frac{Q(\kappa-1)}{\kappa}$$

Apply the definition of surface charge density to express $\sigma_b(a)$ and $\sigma_b(b)$:

$$\sigma_b(a) = \frac{Q_b(a)}{A} = \frac{\dfrac{-Q(\kappa-1)}{\kappa}}{2\pi aL}$$

$$= \boxed{\frac{-Q(\kappa-1)}{2\pi aL\kappa}}$$

and

$$\sigma_b(b) = \frac{Q_b(b)}{A} = \frac{\dfrac{Q(\kappa-1)}{\kappa}}{2\pi bL}$$

$$= \boxed{\frac{Q(\kappa-1)}{2\pi bL\kappa}}$$

(*d*) Express the total stored electrostatic energy in terms of the charge stored and the potential difference between the cylinders:

$$U = \tfrac{1}{2}QV = \tfrac{1}{2}Q\left[\frac{2kQ\ln(b/a)}{\kappa L}\right]$$

$$= \boxed{\frac{kQ^2\ln(b/a)}{\kappa L}}$$

(*e*) Express the work required to remove the dielectric cylindrical shell in terms of the change in the electrostatic potential energy of the system:

$$W = \Delta U = U' - U$$

where $U' = \kappa U$ is the electrostatic potential energy of the system with the dielectric shell in place.

Substitute for U and U' to obtain:

$$W = \kappa U - U = U(\kappa - 1)$$

$$= \boxed{\frac{kQ^2(\kappa-1)\ln(b/a)}{\kappa L}}$$

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Picture the Problem Let the numeral 1 denote the 35-μF capacitor and the numeral 2 the 10-μF capacitor. We can use $U=\frac{1}{2}C_{\text{eq}}V^2$ to find the energy initially stored in the system and the definition of capacitance to find the charges on the two capacitors. When the dielectric is removed from the capacitor the two capacitors will share the total charge stored equally. Finally, we can find the final stored energy from the total charge stored and the equivalent capacitance of the two equal capacitors in parallel.

(*a*) Express the stored energy of the system in terms of the equivalent capacitance and the charging potential:

$$U=\tfrac{1}{2}C_{\text{eq}}V^2$$

Express the equivalent capacitance:

$$C_{\text{eq}}=C_1+C_2$$

Substitute to obtain:

$$U=\tfrac{1}{2}(C_1+C_2)V^2$$

Substitute numerical values and evaluate *U*:

$$U=\tfrac{1}{2}(35\,\mu\text{F}+10\,\mu\text{F})(100\,\text{V})^2=\boxed{0.225\,\text{J}}$$

(*b*) Use the definition of capacitance to find the charges on the two capacitors:

$$Q_1=C_1V=(35\,\mu\text{F})(100\,\text{V})=\boxed{3.50\,\text{mC}}$$

and

$$Q_2=C_2V=(10\,\mu\text{F})(100\,\text{V})=\boxed{1.00\,\text{mC}}$$

(*c*) Because the capacitors are connected in parallel, when the dielectric is removed their charges will be equal; as will be their capacitances and:

$$Q_1=Q_2=\tfrac{1}{2}Q=\tfrac{1}{2}(3.5\,\text{mC}+1\,\text{mC})=\boxed{2.25\,\text{mC}}$$

(*d*) Express the final stored energy in terms of the total charge stored and the equivalent capacitance:

$$U_{\text{f}}=\frac{1}{2}\frac{Q_{\text{tot}}^2}{Q_{\text{eq}}}$$

Substitute numerical values and evaluate U_{f}:

$$U_{\text{f}}=\frac{1}{2}\frac{(4.5\,\text{mC})^2}{2(10\,\mu\text{F})}=\boxed{0.506\,\text{J}}$$

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Picture the Problem We can express the two conditions on the voltage in terms of the charges Q_1 and Q_2 and the capacitances C_1 and C_2 and solve the equations simultaneously to find Q_1 and Q_2. We can then use the definition of capacitance to find the initial voltages V_1 and V_2.

Express the condition for the series connection:

$$V_1 + V_2 = 80\,\text{V}$$

or

$$\frac{Q_1}{C_1} + \frac{Q_2}{C_2} = 80\,\text{V}$$

Substitute numerical values to obtain:

$$\frac{Q_1}{0.4\,\mu\text{F}} + \frac{Q_2}{1.2\,\mu\text{F}} = 80\,\text{V}$$

or

$$3Q_1 + Q_2 = 96\,\mu\text{C} \qquad (1)$$

Use the definition of capacitance to express the condition for the parallel connection:

$$\frac{Q_1 + Q_2}{C_{\text{eq}}} = 20\,\text{V}$$

Because the capacitors are now connected in parallel:

$$C_{\text{eq}} = C_1 + C_2 = 0.4\,\mu\text{F} + 1.2\,\mu\text{F} = 1.6\,\mu\text{F}$$

Substitute to obtain:

$$\frac{Q_1 + Q_2}{1.6\,\mu\text{F}} = 20\,\text{V}$$

or

$$Q_1 + Q_2 = 32\,\mu\text{C} \qquad (2)$$

Solve equations (1) and (2) simultaneously to obtain:

$$Q_1 = 32\,\mu\text{C} \text{ and } Q_2 = 0$$

Use the definition of capacitance to obtain:

$$V_1 = \frac{Q_1}{C_1} = \frac{32\,\mu\text{C}}{0.4\,\mu\text{F}} = \boxed{80.0\,\text{V}}$$

and

$$V_2 = \frac{Q_2}{C_2} = \frac{0}{0.4\,\mu\text{F}} = \boxed{0}$$

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Picture the Problem Note that, with switch S closed, C_1 and C_2 are in parallel and we can use $U_{\text{closed}} = \frac{1}{2} C_{\text{eq}} V^2$ and $C_{\text{eq}} = C_1 + C_2$ to obtain an equation we can solve for C_2.

We can use the definition of capacitance to express Q_2 in terms of V_2 and C_2 and $U_{\text{open}} = \frac{1}{2}C_1V_1^2 + \frac{1}{2}C_2V_2^2$ to obtain an equation from which we can determine V_2.

Express the energy stored in the capacitors after the switch is closed:

$$U_{\text{closed}} = \tfrac{1}{2}C_{\text{eq}}V^2$$

Express the equivalent capacitance of C_1 and C_2 in parallel:

$$C_{\text{eq}} = C_1 + C_2$$

Substitute to obtain:

$$U_{\text{closed}} = \tfrac{1}{2}(C_1 + C_2)V^2$$

Solve for C_2:

$$C_2 = \frac{2U_{\text{closed}}}{V^2} - C_1$$

Substitute numerical values and evaluate C_2:

$$C_2 = \frac{2(960\ \mu\text{J})}{(80\ \text{V})^2} - 0.2\ \mu\text{F} = \boxed{0.100\ \mu\text{F}}$$

Express the charge on C_2 when the switch is open:

$$Q_2 = C_2V_2 \qquad (1)$$

Express the energy stored in the capacitors with the switch open:

$$U_{\text{open}} = \tfrac{1}{2}C_1V_1^2 + \tfrac{1}{2}C_2V_2^2$$

Solve for V_2 to obtain:

$$V_2 = \sqrt{\frac{2U_{\text{open}} - C_1V_1^2}{C_2}}$$

Substitute in equation (1) to obtain:

$$Q_2 = C_2\sqrt{\frac{2U_{\text{open}} - C_1V_1^2}{C_2}} = \sqrt{C_2\left(2U_{\text{open}} - C_1V_1^2\right)}$$

Substitute numerical values and evaluate Q_2:

$$Q_2 = \sqrt{(0.1\ \mu\text{F})\left[2(1440\ \mu\text{J}) - (0.2\ \mu\text{F})(40\ \text{V})^2\right]} = \boxed{16.0\ \mu\text{C}}$$

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Picture the Problem We can express the electric fields in the dielectric and in the free space in terms of the charge densities and then use the fact that the electric field has the same value inside the dielectric as in the free space between the plates to establish that $\sigma_1 = 2\sigma_2$. In part (c) we can model the system as two capacitors in parallel to show that the equivalent capacitance is $3\varepsilon_0 A/2d$ and then use the definition of capacitance to show that the new potential difference is $\frac{2}{3}V$.

(*a*) $\boxed{\text{The potential difference between the plates is the same for both halves (the plates are equipotential surfaces). Therefore, } E = V/d \text{ must be the same everywhere between the plates.}}$

(*b*) Relate the electric field in each region to σ and κ:

$$E = \frac{\sigma}{\kappa \in_0}$$

Solve for σ:

$$\sigma = \kappa \in_0 E$$

Express σ_1 and σ_2:

$$\sigma_1 = \kappa_1 \in_0 E_1 = 2 \in_0 E_1$$

and

$$\sigma_2 = \kappa_2 \in_0 E_2 = \in_0 E_1$$

Divide the 1st of these equations by the 2nd and simplify to obtain:

$$\boxed{\sigma_1 = 2\sigma_2}$$

(*c*) Model the partially dielectric-filled capacitor as two capacitors in parallel to obtain:

$$C_{\text{eq}} = C_1 + C_2$$

where

$$C_1 = \frac{\kappa \in_0 \left(\frac{1}{2} A\right)}{d} = \frac{\kappa \in_0 A}{2d}$$

and

$$C_2 = \frac{\in_0 \left(\frac{1}{2} A\right)}{d} = \frac{\in_0 A}{2d}$$

Substitute and simplify to obtain:

$$C_{\text{eq}} = \frac{\kappa \in_0 A}{2d} + \frac{\in_0 A}{2d} = \frac{2 \in_0 A}{2d} + \frac{\in_0 A}{2d} = \boxed{\frac{3 \in_0 A}{2d}}$$

Use the definition of capacitance to relate V_f, Q_f, and C_f:

$$V_\text{f} = \frac{Q_\text{f}}{C_\text{f}}$$

Because the capacitors are in parallel:

$$Q_\text{f} = Q_\text{i} = VC_\text{i} = \frac{V \in_0 A}{d}$$

Substitute to obtain:

$$V_{\mathrm{f}}=\frac{V\in_0 A}{C_{\mathrm{f}}d}=\frac{V\in_0 A}{\left(\frac{3\in_0 A}{2d}\right)d}=\boxed{\tfrac{2}{3}V}$$

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Picture the Problem Note that when the capacitors are connected in the manner described they are in parallel with each other. Let the numeral one refer to the capacitor with the air gap and the numeral 2 to the capacitor that receives the dielectric and let primes denote physical quantitities after the insertion of the dielectric. We can find the energy stored in the system from our knowledge of the charge on and capacitance of each capacitor. In part (*b*) we can find the final charges on the two capacitors by first finding the equivalent capacitance and the potential difference across the modified system of capacitors. We can use the final potential difference across the system and our knowledge of the stored charge to find the final stored energy of the system.

(*a*) Express the stored energy in the system as the sum of the energy stored in the two capacitors:

$$U=\frac{1}{2}\frac{Q_1^2}{C_1}+\frac{1}{2}\frac{Q_1^2}{C_1}=\frac{Q_1^2}{C_1}$$

Substitute numerical values and evaluate U:

$$U=\frac{(100\,\mu\mathrm{C})^2}{10\,\mu\mathrm{F}}=\boxed{1.00\,\mathrm{mJ}}$$

(*b*) Relate the final charges Q_1' and Q_2' to the total charge stored by the capacitors:

$$Q_1'+Q_2'=200\,\mu\mathrm{C}$$

Express the common potential difference across the capacitors:

$$V=\frac{Q_1'+Q_2'}{C_{\mathrm{eq}}}$$

Express the equivalent capacitance when the dielectric is inserted between the plates of capacitor 2:

$$C_{\mathrm{eq}}=C_1+C_2=C_1+\kappa C_1=C_1(1+\kappa)$$

Substitute to obtain:

$$V=\frac{Q_1'+Q_2'}{C_1(1+\kappa)}$$

Substitute numerical values and evaluate V:

$$V=\frac{200\,\mu\mathrm{C}}{(10\,\mu\mathrm{F})(1+3.2)}=4.76\,\mathrm{V}$$

Use the definition of capacitance to find Q_1' and Q_2':

$$Q_1'=VC_1'=(4.76\,\mathrm{V})(10\,\mu\mathrm{F})=\boxed{47.6\,\mu\mathrm{C}}$$

and

$$Q_2' = VC_2' = V\kappa C_2$$
$$= (4.76\,\text{V})(3.2)(10\,\mu\text{F}) = \boxed{152\,\mu\text{C}}$$

(*c*) Express the final stored energy of the system in terms of the total charge stored and the final potential difference across the capacitors connected in parallel:

$$U_\text{f} = \tfrac{1}{2}QV = \tfrac{1}{2}(200\,\mu\text{C})(4.76\,\text{V})$$
$$= \boxed{0.476\,\text{mJ}}$$

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Picture the Problem Choose a coordinate system in which the positive *x* direction is the right and the origin is at the left edge of the capacitor. We can express an element of capacitance *dC* and then integrate this expression to find *C* for this capacitor.

Express an element of capacitance *dC* of length *b*, width *dx* and separation $d = y_0 + (y_0/a)x$:

$$dC = \frac{\epsilon_0\, b}{d}dx = \frac{\epsilon_0\, b}{y_0(1 + x/a)}dx$$

These elements are all in parallel, so the total capacitance is obtained by integration:

$$C = \frac{\epsilon_0\, b}{y_0}\int_0^{y_0}\frac{1}{1 + x/a}dx = \boxed{\frac{\epsilon_0\, ab}{y_0}\ln 2}$$

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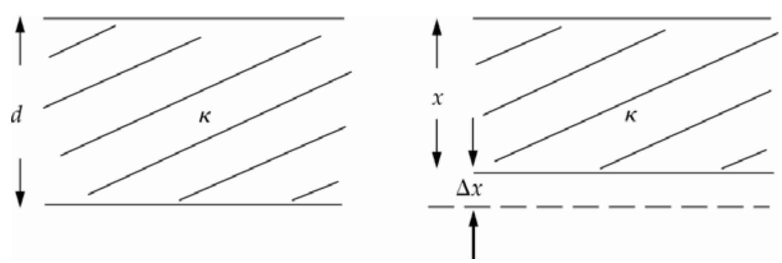
Picture the Problem The diagram to the left shows the dielectric-filled parallel-plate capacitor before compression and the diagram to the right shows the capacitor when the plate separation has been reduced to *x*. We can use the definition of capacitance and the expression for the capacitance of a parallel-plate capacitor to derive an expression for the capacitance as a function of voltage across the capacitor. We can find the maximum voltage that can be applied from the dielectric strength of the dielectric and the separation of the plates. In part (*c*) we can find the fraction of the total energy that is electrostatic field energy and the fraction that is mechanical stress energy by expressing either of these as a fraction of their sum.



(*a*) Use its definition to express the capacitance as a function of the voltage across the capacitor:

$$C(V)=\frac{Q}{V} \qquad (1)$$

The limiting value of the capacitance is:

$$C_0=\frac{\kappa \in_0 A}{d}$$

Substitute numerical values and evaluate C_0:

$$C_0=\frac{3\left(8.85\times10^{-12}\ \mathrm{C^2/N\cdot m^2}\right)A}{0.2\,\mathrm{mm}}$$
$$=0.133A\,\mathrm{C^2/N\cdot m^3}$$

Let x be the variable separation. Because κ is independent of x:

$$C(x)=\frac{\kappa \in_0 A}{x}$$

and

$$Q(x)=C(x)V=\frac{\kappa \in_0 A}{x}V$$

Substitute in equation (1) to obtain:

$$C(V)=\frac{\dfrac{\kappa \in_0 A}{x}V}{V}=\frac{\kappa \in_0 A}{x} \qquad (2)$$
$$=\frac{\kappa \in_0 A}{d-\Delta x}$$

The force of attraction between the plates is given in Problem 95*c*:

$$F=\frac{Q^2(x)}{2\kappa \in_0 A}$$

Substitute to obtain:

$$F=\frac{\left(\dfrac{\kappa \in_0 A}{x}V\right)^2}{2\kappa \in_0 A}=-\frac{\kappa \in_0 AV^2}{2x^2}$$

where the minus sign is used to indicate that the force acts to decrease the plate separation x.

Apply Hooke's law to relate the stress to the strain:

$$Y=\frac{F/A}{\Delta x/x}$$

or

$$\frac{\Delta x}{x}=\frac{F}{YA}$$

Substitute for F to obtain:

$$\frac{\Delta x}{x}=-\frac{\kappa \in_0 V^2}{2Yx^2}$$

and

$$\Delta x = -\frac{\kappa \in_0 V^2}{2Yx} = -\frac{\kappa \in_0 V^2}{2Yd} \qquad (3)$$

provided $\Delta x << d$

The voltage across the capacitor is:

$$V = E_{\max} d = (40\,\text{kV/mm})(0.2\,\text{mm}) = 8.00\,\text{kV}$$

Substitute numerical values in equation (3) and evaluate Δx:

$$\Delta x = -\frac{3\left(8.85\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)(8\,\text{kV})^2}{2\left(5\times10^6\ \text{N/m}^2\right)(0.2\,\text{mm})} = 8.50\times10^{-7}\ \text{m} = 8.50\times10^{-4}\ \text{mm}$$

Substitute in equation (2) to obtain:

$$C(V) = \frac{\kappa \in_0 A}{d - \dfrac{\kappa \in_0 V^2}{2Yd}} = \frac{\kappa \in_0 A}{d\left(1 - \dfrac{\kappa \in_0 V^2}{2Yd^2}\right)} = C_0\left(1 - \frac{\kappa \in_0 V^2}{2Yd^2}\right)^{-1} \approx \boxed{C_0\left(1 + \frac{\kappa \in_0 V^2}{2Yd^2}\right)}$$

provided $\Delta x << d$.

(b) Express the maximum voltage that can be applied in terms of the maximum electric field:

$$V_{\max} = E_{\max}(d - \Delta x) = (40\,\text{kV/mm})\left(0.2\,\text{mm} - 8.5\times10^{-4}\ \text{mm}\right) = \boxed{7.97\,\text{kV}}$$

(c) The fraction of the total energy of the capacitor that is mechanical stress energy is:

$$f = \frac{U_{\text{M}}}{U_{\text{M}} + U_{\text{E}}} \qquad (4)$$

Express the maximum electric field energy:

$$U_{\text{E,max}} = \tfrac{1}{2} C(V_{\max}) V_{\max}^2$$

Evaluate $C(V_{\max})$:

$$C(7.97\,\text{kV}) = (0.133A)\left[1 + \left(6.64\times10^{-11}\right)(7.97\,\text{kV})^2\right]\mu\text{F}/\text{m}^2 = 0.134A\,\mu\text{F}/\text{m}^2$$

Substitute for $C(V_{\max})$ and evaluate $U_{\text{E,max}}$:

$$U_{\text{E,max}} = \tfrac{1}{2}\left(0.134A\,\mu\text{F/m}^2\right)(7.96\,\text{kV})^2 = 4.25\,\text{J/m}^2$$

The mechanical stress energy is given by:

$$U_M = \frac{1}{2}\frac{(\Delta x)^2 Y}{d}$$

Substitute numerical values and evaluate U_M:

$$U_M = \frac{1}{2}\frac{(8.5\times10^{-4}\text{ mm})^2(5\times10^6\text{ N/m}^2)}{0.2\,\text{mm}} = 8.92\,\text{mJ}$$

Substitute numerical values in equation (4) and evaluate f:

$$f = \frac{8.92\,\text{mJ}}{8.92\,\text{mJ}+4.25\,\text{J/m}^2} = \boxed{0.209\%}$$

and the fraction of the total energy that is electrostatic field energy is

$$1-f = 1-0.209\% = \boxed{99.8\%}$$

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Picture the Problem Note that, due to symmetry, the electric field, wherever it exists, will be radial. We can integrate the electric flux over spherical Gaussian surfaces with radii $r < R_1$, $R_1 < r < R_2$, and $r > R_2$ to find the electric field everywhere in space. Once we know the electric field everywhere we can find the potential of the conducting sphere by using $dV = -Edr$ and integrating E in the regions $R_1 < r < R_2$ and $r > R_2$. Finally, knowing the electrostatic potential at the surface of the conducting sphere we can use $U_{\text{tot}} = \frac{1}{2}QV(R_1)$ to find the total electrostatic potential energy of the system.

(*a*) Integrate the electric flux over a spherical Gaussian surface with radius $r < R_1$ to obtain:

$$E_r(4\pi r^2) = \frac{Q_{\text{inside}}}{\epsilon_0} = 0$$

because Q_{inside} the conducting surface is zero.

Solve for $E_r(r < R_1)$ to obtain:

$$E_r(r < R_1) = \boxed{0}$$

Integrate the electric flux over a spherical Gaussian surface with radius $R_1 < r < R_2$ to obtain:

$$E_r(4\pi r^2) = \frac{Q_{\text{inside}}}{\kappa\epsilon_0} = \frac{Q}{\kappa\epsilon_0}$$

Solve for $E_r(R_1 < r < R_2)$ to obtain:

$$E_r(R_1 < r < R_2) = \frac{Q}{4\pi\epsilon_0\kappa r^2} = \boxed{\frac{kQ}{\kappa r^2}}$$

Integrate the electric flux over a spherical Gaussian surface with radius $r > R_2$ to obtain:

$$E_r(4\pi r^2) = \frac{Q_{\text{inside}}}{\epsilon_0} = \frac{Q}{\epsilon_0}$$

Solve for $E_r(r > R_2)$ to obtain:

$$E_r(r > R_2) = \frac{Q}{4\pi \in_0 r^2} = \boxed{\frac{kQ}{r^2}}$$

(*b*) Express the potential at the surface of the conducting sphere in terms of the electric fields $E_r(R_1 < r < R_2)$ and $E_r(r > R_2)$:

$$V(R_1) = -\int_{\infty}^{R_1} E dr$$

$$= -kQ\int_{\infty}^{R_2}\frac{1}{r^2}dr - \frac{kQ}{\kappa}\int_{R_2}^{R_1}\frac{1}{r^2}dr$$

$$= \boxed{\frac{kQ}{\kappa}\left(\frac{R_1(\kappa-1)+R_2}{R_1R_2}\right)}$$

(c) Express the total electrostatic potential energy of the system in terms of $V(R_1)$ and Q:

$$U_{\text{tot}} = \tfrac{1}{2}QV(R_1)$$

$$= \tfrac{1}{2}Q\left(\frac{kQ}{\kappa}\left(\frac{R_1(\kappa-1)+R_2}{R_1R_2}\right)\right)$$

$$= \boxed{\frac{kQ^2}{2\kappa}\left(\frac{R_1(\kappa-1)+R_2}{R_1R_2}\right)}$$