Chapter 35
Applications of the Schrödinger Equation

Conceptual Problems

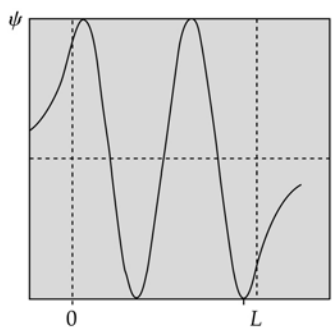
1 •

True

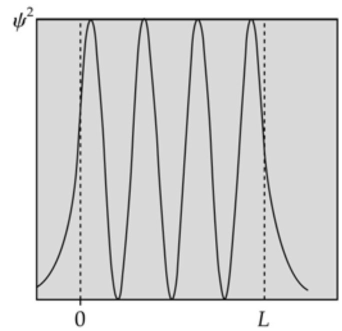
2 •

Determine the Concept Looking at the graphs in the text for the n = 1, 2, and 3 states, we note that the n = 4 state graph of the wave function must have four extrema in the region $0 < x < L$ and decay in toward zero in the regions $x < 0$ and $x > L$.

(*a*)



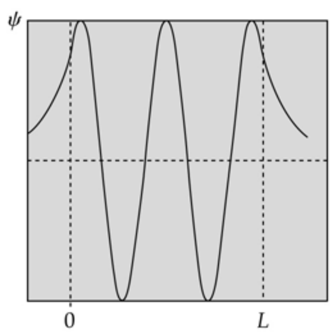
(*b*)



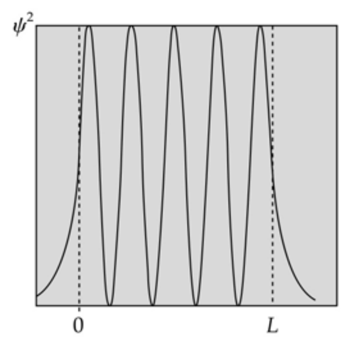
3 •

Determine the Concept Looking at the graphs in the text for the n = 1, 2, and 3 states, we note that the n = 5 state graph of the wave function must have five extrema in the region $0 < x < L$ and decay in toward zero in the regions $x < 0$ and $x > L$.

(*a*)



(*b*)



Estimation and Approximation

***4 •**

Picture the Problem Assume a mass of 150 g for the baseball, 30 cm for the width of the locker, and 1 cm/s for the speed of the ball, and equate the kinetic energy of the ball and the quantum-mechanical energy and solve for the quantum number *n*.

The allowed energy states of a particle of mass *m* in a 1-dimensional infinite potential well of width *L* are given by:

$$E_n = n^2\left(\frac{h^2}{8mL^2}\right)$$

The kinetic energy of the ball is:

$$K = \frac{1}{2}mv^2$$

For $E_n = K$:

$$n^2\left(\frac{h^2}{8mL^2}\right) = \frac{1}{2}mv^2$$

Solve for the quantum number *n*:

$$n = \frac{2mvL}{h}$$

Substitute numerical values and evaluate *n*:

$$n = \frac{2(0.15\,\text{kg})(0.01\,\text{m/s})(0.3\,\text{m})}{6.63\times10^{-34}\,\text{J}\cdot\text{s}}$$
$$= 1.36\times10^{30} \approx \boxed{10^{30}}$$

The Schrödinger Equation

5 ••

Picture the Problem We can show that $\psi_3(x)$ is a solution to the time-independent Schrödinger equation by differentiating it twice and substituting in Equation 35-4.

Equation 35-4 is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$$

Because $\psi_1(x)$ and $\psi_2(x)$ are solutions of Equation 35-4:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_1(x)}{dx^2} + U(x)\psi_1(x) = E\psi_1(x)$$

and

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_2(x)}{dx^2} + U(x)\psi_2(x) = E\psi_2(x)$$

Add these equations to obtain:

$$-\frac{\hbar^2}{2m}\left[\frac{d^2\psi_1(x)}{dx^2}+\frac{d^2\psi_2(x)}{dx^2}\right]+U(x)[\psi_1(x)+\psi_2(x)]=E[\psi_1(x)+\psi_2(x)] \quad (1)$$

Differentiate $\psi_3(x)=\psi_1(x)+\psi_2(x)$ twice with respect to x to obtain:

$$\frac{d\psi_3(x)}{dx}=\frac{d\psi_1(x)}{dx}+\frac{d\psi_2(x)}{dx}$$

and

$$\frac{d^2\psi_3(x)}{dx^2}=\frac{d^2\psi_1(x)}{dx^2}+\frac{d^2\psi_2(x)}{dx^2}$$

Substitute in equation (1) to obtain:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_3(x)}{dx^2}+U(x)\psi_3(x)=E\psi_3(x)$$

which shows that $\psi_3(x)=\psi_1(x)+\psi_2(x)$ satisfies Equation 35-4.

The Harmonic Oscillator

6 ••

Picture the Problem We can relate the spring constant to the mass of the hydrogen atom and its angular frequency and then use the relationship between the allowed energy levels and the angular frequency ω to derive an expression for the spring constant k.

The spring constant k is related to the mass m of the hydrogen molecule and its angular frequency ω:

$$k=m\omega^2 \quad (1)$$

Relate the energy spacing ΔE to the angular frequency ω:

$$\Delta E=hf=\frac{h\omega}{2\pi}=\hbar\omega$$

Solve for ω:

$$\omega=\frac{\Delta E}{\hbar}$$

Substitute for ω in equation (1) to obtain:

$$k=m\left(\frac{\Delta E}{\hbar}\right)^2$$

Substitute numerical values and evaluate k:

$$k = \left(1\,\text{u} \times \frac{1.66\times10^{-27}\ \text{kg}}{\text{u}}\right)\left(\frac{8.7\times10^{-20}\ \text{J}}{1.05\times10^{-34}\ \text{J}\cdot\text{s}}\right)^2 = \boxed{1.14\,\text{kN/m}}$$

Remarks: Our result is very similar to the stiffness constant of typical macroscopic springs. Note that strictly speaking one should use the reduced mass of a hydrogen molecule rather than the simpler model of a single atom attached to a fixed point.

7 ••

Determine the Concept The integral $\langle x\rangle = \int x|\psi|^2 dx = 0$ because the integrand is an odd function of x for the ground state as well as any excited state of the harmonic oscillator.

***8** ••

Picture the Problem We can differentiate $\psi(x)$ twice and substitute in the Schrödinger equation for the harmonic oscillator. Substitution of the given value for a will lead us to an expression for E_1.

The wave function for the first excited state of the harmonic oscillator is:

$$\psi_1(x) = A_1 x e^{-ax^2}$$

Compute $d\psi_1(x)/dx$:

$$\frac{d\psi_1(x)}{dx} = \frac{d}{dx}\left[A_1 x e^{-ax^2}\right] = A_1 e^{-ax^2}$$

Compute $d^2\psi_1(x)/dx^2$:

$$\frac{d^2\psi_1(x)}{dx^2} = \frac{d}{dx}\left[A_1 e^{-ax^2}\right] = -2axA_1e^{-ax^2} - 4axA_1e^{-ax^2} + 4a^2x^3A_1e^{-ax^2}$$
$$= \left(4a^2x^3 - 6ax\right)A_1e^{-ax^2}$$

Substitute in the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\left[\left(4a^2x^3 - 6ax\right)A_1e^{-ax^2}\right] + \tfrac{1}{2}m\omega_0^2x^2A_1xe^{-ax^2} = E_1A_1xe^{-ax^2}$$

Divide out $A_1e^{-ax^2}$ to obtain:

$$-\frac{\hbar^2}{2m}\left[\left(4a^2x^3 - 6ax\right)\right] + \tfrac{1}{2}m\omega_0^2x^3 = E_1x$$

or

$$-\frac{\hbar^2}{2m}\left(4a^2x^3\right)+\frac{\hbar^2}{2m}(6ax)+\tfrac{1}{2}m\omega_0^2x^3=E_1x$$

Substitute for a to obtain:

$$-\frac{\hbar^2}{2m}4\left(\frac{m\omega_0}{2\hbar}\right)^2x^3+\frac{\hbar^2}{2m}6\left(\frac{m\omega_0}{2\hbar}\right)x+\tfrac{1}{2}m\omega_0^2x^3=E_1x$$

Solve for E_1 to obtain:

$$E_1=\boxed{\tfrac{3}{2}\hbar\omega_0}=3E_0$$

9 •••

Picture the Problem We must show that, with $A_0 = (2m\omega_0/h)^{1/4}$, the normalization condition $\int_{-\infty}^{\infty}\left|\psi_0(x)\right|^2dx=\int_{-\infty}^{\infty}\left|A_0e^{-ax^2}\right|^2dx=1$ is satisfied.

We need to show that:

$$\int_{-\infty}^{\infty}\left|\psi_0(x)\right|^2dx=\int_{-\infty}^{\infty}\left|A_0e^{-ax^2}\right|^2dx=1$$

With $A_0=\left(\frac{2m\omega_0}{h}\right)^{1/4}=\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}$, the normalization condition becomes:

$$\int_{-\infty}^{\infty}\left|\psi_0(x)\right|^2dx=\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2}\int_{-\infty}^{\infty}e^{-2ax^2}dx$$

In Example 35-1 it is shown that:

$$a=\frac{m\omega_0}{2\hbar}$$

Substitute to obtain:

$$\int_{-\infty}^{\infty}\left|\psi_0(x)\right|^2dx=\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2}\int_{-\infty}^{\infty}e^{-\frac{m\omega_0}{\hbar}x^2}dx$$

Let $s=\left(\frac{m\omega_0}{\hbar}\right)^{1/2}x$. Then:

$$ds=\left(\frac{m\omega_0}{\hbar}\right)^{1/2}dx$$

and

$$dx=\left(\frac{m\omega_0}{\hbar}\right)^{-1/2}ds$$

Substitute for dx and simplify to obtain:

$$\int_{-\infty}^{\infty}\left|\psi_0(x)\right|^2dx=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-s^2}ds$$

From integral tables (see Table D-5):

$$\int_{-\infty}^{\infty} e^{-s^2}\,ds = \sqrt{\pi}$$

Therefore:

$$\int_{-\infty}^{\infty} \left|\psi_0(x)\right|^2 dx = 1 \text{ provided } A_0 = \left(\frac{2m\omega_0}{h}\right)^{1/4}$$

10 •••

Picture the Problem We are required to evaluate $\left\langle x^2\right\rangle = \int x^2\left|\psi_0(x)\right|^2$ with $\psi_0(x) = A_0 e^{-ax^2}$, where $a = \dfrac{m\omega_0}{2\hbar}$. We can then use $U_{\text{av}} = \frac{1}{2}m\omega_0^2\left\langle x^2\right\rangle$ to find the average potential energy of the harmonic oscillator.

We need to evaluate:

$$\left\langle x^2\right\rangle = \int_{-\infty}^{+\infty} x^2\left|\psi\right|^2 dx$$

For the ground state of the harmonic oscillator:

$$\left|\psi\right|^2 = A_0^2 e^{-2ax^2}$$

Substitute for $\left|\psi\right|^2$ to obtain:

$$\left\langle x^2\right\rangle = A_0^2\int_{-\infty}^{+\infty} x^2 e^{-2ax^2}\,dx = 2A_0^2\int_{0}^{+\infty} x^2 e^{-2ax^2}\,dx$$

Use the appropriate integral from the inside of the back cover of the text to obtain:

$$\left\langle x^2\right\rangle = 2A_0^2\,\frac{1}{4}\sqrt{\frac{\pi}{(2a)^2}} = \frac{A_0^2}{4a}\sqrt{\frac{\pi}{2a}} \qquad (1)$$

The normalization condition is:

$$1 = \int_{-\infty}^{+\infty}\left|\psi\right|^2 dx = A_0^2\int_{-\infty}^{+\infty} e^{-2ax^2}\,dx = 2A_0^2\int_{0}^{+\infty} e^{-2ax^2}\,dx$$

Again, use the appropriate integral from the inside of the back cover of the text to obtain:

$$1 = 2A_0^2\,\frac{1}{2}\sqrt{\frac{\pi}{2a}} = A_0^2\sqrt{\frac{\pi}{2a}}$$

Solve for A_0^2 to obtain:

$$A_0^2 = \sqrt{\frac{2a}{\pi}}$$

Substitute in equation (1) to obtain:

$$\langle x^2\rangle = \frac{1}{4a}\sqrt{\frac{2a}{\pi}}\sqrt{\frac{\pi}{2a}} = \boxed{\frac{1}{4a}} \qquad (2)$$

From Example 36-1:

$$a = \frac{m\omega_0}{2\hbar}$$

Substitute for a in equation (2) to obtain:

$$\langle x^2\rangle = \frac{2\hbar}{4m\omega_0} = \boxed{\frac{\hbar}{2m\omega_0}}$$

The average potential energy of the oscillator is:

$$U_{\text{av}} = \tfrac{1}{2}m\omega_0^2\langle x^2\rangle$$

Substitute for $\langle x^2\rangle$ and simplify:

$$U_{\text{av}} = \tfrac{1}{2}m\omega_0^2\left(\frac{\hbar}{2m\omega_0}\right) = \boxed{\tfrac{1}{4}\hbar\omega_0}$$
$$= \tfrac{1}{2}E_0$$

11 ••

Picture the Problem We can combine the result for $\sqrt{\langle x^2\rangle}$ from Problem 10 and the result for $\langle x\rangle$ from Problem 7 to obtain an expression for $\sqrt{\langle x^2\rangle - \langle x\rangle^2}$. The lowest energy of the electron in an infinite potential well is given by $E_1 = \dfrac{h^2}{8mL^2}$.

(*a*) From Problem 10 we have:

$$\langle x^2\rangle = \frac{\hbar}{2m\omega_0} \qquad (1)$$

The ground-state energy is given by:

$$E_0 = \frac{1}{2}\hbar\omega_0$$

Solve for ω_0 to obtain:

$$\omega_0 = \frac{2E_0}{\hbar}$$

Substitute in equation (1) and simplify to obtain:

$$\langle x^2\rangle = \frac{\hbar^2}{4mE_0}$$

From Problem 7 we have:

$$\langle x\rangle = 0 \qquad (2)$$

Substitute equations (1) and (2) in the expression $\sqrt{\langle x^2\rangle - \langle x\rangle^2}$ to obtain:

$$\sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar^2}{4mE_0} - 0} = \frac{\hbar}{2}\sqrt{\frac{1}{mE_0}}$$

Substitute numerical values and evaluate $\sqrt{\langle x^2\rangle - \langle x\rangle^2}$:

$$\sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{1.05\times10^{-34}\ \mathrm{J\cdot s}}{2}\sqrt{\frac{1}{\left(9.11\times10^{-31}\ \mathrm{kg}\right)\left(2.1\times10^{-4}\ \mathrm{eV}\times\frac{1.6\times10^{-19}\ \mathrm{J}}{\mathrm{eV}}\right)}}$$

$$= \boxed{9.49\ \mathrm{nm}}$$

(*b*) The lowest energy of an electron trapped in an infinite potential well is:

$$E_1 = \frac{h^2}{8mL^2}$$

Letting $L = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$ yields:

$$E_1 = \frac{h^2}{8m\sqrt{\langle x^2\rangle - \langle x\rangle^2}^{\,2}}$$

Substitute numerical values and evaluate E_1:

$$E_1 = \frac{\left(6.63\times10^{-34}\ \mathrm{J\cdot s}\right)^2}{8\left(9.11\times10^{-31}\ \mathrm{kg}\right)\left(9.49\times10^{-9}\ \mathrm{m}\right)^2}\frac{1\,\mathrm{eV}}{1.6\times10^{-19}\ \mathrm{J}} = \boxed{4.19\ \mathrm{meV}}$$

12 •••

Picture the Problem We can begin by equating the average kinetic energy of the harmonic oscillator and its average potential energy and solving for $\langle p^2\rangle$ and then evaluating and substituting for $\langle x^2\rangle$.

According to the problem statement:

$$\frac{\langle p^2\rangle}{2m} = \frac{1}{2}k\langle x^2\rangle$$

Solve for $\langle p^2\rangle$:

$$\langle p^2\rangle = mk\langle x^2\rangle$$

or, because $\omega_0^2 = \dfrac{k}{m}$,

$$\langle p^2\rangle = m^2\omega_0^2\langle x^2\rangle \qquad (1)$$

We need to evaluate:

$$\langle x^2\rangle = \int x^2|\psi|^2 = A_0^2\int_{-\infty}^{\infty} x^2 e^{-2ax^2}\,dx \qquad (2)$$

where $a = \dfrac{m\omega_0}{2\hbar}$

Let $y^2 = 2ax^2$. Then $x^2 = \dfrac{y^2}{2a}$ and:

$$dx = \frac{ydy}{2ax}$$

With appropriate substitutions, the integral becomes:

$$\int_{-\infty}^{\infty} \frac{y^2}{2a} e^{-y^2} \frac{ydy}{2ax} = \int_{-\infty}^{\infty} \frac{y^3}{2a} e^{-y^2} \frac{\sqrt{2a}dy}{2ay} = \frac{1}{(2a)^{3/2}} \int_{-\infty}^{\infty} y^2 e^{-y^2} dy$$

From integral tables (see Table D-5):

$$\int_{-\infty}^{\infty} y^2 e^{-y^2} dy = \frac{1}{2}\sqrt{\pi}$$

In Problem 35-9 it was given that:

$$A_0 = \left(\frac{2m\omega_0}{h}\right)^{1/4}$$

Substitute in equation (2) to obtain:

$$\left\langle x^2 \right\rangle = \frac{1}{(2a)^{3/2}} \left(\frac{2m\omega_0}{h}\right)^{1/2} \left(\frac{1}{2}\sqrt{\pi}\right)$$

Substitute for a and simplify:

$$\left\langle x^2 \right\rangle = \frac{1}{\left(\dfrac{m\omega_0}{\hbar}\right)^{3/2}} \left(\frac{2m\omega_0}{h}\right)^{1/2} \left(\frac{1}{2}\sqrt{\pi}\right) = \frac{\hbar}{2m\omega_0}$$

Substitute for $\left\langle x^2 \right\rangle$ in equation (1):

$$\left\langle p^2 \right\rangle = m^2\omega_0^2 \left(\frac{\hbar}{2m\omega_0}\right) = \boxed{\tfrac{1}{2}\hbar m\omega_0}$$

13 •••

Picture the Problem We can use the definition of the standard deviation of Δx and Δp and the results of Problems 7, 10, and 12 to determine the uncertainty product $\Delta x\,\Delta p$ for the ground state of the harmonic oscillator.

Express the standard deviation of Δp (see Equation 17-35*a*):

$$(\Delta p)^2 = \left[(p - p_{\text{av}})^2\right]_{\text{av}} = \left[p^2 - 2pp_{\text{av}} - p_{\text{av}}^2\right]_{\text{av}}$$

Because $p_{\text{av}} = 0$:

$$(\Delta p)^2 = \left(p^2\right)_{\text{av}}$$

Express the standard deviation of Δx (see Equation 17-35*a*):

$$(\Delta x)^2 = \left[(x - x_{av})^2\right]_{av}$$
$$= \left[x^2 - 2xx_{av} - x_{av}^2\right]_{av}$$

Because $x_{av} = 0$:

$$(\Delta x)^2 = (x^2)_{av}$$

We have, from Problems 10 and 12, for the ground state of the harmonic oscillator:

$$\langle x^2 \rangle = (\Delta x)^2 = \frac{\hbar}{2m\omega_0}$$

and

$$\langle p^2 \rangle = (\Delta p)^2 = \frac{\hbar m\omega_0}{2}$$

Express the product of $(\Delta x)^2$ and $(\Delta p)^2$:

$$(\Delta x)^2 (\Delta p)^2 = \left(\frac{\hbar}{2m\omega_0}\right)\left(\frac{\hbar m\omega_0}{2}\right) = \frac{\hbar^2}{4}$$

Take the square root of both sides of the equation to obtain:

$$\boxed{\Delta x \Delta p = \frac{\hbar}{2}}$$

Reflection and Transmission of Electron Waves: Barrier Penetration

***14** ••

Picture the Problem We can use the total energy of the particle in the region $x > 0$ to express k_2 in terms of α and k_1. Knowing k_2 in terms of k_1, we can use $R = \dfrac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$ to find R and $T = 1 - R$ to determine the transmission coefficient T.

(*a*) Using conservation of energy, express the energy of the particle in the region $x > 0$:

$$\frac{\hbar^2 k_2^2}{2m} + U_0 = \alpha U_0$$

Solve for k_2:

$$k_2 = \frac{\sqrt{2mU_0(\alpha - 1)}}{\hbar}$$

From the equation for the total energy of the particle:

$$k_1 = \frac{\sqrt{2m\alpha U_0}}{\hbar}$$

Express the ratio of k_2 to k_1:

$$\frac{k_2}{k_1}=\frac{\dfrac{\sqrt{2mU_0(\alpha-1)}}{\hbar}}{\dfrac{\sqrt{2m\alpha U_0}}{\hbar}}=\sqrt{\frac{\alpha-1}{\alpha}}$$

and $k_2=\boxed{\sqrt{\dfrac{\alpha-1}{\alpha}}k_1}$

(*b*) The reflection coefficient R is given by:

$$R=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}$$

Factor k_1 from the numerator and denominator to obtain:

$$R=\frac{\left(1-\dfrac{k_2}{k_1}\right)^2}{\left(1+\dfrac{k_2}{k_1}\right)^2}$$

Substitute our result from (*a*) for k_2/k_1:

$$R=\frac{\left(1-\sqrt{\dfrac{\alpha-1}{\alpha}}\right)^2}{\left(1+\sqrt{\dfrac{\alpha-1}{\alpha}}\right)^2}=\left(\frac{1-\sqrt{\dfrac{\alpha-1}{\alpha}}}{1+\sqrt{\dfrac{\alpha-1}{\alpha}}}\right)^2$$

The transmission coefficient is given by:

$$T=1-R=1-\left(\frac{1-\sqrt{\dfrac{\alpha-1}{\alpha}}}{1+\sqrt{\dfrac{\alpha-1}{\alpha}}}\right)^2$$

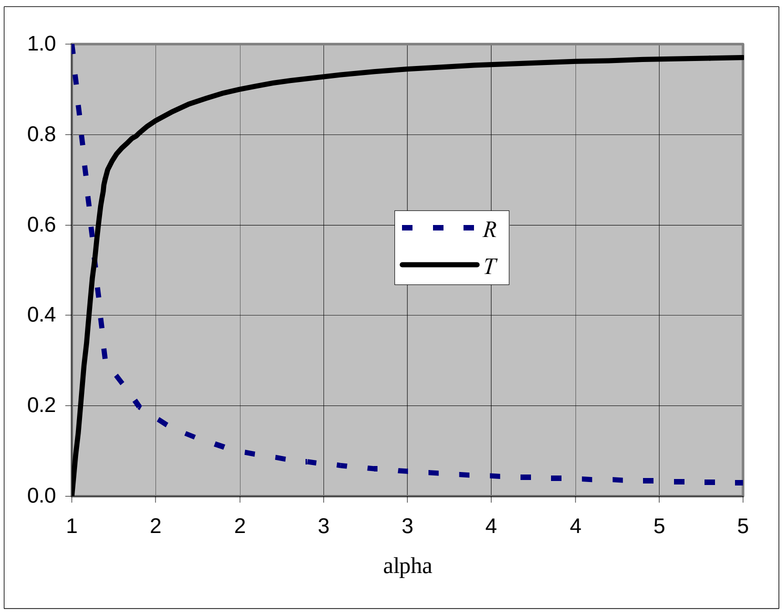
A spreadsheet program to plot R and T as functions of α is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Content/Formula	Algebraic Form
A2	1.0	α
B2	(1–SQRT((A2–1)/A2))/ (1+SQRT((A2–1)/A2))^2	$\left(\frac{1-\sqrt{\frac{\alpha-1}{\alpha}}}{1+\sqrt{\frac{\alpha-1}{\alpha}}}\right)^2$

C2	1–B2	$1-\left(\dfrac{1-\sqrt{\dfrac{\alpha-1}{\alpha}}}{1+\sqrt{\dfrac{\alpha-1}{\alpha}}}\right)^2$

	A	B	C
1	alpha	R	T
2	1.0	1.000	0.000
3	1.2	0.298	0.702
4	1.4	0.198	0.802
5	1.6	0.149	0.851
18	4.2	0.036	0.964
19	4.4	0.034	0.966
20	4.6	0.032	0.968
21	4.8	0.031	0.969
22	5.0	0.029	0.971

The following graph was plotted using the data in the above table:



15 ••

Picture the Problem We can use the total energy of the particle in the region $x > 0$ to find k_2. Knowing k_2, we can use $R = \dfrac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$ to find R and $T = 1 - R$ to determine the transmission coefficient T.

(*a*) Using conservation of energy, express the particle in the region $x > 0$:

$$\frac{\hbar^2 k_2^2}{2m} - U_0 = 2U_0$$

Solve for k_2:

$$k_2 = \frac{\sqrt{6mU_0}}{\hbar}$$

From the equation for the total energy of the particle:

$$k_1 = \frac{\sqrt{4mU_0}}{\hbar}$$

Express the ratio of k_2 to k_1:

$$\frac{k_2}{k_1} = \frac{\dfrac{\sqrt{6mU_0}}{\hbar}}{\dfrac{\sqrt{4mU_0}}{\hbar}} = \sqrt{\frac{3}{2}} \Rightarrow k_2 = \boxed{\sqrt{\frac{3}{2}}k_1}$$

(*b*) The reflection coefficient R is given by:

$$R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} = \frac{\left(1 - \dfrac{k_2}{k_1}\right)^2}{\left(1 + \dfrac{k_2}{k_1}\right)^2}$$

Substitute for k_2/k_1 and evaluate R:

$$R = \frac{\left(1 - \sqrt{\dfrac{3}{2}}\right)^2}{\left(1 + \sqrt{\dfrac{3}{2}}\right)^2} = \boxed{0.0102}$$

(*c*) Because $R + T = 1$:

$$T = 1 - R = 1 - 0.0102 = \boxed{0.990}$$

(*d*) If we let N_0 represent the number of particles incident upon the potential step, then the number that continue beyond is:

$$N_0 T = 10^6 \times 0.990 = \boxed{9.90 \times 10^5}$$

$$\boxed{\begin{array}{l}\text{Classically, all } 10^6 \text{ would continue} \\ \text{to move past the potential step.}\end{array}}$$

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Picture the Problem We can use the energies in the regions $U = 0$ and $U = U_0$ to express the ratio of the potential energy to the total energy in terms of the ratio of the wave numbers. We can also express this ratio in terms of the reflection coefficient R to obtain an expression for the ratio of E to U in terms of R.

In the region $U = 0$:

$$E = \frac{\hbar^2 k_1^2}{2m} \Rightarrow k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

In the region $U = U_0$:

$$E - U_0 = \frac{\hbar^2 k_2^2}{2m} \Rightarrow k_2 = \sqrt{\frac{2m(E - U_0)}{\hbar^2}}$$

Let r equal the ratio of k_2 to k_1:

$$r = \frac{k_2}{k_1} = \frac{\sqrt{\frac{2m(E-U_0)}{\hbar^2}}}{\sqrt{\frac{2mE}{\hbar^2}}} = \sqrt{1 - \frac{U_0}{E}}$$

Letting $U_0 = U$, solve for U/E:

$$\frac{U}{E} = 1 - r^2 \qquad (1)$$

Write the reflection coefficient R as a function of $r = \frac{k_2}{k_1}$:

$$R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} = \frac{\left(1 - \frac{k_2}{k_1}\right)^2}{\left(1 + \frac{k_2}{k_1}\right)^2} = \frac{(1-r)^2}{(1+r)^2}$$

Solve for r to obtain:

$$r = \frac{1 - \sqrt{R}}{1 + \sqrt{R}}$$

Substitute for r in equation (1):

$$\frac{U}{E} = 1 - \left(\frac{1 - \sqrt{R}}{1 + \sqrt{R}}\right)^2$$

and

$$\frac{E}{U} = \left[1 - \left(\frac{1 - \sqrt{R}}{1 + \sqrt{R}}\right)^2\right]^{-1}$$

Substitute a numerical value for R and evaluate E/U:

$$\frac{E}{U} = \left[1 - \left(\frac{1 - \sqrt{0.5}}{1 + \sqrt{0.5}}\right)^2\right]^{-1} = \boxed{1.03}$$

17 ••

Picture the Problem The probability that a proton will tunnel out of a nucleus in one collision with a nuclear barrier if it has a given energy is given by Equation 35-29.

Equation 35-29 is:

$$T = e^{-2\alpha a}$$

where

$$\alpha = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}} = \frac{\sqrt{2m(U_0 - E)}}{\hbar}$$

Multiply the numerator and denominator of α by c to obtain:

$$\alpha = \frac{\sqrt{2mc^2(U_0 - E)}}{\hbar c}$$

where

$$\hbar c = 1.974 \times 10^{-13}\ \text{MeV} \cdot \text{m}$$

Using $m_\text{p}c^2 = 938\,\text{MeV}$, evaluate T:

$$T = \exp\left\{-2\left(10^{-15}\ \text{m}\right)\frac{\sqrt{2(938\,\text{MeV})(6\,\text{MeV})}}{1.974 \times 10^{-13}\ \text{MeV} \cdot \text{m}}\right\} = \boxed{0.341}$$

***18** ••

Picture the Problem The probability that the electron with a given energy will tunnel through the given barrier is given by Equation 35-29.

(*a*) Equation 35-29 is:

$$T = e^{-2\alpha a}$$

where

$$\alpha = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}} = \frac{\sqrt{2m(U_0 - E)}}{\hbar}$$

Multiply the numerator and denominator of α by c to obtain:

$$\alpha = \frac{\sqrt{2mc^2(U_0 - E)}}{\hbar c}$$

where

$$\hbar c = 1.974 \times 10^{-13}\ \text{MeV} \cdot \text{m}$$

Using $m_\text{e}c^2 = 511\,\text{keV}$, evaluate T:

$$T = \exp\left\{-2\left(10^{-9}\ \text{m}\right)\frac{\sqrt{2(511\,\text{keV})(25\,\text{eV} - 10\,\text{eV})}}{1.974 \times 10^{-13}\ \text{MeV} \cdot \text{m}}\right\} = \boxed{5.91 \times 10^{-18}}$$

(*b*) Repeat with $a = 10^{-10}$ m:

$$T = \exp\left\{-2\left(10^{-10}\text{ m}\right)\frac{\sqrt{2(511\text{ keV})(25\text{ eV} - 10\text{ eV})}}{1.974\times10^{-13}\text{ MeV}\cdot\text{m}}\right\} = \boxed{1.89\times10^{-2}}$$

19 •••

Picture the Problems We can find the distance of closest approach by equating the kinetic energy of the alpha particle and the Coulomb potential energy. The probability that the electron with a given energy will tunnel through the given barrier is given by $T = e^{-2\alpha a}$, where α is the transmission coefficient and depends on ΔE.

(*a*) The distance of closest approach is related to the kinetic energy E of the alpha particles:

$$E = \frac{k2eZe}{r_1}$$

Solve for r_1:

$$r_1 = \frac{2kZe^2}{E}$$

For E = 4 MeV:

$$r_{1,4\text{ MeV}} = \frac{2\left(8.99\times10^9\text{ N}\cdot\text{m}^2/\text{C}^2\right)(92)\left(1.6\times10^{-19}\text{ C}\right)^2}{4\text{ MeV}\times\dfrac{1.6\times10^{-19}\text{ J}}{\text{eV}}} = \boxed{6.62\times10^{-14}\text{ m}}$$

For K = 7 MeV:

$$r_{1,7\text{ MeV}} = \frac{2\left(8.99\times10^9\text{ N}\cdot\text{m}^2/\text{C}^2\right)(92)\left(1.6\times10^{-19}\text{ C}\right)^2}{7\text{ MeV}\times\dfrac{1.6\times10^{-19}\text{ J}}{\text{eV}}} = \boxed{3.78\times10^{-14}\text{ m}}$$

(*b*) The transmission coefficient T is given by:

$$T = e^{-2\alpha a} \qquad (1)$$

where

$$\alpha = \sqrt{\frac{2m\Delta E}{\hbar^2}} = \frac{\sqrt{2m\Delta E}}{\hbar}$$

Evaluate $\alpha_{4\text{ MeV}}$ for ΔE = 4 MeV:

$$\alpha_{4\text{ MeV}} = \frac{\sqrt{2\left(4\text{ u}\times\dfrac{1.66\times10^{-27}\text{ kg}}{\text{u}}\right)\left(4\text{ MeV}\times\dfrac{1.6\times10^{-19}\text{ J}}{\text{eV}}\right)}}{1.05\times10^{-34}\text{ J}\cdot\text{s}} = 8.78\times10^{14}\text{ m}^{-1}$$

Evaluate $\alpha_{7\,\text{MeV}}$ for ΔE = 7 MeV:

$$\alpha_{7\,\text{MeV}}=\frac{\sqrt{2\left(4\,\text{u}\times\frac{1.66\times10^{-27}\,\text{kg}}{\text{u}}\right)\left(7\,\text{MeV}\times\frac{1.6\times10^{-19}\,\text{J}}{\text{eV}}\right)}}{1.05\times10^{-34}\,\text{J}\cdot\text{s}}=1.16\times10^{15}\,\text{m}^{-1}$$

Substitute numerical values in equation (1) and evaluate $T_{4\,\text{MeV}}$:

$$T_{4\,\text{MeV}}=e^{-2\left(8.78\times10^{14}\,\text{m}^{-1}\right)\left(6.62\times10^{-14}\,\text{m}\right)}=\boxed{3.27\times10^{-51}}$$

Substitute numerical values in equation (1) and evaluate $T_{7\,\text{MeV}}$:

$$T_{7\,\text{MeV}}=e^{-2\left(1.16\times10^{15}\,\text{m}^{-1}\right)\left(3.78\times10^{-14}\,\text{m}\right)}=\boxed{8.21\times10^{-39}}$$

The Schrödinger Equation in Three Dimensions

20 •

Picture the Problem We can use $E_{n_1,n_2,n_3}=\frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{L_1^2}+\frac{n_2^2}{L_2^2}+\frac{n_3^2}{L_3^2}\right)$ with the given sides of the box to find the quantum numbers n_1, n_2, n_3 that correspond to the lowest ten quantum states of this box.

The energies of the quantum states are given by Equation 35-34:

$$E_{n_1,n_2,n_3}=\frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{L_1^2}+\frac{n_2^2}{L_2^2}+\frac{n_3^2}{L_3^2}\right)$$

For a box with sides L_1, $L_2 = 2L_1$, and $L_3 = 3L_1$:

$$E_{n_1,n_2,n_3}=\frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{L_1^2}+\frac{n_2^2}{4L_1^2}+\frac{n_3^2}{9L_1^2}\right)=\frac{h^2}{8mL_1^2}\left(n_1^2+\frac{n_2^2}{4}+\frac{n_3^2}{9}\right)=\frac{h^2}{288mL_1^2}\left(36n_1^2+9n_2^2+n_3^2\right)$$

The energies in units of $\frac{h^2}{288mL_1^2}$ are listed in the following table:

n_1	n_2	n_3	E
1	1	1	49
1	1	2	61

1	2	1	76
1	1	3	81
1	2	2	88
1	2	3	108
1	1	4	109
1	3	1	121
1	3	2	133
1	2	4	136

21 •

Picture the Problem The wave functions are of the form

$$\psi = A\sin\left(\frac{n_1\pi}{L_1}x\right)\sin\left(\frac{n_2\pi}{2L_1}y\right)\sin\left(\frac{n_3\pi}{3L_1}z\right)$$

22 •

Picture the Problem We can use $E_{n_1,n_2,n_3} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{L_1^2}+\frac{n_2^2}{L_2^2}+\frac{n_3^2}{L_3^2}\right)$ with the given sides of the box to find the quantum numbers n_1, n_2, n_3 that correspond to the lowest ten quantum states of this box.

(a) The energies of the quantum states are given by Equation 35-34:

$$E_{n_1,n_2,n_3} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{L_1^2}+\frac{n_2^2}{L_2^2}+\frac{n_3^2}{L_3^2}\right)$$

For a box with sides L_1, $L_2 = 2L_1$, and $L_3 = 4L_1$:

$$E_{n_1,n_2,n_3} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{L_1^2}+\frac{n_2^2}{4L_1^2}+\frac{n_3^2}{16L_1^2}\right) = \frac{h^2}{8mL_1^2}\left(n_1^2+\frac{n_2^2}{4}+\frac{n_3^2}{16}\right) = \frac{h^2}{128mL_1^2}\left(16n_1^2+4n_2^2+n_3^2\right)$$

The energies in units of $\frac{h^2}{128mL_1^2}$ are listed in the following table:

n_1	n_2	n_3	E
1	1	1	21
1	1	2	24
1	1	3	29
1	2	1	33

1	1	4	36
1	2	2	36
1	2	3	41
1	1	5	45
1	2	4	48
1	3	1	53
1	1	6	56
1	3	2	56

Referring to the table, we see that there are two degenerate levels:

$$\boxed{(1,1,4)\text{and}(1,2,2)}$$

and

$$\boxed{(1,1,6)\text{and}(1,3,2)}$$

***23** •

Picture the Problem The wave functions are of the form

$$\psi = A\sin\left(\frac{n_1\pi}{L_1}x\right)\sin\left(\frac{n_2\pi}{2L_1}y\right)\sin\left(\frac{n_3\pi}{4L_1}z\right)$$

24 •

Picture the Problem The boundary conditions in the y and z directions are the same those in Figure 35-1. In the x direction, we'll require the $\psi = 0$ at $-L/2$ and $L/2$.

(a) The boundary conditions in the x direction are:

$$\psi\left(-\tfrac{1}{2}L\right) = \psi\left(\tfrac{1}{2}L\right) = 0$$

The general solution of the time-independent Schrödinger equation is:

$$\psi(x) = A\sin kx + B\cos kx$$

Apply the boundary conditions to obtain:

$$\psi\left(-\tfrac{1}{2}L\right) = -A\sin\frac{kL}{2} + B\cos\frac{kL}{2} = 0$$

and

$$\psi\left(\tfrac{1}{2}L\right) = A\sin\frac{kL}{2} + B\cos\frac{kL}{2} = 0$$

Eliminate the terms in B by subtracting the equations:

$$A\sin\frac{kL}{2} = 0$$

For $A \neq 0$:

$$\frac{kL}{2} = \sin^{-1}0 = 0, \pi, 2\pi, \ldots$$

or

$$k=\frac{n_1\pi}{L}, n_1=0,2,4,\ ...$$

Eliminate the terms in A by adding the equations:

$$B\sin\frac{kL}{2}=0$$

For $B\neq 0$:

$$\frac{kL}{2}=\cos^{-1}0=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\ ...$$

or

$$k=\frac{n_2\pi}{L}, n_2=1,3,5,\ ...$$

Thus:

$$\psi(x,y,z)=B\cos\left(\frac{n_1\pi}{L}x\right)\sin\left(\frac{n_2\pi}{L}y\right)\sin\left(\frac{n_3\pi}{L}z\right), n_1=2n+1$$

and

$$\psi(x,y,z)=A\sin\left(\frac{n_1\pi}{L}x\right)\sin\left(\frac{n_2\pi}{L}y\right)\sin\left(\frac{n_3\pi}{L}z\right),\ n_1=2n$$

The ground-state wave function is:

$$\boxed{\psi(1,1,1)=A\cos\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{L}y\right)\cos\left(\frac{\pi}{L}z\right)}$$

(b) $\boxed{\text{The allowed energies are the same as those for a well with } U=0 \text{ for } 0<x<\text{L}.}$

25 ••

Picture the Problem We can apply the solution to the time-independent Schrödinger equation in three dimensions to obtain the wave function and the allowed energies for the given two-dimensional region. In (c), we must find three different sets of quantum numbers (m,n) for which the sum of the squares are the same.

(a) The solution to the time-independent Schrödinger equation in two-dimensions is:

$$\psi(x,y)=A\sin k_1x\sin k_2y$$
$$=\boxed{A\sin\frac{n\pi}{L}x\sin\frac{m\pi}{L}y}$$

where n and m are integers.

(*b*) The energy is quantized to the values:

$$E_{n,m}=\boxed{\frac{h^2}{8mL^2}\left(n^2+m^2\right)}$$

(*c*) The lowest two states that are degenerate are:

$$\boxed{E_{1,2}=E_{2,1}=\frac{5h^2}{8mL^2}}$$

(*d*) The energies of three lowest states that have the same energies (in units of $\frac{h^2}{8mL^2}$) are listed in the table to the right:

n	m	$E_{n,m}$
1	7	50
7	1	50
5	5	50

The quantum numbers for the three states are:

$\boxed{(1,7),}$ $\boxed{(7,1),}$ and $\boxed{(5,5)}$

and their energies are

$$E=(50)\frac{h^2}{8mL^2}=\boxed{\frac{25h^2}{4mL^2}}$$

The Schrödinger Equation for Two Identical Particles

26 •

Picture the Problem We must differentiate Equation 35-37 twice and substitute these derivatives in this equation to show that it is a solution.

With $U = 0$, Equation 35-35 becomes:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x_1,x_2)}{\partial x_1^2}-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x_1,x_2)}{\partial x_2^2}=E\psi(x_1,x_2) \quad (1)$$

Differentiate Equation 35-7 with respect to x_1:

$$\frac{\partial\psi}{\partial x_1}=\frac{\partial}{\partial x_1}\left[A\sin\frac{\pi x_1}{L}\sin\frac{2\pi x_2}{L}\right]=\frac{A\pi}{L}\cos\frac{\pi x_1}{L}\sin\frac{2\pi x_2}{L}$$

Compute the second derivative with respect to x_1:

$$\frac{\partial^2\psi}{\partial x_1^2}=\frac{\partial}{\partial x_1}\left[\frac{A\pi}{L}\cos\frac{\pi x_1}{L}\sin\frac{2\pi x_2}{L}\right]=-\frac{A\pi^2}{L^2}\sin\frac{\pi x_1}{L}\sin\frac{2\pi x_2}{L}$$

Differentiate Equation 35-7 with respect to x_2:

$$\frac{\partial\psi}{\partial x_2}=\frac{\partial}{\partial x_2}\left[A\sin\frac{\pi x_1}{L}\sin\frac{2\pi x_2}{L}\right]$$
$$=\frac{2A\pi}{L}\sin\frac{\pi x_1}{L}\cos\frac{2\pi x_2}{L}$$

Compute the second derivative with respect to x_2:

$$\frac{\partial^2\psi}{\partial x_2^2}=\frac{\partial}{\partial x_2}\left[\frac{2A\pi}{L}\sin\frac{\pi x_1}{L}\cos\frac{2\pi x_2}{L}\right]$$
$$=-\frac{4A\pi^2}{L^2}\sin\frac{\pi x_1}{L}\sin\frac{2\pi x_2}{L}$$

Substitute in equation (1) to obtain:

$$-\frac{\hbar^2}{2m}\left[-\frac{A\pi^2}{L^2}\sin\frac{\pi x_1}{L}\sin\frac{2\pi x_2}{L}\right]-\frac{\hbar^2}{2m}\left[-\frac{4A\pi^2}{L^2}\sin\frac{\pi x_1}{L}\sin\frac{2\pi x_2}{L}\right]$$
$$=EA\sin\frac{\pi x_1}{L}\sin\frac{2\pi x_2}{L}$$

Solve for E to obtain:

$$E=\frac{5\hbar^2\pi^2}{2mL^2}$$

$$\boxed{\text{Thus we've shown that Equation 35-37 satisfies Equation 35-35 provided } E=\frac{5\hbar^2\pi^2}{2mL^2}.}$$

27 •

Picture the Problem Because bosons have symmetric wave functions and do not obey the Pauli exclusion principle, they can occupy the same ground state.

The ground-state energy of a single particle in a one-dimensional box of length L is:

$$E_{0,1\,\text{particle}}=\frac{h^2}{8mL^2}$$

For 10 bosons:

$$E_{0,10\,\text{bosons}}=\frac{10h^2}{8mL^2}=\boxed{\frac{5h^2}{4mL^2}}$$

***28** •

Picture the Problem For fermions, such as neutrons for which the spin quantum number is ½, two particles can occupy the same spatial state.

The lowest total energy for the 10 fermions is:

$$E = 2E_1\left(1^2 + 2^2 + 3^2 + 4^2 + 5^2\right)$$
$$= 2\left(\frac{h^2}{8mL^2}\right)(55)$$
$$= \boxed{\frac{55h^2}{4mL^2}}$$

Orthogonality of Wave Functions

29 ••

Picture the Problem We need to show that $\int_{-\infty}^{\infty} \psi_0(x)\psi_1(x)dx = 0$, where $\psi_0(x)$ and $\psi_1(x)$ are given by Equations 35-23 and 35-25, respectively.

Equations 35-23 and 35-25 are:

$$\psi_0(x) = A_0 e^{-ax^2} \qquad \text{35-23}$$

and

$$\psi_1(x) = A_1 x e^{-ax^2} \qquad \text{35-25}$$

Note that $\psi_1(x)$ is antisymmetric, whereas $\psi_0(x)$ is symmetric. Because the product of an antisymmetric function and a symmetric function is antisymmetric:

$\psi_0(x)\psi_1(x)$ is antisymmetric

Because the integral of an antisymmetric function over symmetric limits is zero:

$$\boxed{\int_{-\infty}^{\infty} \psi_0(x)\psi_1(x)dx = 0}$$

30 ••

Picture the Problem We need to show that $\int_{-\infty}^{\infty} \psi_1(x)\psi_2(x)dx = 0$, where $\psi_2(x)$ is given in the problem statement and $\psi_1(x) = A_1 x e^{-ax^2}$.

Note that $\psi_1(x)$ is antisymmetric, whereas $\psi_2(x)$ is symmetric. Because the product of a symmetric function and an antisymmetric function is antisymmetric:

$\psi_1(x)\psi_2(x)$ is antisymmetric.

Because the integral of an antisymmetric function over symmetric limits is zero:

$$\boxed{\int_{-\infty}^{\infty}\psi_1(x)\psi_2(x)dx = 0}$$

31 ••

Picture the Problem We need to show that $\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 0.$

Use the trigonometric identity $(\sin a\alpha)(\sin b\alpha) = \frac{1}{2}\{\cos[(a-b)\alpha] - \cos[(a+b)\alpha]\}$ to rewrite the product of the two sine functions as the difference of two cosine functions:

$$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \frac{1}{2}\left\{\cos\left[(n-m)\frac{\pi x}{L}\right] - \cos\left[(n+m)\frac{\pi x}{L}\right]\right\}$$

Substitute for $\sin\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{m\pi x}{L}\right)$ and evaluate $\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$:

$$\int_{-\infty}^{\infty}\frac{1}{2}\left\{\cos\left[(n-m)\frac{\pi x}{L}\right] - \cos\left[(n+m)\frac{\pi x}{L}\right]\right\}dx = \frac{L}{\pi}\frac{\sin\left[(n-m)\frac{\pi x}{L}\right]}{n-m} - \frac{L}{\pi}\frac{\sin\left[(n+m)\frac{\pi x}{L}\right]}{n+m}$$

Because n and m are integers and $n \neq m$, the sine functions vanish at the two limits $x = 0$ and $x = L$. Therefore, $\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 0$ for $n \neq m$.

General Problems

32 ••

Picture the Problem We can use the wave functions $\psi_1(x)$ and $\psi_2(x)$ and the definitions of $\langle x\rangle$ and $\langle x^2\rangle$ to evaluate these quantities and the wave functions at $x = 0$.

(*a*) The wave functions $\psi_1(x)$ and $\psi_2(x)$ are:

$$\psi_m = \sqrt{\frac{2}{L}}\sin\left(\frac{m\pi}{L}x\right), m = 2n$$

and

$$\psi_m = \sqrt{\frac{2}{L}}\cos\left(\frac{m\pi}{L}x\right), m = 2n+1$$

where $n = 0, 1, 2, \ldots$

Evaluate these functions at $x = 0$ to obtain:

$$\psi_1(0) = \sqrt{\frac{2}{L}}\sin\left[\frac{\pi}{L}(0)\right] = \boxed{0}$$

and

$$\psi_2(0) = \sqrt{\frac{2}{L}}\cos\left[\frac{\pi}{L}(0)\right] = \boxed{\sqrt{\frac{2}{L}}}$$

(*b*) Because $\left|\psi_m(x)\right|^2$ is an even function of x in all cases, $x\psi_m^2(x)$ is an odd function of x and:

$$\langle x\rangle = \int_{-L/2}^{L/2} x\psi_m^2(x)\,dx = \boxed{0}$$

(*c*) For $n = 1$:

$$\langle x^2\rangle = \frac{2}{L}\int_{-L/2}^{L/2} x^2\sin^2\frac{\pi}{L}x\,dx$$

From integral tables:

$$\int x^2\sin^2(ax)\,dx = \frac{x^3}{6} - \left(\frac{x^3}{4a} - \frac{1}{8a^3}\right)\sin(2ax) - \frac{x\cos(2ax)}{4a^2}$$

Use this integral with $a = \pi/L$ to obtain:

$$\langle x^2\rangle = \boxed{\frac{L^2}{12}\left(1 + \frac{6}{\pi^2}\right)}$$

For $n = 2$:

$$\langle x^2\rangle = \frac{2}{L}\int_{-L/2}^{L/2} x^2\cos^2\frac{2\pi}{L}x\,dx$$

From integral tables:

$$\int x^2\cos^2(ax)\,dx = \frac{x^3}{6} - \left(\frac{x^3}{4a} - \frac{1}{8a^3}\right)\sin(2ax) + \frac{x\cos(2ax)}{4a^2}$$

Use this integral with $a = 2\pi/L$ to obtain:

$$\langle x^2\rangle = \boxed{\frac{L^2}{12}\left(1 + \frac{3}{2\pi^2}\right)}$$

Remarks: Note that for any value of *m*, $\langle x^2\rangle = \frac{L^2}{12}\left(1 + \frac{6}{m^2\pi^2}\right)$.

***33** ••

Picture the Problem We can determine the energies of the state by identifying the four lowest quantum states that are occupied in the ground state and computing their combined energies. We can then find the energy difference between the ground state and the first excited state and use this information to find the energy of the excited state.

Each *n*, *m* state can accommodate only 2 particles. Therefore, in the ground state of the system of 8 fermions, the four lowest quantum states are occupied. These are:

(1,1), (1,2), (2,1) and (2,2)

Note that the states (1,2) and (2,1) are distinctly different states because the *x* and *y* directions are distinguishable.

The energies are quantized to the values given by:

$$E_{n_1,n_2} = 2\left(\frac{h^2}{8mL^2}\right)\left(n_1^2 + n_2^2\right)$$

The energy of the ground state is the sum of the energies of the four lowest quantum states:

$$\begin{aligned}E_0 &= E_{1,1} + E_{1,2} + E_{2,1} + E_{2,2}\\ &= 2\left(\frac{h^2}{8mL^2}\right)\left(1^2+1^2\right) + 2\left(\frac{h^2}{8mL^2}\right)\left(1^2+2^2\right) + 2\left(\frac{h^2}{8mL^2}\right)\left(2^2+1^2\right) + 2\left(\frac{h^2}{8mL^2}\right)\left(2^2+2^2\right)\\ &= 2\left(\frac{h^2}{8mL^2}\right)(2+5+5+8)\\ &= \frac{5h^2}{mL^2}\end{aligned}$$

The next higher state is achieved by taking one fermion from the (2,2) state and raising it to the next higher unoccupied state. That state is the (1,3) state. The energy difference between the ground state and this state is:

$$\begin{aligned}\Delta E &= E_{1,3} - E_{2,2}\\ &= \frac{h^2}{8mL^2}\left(1^2+3^2\right) - \frac{h^2}{8mL^2}\left(2^2+2^2\right)\\ &= \frac{h^2}{8mL^2}(10-8) = \frac{h^2}{4mL^2}\end{aligned}$$

Hence, the energies of the degenerate states (1,3) and (3,1) are:

$$\begin{aligned}E_{1,3} = E_{3,1} &= E_0 + \Delta E\\ &= \frac{5h^2}{mL^2} + \frac{h^2}{4mL^2} = \frac{21h^2}{4mL^2}\end{aligned}$$

The three lowest energy levels are therefore:

$$E_0 = \boxed{\frac{5h^2}{mL^2}}$$

and two states of energy

$$E_1 = E_2 = \boxed{\frac{21h^2}{4mL^2}}$$

34 ••

Picture the Problem The energy levels are the same as for a two-dimensional box of widths L and $3L$.

(*a*) The energies of the bound states are given by:

$$E_{n,m} = \frac{h^2}{8m}\left(\frac{n^2}{L^2} + \frac{m^2}{9L^2}\right) = \frac{h^2}{72mL^2}\left(9n^2 + m^2\right)$$

The three lowest energy states are:

$$E_{1,1} = \frac{h^2}{72mL^2}(9+1) = \boxed{\frac{5h^2}{36mL^2}}$$

$$E_{1,2} = \frac{h^2}{72mL^2}(9+4) = \boxed{\frac{13h^2}{72mL^2}}$$

and

$$E_{1,3} = \frac{h^2}{72mL^2}(9+9) = \boxed{\frac{h^2}{4mL^2}}$$

$$\boxed{\text{None of these states are degenerate.}}$$

(*b*) Express the condition that must be satisfied for two states to be degenerate:

$$9\left(n_1^2 - n_2^2\right) = m_2^2 - m_1^2$$

This condition is first satisfied for:

$$n_1 = 2, m_1 = 3, \text{and } n_2 = 1 \text{ and } m_2 = 6$$

Find the energy of this doubly degenerate state:

$$E_{2,3} = \frac{h^2}{72mL^2}(36+9) = \boxed{\frac{5h^2}{8mL^2}}$$

35 •••

Picture the Problem We can use the definition of the classical expectation value (average value) to show that the classical expectation value of x^2 for a particle in a one-dimensional box of length L centered at the origin is $L^2/12$. In (*b*) we'll proceed as in (*a*) using the definition of the quantum expectation value of x^2.

(*a*) The classical expectation value is given by:

$$\left(x^2\right)_{\text{av}} = \frac{1}{\frac{1}{2}L-\left(-\frac{1}{2}L\right)}\int_{-L/2}^{L/2} x^2 dx$$

$$= \frac{1}{L}\left[\frac{x^3}{3}\right]_{-L/2}^{L/2} = \frac{1}{L}\left(\frac{L^3}{12}\right)$$

$$= \boxed{\frac{L^2}{12}}$$

(*b*) For a particle in the *n*th state in a one-dimensional box:

$$\left\langle x^2\right\rangle = \frac{2}{L}\int_{-L/2}^{L/2} x^2 \sin^2\frac{n\pi}{L}x dx$$

From integral tables:

$$\int x^2\sin^2(ax)dx = \frac{x^3}{6}-\left(\frac{x^3}{4a}-\frac{1}{8a^3}\right)\sin(2ax)-\frac{x\cos(2ax)}{4a^2}$$

In the limit $n >> 1$:

$$\left\langle x^2\right\rangle = \boxed{\frac{L^2}{12}}$$

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Picture the Problem We can solve Equation 35-28 for T and substitute for R using Equation 35-27. Letting $r = k_2/k_1$ and simplifying will lead to the given result.

Equation 35-28 is:

$$T + R = 1 \Rightarrow T = 1 - R$$

From Equation 35-27:

$$R = \frac{(k_1-k_2)^2}{(k_1+k_2)^2} = \frac{\left(1-\frac{k_2}{k_1}\right)^2}{\left(1+\frac{k_2}{k_1}\right)^2} = \frac{(1-r)^2}{(1+r)^2}$$

where $r = k_2/k_1$

Substitute for R to obtain:

$$T = 1-\frac{(1-r)^2}{(1+r)^2} = \frac{(1+r)^2-(1-r)^2}{(1+r)^2}$$

$$= \boxed{\frac{4r}{(1+r)^2}}$$

Substitute for k_2/k_1 for r and simplify to obtain:

$$T = \boxed{\frac{4k_1k_2}{(k_1+k_2)^2}}$$

37 ••

Picture the Problem We can use the energies in the regions $U = 0$ and $U = U_0$ to express the ratio of the wave numbers k_1 and k_2 in these regions in terms of E and U_0 and the definition of the reflection coefficient R to show that $R = \frac{(1-r)^2}{(1+r)^2}$.

In the region $U_0 = 0$:

$$E = \frac{\hbar^2 k_1^2}{2m} \Rightarrow k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

In the region $U = U_0$:

$$E - U_0 = \frac{\hbar^2 k_2^2}{2m} \Rightarrow k_2 = \sqrt{\frac{2m(E-U_0)}{\hbar^2}}$$

Let r equal the ratio of k_2 to k_1:

$$r = \frac{k_2}{k_1} = \frac{\sqrt{\frac{2m(E-U_0)}{\hbar^2}}}{\sqrt{\frac{2mE}{\hbar^2}}} = \boxed{\sqrt{1-\frac{U_0}{E}}}$$

The reflection coefficient R is given by:

$$R = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}$$

Factor k_1 from the numerator and denominator to obtain:

$$R = \frac{\left(1-\frac{k_2}{k_1}\right)^2}{\left(1+\frac{k_2}{k_1}\right)^2}$$

Substitute for k_2/k_1 to obtain:

$$R = \boxed{\frac{(1-r)^2}{(1+r)^2}}$$

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Picture the Problem

(*a*) From Problem 37 we have:

$$R = \frac{(1-r)^2}{(1+r)^2}\text{, where } r = \sqrt{1-\frac{U_0}{E}}$$

Because $E = \alpha U_0$, R can be written:

$$r = \sqrt{1-\frac{1}{\alpha}}$$

From Problem 36 we have:

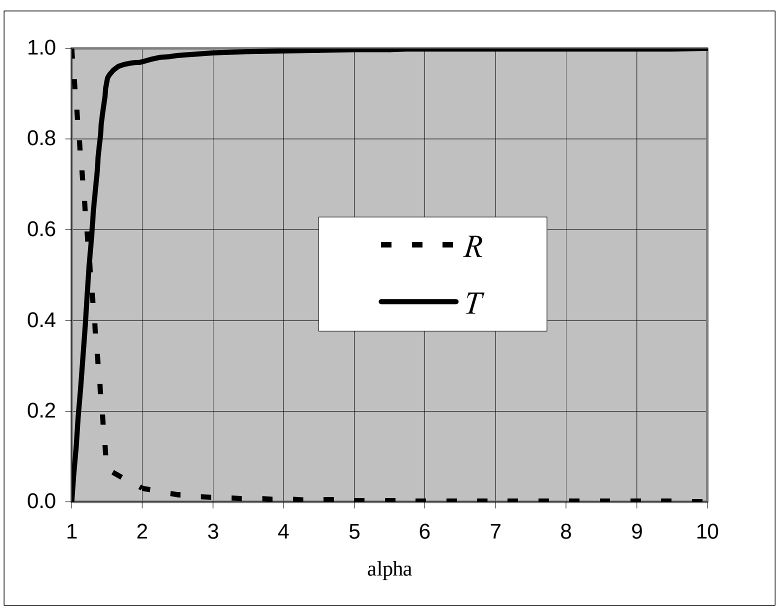
$$T = \frac{4r}{(1+r)^2}$$

A spreadsheet program to plot R and T as functions of α is shown below. The formulas used to calculate the quantities in the columns are as follows:

Cell	Content/Formula	Algebraic Form
A2	1.0	α
B2	SQRT(1–1/A2)	$\sqrt{1-\frac{1}{\alpha}}$
C2	(1–B2)^2/(1+B2)^2	$\frac{(1-r)^2}{(1+r)^2}$
D2	4*B2//(1+B2)^2	$\frac{4r}{(1+r)^2}$

	A	B	C	D
1	alpha	r	R	T
2	1.0	0.000	1.000	0.000
3	1.5	0.577	0.072	0.928
4	2.0	0.707	0.029	0.971
5	2.5	0.775	0.016	0.984
16	8.0	0.935	0.001	0.999
17	8.5	0.939	0.001	0.999
18	9.0	0.943	0.001	0.999
19	9.5	0.946	0.001	0.999
20	10.0	0.949	0.001	0.999

The following graph of R and T as functions of α was plotted using the data in the table:



(*b*) $\boxed{\text{From the graph, we note that, as } \alpha \to \infty, T \to 1 \text{ and } R \to 0. \text{ The graph also shows that, as } \alpha \to 1, T \to 0 \text{ and } R \to 1.}$

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Picture the Problem We require that

$$A_2^2\int_{-\infty}^{\infty}\left(2ax^2-\tfrac{1}{2}\right)^2 e^{-2ax^2}dx = 2A_2^2\int_0^{\infty}\left(2ax^2-\tfrac{1}{2}\right)^2 e^{-2ax^2}dx = 1.$$

Expand the integrand to obtain:

$$\left(2ax^2-\tfrac{1}{2}\right)^2 e^{-2ax^2} = \left(4a^2x^4-2ax^2+\tfrac{1}{4}\right)e^{-2ax^2} = 4a^2x^4e^{-2ax^2}-2ax^2e^{-2ax^2}+\tfrac{1}{4}e^{-2ax^2}$$

Substitute in the integral expression:

$$2A_2^2\int_0^{\infty}\left(4a^2x^4e^{-2ax^2}-2ax^2e^{-2ax^2}+\tfrac{1}{4}e^{-2ax^2}\right)dx = 1$$

or

$$8a^2A_2^2\int_0^{\infty}x^4e^{-2ax^2}dx - 4aA_2^2\int_0^{\infty}x^2e^{-2ax^2}dx + \tfrac{1}{2}A_2^2\int_0^{\infty}e^{-2ax^2}dx = 1 \qquad (1)$$

Use the definite integrals $\int_0^{\infty}e^{-bx^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{b}}$ and $\int_0^{\infty}x^{2n}e^{-bx^2}dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}b^n}\sqrt{\frac{\pi}{b}}, n \geq 1$ (see Table D-5) to integrate equation (1) term by term:

$$8a^2A_2^2\left[\frac{3}{2^3(2a)^2}\sqrt{\frac{\pi}{2a}}\right] - 4aA_2^2\left[\frac{1}{2^2(2a)}\sqrt{\frac{\pi}{2a}}\right] + \tfrac{1}{2}A_2^2\left[\frac{1}{2}\sqrt{\frac{\pi}{2a}}\right] = 1$$

or

$$A_2^2\left[\frac{3}{4}\sqrt{\frac{\pi}{2a}} - \frac{1}{2}\sqrt{\frac{\pi}{2a}} + \frac{1}{4}\sqrt{\frac{\pi}{2a}}\right] = 1$$

or

$$A_2^2\left[\frac{1}{2}\sqrt{\frac{\pi}{2a}}\right] = 1$$

Solve for A_2:

$$A_2 = \sqrt{2\sqrt{\frac{2a}{\pi}}}$$

Because $a = \frac{m\omega_0}{2\hbar} = \frac{m\omega_0\pi}{h}$ $\qquad$ $A_2 = \boxed{\sqrt[4]{\frac{8m\omega_0}{h}}}$

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(*a*) Let $x = -x$. The second derivative is an even operator, that is, $d^2\psi(-x)/d(-x)^2 = d^2\psi(x)/dx^2$. Therefore, if $U(-x) = U(x)$, the Schrödinger equation for $\psi(-x) = \psi(x)$ and must give the same values for the energy E. If $\psi(-x)$ differs from $\psi(x)$, the ratio $\psi(-x)/\psi(x)$ cannot be a function of x and must be a constant. Hence, $\psi(x) = C\psi(-x)$.

(*b*) The previous result means that replacing the argument of the wave function by its negative is equivalent to multiplication by C. Thus, if $C\psi(-x)$ is a good wave function and we replace its argument by its negative, that is, by x, we must multiply by C again. Thus, $\psi(x) = C^2\psi(x)$, $C^2 = 1$, and $C = \pm 1$.

***41** •••

Picture the Problem We can follow the step-by-step procedure outlined in the problem statement to show that $(E_{\text{av}})_{\text{min}} = +\frac{1}{2}\hbar\omega$.

1. The total classical energy is:

$$E_{\text{av}} = U_{\text{av}} + K_{\text{av}} = \tfrac{1}{2}m\omega^2\left(x^2\right)_{\text{av}} + \frac{\left(p^2\right)_{\text{av}}}{2m} \qquad (1)$$

2. Express the standard deviation of Δp:

$$(\Delta p)^2 = \left[(p - p_{\text{av}})^2\right]_{\text{av}} = \left[p^2 - 2pp_{\text{av}} - p_{\text{av}}^2\right]_{\text{av}}$$

Because $p_{\text{av}} = 0$:

$$(\Delta p)^2 = \left(p^2\right)_{\text{av}}$$

3. Express the standard deviation of Δx:

$$(\Delta x)^2 = \left[(x - x_{\text{av}})^2\right]_{\text{av}} = \left[x^2 - 2xx_{\text{av}} - x_{\text{av}}^2\right]_{\text{av}}$$

Because $x_{\text{av}} = 0$:

$$(\Delta x)^2 = \left(x^2\right)_{\text{av}}$$

4. Use the uncertainty principle $\Delta p = \hbar/2\Delta x$ to eliminate $(p^2)_{av}$ from the average energy in equation (1):

$$E_{av} = \tfrac{1}{2}m\omega^2\left(x^2\right)_{av} + \frac{\left(\Delta p^2\right)}{2m}$$
$$= \tfrac{1}{2}m\omega^2\left(x^2\right)_{av} + \frac{1}{2m}\left[\frac{\hbar^2}{4(\Delta x)^2}\right]$$
$$= \tfrac{1}{2}m\omega^2\left(x^2\right)_{av} + \frac{\hbar^2}{8m\left(x^2\right)_{av}}$$

Let $Z = (x^2)_{av}$ to obtain:

$$E_{av} = \tfrac{1}{2}m\omega^2 Z + \frac{\hbar^2}{8mZ}$$

5. Differentiate E_{av} with respect to Z and set this derivative equal to zero:

$$\frac{dE_{av}}{dZ} = \frac{d}{dZ}\left[\tfrac{1}{2}m\omega^2 Z + \frac{\hbar^2}{8mZ}\right]$$
$$= \tfrac{1}{2}m\omega^2 - \frac{\hbar^2}{8mZ^2} = 0 \text{ for extrema}$$

Solve for Z to find the value of Z that minimizes E_{av} (see the remark below):

$$Z = \frac{\hbar}{2m\omega}$$

6. Evaluate E_{av} when $Z = \hbar/2m\omega$:

$$\left(E_{av}\right)_{min} = \tfrac{1}{2}m\omega^2\left(\frac{\hbar}{2m\omega}\right) + \frac{\hbar^2}{8m}\left(\frac{2m\omega}{\hbar}\right)$$
$$= \boxed{\tfrac{1}{2}\hbar\omega}$$

Remarks: All we've shown is that $Z = \hbar/2m\omega$ is an extreme value, i.e., either a *maximum* or a *minimum*. To show that $Z = \hbar/2m\omega$ minimizes E_{av}, we must either 1) show that the second derivative of E_{av} with respect to Z evaluated at $Z = \hbar/2m\omega$ is positive, or 2) confirm that the graph of E_{av} as a function of Z opens upward at $Z = \hbar/2m\omega$.

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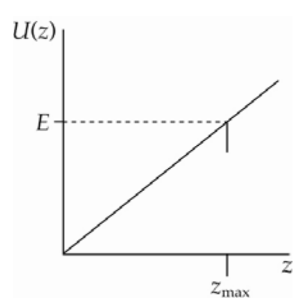
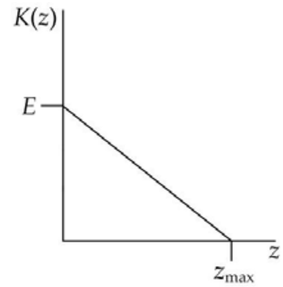
Picture the Problem

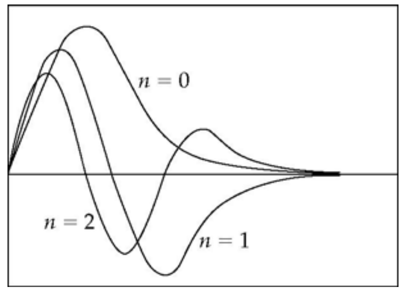
The classically allowed region is for $E \geq U(z)$. In the figure below, this region extends from $z = 0$ to $z = z_{max}$.



The kinetic energy is $E - U(z)$. In this case, $K(z)$ is a straight line extending from E at $z = 0$ to 0 at $z = z_{\text{max}}$.



A sketch of the wave functions for the lowest three energy states is shown to the right:



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Picture the Problem If $f(x) = 0$ everywhere on the interval $1 < x < 2$, then the slope of $f(x)$ is zero everywhere on the interval; and if the slope remains zero everywhere on the interval, then the rate of change of the slope (with respect to x) also remains zero everywhere on the interval; the rate of change of slope remains zero everywhere on the interval, then the rate of change of the rate of change of the slope also remains zero everywhere on the interval; and so on. More concisely, if $f(x) = 0$ everywhere on the interval $1 < x < 2$, then derivatives of $f(x)$ with respect to x of order 1, 2, 3, … are each equal to zero everywhere on the interval.

Calculating the first three derivatives of f we obtain:

$$\frac{df}{dx} = 3Ax^2 + 2Bx + Cx$$

$$\frac{d^2f}{dx^2} = 6Ax + 2B$$

and

$$\frac{d^3f}{dx^3} = 6A$$

Using $d^3f/df^3 = 0$ and solving for A one obtains:

$$A = 0$$

Substituting 0 for A in the expression for d^2f/dx^2 gives:

$$\frac{d^2f}{dx^2} = 0 + 2B = 2B$$

Using $d^2f/dx^2 = 0$ and solving for B yields:

$$B = 0$$

Substituting 0 for both A and B in the expression for df/dx yields:

$$\frac{df}{dx} = 0 + 0 + Cx = Cx$$

Using $df/dx = 0$ and solving for C

$$C = 0$$

one obtains:

Substituting 0 for A, B, and C in the expression for f gives:	$f = 0 + 0 + 0 + D = D$
Using $f = 0$ and solving for D gives:	$D = 0$

Thus, we've shown that if $f(x) = \mathrm{A}x^3 + Bx^2 + \mathrm{C}x + D = 0$ everywhere on the interval $1 < x < 2$, it follows that $A = B = C = D = 0$.