# Chapter 31
# Properties of Light

## Conceptual Problems

**1** •

**Determine the Concept** The population inversion between the state $E_{2,\mathrm{Ne}}$ and the state 1.96 eV below it (see Figure 31-9) is achieved by inelastic collisions between neon atoms and helium atoms excited to the state $E_{2,\mathrm{He}}$.

**2** ••

**Determine the Concept** Although the excited atoms emit the light of the same frequency on returning to the ground state, the light is emitted in a random direction, not exclusively in the direction of the incident beam. Consequently, the beam intensity is greatly diminished.

**3** •

**Determine the Concept** The layer of water greatly reduces the light reflected back from the car's headlights, but increases the light reflected by the road of light from the headlights of oncoming cars.

**4** •

**Determine the Concept** When light passes from air into water its wavelength changes ($\lambda_{\mathrm{water}} = \lambda_{\mathrm{air}}/n_{\mathrm{water}}$), its speed changes ($v_{\mathrm{water}} = c/n_{\mathrm{water}}$), and the direction of its propagation changes in accordance with Snell's law. $\boxed{(c)\text{ is correct.}}$

***5** ••

**Determine the Concept** The change in atmospheric density results in refraction of the light from the sun, bending it toward the earth. Consequently, the sun can be seen even after it is just below the horizon. Also, the light from the lower portion of the sun is refracted more than that from the upper portion, so the lower part appears to be slightly higher in the sky. The effect is an apparent flattening of the disk into an ellipse.

**6** •

**Determine the Concept** (*a*) Yes. (*b*) Her procedure is based on Fermat's principle in that, since the ball presumably travels at constant speed, the path that requires the least time of travel corresponds to the shortest distance of travel.

**7** •

**Determine the Concept** Because she can run faster than she can swim, she should choose the path that will maximize her running distance. Path *LES* is the path that satisfies this criterion.

**8** •

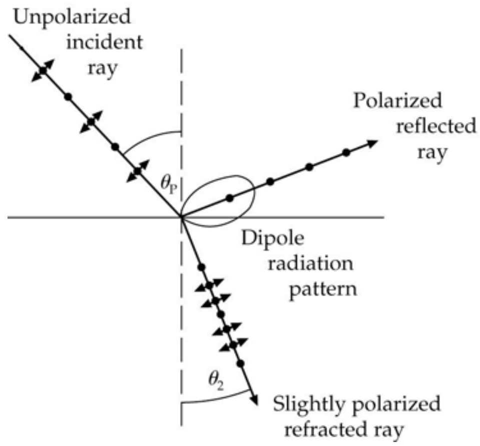
**Picture the Problem** The intensity of the light transmitted by the second polarizer is given by $I_{\text{trans}} = I_0 \cos^2 \theta$, where $I_0 = \frac{1}{2} I$. Therefore, $I_{\text{trans}} = \frac{1}{2} I \cos^2 \theta$ and $\boxed{(b) \text{ is correct.}}$

**9** •

**Picture the Problem** Polarized light can be produced from unpolarized light by absorption, reflection, birefringence, and scattering. Therefore, $\boxed{(d) \text{ is correct.}}$

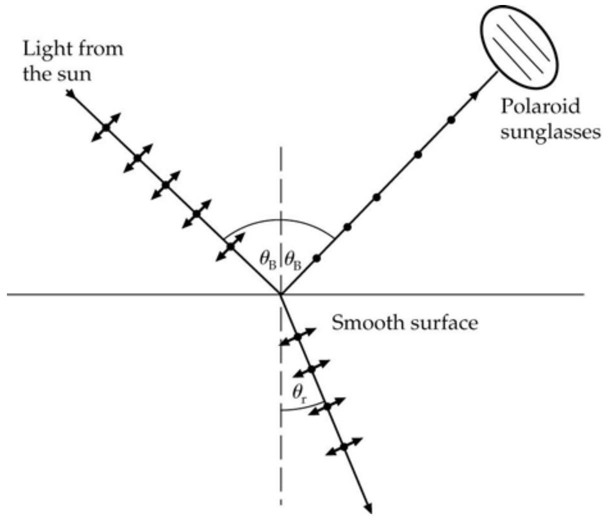
***10** ••

**Determine the Concept** The diagram shows that the radiated intensity for a dipole is zero in the direction of the dipole moment. Because the dipole axis is in the same direction as the polarization, for light polarized parallel to plane of incidence, the dipole axis will point in the same direction as the reflected wave, i.e., in the direction described by Brewster's law. As the diagram indicates, there is zero field in the direction of the refracted ray. On the other hand, if the incoming wave is polarized perpendicular to the plane of incidence, the dipole axis will never point along the direction of propagation for the reflected or refracted wave.




**11** ••

**Determine the Concept** The diagram shows unpolarized light from the sun incident on the smooth surface at the polarizing angle for that particular surface. The reflected light is polarized perpendicular to the plane of incidence, i.e., in the horizontal direction. The sunglasses are shown in the correct orientation to pass vertically polarized light and block the reflected sunlight.




**12** •

(*a*) True.

(*b*) False. Most of the light incident normally on an air–glass interface is transmitted.

(*c*) False. The relationship between the angles of incidence and refraction depends on the indices of refraction on both sides of the interface.

(*d*) False. The index of refraction of water is a function of the wavelength of light.

(*e*) True.

**13** ••

**Picture the Problem** Because the speed of light in a given medium is inversely proportional to the index of refraction of the medium, we can decide which of the statements are true by referring to Figure 31-26.

(*a*) The graphs of *n* vs. $\lambda$ are not horizontal lines and so the speed of light is a function of its wavelength.

(*b*) Because the index of refraction decreases with wavelength, violet light has the lowest speed and red light the highest speed.

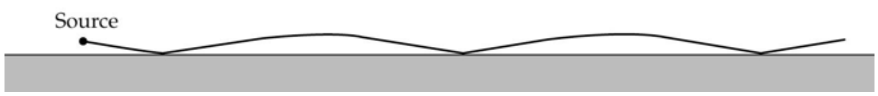
(*c*) Because the index of refraction decreases with wavelength, violet light has the lowest speed and red light the highest speed. $\boxed{(c)\text{ is correct.}}$

(*d*) Examination of Figure 31-26 tells us that this statement is false.

(*e*) Examination of Figure 31-26 tells us that this statement is false.

***14 ••**

**Picture the Problem** The sound is reflected specularly from the surface of the water (we assume it is calm). It is then refracted back toward the water in the region above the water because the speed of sound depends on the temperature of the air and is greater at the higher temperature. The pattern of the sound wave is shown schematically below.




**15 •**

**Determine the Concept** In resonance absorption, the molecules respond to the frequency of the light through the Einstein photon relation $E = hf$. Thus, the color appears to be the same in spite of the fact that the wavelength has changed.

## Estimation and Approximation

**16 •**

**Picture the Problem** We can use the distance, rate, and time relationship to estimate the time required to travel 6 km (see Problem 31-14).

Express the distance $D$ to light traveled in terms of its speed $c$ and the elapsed time $\Delta t$:

$$D = c\Delta t$$

Solve for $\Delta t$:

$$\Delta t = \frac{D}{c}$$

Substitute numerical values and evaluate $\Delta t$:

$$\Delta t = \frac{6\,\text{km}}{2.998\times 10^{8}\ \text{m/s}} = \boxed{20.0\,\mu\text{s}}$$

**17 •**

**Picture the Problem** We can use the period of Io's motion and the position of the earth at $B$ to find the number of eclipses of Io during the earth's movement and then use this information to find the number of days before a night-time eclipse. During the 42.5 h between eclipses of Jupiter's moon, the earth moves from $A$ to $B$, increasing the distance from Jupiter by approximately the distance from the earth to the Sun, making the path for the light longer and introducing a delay in the onset of the eclipse.

(*a*) Find the time it takes the earth to travel from point *A* to point *B*:

$$t_{A\to B} = \frac{T_{\text{earth}}}{4} = \frac{365.24\,\text{d}}{4} \times \frac{24\,\text{h}}{\text{d}} = 2191\,\text{h}$$

Because there are 42.5 h between eclipses of Io, the number of eclipses *N* occurring in the time it takes for the earth to move from *A* to *B* is:

$$N = \frac{t_{A\to B}}{T_{\text{Io}}} = \frac{2191\,\text{h}}{42.5\,\text{h}} = 51.55$$

Hence, in one-fourth of a year, there will be 51.55 eclipses. Because we want to find the next occurrence that happens in the evening hours, we'll use 52 as the number of eclipses. We'll also assume that Jupiter is visible so that the eclipse of Io can be observed at the time we determine.

Relate the time $t(N)$ at which the $N$th eclipse occurs to *N* and the period $T_{\text{Io}}$ of Io:

$$t(N) = NT_{\text{Io}}$$

Evaluate $t(52)$ to obtain:

$$t(52) = (52)\left(42.5\,\text{h} \times \frac{1\text{d}}{24\,\text{h}}\right) = 92.083\,\text{d}$$

Subtract the number of whole days to find the clock time *t*:

$$t = t(52) - 92\,\text{d} = 92.083\,\text{d} - 92\,\text{d} = 0.083\,\text{d} \times \frac{24\,\text{h}}{\text{d}} = 1.992\,\text{h} \approx \boxed{2\,\text{am}}$$

Because June, July, and August have 30, 31, and 31 d, respectively, the date is:

$$\boxed{\text{September } 1}$$

(*b*) Express the time delay $\Delta t$ in the arrival of light from Io due to the earth's location at *B*:

$$\Delta t = \frac{r_{\text{earth-sun}}}{c}$$

Substitute numerical values and evaluate $\Delta t$:

$$\Delta t = \frac{1.5\times10^{11}\text{ m}}{2.998\times10^{8}\text{ m/s}} = 500\,\text{s}$$
$$= 8.33\,\text{min}$$

$$\boxed{\text{Hence, the eclipse will actually occur at 2:08 pm.}}$$

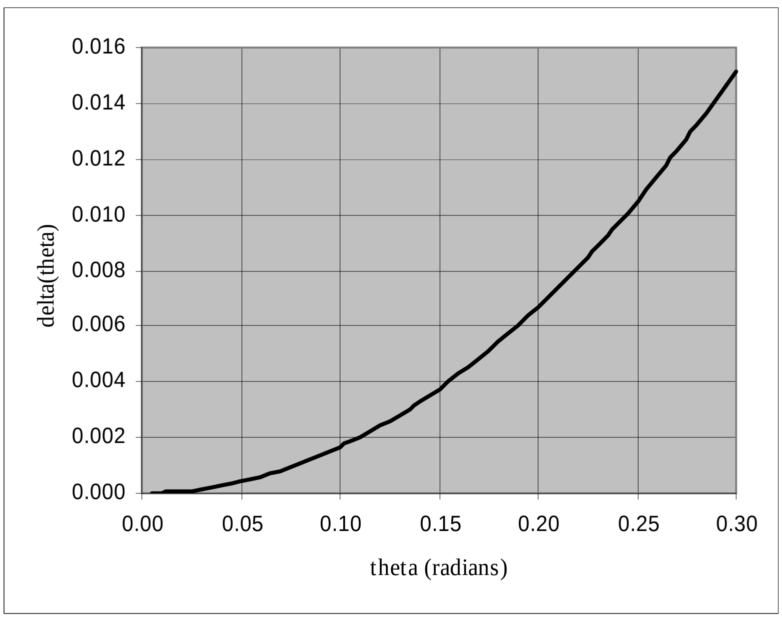
**18** ••

**Picture the Problem** We can express the relative error in using the small angle approximation and then either 1) use trial-and-error methods, 2) use a spreadsheet program, or 3) use the Solver capability of a scientific calculator to solve the transcendental equation the results from setting the error function equal to 0.01.

Express the relative error $\delta$ in using the small angle approximation:

$$\delta(\theta) = \frac{\theta - \sin\theta}{\sin\theta} = \frac{\theta}{\sin\theta} - 1$$

A spreadsheet program was used to plot the following graph of $\delta(\theta)$.




From the graph, we can see that $\delta(\theta) < 1\%$ for $\theta \leq 0.24$ radians. In degree measure, $\theta \leq \boxed{14°}$

**Remarks: Using the Solver program on a TI-85 gave $\theta$ = 0.244 radians.**

## Sources of Light

**19** •

**Picture the Problem** We can use the definition of power to find the total energy of the pulse. The ratio of the total energy to the energy per photon will yield the number of photons emitted in the pulse.

(*a*) Use the definition of power to obtain:

$$E = P\Delta t$$

Substitute numerical values and evaluate *E*:

$$E = (10\,\text{MW})(1.5\,\text{ns}) = \boxed{15.0\,\text{mJ}}$$

(*b*) Relate the number of photons *N* to the total energy in the pulse and the energy of a single photon $E_{\text{photon}}$:

$$N = \frac{E}{E_{\text{photon}}}$$

The energy of a photon is given by:

$$E_{\text{photon}} = \frac{hc}{\lambda}$$

Substitute for $E_{\text{photon}}$ to obtain:

$$N = \frac{\lambda E}{hc}$$

Substitute numerical values (the wavelength of light emitted by a ruby laser is 694.3 nm) and evaluate *N*:

$$N = \frac{(694.3\,\text{nm})(15.0\,\text{mJ})}{1240\,\text{eV}\cdot\text{nm}}\frac{1\,\text{eV}}{1.60\times10^{-19}\,\text{J}} = \boxed{5.25\times10^{16}}$$

**20** •

**Picture the Problem** We can express the number of photons emitted per second as the ratio of the power output of the laser and energy of a single photon.

Relate the number of photons per second *n* to the power output of the pulse and the energy of a single photon $E_{\text{photon}}$:

$$n = \frac{P}{E_{\text{photon}}}$$

The energy of a photon is given by:

$$E_{\text{photon}} = \frac{hc}{\lambda}$$

Substitute for $E_{\text{photon}}$ to obtain:

$$n = \frac{\lambda P}{hc}$$

Substitute numerical values and evaluate $n$:

$$n = \frac{(632.8\,\text{nm})(4\,\text{mW})}{1240\,\text{eV}\cdot\text{nm}}\frac{1\,\text{eV}}{1.60\times10^{-19}\,\text{J}} = \boxed{1.28\times10^{16}\ \text{photons/s}}$$

**21** •

**Picture the Problem** We can use the Einstein equation for photon energy to find the wavelength of the radiation for resonance absorption. We can use the same relationship, with $E_{\text{Raman}} = E_{\text{inc}} - \Delta E$ where $\Delta E$ is the energy for resonance absorption, to find the wavelength of the Raman scattered light.

(*a*) Use the Einstein equation for photon energy to relate the wavelength of the radiation to energy of the first excited state:

$$\lambda = \frac{hc}{E}$$

Substitute numerical values and evaluate $\lambda$:

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{2.85\,\text{eV}} = \boxed{435\,\text{nm}}$$

(*b*) The wavelength of the Raman scattered light is given by:

$$\lambda_{\text{Raman}} = \frac{1240\,\text{eV}\cdot\text{nm}}{E_{\text{Raman}}}$$

Relate the energy of the Raman scattered light $E_{\text{Raman}}$ to the energy of the incident light $E_{\text{inc}}$:

$$\begin{aligned} E_{\text{Raman}} &= E_{\text{inc}} - \Delta E \\ &= \frac{1240\,\text{eV}\cdot\text{nm}}{320\,\text{nm}} - 2.85\,\text{eV} \\ &= 1.025\,\text{eV} \end{aligned}$$

Substitute numerical values and evaluate $\lambda_{\text{Raman}}$:

$$\lambda_{\text{Raman}} = \frac{1240\,\text{eV}\cdot\text{nm}}{1.025\,\text{eV}} = \boxed{1210\,\text{nm}}$$

**22** ••

**Picture the Problem** The incident radiation will excite atoms of the gas to higher energy states. The scattered light that is observed is a consequence of these atoms returning to their ground state. The energy difference between the ground state and the atomic state excited by the irradiation is given by $\Delta E = hf = \dfrac{hc}{\lambda}$.

The energy difference between the ground state and the atomic state excited by the irradiation is given by:

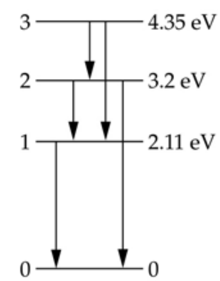
$$\Delta E = hf = \frac{hc}{\lambda} = \frac{1240\,\text{eV}\cdot\text{fm}}{\lambda}$$

Substitute 368 nm for $\lambda$ and evaluate $\Delta E$:

$$\Delta E = \frac{1240\,\text{eV}\cdot\text{fm}}{368\,\text{nm}} = \boxed{3.37\,\text{eV}}$$

**23** ••

**Picture the Problem** The ground state and the three excited energy levels are shown in the diagram to the right. Because the wavelength is related to the energy of a photon by $\lambda = hc/\Delta E$, longer wavelengths correspond to smaller energy differences.




(*a*) The maximum wavelength of radiation that will result in resonance fluorescence corresponds to an excitation to the 3.2 eV level followed by decays to the 2.11 eV level and the ground state:

$$\lambda_{max} = \frac{1240\,\text{eV}\cdot\text{fm}}{3.2\,\text{eV}} = \boxed{387.5\,\text{nm}}$$

The fluorescence wavelengths are:

$$\lambda_{21} = \frac{1240\,\text{eV}\cdot\text{fm}}{3.2\,\text{eV} - 2.11\,\text{eV}} = \boxed{1138\,\text{nm}}$$

and

$$\lambda_{10} = \frac{1240\,\text{eV}\cdot\text{fm}}{2.11\,\text{eV} - 0} = \boxed{587.7\,\text{nm}}$$

(*b*) For excitation:

$$\lambda_{03} = \frac{1240\,\text{eV}\cdot\text{fm}}{4.35\,\text{eV}} = \boxed{285.1\,\text{nm}}$$

The fluorescence wavelengths corresponding to the possible transitions are:

$$\lambda_{32} = \frac{1240\,\text{eV}\cdot\text{fm}}{4.35\,\text{eV} - 3.2\,\text{eV}} = \boxed{1078\,\text{nm}}$$

$$\lambda_{21} = \frac{1240\,\text{eV}\cdot\text{fm}}{3.2\,\text{eV} - 2.11\,\text{eV}} = \boxed{1138\,\text{nm}}$$

$$\lambda_{10} = \frac{1240\,\text{eV}\cdot\text{fm}}{2.11\,\text{eV} - 0} = \boxed{587.7\,\text{nm}}$$

$$\lambda_{31} = \frac{1240\,\text{eV}\cdot\text{fm}}{4.35\,\text{eV} - 2.11\,\text{eV}} = \boxed{553.6\,\text{nm}}$$

and

$$\lambda_{20} = \frac{1240\,\text{eV}\cdot\text{fm}}{3.2\,\text{eV} - 0} = \boxed{387.5\,\text{nm}}$$

***24 ••**

**Determine the Concept** The energy difference between the ground state and the first excited state is $3E_0$ = 40.8 eV, corresponding to a wavelength of 30.4 nm. This is in the far ultraviolet, well outside the visible range of wavelengths. There will be no dark lines in the transmitted radiation.

## The Speed of Light

**25 •**

**Picture the Problem** We can use the distance, rate, and time relationship to find the distance to the spaceship.

Relate the distance $D$ to the spaceship to the speed of electromagnetic radiation in a vacuum and to the time for the message to reach the astronauts:

$$D = c\Delta t$$

Noting that the time for the message to reach the astronauts is half the time for Mission Control to hear their response, substitute numerical values and evaluate $D$:

$$D = \left(2.998\times10^{8}\ \text{m/s}\right)\left(2.5\,\text{s}\right) = 7.50\times10^{8}\ \text{m}$$

and

$$\boxed{(a)\ \text{is correct.}}$$

**26 •**

**Picture the Problem** We can use the conversion factor, found in EP-3, to convert a distance in km into $c{\cdot}y$:

Convert $D = 2\times10^{19}$ km into light-years:

$$D = 2\times10^{19}\ \text{km}\times\frac{1c\cdot y}{9.46\times10^{15}\ \text{m}} = \boxed{2.11\times10^{6}\ c\cdot y}$$

**27 •**

**Picture the Problem** We can use the distance, rate, and time relationship to find the time delay between sending the signal from the earth and receiving it on Mars.

Relate the distance $D$ to Mars to the

$$D = c\Delta t$$

speed of electromagnetic radiation in a vacuum and to the travel time for the signal:

Solve for $\Delta t$:

$$\Delta t = \frac{D}{c}$$

Substitute numerical values and evaluate $\Delta t$:

$$\Delta t = \frac{9.7\times10^{10}\text{ m}}{2.998\times10^{8}\text{ m/s}} = 324\,\text{s}$$

$$= \boxed{5\,\text{min}\,23\,\text{s}}$$

**28** •

**Picture the Problem** We can use the given information that the uncertainty in the measured distance $\Delta x$ is related to the uncertainty in the time $\Delta t$ by $\Delta x = c\Delta t$ to evaluate $\Delta x$.

The uncertainty in the distance is:

$$\Delta x = \pm c\Delta t$$

Substitute numerical values and evaluate $\Delta t$:

$$\Delta x = \pm\left(2.998\times10^{8}\text{ m/s}\right)\left(1.0\,\text{ns}\right)$$

$$= \boxed{\pm 30.0\,\text{cm}}$$

***29** ••

**Picture the Problem** We can use the distance, rate, and time relationship to find the time difference Galileo would need to be able to measure the speed of light successfully.

(*a*) Relate the distance separating Galileo and his assistant to the speed of light and the time required for it travel to the assistant and back to Galileo:

$$D = c\Delta t$$

Solve for $\Delta t$:

$$\Delta t = \frac{D}{c}$$

Substitute numerical values and evaluate $\Delta t$:

$$\Delta t = \frac{2\left(3\,\text{km}\right)}{3\times10^{8}\text{ m/s}} = \boxed{20.0\,\mu\text{s}}$$

(*b*) Express the ratio of the human reaction time to the transit time for the light:

$$\frac{\Delta t_{\text{reaction}}}{\Delta t} = \frac{0.2\,\text{s}}{20\,\mu\text{s}} = 10^{4}$$

or

$$\Delta t_{\text{reaction}} = \boxed{10^4 \Delta t}$$

## Reflection and Refraction

**30** •

**Picture the Problem** Let the subscript 1 refer to air and the subscript 2 to water and use the equation relating the intensity of reflected light at normal incidence to the intensity of the incident light and the indices of refraction of the media on either side of the interface.

Express the intensity $I$ of the light reflected from an air-water interface at normal incidence in terms of the indices of refraction and the intensity $I_0$ of the incident light:

$$I = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 I_0$$

Solve for the ratio $I/I_0$:

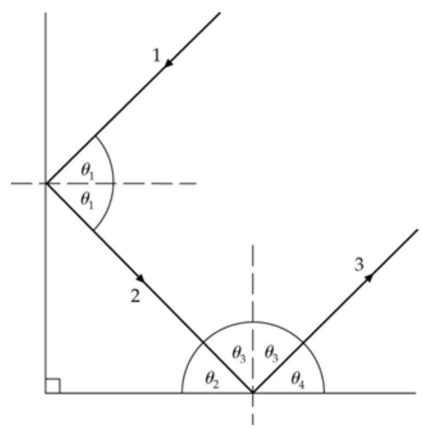
$$\frac{I}{I_0} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

Substitute numerical values and evaluate $I/I_0$:

$$\frac{I}{I_0} = \left(\frac{1 - 1.33}{1 + 1.33}\right)^2 = 0.0201 = \boxed{2.01\%}$$

***31** ••

**Picture the Problem** The diagram shows ray 1 incident on the vertical surface at an angle $\theta_1$, reflected as ray 2, and incident on the horizontal surface at an angle of incidence $\theta_3$. We'll prove that rays 1 and 3 are parallel by showing that $\theta_1 = \theta_4$, i.e., by showing that they make equal angles with the horizontal. Note that the law of reflection has been used in identifying equal angles of incidence and reflection.




We know that the angles of the right triangle formed by ray 2 and the two mirror surfaces add up to 180°:

$$\theta_2 + 90° + 90° - \theta_1 = 180°$$

or

$$\theta_1 = \theta_2$$

The sum of $\theta_2$ and $\theta_3$ is 90°:

$$\theta_3 = 90° - \theta_2$$

Because $\theta_1 = \theta_2$:

$$\theta_3 = 90° - \theta_1$$

The sum of $\theta_4$ and $\theta_3$ is 90°:

$$\theta_3 + \theta_4 = 90°$$

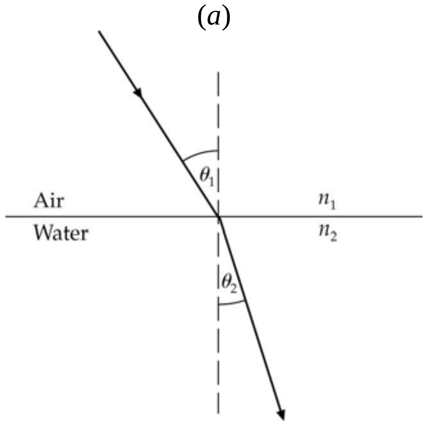
Substitute for $\theta_3$ to obtain:

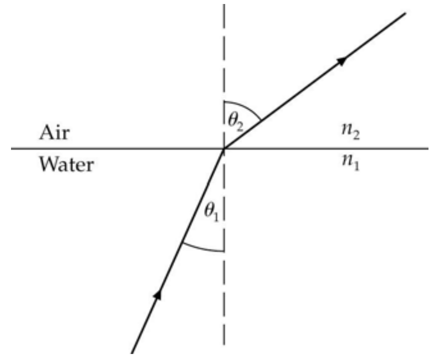
$$90° - \theta_1 + \theta_4 = 90° \Rightarrow \theta_1 = \boxed{\theta_4}$$

**32** ••

**Picture the Problem** Diagrams showing the light rays for the two cases are shown below. In (a) the light travels from air into water and in (*b*) it travels from water into air.

(*a*) (*b*)




(*a*) Apply Snell's law to the air-water interface to obtain:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

where the angles of incidence and refraction are $\theta_1$ and $\theta_2$, respectively.

Solve for $\theta_2$:

$$\theta_2 = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_1\right)$$

A spreadsheet program to graph $\theta_2$ as a function of $\theta_1$ is shown below. The formulas used to calculate the quantities in the columns are as follows:

| Cell | Content/Formula | Algebraic Form |
|---|---|---|
| B1 | 1 | $n_1$ |
| B2 | 1.33333 | $n_2$ |
| A6 | 0 | $\theta_1$ (deg) |
| A7 | A6 + 5 | $\theta_1 + \Delta\theta$ |
| B6 | A6*PI()/180 | $\theta_1 \times \frac{\pi}{180}$ |
| C6 | ASIN(($B$1/$B$2)*SIN(B6)) | $\sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_1\right)$ |
| D6 | C6*180/PI() | $\theta_2 \times \frac{180}{\pi}$ |

| | A | B | C | D |
|---|---|---|---|---|
| 1 | n1= | 1 | | |
| 2 | n2= | 1.33333 | | |
| 3 | | | | |
| 4 | theta1 | theta1 | theta2 | theta2 |
| 5 | (deg) | (rad) | (rad) | (deg) |
| 6 | 0 | 0.00 | 0.000 | 0.00 |
| 7 | 1 | 0.02 | 0.013 | 0.75 |
| 8 | 2 | 0.03 | 0.026 | 1.50 |
| 9 | 3 | 0.05 | 0.039 | 2.25 |
| | | | | |
| 21 | 87 | 1.52 | 0.847 | 48.50 |
| 22 | 88 | 1.54 | 0.847 | 48.55 |
| 23 | 89 | 1.55 | 0.848 | 48.58 |
| 24 | 90 | 1.57 | 0.848 | 48.59 |

A graph of $\theta_2$ as a function of $\theta_1$ follows:


(*b*) Change the contents of cell B1 to 1.33333 and the contents of cell B2 to 1 to obtain the following graph:


Note that as the angle of incidence approaches the critical angle for a water-air interface (48.6°), the angle of refraction approaches 90°. No light will be refracted into the air if the angle of incidence is greater than 48.6°.

**33** •

**Picture the Problem** We can use the definition of the index of refraction to find the speed of light in water and in glass.

The definition of the index of refraction is:

$$n = \frac{c}{v}$$

Solve for $v$ to obtain:

$$v = \frac{c}{n}$$

Substitute numerical values and evaluate $v_{\text{water}}$:

$$v_{\text{water}} = \frac{3\times10^8\ \text{m/s}}{1.33} = \boxed{2.25\times10^8\ \text{m/s}}$$

Substitute numerical values and evaluate $v_{\text{glass}}$:

$$v_{\text{glass}} = \frac{3\times10^8\ \text{m/s}}{1.5} = \boxed{2.00\times10^8\ \text{m/s}}$$

**34** •

**Picture the Problem** Let the subscript 1 refer to the air and the subscript 2 to the silicate glass and apply Snell's law to the air-glass interface.

Apply Snell's law to the air-glass interface to obtain:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Solve for $\theta_2$:

$$\theta_2 = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_1\right)$$

Substitute numerical values for the light of wavelength 400 nm and evaluate $\theta_{2,\,400\text{ nm}}$:

$$\theta_{2,\,400\,\text{nm}} = \sin^{-1}\left(\frac{1}{1.66}\sin 45°\right) = \boxed{25.2°}$$

Substitute numerical values for the light of wavelength 700 nm and evaluate $\theta_{2,\,700\text{ nm}}$:

$$\theta_{2,\,700\,\text{nm}} = \sin^{-1}\left(\frac{1}{1.61}\sin 45°\right) = \boxed{26.1°}$$

**35** ••

**Picture the Problem** Let the subscript 1 refer to the water and the subscript 2 to the glass and apply Snell's law to the water-glass interface.

Apply Snell's law to the water-glass interface to obtain:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Solve for $\theta_2$:

$$\theta_2 = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_1\right)$$

(*a*) Evaluate $\theta_2$ for $\theta_1$ = 60°:

$$\theta_2 = \sin^{-1}\left(\frac{1.33}{1.5}\sin 60°\right) = \boxed{50.2°}$$

(*b*) Evaluate $\theta_2$ for $\theta_1$ = 45°:

$$\theta_2 = \sin^{-1}\left(\frac{1.33}{1.5}\sin 45°\right) = \boxed{38.8°}$$

(*c*) Evaluate $\theta_2$ for $\theta_1$ = 30°:

$$\theta_2 = \sin^{-1}\left(\frac{1.33}{1.5}\sin 30°\right) = \boxed{26.3°}$$

**36** ••

**Picture the Problem** Let the subscript 1 refer to the glass and the subscript 2 to the water and apply Snell's law to the glass-water interface.

Apply Snell's law to the water-glass interface to obtain:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Solve for $\theta_2$:

$$\theta_2 = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_1\right)$$

(*a*) Evaluate $\theta_2$ for $\theta_1 = 60°$:

$$\theta_2 = \sin^{-1}\left(\frac{1.5}{1.33}\sin 60°\right) = \boxed{77.6°}$$

(*b*) Evaluate $\theta_2$ for $\theta_1 = 45°$:

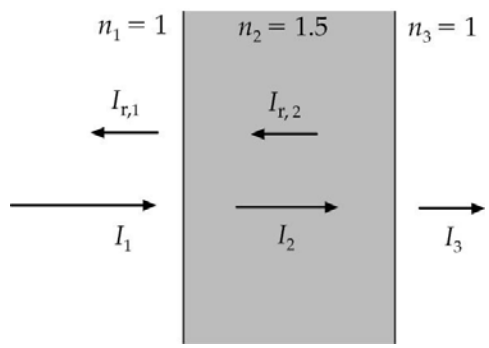
$$\theta_2 = \sin^{-1}\left(\frac{1.5}{1.33}\sin 45°\right) = \boxed{52.9°}$$

(*c*) Evaluate $\theta_2$ for $\theta_1 = 30°$:

$$\theta_2 = \sin^{-1}\left(\frac{1.5}{1.33}\sin 30°\right) = \boxed{34.3°}$$

***37 ••**

**Picture the Problem** Let the subscript 1 refer to the medium to the left (air) of the first interface, the subscript 2 to glass, and the subscript 3 to the medium (air) to the right of the second interface. Apply the equation relating the intensity of reflected light at normal incidence to the intensity of the incident light and the indices of refraction of the media on either side of the interface to both interfaces. We'll neglect multiple reflections at glass-air interfaces.




Express the intensity of the transmitted light in the second medium:

$$I_2 = I_1 - I_{r,1} = I_1 - \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 I_1$$

$$= I_1\left[1 - \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2\right]$$

Express the intensity of the transmitted light in the third medium:

$$I_3 = I_2 - I_{r,2} = I_2 - \left(\frac{n_2 - n_3}{n_2 + n_3}\right)^2 I_2$$

$$= I_2\left[1 - \left(\frac{n_2 - n_3}{n_2 + n_3}\right)^2\right]$$

Substitute for $I_2$ to obtain:

$$I_3 = I_1\left[1 - \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2\right]\left[1 - \left(\frac{n_2 - n_3}{n_2 + n_3}\right)^2\right]$$

Solve for the ratio $I_3/I_1$:

$$\frac{I_3}{I_1} = \left[1 - \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2\right]\left[1 - \left(\frac{n_2 - n_3}{n_2 + n_3}\right)^2\right]$$

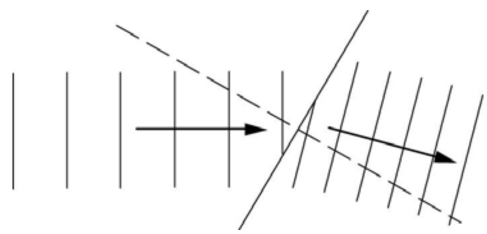
Substitute numerical values and evaluate $I_3/I_1$:

$$\frac{I_3}{I_1}=\left[1-\left(\frac{1-1.5}{1+1.5}\right)^2\right]\left[1-\left(\frac{1.5-1}{1.5+1}\right)^2\right]$$
$$=0.922=\boxed{92.2\%}$$

**38** ••

**Picture the Problem** As the line enters the muddy field, its speed is reduced by half and the direction of the forward motion of the line is changed. In this case, the forward motion in the muddy field makes an angle of 14.5° with respect to the normal of the boundary line. Note that the separation between successive lines in the muddy field is half that in the dry field.

**Picture the Problem** As the line enters the muddy field, its speed is reduced by half and the direction of the forward motion of the line is changed. In this case, the forward motion in the muddy field makes an angle of 14.5° with respect to the normal of the boundary line. Note that the separation between successive lines in the muddy field is half that in the dry field.



**39** ••

**Picture the Problem** We can apply Snell's law consecutively, first to the $n_1$-$n_2$ interface and then to the $n_2$-$n_3$ interface.

Apply Snell's law to the $n_1$-$n_2$ interface:

$$n_1\sin\theta_1=n_2\sin\theta_2$$

Apply Snell's law to the $n_2$-$n_3$ interface:

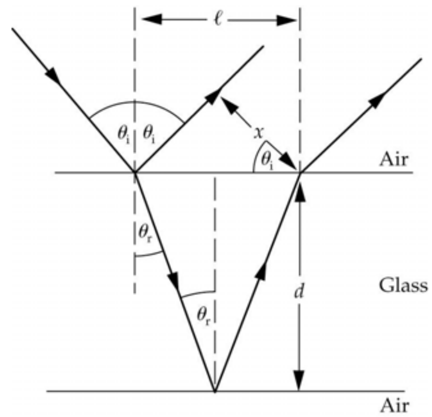
$$n_2\sin\theta_2=n_3\sin\theta_3$$

Equate the two expressions for $n_2\sin\theta_2$ to obtain:

$$\boxed{n_1\sin\theta_1=n_3\sin\theta_3}$$

***40** •••

**Picture the Problem** Let $x$ be the perpendicular separation between the two rays and let $\ell$ be the separation between the points of emergence of the two rays on the glass surface. We can use the geometry of the refracted and reflected rays to express $x$ as a function of $\ell$, $d$, $\theta_r$, and $\theta_i$. Setting the derivative of the resulting equation equal to zero will yield the value of $\theta_i$ that maximizes $x$.




(*a*) Express $\ell$ in terms of $d$ and the angle of refraction $\theta_r$:

$$\ell = 2d\tan\theta_r$$

Express $x$ as a function of $\ell$, $d$, $\theta_r$, and $\theta_i$:

$$x = 2d\tan\theta_r\cos\theta_i$$

Differentiate $x$ with respect to $\theta_i$:

$$\frac{dx}{d\theta_i} = 2d\frac{d}{d\theta_i}\left(\tan\theta_r\cos\theta_i\right) = 2d\left(-\tan\theta_r\sin\theta_i + \sec^2\theta_r\cos\theta_i\frac{d\theta_r}{d\theta_i}\right) \quad (1)$$

Apply Snell's law to the air-glass interface:

$$n_1\sin\theta_i = n_2\sin\theta_r \quad (2)$$

or, since $n_1 = 1$ and $n_2 = n$,

$$\sin\theta_i = n\sin\theta_r$$

Differentiate implicitly with respect to $\theta_i$ to obtain:

$$\cos\theta_i d\theta_i = n\cos\theta_r d\theta_r$$

or

$$\frac{d\theta_r}{d\theta_i} = \frac{1}{n}\frac{\cos\theta_i}{\cos\theta_r}$$

Substitute in equation (1) to obtain:

$$\frac{dx}{d\theta_i} = 2d\left(-\frac{\sin\theta_r}{\cos\theta_r}\sin\theta_i + \frac{1}{n}\frac{\cos\theta_i}{\cos^2\theta_r}\frac{\cos\theta_i}{\cos\theta_r}\right) = 2d\left(\frac{1}{n}\frac{\cos^2\theta_i}{\cos^3\theta_r} - \frac{\sin\theta_r\sin\theta_i}{\cos\theta_r}\right)$$

Substitute $1-\sin^2\theta_i$ for $\cos^2\theta_i$ and $\frac{1}{n}\sin\theta_i$ for $\sin\theta_r$ to obtain:

$$\frac{dx}{d\theta_i}=2d\left(\frac{1-\sin^2\theta_i}{n\cos^3\theta_r}-\frac{\sin^2\theta_i}{n\cos\theta_r}\right)$$

Multiply the second term in parentheses by $\cos^2\theta_r/\cos^2\theta_r$ and simplify to obtain:

$$\frac{dx}{d\theta_i}=2d\left(\frac{1-\sin^2\theta_i}{n\cos^3\theta_r}-\frac{\sin^2\theta_i\cos^2\theta_r}{n\cos^3\theta_r}\right)=\frac{2d}{n\cos^3\theta_r}\left(1-\sin^2\theta_i-\sin^2\theta_i\cos^2\theta_r\right)$$

Substitute $1-\sin^2\theta_r$ for $\cos^2\theta_r$:

$$\frac{dx}{d\theta_i}=\frac{2d}{n\cos^3\theta_r}\left[1-\sin^2\theta_i-\sin^2\theta_i\left(1-\sin^2\theta_r\right)\right]$$

Substitute $\frac{1}{n}\sin\theta_i$ for $\sin\theta_r$ to obtain:

$$\frac{dx}{d\theta_i}=\frac{2d}{n\cos^3\theta_r}\left[1-\sin^2\theta_i-\sin^2\theta_i\left(1-\frac{1}{n^2}\sin^2\theta_i\right)\right]$$

Factor out $1/n^2$, simplify, and set equal to zero to obtain:

$$\frac{dx}{d\theta_i}=\frac{2d}{n^3\cos^3\theta_r}\left[\sin^4\theta_i-2n^2\sin^2\theta_i+n^2\right]=0 \text{ for extrema}$$

If $dx/d\theta_1=0$, then it must be true that:

$$\sin^4\theta_i-2n^2\sin^2\theta_i+n^2=0$$

Solve this quartic equation for $\theta_i$ to obtain:

$$\theta_i=\boxed{\sin^{-1}\left(n\sqrt{1-\sqrt{1-\frac{1}{n^2}}}\right)}$$

(*b*) Evaluate $\theta_1$ for $n = 1.60$:

$$\theta_i=\sin^{-1}\left(1.6\sqrt{1-\sqrt{1-\frac{1}{(1.6)^2}}}\right)=\boxed{48.5°}$$

In (*a*) we showed that:

$$x=2d\tan\theta_r\cos\theta_i$$

Solve equation (2) for $\theta_r$:

$$\theta_r = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_i\right)$$

Substitute numerical values and evaluate $\theta_r$:

$$\theta_r = \sin^{-1}\left(\frac{1}{1.6}\sin 48.5°\right) = 27.9°$$

Substitute numerical values and evaluate $x$:

$$x = 2(4\,\text{cm})\tan 27.9°\cos 48.5° = \boxed{2.81\,\text{cm}}$$

## Total Internal Reflection

**41** •

**Picture the Problem** Let the subscript 1 refer to the glass and the subscript 2 to the water and use Snell's law under total internal reflection conditions.

Use Snell's law to obtain:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

When there is total internal reflection:

$$\theta_1 = \theta_c \text{ and } \theta_2 = 90°$$

Substitute to obtain:

$$n_1 \sin\theta_c = n_2 \sin 90° = n_2$$

Solve for $\theta_c$:

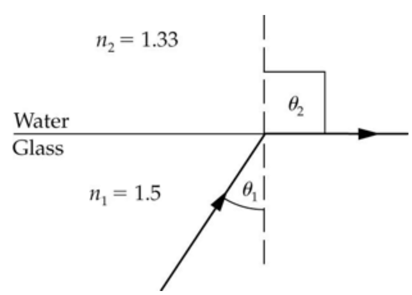
$$\theta_c = \sin^{-1}\frac{n_2}{n_1}$$

Substitute numerical values and evaluate $\theta_c$:

$$\theta_c = \sin^{-1}\frac{1.33}{1.5} = \boxed{62.5°}$$

**42** ••

**Picture the Problem** Let the index of refraction of glass be represented by $n_1$ and the index of refraction of water by $n_2$ and apply Snell's law to the glass-water interface under total internal reflection conditions.




Apply Snell's law to the glass-water interface:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

At the critical angle, $\theta_1 = \theta_c$ and

$$n_1 \sin\theta_c = n_2 \sin 90°$$

$\theta_2 = 90°$:

Solve for $\theta_c$:

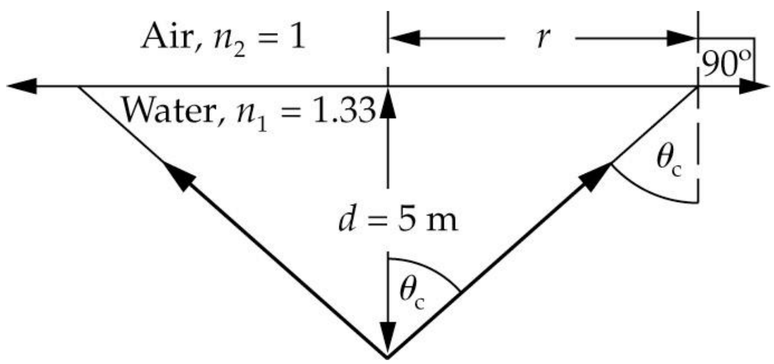
$$\theta_c = \sin^{-1}\left[\frac{n_2}{n_1}\sin 90°\right]$$

Substitute numerical values and evaluate $\theta_c$:

$$\theta_c = \sin^{-1}\left[\frac{1.33}{1.5}\sin 90°\right] = \boxed{62.5°}$$

**43** ••

**Picture the Problem** We can apply Snell's law to the water-air interface to express the critical angle $\theta_c$ in terms of the indices of refraction of water ($n_1$) and air ($n_2$) and then relate the radius of the circle to the depth $d$ of the point source and $\theta_c$.




Express the area of the circle whose radius is $r$:

$$A = \pi r^2$$

Relate the radius of the circle to the depth $d$ of the point source and the critical angle $\theta_c$:

$$r = d\tan\theta_c$$

Apply Snell's law to the water-air interface to obtain:

$$n_1\sin\theta_c = n_2\sin 90° = n_2$$

Solve for $\theta_c$:

$$\theta_c = \sin^{-1}\frac{n_2}{n_1}$$

Substitute for $r$ and $\theta_c$ to obtain:

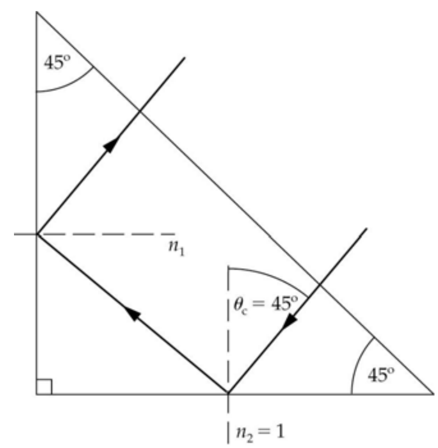
$$A = \pi\left[d\tan\theta_c\right]^2 = \pi\left[d\tan\left(\sin^{-1}\frac{n_2}{n_1}\right)\right]^2$$

Substitute numerical values and evaluate $A$:

$$A = \pi\left[(5\,\text{m})\tan\left(\sin^{-1}\frac{1}{1.33}\right)\right]^2 = \boxed{102\,\text{m}^2}$$

**44** ••

**Picture the Problem** We can use the definition of the index of refraction to express the speed of light in the prism in terms of the index of refraction $n_1$ of the prism. The application of Snell's law at the glass-air interface will allow us to relate the index of refraction of the prism to the critical angle for total internal reflection. Finally, we can use the geometry of the isosceles-right-triangle prism to conclude that $\theta_c = 45°$.




Express the speed of light $v$ in the prism in terms of its index of refraction $n_1$:

$$v = \frac{c}{n_1}$$

Apply Snell's law to the glass-air interface to obtain:

$$n_1 \sin\theta_c = n_2 \sin 90° = 1$$

Solve for $n_1$:

$$n_1 = \frac{1}{\sin\theta_c}$$

Substitute to obtain:

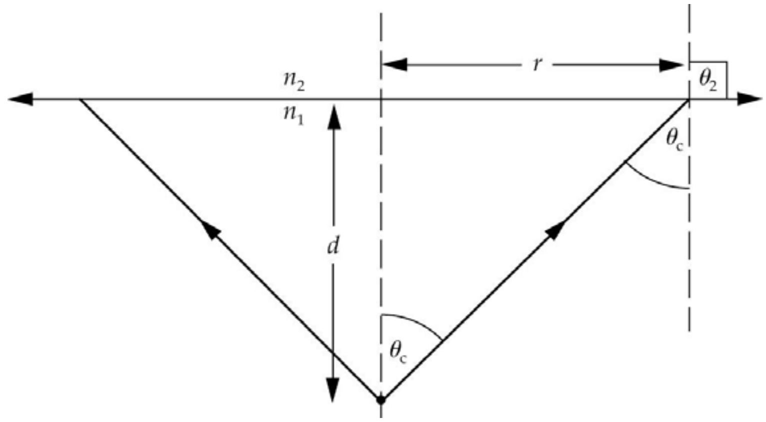
$$v = c\sin\theta_c$$

Substitute numerical values and evaluate $v$:

$$v = (2.998\times10^8\ \text{m/s})\sin 45° = \boxed{2.12\times10^8\ \text{m/s}}$$

**45** ••

**Picture the Problem** The observer above the surface of the fluid will not see any light until the angle of incidence of the light at the fluid-air interface is less than or equal to the critical angle for the two media. We can use Snell's law to express the index of refraction of the fluid in terms of the critical angle and use the geometry of card and light source to express the critical angle.




Apply Snell's law to the fluid-air interface to obtain:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Light is seen by the observer when $\theta_1 = \theta_c$ and $\theta_2 = 90°$:

$$n_1 \sin\theta_c = n_2 \sin 90° = n_2$$

Because the medium above the interface is air, $n_2 = 1$. Solve for $n_1$ to obtain:

$$n_1 = \frac{1}{\sin\theta_c}$$

From the geometry of the diagram:

$$\tan\theta_c = \frac{r}{d} \Rightarrow \theta_c = \tan^{-1}\frac{r}{d}$$

Substitute to obtain:

$$n_1 = \frac{1}{\sin\left[\tan^{-1}\frac{r}{d}\right]}$$

Substitute numerical values and evaluate $n_1$:

$$n_1 = \frac{1}{\sin\left[\tan^{-1}\frac{6\,\text{cm}}{5\,\text{cm}}\right]} = \boxed{1.30}$$

***46** ••

**Picture the Problem** We can use the geometry of the figure, the law of refraction at the air-$n_1$ interface, and the condition for total internal reflection at the $n_1$-$n_2$ interface to show that the numerical aperture is given by $\sqrt{n_2^2 - n_3^2}$ .

Referring to the figure, note that:

$$\sin\theta_c = \frac{n_3}{n_2} = \frac{a}{c}$$

and

$$\sin\theta_2 = \frac{b}{c}$$

Apply the Pythagorean theorem to the right triangle to obtain:

$$a^2 + b^2 = c^2$$
or
$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$$

Solve for $\frac{b}{c}$:

$$\frac{b}{c} = \sqrt{1 - \frac{a^2}{c^2}}$$

Substitute for $\frac{a}{c}$ and $\frac{b}{c}$ to obtain:

$$\sin\theta_2 = \sqrt{1 - \frac{n_3^2}{n_2^2}}$$

Use the law of refraction to relate $\theta_1$ and $\theta_2$:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Substitute for $\sin\theta_2$ and let $n_1 = 1$ (air) to obtain:

$$\sin\theta_1 = n_2\sqrt{1 - \frac{n_3^2}{n_2^2}} = \boxed{\sqrt{n_2^2 - n_3^2}}$$

**47** •

**Picture the Problem** We can use the result of Problem 46 to find the maximum angle of incidence under the given conditions.

From Problem 46:

$$\sin\theta_0 = \sqrt{n_1^2 - n_2^2}$$

Solve for $\theta_0$:

$$\theta_0 = \sin^{-1}\left(\sqrt{n_1^2 - n_2^2}\right)$$

Substitute numerical values and evaluate $\theta_0$:

$$\theta_0 = \sin^{-1}\left(\sqrt{(1.492)^2 - (1.489)^2}\right) = \boxed{5.43°}$$

**48** ••

**Picture the Problem** Examination of the figure reveals that, if the length of the tube is *L*, the distance traveled by the pulse that enters at an angle $\theta_0$ is the ratio of *a* to *b* multiplied by *L*. Let the subscripts 1 and 2 denote the pulses entering the tube normally and at an angle $\theta_0$, respectively.

Express the difference in time $\Delta t$ needed for the two pulses to travel a distance $L$:

$$\Delta t = t_2 - t_1 = \frac{L\frac{b}{a}}{\frac{c}{n_1}} - \frac{L}{\frac{c}{n_1}}$$

Substitute for $t_2$ and $t_1$ and simplify to obtain:

$$\Delta t = \frac{L\frac{b}{a}}{\frac{c}{n_1}} - \frac{L}{\frac{c}{n_1}} = \frac{n_1 L}{c}\left(\frac{b}{a} - 1\right) \qquad (1)$$

Referring to the figure, note that:

$$\sin\theta_c = \frac{b}{a}$$

From Snell's law, the sine of the critical angle is also given by:

$$\sin\theta_c = \frac{n_2}{n_1} \Rightarrow \frac{b}{a} = \frac{n_2}{n_1}$$

Substitute for $b/a$ in equation (1) and simplify to obtain:

$$\Delta t = \frac{n_1 L}{c}\left(\frac{n_2}{n_1} - 1\right) = \frac{L}{c}(n_2 - n_1)$$

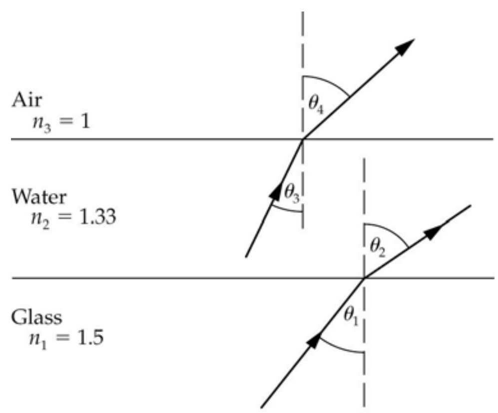
Substitute numerical values and evaluate $\Delta t$:

$$\Delta t = \frac{15\,\text{km}}{3\times10^8\ \text{m/s}}(1.492 - 1.489)$$

$$= \boxed{150\,\text{ns}}$$

**49** •••

**Picture the Problem** Let the index of refraction of glass be represented by $n_1$, the index of refraction of water by $n_2$, and the index of refraction of air by $n_3$. We can apply Snell's law to the glass-water interface under total internal reflection conditions to find the critical angle for total internal reflection. The application of Snell's law to glass-air and glass-water interfaces will allow us to decide whether there are angles of incidence greater than $\theta_c$ for glass-to-air refraction for which light rays will leave the glass and the water and pass into the air.




(*a*) Apply Snell's law to the glass-water interface:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

At the critical angle, $\theta_1 = \theta_c$ and $\theta_2 = 90°$:

$$n_1 \sin\theta_c = n_2 \sin 90°$$

Solve for $\theta_c$:

$$\theta_c = \sin^{-1}\left[\frac{n_2}{n_1}\sin 90°\right]$$

Substitute numerical values and evaluate $\theta_c$:

$$\theta_c = \sin^{-1}\left[\frac{1.33}{1.5}\sin 90°\right] = \boxed{62.5°}$$

(*b*) Apply Snell's law to a glass-air interface:

$$n_1 \sin\theta_c = n_3 \sin 90°$$

or

$$1.5\sin\theta_c = \sin 90° = 1$$

Solve for $\theta_c$:

$$\theta_c = \sin^{-1}\left(\frac{1}{1.5}\right) = 41.8°$$

Apply Snell's law to a ray incident at the critical angle for a glass-water interface:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

or

$$1.5\sin 41.8° = 1.33\sin\theta_2$$

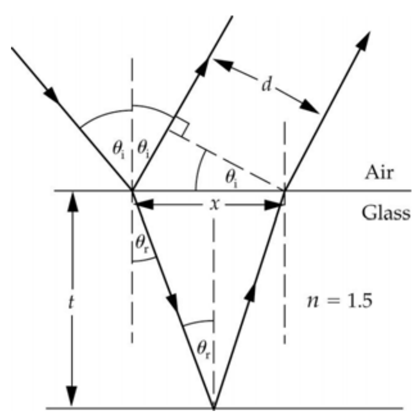
Solve for $\theta_2$:

$$\theta_2 = \sin^{-1}\left(\frac{1.5\sin 41.8°}{1.33}\right) = 48.7°$$

$$\boxed{\text{Note that } \theta_2 \text{ equals the critical angle for a water-air interface. Therefore, the ray will not leave the water for } \theta_1 \geq 41.8°.}$$

**50** •••

**Picture the Problem** The situation is shown in the adjacent figure. We can use the geometry of the diagram and trigonometric relationships to derive an expression for $d$ in terms of the angles of incidence and refraction. Applying Snell's law will yield $\theta_r$.




Express the distance $x$ in terms of $t$ and $\theta_r$:

$$x = 2t\tan\theta_r$$

The separation of the reflected rays is:

$$d = x\cos\theta_{\mathrm{i}}$$

Substitute to obtain:

$$d = 2t\tan\theta_{\mathrm{r}}\cos\theta_{\mathrm{i}} \qquad (1)$$

Apply Snell's law at the air-glass interface to obtain:

$$\sin\theta_{\mathrm{i}} = n\sin\theta_{\mathrm{r}}$$

Solve for $\theta_{\mathrm{r}}$:

$$\theta_{\mathrm{r}} = \sin^{-1}\left(\frac{\sin\theta_{\mathrm{i}}}{n}\right)$$

Substitute in equation (1) to obtain:

$$d = 2t\tan\left[\sin^{-1}\left(\frac{\sin\theta_{\mathrm{i}}}{n}\right)\right]\cos\theta_{\mathrm{i}}$$

Substitute numerical values and evaluate $d$:

$$d = 2(3\,\mathrm{cm})\tan\left[\sin^{-1}\left(\frac{\sin 40°}{1.5}\right)\right]\cos 40° = \boxed{2.18\,\mathrm{cm}}$$

## Dispersion

***51 ••**

**Picture the Problem** We can apply Snell's law of refraction to express the angles of refraction for red and violet light in silicate flint glass.

Express the difference between the angle of refraction for violet light and for red light:

$$\Delta\theta = \theta_{\mathrm{r,red}} - \theta_{\mathrm{r,violet}} \qquad (1)$$

Apply Snell's law of refraction to the interface to obtain:

$$\sin 45° = n\sin\theta_{\mathrm{r}}$$

Solve for $\theta_{\mathrm{r}}$:

$$\theta_{\mathrm{r}} = \sin^{-1}\left(\frac{1}{\sqrt{2}n}\right)$$

Substitute in equation (1):

$$\Delta\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}n_{\mathrm{red}}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{2}n_{\mathrm{violet}}}\right)$$

Substitute numerical values and evaluate $\Delta\theta$:

$$\Delta\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}(1.60)}\right) - \sin^{-1}\left(\frac{1}{\sqrt{2}(1.66)}\right) = 26.23° - 25.21° = \boxed{1.02°}$$

**52 ••**

**Picture the Problem** The transit times will be different because the speed with which light of various wavelengths propagates in silicate crown glass is dependent on the index of refraction. We can use Table 31-26 to estimate the indices of refraction for pulses of wavelengths 500 and 700 nm.

Express the difference in time needed for two short pulses of light to travel a distance $L$ in the fiber:

$$\Delta t = \frac{L}{v_{500}} - \frac{L}{v_{700}}$$

Substitute for $L$, $v_{500}$, and $v_{700}$ and simplify to obtain:

$$\Delta t = \frac{n_{500}L}{c} - \frac{n_{700}L}{c} = \frac{L}{c}\left(n_{500} - n_{700}\right)$$

Use Table 31-26 to find the indices of refraction of silicate crown glass for the two wavelengths:

$$n_{500} \approx 1.55$$

and

$$n_{700} \approx 1.50$$

Substitute numerical values and evaluate $\Delta t$:

$$\Delta t = \frac{15\,\text{km}}{2.998\times10^{8}\ \text{m/s}}(1.55-1.50) = \boxed{2.50\,\mu\text{s}}$$

## Polarization

**53 •**

**Picture the Problem** The polarizing angle is given by Brewster's law: $\tan\theta_{\text{p}} = n_2/n_1$ where $n_1$ and $n_2$ are the indices of refraction on the near and far sides of the interface, respectively.

Use Brewster's law to obtain:

$$\theta_{\text{p}} = \tan^{-1}\left(\frac{n_2}{n_1}\right)$$

($a$) For $n_1 = 1$ and $n_2 = 1.33$:

$$\theta_{\text{p}} = \tan^{-1}\left(\frac{1.33}{1}\right) = \boxed{53.1^\circ}$$

($b$) For $n_1 = 1$ and $n_2 = 1.50$:

$$\theta_{\text{p}} = \tan^{-1}\left(\frac{1.50}{1}\right) = \boxed{56.3^\circ}$$

**54 •**

**Picture the Problem** The intensity of the transmitted light $I$ is related to the intensity of the incident light $I_0$ and the angle the transmission axis makes with the horizontal $\theta$

according to $I = I_0 \cos^2 \theta$.

Express the intensity of the transmitted light in terms of the intensity of the incident light and the angle the transmission axis makes with the horizontal:

$$I = I_0 \cos^2 \theta$$

Solve for $\theta$:

$$\theta = \cos^{-1} \sqrt{\frac{I}{I_0}}$$

Substitute numerical values and evaluate $\theta$:

$$\theta = \cos^{-1} \sqrt{0.15} = 67.2°$$

and $\boxed{(d) \text{ is correct.}}$

**55** •

**Picture the Problem** Let $I_n$ be the intensity after the *n*th polarizing sheet and use $I = I_0 \cos^2 \theta$ to find the intensity of the light transmitted through all three sheets for $\theta = 45°$ and $\theta = 30°$.

(*a*) Express the intensity of the light between the first and second sheets:

$$I_1 = \tfrac{1}{2} I_0$$

Express the intensity of the light between the second and third sheets:

$$I_2 = I_1 \cos^2 \theta_{1,2} = \tfrac{1}{2} I_0 \cos^2 45° = \tfrac{1}{4} I_0$$

Express the intensity of the light that has passed through the third sheet:

$$I_3 = I_2 \cos^2 \theta_{2,3} = \tfrac{1}{4} I_0 \cos^2 45° = \boxed{\tfrac{1}{8} I_0}$$

(*b*) Express the intensity of the light between the first and second sheets:

$$I_1 = \tfrac{1}{2} I_0$$

Express the intensity of the light between the second and third sheets:

$$I_2 = I_1 \cos^2 \theta_{1,2} = \tfrac{1}{2} I_0 \cos^2 30° = \tfrac{3}{8} I_0$$

Express the intensity of the light that has passed through the third sheet:

$$I_3 = I_2 \cos^2 \theta_{2,3} = \tfrac{3}{8} I_0 \cos^2 60° = \boxed{\tfrac{3}{32} I_0}$$

**56** ••

**Picture the Problem** Because the light is polarized in the vertical direction and the first polarizer is also vertically polarized, no loss of intensity results from the first transmission. We can use Malus's law to find the intensity of the light after it has passed through the second polarizer.

The intensity of the beam is the ratio of its power to cross-sectional area:

$$I = \frac{P}{A}$$

Express the intensity of the light between the first and second polarizers:

$$I_1 = I_0 \text{ and } P_1 = P_0$$

Express Malus's law in terms of the power of the beam:

$$\frac{P}{A} = \frac{P_0}{A}\cos^2\theta \Rightarrow P = P_0\cos^2\theta$$

Express the power of the beam after the second transmission:

$$P_2 = P_1\cos^2\theta_{1,2} = P_0\cos^2\theta_{12}$$

Substitute numerical values and evaluate $I_2$:

$$P_2 = (5\,\text{mW})\cos^2 27° = \boxed{3.97\,\text{mW}}$$

**57** ••

**Picture the Problem** Assume that light is incident in air ($n_1$ = 1). We can use the relationship between the polarizing angle and the angle of refraction to determine the latter and Brewster's law to find the index of refraction of the substance.

(*a*) At the polarizing angle, the sum of the angles of polarization and refraction is 90°:

$$\theta_\text{p} + \theta_\text{r} = 90°$$

Solve for $\theta_\text{r}$:

$$\theta_\text{r} = 90° - \theta_\text{p}$$

Substitute for $\theta_\text{p}$ to obtain:

$$\theta_\text{r} = 90° - 60° = \boxed{30.0°}$$

(*b*) From Brewster's law we have:

$$\tan\theta_\text{p} = \frac{n_2}{n_1}$$

or, because $n_1$ = 1,

$$n_2 = \tan\theta_\text{p}$$

Substitute for $\theta_\text{p}$ and evaluate $n_2$:

$$n_2 = \tan 60° = \boxed{1.73}$$

**58** ••

**Picture the Problem** Let $I_n$ be the intensity after the $n$th polarizing sheet and use $I = I_0 \cos^2\theta$ to find the intensity of the light transmitted through the three sheets.

Express the intensity of the light between the first and second sheets:

$$I_1 = \tfrac{1}{2} I_0$$

Express the intensity of the light between the second and third sheets:

$$I_2 = I_1 \cos^2\theta_{1,2} = \tfrac{1}{2} I_0 \cos^2\theta$$

Express the intensity of the light that has passed through the third sheet and simplify to obtain:

$$\begin{aligned} I_3 &= I_2 \cos^2\theta_{2,3} \\ &= \tfrac{1}{2} I_0 \cos^2\theta \cos^2(90° - \theta) \\ &= \tfrac{1}{2} I_0 \cos^2\theta \sin^2\theta \\ &= \tfrac{1}{8} I_0 (2\cos\theta \sin\theta)^2 \\ &= \boxed{\tfrac{1}{8} I_0 \sin^2 2\theta} \end{aligned}$$

Because the sine function is a maximum when its argument is 90°, the maximum value of $I_3$ occurs when:

$$\boxed{\theta = 45.0°}$$

**59** ••

**Picture the Problem** Let $I_n$ be the intensity after the $n$th polarizing sheet, use $I = I_0 \cos^2\theta$ to find the intensity of the light transmitted through each sheet, and replace $\theta$ with $\omega t$.

Express the intensity of the light between the first and second sheets:

$$I_1 = \tfrac{1}{2} I_0$$

Express the intensity of the light between the second and third sheets:

$$I_2 = I_1 \cos^2\theta_{1,2} = \tfrac{1}{2} I_0 \cos^2\omega t$$

Express the intensity of the light that has passed through the third sheet and simplify to obtain:

$$\begin{aligned} I_3 &= I_2 \cos^2\theta_{2,3} \\ &= \tfrac{1}{2} I_0 \cos^2\omega t \cos^2(90° - \omega t) \\ &= \tfrac{1}{2} I_0 \cos^2\omega t \sin^2\omega t \\ &= \tfrac{1}{8} I_0 (2\cos\omega t \sin\omega t)^2 \\ &= \boxed{\tfrac{1}{8} I_0 \sin^2 2\omega t} \end{aligned}$$

***60** ••

**Picture the Problem** Let $I_n$ be the intensity after the $n$th polarizing sheet and use $I = I_0 \cos^2\theta$ to find the ratio of $I_{n+1}$ to $I_n$.

($a$) Find the ratio of $I_{n+1}$ to $I_n$:

$$\frac{I_{n+1}}{I_n} = \cos^2\frac{\pi}{2N}$$

Because there are $N$ such reductions of intensity:

$$\frac{I_{N+1}}{I_1} = \frac{I_{N+1}}{I_0} = \cos^{2N}\left(\frac{\pi}{2N}\right)$$

and

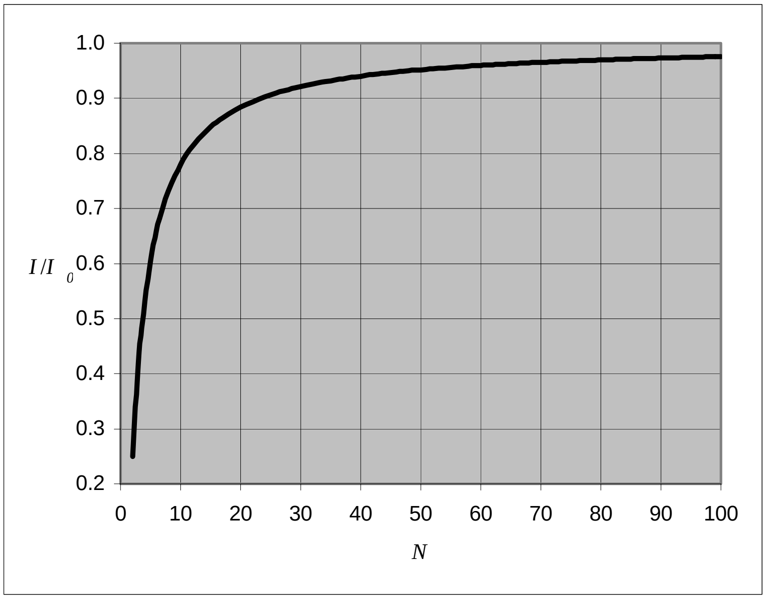
$$I_{N+1} = \boxed{I_0 \cos^{2N}\left(\frac{\pi}{2N}\right)}$$

($b$) A spreadsheet program to graph $I_{N+1}/I_0$ as a function of $N$ is shown below. The formulas used to calculate the quantities in the columns are as follows:

| Cell | Content/Formula | Algebraic Form |
|---|---|---|
| A2 | 2 | $N$ |
| A3 | A2 + 1 | $N + 1$ |
| B2 | (cos(PI()/(2*A2))^(2*A2) | $\cos^{2N}\left(\frac{\pi}{2N}\right)$ |

| | A | B |
|---|---|---|
| 1 | $N$ | $I/I_0$ |
| 2 | 2 | 0.250 |
| 3 | 3 | 0.422 |
| 4 | 4 | 0.531 |
| 5 | 5 | 0.605 |
| | | |
| 95 | 95 | 0.974 |
| 96 | 96 | 0.975 |
| 97 | 97 | 0.975 |
| 98 | 98 | 0.975 |
| 99 | 99 | 0.975 |
| 100 | 100 | 0.976 |

A graph of $I/I_0$ as a function of $N$ follows.




(*c*) $\boxed{\text{In each case, the polarization of the transmitted beam is perpendicular to that of the incident beam.}}$

**61** ••

**Picture the Problem** Let $I_n$ be the intensity after the *n*th polarizing sheet and use $I = I_0 \cos^2\theta$ to find the ratio of $I_{n+1}$ to $I_n$. Because each sheet introduces a 2% loss of intensity, the net transmission after *N* sheets $(0.98)^N$.

Find the ratio of $I_{n+1}$ to $I_n$:

$$\frac{I_{n+1}}{I_n} = (0.98)\cos^2\frac{\pi}{2N}$$

Because there are *N* such reductions of intensity:

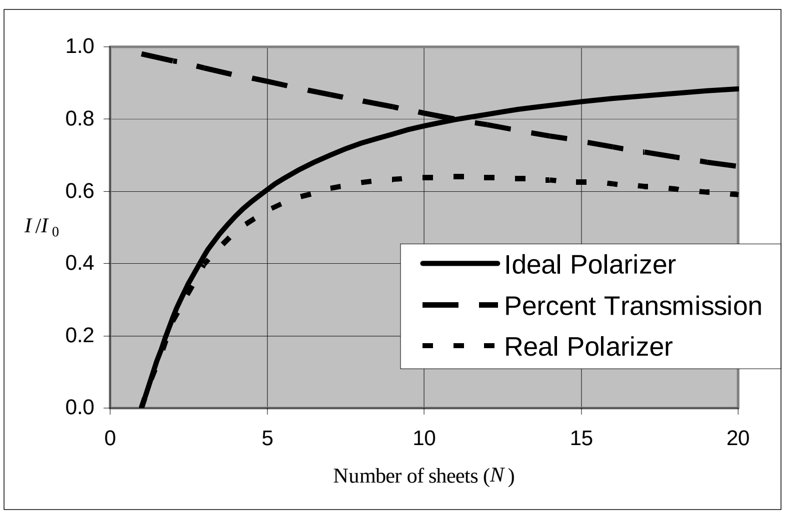
$$\frac{I_{N+1}}{I_0} = (0.98)^N \cos^{2N}\left(\frac{\pi}{2N}\right)$$

(*b*) A spreadsheet program to graph $I_{N+1}/I_0$ for an ideal polarizer as a function of *N*, the percent transmission, and $I_{N+1}/I_0$ for a real polarizer as a function of *N* is shown below. The formulas used to calculate the quantities in the columns are as follows:

| Cell | Content/Formula | Algebraic Form |
|---|---|---|
| A3 | 1 | $N$ |
| B2 | (cos(PI()/(2*A2))^(2*A2) | $\cos^{2N}\left(\frac{\pi}{2N}\right)$ |
| C3 | (0.98)^A3 | $(0.98)^N$ |
| D4 | B3*C3 | $(0.98)^N\cos^{2N}\left(\frac{\pi}{2N}\right)$ |

| | A | B | C | D |
|---|---|---|---|---|
| 1 | | Ideal | Percent | Real |
| 2 | N | Polarizer | Transmission | Polarizer |
| 3 | 1 | 0.000 | 0.980 | 0.000 |
| 4 | 2 | 0.250 | 0.960 | 0.240 |
| 5 | 3 | 0.422 | 0.941 | 0.397 |
| 6 | 4 | 0.531 | 0.922 | 0.490 |
| 7 | 5 | 0.605 | 0.904 | 0.547 |
| 8 | 6 | 0.660 | 0.886 | 0.584 |
| 9 | 7 | 0.701 | 0.868 | 0.608 |
| 10 | 8 | 0.733 | 0.851 | 0.624 |
| 11 | 9 | 0.759 | 0.834 | 0.633 |
| 12 | 10 | 0.781 | 0.817 | 0.638 |
| 13 | 11 | 0.798 | 0.801 | 0.639 |
| 14 | 12 | 0.814 | 0.785 | 0.638 |
| 15 | 13 | 0.827 | 0.769 | 0.636 |
| 16 | 14 | 0.838 | 0.754 | 0.632 |
| 17 | 15 | 0.848 | 0.739 | 0.626 |
| 18 | 16 | 0.857 | 0.724 | 0.620 |
| 19 | 17 | 0.865 | 0.709 | 0.613 |
| 20 | 18 | 0.872 | 0.695 | 0.606 |
| 21 | 19 | 0.878 | 0.681 | 0.598 |
| 22 | 20 | 0.884 | 0.668 | 0.590 |

A graph of $I/I_0$ as a function of $N$ for the quantities described above follows:




Inspection of the table, as well as of the graph, tells us that the optimum number of sheets is $\boxed{13.}$

***62** ••

**Picture the Problem** A circularly polarized wave is said to be *right circularly polarized* if the electric and magnetic fields rotate clockwise when viewed along the direction of propagation and *left circularly polarized* if the fields rotate counterclockwise.

For a circularly polarized wave, the $x$ and $y$ components of the electric field are given by:

$$E_x = E_0 \cos \omega t$$

and

$$E_y = E_0 \sin \omega t \text{ or } E_y = -E_0 \sin \omega t$$

for left and right circular polarization, respectively.

For a wave polarized along the $x$ axis:

$$\vec{E}_{\text{right}} + \vec{E}_{\text{left}} = E_0 \cos \omega t \, \hat{i} + E_0 \cos \omega t \, \hat{i}$$

$$= \boxed{2E_0 \cos \omega t \, \hat{i}}$$

**63** ••

**Picture the Problem** Let $I_n$ be the intensity after the $n$th polarizing sheet and use $I = I_0 \cos^2 \theta$ to find the intensity of the light transmitted by the four sheets.

($a$) Express the intensity of the light between the first and second sheets:

$$I_1 = \tfrac{1}{2} I_0$$

Express the intensity of the light between the second and third sheets:

$$I_2 = I_1 \cos^2 \theta_{1,2} = \tfrac{1}{2} I_0 \cos^2 30° = \tfrac{3}{8} I_0$$

Express the intensity of the light between the third and fourth sheets:

$$I_3 = I_2 \cos^2 \theta_{2,3} = \tfrac{3}{8} I_0 \cos^2 30° = \tfrac{9}{32} I_0$$

Express the intensity of the light to the right of the fourth sheet:

$$I_4 = I_3 \cos^2 \theta_{3,4} = \tfrac{9}{32} I_0 \cos^2 30° = \tfrac{27}{128} I_0$$

$$= \boxed{0.211 I_0}$$

Note that, for the single sheet between the two end sheets at $\theta = 45^\circ$, $I = 0.125 I_0$. Using two sheets at relative angles of $30^\circ$ increases the transmitted intensity.

**Remarks: We could also apply the result obtained in Problem 60(*a*) to solve this problem.**

***64** ••

**Picture the Problem** We can use the components of $\vec{E}$ to show that $\vec{E}$ is constant in time and rotates with angular frequency $\omega$.

Express the magnitude of $\vec{E}$ in terms of its components:

$$E = \sqrt{E_x^2 + E_y^2}$$

Substitute for $E_x$ and $E_y$ to obtain:

$$E = \sqrt{\left[E_0 \sin(kx - \omega t)\right]^2 + \left[E_0 \cos(kx - \omega t)\right]^2} = \sqrt{E_0^2\left[\sin^2(kx - \omega t) + \cos^2(kx - \omega t)\right]}$$
$$= E_0$$

and the $\vec{E}$ vector rotates in the $yz$ plane with angular frequency $\omega$.

**65** ••

**Picture the Problem** We can apply the given definitions of right and left circular polarization to the electric field and magnetic fields of the wave.

The electric field of the wave in Problem 64 is:

$$\vec{E} = E_0 \sin(kx - \omega t)\hat{j} + E_0 \cos(kx - \omega t)\hat{k}$$

The corresponding magnetic field is:

$$\vec{B} = B_0 \sin(kx - \omega t)\hat{k} - B_0 \cos(kx - \omega t)\hat{j}$$

Because these fields rotate clockwise when viewed along the direction of propagation, the wave is $\boxed{\text{right circularly polarized.}}$

For a left circularly polarized wave traveling in the opposite direction:

$$\vec{E} = \boxed{E_0 \sin(kx + \omega t)\hat{j} - E_0 \cos(kx + \omega t)\hat{k}}$$

## General Problems

**66** •

**Picture the Problem** We can use $v = f\lambda$ and the definition of the index of refraction to relate the wavelength of light in a medium whose index of refraction is $n$ to the wavelength of light in air.

($a$) The wavelength $\lambda_n$ of light in a medium whose index of refraction is $n$ is given by:

$$\lambda_n = \frac{v}{f} = \frac{c}{nf} = \frac{\lambda_0}{n}$$

Substitute numerical values and evaluate $\lambda_{\text{water}}$:

$$\lambda_{\text{water}} = \frac{700\,\text{nm}}{n_{\text{water}}} = \frac{700\,\text{nm}}{1.33} = \boxed{526\,\text{nm}}$$

(*b*) $\boxed{\text{Because the color observed depends on the frequency of the light, a swimmer observes the same color in air and in water.}}$

**67** ••

**Picture the Problem** We can use Snell's law, under critical angle and polarization conditions, to relate the polarizing angle of the substance to the critical angle for internal reflection.

Apply Snell's law, under critical angle conditions, to the interface:

$$n_1 \sin\theta_c = n_2 \qquad (1)$$

Apply Snell's law, under polarization conditions, to the interface:

$$n_1 \sin\theta_p = n_2 \sin(90° - \theta_p) = n_2 \cos\theta_p$$

or

$$\tan\theta_p = \frac{n_2}{n_1}$$

Solve for $\theta_p$:

$$\theta_p = \tan^{-1}\left(\frac{n_2}{n_1}\right) \qquad (2)$$

Solve equation (1) for the ratio of $n_2$ to $n_1$:

$$\frac{n_2}{n_1} = \sin\theta_c$$

Substitute for $n_2/n_1$ in equation (2) to obtain:

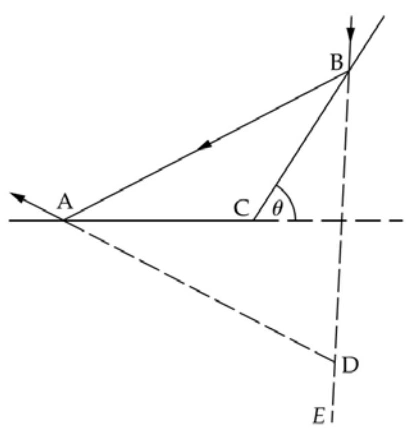
$$\theta_p = \tan^{-1}(\sin\theta_c)$$

Substitute numerical values and evaluate $\theta_p$:

$$\theta_p = \tan^{-1}(\sin 45°) = \boxed{35.3°}$$

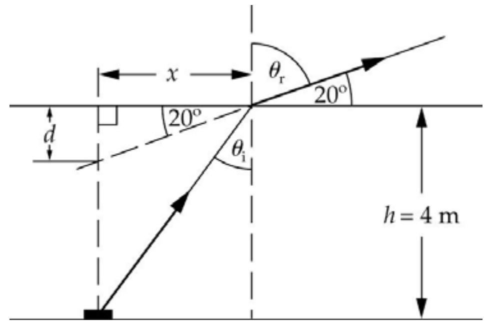
***68** ••

**Picture the Problem** Angle *ADE* is the angle between the direction of the incoming ray and that reflected by the two mirror surfaces. Note that triangle *ABC* is isosceles and that angles *CAB* and *ABC* are equal and their sum equals $\theta$. Also from the law of reflection, angles *CAD* and *CBD* equal angle *ABC*. Because angle *BAD* is twice *BAC* and angle *DBA* is twice *CBA*, angle *ADE* is twice the angle $\theta$.



**69** ••

**Picture the Problem** The sketch shows the ray from the coin passing through the water to the eye of the observer. We can use trigonometry to express the apparent depth *d* in terms of the depth *h* of the water, the 20° angle, and the angle of incidence $\theta_i$. The application of Snell's law at the interface will yield an expression for $\theta_i$.




Express the apparent depth *d* in terms of the distance *x*:

$$d = x\tan 20° \qquad (1)$$

Relate the distance *x* to the depth of the water and the angle $\theta_i$:

$$x = h\tan\theta_i$$

Substitute for *x* in equation (1) to obtain:

$$d = h\tan\theta_i\tan 20° \qquad (2)$$

Apply Snell's law to the water-air interface:

$$n_1\sin\theta_i = n_2\sin\theta_r$$

Solve for $\theta_i$:

$$\theta_i = \sin^{-1}\left(\frac{n_2}{n_1}\sin\theta_r\right)$$

Substitute for $\theta_i$ in equation (2) to obtain:

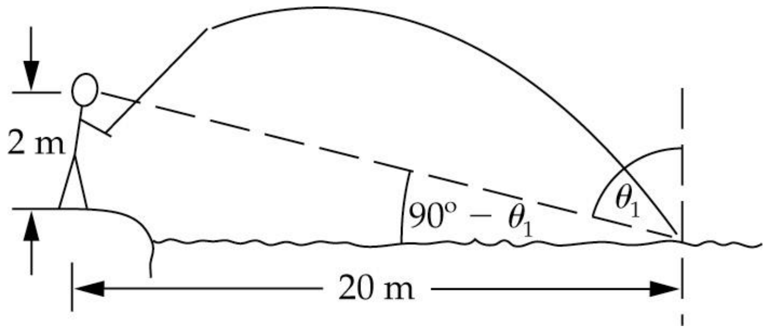
$$d = h\tan\left[\sin^{-1}\left(\frac{n_2}{n_1}\sin\theta_r\right)\right]\tan 20°$$

Substitute numerical values and evaluate *d*:

$$d = (4\,\text{m})\tan\left[\sin^{-1}\left(\frac{1}{1.33}\sin 70°\right)\right]\tan 20° = \boxed{1.45\,\text{m}}$$

**70** ••

**Picture the Problem** Assume that the sound source is the voice of the fisherman and that the fisherman's mouth is 2 m from the surface of the water as shown below. We can apply Snell's law at the air-water interface to find $\theta_c$ and use trigonometry to find $\theta_1$. If we can show that $\theta_1 > \theta_c$, then we can conclude that the noise on shore cannot possibly be sensed by fish 20 m from shore.




Apply Snell's law at the air-water interface 20 m from the shore:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

For $\theta_1 = \theta_c$:

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) = \sin^{-1}\left(\frac{v_1}{v_2}\right)$$

Substitute numerical values and evaluate $\theta_c$:

$$\theta_c = \sin^{-1}\left(\frac{330\ \text{m/s}}{1450\ \text{m/s}}\right) = 13.2^\circ$$

Relate $\theta_1$ to the distance from the shore and the distance from the surface of the water to the fisherman's mouth:

$$\tan(90^\circ - \theta_1) = \frac{2\ \text{m}}{20\ \text{m}}$$

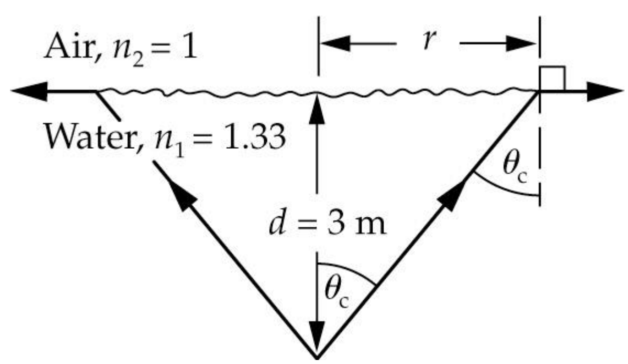
Solve for and evaluate $\theta_1$:

$$\theta_1 = 90^\circ - \tan^{-1}(0.1) = 84.3^\circ$$

$$\boxed{\text{Because } \theta_1 > \theta_c, \text{ all the sound is reflected at air - water interface.}}$$

***71 ••**

**Picture the Problem** We can apply Snell's law to the water-air interface to express the critical angle $\theta_c$ in terms of the indices of refraction of water ($n_1$) and air ($n_2$) and then relate the radius of the circle to the depth $d$ of the swimmer and $\theta_c$.



Relate the radius of the circle to the depth $d$ of the point source and the critical angle $\theta_c$:

$$r = d\tan\theta_c$$

Apply Snell's law to the water-air interface to obtain:

$$n_1\sin\theta_c = n_2\sin 90° = n_2$$

Solve for $\theta_c$:

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

Substitute for $\theta_c$ to obtain:

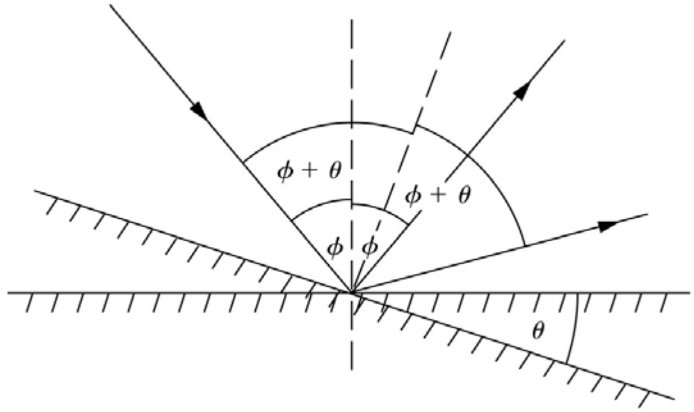
$$r = d\tan\left[\sin^{-1}\left(\frac{n_2}{n_1}\right)\right]$$

Substitute numerical values and evaluate $r$:

$$r = (3\,\text{m})\tan\left[\sin^{-1}\left(\frac{1}{1.33}\right)\right] = \boxed{3.42\,\text{m}}$$

**72** ••

**Picture the Problem** Let $\phi$ be the initial angle of incidence. Since the angle of reflection with the normal to the mirror is also $\phi$, the angle between incident and reflected rays is $2\phi$. If the mirror is now rotated by a further angle $\theta$, the angle of incidence is increased by $\theta$ to $\phi + \theta$, and so is the angle of reflection. Consequently, the reflected beam is rotated by $2\theta$ relative to the incident beam.




**73** ••

**Picture the Problem** We can apply Snell's law at the glass-air interface to express $\theta_c$ in terms of the index of refraction of the glass and use Figure 31-25 to find the index of refraction of the glass for the given wavelengths of light.

Apply Snell's law at the glass-air interface:

$$n_1\sin\theta_1 = n_2\sin\theta_2$$

If $\theta_1 = \theta_c$ and $n_2 = 1$:

$$n_1 \sin\theta_c = \sin 90° = 1$$

and

$$\theta_c = \sin^{-1}\left(\frac{1}{n_1}\right)$$

(*a*) For violet light of wavelength 400 nm, $n_2 = 1.67$:

$$\theta_c = \sin^{-1}\left(\frac{1}{1.67}\right) = \boxed{36.8°}$$

(*b*) For red light of wavelength 700 nm, $n_2 = 1.60$:

$$\theta_c = \sin^{-1}\left(\frac{1}{1.60}\right) = \boxed{38.7°}$$

**74** ••

**Picture the Problem** We'll neglect multiple reflections at the glass-air interfaces. We can use the expression (Equation 31-11) for the reflected intensity at an interface to express the intensity of the light in the glass slab as the difference between the intensity of the incident beam and the reflected beam. Repeating this analysis at the glass-air interface will lead to the desired result.

Express the intensity of the light transmitted into the glass:

$$I_{glass} = I_0 - I_{R,1}$$

where $I_{R,1}$ is the intensity of the light reflected at the air-glass interface.

The intensity of the light reflected at the air-glass interface is:

$$I_{R,1} = \left(\frac{1-n}{1+n}\right)^2 I_0$$

Substitute and simplify to obtain:

$$I_{glass} = I_0 - \left(\frac{1-n}{1+n}\right)^2 I_0 = I_0\left[1 - \left(\frac{1-n}{1+n}\right)^2\right] = I_0\left[\frac{4n}{(1+n)^2}\right]$$

Express the intensity of the light transmitted at the glass-air interface:

$$I_T = I_{glass} - I_{R,2}$$

where $I_{R,2}$ is the intensity of the light reflected at the glass-air interface.

The intensity of the light reflected at the glass-air interface is:

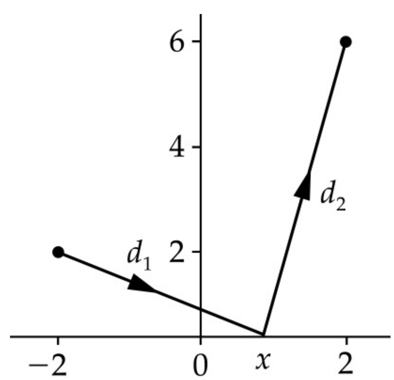
$$I_{R,2} = \left(\frac{1-n}{1+n}\right)^2 I_{glass} = \left(\frac{1-n}{1+n}\right)^2\left[\frac{4n}{(1+n)^2}\right]I_0$$

Substitute and simplify to obtain:

$$I_T = I_0\left[\frac{4n}{(1+n)^2}\right] - \left(\frac{1-n}{1+n}\right)^2\left[\frac{4n}{(1+n)^2}\right]I_0$$
$$= I_0\left[1-\left(\frac{1-n}{1+n}\right)^2\right]\left[\frac{4n}{(1+n)^2}\right]$$
$$= I_0\left[\frac{4n}{(1+n)^2}\right]\left[\frac{4n}{(1+n)^2}\right]$$
$$= \boxed{I_0\left[\frac{4n}{(1+n)^2}\right]^2}$$

**75** ••

**Picture the Problem** We can write an expression for the total distance traveled by the light as a function of *x* and set the derivative of this expression equal to zero to find the value of *x* that minimizes the distance traveled by the light. The adjacent figure shows the two points and the reflecting surface. The *x* and *y* coordinates are in meters.




(*a*) Express the total distance *D* traveled by the light:

$$D = d_1 + d_2 = \sqrt{(x+2)^2+4} + \sqrt{(2-x)^2+36}$$

Differentiate *D* with respect to *x*:

$$\frac{dD}{dx} = \frac{d}{dx}\left[\sqrt{(x+2)^2+4} + \sqrt{(2-x)^2+36}\right]$$
$$= \tfrac{1}{2}\left[(x+2)^2+4\right]^{-\frac{1}{2}}2(x+2) + \tfrac{1}{2}\left[(2-x)^2+36\right]^{-\frac{1}{2}}2(2-x)(-1) = 0 \text{ for extrema}$$

Simplify this expression to obtain:

$$\frac{x+2}{\sqrt{(x+2)^2+4}}-\frac{2-x}{\sqrt{(2-x)^2+36}}=0$$

Solve for $x$ to obtain:

$$x=\boxed{-1.00\,\text{m}}$$

(*b*) With $x$ = –1 m:

$$\theta_{\text{i}}=\tan^{-1}\left[\frac{-2-(-1)}{0-2}\right]$$

$$=\tan^{-1}\left(\frac{1}{2}\right)=\boxed{26.6^\circ}$$

and

$$\theta_{\text{r}}=\tan^{-1}\left[\frac{-1-(2)}{0-6}\right]$$

$$=\tan^{-1}\left(\frac{3}{6}\right)=\boxed{26.6^\circ}$$

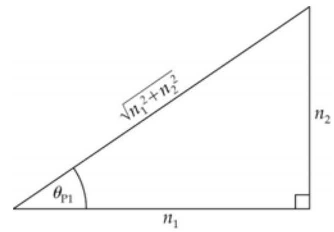
***76 ••**

**Picture the Problem** Let the angle of refraction at the first interface by $\theta_1$ and the angle of refraction at the second interface be $\theta_2$. We can apply Snell's law at each interface and eliminate $\theta_1$ and $n_2$ to show that $\theta_2 = \theta_{P2}$.

Apply Snell's Brewster's law at the $n_1$-$n_2$ interface:

$$\tan\theta_{\text{P1}}=\frac{n_2}{n_1}$$

Draw a reference triangle consistent with Brewster's law:




Apply Snell's law at the $n_1$-$n_2$ interface:

$$n_1\sin\theta_{\text{P1}}=n_2\sin\theta_1$$

Solve for $\theta_1$ to obtain:

$$\theta_1=\sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_{\text{P1}}\right)$$

Referring to the reference triangle we note that:

$$\theta_1=\sin^{-1}\left(\frac{n_1}{n_2}\frac{n_2}{\sqrt{n_1^2+n_2^2}}\right)$$

$$=\sin^{-1}\left(\frac{n_1}{\sqrt{n_1^2+n_2^2}}\right)$$

i.e., $\theta_1$ is the complement of $\theta_{p1}$.

Apply Snell's law at the $n_2$-$n_1$ interface:

$$n_2 \sin\theta_1 = n_1 \sin\theta_2$$

Solve for $\theta_2$ to obtain:

$$\theta_2 = \sin^{-1}\left(\frac{n_2}{n_1}\sin\theta_1\right)$$

Refer to the reference triangle again to obtain:

$$\theta_2 = \sin^{-1}\left(\frac{n_2}{n_1}\frac{n_1}{\sqrt{n_1^2 + n_2^2}}\right)$$

$$= \sin^{-1}\left(\frac{n_2}{\sqrt{n_1^2 + n_2^2}}\right) = \boxed{\theta_{P2}}$$

Equate these expressions for $n_2 \sin\theta_1$ to obtain:

$$n_1 \sin\theta_P = n_1 \sin\theta_2 \Rightarrow \theta_2 = \boxed{\theta_P}$$

**77** ••

**Picture the Problem** We can use Brewster's law in conjunction with index of refraction data from Figure 31-29 to calculate the polarization angles for the air-glass interface.

From Brewster's law we have:

$$\theta_p = \tan^{-1}\left(\frac{n_2}{n_1}\right)$$

or, for $n_1 = 1$,

$$\theta_p = \tan^{-1} n_2$$

For silicate flint glass, $n_2 \approx 1.62$ and:

$$\theta_p = \tan^{-1}(1.62) = \boxed{58.3°}$$

For borate flint glass, $n_2 \approx 1.57$ and:

$$\theta_p = \tan^{-1}(1.57) = \boxed{57.5°}$$

For quartz glass, $n_2 \approx 1.54$ and:

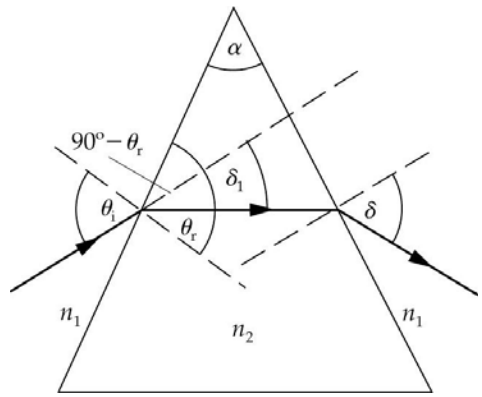
$$\theta_p = \tan^{-1}(1.54) = \boxed{57.0°}$$

For silicate crown glass, $n_2 \approx 1.51$ and:

$$\theta_p = \tan^{-1}(1.51) = \boxed{56.5°}$$

**78** •••

**Picture the Problem** The diagram to the right shows the angles of incidence, refraction, and deviation at the first interface. We can use the geometry of this symmetric passage of the light to express $\theta_r$ in terms of $\alpha$ and $\delta_1$ in terms of $\theta_r$ and $\alpha$. We can then use a symmetry argument to express the deviation at the second interface and the total deviation $\delta$. Finally, we can apply Snell's law at the first interface to complete the derivation of the given expression.




(*a*) With respect to the normal to the left face of the prism, let the angle of incidence be $\theta_i$ and the angle of refraction be $\theta_r$. From the geometry of the figure, it is evident that:

$$\theta_r = \tfrac{1}{2}\alpha$$

Express the angle of deviation at the refracting surface:

$$\delta_1 = \theta_i - \theta_r = \theta_i - \tfrac{1}{2}\alpha$$

By symmetry, the angle of deviation at the second refracting surface is also of this magnitude. Thus:

$$\delta = 2\delta_1 = 2\theta_i - \alpha$$

Solve for $\theta_i$:

$$\theta_i = \tfrac{1}{2}(\alpha + \delta)$$

Apply Snell's law, with $n_1 = 1$ and $n_2 = n$, to the first interface:

$$\sin\theta_i = n\sin\tfrac{1}{2}\alpha$$

Substitute for $\theta_i$ to obtain:

$$\boxed{\sin\frac{\alpha+\delta}{2} = n\sin\frac{\alpha}{2}} \qquad (1)$$

(*b*) The angular separation is:

$$\Delta\delta = \delta_{\text{violet}} - \delta_{\text{red}}$$

Solve equation (1) for $\delta$:

$$\delta = 2\sin^{-1}\left[n\sin\frac{\alpha}{2}\right] - \alpha$$

Substitute to obtain:

$$\begin{aligned}\Delta\delta &= 2\sin^{-1}\left[n_{\text{violet}}\sin\frac{\alpha}{2}\right]-\alpha-\left\{2\sin^{-1}\left[n_{\text{red}}\sin\frac{\alpha}{2}\right]-\alpha\right\}\\ &= 2\sin^{-1}\left[n_{\text{violet}}\sin\frac{\alpha}{2}\right]-2\sin^{-1}\left[n_{\text{red}}\sin\frac{\alpha}{2}\right]\end{aligned}$$

Substitute numerical values and evaluate $\Delta\delta$:

$$\Delta\delta = 2\sin^{-1}\left[1.52\sin\frac{60^\circ}{2}\right]-2\sin^{-1}\left[1.48\sin\frac{60^\circ}{2}\right]=\boxed{3.47^\circ}$$

***79** ••

**Picture the Problem** We can apply Snell's law at the critical angle and the polarizing angle to show that tan $\theta_p$ = sin $\theta_c$.

(*a*) Apply Snell's law at the medium-vacuum interface:

$$n_1\sin\theta_1 = n_2\sin\theta_r$$

For $\theta_1 = \theta_c$, $n_1 = n$, and $n_2 = 1$:

$$n\sin\theta_c = \sin 90^\circ = 1$$

For $\theta_1 = \theta_p$, $n_1 = n$, and $n_2 = 1$:

$$\tan\theta_p = \frac{n_2}{n_1}=\frac{1}{n} \Rightarrow n\tan\theta_p = 1$$

Because both expressions equal one:

$$\boxed{\tan\theta_p = \sin\theta_c}$$

(*b*) For any value of $\theta$:

$$\tan\theta > \sin\theta \Rightarrow \boxed{\theta_p > \theta_c}$$

**80** ••

**Picture the Problem** Let the numeral 1 refer to the side of the interface from which the light is incident and the numeral 2 to the refraction side of the interface. We can apply Snell's law, under the conditions described in the problem statement, at the interface to derive an expression for *n* as a function of the angle of incidence (also the polarizing angle).

(*a*) Apply Snell's law at the air-medium interface:

$$\sin\theta_1 = n\sin\theta_2$$

Because the reflected and refracted rays are mutually perpendicular:

$$\theta_1+\theta_2 = 90^\circ \Rightarrow \theta_2 = 90^\circ-\theta_1$$

Substitute for $\theta_2$ to obtain:

$$\sin\theta_1 = n\sin(90° - \theta_1) = n\cos\theta_1$$

or

$$n = \tan\theta_1 = \tan\theta_p$$

Substitute for $\theta_p$ and evaluate $n$:

$$n = \tan 58° = \boxed{1.60}$$

(*b*) Apply Snell's law at the interface under conditions of total internal reflection:

$$n_2 \sin\theta_c = n_1 \sin 90° = n_1$$

Because $n_1 = 1$:

$$\theta_c = \sin^{-1}\left(\frac{1}{n_2}\right) = \sin^{-1}\left(\frac{1}{n}\right)$$

Substitute for $n$ and evaluate $\theta_c$:

$$\theta_c = \sin^{-1}\left(\frac{1}{1.6}\right) = \boxed{38.7°}$$

**81** ••

**Picture the Problem** We can apply Snell's law at the glass–liquid and liquid–air interfaces to find the refractive index of the unknown liquid, the angle of incidence (glass-air interface) for total internal reflection, and the angle of refraction of a ray into the liquid film.

(*a*) Apply Snell's law, under critical-angle conditions, at the glass–liquid interface:

$$\sin\theta_c = \frac{n_{\text{liquid}}}{n_{\text{glass}}}$$

Solve for $n_{\text{liquid}}$:

$$n_{\text{liquid}} = n_{\text{glass}} \sin\theta_c$$

Substitute numerical values and evaluate $n_{\text{liquid}}$:

$$n_{\text{liquid}} = (1.655)\sin 53.7° = \boxed{1.33}$$

(*b*) With the liquid removed:

$$\theta_c = \sin^{-1}\left(\frac{1}{n_{\text{glass}}}\right)$$

Substitute numerical values and evaluate $\theta_c$:

$$\theta_c = \sin^{-1}\left(\frac{1}{1.655}\right) = \boxed{37.2°}$$

(*c*) Apply Snell's law at the glass–liquid interface:

$$n_{\text{glass}} \sin\theta_1 = n_{\text{liquid}} \sin\theta_2$$

Solve for $\theta_2$:

$$\theta_2 = \sin^{-1}\left[\frac{n_{\text{glass}}}{n_{\text{liquid}}}\sin\theta_1\right]$$

Substitute numerical values and evaluate $\theta_2$:

$$\theta_2 = \sin^{-1}\left[\frac{1.655}{1.33}\sin 37.2°\right] = \boxed{48.8°}$$

$$\boxed{\begin{array}{l}\text{Because } \theta_2 \text{ is also the angle of incidence at the liquid} - \text{air interface and because} \\ \text{it is larger than the critical angle for total internal reflection at this interface, no} \\ \text{light will emerge.}\end{array}}$$

**82** ••

**Picture the Problem** We can use Equation 31-18 and the result of Problem 86 to find the angular separation of these colors in the primary rainbow.

Express the angular separation $\Delta\phi$ of the colors:

$$\Delta\phi = \phi_{\text{d,blue}} - \phi_{\text{d,red}} \qquad (1)$$

From Equation 31-18, with $n_{\text{air}} = 1$ and $n_{\text{water}} = n$:

$$\phi_{\text{d}} = \pi + 2\theta_1 - 4\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)$$

From Problem 86:

$$\cos\theta_{1\text{m}} = \sqrt{\frac{n^2-1}{3}}$$

or

$$\theta_{1\text{m}} = \cos^{-1}\left[\sqrt{\frac{n^2-1}{3}}\right]$$

Substitute to obtain:

$$\phi_{\text{d}} = \pi + 2\cos^{-1}\left[\sqrt{\frac{n^2-1}{3}}\right] - 4\sin^{-1}\left(\frac{\sin\left\{\cos^{-1}\left[\sqrt{\frac{n^2-1}{3}}\right]\right\}}{n}\right)$$

Evaluate $\phi_{\text{d}}$ for blue light in water:

$$\phi_{\text{d,blue}} = \pi + 2\cos^{-1}\left[\sqrt{\frac{(1.3435)^2-1}{3}}\right] - 4\sin^{-1}\left(\frac{\sin\left\{\cos^{-1}\left[\sqrt{\frac{(1.3435)^2-1}{3}}\right]\right\}}{1.3435}\right)$$

$$= 139.42°$$

Evaluate $\phi_d$ for red light in water:

$$\phi_{\text{d,red}} = \pi + 2\cos^{-1}\left[\sqrt{\frac{(1.3318)^2-1}{3}}\right] - 4\sin^{-1}\left(\frac{\sin\left\{\cos^{-1}\left[\sqrt{\frac{(1.3318)^2-1}{3}}\right]\right\}}{1.3318}\right)$$

$$= 137.75°$$

Substitute in equation (1) and evaluate $\Delta\phi$:

$$\Delta\phi = 139.42 - 137.75° = \boxed{1.67°}$$

**83** ••

**Picture the Problem** We can use the result, obtained in Problem 74, that each slab reduces the intensity of the transmitted light by $\left[\frac{4n}{(n+1)^2}\right]^2$, to find the ratio of the transmitted intensity to the incident intensity through $N$ parallel slabs of glass for light of normal incidence.

(*a*) From Problem 74, each slab reduces the intensity by the factor:

$$\left[\frac{4n}{(n+1)^2}\right]^2$$

For $N$ slabs:

$$I_t = I_0\left[\frac{4n}{(n+1)^2}\right]^{2N}$$

and

$$\frac{I_t}{I_0} = \boxed{\left[\frac{4n}{(n+1)^2}\right]^{2N}} \qquad (1)$$

(*b*) Evaluate equation (1) with $N = 3$ and $n = 1.5$:

$$\frac{I_t}{I_0} = \left[\frac{4(1.5)}{(1.5+1)^2}\right]^{2(3)} = \boxed{0.783}$$

(*c*) Begin the solution of equation (1) for *N* by taking the logarithm (arbitrarily to base 10) of both sides of the equation:

$$\log\left(\frac{I_t}{I_0}\right) = \log\left[\frac{4n}{(n+1)^2}\right]^{2N} = 2N\log\left[\frac{4n}{(n+1)^2}\right]$$

Solve for *N*:

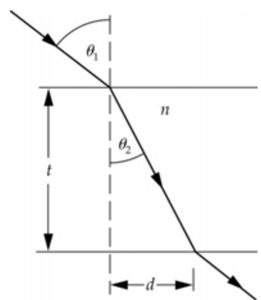
$$N = \frac{\log\left(\frac{I_t}{I_0}\right)}{2\log\left[\frac{4n}{(n+1)^2}\right]}$$

Substitute numerical values and evaluate *N*:

$$N = \frac{\log(0.1)}{2\log\left[\frac{4(1.5)}{(1.5+1)^2}\right]} = 28.2 \approx \boxed{28}$$

**84** ••

**Picture the Problem** We can apply Snell's law at the air-slab interface to express the index of refraction *n* in terms of $\theta_1$ and $\theta_2$ and then use the geometry of the figure to relate $\theta_2$ to *t* and *d*.




Apply Snell's law to the first interface:

$$\sin\theta_1 = n\sin\theta_2$$

Solve for *n*:

$$n = \frac{\sin\theta_1}{\sin\theta_2}$$

From the diagram:

$$d = t\tan\theta_2 \Rightarrow \theta_2 = \tan^{-1}\left(\frac{d}{t}\right)$$

Substitute to obtain:

$$n = \boxed{\frac{\sin\theta_1}{\sin\left[\tan^{-1}\left(\frac{d}{t}\right)\right]}}$$

***85** ••

**Picture the Problem** The angle that the rain appears to make with the vertical, according to the marathoner, is the angle whose tangent is the ratio of $v_{\text{runner}}$ to $v_{\text{rain}}$. The circular motion of the star is analogous to the circular motion of the cloud with $v_{\text{runner}} = v_{\text{earth}}$ and $v_{\text{rain}} = c$.

(*a*) The angle that the rain appears to make with the vertical to the marathoner is given by:

$$\theta = \tan^{-1}\left(\frac{v_{\text{runner}}}{v_{\text{rain}}}\right)$$

Substitute numerical values and evaluate $\theta$:

$$\theta = \tan^{-1}\left(\frac{4\,\text{m/s}}{9\,\text{m/s}}\right) = \boxed{24.0°}$$

(*b*) The cloud moves in a circle whose radius is given by:

$$R = H\tan\theta$$

Substitute numerical values and evaluate $R$:

$$R = (10\,\text{km})\tan 24° = \boxed{4.45\,\text{km}}$$

(*c*) Here $v_{\text{runner}} = v_{\text{earth}}$ and $v_{\text{rain}} = c$:

$$\theta = \boxed{\tan^{-1}\left(\frac{v_{\text{earth}}}{c}\right)} \qquad (1)$$

where $\theta = \frac{1}{2}(\text{angular diameter})$

(*d*) From equation (1):

$$c = \frac{v_{\text{earth}}}{\tan\theta} = \frac{2\pi R_{\text{earth-sun}}}{T_{\text{earth}}\tan\theta}$$

Convert 20.6″ to degrees:

$$20.6'' = 20.6'' \times \frac{1'}{60''} \times \frac{1°}{60'} = 5.722\times10^{-3\circ}$$

Substitute numerical values and evaluate $c$:

$$c = \frac{2\pi(1.5\times10^{11}\text{ m})}{(1\,\text{y})(3.156\times10^{7}\text{ s/y})\tan(20.6'')} = \boxed{2.99\times10^{8}\text{ m/s}}$$

Substitute numerical values and evaluate $c$:

$$c = \frac{2\pi(1.5\times10^{11}\text{ m})}{(1\,\text{y})(3.156\times10^{7}\text{ s/y})\tan(5.722\times10^{-3\circ})} = \boxed{2.99\times10^{8}\text{ m/s}}$$

**86** •••

**Picture the Problem** We can follow the directions given in the problem statement and use the hint to establish the given result.

($a$) Equation 31-18 is:

$$\phi_d = \pi + 2\theta_1 - 4\sin^{-1}\left(\frac{n_{air}\sin\theta_1}{n_{water}}\right)$$

For $n_{air} = 1$and $n_{water} = n$:

$$\phi_d = \pi + 2\theta_1 - 4\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)$$

Use the hint to differentiate $\phi_d$ with respect to $\theta_1$:

$$\frac{d\phi_d}{d\theta_1} = \frac{d}{d\theta_1}\left[\pi + 2\theta_1 - 4\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]$$
$$= \boxed{2 - \frac{4\cos\theta_1}{\sqrt{n^2 - \sin^2\theta_1}}}$$

($b$) Set $d\phi_d/d\theta_1 = 0$:

$$2 - \frac{4\cos\theta_1}{\sqrt{n^2 - \sin^2\theta_1}} = 0 \text{ for extrema}$$

Simplify to obtain:

$$16\cos^2\theta_1 = 4\left(n^2 - \sin^2\theta_1\right)$$

Replace $\sin^2\theta_1$ with $1 - \cos^2\theta_1$ and simplify:

$$12\cos^2\theta_1 = 4n^2 - 4$$

Solve for $\cos\theta_1 = \cos\theta_{1m}$:

$$\cos\theta_{1m} = \boxed{\sqrt{\frac{n^2 - 1}{3}}}$$

and

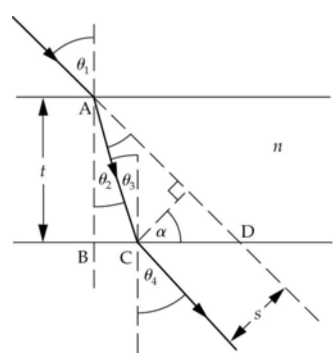
$$\theta_{1m} = \cos^{-1}\left[\sqrt{\frac{n^2 - 1}{3}}\right]$$

Evaluate $\theta_{1m}$ for $n = 1.33$:

$$\theta_{1m} = \cos^{-1}\left[\sqrt{\frac{(1.33)^2 - 1}{3}}\right] = \boxed{59.6°}$$

**87** •••

**Picture the Problem** Let the thickness of the slab be $t$ and the separation of the incident and emerging rays be $ds$. We can apply Snell's law at both interfaces and use the geometry of the diagram and trigonometric relationships to show that the emerging ray and incident ray are parallel and to derive an expression for $d$.



Apply Snell's law at the two interfaces to obtain:

$$\sin\theta_1 = n\sin\theta_2 \qquad (1)$$

and

$$n\sin\theta_3 = \sin\theta_4$$

Because $\theta_2$ and $\theta_3$ are equal (they are alternate interior angles formed by parallel lines and a transversal):

$$\sin\theta_1 = n\sin\theta_3$$

and

$$n\sin\theta_3 = \sin\theta_4$$

Substitute for $n\sin\theta_3$ in the first of these equations to obtain:

$\sin\theta_1 = \sin\theta_4 \Rightarrow \theta_1 = \theta_4$ and $\boxed{\text{the emerging ray and incident ray are parallel.}}$

Express the distance $d_{BD}$ in terms of $t$ and $\theta_1$:

$$d_{BD} = t\tan\theta_1$$

The distance $d_{BC}$ is:

$$d_{BC} = t\tan\theta_2$$

Use the distances $d_{BD}$ and $d_{BC}$ to express the distance $d_{CD}$:

$$d_{CD} = d_{BD} - d_{BC} = t(\tan\theta_1 - \tan\theta_2)$$

Because $\alpha$ and $\theta_1$ have their right and left sides mutually perpendicular, they are equal and:

$$\begin{aligned} s &= t(\tan\theta_1 - \tan\theta_2)\cos\alpha \\ &= t(\tan\theta_1 - \tan\theta_2)\cos\theta_1 \end{aligned} \qquad (2)$$

Substitute for $\tan\theta_1$ and $\tan\theta_2$ and simplify to obtain:Solve equation (1) for $\theta_2$:

$$\begin{aligned} s &= t\left(\frac{\sin\theta_1}{\cos\theta_1} - \frac{\sin\theta_2}{\cos\theta_2}\right)\cos\theta_1 \\ &= t\left(\sin\theta_1 - \frac{\sin\theta_2\cos\theta_1}{\cos\theta_2}\right) \\ &= \frac{t(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)}{\cos\theta_2} \\ &= \boxed{\frac{t\sin(\theta_1 - \theta_2)}{\cos\theta_2}} \end{aligned}$$

**Remarks: One can also derive this expression using the law of sines.**

**88** ••

**Picture the Problem** We can use Snell's law to determine $\theta_2$ and then apply the result of Problem 87 to find $s$.

From Problem 87 we have:
$$s = \frac{t\sin(\theta_1 - \theta_2)}{\cos\theta_2}$$

Apply Snell's law to the first interface to obtain:
$$\sin\theta_1 = n\sin\theta_2$$

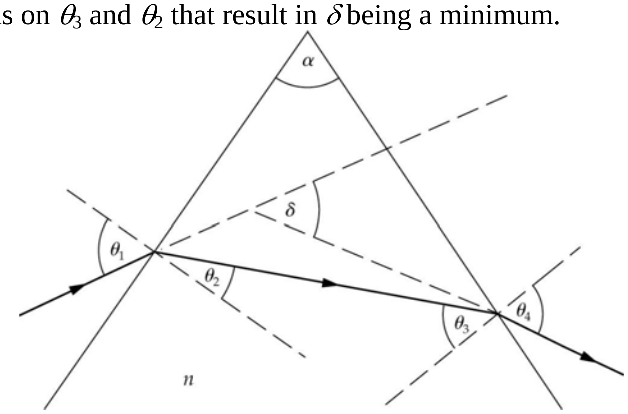
Solve for $\theta_2$:
$$\theta_2 = \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)$$

Substitute numerical values and evaluate $\theta_2$:
$$\theta_2 = \sin^{-1}\left(\frac{\sin 30°}{1.5}\right) = 19.47°$$

Substitute numerical values and evaluate $s$:
$$s = \frac{(15\,\text{mm})\sin(30° - 19.47°)}{\cos(19.47°)} = \boxed{2.91\,\text{mm}}$$

**89** •••

**Picture the Problem** The figure below shows the prism and the path of the ray through it. The dashed lines are the normals to the prism faces. The triangle formed by the interior ray and the prism faces has interior angles of $\alpha$, 90° – $\theta_2$, and 90° – $\theta_3$. Consequently, $\theta_2 + \theta_3 = \alpha$. We can apply Snell's law at both interfaces to express the angle of deviation $\delta$ as a function of $\theta_3$ and then set the derivative of this function equal to zero to find the conditions on $\theta_3$ and $\theta_2$ that result in $\delta$ being a minimum.




Express the angle of deviation:
$$\delta = \theta_1 + \theta_2 - \alpha \quad (1)$$

Apply Snell's law to relate $\theta_1$ to $\theta_2$ and $\theta_3$ to $\theta_4$:
$$\sin\theta_1 = n\sin\theta_2 \quad (2)$$
and
$$n\sin\theta_3 = \sin\theta_4 \quad (3)$$

Solve equation (2) for $\theta_1$ and
$$\theta_1 = \sin^{-1}(n\sin\theta_2)$$

equation (3) for $\theta_4$: and

$$\theta_4 = \sin^{-1}(n \sin\theta_3)$$

Substitute in equation (1) to obtain:

$$\delta = \sin^{-1}(n\sin\theta_2) + \sin^{-1}(n\sin\theta_3) - \alpha = \sin^{-1}[n\sin(\theta_3 - \alpha)] + \sin^{-1}(n\sin\theta_3) - \alpha$$

Note that the only variable in this expression is $\theta_3$. To determine the condition that minimizes $\delta$, take the derivative of $\delta$ with respect to $\theta_3$ and set it equal to zero.

$$\begin{aligned}\frac{d\delta}{d\theta_3} &= \frac{d}{d\theta_3}\left\{\sin^{-1}[n\sin(\theta_3 - \alpha)] + \sin^{-1}(n\sin\theta_3) - \alpha\right\} \\ &= -\frac{n\cos(\alpha - \theta_3)}{\sqrt{1 - [n\sin(\alpha - \theta_3)]^2}} + \frac{n\cos\theta_3}{\sqrt{1 - (n\sin\theta_3)^2}} = 0 \text{ for extrema}\end{aligned}$$

This equation is satisfied provided: $\alpha - \theta_3 = \theta_3 \Rightarrow \theta_3 = \frac{1}{2}\alpha$

Because $\theta_2 = \alpha - \theta_3$: $\theta_2 = \alpha - \frac{1}{2}\alpha = \frac{1}{2}\alpha$

Because $\theta_2 = \theta_3$, we can conclude that the deviation angle is a minimum if the ray passes through the prism symmetrically.

**Remarks: Setting $d\delta/d\theta_3 = 0$ establishes the condition on $\theta_3$ that $\delta$ is either a maximum or a minimum. To establish that $\delta$ is indeed a minimum when $\theta_3 = \theta_2 = \frac{1}{2}\alpha$, we can either show that $d^2\delta/d\theta_3^2$, evaluated at $\theta_3 = \frac{1}{2}\alpha$, is positive or, alternatively, plot a graph of $\delta(\theta_3)$ to show that it is concave upward at $\theta_3 = \frac{1}{2}\alpha$.**