Chapter 39
Relativity

Conceptual Problems

***1** •

Picture the Problem The total relativistic energy E of a particle is defined to be the sum of its kinetic and rest energies.

The total relativistic energy of a particle is given by:

$$E = K + mc^2 = \tfrac{1}{2}mu^2 + mc^2$$

and $\boxed{(a)\text{ is correct.}}$

***2** •

Determine the Concept The gravitational field of the earth is slightly greater in the basement of the office building than it is at the top floor. Because clocks run more slowly in regions of low gravitational potential, clocks in the basement will run more slowly than clocks on the top floor. Hence, the twin who works on the top floor will age more quickly. $\boxed{(b)\text{ is correct.}}$

3 •

(*a*) True

(*b*) True

(*c*) False. The shortening of the length of an object in the direction in which it is moving is independent of the velocity of the frame of reference from which it is observed.

(*d*) True

(*e*) False. Consider two explosions equidistant, but in opposite directions, from an observer in the observer's frame of reference.

(*f*) False. Whether events appear to be simultaneous depends on the motion of the observer.

(*g*) True

4 •

Determine the Concept Because the clock is moving with respect to the first observer, a time interval will be longer for this observer than for the observer moving with the spring-and-mass oscillator. Hence, the observer moving with the system will measure a period that is less than T. $\boxed{(b)\text{ is correct.}}$

5 •

Determine the Concept Although $\Delta y = \Delta y'$, $\Delta t \neq \Delta t'$. Consequently, $u_y = \Delta y/\Delta t' \neq \Delta y'/\Delta t' = u_y'$.

Estimation and Approximation

6 ••

Picture the Problem We can calculate the sun's loss of mass per day from the number of reactions per second and the loss of mass per reaction.

Express the rate at which the sun loses mass:

$$\frac{\Delta M}{\Delta t} = N\Delta m$$

where N is the number of reactions per second and Δm is the loss of mass per reaction.

Solve for ΔM:

$$\Delta M = N\Delta m\Delta t \qquad (1)$$

Find the number of reactions per second, N:

$$N = \frac{P}{E/\text{reaction}} = \frac{4\times10^{26}\text{ J/s}}{25\frac{\text{MeV}}{\text{reaction}}\times1.60\times10^{-19}\frac{\text{J}}{\text{eV}}} = 10^{38}\text{ s}^{-1}$$

The loss of mass per reaction Δm is:

$$\Delta m = \frac{E/\text{reaction}}{c^2} = \frac{25\frac{\text{MeV}}{\text{reaction}}\times1.60\times10^{-19}\frac{\text{J}}{\text{eV}}}{\left(3\times10^{8}\text{ m/s}\right)^2} = 4.44\times10^{-29}\text{ kg}$$

Substitute numerical values in equation (1) and evaluate ΔM:

$$\Delta M = \left(10^{38}\text{ s}^{-1}\right)\left(4.44\times10^{-29}\text{ kg}\right)(1\text{d})(86.4\text{ ks/d}) = \boxed{3.84\times10^{14}\text{ kg}}$$

***7** ••

Picture the Problem We can use the result from Problem 30, for light that is Doppler-shifted with respect to an observer, $v = c\left(\frac{u^2-1}{u^2+1}\right)$, where $u = z + 1$ and z is the red-shift parameter, to find the ratio of v to c. In (*b*) we can solve Hubble's law for x and substitute our result from (*a*) to estimate the distance to the galaxy.

(*a*) Use the result of Problem 30 to express v/c as a function of z:

$$\frac{v}{c} = \frac{(z+1)^2-1}{(z+1)^2+1}$$

Substitute for z and evaluate v/c:

$$\frac{v}{c} = \frac{(5+1)^2-1}{(5+1)^2+1} = \boxed{0.946}$$

(*b*) Solve Hubble's law for x:

$$x = \frac{v}{H}$$

Substitute numerical values and evaluate x:

$$x = \frac{0.946c}{H} = \frac{0.946(3\times10^5\ \text{km/s})}{75\,\dfrac{\text{km/s}}{\text{Mpc}}}$$

$$= 3.78\times10^3\ \text{Mpc}\times\frac{3.26\times10^6\ c\cdot\text{y}}{\text{Mpc}}$$

$$= \boxed{12.3\,\text{G}c\cdot\text{y}}$$

Time Dilation and Length Contraction

8 •

Picture the Problem We can find the mean lifetime of a muon as measured in the laboratory using $t' = \gamma t$ where $\gamma = 1\Big/\sqrt{1-(v/c)^2}$ and t is the proper mean lifetime of the muon. The distance L that the muon travels is the product of its speed and its mean lifetime in the laboratory.

(*a*) The mean lifetime of the muon, as measured in the laboratory, is given by:

$$t' = \frac{t}{\sqrt{1-\left(\dfrac{v}{c}\right)^2}}$$

Substitute numerical values and evaluate t':

$$t' = \frac{2\,\mu\text{s}}{\sqrt{1-\left(\dfrac{0.95c}{c}\right)^2}} = \boxed{6.41\,\mu\text{s}}$$

(*b*) The distance L that the muon travels is related to its mean lifetime in the laboratory:

$$L = vt'$$

Substitute numerical values and evaluate L:

$$L = 0.95ct'$$
$$= 0.95\left(3\times10^{8}\text{ m/s}\right)\left(6.41\,\mu\text{s}\right)$$
$$= \boxed{1.83\,\text{km}}$$

9 ••

Picture the Problem The proper length L_p of the beam is its length as measured in a reference frame in which it is not moving. The proper length is related to its length in the frame in which it is measured by $L_p = \gamma L$.

(*a*) Relate the proper length L_p of the beam to its length L in the laboratory frame of reference:

$$L_p = \gamma L$$

The energy of the beam also depends on γ:

$$E = \gamma\, mc^2$$

Solve for and evaluate γ:

$$\gamma = \frac{E}{mc^2} = \frac{50\,\text{GeV}}{0.511\,\text{MeV}} = 9.785\times10^{4}$$

Substitute numerical values and evaluate L_p:

$$L_p = \left(9.785\times10^{4}\right)\left(1\,\text{cm}\right) = \boxed{978.5\,\text{m}}$$

and

[The width w of the beam is unchanged.]

(*b*) Express the length of the accelerator in the electron beam's frame of reference:

$$L_{acc} = \frac{L_{acc,p}}{\gamma}$$

Set $L_{acc} = L_p$:

$$L_p = \frac{L_{acc,p}}{\gamma}$$

Solve for $L_{acc,p}$:

$$L_{acc,p} = \gamma L_p$$

Substitute numerical values and evaluate L_p:

$$L_{acc,p} = (9.785\times10^4)(978.5\,\text{m}) = \boxed{9.57\times10^7\ \text{m}}$$

(*c*) The length of the positron bundle in the electron's frame of reference is:

$$L_{pos} = \frac{L}{\gamma}$$

Substitute numerical values and evaluate L_{pos}:

$$L_{pos} = \frac{1\text{cm}}{9.785\times10^4} = \boxed{0.102\,\mu\text{m}}$$

***10** ••

Picture the Problem The time required for the particles to reach the detector, as measured in the laboratory frame of reference is the ratio of the distance they must travel to their speed. The half life of the particles is the trip time as measured in a frame traveling with the particles. We can find the speed at which the particles must move if they are to reach the more distant detector by equating their half life to the ratio of the distance to the detector in the particle's frame of reference to their speed.

(*a*) The time required to reach the detector is the ratio of the distance to the detector and the speed with which the particles are traveling:

$$\Delta t = \frac{\Delta x}{v} = \frac{\Delta x}{0.866c}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{1000\,\text{m}}{0.866(3\times10^8\ \text{m/s})} = \boxed{3.85\,\mu\text{s}}$$

(*b*) The half life is the trip time as measured in a frame traveling with the particles:

$$\Delta t' = \frac{\Delta t}{\gamma} = \Delta t\sqrt{1-\left(\frac{v}{c}\right)^2}$$

Substitute numerical values and evaluate $\Delta t'$:

$$\Delta t' = 3.85\,\mu\text{s}\sqrt{1-\left(\frac{0.866c}{c}\right)^2} = \boxed{1.93\,\mu\text{s}}$$

(*c*) In order for half the particles to reach the detector:

$$\Delta t' = \frac{\Delta x'}{\gamma v} = \frac{\Delta x'\sqrt{1-\left(\frac{v}{c}\right)^2}}{v}$$

where $\Delta x'$ is the distance to the new detector.

Rewrite this expression to obtain:

$$\frac{v}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{\Delta x'}{\Delta t'}$$

Squaring both sides of the equation yields:

$$\frac{v^2}{1-\left(\frac{v}{c}\right)^2} = \left(\frac{\Delta x'}{\Delta t'}\right)^2$$

Substitute numerical values for $\Delta x'$ and $\Delta t'$ and simplify to obtain:

$$\frac{v^2}{1-\left(\frac{v}{c}\right)^2} = \left(\frac{10^4\ \text{m}}{1.93\ \mu\text{s}}\right)^2 = (17.3c)^2$$

Divide both sides of the equation by c^2 to obtain:

$$\frac{\frac{v^2}{c^2}}{1-\left(\frac{v}{c}\right)^2} = (17.3)^2$$

Solve this equation for v^2/c^2:

$$\frac{v^2}{c^2} = \frac{(17.3)^2}{1+(17.3)^2} = 0.9967$$

Finally, solving for v yields:

$$v = \boxed{0.998c}$$

11 ••

Picture the Problem We can use the time-dilation relationship to find the speed of the spacecraft. The distance to the second star is the product of the new gamma factor, the speed of the spacecraft, and the elapsed time. Finally, the time that has elapsed on earth (your age) is the sum of the elapsed times for the three legs of the journey.

(*a*) From the point of view of an observer on earth, the time for the trip will be:

$$\Delta t = \frac{L}{v}$$

From the point of view of an observer on the spaceship, the time for the trip will be:

$$\Delta t' = \frac{\Delta t}{\gamma} = \frac{L}{\gamma v}$$

Substitute for γ to obtain:

$$\Delta t' = \frac{L}{v}\sqrt{1-\frac{v^2}{c^2}}$$

Solve for v:

$$v = \frac{Lc}{\sqrt{L^2 + c^2(\Delta t')^2}}$$

Substitute numerical values and evaluate v:

$$v = \frac{(27\,c\cdot\text{y})c}{\sqrt{(27\,c\cdot\text{y})^2 + c^2(12\,\text{y})^2}} = \boxed{0.914c}$$

Note that from the point of view of an earth observer, this part of the trip has taken 27 $c\cdot$ y/0.914c = 29.5 y.

(b) The distance the ship travels, from the point of view of an earth observer, in 5 y is:

$$\Delta L' = 2\gamma\Delta L = 2\gamma v\Delta t$$

where γ is the gamma factor for the first part of the trip.

The gamma factor in Part (a) is:

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{1}{\sqrt{1-\left(\frac{0.914c}{c}\right)^2}} = 2.46$$

Substitute numerical values and evaluate $\Delta L'$:

$$\Delta L' = 2(2.46)(0.914c)(5\,\text{y}) = \boxed{22.5\,c\cdot\text{y}}$$

(c) The elapsed time Δt on earth (your age) is the sum of the times for the spacecraft to travel to the star 27 $c\cdot$ y away, l to the second star, and to return home from the second star:

$$\Delta t = 29.5\,\text{y} + 22.5\,\text{y} + \Delta t_{\text{returning home}}$$

The elapsed time on earth while the spacecraft is returning to earth is:

$$\Delta t_{\text{returning home}} = 2\gamma\Delta t_{\text{ship's time}} = 2(2.46)(10\,\text{y}) = 49.2\,\text{y}$$

Substitute for $\Delta t_{\text{returning}}$ home and evaluate Δt:

$$\Delta t = 29.5\,\text{y} + 22.5\,\text{y} + 49.2\,\text{y} = \boxed{101\,\text{y}}$$

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Picture the Problem We can use $\Delta t = L/v$, where L is the distance to the star and v is the speed of the spaceship to find the time Δt for the trip as measured on earth. The travel time as measured by a passenger on the spaceship can be found using $\Delta t' = \Delta t/\gamma$.

(a) The travel time as measured on earth is the ratio of the distance

$$\Delta t = \frac{L}{v}$$

traveled L to speed of the spaceship:

Substitute numerical values and evaluate Δt:

$$\Delta t=\frac{35c\cdot\text{y}}{2.7\times10^{8}\text{ m/s}}=\frac{35\text{ y}}{\frac{2.7\times10^{8}\text{ m/s}}{c}}$$

$$=\frac{35\text{ y}}{0.9}=\boxed{38.9\text{ y}}$$

(a) The travel time as measured by a passenger on the spaceship is given by:

$$\Delta t'=\frac{\Delta t}{\gamma}=\Delta t\sqrt{1-\left(\frac{v}{c}\right)^{2}}$$

Substitute numerical values and evaluate $\Delta t'$:

$$\Delta t'=(38.9\text{ y})\sqrt{1-(0.9)^{2}}=\boxed{17.0\text{ y}}$$

13 •

Picture the Problem We can use the definition of γ and the binomial expansion of $(1 + x)^n$ to show that each of these relationships holds provided $v << c$.

(a) Express the gamma factor:

$$\gamma=\frac{1}{\sqrt{1-\frac{v^{2}}{c^{2}}}}=\left(1-\frac{v^{2}}{c^{2}}\right)^{-1/2}$$

Expand the radical factor binomially to obtain:

$$\gamma=\left(1-\frac{v^{2}}{c^{2}}\right)^{-1/2}$$

$$=1+\left(-\frac{1}{2}\right)\left(-\frac{v^{2}}{c^{2}}\right)+\text{higher order terms}$$

For $v << c$:

$$\gamma\approx\boxed{1+\frac{1}{2}\frac{v^{2}}{c^{2}}}$$

(b) Express the reciprocal of γ:

$$\frac{1}{\gamma}=\sqrt{1-\frac{v^{2}}{c^{2}}}$$

Expand the radical binomially to obtain:

$$\frac{1}{\gamma}=\left(1-\frac{v^{2}}{c^{2}}\right)^{1/2}$$

$$=1+\left(\frac{1}{2}\right)\left(-\frac{v^{2}}{c^{2}}\right)+\text{higher order terms}$$

For $v << c$:

$$\frac{1}{\gamma} \approx \boxed{1 - \frac{1}{2}\frac{v^2}{c^2}}$$

(*c*) Express the gamma factor:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

Subtract one from both sides of the equation to obtain:

$$\gamma - 1 = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1$$

Expand the radical binomially to obtain:

$$\gamma - 1 = 1 + \left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) - 1 + \text{higher order terms}$$

For $v << c$:

$$\gamma - 1 \approx \boxed{\frac{1}{2}\frac{v^2}{c^2}}$$

14 ••

Picture the Problem We can express the fractional difference in your time-interval measurements as a function of γ and solve the resulting equation for the relative speed of the two spaceships.

Express the fractional difference in the time-interval measurements of the two observers:

$$\frac{\Delta t - \Delta t'}{\Delta t} = 1 - \frac{\Delta t'}{\Delta t} = 0.01$$

Since $\Delta t'/\Delta t = 1/\gamma$:

$$\frac{\Delta t - \Delta t'}{\Delta t} = 1 - \frac{1}{\gamma} = 0.01$$

From Problem 13(*b*) we have:

$$\frac{1}{\gamma} \approx 1 - \frac{1}{2}\frac{v^2}{c^2}$$

Substitute to obtain:

$$1 - \frac{1}{\gamma} \approx 1 - \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) = 0.01$$

or

$$\frac{1}{2}\frac{v^2}{c^2} = 0.01$$

Solve for v to obtain:

$$v = \sqrt{0.02}c = 0.141c = \boxed{4.23\times10^{7}\ \text{m/s}}$$

15 ••

Picture the Problem We can use the time dilation equation to relate the time lost by the clock to the speed of the plane and the time it must fly.

Express the time δt lost by the clock:

$$\delta t = \Delta t - \Delta t_{\text{p}} = \Delta t - \frac{\Delta t}{\gamma} = \Delta t\left(1-\frac{1}{\gamma}\right)$$

Because $V << c$, we can use part (*b*) of Problem 13:

$$\frac{1}{\gamma} \approx 1-\frac{1}{2}\frac{V^2}{c^2}$$

Substitute to obtain:

$$\delta t = \Delta t\left[1-\left(1-\frac{1}{2}\frac{V^2}{c^2}\right)\right] = \frac{1}{2}\frac{V^2}{c^2}\Delta t$$

Solve for Δt:

$$\Delta t = \frac{2\delta t\, c^2}{V^2}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{2(1\,\text{s})(3\times10^{8}\ \text{m/s})^2}{(2000\,\text{km/h}\times1\text{h}/3600\,\text{s})^2}$$

$$= 5.83\times10^{11}\,\text{s}\times\frac{1\,\text{y}}{31.56\,\text{Ms}}$$

$$= \boxed{1.85\times10^{4}\ \text{y}}$$

The Lorentz Transformation, Clock Synchronization, and Simultaneity

16 ••

Picture the Problem We can use the inverse Lorentz transformations and the result of Problem 13(*c*) to show that when $u << c$ the transformation equations for x, t, and u reduce to the Galilean equations.

The inverse transformation for x is:

$$x' = \gamma(x - vt)$$

From Problem 13(*c*):

$$\gamma = 1+\frac{1}{2}\frac{v^2}{c^2}$$

Substitute for γ and expand to obtain:

$$x' = \left(1+\frac{1}{2}\frac{v^2}{c^2}\right)(x-vt)$$

$$= x - vt + \frac{1}{2}\frac{v^2}{c^2}x - \frac{1}{2}\frac{v^3}{c^2}$$

When $v << c$:

$$\boxed{x' \approx x - vt}$$

The inverse transformation for t is:

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

Substitute for γ and expand to obtain:

$$t' = \left(1+\frac{1}{2}\frac{v^2}{c^2}\right)\left(t - \frac{vx}{c^2}\right)$$

$$= t - \frac{vx}{c^2} + \frac{1}{2}\frac{v^2}{c^2}t - \frac{1}{2}\frac{v^3}{c^4}x$$

When $v << c$:

$$\boxed{t' \approx t}$$

The inverse velocity transformation for motion in the x direction is:

$$u_x' = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

When $v << c$:

$$\boxed{u_x' \approx u_x - v}$$

The inverse velocity transformation for motion in the y direction is:

$$u_y' = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

Substitute for γ and expand to obtain:

$$u_y' = \frac{u_y}{\left(1+\frac{1}{2}\frac{v^2}{c^2}\right)\left(1 - \frac{vu_x}{c^2}\right)}$$

$$= \frac{u_y}{1 - \frac{vu_x}{c^2} + \frac{1}{2}\frac{v^2}{c^2} - \frac{1}{2}\frac{v^3u_x}{c^4}}$$

When $v << c$:

$$\boxed{u_y' \approx u_y}$$

Proceed similarly to show that:

$$\boxed{u_z' \approx u_z}$$

***17 ••**

Picture the Problem Let S be the reference frame of the spaceship and S' be that of the earth (transmitter station). Let event A be the emission of the light pulse and event B the reception of the light pulse at the nose of the spaceship. In (*a*) and (*c*) we can use the classical distance, rate, and time relationship and in (*b*) and (*d*) we can apply the inverse Lorentz transformations.

(*a*) In both *S* and *S*′ the pulse travels at the speed *c*. Thus:

$$t_A = \frac{L_p}{v} = \frac{400\,\text{m}}{0.76\text{c}} = \boxed{1.76\,\mu\text{s}}$$

(*c*) The elapsed time, according to the clock on the ship is:

$$t_B = t_{\text{pulse to travel length of ship}} + t_\text{A}$$

Find the time of travel of the pulse to the nose of the ship:

$$t_{\text{pulse to travel length of ship}} = \frac{400\,\text{m}}{2.998\times10^8\ \text{m/s}} = 1.33\,\mu\text{s}$$

Substitute numerical values and evaluate t_B:

$$t_B = 1.33\,\mu\text{s} + 1.76\,\mu\text{s} = \boxed{3.09\,\mu\text{s}}$$

(*b*) The inverse time transformation is:

$$t_B' = \gamma\left(t - \frac{vx}{c^2}\right)$$

where

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{(0.76c)^2}{c^2}}} = 1.54$$

Substitute numerical values and evaluate t'_B:

$$t_B' = (1.54)\left(3.09\,\mu\text{s} - \frac{(-0.76c)(400\,\text{m})}{c^2}\right)$$

$$= (1.54)\left(3.09\,\mu\text{s} - \frac{(-0.76)(400\,\text{m})}{3\times10^8\ \text{m/s}}\right)$$

$$= \boxed{6.32\,\mu\text{s}}$$

(*d*) The inverse transformation for *x* is:

$$x' = \gamma(x - vt)$$

Substitute numerical values and evaluate x':

$$x' = (1.54)\left[400\,\text{m} - (-0.76)(3\times10^8\ \text{m/s})(3.09\times10^{-6}\ \text{s})\right] = \boxed{1.70\,\text{km}}$$

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Picture the Problem We can use Equation 39-12, the inverse time transformation equation, to find the required speed of the observer.

Use Equation 39-12 to obtain:

$$t_B' - t_A' = \gamma\left[(t_B - t_A) - \frac{v}{c^2}(x_B - x_A)\right]$$

$$= \boxed{\gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)}$$

where $\Delta t = t_B - t_A$ and $\Delta x = x_B - x_A$.

Events A and B are simultaneous if:

$$\Delta t - \frac{v\Delta x}{c^2} = 0$$

Solve for v:

$$v = \frac{c^2\Delta t}{\Delta x}$$

Substitute numerical values and evaluate v:

$$v = \frac{(3\times10^8\ \text{m/s})^2(2\ \mu\text{s})}{1.5\ \text{km}}$$

$$= 1.20\times10^8\ \text{m/s} = \boxed{0.4\text{c}}$$

$$\boxed{\text{Yes, } t_B' \text{ will be less than } t_A' \text{ if } V > 0.4c.}$$

19 ••

Picture the Problem We can use Equation 39-12, the inverse time transformation equation, to express the separation in time between the two explosions as measured in S' as a function of the speed of the observer and Equation 39-11, the inverse position transformation equation, to find the speed of the observer.

Use Equation 39-12 to obtain:

$$\Delta t' = \gamma\left[\Delta t - \frac{v}{c^2}\Delta x\right]$$

$$= \frac{\Delta t - \frac{v}{c^2}\Delta x}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \qquad (1)$$

From Equation 39-11:

$$\Delta x' = \gamma(\Delta x - v\Delta t)$$

Because the explosions occur at the same point in space, $\Delta x' = 0$:

$$0 = \gamma(\Delta x - v\Delta t)$$

Solve for v:

$$v = \frac{\Delta x}{\Delta t}$$

Substitute numerical values and evaluate v:

$$v = \frac{1200\,\text{m} - 480\,\text{m}}{5\,\mu\text{s}} = 1.44\times10^{8}\ \text{m/s}$$

Substitute numerical values in equation (1) and evaluate $\Delta t'$:

$$\Delta t' = \frac{5\,\mu\text{s} - \dfrac{1.44\times10^{8}\ \text{m/s}}{\left(3\times10^{8}\ \text{m/s}\right)^{2}}\left(1200\,\text{m} - 480\,\text{m}\right)}{\sqrt{1-\left(\dfrac{1.44\times10^{8}\ \text{m/s}}{3\times10^{8}\ \text{m/s}}\right)^{2}}} = \boxed{4.39\,\mu\text{s}}$$

20 •••

Picture the Problem We can use Equation 39-12, the inverse time transformation equation, to establish the results called for in this problem.

(*a*) Use Equation 39-12 to obtain:

$$t_2' - t_1' = \gamma\left[\left(t_2 - t_1\right) - \frac{v}{c^2}\left(x_2 - x_1\right)\right]$$

$$= \boxed{\gamma\left(T - \frac{vD}{c^2}\right)}$$

where $T = t_2 - t_1$ and $D = x_2 - x_1$.

(*b*) Events 1 and 2 are simultaneous in S' if:

$$t_2' = t_1'$$

or

$$T - \frac{vD}{c^2} = 0 \Rightarrow D = \frac{c^2T}{v}$$

Because $v \le c$:

$$D \ge \boxed{cT}$$

(*c*) $\boxed{\text{If } D < cT\text{, then } t_2' > t_1' \text{ and the events are not simultaneous in } S'.}$

(*d*) If $D = c'T > cT$, then:

$$T - \frac{vD}{c^2} = T\left[1 - \frac{v}{c}\frac{c'}{c}\right] = t_2' - t_1'$$

$\boxed{\text{In this case, } t_2' - t_1' \text{ could be negative; i.e., } t_2' \text{ could be less than } t_1'\text{, or the effect could precede the cause.}}$

21 •••

Picture the Problem Let S be the ground reference frame, S' the reference frame of the rocket, and $v = 0.9c$ be the speed of the rocket relative to S. Denote the tail and nose of the rocket by T and N, respectively. The initial conditions in S' are $t_N' = 0$, $x_N' = 0$, $x_T' = 0$, and $x_T' = -L' = -700\,\text{m}$.

(a) The reading of the tail clock is given by:

$$t_T' = \gamma\left(t_T - \frac{vx_T}{c^2}\right) = -\frac{\gamma v x_T}{c^2}$$

because $t_T = 0$

We can find x_T using the length contraction equation:

$$x_T = -\frac{L'}{\gamma}$$

Substitute to obtain:

$$t_T' = \frac{vL'}{c^2}$$

Substitute numerical values and evaluate t_T':

$$t_T' = \frac{(0.9)(700\,\text{m})}{3\times10^8\ \text{m/s}} = \boxed{2.10\,\mu\text{s}}$$

(b) The time for the rocket to move a distance L' is given by:

$$t_T' = \frac{L'}{v} = \frac{L'}{0.9c}$$

Substitute numerical values and evaluate t_T':

$$t_T' = \frac{700\,\text{m}}{0.9(3\times10^8\ \text{m/s})} = \boxed{2.59\,\mu\text{s}}$$

(c) As seen by an observer on the ground:

$$t_N = \Delta t' = 2.59\,\mu\text{s} - 2.10\,\mu\text{s} = \boxed{0.49\,\mu\text{s}}$$

(d) Because the clocks are synchronized in S':

$$t_N' = t_T' = \boxed{2.59\,\mu\text{s}}$$

(e) The time the signal is received on the ground is the sum of the time when the signal is sent and the time for it to travel to the ground:

$$t_{\text{rec}} = \Delta t + \Delta t_{\text{travel}}$$

Find Δt, the time the signal is sent:

$$\Delta t = \gamma\Delta t_{\text{p}} = \frac{1\,\text{h}}{\sqrt{1-\frac{(0.9c)^2}{c^2}}} = 2.294\,\text{h}$$

Find Δt_{travel}, the time for the signal to travel to the ground:

$$\Delta t_{\text{travel}} = \frac{\Delta x}{c} = \frac{(2.294\,\text{h})(0.9c)}{c} = 2.065\,\text{h}$$

Substitute for Δt and Δt_{travel} and evaluate t_{rec}:

$$t_{rec} = 2.294\,\text{h} + 2.065\,\text{h} = \boxed{4.36\,\text{h}}$$

(*f*) Find Δx when the signal is sent:

$$\Delta x = (4.36\,\text{h})(0.9c) = 3.924\,c\cdot\text{h}$$

In *S*, the signal arrives at $0.1c$ relative to the rocket. The time required for the signal to travel to the rocket is:

$$\Delta t = \frac{\Delta x}{0.1c} = \frac{3.924\,c\cdot\text{h}}{0.1c} = 39.24\,\text{h}$$

Find the time when the signal reaches the rocket:

$$t = 39.24\,\text{h} + 3.924\,\text{h} = 43.16\,\text{h}$$

Finally, use the time dilation equation to find t_N':

$$t_N' = \frac{t}{\gamma} = (43.16\,\text{h})\sqrt{1-\frac{(0.9c)^2}{c^2}}$$
$$= \boxed{18.8\,\text{h}}$$

***22 •••**

Picture the Problem We can use the inverse time dilation equation to derive an expression for the elapsed time between the flashes in *S′* in terms of the elapsed time between the flashes in *S,* their separation in space, and the speed *v* with which *S′* is moving.

From the inverse time transformation we have:

$$\Delta t' = \gamma\left[\Delta t - \frac{v}{c^2}\Delta x\right]$$

where $\Delta t'$ is the time between the flashes in *S′* and Δt and Δx are the elapsed time between the flashes and their separation in *S.*

Set $\Delta t' = -\Delta t$ to obtain:

$$\frac{-\Delta t}{\gamma} = \Delta t - \frac{v}{c^2}\Delta x$$

or

$$-\Delta t\sqrt{1-\frac{v^2}{c^2}} = \Delta t - \frac{v}{c^2}\Delta x$$

Square both sides of the equation:

$$(\Delta t)^2 - \frac{v^2}{c^2}(\Delta t)^2 = (\Delta t)^2 - 2\frac{v}{c^2}\Delta x\Delta t + \frac{v^2}{c^4}(\Delta x)^2$$

Simplify to obtain:

$$-v(\Delta t)^2 = -2\Delta x\Delta t + \frac{v}{c^2}(\Delta x)^2$$

Solve for v:

$$v = \frac{2\dfrac{\Delta x}{\Delta t}}{1+\left(\dfrac{1}{c}\dfrac{\Delta x}{\Delta t}\right)^2}$$

Substitute numerical values and evaluate v:

$$v = \frac{2\left(\dfrac{2400\,\text{m}}{5\,\mu\text{s}}\right)}{1+\left[\dfrac{1}{3\times10^8\,\text{m/s}}\left(\dfrac{2400\,\text{m}}{5\,\mu\text{s}}\right)\right]^2} = 2.697\times10^8\,\text{m/s} = \boxed{0.899c}$$

Because v is positive, S' is moving in the positive x direction.

The Velocity Transformation

23 ••

Picture the Problem We can make the substitutions given in the hint in Equation 39-18*a* and simplify the resulting expression to show that $u_x < c$.

Equation 39-18*a* gives the x direction relativistic velocity transformation:

$$u_x = \frac{u_x' + v}{1+\dfrac{vu_x'}{c^2}} \text{ or } \frac{u_x}{c} = \frac{u_x' + v}{c+\dfrac{vu_x'}{c}}$$

Make the substitutions given in the hint to obtain:

$$\frac{u_x}{c} = \frac{(1-\varepsilon_1)c + (1-\varepsilon_2)c}{c+\dfrac{(1-\varepsilon_2)c(1-\varepsilon_1)c}{c}} = \frac{2-(\varepsilon_1+\varepsilon_2)}{1+(1-\varepsilon_2)(1-\varepsilon_1)} = \frac{2-(\varepsilon_1+\varepsilon_2)}{2-(\varepsilon_1+\varepsilon_2)+\varepsilon_1\varepsilon_2}$$

Because ε_1 and ε_2 are small positive numbers that are less than 1:

$$\frac{u_x}{c} < 1 \Rightarrow \boxed{u_x < c}$$

***24** ••

Picture the Problem We'll let the velocity (in S) of the spaceship after the ith boost be v_i and derive an expression for the ratio of v to c after the spaceship's (i + 1)th boost as a function of N. We can use the definition of γ, in terms of v/c to plot γ as a function of N.

(*a*) and (*b*) The velocity of the spaceship after the (i + 1)th boost is given by relativistic velocity addition equation:

$$v_{i+1} = \frac{v_i + 0.5c}{1 + \frac{(0.5c)v_i}{c^2}}$$

Factor c from both the numerator and denominator to obtain:

$$v_{i+1} = \frac{\frac{v_i}{c} + 0.5}{1 + 0.5\frac{v_i}{c}}$$

γ_i is given by:

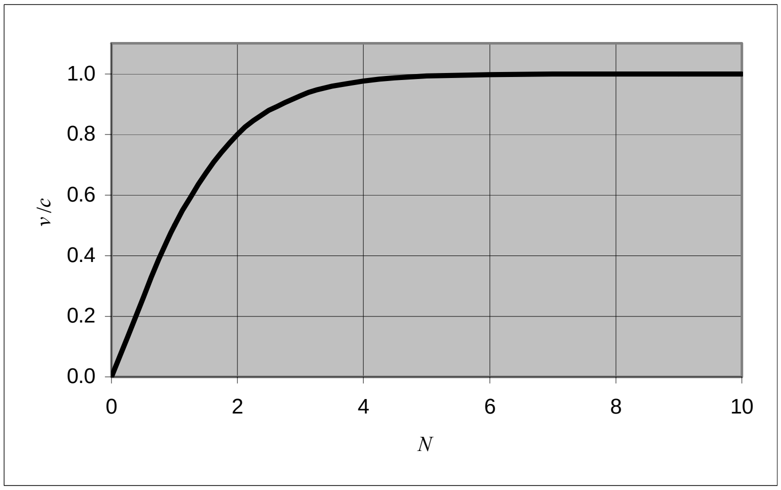
$$\gamma_i = \frac{1}{\sqrt{1 - \left(\frac{v_i}{c}\right)}}$$

A spreadsheet program to calculate v/c and γ as functions of the number of boosts N is shown below. The formulas used to calculate the quantities in the columns are as follows:

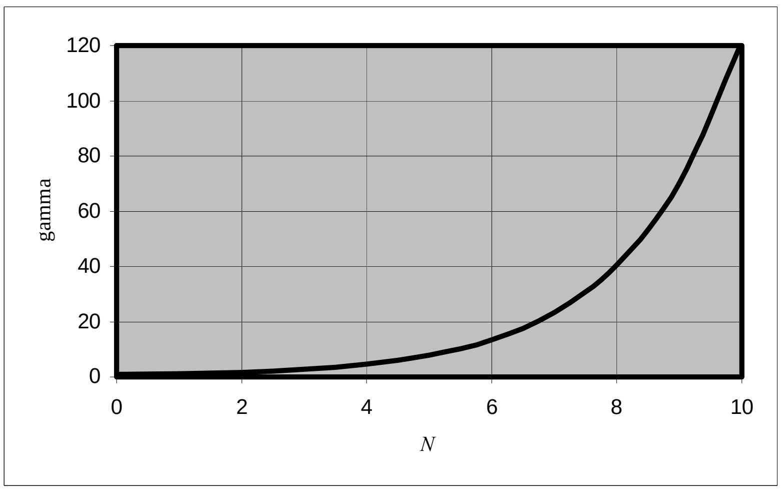
Cell	Content/Formula	Algebraic Form
A3	0	N
B2	0	v_0
B3	(B2+0.5)/(1+0.5*B2)	v_{i+1}
C1	1/(1–B2^2)^0.5	γ

	A	B	C
1	boost	v/c	gamma
2	0	0.000	1.00
3	1	0.500	1.15
4	2	0.800	1.67
5	3	0.929	2.69
6	4	0.976	4.56
7	5	0.992	7.83
8	6	0.997	13.52
9	7	0.999	23.39
10	8	1.000	40.51
11	9	1.000	70.15
12	10	1.000	121.50

A graph of v/c as a function of N is shown below:



A graph of γ as a function of N is shown below:



(c) Examination of the spreadsheet or of the graph of v/c as a function of N indicates that, after 8 boosts, the velocity of the spaceship is greater than $0.999c$.

(d) After 5 boosts, the spaceship has traveled a distance Δx, measured in the earth frame of reference (S), given by:

$$\begin{aligned}\Delta x &= \Delta x_{1\to 2} + \Delta x_{2\to 3} + \Delta x_{3\to 4} + \Delta x_{4\to 5} \\ &= (0.5c)(10\,\text{s})\gamma_{1\to 2} + (0.8c)(10\,\text{s})\gamma_{2\to 3} + (0.929c)(10\,\text{s})\gamma_{3\to 4} + (0.976c)(10\,\text{s})\gamma_{4\to 5} \\ &\quad + (0.992c)(10\,\text{s})\gamma_{5\to 6} \\ &= (0.5c)(10\,\text{s})(1.15) + (0.8c)(10\,\text{s})(1.67) + (0.929c)(10\,\text{s})(2.69) \\ &\quad + (0.976c)(10\,\text{s})(4.56) + (0.992c)(10\,\text{s})(7.83) \\ &= \boxed{166\,c\cdot\text{s}}\end{aligned}$$

The average speed of the spaceship, between boost 1 and boost 5, as measured in S is given by:

$$v_{\text{av}} = \frac{\Delta x}{\Delta t}$$

where Δt is the travel time as measured in the earth frame of reference.

Express Δt as the sum of the times the spaceship travels during each 10-s interval following a boost in its speed:

$$\begin{aligned}\Delta t &= \Delta t_{1\to 2} + \Delta t_{2\to 3} + \Delta t_{3\to 4} + \Delta t_{4\to 5}\\ &= (10\,\text{s})\gamma_{1\to 2} + (10\,\text{s})\gamma_{2\to 3} + (10\,\text{s})\gamma_{3\to 4} + (10\,\text{s})\gamma_{4\to 5} + (10\,\text{s})\gamma_{5\to 6}\\ &= (10\,\text{s})(\gamma_{1\to 2} + \gamma_{2\to 3} + \gamma_{3\to 4} + \gamma_{4\to 5} + \gamma_{5\to 6})\end{aligned}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = (10\,\text{s})(1.15 + 1.67 + 2.69 + 4.56 + 7.83) = 179\,\text{s}$$

Substitute for Δx and Δt and evaluate v_{av}:

$$v_{\text{av}} = \frac{166\,c\cdot\text{s}}{179\,\text{s}} = \boxed{0.927c}$$

Remarks: This result seems to be reasonable. Relativistic time dilation implies that the spacecraft will be spending larger amounts of time at high speed (as seen in reference frame *S*).

The Relativistic Doppler Shift

25 •

Picture the Problem We can substitute, using $v = f\lambda$, in the relativistic Doppler shift equation and solve for the speed of the source.

Using the expression for the relativistic Doppler shift, express f' as a function of v:

$$f' = f\sqrt{\frac{1 + v/c}{1 - v/c}}$$

Substitute using $v = f\lambda$ and simplify to obtain:

$$\frac{v}{\lambda'} = \frac{v}{\lambda}\sqrt{\frac{1 + v/c}{1 - v/c}}$$

$$\frac{\lambda}{\lambda'} = \sqrt{\frac{1 + v/c}{1 - v/c}}$$

or

$$\left(\frac{\lambda}{\lambda'}\right)^2 = \frac{1 + v/c}{1 - v/c}$$

Solve for v to obtain:

$$v = \left[\frac{\left(\frac{\lambda}{\lambda'}\right)^2 - 1}{\left(\frac{\lambda}{\lambda'}\right)^2 + 1}\right]c$$

Substitute numerical values and evaluate v:

$$v = \left[\frac{\left(\frac{589\,\text{nm}}{547\,\text{nm}}\right)^2 - 1}{\left(\frac{589\,\text{nm}}{547\,\text{nm}}\right)^2 + 1}\right]\left(3\times10^8\ \text{m/s}\right) = \boxed{2.22\times10^7\ \text{m/s}}$$

26 •

Picture the Problem We can use the relativistic Doppler shift, when the source and the receiver are receding, to relate the frequencies of the two wavelengths and $c = f\lambda$ to express the ratio of the wavelengths as a function of the speed of the galaxy.

When the source and receiver are moving away from each other, the relativistic Doppler shift is given by:

$$f' = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\,f_0$$

Use the relationship between the wavelength and frequency to obtain:

$$\frac{c}{\lambda'} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\,\frac{c}{\lambda_0} \Rightarrow \frac{\lambda_0}{\lambda'} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

Solve for λ'/λ_0:

$$\frac{\lambda'}{\lambda_0} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

Express the fractional redshift:

$$\frac{\lambda'-\lambda_0}{\lambda_0} = \frac{\lambda'}{\lambda_0} - 1 = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} - 1$$

Substitute numerical values and evaluate $(\lambda'-\lambda_0)/\lambda_0$:

$$\frac{\lambda'-\lambda_0}{\lambda_0}=\sqrt{\frac{1+\dfrac{1.85\times10^7\ \text{m/s}}{2.998\times10^8\ \text{m/s}}}{1-\dfrac{1.85\times10^7\ \text{m/s}}{2.998\times10^8\ \text{m/s}}}}-1=\boxed{0.0637}$$

27 ••

Picture the Problem We can begin the derivation by expressing the number of waves encountered by the observer, in the rest frame of the source, in a time interval Δt. We can then relate this time interval to the time interval in the rest frame of the observer to complete the derivation of Equation 39-16a.

Express the number of waves n encountered by the observer, in the rest frame of the source, in a time interval Δt_s:

$$n=\frac{(c+v)\Delta t_s}{\lambda}=\frac{(c+v)f_o\Delta t_s}{c}=f_o\left(1+\frac{v}{c}\right)\Delta t_s$$

This time interval in the rest frame of the observer is given by:

$$\Delta t_o=\frac{\Delta t_s}{\gamma}$$

Express the frequency heard by the observer and simplify to obtain:

$$f_o=\frac{n}{\Delta t_o}=\gamma\left(1+\frac{v}{c}\right)f_o=\frac{1+\dfrac{v}{c}}{\sqrt{1-\dfrac{v^2}{c^2}}}f_o=\sqrt{\frac{1+\dfrac{v}{c}}{1-\dfrac{v}{c}}}f_o=\boxed{\frac{\sqrt{1-\dfrac{v^2}{c^2}}}{1-\dfrac{v}{c}}f_o}$$

28 •

Picture the Problem We can use the expression for the relativistic Doppler shift to show that, to a good approximation, $\Delta f/f\approx\pm v/c$.

Express the fractional Doppler shift in terms of f and f_0:

$$\frac{\Delta f}{f_0}=\frac{f-f_0}{f_0}=\frac{f}{f_0}-1$$

When the source and receiver are approaching each other, the relativistic Doppler shift is given by:

$$f=\sqrt{\frac{1+\dfrac{v}{c}}{1-\dfrac{v}{c}}}f_0\Rightarrow\frac{f}{f_0}=\sqrt{\frac{1+\dfrac{v}{c}}{1-\dfrac{v}{c}}}$$

Substitute in the expression for $\Delta f/f_0$ to obtain:

$$\frac{\Delta f}{f_0}=\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}-1$$

$$=\left(1+\frac{v}{c}\right)^{1/2}\left(1-\frac{v}{c}\right)^{-1/2}-1$$

Keeping just the lowest order terms in v/c, expand binomially to obtain:

$$\frac{\Delta f}{f_0}=\left(1+\frac{1}{2}\frac{v}{c}\right)\left(1+\frac{1}{2}\frac{v}{c}\right)-1$$

$$\approx 1+\frac{v}{c}-1=\boxed{\frac{v}{c}}$$

The sign depends on whether the source and receiver are approaching or receding. Here we have assumed that they are approaching.

***29 ••**

Picture the Problem Due to its motion, the orbiting clock will run more slowly than the earth-bound clock. We can use Kepler's third law to find the radius of the satellite's orbit in terms of its period, the definition of speed to find the orbital speed of the satellite from the radius of its orbit, and the time dilation equation to find the difference δ in the readings of the two clocks.

Express the time δ lost by the clock:

$$\delta=\Delta t-\Delta t_{\rm p}=\Delta t-\frac{\Delta t}{\gamma}=\Delta t\left(1-\frac{1}{\gamma}\right)$$

Because $v \ll c$, we can use part (*b*) of Problem 13:

$$\frac{1}{\gamma}\approx 1-\frac{1}{2}\frac{v^2}{c^2}$$

Substitute to obtain:

$$\delta=\Delta t\left[1-\left(1-\frac{1}{2}\frac{v^2}{c^2}\right)\right]=\frac{1}{2}\frac{v^2}{c^2}\Delta t \qquad (1)$$

Express the square of the speed of the satellite in its orbit:

$$v^2=\left(\frac{2\pi r}{T}\right)^2=\frac{4\pi^2 r^2}{T^2} \qquad (2)$$

where T is its period and r is the radius of its (assumed) circular orbit.

Use Kepler's third law to relate the period of the satellite to the radius of its orbit about the earth:

$$T^2=\frac{4\pi^2}{GM_{\rm e}}r^3=\frac{4\pi^2}{gR_{\rm e}^2}r^3$$

Solve for r:

$$r = \sqrt[3]{\frac{gR_e^2T^2}{4\pi^2}}$$

Substitute numerical values and evaluate r:

$$r = \sqrt[3]{\frac{(9.81\,\text{m/s}^2)(6370\,\text{km})^2(90\,\text{min}\times 60\,\text{s/min})^2}{4\pi^2}} = 6.65\times 10^6\ \text{m}$$

Substitute numerical values in equation (2) and evaluate v^2:

$$v^2 = \frac{4\pi^2(6.65\times 10^6\ \text{m})^2}{(90\,\text{min}\times 60\,\text{s/min})^2} = 5.99\times 10^7\ \text{m}^2/\text{s}^2$$

Finally, substitute for v^2 in equation (1) and evaluate δ:

$$\delta = \frac{1}{2}\frac{(5.99\times 10^7\ \text{m}^2/\text{s}^2)(1\,\text{y}\times 31.56\,\text{Ms/y})}{(3\times 10^8\ \text{m/s})^2} = \boxed{10.5\,\text{ms}}$$

30 ••

Picture the Problem We can use the definition of the redshift parameter and the relativistic Doppler shift equation to show that $v = c\left(\frac{u^2-1}{u^2+1}\right)$, where $u = z + 1$.

The red-shift parameter is defined to be:

$$z = \frac{f - f'}{f'}$$

The relativistic Doppler shift is given by:

$$f' = f\sqrt{\frac{1+v/c}{1-v/c}}$$

Substitute to obtain:

$$z = \frac{f - f\sqrt{\frac{1+v/c}{1-v/c}}}{f\sqrt{\frac{1+v/c}{1-v/c}}} = \frac{1-\sqrt{\frac{1+v/c}{1-v/c}}}{\sqrt{\frac{1+v/c}{1-v/c}}} = \sqrt{\frac{1+v/c}{1-v/c}} - 1$$

Letting $u = z + 1$:

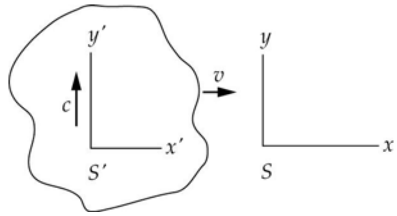
$$u = z + 1 = \sqrt{\frac{1+v/c}{1-v/c}}$$

Solve for v to obtain:

$$v = \boxed{c\left(\frac{u^2-1}{u^2+1}\right)}$$

31 •

Picture the Problem We can use the velocity transformation equations for the x and y directions to express the x and y components of the velocity of the light beam in frame S.



(a) The x and y components of the velocity of the light beam in frame S are:

$$u_x = \frac{u_x' + v}{1 + \frac{vu_x'}{c^2}}$$

and

$$u_y = \frac{u_y'}{\gamma\left(1 + \frac{vu_x'}{c^2}\right)}$$

Because $u_x' = 0$:

$$u_x = \boxed{v} \text{ and } u_y = \boxed{\frac{c}{\gamma}}$$

(b) The magnitude of the velocity of the light beam in S is given by:

$$u = \sqrt{u_x^2 + u_y^2} = \sqrt{v^2 + \frac{c^2}{\gamma^2}}$$

$$= \sqrt{v^2 + \left(1 - \frac{v^2}{c^2}\right)c^2} = \boxed{c}$$

32 •

Picture the Problem Let S be the earth reference frame and S' be that of the ship traveling east (positive x direction). Then in the reference frame S', the velocity of S is directed west, i.e., $v = -u_x$. We can apply the inverse velocity transformation equation to express u'_x in terms of u_x.

Apply the inverse velocity transformation equation to obtain:

$$u_x' = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

Substitute for v:

$$u_x' = \frac{u_x + u_x}{1 + \frac{u_x^2}{c^2}} = \frac{2u_x}{1 + \frac{u_x^2}{c^2}}$$

Because $u_x = 0.90c$:

$$u_x' = \frac{2(0.90c)}{1+\frac{(0.90c)^2}{c^2}} = \boxed{0.994c}$$

Picture the Problem We can apply the inverse velocity transformation equation to express the speed of the particle relative to both frames of reference.

(*a*) Express u_x' in terms of u_x'':

$$u_x' = \frac{u_x'' + v}{1+\frac{vu_x''}{c^2}}$$

where V of S', relative to S'', is $0.8c$.

Substitute numerical values and evaluate u_x':

$$u_x' = \frac{0.8c + 0.8c}{1+\frac{(0.8c)^2}{c^2}} = \frac{1.6c}{1.64} = \boxed{0.976c}$$

(*b*) Express u_x in terms of u_x':

$$u_x = \frac{u_x' + v}{1+\frac{vu_x'}{c^2}}$$

where v of S, relative to S', is $0.8c$.

Substitute numerical values and evaluate u_x:

$$u_x = \frac{0.976c + 0.8c}{1+\frac{(0.8c)(0.976c)}{c^2}} = \frac{1.776c}{1.781}$$

$$= \boxed{0.997c}$$

Relativistic Momentum and Relativistic Energy

***34** •

Picture the Problem We can use the relation for the total energy, momentum, and rest energy to find the momentum of the proton and Equation 39-26 to relate the speed of the proton to its energy and momentum.

Relate the energy of the proton to its momentum:

$$E^2 = p^2c^2 + \left(mc^2\right)^2$$

(*b*) Solve for p to obtain:

$$p = \sqrt{\frac{E^2 - \left(mc^2\right)^2}{c^2}}$$

Substitute numerical values and evaluate p:

$$p = \frac{\sqrt{(2200\,\text{MeV})^2 - (938\,\text{MeV})^2}}{c} = \boxed{1.99\,\frac{\text{GeV}}{c}}$$

(a) From Equation 39-26 we have:

$$\frac{v}{c} = \frac{pc}{E}$$

Solve for v to obtain:

$$v = \frac{pc}{E}c$$

Substitute numerical values and evaluate v:

$$v = \frac{1.99\,\text{GeV}}{2200\,\text{MeV}}c = \boxed{0.905c}$$

35 •

Picture the Problem We can use $E^2 = p^2c^2 + (mc^2)^2$ (from Problem R-37) to find the relativistic momentum of the particle in terms of γ and the fact that the kinetic energy of the particle equals twice its rest energy to find the error made in using mv for the momentum of the particle.

Express the error e in using $p' = mv$ for the momentum of the particle:

$$e = \frac{p - p'}{p} = 1 - \frac{p'}{p} \qquad (1)$$

From Problem R-37, the relationship between the total energy E, momentum p, and rest energy mc^2 of the particle is:

$$E^2 = p^2c^2 + (mc^2)^2$$

Solve for p to obtain:

$$p = \sqrt{\frac{E^2}{c^2} - m^2c^2} = \sqrt{m^2c^2\left(\frac{E^2}{m^2c^4} - 1\right)}$$

Because $E = \gamma mc^2$:

$$p = mc\sqrt{\gamma^2 - 1}$$

Substitute for p and p' in equation (1) to obtain:

$$e = 1 - \frac{mu}{mc\sqrt{\gamma^2 - 1}} = 1 - \frac{u}{c\sqrt{\gamma^2 - 1}} \qquad (2)$$

From the definition of γ:

$$\frac{u}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$$

Eliminate v/c in equation (2) to obtain:

$$e = 1 - \frac{1}{\sqrt{\gamma^2 - 1}}\sqrt{1 - \frac{1}{\gamma^2}} = 1 - \frac{1}{\gamma} \qquad (3)$$

The kinetic energy of the particle is related to its rest energy:

$$K = (\gamma - 1)mc^2$$

Solve for γ to obtain:

$$\gamma = 1 + \frac{K}{mc^2}$$

Because the kinetic energy of the particle is twice its rest energy:

$$\gamma = 1 + \frac{2mc^2}{mc^2} = 3$$

Substitute for γ in equation (3) and evaluate e:

$$e = 1 - \frac{1}{3} = 0.667 = \boxed{66.7\%}$$

36 ••

Picture the Problem We can use the result of Problem R-37 to find the energy of the particle and its energy in a reference frame in which its momentum is 4 MeV/c. We can apply the inverse velocity transformation equation to find the relative velocities of the two reference frames.

(a) From Problem R-37 we have:

$$E^2 = p^2c^2 + m_0^2c^4 = p^2c^2 + E_0^2$$

Solve for E_0:

$$E_0 = \sqrt{E^2 - p^2c^2}$$

Substitute numerical values and evaluate E_0:

$$E_0 = \sqrt{(8\,\text{MeV})^2 - (6\,\text{MeV}/c)^2c^2} = \sqrt{(8\,\text{MeV})^2 - (6\,\text{MeV})^2} = \boxed{5.29\,\text{MeV}}$$

(b) Because E_0 is independent of the reference frame:

$$E = \sqrt{p^2c^2 + E_0^2}$$

Substitute numerical values and evaluate E:

$$E = \sqrt{(4\,\text{MeV}/c)^2c^2 + (5.29\,\text{MeV})^2} = \boxed{6.63\,\text{MeV}}$$

(c) The inverse velocity transformation is:

$$u_b = \frac{u_a - v}{1 - \frac{vu_a}{c^2}}$$

where the subscripts refer to the velocities in parts (a) and (b) of the problem.

Solve for v to obtain:

$$v = \frac{u_a - u_b}{1 - \frac{u_a u_b}{c^2}} \qquad (1)$$

Relate the relativistic energy of the particle in (a) to its velocity:

$$E = \frac{E_0}{\sqrt{1 - \frac{u_a^2}{c^2}}} \Rightarrow u_a = c\sqrt{1 - \left(\frac{E_0}{E}\right)^2}$$

Substitute numerical values and evaluate u_a:

$$u_a = c\sqrt{1 - \left(\frac{5.29\,\text{MeV}}{8\,\text{MeV}}\right)^2} = 0.750c$$

Relate the relativistic momentum of the particle in (b) to its velocity:

$$p = \frac{m_0 u_b}{\sqrt{1 - \frac{u_b^2}{c^2}}} \Rightarrow u_b = \frac{pc}{\sqrt{p^2 + m_0 c^2}}$$

Substitute numerical values and evaluate u_b:

$$u_b = \frac{(4\,\text{MeV}/c)c}{\sqrt{(4\,\text{MeV}/c)^2 + (5.29\,\text{MeV}/c)^2}} = 0.603c$$

Substitute in equation (1) and evaluate V:

$$V = \frac{0.750c - 0.603c}{1 - \frac{(0.750c)(0.603c)}{c^2}} = \boxed{0.268c}$$

37 ••

Picture the Problem We can use the rule for the derivative of a quotient to establish the result given in the problem statement.

Use the expression for the derivative of a quotient to obtain:

$$\frac{d}{du}\left(\frac{mu}{\sqrt{1 - u^2/c^2}}\right) = \frac{\sqrt{1 - \frac{u^2}{c^2}}\,m + \frac{mu^2}{c^2}\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}}{1 - \frac{u^2}{c^2}}$$

Multiply the numerator and denominator of the right-hand side of this expression by

$\sqrt{1-\frac{u^2}{c^2}}$ and simplify to obtain:

$$\frac{d}{du}\left(\frac{mu}{\sqrt{1-u^2/c^2}}\right)=\frac{\sqrt{1-\frac{u^2}{c^2}}\sqrt{1-\frac{u^2}{c^2}}m+\frac{mu^2}{c^2}\frac{\sqrt{1-\frac{u^2}{c^2}}}{\sqrt{1-\frac{u^2}{c^2}}}}{\left(1-\frac{u^2}{c^2}\right)\sqrt{1-\frac{u^2}{c^2}}}=\frac{\left(1-\frac{u^2}{c^2}\right)m+\frac{mu^2}{c^2}}{\left(1-\frac{u^2}{c^2}\right)^{3/2}}$$

$$=m\left(1-\frac{u^2}{c^2}\right)^{-3/2}$$

and

$$d\left(\frac{mu}{\sqrt{1-u^2/c^2}}\right)=\boxed{m\left(1-\frac{u^2}{c^2}\right)^{-3/2}du}$$

38 ••

Picture the Problem We will first consider the decay process in the center of mass reference frame and then transform to the laboratory reference frame in which one of the pions is at rest.

Apply energy conservation in the center of mass frame of reference to obtain:

$$m_{K_0}c^2=2m_{\pi_0}\gamma c^2$$

Solve for γ:

$$\gamma=\frac{m_{K_0}}{2m_{\pi_0}}$$

Substitute numerical values and evaluate γ:

$$\gamma=\frac{497.7\,\text{MeV}/c^2}{2(139.6\,\text{MeV}/c^2)}=1.78$$

Because one of the pions is at rest in the laboratory frame, γ = 1.78 for the transformation to the laboratory frame. The kinetic energy of the K^0 particle is:

$$K_{K^0}=(\gamma-1)E=(1.78-1)(497.7\,\text{MeV})=\boxed{388.2\,\text{MeV}}$$

The total initial energy in the laboratory frame is:

$$E=497.7\,\text{MeV}+388.2\,\text{MeV}=885.9\,\text{MeV}$$

Express the energy of the other pion:

$$E_\pi = E - 2m_{0\pi}c^2$$

Substitute numerical values and evaluate E_π:

$$E_\pi = 885.9\,\text{MeV} - 2(139.6\,\text{MeV})$$
$$= \boxed{607\,\text{MeV}}$$

***39 ••**

Picture the Problem The total kinetic energy of the two protons in part (*a*) is the sum of their kinetic energies and is given by $K = 2(\gamma - 1)E_0$. Part (*b*) differs from part (*a*) in that we need to find the speed of the moving proton relative to frame *S*.

(*a*) The total kinetic energy of the protons in frame *S′* is given by:

$$K = 2(\gamma - 1)E_0$$

Substitute for γ and E_0 and evaluate *K*:

$$K = 2\left(\frac{1}{\sqrt{1 - \frac{(0.5c)^2}{c^2}}} - 1\right)(938.28\,\text{MeV})$$
$$= \boxed{290\,\text{MeV}}$$

(*b*) The kinetic energy of the moving proton in frame *S* is given by:

$$K = (\gamma - 1)E_0 \qquad (1)$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{uv}{c^2}}}$$

Express the speed *u* of the proton in frame *S*:

$$u = \frac{u_x' + v}{1 + \frac{vu_x'}{c^2}}$$

Substitute numerical values and evaluate *u*:

$$u = \frac{0.5c + 0.5c}{1 + \frac{(0.5c)(0.5c)}{c^2}} = 0.800c$$

Evaluate γ:

$$\gamma = \frac{1}{\sqrt{1 - \frac{(0.8c)(0.8c)}{c^2}}} = 1.67$$

Substitute numerical values in equation (1) and evaluate *K*:

$$K = (1.67 - 1)(938.28\,\text{MeV}) = \boxed{629\,\text{MeV}}$$

40 ••

Picture the Problem We can find the speed of each proton by equating their total relativistic kinetic energy to $2mc^2$. In (*b*) we can use the inverse velocity transformation with $V = u$ and $u_x = -u$ to find u_x'. In part (*c*) we'll need to evaluate γ' for the kinetic energy transformation $K_\text{L} = (\gamma' - 1)E_0$.

(*a*) Set the relativistic kinetic energy of the protons equal to $2mc^2$ to obtain:

$$2(\gamma - 1)E_0 = 2mc^2 \Rightarrow \gamma = 2$$

Substitute for γ:

$$\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} = 2$$

Solve for *u* to obtain:

$$u = \frac{\sqrt{3}}{2}c = \boxed{0.866c}$$

(*b*) Use the inverse velocity transformation with $v = u$ and $u_x = -u$ to find u_x':

$$u_x' = \frac{u_x - v}{1 - \frac{vu_x}{c^2}} = \frac{-\frac{\sqrt{3}}{2}c - \frac{\sqrt{3}}{2}c}{1 - \frac{\left(-\frac{\sqrt{3}}{2}c\right)\left(\frac{\sqrt{3}}{2}c\right)}{c^2}}$$

$$= \frac{-4\sqrt{3}c}{7} = \boxed{-0.990c}$$

(*c*) The kinetic energy of the moving proton in the laboratory's frame is given by:

$$K_\text{L} = (\gamma' - 1)E_0$$

where

$$\gamma' = \frac{1}{\sqrt{1 - \frac{(u_x')^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{\left(-\frac{4\sqrt{3}}{7}c\right)^2}{c^2}}} = 7$$

Substitute for γ' and E_0 and evaluate K_L:

$$K_\text{L} = (7 - 1)mc^2 = \boxed{6mc^2}$$

41 •••

Picture the Problem (*a*) and (*b*) The initial speed of the particle can be found from its total energy and its total energy found using $E = K + E_0 = \gamma E_0$. (*c*) We can solve $E^2 = p^2c^2 + (mc^2)^2$ for the initial momentum of the system. In (*d*) and (*e*) we can use conservation of energy and conservation of momentum to find the total kinetic energy after the collision and the mass of the system after the collision.

(*a*) Express the total energy of the particle:

$$E = K + E_0 = \gamma E_0$$

Because the kinetic energy of the particle is twice its energy:

$$2E_0 + E_0 = \gamma E_0 \text{ and } \gamma = 3$$

Solve the factor γ for u:

$$u = c\sqrt{1 - \frac{1}{\gamma^2}}$$

Substitute for γ and evaluate u:

$$u = c\sqrt{1 - \frac{1}{3^2}} = c\sqrt{\frac{8}{9}} = \boxed{0.943c}$$

(*b*) The total energy of the particle is:

$$E = K + E_0 = \gamma E_0 = 3E_0$$

Substitute for E_0 and evaluate E:

$$E = 3(1\,\text{MeV}) = \boxed{3\,\text{MeV}}$$

(*c*) The initial momentum of the incoming particle is related to its energy and mass according to:

$$E^2 = p^2c^2 + (mc^2)^2$$

Solve for p:

$$p = \frac{1}{c}\sqrt{E^2 - (mc^2)^2}$$

Substitute for E and mc^2 and simplify to obtain:

$$p = \frac{1}{c}\sqrt{(3E_0)^2 - (E_0)^2} = \frac{\sqrt{8}E_0}{c}$$

Substitute for E_0 and evaluate p:

$$p = \frac{\sqrt{8}(1\,\text{MeV})}{c} = \boxed{2.83\,\text{MeV}/c}$$

(*d*) and (*e*) From conservation of energy we have:

$$E_f = E_i = 5\,\text{MeV}$$

From conservation of momentum we have:

$$p_f = p_i$$

The final momentum of the system is related to its energy and mass according to:

$$E_f^2 = p_f^2 c^2 + E_{f0}^2$$

Solve for E_{f0}:

$$E_{f0} = \sqrt{E_f^2 - p_f^2 c^2}$$

Substitute numerical values and evaluate E_{f0}:

$$E_{f0} = \sqrt{(5\,\text{MeV})^2 - (2.83\,\text{MeV}/c)^2 c^2} = 4.122\,\text{MeV}$$

Because $E_{f0} = m_{f0}c^2$:

$$m_{f0} = \frac{E_{f0}}{c^2} = \boxed{4.12\,\text{MeV}/c^2}$$

The total kinetic energy after the collision is given by:

$$K_f = E_f - E_{f0} = 5\,\text{MeV} - 4.122\,\text{MeV} = \boxed{0.878\,\text{MeV}}$$

General Relativity

***42 ••**

Picture the Problem Let *m* represent the mass equivalent of a photon. We can equate the change in the gravitational potential energy of a photon as it rises a distance *L* in the gravitational field to $h\Delta f$ and then express the wavelength shift in terms of the frequency shift.

The speed of the photons in the light beam are related to their frequency and wavelength:

$$c = f\lambda \Rightarrow f = \frac{c}{\lambda}$$

Differentiate this expression with respect to λ to obtain:

$$\frac{df}{d\lambda} = -c\lambda^{-2} = -\frac{c}{\lambda^2}$$

Approximate $df/d\lambda$ by $\Delta f/\Delta\lambda$ and solve for Δf:

$$\Delta f = -\frac{c}{\lambda^2}\Delta\lambda$$

Divide both sides of this equation by *f* to obtain:

$$\frac{\Delta f}{f} = \frac{-\frac{c}{\lambda^2}\Delta\lambda}{\frac{c}{\lambda}} = -\frac{\Delta\lambda}{\lambda}$$

Solve for $\Delta\lambda$:

$$\Delta\lambda = -\lambda\frac{\Delta f}{f} \qquad (1)$$

The change in the energy of the photon as it rises a distance L in a gravitational field is given by:

$$\Delta E = \Delta U = mgL$$

Because $\Delta E = h\Delta f$:

$$h\Delta f = mgL \qquad (2)$$

Letting m represent the mass equivalent of the photon:

$$E = mc^2 = hf \qquad (3)$$

Divide equation (2) by equation (3) to obtain:

$$\frac{h\Delta f}{hf} = \frac{mgL}{mc^2} \Rightarrow \frac{\Delta f}{f} = \frac{gL}{c^2}$$

Substitute for $\Delta f/f$ in equation (1):

$$\Delta\lambda = -\frac{gL\lambda}{c^2}$$

Substitute numerical values and evaluate $\Delta\lambda$:

$$\Delta\lambda = -\frac{(9.81\,\text{m/s}^2)(100\,\text{m})(632.8\,\text{nm})}{(3\times10^8\,\text{m/s})^2} = \boxed{-6.90\times10^{-12}\,\text{nm}}$$

43 ••

Picture the Problem In a freely falling reference frame, both cannonballs travel along straight lines, so they must hit each other, as they were pointed at each other when they were fired.

44 •••

Picture the Problem Consider the turntable to be a giant hollow cylinder in space that is spinning about its axis. Someone on the inside surface of the cylinder would experience a centripetal acceleration caused by the normal force of the surface pushing them toward the rotation axis. Alternatively, they can consider that they are not accelerating but a gravitational field $\vec{g} = \omega^2 r\hat{\boldsymbol{r}}$ is pushing them away from the axis ($\hat{\boldsymbol{r}}$ is away from the axis). This is the principle of equivalence. From this perspective, up is toward the axis and the points closer to the axis are at the higher gravitational potential. (Just like the electric field points in the direction of decreasing electric potential, the gravitational field points in the direction of decreasing gravitational potential.)

(*a*) From the time dilation equation we have:

$$\Delta t_{\text{r}} = \frac{\Delta t_0}{\gamma}$$

and

$$\frac{\Delta t_{\text{r}} - \Delta t_0}{\Delta t_0} = \frac{1}{\gamma} - 1$$

Because $r\omega/c << 1$ (see Problem 14):

$$\frac{1}{\gamma} \approx 1 - \frac{1}{2}\frac{u^2}{c^2} = 1 - \frac{r^2\omega^2}{2c^2}$$

Substitute to obtain:

$$\frac{\Delta t_{\text{r}} - \Delta t_0}{\Delta t_0} = 1 - \frac{r^2\omega^2}{2c^2} - 1 = \boxed{-\frac{r^2\omega^2}{2c^2}}$$

(*b*) The pseudoforce is given by:

$$F_{\text{P}} = -ma$$

where a is the acceleration of the non-inertial reference frame.

In this case a is the centripetal acceleration:

$$a = -r\omega^2 \Rightarrow F_{\text{P}} = F_r = \boxed{mr\omega^2}$$

To relate this problem to Equation 39-31, point 2 is a distance r from the axis and point 1 is on the axis. The term in parentheses on the right hand side of Equation 2 is $\phi_2 - \phi_1$, which translates to $\phi_{\text{r}} - \phi_0$. Because ϕ_{r} is at a lower potential than ϕ_0, this term is negative. Hence:

$$\phi_r - \phi_0 = -\int_0^r \vec{g} \cdot d\vec{\ell} = -\int_0^r \omega^2 r\,\hat{\boldsymbol{r}} \cdot d\vec{\ell} = -\int_0^r \omega^2 r\,dr = \boxed{-\tfrac{1}{2}r^2\omega^2}$$

From Equation 39-31:

$$\frac{\Delta t_{\text{r}} - \Delta t_0}{\Delta t_0} = \frac{1}{c^2}\left(\phi_r - \phi_0\right) = \frac{1}{c^2}\left(-\tfrac{1}{2}r^2\omega^2\right) = \boxed{-\frac{r^2\omega^2}{2c^2}}$$

General Problems

45 •

Picture the Problem We can use the definition of γ and the time dilation equation to find the speed of the muon.

(*a*) From the definition of γ we have:

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}}$$

Solve for *u*/*c*:

$$\frac{u}{c} = \sqrt{1-\frac{1}{\gamma^2}}$$

Relate the mean lifetime of the muon to its proper lifetime:

$$\Delta t = \gamma\Delta t_{\mathrm{p}} \Rightarrow \gamma = \frac{\Delta t}{\Delta t_{\mathrm{p}}}$$

Substitute in the expression for *u*/*c* to obtain:

$$\frac{u}{c} = \sqrt{1-\left(\frac{\Delta t_{\mathrm{p}}}{\Delta t}\right)^2}$$

Substitute numerical values and evaluate *u*/*c*:

$$\frac{u}{c} = \sqrt{1-\left(\frac{2\ \mu\mathrm{s}}{46\ \mu\mathrm{s}}\right)^2} = 0.999$$

or

$$u = \boxed{0.999c}$$

46 •

Picture the Problem We can use the relativistic Doppler shift, when the source and the receiver are receding, to relate the frequencies of the two wavelengths and $c = f\lambda$ to express the ratio of the wavelengths as a function of the speed of the galaxy.

When the source and receiver are moving away from each other, the relativistic Doppler shift is given by:

$$f' = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\,f_0$$

Use the relationship between the wavelength and frequency to obtain:

$$\frac{c}{\lambda'} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\,\frac{c}{\lambda_0} \Rightarrow \frac{\lambda_0}{\lambda'} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

Because $\lambda' = 2\lambda_0$:

$$\frac{\lambda_0}{2\lambda_0} = \frac{1}{2} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

Solve for v/c:

$$\frac{v}{c} = \frac{3}{5} \Rightarrow v = \boxed{0.600c}$$

***47 ••**

Picture the Problem We can use Equation 39-12, the inverse time transformation equation, to relate the elapsed times and separations of the events in the two systems to the velocity of S' relative to S. We can use this same relationship in (*b*) to find the time at which these events occur as measured in S'.

(*a*) Use Equation 39-12 to obtain:

$$\Delta t' = t_2' - t_1' = \gamma\left[(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)\right]$$
$$= \gamma\left[\Delta t - \frac{v}{c^2}\Delta x\right]$$

Because the events occur simultaneously in frame S', $\Delta t' = 0$ and:

$$0 = \Delta t - \frac{v}{c^2}\Delta x$$

Solve for v to obtain:

$$v = \frac{c^2 \Delta t}{\Delta x}$$

Substitute for Δt and Δx and evaluate V:

$$v = \frac{c^2(0.5\,\mathrm{y} - 1\,\mathrm{y})}{2.0\,c\cdot\mathrm{y} - 1.0\,c\cdot\mathrm{y}} = \boxed{-0.5c}$$

Because $\Delta t = t_2 - t_1 = -0.5\,\mathrm{y}$:

$$\boxed{S' \text{ moves in the negative } x \text{ direction.}}$$

(*b*) Use the inverse time transformation to obtain:

$$t_2' = \gamma\left(t_2 - \frac{vx_2}{c^2}\right) = \frac{t_2 - \dfrac{vx_2}{c^2}}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

Substitute numerical values and evaluate t_2' and t_1':

$$t_2' = t_1' = \frac{0.5\,\mathrm{y} - \dfrac{(-0.5c)(2.0\,c\cdot\mathrm{y})}{c^2}}{\sqrt{1 - \dfrac{(-0.5c)^2}{c^2}}}$$
$$= \boxed{1.73\,\mathrm{y}}$$

48 ••

Picture the Problem We can use the relationship between distance, speed, and time and the length contraction relationship to find the speed of the ship relative to the earth. The elapsed time between the departure of the spaceship and the receipt of the signal at earth is the sum of the travel time to the distant star system and the time it takes the signal to return to earth.

(*a*) Express the travel time as measured on the spaceship:

$$\Delta t' = \frac{L'}{u} = \frac{L}{\gamma u}$$

Solve for γu:

$$\gamma u = \frac{L}{\Delta t'}$$

Substitute numerical values and evaluate γu:

$$\gamma u = \frac{12\,c\cdot\text{y}}{15\,\text{y}} = 0.8c$$

or

$$\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{0.8c}{u}$$

Solve for u to obtain:

$$u = \boxed{0.625c}$$

(*b*) The elapsed time T before earth receives the signal is the sum of the travel time to the distant star system and the time it takes the signal to return:

$$T = \frac{L}{u} + \frac{L}{c}$$

Substitute numerical values and evaluate T:

$$T = \frac{12\,c\cdot\text{y}}{0.625c} + \frac{12\,c\cdot\text{y}}{c} = \boxed{31.2\,\text{y}}$$

49 ••

Picture the Problem We can use conservation of energy to find γ in the CM frame of reference and then use the definition of γ to find the speed u of the projectile proton. We can then use the velocity transformation equation to find the speed and kinetic energy of this proton in the laboratory frame of reference.

Use conservation of energy to find γ in the CM frame of reference:

$$\gamma E_\text{i} = E_\text{f} \Rightarrow \gamma = \frac{E_\text{f}}{E_\text{i}}$$

E_i and E_f are:

$$E_\text{i} = 938\,\text{Mev} + 938\,\text{MeV} = 1876\,\text{MeV}$$

and

$$E_f = 938\,\text{Mev} + 938\,\text{MeV} + 135\,\text{MeV} = 2011\,\text{MeV}$$

Substitute E_i and E_f and evaluate γ:

$$\gamma = \frac{2011\,\text{MeV}}{1876\,\text{MeV}} = 1.072$$

Express γ as a function of the speed u of the projectile proton:

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Solve for u to obtain:

$$u = c\sqrt{1 - \frac{1}{\gamma^2}}$$

Substitute for γ and evaluate u:

$$u = c\sqrt{1 - \frac{1}{(1.072)^2}} = 0.360c$$

Transform to the laboratory frame and find u':

$$u' = \frac{u - v}{1 - \frac{vu}{c^2}} = \frac{0.360c - (-0.360c)}{1 - \frac{(0.360c)(-0.360\text{c})}{c^2}} = 0.637c$$

The kinetic energy of the moving proton in the laboratory's frame is given by:

$$K_L = (\gamma_L - 1)E_0$$

where

$$\gamma_L = \frac{1}{\sqrt{1 - \frac{(u')^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{(0.637c)^2}{c^2}}} = 1.30$$

Substitute for γ_L and E_0 and evaluate K_L:

$$K_L = (1.30 - 1)(938\,\text{MeV}) = \boxed{281\,\text{MeV}}$$

Remarks: In Problem 55 we show that the threshold kinetic energy of the projectile is given by $\boldsymbol{K_{th} = \frac{\left(\sum m_{in} + \sum m_{fin}\right)\left(\sum m_{fin} - \sum m_{in}\right)c^2}{2m_{target}}}$.

50 ••

Picture the Problem We can use $\Delta t_{\text{p}} = L_{\text{p}}/u$, where u is the speed of the bullet relative to the rocket, to find the elapsed time in the frame of the rocket. In (*b*) and (*c*) we can proceed similarly, finding the speed of the bullet relative to the rocket as seen from the ground frame in (*b*) and, in (*c*), using the speed of the bullet relative to the rocket.

(*a*) In the rocket frame:

$$\Delta t = \Delta t_{\text{p}} = \frac{L_{\text{p}}}{u} = \frac{L_{\text{p}}}{0.8c}$$

Substitute numerical values and evaluate Δt:

$$\Delta t = \frac{1000\,\text{m}}{0.8\left(2.998\times10^{8}\ \text{m/s}\right)} = \boxed{4.17\ \mu\text{s}}$$

(*b*) In the ground frame of reference, the elapsed time is given by:

$$\Delta t_{\text{ground}} = \frac{L}{u'} \qquad (1)$$

where u' is the speed of the bullet relative to the rocket as seen from the ground.

The speed of the ground is given by:

$$u_{\text{ground}} = \frac{u_{\text{rocket}} + V}{1 + \frac{V u_{\text{ground}}}{c^2}}$$

Substitute for u_{rocket} and V and evaluate u_{ground}:

$$u_{\text{ground}} = \frac{0.6c + 0.8c}{1 + \frac{(0.8c)(0.6c)}{c^2}} = 0.946c$$

The speed of the bullet relative to the rocket as seen from the ground is:

$$u' = 0.946c - 0.6c = 0.346c$$

Relate L_{ground} to L_{p}:

$$L_{\text{ground}} = \frac{L_{\text{p}}}{\gamma} = L_{\text{p}}\sqrt{1 - \frac{u_{\text{ground}}^2}{\text{c}^2}}$$

Substitute for L_{p} and u_{ground} and evaluate L_{ground}:

$$L_{\text{ground}} = (1000\,\text{m})\sqrt{1 - \frac{(0.6c)^2}{\text{c}^2}} = 800\,\text{m}$$

Substitute numerical values in equation (1) and evaluate Δt_{ground}:

$$\Delta t_{\text{ground}} = \frac{800\,\text{m}}{0.346\left(2.998\times10^{8}\ \text{m/s}\right)} = \boxed{7.71\ \mu\text{s}}$$

(*c*) In the bullet's frame of reference, the elapsed time is given by:

$$\Delta t_{\text{bullet}} = \frac{L'}{u_{\text{bullet}}} \qquad (2)$$

The length L' of the rocket in the bullet's frame is given by:

$$L' = \frac{L_{\text{p}}}{\gamma} = L_{\text{p}}\sqrt{1-\frac{u_{\text{bullet}}^2}{c^2}}$$

Substitute in equation (2) to obtain:

$$\Delta t_{\text{bullet}} = \frac{L_{\text{p}}}{u_{\text{bullet}}}\sqrt{1-\frac{u_{\text{bullet}}^2}{c^2}}$$

Substitute numerical values and evaluate Δt_{bullet}:

$$\Delta t_{\text{bullet}} = \frac{1000\,\text{m}}{0.8\left(2.998\times10^{8}\ \text{m/s}\right)}\sqrt{1-\frac{(0.8c)^2}{c^2}} = \boxed{2.50\,\mu\text{s}}$$

***51 •••**

Picture the Problem We can use conservation of energy to express the recoil velocity of the box and the relationship between distance, speed, and time to find the distance traveled by the box in time $\Delta t = L/c$. Equating the initial and final locations of the center of mass will allow us to show that the radiation must carry mass $m = E/c^2$.

(*a*) Apply conservation of momentum to obtain:

$$\frac{E}{c} + Mv = p_{\text{i}} = 0$$

Solve for v:

$$v = \boxed{-\frac{E}{Mc}}$$

(*b*) The distance traveled by the box in time $\Delta t = L/c$ is:

$$d = v\Delta t = \frac{vL}{c}$$

Substitute for v from (*a*):

$$d = \frac{L}{c}\left(-\frac{E}{Mc}\right) = \boxed{-\frac{LE}{Mc^2}}$$

(*c*) Let $x = 0$ be at the center of the box and let the mass of the photon be m. Then initially the center of mass is at:

$$x_{\text{CM}} = \frac{-\frac{1}{2}mL}{M+m}$$

When the photon is absorbed at the other end of the box, the center of mass is at:

$$x_{\text{CM}} = \frac{\left[\dfrac{-MEL}{Mc^2} + m\left(\tfrac{1}{2}L - \dfrac{EL}{Mc^2}\right)\right]}{M+m}$$

Because no external forces act on the system, these expressions for x_{CM} must be equal:

$$\frac{-\tfrac{1}{2}mL}{M+m} = \frac{\left[\dfrac{-MEL}{Mc^2} + m\left(\tfrac{1}{2}L - \dfrac{EL}{Mc^2}\right)\right]}{M+m}$$

Solve for m to obtain:

$$m = \frac{E}{c^2\left(1 - \dfrac{E}{Mc^2}\right)}$$

Because Mc^2 is of the order of 10^{16} J and $E = hf$ is of the order of 1 J for reasonable values of f, $E/Mc^2 << 1$ and:

$$m = \boxed{\frac{E}{c^2}}$$

52 •••

Picture the Problem We can apply a velocity transformation equation to find the speed of the particle and use the distance and time transformation equations to find the distance and direction the particle traveled from t'_1 to t'_2 and the time the particle traveled as observed in frame S.

(*a*) The velocity transformation equation for motion at speed v along the x axis is:

$$u_x = \frac{u_x' + v}{1 + \dfrac{vu_x'}{c^2}}$$

Evaluate u_x for $u_x' = -c/3$ and $v = 0.6c$:

$$u_x = \frac{-\dfrac{1}{3}c + \dfrac{3}{5}c}{1 + \dfrac{\left(\dfrac{3}{5}c\right)\left(-\dfrac{1}{3}c\right)}{c^2}} = \boxed{\tfrac{1}{3}c}$$

(*b*) The distance traveled by the particle from t'_1 to t'_2 is given by:

$$\Delta x = x_2 - x_1 \qquad (1)$$

To find x_2, we must first find x_2' and $\Delta t'$:

$$x_2' = 10\,\text{m} - (60\,\text{m}/c)\left(\frac{c}{3}\right) = -10\,\text{m}$$

and

$$\Delta t' = t_2' - t_1' = 60\,\text{m}/c = 200\,\text{ns}$$

x_2 is related to x_2' through the relativistic transformation:

$$x_2 = \gamma\left(x_2' + v\Delta t'\right) = \frac{x_2' + v\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}}$$

Substitute numerical values and evaluate x_2:

$$x_2 = \frac{-10\,\text{m} + (0.6c)(200\,\text{ns})}{\sqrt{1-\frac{(0.6c)^2}{c^2}}} = 32.5\,\text{m}$$

x_1 is given by:

$$x_1 = \gamma x_1' = \frac{x_1'}{\sqrt{1-\frac{v^2}{c^2}}}$$

Substitute numerical values and evaluate x_1:

$$x_1 = \frac{10\,\text{m}}{\sqrt{1-\frac{(0.6c)^2}{c^2}}} = 12.5\,\text{m}$$

Substitute numerical values in equation (1) and evaluate Δx:

$$\Delta x = 32.5\,\text{m} - 12.5\,\text{m} = \boxed{20.0\,\text{m}}$$

(c) The time the particle traveled is given by:

$$\Delta t = t_2 - t_1 \qquad (2)$$

Express and evaluate t_1:

$$t_1 = \gamma t_1' = \frac{t_1'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{6\,\text{m}/c}{\sqrt{1-\frac{(0.6c)^2}{c^2}}} = 7.50\,\text{m}/c = 25.0\,\text{ns}$$

Express and evaluate t_2:

$$t_2 = \gamma\left(t' + \frac{vt'}{c^2}\right) = \frac{t' + \frac{vt'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{60\,\text{m}/c + (-6\,\text{m}/c)}{\sqrt{1-\frac{(0.6c)^2}{c^2}}} = 225\,\text{ns}$$

Substitute in equation (2) and evaluate Δt:

$$\Delta t = 225\,\text{ns} - 25\,\text{ns} = \boxed{200\,\text{ns}}$$

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Picture the Problem We can evaluate the differentials of Equations 39-19*a, b,* and *c* and 39-10 and express their ratio to obtain expressions for a_x', a_y', and a_z'.

From Equation 39-19*a* we have:

$$du_x' = d\left(\frac{u_x - v}{1-\frac{vu_x}{c^2}}\right) = \frac{\left(1-\frac{vu_x}{c^2}\right)du_x + (u_x - v)\left(\frac{v}{c^2}\right)du_x}{\left(1-\frac{vu_x}{c^2}\right)^2} = \frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{vu_x}{c^2}\right)^2}du_x$$

From Equation 39-10:

$$dt' = \gamma d\left(t - \frac{vx}{c^2}\right) = \gamma dt - \frac{\gamma v}{c^2}dx = \frac{\gamma v}{c^2}\frac{dx}{dt}dt = \gamma\left(1-\frac{vu_x}{c^2}\right)dt$$

Divide du_x' by dt' to obtain:

$$a_x' = \frac{du_x'}{dt'} = \frac{\frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{vu_x}{c^2}\right)^2}du_x}{\gamma\left(1-\frac{vu_x}{c^2}\right)dt} = \frac{\left(1-\frac{v^2}{c^2}\right)}{\gamma\left(1-\frac{vu_x}{c^2}\right)^3}\frac{du_x}{dt} = \boxed{\frac{1}{\gamma^3\delta^3}a_x}$$

where $\delta = 1 - \frac{vu_x}{c^2}$

Proceeding in exactly the same manner, one obtains:

$$a_y' = \boxed{\frac{1}{\gamma^2\delta^2}a_y + \frac{vu_y}{\gamma^3\delta^3c^2}a_x}$$

and an identical expression for a_z' with *z* replacing *y.*

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Picture the Problem Without loss of generality, we'll consider the absorption case. We'll assume that the electron is initially at rest and that it travels with a speed *v* after it absorbs the photon. Applying the conservation of energy and the conservation of momentum will lead us to an absurd conclusion that, in turn, will force us to abandon our initial assumption that an electron can absorb a photon. Such an argument is known as a *reductio ad absurdum* argument.

When the electron absorbs a photon, the conservation of relativistic momentum requires that its momentum become:

$$p = \gamma m v$$

From the conservation of energy:

$$mc^2 + pc = \gamma mc^2 \Rightarrow p = (\gamma - 1)mc$$

Equate these expression for p to obtain:

$$\gamma m v = (\gamma - 1)mc$$

Solving for v yields:

$$v = \left(\frac{\gamma - 1}{\gamma}\right)c \qquad (1)$$

Square both sides of the equation to obtain:

$$v^2 = \left(\frac{\gamma - 1}{\gamma}\right)^2 c^2 = \frac{\gamma^2 - 2\gamma + 1}{\gamma^2}c^2 \qquad (2)$$

From the definition of γ:

$$\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}} = \frac{c^2}{c^2 - v^2}$$

Solve for v^2 to obtain:

$$v^2 = \frac{\gamma^2 - 1}{\gamma^2}c^2$$

Substitute for v^2 in equation (1) and simplify to obtain:

$$\frac{\gamma^2 - 1}{\gamma^2}c^2 = \frac{\gamma^2 - 2\gamma + 1}{\gamma^2}c^2$$

or

$$-1 = -2\gamma + 1 \Rightarrow \gamma = 1$$

Substitute for γ in equation (1) and evaluate v:

$$v = \left(\frac{1 - 1}{1}\right)c = 0$$

Our assumption that an electron can absorb a photon has led to the contradictory conclusion that its speed after the absorption is zero. Hence, we must conclude that the electron cannot absorb a photon.

***55** •••

Picture the Problem Let m_i denote the mass of the incident (projectile) particle. Then $\sum m_{in} = m_i + m_{target}$ and we can use this expression to determine the threshold kinetic energy of protons incident on a stationary proton target for the production of a proton–antiproton pair.

Consider the situation in the center of mass reference frame. At threshold we have:

$$E^2 - p^2c^2 = \sum m_{\text{fin}}c^2$$

Note that this is a relativistically invariant expression.

In the laboratory frame, the target is at rest so:

$$E_{\text{target}} = E_{\text{t}} = E_{\text{t,0}}$$

We can, therefore, write:

$$\left(E_{\text{i}} + E_{\text{t,0}}\right)^2 - p_{\text{i}}^2c^2 = \left(\sum m_{\text{fin}}c^2\right)^2$$

For the incident particle:

$$E_{\text{i}}^2 - p_{\text{i}}^2c^2 = E_{\text{i,0}}^2$$

and

$$E_{\text{i}} = E_{\text{i,0}} + K_{\text{th}}$$

where K_{th} is the threshold kinetic energy of tl incident particle in the laboratory frame.

Express K_{th} in terms of the rest energies:

$$\left(E_{\text{t,0}} + E_{\text{i.0}}\right)^2 + 2K_{\text{th}}E_{\text{t,0}} = \left(\sum m_{\text{fin}}c^2\right)^2$$

where

$$E_{\text{t,0}} + E_{\text{i.0}} = \sum m_{\text{fin}}c^2$$

and

$$E_{\text{t,0}} = m_{\text{target}}c^2$$

Substitute to obtain:

$$\left(\sum m_{\text{fin}}c^2\right)^2 + 2K_{\text{th}}m_{\text{target}}c^2 = \left(\sum m_{\text{fin}}c^2\right)^2$$

Solve for K_{th} to obtain:

$$K_{\text{th}} = \boxed{\frac{\left(\sum m_{\text{in}} + \sum m_{\text{fin}}\right)\left(\sum m_{\text{fin}} - \sum m_{\text{in}}\right)c^2}{2m_{\text{target}}}}$$

For the creation of a proton-antiproton pair in a proton-proton collision:

$$\sum m_{\text{in}} = 2m_{\text{p}}$$

$$\sum m_{\text{fin}} = 4m_{\text{p}}$$

and

$$m_{\text{target}} = m_{\text{p}}$$

Substitute to obtain:

$$K_{\text{th}} = \frac{\left(2m_{\text{p}} + 4m_{\text{p}}\right)\left(4m_{\text{p}} - 2m_{\text{p}}\right)c^2}{2m_{\text{p}}} = \frac{\left(6m_{\text{p}}\right)\left(2m_{\text{p}}\right)c^2}{2m_{\text{p}}} = \boxed{6m_{\text{p}}c^2}$$

in agreement with Problem 40.

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Picture the Problem We'll solve the problem for the general case of a particle of rest mass *M* decaying into two identical particles each of rest mass *m*.

In the center of mass reference frame:

$$Mc^2 = 2mc^2 = 2\gamma mc^2$$

Solve for u/c to obtain:

$$\frac{u}{c} = \sqrt{1-\left(\frac{2m}{M}\right)^2}$$

where u is the speed of each of the decay particles in the CM frame.

Next we determine the speed v of the laboratory frame relative to the CM frame. The energy of the particle of rest mass M is:

$$\gamma_{\mathrm{CM}} Mc^2$$

where

$$\gamma_{\mathrm{CM}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

and

$$\frac{v}{c} = \beta_{\mathrm{CM}} = \sqrt{1-\frac{1}{\gamma_{\mathrm{CM}}^2}}$$

Use Equation 39-18*a* to express u_{lab}, the speeds of the decay products in the laboratory reference frame:

$$u_{\mathrm{lab}} = \frac{\beta_{\mathrm{CM}} \pm \frac{u}{c}}{1 \pm \beta_{\mathrm{CM}} \frac{u}{c}} c$$

where ± refers to the fact that one of the decay particles will travel in the direction of *M*, and the other in the direction opposite to that of *M*.

In this problem we have:

$$\gamma_{\mathrm{CM}} = 4\,,\ \beta_{\mathrm{CM}} = 0.968\,,\ \frac{2m_0}{M_0} = 0.6\,,$$

and $\frac{u}{c} = 0.8$

Substitute to obtain:

$$u_{\mathrm{lab}} = \frac{0.968+0.8}{1+(0.968)(0.8)} c = \boxed{0.996c}$$

and

$$u_{\text{lab}} = \frac{0.968 - 0.8}{1 - (0.968)(0.8)}c = \boxed{0.745c}$$

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Picture the Problem We can write the components of the stick in its reference frame and then apply the Lorentz length contraction equation to obtain the given result.

In its reference frame, the stick has x and y components:

$$L_{px} = L_p \cos\theta$$

and

$$L_{py} = L_p \sin\theta$$

Only L_{px} is Lorentz contracted to:

$$L_x' = \frac{L_{px}}{\gamma}$$

Hence, the length in the reference frame S' is:

$$L' = \left[(L_x')^2 + (L_y')^2\right]^{1/2}$$

$$= \boxed{L_p\left(\frac{\cos^2\theta}{\gamma^2} + \sin^2\theta\right)^{1/2}}$$

The angle that L' makes with the x' axis is given by:

$$\tan\theta' = \frac{L_y'}{L_x'} = \frac{\sin\theta}{\frac{\cos\theta}{\gamma}} = \boxed{\gamma\tan\theta}$$

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Picture the Problem We can express the tangent of the angle u' makes with the x' axis and then use the velocity transformation equations to obtain the given result.

Express the tangent of the angle u' makes with the x' axis:

$$\tan\theta' = \frac{u_y'}{u_x'}$$

Substitute for u_y' and u_x':

$$\tan\theta' = \frac{\dfrac{u_y}{\gamma\left(1 - \dfrac{vu_x}{c^2}\right)}}{\dfrac{u_x - v}{1 - \dfrac{vu_x}{c^2}}} = \frac{u_y}{\gamma(u_x - v)}$$

Substitute for u_y and u_x and simplify to obtain:

$$\tan\theta' = \frac{u\sin\theta}{\gamma(u\cos\theta - v)} = \boxed{\frac{\sin\theta}{\gamma(\cos\theta - v/u)}}$$

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Picture the Problem We can use the expressions for $\vec{p}$ and E in S together with the relations we wish to verify and the inverse velocity transformation equations to establish the condition $u'^2 = (u_x')^2 + (u_y')^2 + (u_z')^2 = v^2 + \frac{u^2}{\gamma^2}$ and then use this result to verify the given expressions for p_x', p_y', p_z' and E'/c.

In any inertial frame the momentum and energy are given by:

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}} \text{ and } E = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}}$$

where $\vec{u}$ is the velocity of the particle and u is its speed.

The components of $\vec{p}$ in S are:

$$p_x = \frac{mu_x}{\sqrt{1-\frac{u^2}{c^2}}},\ p_y = \frac{mu_y}{\sqrt{1-\frac{u^2}{c^2}}}, \text{ and}$$

$$p_z = \frac{mu_z}{\sqrt{1-\frac{u^2}{c^2}}}$$

Because $u_x = u_z = 0$ and $u_y = u$:

$$p_x = p_z = 0$$

and

$$p_y = \frac{mu}{\sqrt{1-\frac{u^2}{c^2}}}$$

Substituting zeros for p_x and p_z in the relations we are trying to show yields:

$$p_x' = \gamma\left(0 - \frac{vE}{c^2}\right) = -\gamma\frac{vE}{c^2},\ p_y' = p_y,$$

$p_z' = 0$, and

$$\frac{E'}{c} = \gamma\left(\frac{E}{c} - 0\right) = \gamma\frac{E}{c}$$

In S' the momentum components are:

$$p_x' = \frac{mu_x'}{\sqrt{1-\frac{u'^2}{c^2}}},\ p_y' = \frac{mu_y'}{\sqrt{1-\frac{u'^2}{c^2}}}, \text{ and}$$

$$p_z' = \frac{mu_z'}{\sqrt{1-\frac{u'^2}{c^2}}}$$

The inverse velocity transformations are:

$$u_x' = \frac{u_x - v}{\sqrt{1 - \frac{vu_x}{c^2}}},\ u_y' = \frac{u_y}{\sqrt{1 - \frac{vu_y}{c^2}}}, \text{ and}$$

$$u_z' = \frac{u_z}{\sqrt{1 - \frac{vu_z}{c^2}}}$$

Substitute $u_x = u_z = 0$ and $u_y = u$ to obtain:

$$u_x' = -v,\ u_y' = \gamma u, \text{ and } u_z' = 0$$

Thus:

$$u'^2 = (u_x')^2 + (u_y')^2 + (u_z')^2$$
$$= v^2 + \frac{u^2}{\gamma^2}$$

First we verify that $p_z' = p_z = 0$:

$$p_z' = \frac{m(0)}{\sqrt{1 - \frac{u'^2}{c^2}}} = p_z = \boxed{0}$$

Next we verify that $p_y' = p_y$:

$$p_y' = \frac{mu_y'}{\sqrt{1 - \frac{u'^2}{c^2}}} = \frac{mu}{\gamma\sqrt{1 - \frac{v^2}{c^2} - \frac{u^2}{\gamma^2 c^2}}} = \frac{mu}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{\sqrt{1 - \frac{u^2}{c^2}}}{\gamma\sqrt{1 - \frac{v^2}{c^2} - \frac{u^2}{\gamma^2 c^2}}}$$

$$= \frac{mu}{\sqrt{1 - \frac{u^2}{c^2}}} \sqrt{\frac{\left(1 - \frac{u^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}{1 - \frac{v^2}{c^2} - \frac{u^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)}} = p_y \sqrt{\frac{1 - \frac{v^2}{c^2} - \frac{u^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)}{1 - \frac{v^2}{c^2} - \frac{u^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)}}$$

$$= \boxed{p_y}$$

Next we verify that $p_x' = \gamma\left(p_x - \frac{vE}{c^2}\right)$:

$$p_x' = \frac{mu_x'}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{-mv}{\gamma\sqrt{1-\frac{v^2}{c^2}-\frac{u^2}{\gamma^2c^2}}} = -\frac{\gamma v}{c^2}\frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}}\frac{\gamma^{-1}\sqrt{1-\frac{u^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}-\frac{u^2}{\gamma^2c^2}}}$$

$$= -\frac{\gamma v}{c^2}E\sqrt{\frac{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{1-\frac{v^2}{c^2}-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}} = -\frac{\gamma v}{c^2}E\sqrt{\frac{1-\frac{v^2}{c^2}-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}{1-\frac{v^2}{c^2}-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}}$$

$$= \boxed{-\frac{\gamma v}{c^2}E}$$

Finally, we verify that $\frac{E'}{c} = \gamma\left(\frac{E}{c}-\frac{vp_x}{c}\right) = \gamma\frac{E}{c}$, or $E' = \gamma E$:

$$E' = \frac{mc^2}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{\gamma mc^2}{\sqrt{1-\frac{u^2}{c^2}}}\frac{\gamma^{-1}\sqrt{1-\frac{u^2}{c^2}}}{\sqrt{1-\frac{u'^2}{c^2}}} = \gamma E\frac{\gamma^{-1}\sqrt{1-\frac{u^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}-\frac{u^2}{\gamma^2c^2}}}$$

$$= \gamma E\sqrt{\frac{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{1-\frac{v^2}{c^2}-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}} = \gamma E\sqrt{\frac{1-\frac{v^2}{c^2}-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}{1-\frac{v^2}{c^2}-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}}$$

$$= \boxed{\gamma E}$$

The *x, y, z,* and *t* transformation equations are:

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$

and

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

The *x, y, z,* and *ct* transformation equations are:

$$x' = \gamma\left(x - \frac{v}{c}ct\right)$$
$$y' = y$$
$$z' = z$$

and

$$ct' = \gamma\left(ct - \frac{v}{c}x\right)$$

The p_x, p_y, p_z, and E/c transformation equations are:

$$p_x' = \gamma\left(p_x - \frac{v}{c}\frac{E}{c}\right)$$

$$p_y' = p_y$$

$$p_z' = p_z$$

and

$$\frac{E'}{c} = \gamma\left(\frac{E}{c} - \frac{v}{c}p_x\right)$$

Note that the transformation equations for x, y, z, and ct and the transformation equations for p_x, p_y, p_z, and E/c are identical.

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Picture the Problem The Lorentz transformation was derived on the basis of the postulate that the speed of light is c in any inertial reference frame. Thus, if the clocks in S and S' are synchronized at $t = t' = 0$, then it follows from the Einstein postulate that $r^2 = c^2t^2$ and $r'^2 = c^2t'^2$ or $r^2 - c^2t^2 = 0 = r'^2 - c^2t'^2$. In other words, the quantity $s^2 = r^2 - c^2t^2 = 0$ is a relativistic invariant, which can also be written as $x^2 + y^2 + z^2 - c^2t^2 = 0$.

Using the Lorentz transformation equations for x, y, z, and t we have:

$$x'^2 + y'^2 + z'^2 - (ct')^2 = \gamma^2(x^2 - 2vxt + v^2t^2) + y^2 + z^2 - \gamma^2(c^2t^2 - 2vxt + v^2x^2/c^2)$$

The terms linear in x cancel and the terms in x^2 combine to give:

$$\gamma^2x^2(1 - v^2/c^2) = x^2$$

The coefficients of the terms in $(ct)^2$ give:

$$\gamma^2(v^2/c^2 - 1) = -1$$

Thus, $r^2 - c^2t^2 = r'^2 - c^2t'^2$ as required by the Einstein postulate.

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Picture the Problem We'll use Equation 39-27 to show that this quantity has the value $-mc^2$ in both the S and S' reference frames.

From Equation 39-27, the relationship between total energy E, momentum p, and rest energy mc^2 is:

$$E^2 = p^2c^2 + (mc^2)^2$$

or

$$p^2c^2 - E^2 = -(mc^2)^2$$

Divide both sides of this equation by c^2 to obtain:

$$p^2 - \left(\frac{E}{c}\right)^2 = -(mc)^2 \qquad (1)$$

We can relate p to p_x, p_y, and p_z:

$$p^2 = p_x^2 + p_y^2 + p_z^2$$

Substitute for p^2 in equation (1) to obtain:

$$p_x^2 + p_y^2 + p_z^2 - \left(\frac{E}{c}\right)^2 = -m^2c^2$$

Because m is the mass of the particle in its rest frame, it is constant. Hence:

$p^2 - \left(\frac{E}{c}\right)^2$ must be a relativistic invariant.

Also, in Problem 59 we saw that the components of p and the quantity E/c transform like the components of r and the quantity ct. In Problem 60 we demonstrated that $r^2 - (ct)^2$ is a relativistic invariant. Consequently, $p^2 - (E/c)^2$ must also be relativistically invariant.

***62** •••

Picture the Problem We can use the inverse Lorentz transformation for time to show that the observer will conclude that the rod is bent into a parabolic shape.

In frame S where the rod is not moving along the x axis, the height of the rod at time t is:

$$y(t) = -\tfrac{1}{2}gt^2$$

The inverse Lorentz time transformation is:

$$t = \gamma\left(t' + \frac{vx}{c^2}\right)$$

Express $y'(t)$ in the moving frame of reference:

$$y'(t) = -\tfrac{1}{2}g\gamma\left(t' + \frac{vx}{c^2}\right)^2$$

Evaluate $y'(t)$ at $t' = 0$ to obtain:

$$y'(t) = -\frac{g\gamma v^2}{2c^2}x^2 \qquad (1)$$

Because equation (1) is the equation of a parabola, we've shown that the moving observer will conclude that the rod is bent into a parabolic shape. Because the coefficient of x^2 is negative, the parabola is concave downward.