Solution Manual to Introduction to Quantum Mechanics

by

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Plus corrections

Errata
Instructor's Solutions Manual
Introduction to Quantum Mechanics
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- Page 3, Prob. 1.6(b): last two lines should read

$$= A\left[\frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}} + 0 + a^2\sqrt{\frac{\pi}{\lambda}}\right] = \boxed{a^2 + \frac{1}{2\lambda}.}$$

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = a^2 + \frac{1}{2\lambda} - a^2 = \frac{1}{2\lambda};\ \boxed{\sigma = \frac{1}{\sqrt{2\lambda}}.}$$

- Page 8, Prob. 2.6(b): in the first box, the argument of the second sine should include an x.
- Page 9, Prob. 2.9: in the last line, $c_1 = 8\sqrt{15}/\pi^3$.
- Page 10, Prob. 2.11: in the third line the proof assumes that g (which in our case will be $a_-\psi$) does not actually *blow up* at $\pm\infty$ faster than f (in our case ψ) goes to zero. I don't know how to fix this defect without appealing to the analytic approach, where we find (Eq. 2.60) that $\psi(x)$ goes asymptotically like $e^{-m\omega x^2/2\hbar}$, and hence so too does $a_-\psi$.
- Page 10, Prob. 2.12: Because of the i and $-i$ inserted in Eqs. 2.52 and 2.53 respectively (see Corrections #2—June 1997), the expression for c at the end of (a) should include a factor of i. Also, add at the end of (a): "(The signs are conventional.)" In part (b), every $\sqrt{\hbar\omega}$ should carry a factor of i (i.e. insert i three times in the first line, i^n three times in the next line, and $(-i)^n$ in the boxed answer).
- Page 11, Prob. 2.14(a): for the same reason, in the third line, remove the i in the expression for ψ_1.
- Page 19, Prob. 2.30: or "$\ldots \tan z \approx z = \sqrt{(z_0/z)^2 - 1} = (1/z)\sqrt{z_0^2 - z^2}$. Now (Eqs. 2.130 and 2.137) $z_0^2 - z^2 = \kappa^2 a^2$, so $z^2 = \kappa a$. But $z_0^2 - z^2 = z^4 \ll 1 \Rightarrow z \approx z_0$, so $\kappa a \approx z_0^2 \ldots$ "
- Page 22, Prob. 2.36: remove box, and continue as follows:

$$\Psi(x,t) = \frac{1}{\sqrt{10}}\left[3\psi_1(x)e^{-iE_1t/\hbar} - \psi_3(x)e^{-iE_3t/\hbar}\right],$$

$$|\Psi(x,t)|^2 = \frac{1}{10}\left[9\psi_1^2 + \psi_3^2 - 6\psi_1\psi_3\cos\left(\frac{E_3 - E_1}{\hbar}t\right)\right],$$

so

$$\begin{aligned}\langle x\rangle &= \int_0^a x|\Psi(x,t)|^2\,dx \\ &= \frac{9}{10}\langle x\rangle_1 + \frac{1}{10}\langle x\rangle_3 - \frac{3}{5}\cos\left(\frac{E_3-E_1}{\hbar}t\right)\int_0^a x\psi_1(x)\psi_3(x)\,dx,\end{aligned}$$

where $\langle x\rangle_n = a/2$ is the expectation value of x in the nth stationary state. The remaining integral is

$$\frac{2}{a}\int_0^a x\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{3\pi x}{a}\right)dx = \frac{1}{a}\int_0^a x\left[\cos\left(\frac{2\pi x}{a}\right)-\cos\left(\frac{4\pi x}{a}\right)\right]dx$$

$$= \frac{1}{a}\left[\left(\frac{a}{2\pi}\right)^2\cos\left(\frac{2\pi x}{a}\right)+\left(\frac{xa}{2\pi}\right)\sin\left(\frac{2\pi x}{a}\right)-\left(\frac{a}{4\pi}\right)^2\cos\left(\frac{4\pi x}{a}\right)\right.$$
$$\left.-\left(\frac{xa}{4\pi}\right)\sin\left(\frac{4\pi x}{a}\right)\right]\Bigg|_0^a = 0.$$

Evidently, then,

$$\langle x\rangle = \frac{9}{10}\left(\frac{a}{2}\right)+\frac{1}{10}\left(\frac{a}{2}\right) = \boxed{\frac{a}{2}.}$$

- Page 23, Prob. 2.37: Because of the i and $-i$ inserted in Eqs. 2.52 and 2.53 respectively (see Corrections #2—June 1997), line 3 on should read as follows:

$$\langle x\rangle = \frac{-i}{\omega\sqrt{2m}}\int \psi_n^*(a_+ - a_-)\psi_n\,dx.$$

But $\left\{\begin{array}{lcll} a_+\psi_n &=& i\sqrt{(n+1)\hbar\omega}\psi_{n+1} & [2.52] \\ a_-\psi_n &=& -i\sqrt{n\hbar\omega}\psi_{n-1} & [2.53]\end{array}\right\}$, so

$$\langle x\rangle = \frac{1}{\omega\sqrt{2m}}\left[\sqrt{(n+1)\hbar\omega}\int\psi_n^*\psi_{n+1}\,dx + \sqrt{n\hbar\omega}\int\psi_n^*\psi_{n-1}\,dx\right] = \boxed{0}$$

(by the orthogonality of $\{\psi_n\}$). Also $\langle p\rangle = m\dfrac{d\langle x\rangle}{dt} = \boxed{0.}$ Meanwhile

$$\hat{x}^2 = \frac{-1}{2m\omega^2}(a_+ - a_-)(a_+ - a_-) = \frac{-1}{2m\omega^2}(a_+^2 - a_+a_- - a_-a_+ + a_-^2),$$

so $\langle x^2\rangle = \dfrac{-1}{2m\omega^2}\displaystyle\int\psi_n^*(a_+^2 - a_+a_- - a_-a_+ + a_-^2)\psi_n\,dx.$ But

$$\left\{\begin{array}{lclcl} a_+^2\psi_n &=& a_+(i\sqrt{(n+1)\hbar\omega}\psi_{n+1}) &=& -\sqrt{(n+1)(n+2)}\hbar\omega\psi_{n+2}. \\ a_+a_-\psi_n &=& a_+(-i\sqrt{n\hbar\omega}\psi_{n-1}) &=& n\hbar\omega\psi_n. \\ a_-a_+\psi_n &=& a_-(i\sqrt{(n+1)\hbar\omega}\psi_{n+1}) &=& (n+1)\hbar\omega\psi_n. \\ a_-^2\psi_n &=& a_-(-i\sqrt{n\hbar\omega}\psi_{n-1}) &=& -\sqrt{n(n-1)}\hbar\omega\psi_{n-2}.\end{array}\right.$$

[The rest is unchanged.]

- Page 32, Prob. 3.2(b): in the odd case the dimension is $(N+1)/2$.
- Pages 49-50, Prob. 3.50: Because of the i and $-i$ inserted in Eqs. 2.52 and 2.53 respectively (see Corrections #2—June 1997), the solution should be changed to read as follows:

$$x = -\frac{i}{\omega\sqrt{2m}}(a_+ - a_-) \quad \text{[Prob. 2.37].} \begin{cases} a_+|n\rangle &= i\sqrt{(n+1)\hbar\omega}|n+1\rangle, \\ a_-|n\rangle &= -i\sqrt{n\hbar\omega}|n-1\rangle. \end{cases}$$

$$\begin{aligned}
\langle n|x|n'\rangle &= \frac{-i}{\omega\sqrt{2m}}\langle n|(a_+ - a_-)|n'\rangle \\
&= \frac{-i}{\omega\sqrt{2m}}\left[i\sqrt{(n'+1)\hbar\omega}\langle n|n'+1\rangle + i\sqrt{n'\hbar\omega}\langle n|n'-1\rangle\right] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n'+1}\delta_{n,n'+1} + \sqrt{n'}\delta_{n,n'-1}\right) \\
&= \boxed{\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\delta_{n',n-1} + \sqrt{n'}\delta_{n,n'-1}\right).}
\end{aligned}$$

$$p = \sqrt{\frac{m}{2}}(a_+ + a_-) \Rightarrow \langle n|p|n'\rangle = \boxed{i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{n}\delta_{n',n-1} - \sqrt{n'}\delta_{n,n'-1}\right).}$$

Noting that n and n' run from *zero* to infinity, the matrices are:

$$\boxed{X = \sqrt{\frac{\hbar}{2m\omega}}\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & 0 & \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 & 0 & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & 0 & \vdots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & 0 & \\ 0 & 0 & 0 & \sqrt{4} & 0 & \sqrt{5} & \\ & & & \cdots & & & \end{pmatrix}}$$

$$\boxed{P = i\sqrt{\frac{m\hbar\omega}{2}}\begin{pmatrix} 0 & -\sqrt{1} & 0 & 0 & 0 & 0 & \\ \sqrt{1} & 0 & -\sqrt{2} & 0 & 0 & 0 & \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & 0 & 0 & \vdots \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{4} & 0 & \\ 0 & 0 & 0 & \sqrt{4} & 0 & -\sqrt{5} & \\ & & & \cdots & & & \end{pmatrix}}$$

Squaring these matrices:

$$X^2 = \frac{\hbar}{2m\omega}\begin{pmatrix} 1 & 0 & \sqrt{1\cdot 2} & 0 & 0 & 0 & \\ 0 & 3 & 0 & \sqrt{2\cdot 3} & 0 & 0 & \\ \sqrt{1\cdot 2} & 0 & 5 & 0 & \sqrt{3\cdot 4} & 0 & \vdots \\ 0 & \sqrt{2\cdot 3} & 0 & 7 & 0 & \sqrt{4\cdot 5} & \\ & & & \cdots & & & \end{pmatrix};$$

$$P^2 = -\frac{m\hbar\omega}{2}\begin{pmatrix} -1 & 0 & \sqrt{1\cdot 2} & 0 & 0 & 0 & \\ 0 & -3 & 0 & \sqrt{2\cdot 3} & 0 & 0 & \\ \sqrt{1\cdot 2} & 0 & -5 & 0 & \sqrt{3\cdot 4} & 0 & \vdots \\ 0 & \sqrt{2\cdot 3} & 0 & -7 & 0 & \sqrt{4\cdot 5} & \\ & & & \cdots & & & \end{pmatrix}.$$

So the Hamiltonian, in matrix form, is

$$\begin{aligned} H &= \frac{1}{2m}P^2 + \frac{m\omega^2}{2}X^2 \\ &= -\frac{\hbar\omega}{4}\begin{pmatrix} -1 & 0 & \sqrt{1\cdot 2} & 0 & 0 & 0 & \\ 0 & -3 & 0 & \sqrt{2\cdot 3} & 0 & 0 & \\ \sqrt{1\cdot 2} & 0 & -5 & 0 & \sqrt{3\cdot 4} & 0 & \vdots \\ 0 & \sqrt{2\cdot 3} & 0 & -7 & 0 & \sqrt{4\cdot 5} & \\ & & & \cdots & & & \end{pmatrix} \\ &\quad + \frac{\hbar\omega}{4}\begin{pmatrix} 1 & 0 & \sqrt{1\cdot 2} & 0 & 0 & 0 & \\ 0 & 3 & 0 & \sqrt{2\cdot 3} & 0 & 0 & \\ \sqrt{1\cdot 2} & 0 & 5 & 0 & \sqrt{3\cdot 4} & 0 & \vdots \\ 0 & \sqrt{2\cdot 3} & 0 & 7 & 0 & \sqrt{4\cdot 5} & \\ & & & \cdots & & & \end{pmatrix} \\ &= \frac{\hbar\omega}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & \\ 0 & 3 & 0 & 0 & \\ 0 & 0 & 5 & 0 & \\ 0 & 0 & 0 & 7 & \\ & & & & \ddots \end{pmatrix}. \end{aligned}$$

The diagonal elements are $H_{nn} = (n + 1/2)\hbar\omega$, as they should be.

- Page 53, Prob. 3.57(a): add the following:

  If $|\gamma\rangle$ is an eigenvector of $\hat{P}$ with eigenvalue λ, then $\hat{P}|\gamma\rangle = \lambda|\gamma\rangle$, and it follows that $\hat{P}^2|\gamma\rangle = \lambda\hat{P}|\gamma\rangle = \lambda^2|\gamma\rangle$. But $\hat{P}^2 = \hat{P}$, and $|\gamma\rangle \neq 0$, so $\lambda^2 = \lambda$, and hence the eigenvalues of $\hat{P}$ are $\boxed{\text{0 and 1.}}$ Any (complex) *multiple* of $|\alpha\rangle$ is an eigenvector of $\hat{P}$, with eigenvalue 1; any vector *orthogonal* to $|\alpha\rangle$ is an eigenvector of $\hat{P}$, with eigenvalue 0.

- Page 84, Prob. 5.6(b): all three minus signs should be plus.

- Page 87, Prob. 5.12: for the configuration given $S = 2$, so we need four unpaired electrons. In the proposed "likely" arrangement of the 30 extra electrons all shells are filled except 2 in the $4f$ state, so this doesn't work. The most probable arrangement is actually

$$(4d)^{10}(5s)^2(5p)^6(4f)^{10}(6s)^2.$$

- Page 88, Prob. 5.15(b): change to read "In this case [5.63] $\Rightarrow$ 0=0, [5.61] holds automatically, and [5.62] gives $ka - (\pm 1)k[A(\pm 1) - 0] = -(2m\alpha/\hbar^2)B \Rightarrow B = 0$." [The rest is unchanged.]

- Page 99, Prob. 6.4(b): in the last line, change "Problem 15(a)" to "Problem 6.2(a)".

- Pages 101-102, Prob. 6.7(b and c): starting with the fourth line of (b), all exponentials of the form

$$e^{-(2\pi n/L)^2 a} \qquad \text{should read} \qquad e^{-(2\pi n a/L)^2}.$$

By my count this happens a total of six times.

- Page 106, Prob. 6.18: using $r = a$ is reasonable, since all we're looking for is a rough estimate, but the magnetic field in the ground state is problematic, so you might prefer to use, say, $n = 2$, $l = 1$; in that case (see Eq. 6.63) the answer is reduced by a factor of 24, to 0.5 T.

- Page 116-117, Prob. 6.31: in later printings the statement of the problem has been corrected, removing the minus signs in the displayed equation, and inserting one in the *Partial Answer*. The solution will match the revised version if the sign of E_{ext} is systematically reversed.

- Pages 121-124, Prob. 6.34: for some reason the solution, which begins on page 121, skips then to page 124, and finishes on page 123.

- Page 145, Prob. 9.1: top line, Table 4.5 should be Table 4.6; second line, the exponent in ψ_{210} should be $-r/2a$.

- Page 146, Prob. 9.3: This is a tricky problem, and I thank Prof. Onuttom Narayan for showing me the correct solution. The safest approach is to represent the delta function as a sequence of rectangles:

$$\delta_\epsilon(t) = \left\{ \begin{array}{ll} (1/2\epsilon), & -\epsilon < t < \epsilon, \\ 0, & \text{otherwise.} \end{array} \right\}$$

We may as well set $t_0 = 0$ (in later printings this is done in the text); then

$$\left\{ \begin{array}{ll} t < -\epsilon: & c_a(t) = 1, \ c_b(t) = 0, \\ & \\ t > \epsilon: & c_a(t) = a, \ c_b(t) = b, \\ & \\ -\epsilon < t < \epsilon: & \left\{ \begin{array}{rcl} \dot{c}_a & = & -\frac{i\alpha}{2\epsilon\hbar} e^{-i\omega_0 t} c_b, \\ & & \\ \dot{c}_b & = & -\frac{i\alpha^*}{2\epsilon\hbar} e^{i\omega_0 t} c_a. \end{array} \right. \end{array} \right\}$$

In the interval $-\epsilon < t < \epsilon$,

$$\begin{aligned}
\frac{d^2c_b}{dt^2} &= -\frac{i\alpha^*}{2\epsilon\hbar}\left[i\omega_0 e^{i\omega_0 t}c_a + e^{i\omega_0 t}\left(\frac{-i\alpha}{2\epsilon\hbar}e^{-i\omega_0 t}c_b\right)\right] \\
&= -\frac{i\alpha^*}{2\epsilon\hbar}\left[i\omega_0\frac{i2\epsilon\hbar}{\alpha^*}\frac{dc_b}{dt} - \frac{i\alpha}{2\epsilon\hbar}c_b\right] = i\omega_0\frac{dc_b}{dt} - \frac{|\alpha|^2}{(2\epsilon\hbar)^2}c_b.
\end{aligned}$$

Thus c_b satisfies a homogeneous linear differential equation with constant coefficients:

$$\frac{d^2c_b}{dt^2} - i\omega_0\frac{dc_b}{dt} + \frac{|\alpha|^2}{(2\epsilon\hbar)^2}c_b = 0.$$

Try a solution of the form $c_b(t) = e^{\lambda t}$:

$$\lambda^2 - i\omega_0\lambda + \frac{|\alpha|^2}{(2\epsilon\hbar)^2} = 0 \Rightarrow \lambda = \frac{i\omega_0 \pm \sqrt{-\omega_0^2 - |\alpha|^2/(\epsilon\hbar)^2}}{2},$$

or

$$\lambda = \frac{i\omega_0}{2} \pm \frac{i\omega}{2}, \text{ where } \omega \equiv \sqrt{\omega_0^2 + |\alpha|^2/(\epsilon\hbar)^2}.$$

The general solution is

$$c_b(t) = e^{i\omega_0 t/2}\left(Ae^{i\omega t/2} + Be^{-i\omega t/2}\right).$$

But

$$c_b(-\epsilon) = 0 \Rightarrow Ae^{-i\omega\epsilon/2} + Be^{i\omega\epsilon/2} = 0 \Rightarrow B = -Ae^{-i\omega\epsilon},$$

so

$$c_b(t) = Ae^{i\omega_0 t/2}\left(e^{i\omega t/2} - e^{-i\omega(\epsilon+t/2)}\right).$$

Meanwhile

$$\begin{aligned}
c_a(t) &= \frac{2i\epsilon\hbar}{\alpha^*}e^{-i\omega_0 t}\dot{c}_b \\
&= \frac{2i\epsilon\hbar}{\alpha^*}e^{-i\omega_0 t/2}A\left[\frac{i\omega_0}{2}\left(e^{i\omega t/2} - e^{-i\omega(\epsilon+t/2)}\right) + \frac{i\omega}{2}\left(e^{i\omega t/2} + e^{-i\omega(\epsilon+t/2)}\right)\right] \\
&= -\frac{\epsilon\hbar}{\alpha^*}e^{-i\omega_0 t/2}A\left[(\omega+\omega_0)e^{i\omega t/2} + (\omega-\omega_0)e^{-i\omega(\epsilon+t/2)}\right].
\end{aligned}$$

But

$$c_a(-\epsilon) = 1 = -\frac{\epsilon\hbar}{\alpha^*}e^{i(\omega_0-\omega)\epsilon/2}A\left[(\omega+\omega_0) + (\omega-\omega_0)\right] = -\frac{2\epsilon\hbar\omega}{\alpha^*}e^{i(\omega_0-\omega)\epsilon/2}A,$$

so $A = -\frac{\alpha^*}{2\epsilon\hbar\omega}e^{i(\omega-\omega_0)\epsilon/2}$. Therefore

$$\begin{aligned}
c_a(t) &= \frac{1}{2\omega}e^{-i\omega_0(t+\epsilon)/2}\left[(\omega+\omega_0)e^{i\omega(t+\epsilon)/2} + (\omega-\omega_0)e^{-i\omega(t+\epsilon)/2}\right] \\
&= e^{-i\omega_0(t+\epsilon)/2}\left\{\cos\left[\frac{\omega(t+\epsilon)}{2}\right] + i\frac{\omega_0}{\omega}\sin\left[\frac{\omega(t+\epsilon)}{2}\right]\right\}; \\
c_b(t) &= -\frac{i\alpha^*}{2\epsilon\hbar\omega}e^{i\omega_0(t-\epsilon)/2}\left[e^{i\omega(t+\epsilon)/2} - e^{-i\omega(t+\epsilon)/2}\right] \\
&= -\frac{i\alpha^*}{\epsilon\hbar\omega}e^{i\omega_0(t-\epsilon)/2}\sin\left[\frac{\omega(t+\epsilon)}{2}\right].
\end{aligned}$$

Thus

$$\begin{aligned}
a &= c_a(\epsilon) = e^{-i\omega_0\epsilon}\left[\cos(\omega\epsilon) + i\frac{\omega_0}{\omega}\sin(\omega\epsilon)\right] \\
b &= c_b(\epsilon) = -\frac{i\alpha^*}{\epsilon\hbar\omega}\sin(\omega\epsilon).
\end{aligned}$$

This is for the rectangular pulse; it remains to take the limit $\epsilon \to 0$: $\omega \to |\alpha|/\epsilon\hbar$, so

$$\begin{aligned}
a &\to \cos\left(\frac{|\alpha|}{\hbar}\right) + i\frac{\omega_0\epsilon\hbar}{|\alpha|}\sin\left(\frac{|\alpha|}{\hbar}\right) \to \cos\left(\frac{|\alpha|}{\hbar}\right), \\
b &\to -\frac{i\alpha^*}{|\alpha|}\sin\left(\frac{|\alpha|}{\hbar}\right),
\end{aligned}$$

and we conclude that for the delta function

$$\boxed{\begin{aligned}
c_a(t) &= \begin{cases} 1, & t<0, \\ \cos(|\alpha|/\hbar), & t>0; \end{cases} \\
c_b(t) &= \begin{cases} 0, & t<0, \\ -i\sqrt{\frac{\alpha^*}{\alpha}}\sin(|\alpha|/\hbar), & t>0. \end{cases}
\end{aligned}}$$

Obviously, $|c_a(t)|^2 + |c_b(t)|^2 = 1$ in both time periods. Finally,

$$\boxed{P_{a\to b} = |b|^2 = \sin^2(|\alpha|/\hbar).}$$

- Page 168, Prob, 11.2: the one-dimensional case should read

$$\psi(x) \approx A\left\{e^{ikx} + f(\mathrm{sign}(x))e^{ik|x|}\right\}.$$

CONTENTS

CHAPTER 1

PROBLEM 1.1 (a) $\langle j^2\rangle = \frac{1}{N}\sum j^2 N(j) = \frac{1}{14}\left[14^2\times1 + 15^2\times1 + 16^2\times3 + 22^2\times2 + 24^2\times2 + 25^2\times5\right]$

$$= \frac{1}{14}(196+225+768+968+1152+3125) = \frac{6434}{14} = \boxed{459.571}.$$

$$\langle j\rangle^2 = 21^2 = \boxed{441}.$$

(b)
$$\left.\begin{array}{l} j=14:\ \Delta j = 14-21 = -7 \\ j=15:\ \Delta j = 15-21 = -6 \\ j=16:\ \Delta j = 16-21 = -5 \\ j=22:\ \Delta j = 22-21 = 1 \\ j=24:\ \Delta j = 24-21 = 3 \\ j=25:\ \Delta j = 25-21 = 4 \end{array}\right\}
\begin{array}{l} \sigma^2 = \frac{1}{N}\sum(\Delta j)^2 N(j) \\ \quad = \frac{1}{14}\left[(-7)^2\cdot1 + (-6)^2\cdot1 + (-5)^2\cdot3 + (1)^2\cdot2 + (3)^2\cdot2 + (4)^2\cdot5\right] \\ \quad = \frac{1}{14}(49+36+75+2+18+80) = \frac{260}{14} = 18.571. \\ \sigma = \sqrt{18.571} = \boxed{4.309} \end{array}$$

(c) $\langle j^2\rangle - \langle j\rangle^2 = 459.571 - 441 = 18.571.$ (Agrees with (b)).

PROBLEM 1.2 From Math Tables: $\pi = 3.14159\ 26535\ 89793\ 23846\ 2643\cdots$

(a)

$P(0)=0$	$P(3)=5/25$	$P(6)=3/25$	$P(9)=3/25$
$P(1)=2/25$	$P(4)=3/25$	$P(7)=1/25$	
$P(2)=3/25$	$P(5)=3/25$	$P(8)=2/25$	

(In general, $P(j)=\frac{N(j)}{N}$.)

(b) Most probable: $\boxed{3}$. Median: 13 are ≤ 4, 12 are ≥ 5, so median is $\boxed{4}$.

Average: $\langle j\rangle = \frac{1}{25}\left[0\cdot0 + 1\cdot2 + 2\cdot3 + 3\cdot5 + 4\cdot3 + 5\cdot3 + 6\cdot3 + 7\cdot1 + 8\cdot2 + 9\cdot3\right]$

$$= \frac{1}{25}(0+2+6+15+12+15+18+7+16+27) = \frac{118}{25} = \boxed{4.72}.$$

(c) $\langle j^2\rangle = \frac{1}{25}\left[0^2\cdot0 + 1^2\cdot2 + 2^2\cdot3 + 3^2\cdot5 + 4^2\cdot3 + 5^2\cdot3 + 6^2\cdot3 + 7^2\cdot1 + 8^2\cdot2 + 9^2\cdot3\right]$

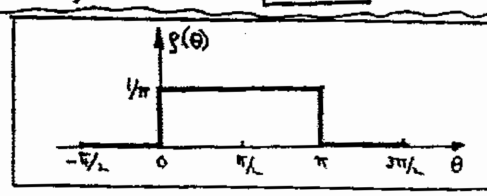
$$= \frac{1}{25}(0+2+12+45+48+75+108+49+128+243) = \frac{710}{25} = 28.4.$$

$$\sigma^2 = \langle j^2\rangle - \langle j\rangle^2 = 28.4 - (4.72)^2 = 28.4 - 22.2784 = 6.1216;\ \sigma = \sqrt{6.1216} = \boxed{2.474}$$

PROBLEM 1.3 (a) Constant for $0 \le \theta \le \pi$, otherwise zero.

In view of [1.16], the constant is $1/\pi$.

$$\boxed{\rho(\theta) = \begin{cases} 1/\pi, & \text{if } 0 \le \theta \le \pi \\ 0, & \text{otherwise} \end{cases}}.$$



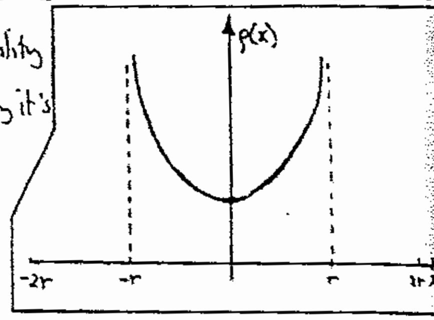
(b) $\langle\theta\rangle = \int \theta\rho(\theta)\,d\theta = \frac{1}{\pi}\int_0^\pi \theta\,d\theta = \frac{1}{\pi}\left(\frac{\theta^2}{2}\right)\Big|_0^\pi = \boxed{\frac{\pi}{2}}$ (of course).

$$\langle\theta^2\rangle = \frac{1}{\pi}\int_0^\pi \theta^2\,d\theta = \frac{1}{\pi}\left(\frac{\theta^3}{3}\right)\Big|_0^\pi = \boxed{\frac{\pi^2}{3}}.\quad \sigma^2 = \langle\theta^2\rangle - \langle\theta\rangle^2 = \frac{\pi^2}{3} - \frac{\pi^2}{4} = \frac{\pi^2}{12};\ \boxed{\sigma = \frac{\pi}{2\sqrt{3}}}.$$

(c) $\langle\sin\theta\rangle = \frac{1}{\pi}\int_0^\pi \sin\theta\,d\theta = \frac{1}{\pi}(-\cos\theta)\Big|_0^\pi = \frac{1}{\pi}(1-(-1)) = \boxed{\frac{2}{\pi}}$; $\langle\cos\theta\rangle = \frac{1}{\pi}\int_0^\pi \cos\theta\,d\theta = \frac{1}{\pi}(\sin\theta)\Big|_0^\pi = \boxed{0}$.

$\langle\cos^2\theta\rangle = \frac{1}{\pi}\int_0^\pi \cos^2\theta\,d\theta = \frac{1}{\pi}\int_0^\pi (1/2)\,d\theta = \boxed{1/2}$. [Because $\sin^2\theta + \cos^2\theta = 1$, and the integrals of $\sin^2$ and $\cos^2$ are equal (over suitable intervals), you can replace them by 1/2 in such integrals.]

<u>PROBLEM 1.4</u> (a) $x = r\cos\theta \Rightarrow dx = -r\sin\theta\,d\theta$. The probability needle lies in range $d\theta$ is $\rho(\theta)\,d\theta = \frac{1}{\pi}\,d\theta$, so probability it's in the range dx is



$$\rho(x)\,dx = \frac{1}{\pi}\frac{dx}{r\sin\theta} = \frac{1}{\pi r}\frac{dx}{\sqrt{1-(x/r)^2}} = \frac{dx}{\pi\sqrt{r^2-x^2}}.$$

$$\therefore\quad \boxed{\rho(x) = \begin{cases} \dfrac{1}{\pi\sqrt{r^2-x^2}}, & \text{if } -r<x<r \\ 0, & \text{otherwise} \end{cases}}$$

[Note: we want the <u>magnitude</u> of the interval dx, here.]

Total: $\int_{-r}^{r} \frac{1}{\pi\sqrt{r^2-x^2}}\,dx = \frac{2}{\pi}\int_0^r \frac{1}{\sqrt{r^2-x^2}}\,dx = \frac{2}{\pi}\sin^{-1}\left(\frac{x}{r}\right)\Big|_0^r = \frac{2}{\pi}\sin^{-1}(1) = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1.$ ✓

(b) $\langle x\rangle = \frac{1}{\pi}\int_{-r}^{r} x\frac{1}{\sqrt{r^2-x^2}}\,dx = \boxed{0}$ (odd integrand, even interval).

$$\langle x^2\rangle = \frac{2}{\pi}\int_0^r \frac{x^2}{\sqrt{r^2-x^2}}\,dx = \frac{2}{\pi}\left[-\frac{x}{2}\sqrt{r^2-x^2} + \frac{r^2}{2}\sin^{-1}\left(\frac{x}{r}\right)\right]\Big|_0^r = \frac{2}{\pi}\frac{r^2}{2}\sin^{-1}(1) = \boxed{\frac{r^2}{2}}$$

$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = r^2/2 \Rightarrow \boxed{\sigma = r/\sqrt{2}}$. To get $\langle x\rangle$ and $\langle x^2\rangle$ from 1.3 c, use $x = r\cos\theta$, so $\langle x\rangle = r\langle\cos\theta\rangle = 0$, $\langle x^2\rangle = r^2\langle\cos^2\theta\rangle = r^2/2$. ✓

<u>PROBLEM 1.5</u> Suppose eye end lands a distance y up from a line $(0\le y<\ell)$, and let x be the projection along that same direction $(-\ell\le x<\ell)$. Needle crosses line above if $y+x\geq\ell$ (i.e. $x\geq\ell-y$), and it crosses the line below if $y+x<0$ (i.e. $x<-y$). So for given y, the probability of crossing (using 1.4) is

$$P(y) = \int_{-\ell}^{-y}\rho(x)\,dx + \int_{\ell-y}^{\ell}\rho(x)\,dx = \frac{1}{\pi}\left\{\int_{-\ell}^{-y}\frac{1}{\sqrt{\ell^2-x^2}}\,dx + \int_{\ell-y}^{\ell}\frac{1}{\sqrt{\ell^2-x^2}}\,dx\right\} = \frac{1}{\pi}\left\{\sin^{-1}\left(\frac{x}{\ell}\right)\Big|_{-\ell}^{-y} + \sin^{-1}\left(\frac{x}{\ell}\right)\Big|_{\ell-y}^{\ell}\right\}$$

$$= \frac{1}{\pi}\left[-\sin^{-1}(y/\ell) + \sin^{-1}(1) + \sin^{-1}(1) - \sin^{-1}(1-y/\ell)\right] = 1 - \frac{\sin^{-1}(y/\ell)}{\pi} - \frac{\sin^{-1}(1-y/\ell)}{\pi}.$$

Now, all values of y are equally likely, so $\rho(y) = 1/\ell$, and hence the probability of crossing is

$$P = \frac{1}{\pi\ell}\int_0^\ell\left[\pi - \sin^{-1}(y/\ell) - \sin^{-1}\left(\frac{\ell-y}{\ell}\right)\right]dy = \frac{1}{\pi\ell}\int_0^\ell\left[\pi - 2\sin^{-1}(y/\ell)\right]dy$$

$$P = \frac{1}{\pi\ell}\left[\pi\ell - 2\left(y\sin^{-1}(y/\ell) + \ell\sqrt{1-(y/\ell)^2}\right)\Big|_0^{\ell}\,\right] = 1 - \frac{2}{\pi\ell}\left(\ell\sin^{-1}(1) - \ell\right) = 1 - 1 + \frac{2}{\pi} = \boxed{\frac{2}{\pi}}.$$

Problem 1.6 (a) $1 = \int_{-\infty}^{\infty} A e^{-\lambda(x-a)^2}dx.$ Let $u \equiv x-a$, $du = dx$, $u: -\infty \to \infty$.

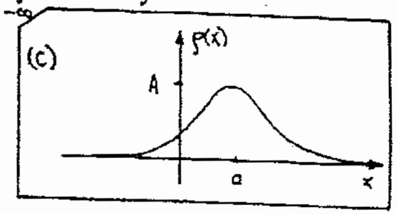
$$= A\int_{-\infty}^{\infty} e^{-\lambda u^2}du = A\sqrt{\pi/\lambda}, \text{ so } \boxed{A = \sqrt{\frac{\lambda}{\pi}}}.$$

(b) $$\langle x\rangle = A\int_{-\infty}^{\infty} x e^{-\lambda(x-a)^2}dx = A\int_{-\infty}^{\infty}(u+a)e^{-\lambda u^2}du = A\left[\int_{-\infty}^{\infty} u e^{-\lambda u^2}du + a\int_{-\infty}^{\infty} e^{-\lambda u^2}du\right] = A\left(0 + a\sqrt{\frac{\pi}{\lambda}}\right) = \boxed{a}$$

$$\langle x^2\rangle = A\int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2}dx = A\left\{\int_{-\infty}^{\infty} u^2 e^{-\lambda u^2}du + 2a\int_{-\infty}^{\infty} u e^{-\lambda u^2}du + a^2\int_{-\infty}^{\infty} e^{-\lambda u^2}du\right\}$$

$$= A\left[\frac{1}{2\sqrt{\lambda}}\sqrt{\frac{\pi}{\lambda}} + 0 + a^2\sqrt{\frac{\pi}{\lambda}}\right] = \boxed{a^2 + \frac{1}{2\sqrt{\lambda}}}$$

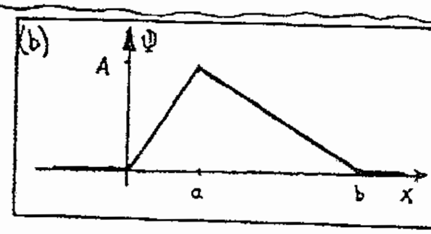
$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = a^2 + \frac{1}{2\sqrt{\lambda}} - a^2 = \frac{1}{2\sqrt{\lambda}}; \quad \boxed{\sigma = (4\lambda)^{-1/4}}$$



Problem 1.7 (a) $$1 = \frac{|A|^2}{a^2}\int_0^a x^2dx + \frac{|A|^2}{(b-a)^2}\int_a^b (b-x)^2dx$$

$$= |A|^2\left\{\frac{1}{a^2}\left(\frac{x^3}{3}\right)\Big|_0^a + \frac{1}{(b-a)^2}\left(-\frac{(b-x)^3}{3}\right)\Big|_a^b\right\}$$

$$= |A|^2\left[\frac{a}{3} + \frac{b-a}{3}\right] = |A|^2\frac{b}{3}; \quad \boxed{A = \sqrt{\frac{3}{b}}}.$$



(c) At $\boxed{x=a}$. (d) $P = \int_0^a |\Psi|^2dx = \frac{|A|^2}{a^2}\int_0^a x^2dx = |A|^2\frac{a}{3} = \boxed{\frac{a}{b}}.$ $\begin{cases}\text{if } b=a, P=1 \checkmark\\ \text{if } b=2a, P=1/2 \checkmark\end{cases}$

(e) $$\langle x\rangle = \int x|\Psi|^2dx = |A|^2\left\{\frac{1}{a^2}\int_0^a x^3dx + \frac{1}{(b-a)^2}\int_a^b x(b-x)^2dx\right\}$$

$$= \frac{3}{b}\left\{\frac{1}{a^2}\left(\frac{x^4}{4}\right)\Big|_0^a + \frac{1}{(b-a)^2}\left(b^2\frac{x^2}{2} - 2b\frac{x^3}{3} + \frac{x^4}{4}\right)\Big|_a^b\right\}$$

$$= \frac{3}{4b(b-a)^2}\left[a^2(b-a)^2 + 2b^4 - 8b^4/3 + b^4 - 2a^2b^2 + 8a^3b/3 - a^4\right]$$

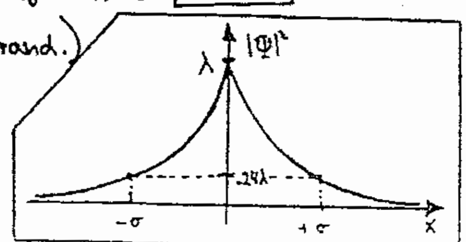
$$= \frac{3}{4b(b-a)^2}\left(\frac{b^4}{3} - a^2b^2 + \frac{2}{3}a^3b\right) = \frac{1}{4(b-a)^2}\left(b^3 - 3a^2b + 2a^3\right) = \frac{(b-a)^2(2a+b)}{4(b-a)^2} = \boxed{\frac{2a+b}{4}}$$

Problem 1.8 (a) $$1 = \int|\Psi|^2dx = 2|A|^2\int_0^{\infty} e^{-2\lambda x}dx = 2|A|^2\left(\frac{e^{-2\lambda x}}{-2\lambda}\right)\Big|_0^{\infty} = \frac{|A|^2}{\lambda}; \quad \boxed{A = \sqrt{\lambda}}$$

(b) $\langle x\rangle = \int x|\Psi|^2dx = |A|^2\int_{-\infty}^{\infty} x e^{-2\lambda|x|}dx = \boxed{0}.$ (Odd integrand.)

$$\langle x^2\rangle = 2|A|^2\int_0^{\infty} x^2e^{-2\lambda x}dx = 2\lambda\left(\frac{2}{(2\lambda)^3}\right) = \boxed{\frac{1}{2\lambda^2}}.$$

(c) $\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{1}{2\lambda^2}; \quad \boxed{\sigma = \frac{1}{\sqrt{2}\lambda}}.$



$$|\Psi(\pm\sigma)|^2 = |A|^2 e^{-2\lambda\sigma} = \lambda e^{-2\lambda/\sqrt{2}\lambda} = \lambda e^{-\sqrt{2}} = 0.2431\,\lambda.$$

Probability outside: $2\int_\sigma^\infty |\Psi|^2 dx = 2|A|^2 \int_\sigma^\infty e^{-2\lambda x}dx = 2\lambda\left(\frac{e^{-2\lambda x}}{-2\lambda}\right)\Big|_\sigma^\infty = e^{-2\lambda\sigma} = \boxed{e^{-\sqrt{2}} = 0.2431}$

PROBLEM 1.9 (a) $P_{ab}(t) = \int_a^b |\Psi(x,t)|^2 dx$, so $\frac{dP_{ab}}{dt} = \int_a^b \frac{\partial}{\partial t}|\Psi|^2 dx$. But (eq. [1.25]):

$$\frac{\partial|\Psi|^2}{\partial t} = \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\right] = -\frac{\partial}{\partial x}J(x,t).$$

$$\therefore \frac{dP_{ab}}{dt} = -\int_a^b \frac{\partial}{\partial x}J(x,t)\,dx = -\left(J(x,t)\right)\Big|_a^b = J(a,t) - J(b,t). \quad \text{QED.}$$

Probability is dimensionless, so J has dimensions $1/\text{time}$, and units $\boxed{\text{seconds}^{-1}}$.

(b) Here $\Psi(x,t) = f(x)e^{-i\omega t}$, where $f(x) \equiv \sqrt{\lambda}\, e^{-\lambda|x|}$, so $\Psi\frac{\partial\Psi^*}{\partial x} = fe^{-i\omega t}\frac{df}{dx}e^{i\omega t} = f\frac{df}{dx}$, and $\Psi^*\frac{\partial\Psi}{\partial x} = f\frac{df}{dx}$, too, so $\boxed{J(x,t) = 0.}$

PROBLEM 1.10 (a) Equation [1.24] now reads $\frac{\partial\Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V^*\Psi^*$, and [1.25] picks up an extra term:

$$\frac{\partial}{\partial t}|\Psi|^2 = \cdots + \frac{i}{\hbar}|\Psi|^2(V^* - V) = \cdots + \frac{i}{\hbar}|\Psi|^2(V_0 + i\Gamma - V_0 + i\Gamma) = \cdots - \frac{2\Gamma}{\hbar}|\Psi|^2,$$

and [1.27] becomes

$$\frac{dP}{dt} = -\frac{2\Gamma}{\hbar}\int_{-\infty}^{\infty}|\Psi|^2 dx = -\frac{2\Gamma}{\hbar}P. \quad \text{QED}$$

(b) $\frac{dP}{P} = -\frac{2\Gamma}{\hbar}dt \Rightarrow \ln P = -\frac{2\Gamma}{\hbar}t + \text{constant} \Rightarrow \boxed{P(t) = P(0)e^{-2\Gamma t/\hbar}}$, so $\boxed{\tau = \frac{\hbar}{2\Gamma}}$

PROBLEM 1.11 For integration by parts the differentiation has to be with respect to the integration variable — in this case the differentiation is with respect to t, but the integration variable is x. It's true that

$$\frac{\partial}{\partial t}(x|\Psi|^2) = \frac{\partial x}{\partial t}|\Psi|^2 + x\frac{\partial}{\partial t}|\Psi|^2 = x\frac{\partial}{\partial t}|\Psi|^2,$$

but this does not allow us to perform the integration:

$$\int_a^b x\frac{\partial}{\partial t}|\Psi|^2 dx = \int_a^b \frac{\partial}{\partial t}(x|\Psi|^2)dx \neq (x|\Psi|^2)\Big|_a^b.$$

PROBLEM 1.12 From [1.33], $\frac{d\langle p\rangle}{dt} = -i\hbar\int\frac{\partial}{\partial t}\left(\Psi^*\frac{\partial\Psi}{\partial x}\right)dx$. But, noting that $\frac{\partial^2\Psi}{\partial x\partial t} = \frac{\partial^2\Psi}{\partial t\partial x}$ and using [1.23]–[1.24]:

$$\frac{\partial}{\partial t}\left(\Psi^*\frac{\partial\Psi}{\partial x}\right) = \frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial x} + \Psi^*\frac{\partial}{\partial x}\left(\frac{\partial\Psi}{\partial t}\right) = \left[-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*\right]\frac{\partial\Psi}{\partial x} + \Psi^*\frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right]$$

$$= \underbrace{\frac{i\hbar}{2m}\left[\Psi^*\frac{\partial^3\Psi}{\partial x^3} - \frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial\Psi}{\partial x}\right]}_{\text{(this term integrates to zero, using integration-by-parts twice)}} + \frac{i}{\hbar}\underbrace{\left(V\Psi^*\frac{\partial\Psi}{\partial x} - \Psi^*\frac{\partial}{\partial x}(V\Psi)\right)}_{V\Psi^*\frac{\partial\Psi}{\partial x} - \Psi^*V\frac{\partial\Psi}{\partial x} - \Psi^*\frac{\partial V}{\partial x}\Psi = -|\Psi|^2\frac{\partial V}{\partial x}}$$

So $\frac{d\langle p\rangle}{dt} = -i\hbar\left(\frac{i}{\hbar}\right)\int -|\Psi|^2\frac{\partial V}{\partial x}dx = \langle -\frac{\partial V}{\partial x}\rangle$. QED

PROBLEM 1.13 Suppose Ψ satisfies the Schrodinger equation without V_0: $i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+V\Psi$.
We want to find the solution (Ψ_0) with V_0: $i\hbar\frac{\partial\Psi_0}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi_0}{\partial x^2}+(V+V_0)\Psi_0$. Claim: $\Psi_0=\Psi e^{-iV_0t/\hbar}$

Proof: $i\hbar\frac{\partial\Psi_0}{\partial t}=i\hbar\frac{\partial\Psi}{\partial t}e^{-iV_0t/\hbar}+i\hbar\Psi\left(-\frac{iV_0}{\hbar}\right)e^{-iV_0t/\hbar}=\left[-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+V\Psi\right]e^{-iV_0t/\hbar}+V_0\Psi e^{-iV_0t/\hbar}$
$=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi_0}{\partial x^2}+(V+V_0)\Psi_0$. QED.

This has no effect on the expectation value of a dynamical variable, since the extra phase factor, being independent of x, cancels out in equation [1.36].

PROBLEM 1.14 (a) $1=2|A|^2\int_0^\infty e^{-2amx^2/\hbar}dx=2|A|^2\frac{1}{2}\sqrt{\frac{\pi}{(2am/\hbar)}}=|A|^2\sqrt{\frac{\pi\hbar}{2am}}$; $\boxed{A=\left(\frac{2am}{\pi\hbar}\right)^{1/4}}$

(b) $\frac{\partial\Psi}{\partial t}=-ia\Psi$; $\frac{\partial\Psi}{\partial x}=-\frac{2amx}{\hbar}\Psi$; $\frac{\partial^2\Psi}{\partial x^2}=-\frac{2am}{\hbar}\left(\Psi+x\frac{\partial\Psi}{\partial x}\right)=-\frac{2am}{\hbar}\left(1-\frac{2amx^2}{\hbar}\right)\Psi$

Plug these into the Schrodinger equation: $i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+V\Psi\Rightarrow$
$V\Psi=i\hbar(-ia)\Psi+\frac{\hbar^2}{2m}\left(-\frac{2am}{\hbar}\right)\left(1-\frac{2amx^2}{\hbar}\right)\Psi=\left[\hbar a-\hbar a\left(1-\frac{2amx^2}{\hbar}\right)\right]\Psi=2a^2mx^2\Psi$, so
$\boxed{V(x)=2ma^2x^2}$. (c) $\langle x\rangle=\int_{-\infty}^{\infty}x|\Psi|^2dx=\boxed{0}$ (odd integrand).

$\langle x^2\rangle=2|A|^2\int_0^\infty x^2e^{-2amx^2/\hbar}dx=2|A|^2\frac{1}{2^2(2am/\hbar)}\sqrt{\frac{\pi\hbar}{2am}}=\boxed{\frac{\hbar}{4am}}$.

$\langle p\rangle=m\frac{d\langle x\rangle}{dt}=\boxed{0}$. $\langle p^2\rangle=\int\Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\Psi dx=-\hbar^2\int\Psi^*\underbrace{\frac{\partial^2\Psi}{\partial x^2}}_{-\frac{2am}{\hbar}\left(1-\frac{2amx^2}{\hbar}\right)\Psi \text{ (see above)}}dx$

$\langle p^2\rangle=2am\hbar\left\{\int|\Psi|^2dx-\frac{2am}{\hbar}\int x^2|\Psi|^2dx\right\}=2am\hbar\left(1-\frac{2am}{\hbar}\langle x^2\rangle\right)=2am\hbar\left(1-\frac{2am}{\hbar}\frac{\hbar}{4am}\right)$
$=2am\hbar\left(\frac{1}{2}\right)=\boxed{am\hbar}$

(e) $\sigma_x^2=\langle x^2\rangle-\langle x\rangle^2=\frac{\hbar}{4am}\Rightarrow\boxed{\sigma_x=\sqrt{\frac{\hbar}{4am}}}$; $\sigma_p^2=\langle p^2\rangle-\langle p\rangle^2=am\hbar\Rightarrow\boxed{\sigma_p=\sqrt{am\hbar}}$.

$\sigma_x\sigma_p=\sqrt{\frac{\hbar}{4am}}\sqrt{am\hbar}=\frac{\hbar}{2}$. This is (just barely) consistent with the uncertainty principle.

CHAPTER 2

PROBLEM 2.1 (a) $\Psi(x,t) = \psi(x)e^{-i(E_0+i\Gamma)t/\hbar} = \psi(x)e^{\Gamma t/\hbar}e^{-iE_0 t/\hbar} \Rightarrow |\Psi|^2 = |\psi|^2 e^{2\Gamma t/\hbar}$.

$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = e^{2\Gamma t/\hbar}\int_{-\infty}^{\infty}|\psi(x)|^2 dx$. The second term is independent of t, so if the product is to be 1 for all time, the first term $(e^{2\Gamma t/\hbar})$ must also be constant, and hence $\Gamma = 0$. QED

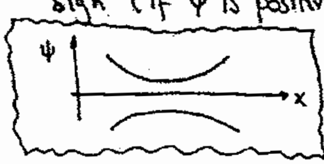
(b) If $\psi(x)$ satisfies [2.4]: $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$, then (taking the complex conjugate and noting that V and E are real): $-\frac{\hbar^2}{2m}\frac{d^2\psi^*}{dx^2} + V\psi^* = E\psi^*$, so ψ^* also satisfies [2.4]. Now, if ψ_1 and ψ_2 satisfy [2.4], so too does any linear combination of them: $\psi_3 = c_1\psi_1 + c_2\psi_2$, for

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_3}{dx^2} + V\psi_3 = -\frac{\hbar^2}{2m}\left(c_1\frac{d^2\psi_1}{dx^2} + c_2\frac{d^2\psi_2}{dx^2}\right) + V(c_1\psi_1 + c_2\psi_2) = c_1\left[-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + V\psi_1\right] + c_2\left[-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} + V\psi_2\right]$$
$$= c_1(E\psi_1) + c_2(E\psi_2) = E(c_1\psi_1 + c_2\psi_2) = E\psi_3.$$

Thus $(\psi + \psi^*)$ and $i(\psi - \psi^*)$ — both of which are real — satisfy [2.4]. Conclusion: from any complex solution I can always construct two real solutions (of course, if ψ is already real, the second one will be zero). In particular, since $\psi = \frac{1}{2}[(\psi+\psi^*) - i(i(\psi-\psi^*))]$, ψ can be expressed as a linear combination of two real solutions. QED

(c) If $\psi(x)$ satisfies [2.4]: $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$, then, changing variables $x \to -x$,
$-\frac{\hbar^2}{2m}\frac{d^2\psi(-x)}{dx^2} + V(-x)\psi(-x) = E\psi(-x)$; so if $V(-x) = V(x)$, then $\psi(-x)$ also satisfies [2.4]. It follows that $\psi_+(x) \equiv \psi(x) + \psi(-x)$ (which is even: $\psi_+(-x) = \psi_+(x)$) and $\psi_- \equiv \psi(x) - \psi(-x)$ (which is odd: $\psi_-(-x) = -\psi_-(x)$) both satisfy [2.4]. But $\psi(x) = \frac{1}{2}(\psi_+(x) + \psi_-(x))$, so any solution can be expressed as a linear combination of even and odd solutions. QED.

PROBLEM 2.2 Given $\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}[V(x) - E]\psi$, if $E < V_{min}$, then ψ'' and ψ always have the same sign (if ψ is positive, so is ψ''; if ψ is negative, so is ψ''). This means that ψ always



curves away from the axis (as shown). But it's got to go to zero at $x \to \pm\infty$ (else it would not be normalizable). If it ever departs from zero (if not, it isn't normalizable anyway), it must be positive and increasing (in which case it can only increase its slope) or negative and decreasing (in which case it can only head downward more steeply) — in either case there is no way for it ever to come back to zero. For the graph to "turn around", and return to the axis, there must come a place where ψ'' is negative when ψ is positive (or vice versa). QED

<u>PROBLEM 2.3</u> [2.16] says $\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$; $\psi(0) = \psi(a) = 0$.

<u>If $E = 0$</u>, $d^2\psi/dx^2 = 0$, so $\psi(x) = A + Bx$; $\psi(0) = A = 0 \Rightarrow \psi = Bx$; $\psi(a) = Ba = 0 \Rightarrow B = 0$, so $\psi = 0$.

<u>If $E < 0$</u>, $d^2\psi/dx^2 = \kappa^2\psi$, with $\kappa \equiv \sqrt{-2mE}/\hbar$ (real), so $\psi(x) = Ae^{\kappa x} + Be^{-\kappa x}$.

$\psi(0) = A + B = 0 \Rightarrow B = -A$, so $\psi = A(e^{\kappa x} - e^{-\kappa x})$; $\psi(a) = A(e^{\kappa a} - e^{-\kappa a}) = 0 \Rightarrow A = 0$, so $\psi = 0$, or else $e^{\kappa a} = e^{-\kappa a}$, or $e^{2\kappa a} = 1$, so $2\kappa a = \ln(1) = 0$, and $\kappa = 0$, whence again $\psi = 0$.

Thus in either case the boundary conditions force $\psi = 0$, which is unacceptable (non-normalizable).

<u>PROBLEM 2.4</u> [2.18] says $\psi(x) = A\sin kx + B\cos kx$, and $\psi(-a/2) = \psi(a/2) = 0$.

$\psi(-a/2) = -A\sin(ka/2) + B\cos(ka/2) = 0$; $\psi(a/2) = A\sin(ka/2) + B\cos(ka/2) = 0$. <u>Add</u> these equations:

$2B\cos(ka/2) = 0 \Rightarrow B = 0$, or else $ka/2 = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2$, etc, or $ka = n\pi$ (n odd).

<u>Subtract</u>: $2A\sin(ka/2) = 0 \Rightarrow A = 0$, or else $ka/2 = 0, \pm\pi, \pm 2\pi, \pm 3\pi$, etc, or $ka = n\pi$ (n even).

(As before – p. 26 – $k = 0$ is not allowed, and negative signs can be absorbed into normalization constant.)

Thus $\boxed{k = n\pi/a\ ,\ n = 1, 2, 3, \ldots}$, same as [2.22], so the allowed energies are same as before.

And $\psi_n = A\sin\left(\frac{n\pi}{a}x\right)$, ($n$ even) ; $\psi_n = B\cos\left(\frac{n\pi}{a}x\right)$, ($n$ odd). Normalizing:

$\int_{-a/2}^{a/2} |A|^2 \sin^2\left(\frac{n\pi}{a}x\right)dx = \frac{1}{2}|A|^2 \int_{-a/2}^{a/2} dx = \frac{a}{2}|A|^2 \Rightarrow A = \sqrt{\frac{2}{a}}$, and same for B. So the wave functions are

$$\boxed{\begin{aligned} \psi_n(x) &= \sqrt{\tfrac{2}{a}}\sin\left(\tfrac{n\pi}{a}x\right),\ n \text{ even} \\ \psi_n(x) &= \sqrt{\tfrac{2}{a}}\cos\left(\tfrac{n\pi}{a}x\right),\ n \text{ odd} \end{aligned}}$$

To check consistency, we put $x \to x - \frac{a}{2}$ into [2.24]: $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x - \frac{n\pi}{2}\right)$.

Now $\sin\left(\theta - \frac{n\pi}{2}\right) = \begin{cases} \sin\theta\ (n = 4, 8, 12, \ldots),\ -\sin\theta\ (n = 2, 6, 10, \ldots) \\ \cos\theta\ (n = 1, 5, 9, \ldots),\ -\cos\theta\ (n = 3, 7, 11, \ldots) \end{cases}$. So apart from the minus signs (which are irrelevant anyway) the two solutions are equivalent.

<u>PROBLEM 2.5</u> $\langle x\rangle = \int x|\psi|^2 dx = \frac{2}{a}\int_0^a x\sin^2\left(\frac{n\pi}{a}x\right)dx$. Let $y \equiv \frac{n\pi}{a}x$, so $dx = \frac{a}{n\pi}dy$; $y: 0 \to n\pi$.

$$= \frac{2}{a}\left(\frac{a}{n\pi}\right)^2 \int_0^{n\pi} y\sin^2 y\, dy = \frac{2a}{n^2\pi^2}\left[\frac{y^2}{4} - \frac{y\sin 2y}{4} - \frac{\cos 2y}{8}\right]_0^{n\pi} = \frac{2a}{n^2\pi^2}\left[\frac{n^2\pi^2}{4} - \frac{\cos 2n\pi}{8} + \frac{1}{8}\right]$$

$= \boxed{a/2}$. (Independent of n).

$$\langle x^2\rangle = \frac{2}{a}\int_0^a x^2\sin^2\left(\frac{n\pi}{a}x\right)dx = \frac{2}{a}\left(\frac{a}{n\pi}\right)^3 \int_0^{n\pi} y^2\sin^2 y\, dy = \frac{2a^2}{(n\pi)^3}\left[\frac{y^3}{6} - \left(\frac{y^2}{4} - \frac{1}{8}\right)\sin 2y - \frac{y\cos 2y}{4}\right]_0^{n\pi}$$

$$= \frac{2a^2}{(n\pi)^3}\left[\frac{(n\pi)^3}{6} - \frac{n\pi\cos(2n\pi)}{4}\right] = \boxed{a^2\left[\frac{1}{3} - \frac{1}{2(n\pi)^2}\right]}.$$

$\langle p\rangle = m\, d\langle x\rangle/dt = \boxed{0}$. [<u>Note</u>: [1.33] is much faster than [1.35].]

$$\langle p^2\rangle = \int \psi_n^*\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2\psi_n dx = -\hbar^2\int \psi_n^*\left(\frac{d^2\psi_n}{dx^2}\right)dx = (-\hbar^2)\left(-\frac{2mE_n}{\hbar^2}\right)\int\psi_n^*\psi_n dx = 2mE_n = \boxed{\left(\frac{n\pi\hbar}{a}\right)^2}.$$

$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = a^2\left(\frac{1}{3} - \frac{1}{2(n\pi)^2} - \frac{1}{4}\right) = \frac{a^2}{4}\left(\frac{1}{3} - \frac{2}{(n\pi)^2}\right)$; $\boxed{\sigma_x = \frac{a}{2}\sqrt{\frac{1}{3} - \frac{2}{(n\pi)^2}}}$

$\sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = \left(\frac{n\pi\hbar}{a}\right)^2$; $\boxed{\sigma_p = \frac{n\pi\hbar}{a}}$. $\sigma_x\sigma_p = \frac{a}{2}\sqrt{\frac{1}{3} - \frac{2}{(n\pi)^2}}\,\frac{n\pi\hbar}{a} = \frac{\hbar}{2}\sqrt{\frac{(n\pi)^2}{3} - 2}$.

The product $\sigma_x\sigma_p$ is $\boxed{\text{smallest for } n=1}$; in that case $\sigma_x\sigma_p = \frac{\hbar}{2}\sqrt{\frac{\pi^2}{3} - 2} = (1.136)\hbar/2 > \frac{\hbar}{2}$.

PROBLEM 2.6 (a) $|\Psi|^2 = \Psi^*\Psi = |A|^2(\psi_1^* + \psi_2^*)(\psi_1 + \psi_2) = |A|^2[\psi_1^*\psi_1 + \psi_1^*\psi_2 + \psi_2^*\psi_1 + \psi_2^*\psi_2]$.

$1 = \int|\Psi|^2dx = |A|^2\int[|\psi_1|^2 + \psi_1^*\psi_2 + \psi_2^*\psi_1 + |\psi_2|^2]dx = |A|^2[1+0+0+1] = 2|A|^2$. $\boxed{A = 1/\sqrt{2}}$.

(b) $\Psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_1 e^{-iE_1t/\hbar} + \psi_2 e^{-iE_2t/\hbar}\right]$. $E_n/\hbar = \frac{n^2\pi^2\hbar}{2ma^2}$. Let $\omega \equiv \frac{\pi^2\hbar}{2ma^2}$. Then $\frac{E_n}{\hbar} = n^2\omega$.

$$= \frac{1}{\sqrt{2}}\sqrt{\frac{2}{a}}\left[\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{-4i\omega t}\right] = \boxed{\frac{1}{\sqrt{a}}e^{-i\omega t}\left[\sin\left(\frac{\pi}{a}x\right) + \sin\left(\frac{2\pi}{a}x\right)e^{-3i\omega t}\right]}.$$

$|\Psi(x,t)|^2 = \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\left(e^{-3i\omega t} + e^{3i\omega t}\right) + \sin^2\left(\frac{2\pi}{a}x\right)\right]$. But $\left(e^{3i\omega t} + e^{-3i\omega t}\right) = 2\cos(3\omega t)$

$$\therefore = \boxed{\frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right) + 2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right]}.$$

(c) $$\langle x\rangle = \int x|\Psi(x,t)|^2dx = \frac{1}{a}\int_0^a x\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right) + 2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right]dx$$

$$\int_0^a x\sin^2\left(\frac{\pi}{a}x\right)dx = \left[\frac{x^2}{4} - \frac{x\sin\left(\frac{2\pi}{a}x\right)}{4\pi/a} - \frac{\cos\left(\frac{2\pi}{a}x\right)}{8(\pi/a)^2}\right]_0^a = \frac{a^2}{4} = \int_0^a x\sin^2\left(\frac{2\pi}{a}x\right)dx.$$

$$\int_0^a x\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)dx = \frac{1}{2}\int_0^a x\left[\cos\left(\frac{\pi}{a}x\right) - \cos\left(\frac{3\pi}{a}x\right)\right]dx = \frac{1}{2}\left\{\frac{a^2}{\pi^2}\cos\left(\frac{\pi}{a}x\right) + \frac{ax}{\pi}\sin\left(\frac{\pi}{a}x\right) - \frac{a^2}{9\pi^2}\cos\left(\frac{3\pi}{a}x\right) - \frac{ax}{3\pi}\sin\left(\frac{3\pi}{a}x\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{a^2}{\pi^2}(\cos(\pi) - \cos(0)) - \frac{a^2}{9\pi^2}(\cos(3\pi) - \cos(0))\right\} = -\frac{a^2}{\pi^2}\left(1 - \frac{1}{9}\right) = -\frac{8a^2}{9\pi^2}.$$

$$\langle x\rangle = \frac{1}{a}\left[\frac{a^2}{4} + \frac{a^2}{4} - \frac{16a^2}{9\pi^2}\cos(3\omega t)\right] = \boxed{\frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t)\right]}.$$

Frequency: $\boxed{\nu = \frac{3\omega}{2\pi} = \frac{3\pi\hbar}{4ma^2}}$; Amplitude: $\boxed{\frac{32}{9\pi^2}\left(\frac{a}{2}\right) = 0.3603\,(a/2)}$.

(d) $\langle p\rangle = m\frac{d\langle x\rangle}{dt} = m\left(\frac{a}{2}\right)\left(-\frac{32}{9\pi^2}\right)(-3\omega)\sin(3\omega t) = \boxed{\frac{8\hbar}{3a}\sin(3\omega t)}$.

(e) From [2.11], $H\psi_1 = E_1\psi_1$, $H\psi_2 = E_2\psi_2$, so $H\Psi = \frac{1}{\sqrt{2}}H\left(\psi_1 e^{-iE_1t/\hbar} + \psi_2 e^{-iE_2t/\hbar}\right)$
$= \frac{1}{\sqrt{2}}\left(E_1\psi_1 e^{-iE_1t/\hbar} + E_2\psi_2 e^{-iE_2t/\hbar}\right)$.

$$\langle H\rangle = \int\Psi^* H\Psi dx = \frac{1}{2}\int\left(\psi_1^* e^{iE_1t/\hbar} + \psi_2^* e^{iE_2t/\hbar}\right)\left(E_1\psi_1 e^{-iE_1t/\hbar} + E_2\psi_2 e^{-iE_2t/\hbar}\right)dx$$

$$= \frac{1}{2}\left\{E_1\int|\psi_1|^2dx + E_2\int|\psi_2|^2dx + E_2 e^{i(E_1-E_2)t/\hbar}\int\psi_1^*\psi_2\,dx + E_1 e^{i(E_2-E_1)t/\hbar}\int\psi_2^*\psi_1\,dx\right\}$$

$= \frac{1}{2}(E_1 + E_2 + 0 + 0) = \boxed{\frac{1}{2}(E_1 + E_2)} = \boxed{\frac{5\pi^2\hbar^2}{4ma^2}}$. (It's the $\boxed{\text{average}}$ of E_1 and E_2.)

(f) $\frac{1}{2}mv^2 = \frac{5\pi^2\hbar^2}{4ma^2} \Rightarrow v = \sqrt{\frac{5}{2}}\frac{\pi\hbar}{ma}$; $\nu_c = \frac{v}{2a} = \boxed{\sqrt{\frac{5}{2}}\frac{\pi\hbar}{2ma^2}} = \sqrt{\frac{10}{9}}\,\nu_q$ .... pretty close.

Problem 2.7 From 2.6, we see that: $\boxed{\Psi(x,t)=\frac{1}{\sqrt{a}}e^{-i\omega t}\left[\sin\left(\frac{\pi}{a}x\right)+\sin\left(\frac{2\pi}{a}x\right)e^{-3i\omega t}e^{i\phi}\right]}$;

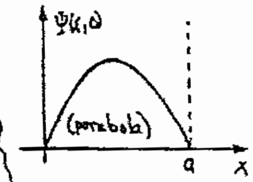
$|\Psi(x,t)|^2=\boxed{\frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right)+\sin^2\left(\frac{2\pi}{a}x\right)+2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t-\phi)\right]}$; and hence

$\boxed{\langle x\rangle=\frac{a}{2}\left[1-\frac{32}{9\pi^2}\cos(3\omega t-\phi)\right]}$. This amounts physically to starting the clock at a different time (i.e. shifting the $t=0$ point).

If $\phi=\frac{\pi}{2}$ [so $\Psi(x,0)=A(\psi_1(x)+i\psi_2(x))$], $\cos(3\omega t-\phi)=\sin(3\omega t)$; $\langle x\rangle$ starts at $a/2$.
If $\phi=\pi$ [so $\Psi(x,0)=A(\psi_1(x)-\psi_2(x))$], $\cos(3\omega t-\phi)=-\cos(3\omega t)$; $\langle x\rangle$ starts at $\frac{a}{2}\left(1+\frac{32}{9\pi^2}\right)$.

Problem 2.8 (a) $1=|A|^2\int_0^a x^2(a-x)^2dx=|A|^2\left\{a^2\int_0^a x^2dx-2a\int_0^a x^3dx+\int_0^a x^4dx\right\}$

$$=|A|^2\left(a^2\frac{a^3}{3}-2a\frac{a^4}{4}+\frac{a^5}{5}\right)=\frac{a^5}{30}|A|^2;\quad \boxed{A=\sqrt{\frac{30}{a^5}}}.$$



Closest to $\psi_1(x)$, so I estimate $\boxed{\langle H\rangle\cong E_1=\pi^2\hbar^2/2ma^2}$.

(b) $\langle x\rangle=\int x|\Psi(x,0)|^2dx=|A|^2\int_0^a x^3(a-x)^2dx=|A|^2\left\{a^2\int_0^a x^3dx-2a\int_0^a x^4dx+\int_0^a x^5dx\right\}$

$$=|A|^2\left(a^2\cdot\frac{a^4}{4}-2a\cdot\frac{a^5}{5}+\frac{a^6}{6}\right)=\frac{a^6}{60}|A|^2(15-24+10)=\frac{a^6}{60}\cdot\frac{30}{a^5}=\boxed{\frac{a}{2}}\ \text{(of course!)}.$$

$\langle p\rangle=\int\Psi^*\left(\frac{\hbar}{i}\frac{d}{dx}\right)\Psi\,dx=\frac{\hbar}{i}|A|^2\int_0^a x(a-x)[a-2x]\,dx=-i\hbar|A|^2\left\{a^2\int_0^a x\,dx-3a\int_0^a x^2dx+2\int_0^a x^3dx\right\}$

$$=-i\hbar|A|^2\left(a^2\cdot\frac{a^2}{2}-3a\cdot\frac{a^3}{3}+2\frac{a^4}{4}\right)=\boxed{0}.$$

$\langle H\rangle=-\frac{\hbar^2}{2m}\int\Psi^*\left(\frac{d}{dx}\right)^2\Psi\,dx=-\frac{\hbar^2}{2m}|A|^2\int_0^a x(a-x)(-2)dx=\frac{\hbar^2}{m}|A|^2\left\{a\int_0^a x\,dx-\int_0^a x^2dx\right\}$

$$=\frac{\hbar^2}{m}|A|^2\left(a\cdot\frac{a^2}{2}-\frac{a^3}{3}\right)=\frac{\hbar^2a^3}{6m}\cdot\frac{30}{a^5}=\boxed{\frac{5\hbar^2}{ma^2}}.$$

[Since $\frac{\pi^2}{2}=4.935$, the estimate in part (a) was very close.]

Problem 2.9 Eq. [2.33] $\Rightarrow$ $c_n=\sqrt{\frac{2}{a}}A\int_0^a x(a-x)\sin\left(\frac{n\pi}{a}x\right)dx.$

$c_n=\sqrt{\frac{2}{a}}A\left\{a\int_0^a x\sin\left(\frac{n\pi}{a}x\right)dx-\int_0^a x^2\sin\left(\frac{n\pi}{a}x\right)dx\right\}$. $\quad\int_0^a x\sin\left(\frac{n\pi}{a}x\right)dx=\left[\left(\frac{a}{n\pi}\right)^2\sin\left(\frac{n\pi}{a}x\right)-\frac{ax}{n\pi}\cos\left(\frac{n\pi}{a}x\right)\right]\Big|_0^a$

$=-\frac{a^2}{n\pi}\cos(n\pi)=-\frac{a^2}{n\pi}(-1)^n$; $\quad\int_0^a x^2\sin\left(\frac{n\pi}{a}x\right)dx=\left[\frac{2a^2x}{(n\pi)^2}\sin\left(\frac{n\pi}{a}x\right)-\frac{(n\pi x/a)^2-2}{(n\pi/a)^3}\cos\left(\frac{n\pi}{a}x\right)\right]\Big|_0^a$

$$=-\frac{(n\pi)^2-2}{(n\pi/a)^3}\cos(n\pi)-\frac{2}{(n\pi/a)^3}\cos(0)=-\left(\frac{a}{n\pi}\right)^3\left(2+[(n\pi)^2-2](-1)^n\right).$$

$$c_n=\sqrt{\frac{2}{a}}\sqrt{\frac{30}{a^5}}\,a^3\left[-\frac{1}{n\pi}(-1)^n+\frac{2}{(n\pi)^3}+\left(\frac{1}{n\pi}-\frac{2}{(n\pi)^3}\right)(-1)^n\right]=2\sqrt{15}\frac{2}{(n\pi)^3}\left[1-(-1)^n\right]=\boxed{\begin{cases}0, & \text{if } n \text{ is even}\\ \frac{8\sqrt{15}}{(n\pi)^3}, & \text{if } n \text{ is odd}\end{cases}}$$

From [2.32]: $\boxed{\Psi(x,t)=\left(\frac{2}{\pi}\right)^3\sqrt{\frac{30}{a}}\sum_{n=1,3,5,\ldots}\frac{1}{n^3}\sin\left(\frac{n\pi}{a}x\right)e^{-i(n^2\pi^2\hbar/2ma^2)t}}$

$c_1=8\sqrt{15}/\pi^3=\boxed{0.99928}$; $c_2=\boxed{0}$; $c_3=c_1/3^3=\boxed{0.03701}$.

Conclusion: This is mainly ψ_1 (as noted visually in 2.8 a). After the measurement $c_3=1$, all others zero.

PROBLEM 2.10 $1=\int \Psi^* \Psi dx = \sum_n c_n^* \sum_m c_m e^{i(E_n-E_m)t/\hbar} \underbrace{\int \psi_n^* \psi_m dx}$

$= \delta_{nm}$, so the only term in the m summation that survives is m=n

$= \sum_{n=1}^{\infty} |c_n|^2$. QED.

$\langle H \rangle = \int \Psi^* H \Psi dx = \sum_n c_n^* \sum_m c_m e^{i(E_n-E_m)t/\hbar} \underbrace{\int \psi_n^* \underbrace{H\psi_m}_{=E_m\psi_m} dx}_{=E_m \int \psi_n^* \psi_m dx = E_m \delta_{nm}} = \sum_{n=1}^{\infty} E_n |c_n|^2$. QED

(again, only m=n survives)

PROBLEM 2.11 Let $f(x)$ and $g(x)$ be arbitrary functions, such that $f \to 0$ at $\pm\infty$.

$$\int_{-\infty}^{\infty} (a_- f)^* g\, dx = \int \frac{1}{\sqrt{2m}}\left(-\frac{\hbar}{i}\frac{df^*}{dx} + im\omega x f^*\right) g\, dx = \frac{1}{\sqrt{2m}}\left\{-\frac{\hbar}{i}\int \frac{df^*}{dx} g\, dx + im\omega \int x f^* g\, dx\right\}$$

But $\int \frac{df^*}{dx} g\, dx = f^* g\Big|_{-\infty}^{\infty} - \int f^* \frac{dg}{dx} dx = -\int f^* \frac{dg}{dx} dx$. So

$\int (a_- f)^* g\, dx = \int f^* \left[\frac{1}{\sqrt{2m}}\left(\frac{\hbar}{i}\frac{d}{dx} + im\omega x\right) g\right] dx = \int f^* (a_+ g)\, dx$. In particular, if ψ is a normalized state of the harmonic oscillator, $\int (a_-\psi)^*(a_-\psi) dx = \int \psi^* (a_+ a_-)\psi\, dx$. But [2.46] $(a_+ a_-)\psi = (E - \frac{1}{2}\hbar\omega)\psi$, so $\int |a_-\psi|^2 dx = (E - \frac{1}{2}\hbar\omega)\int |\psi|^2 dx = E - \frac{1}{2}\hbar\omega < \infty$. QED

PROBLEM 2.12 (a) From Problem 2.11: $\int |a_-\psi_n|^2 dx = E_n - \frac{1}{2}\hbar\omega = (n+\frac{1}{2})\hbar\omega - \frac{1}{2}\hbar\omega = n\hbar\omega$. The same argument shows that $\int (a_+\psi)^*(a_+\psi) dx = \int \psi^*(a_- a_+)\psi\, dx$. But [2.43] $\Rightarrow a_- a_+ \psi = (E + \frac{1}{2}\hbar\omega)\psi$. So $\int |a_+\psi_n|^2 dx = E_n + \frac{1}{2}\hbar\omega = (n+\frac{1}{2})\hbar\omega + \frac{1}{2}\hbar\omega = (n+1)\hbar\omega$. We know that $a_+\psi_n = c\,\psi_{n+1}$, for some constant c. With ψ_n and ψ_{n+1} normalized,

$\int |a_+\psi_n|^2 dx = |c|^2 \int |\psi_{n+1}|^2 dx = |c|^2 = (n+1)\hbar\omega$, so $c = \sqrt{(n+1)\hbar\omega}$, confirming [2.52] and the same reasoning confirms [2.53].

(b) From [2.48] and [2.50], $\psi_n = \frac{A_n}{A_0}(a_+)^n \psi_0 = \frac{A_n}{A_0}(a_+)^{n-1}\underbrace{(a_+\psi_0)}_{\sqrt{\hbar\omega}\,\psi_1} = \frac{A_n}{A_0}\sqrt{\hbar\omega}(a_+)^{n-2}\underbrace{(a_+\psi_1)}_{\sqrt{2\hbar\omega}\,\psi_2} = \cdots$

$$\psi_n = \frac{A_n}{A_0}\sqrt{\hbar\omega}\sqrt{2\hbar\omega}\sqrt{3\hbar\omega}\cdots\sqrt{n\hbar\omega}\,\psi_n = \frac{A_n}{A_0}\sqrt{n!(\hbar\omega)^n}\,\psi_n. \therefore A_n = \frac{1}{\sqrt{n!(\hbar\omega)^n}} A_0.$$

Normalizing ψ_0: $1 = |A_0|^2 \int_{-\infty}^{\infty} e^{-m\omega x^2/\hbar} dx = |A_0|^2 \sqrt{\frac{\pi\hbar}{m\omega}} \Rightarrow A_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$. So $\boxed{A_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{n!(\hbar\omega)^n}}}$

PROBLEM 2.13 (a) $1 = |A_1|^2 2m\omega^2 \int_{-\infty}^{\infty} x^2 e^{-m\omega x^2/\hbar} dx = |A_1|^2 2m\omega^2 \frac{\hbar}{2m\omega}\sqrt{\frac{\pi\hbar}{m\omega}} = |A_1|^2 \hbar\omega\sqrt{\frac{\pi\hbar}{m\omega}}$.

$\therefore \boxed{A_1 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{\hbar\omega}}}$ (which matches [2.54] for n=1). (b) $\psi_2 = A_2 (a_+)^2 e^{-\frac{m\omega}{2\hbar}x^2} =$ (using [2.5…

$= A_2 a_+ \left[i\omega\sqrt{2m}\, x e^{-m\omega x^2/2\hbar}\right] = (A_2 i\omega\sqrt{2m})\frac{1}{\sqrt{2m}}\left[\frac{\hbar}{i}\frac{d}{dx} + im\omega x\right] x e^{-m\omega x^2/2\hbar}$

$$\psi_2 = A_2\hbar\omega\left(\frac{d}{dx} - \frac{m\omega}{\hbar}x\right)x e^{-m\omega x^2/2\hbar} = A_2\hbar\omega\left\{e^{-m\omega x^2/2\hbar} - \frac{m\omega}{\hbar}x^2 e^{-m\omega x^2/2\hbar} - \frac{m\omega}{\hbar}x^2 e^{-m\omega x^2/2\hbar}\right\}$$

$$= \boxed{A_2\hbar\omega\left(1 - \frac{2m\omega}{\hbar}x^2\right)e^{-m\omega x^2/2\hbar}}.$$

(c) [See Figure 2.5 a.] (d) ψ_0 and ψ_2 are even, ψ_1 is odd.

So $\int \psi_0^* \psi_1\, dx = \int \psi_2^* \psi_1\, dx = 0$ automatically. The only one we need to check is $\int \psi_2^* \psi_0\, dx$.

$$\int \psi_2^* \psi_0\, dx = (A_2\hbar\omega)(A_0)\int_{-\infty}^{\infty}\left(1 - \frac{2m\omega}{\hbar}x^2\right)e^{-m\omega x^2/\hbar}dx = A_0 A_2\hbar\omega\left\{\int_{-\infty}^{\infty} e^{-m\omega x^2/\hbar}dx - \frac{2m\omega}{\hbar}\int_{-\infty}^{\infty} x^2 e^{-m\omega x^2/\hbar}dx\right\}$$

$$= A_0 A_2\hbar\omega\left\{\sqrt{\frac{\pi\hbar}{m\omega}} - \frac{2m\omega}{\hbar}\,\frac{\hbar}{2m\omega}\sqrt{\frac{\pi\hbar}{m\omega}}\right\} = 0. \checkmark$$

Problem 2.14 (a) ψ_0 is even, ψ_1 odd, in x — in either case $|\psi_n|^2$ is even, so $\langle x\rangle = \int x|\psi_n|^2 dx = \boxed{0}$.

$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{0}$. (These results hold for any stationary state of the harmonic oscillator.)

In terms of the variable $\xi \equiv \sqrt{\frac{m\omega}{\hbar}}\,x$ and the constant $\alpha \equiv \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$, $\psi_0 = \alpha e^{-\xi^2/2}$, $\psi_1 = i\sqrt{2}\,\alpha\xi e^{-\xi^2/2}$.

$\underline{n=0}$: $$\langle x^2\rangle = \alpha^2\int_{-\infty}^{\infty} x^2 e^{-\xi^2}dx = \alpha^2\left(\frac{\hbar}{m\omega}\right)^{3/2}\int_{-\infty}^{\infty}\xi^2 e^{-\xi^2}d\xi = \frac{1}{\sqrt{\pi}}\left(\frac{\hbar}{m\omega}\right)\frac{\sqrt{\pi}}{2} = \boxed{\frac{\hbar}{2m\omega}}.$$

$$\langle p^2\rangle = \int \psi_0\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2\psi_0\, dx = -\hbar^2\alpha^2\sqrt{\frac{m\omega}{\hbar}}\int_{-\infty}^{\infty} e^{-\xi^2/2}\underbrace{\left(\frac{d^2}{d\xi^2}e^{-\xi^2/2}\right)}_{\frac{d}{d\xi}\left(-\xi e^{-\xi^2/2}\right) = (\xi^2-1)e^{-\xi^2/2}} d\xi = -\frac{m\hbar\omega}{\sqrt{\pi}}\int_{-\infty}^{\infty}(\xi^2-1)e^{-\xi^2}d\xi$$

$$= -\frac{m\hbar\omega}{\sqrt{\pi}}\left\{\frac{\sqrt{\pi}}{2} - \sqrt{\pi}\right\} = \boxed{\frac{m\hbar\omega}{2}}.$$

$\underline{n=1}$: $$\langle x^2\rangle = 2\alpha^2\int_{-\infty}^{\infty} x^2\xi^2 e^{-\xi^2}dx = 2\alpha^2\left(\frac{\hbar}{m\omega}\right)^{3/2}\int_{-\infty}^{\infty}\xi^4 e^{-\xi^2}d\xi = \frac{2\hbar}{\sqrt{\pi}\,m\omega}\cdot\frac{3\sqrt{\pi}}{4} = \boxed{\frac{3\hbar}{2m\omega}}.$$

$$\langle p^2\rangle = -\hbar^2 2\alpha^2\sqrt{\frac{m\omega}{\hbar}}\int_{-\infty}^{\infty}\xi e^{-\xi^2/2}\underbrace{\left(\frac{d^2}{d\xi^2}\left(\xi e^{-\xi^2/2}\right)\right)}_{\frac{d}{d\xi}\left((1-\xi^2)e^{-\xi^2/2}\right) = (\xi^3-3\xi)e^{-\xi^2/2}} d\xi = -\frac{2m\omega\hbar}{\sqrt{\pi}}\int_{-\infty}^{\infty}(\xi^4 - 3\xi^2)e^{-\xi^2}d\xi$$

$$= -\frac{2m\omega\hbar}{\sqrt{\pi}}\left[\frac{3}{4}\sqrt{\pi} - 3\frac{\sqrt{\pi}}{2}\right] = \boxed{\frac{3m\hbar\omega}{2}}.$$

(b) $\underline{n=0}$: $\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{2m\omega} \Rightarrow \sigma_x = \sqrt{\frac{\hbar}{2m\omega}}$; $\sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = \frac{m\hbar\omega}{2} \Rightarrow \sigma_p = \sqrt{\frac{m\hbar\omega}{2}}$.

$\therefore \sigma_x\sigma_p = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{m\omega\hbar}{2}} = \frac{\hbar}{2}$. (Right at the uncertainty limit.)

$\underline{n=1}$: $\sigma_x = \sqrt{\frac{3\hbar}{2m\omega}}$; $\sigma_p = \sqrt{\frac{3m\hbar\omega}{2}}$; $\sigma_x\sigma_p = 3\frac{\hbar}{2} > \frac{\hbar}{2}$ $\checkmark$.

(c) $\langle T\rangle = \frac{1}{2m}\langle p^2\rangle = \boxed{\begin{Bmatrix}\frac{1}{4}\hbar\omega & (n=0)\\ \frac{3}{4}\hbar\omega & (n=1)\end{Bmatrix}}$; $\langle V\rangle = \frac{1}{2}m\omega^2\langle x^2\rangle = \boxed{\begin{Bmatrix}\frac{1}{4}\hbar\omega & (n=0)\\ \frac{3}{4}\hbar\omega & (n=1)\end{Bmatrix}}$.

Sum: $\langle T\rangle + \langle V\rangle = \langle H\rangle = \begin{Bmatrix}\frac{1}{2}\hbar\omega & (n=0) = E_0\\ \frac{3}{2}\hbar\omega & (n=1) = E_1\end{Bmatrix}$, as expected.

Problem 2.15 $\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2}$, so $P = 2\sqrt{\frac{m\omega}{\pi\hbar}} \int_{x_0}^{\infty} e^{-\xi^2} dx = 2\sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{\hbar}{m\omega}} \int_{\xi_0}^{\infty} e^{-\xi^2} d\xi.$

Classically allowed region extends out to: $\frac{1}{2}m\omega^2 x_0^2 = E_0 = \frac{1}{2}\hbar\omega$, or $x_0 = \sqrt{\frac{\hbar}{m\omega}}$, so $\xi_0 = 1$.

$\therefore P = \frac{2}{\sqrt{\pi}} \int_1^{\infty} e^{-\xi^2} d\xi = 2(1 - F(\sqrt{2}))$ (in notation of CRC Table) $= \boxed{0.157}$.

Problem 2.16 $\underline{n=5}$: $j=1 \Rightarrow a_3 = \frac{-2(5-1)}{(1+1)(1+2)} a_1 = -\frac{4}{3}a_1$; $j=3 \Rightarrow a_5 = \frac{-2(5-3)}{(3+1)(3+2)} a_3 = -\frac{1}{5}a_3 = \frac{4}{15}a_1$

$j=5 \Rightarrow a_7 = 0$. So $H_5(\xi) = a_1\xi - \frac{4}{3}a_1\xi^3 + \frac{4}{15}a_1\xi^5 = \frac{a_1}{15}(15\xi - 20\xi^3 + 4\xi^5)$. By convention (p.41) the coefficient of ξ^5 is 2^5, so $a_1 = 15\cdot 8$, and $\boxed{H_5(\xi) = 120\xi - 160\xi^3 + 32\xi^5}$ (agrees with Ta

$\underline{n=6}$: $j=0 \Rightarrow a_2 = \frac{-2(6-0)}{(0+1)(0+2)} a_0 = -6a_0$; $j=2 \Rightarrow a_4 = \frac{-2(6-2)}{(2+1)(2+2)} a_2 = -\frac{2}{3}a_2 = 4a_0$;

$j=4 \Rightarrow a_6 = \frac{-2(6-4)}{(4+1)(4+2)} a_4 = -\frac{2}{15}a_4 = -\frac{8}{15}a_0$; $j=6 \Rightarrow a_8 = 0$. So $H_6(\xi) = a_0 - 6a_0\xi^2 + 4a_0\xi^4$

Coefficient of ξ^6 is 2^6, so $2^6 = -\frac{8}{15}a_0$, so $a_0 = -15\cdot 8 = -120$. $\boxed{H_6(\xi) = -120 + 720\xi^2 - 480\xi^4 + 64}$

Problem 2.17 (a) $1 = |A|^2 \int (\psi_0^* + \psi_1^*)(\psi_0 + \psi_1)dx = |A|^2 \left\{ \int |\psi_0|^2 dx + \int \psi_0^*\psi_1 dx + \int \psi_1^*\psi_0 dx + \int |\psi_1|^2 dx \right\}$

$= |A|^2(1 + 0 + 0 + 1) = 2|A|^2 \Rightarrow \boxed{A = 1/\sqrt{2}}$.

(b) $\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_0 e^{-iE_0t/\hbar} + \psi_1 e^{-iE_1t/\hbar}\right)$, but $E_0 = \frac{1}{2}\hbar\omega$, $E_1 = \frac{3}{2}\hbar\omega$, so

$= \frac{1}{\sqrt{2}}\left(\psi_0 e^{-i\omega t/2} + \psi_1 e^{-i3\omega t/2}\right) = \frac{1}{\sqrt{2}} e^{-i\omega t/2}\left(\psi_0 + \psi_1 e^{-i\omega t}\right)$. But

$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2}$ and $\psi_1 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2}} 2\xi e^{-\xi^2/2}$.

$$\therefore \boxed{\Psi(x,t) = \frac{1}{\sqrt{2}} e^{-i\omega t/2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2}\left(1 + \sqrt{2}\,\xi e^{-i\omega t}\right)}.$$

Now $|1 + \sqrt{2}\xi e^{-i\omega t}|^2 = (1 + \sqrt{2}\xi e^{+i\omega t})(1 + \sqrt{2}\xi e^{-i\omega t}) = 1 + \sqrt{2}\xi(e^{i\omega t} + e^{-i\omega t}) + 2\xi^2$

$= 1 + 2\xi^2 + 2\sqrt{2}\xi\cos\omega t$. So

$$\boxed{|\Psi(x,t)|^2 = \frac{1}{2}\sqrt{\frac{m\omega}{\pi\hbar}}\, e^{-\xi^2}(1 + 2\xi^2 + 2\sqrt{2}\xi\cos\omega t)}.$$

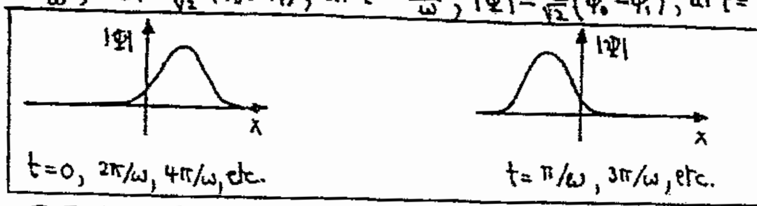
(c) $\langle x\rangle = \int x|\Psi|^2 dx = \frac{1}{2}\sqrt{\frac{m\omega}{\pi\hbar}}\,\frac{\hbar}{m\omega}\int_{-\infty}^{\infty} \xi(1 + 2\xi^2 + 2\sqrt{2}\xi\cos\omega t)e^{-\xi^2}d\xi = \frac{1}{2}\sqrt{\frac{\hbar}{\pi m\omega}}\, 2\sqrt{2}\cos\omega t \int_{-\infty}^{\infty} \xi^2 e^{-\xi^2} d\xi$

$= \sqrt{\frac{2\hbar}{\pi m\omega}}\cos\omega t\left(\frac{\sqrt{\pi}}{2}\right) = \boxed{\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)}$. Amplitude: $\boxed{\sqrt{\frac{\hbar}{2m\omega}}}$; angular frequency: $\boxed{\omega}$.

(d) $\langle p\rangle = m\frac{d\langle x\rangle}{dt} = m\sqrt{\frac{\hbar}{2m\omega}}(-\omega\sin\omega t) = \boxed{-\sqrt{\frac{\hbar m\omega}{2}}\sin(\omega t)}$. $\frac{d\langle p\rangle}{dt} = -\sqrt{\frac{\hbar m\omega}{2}}\,\omega\cos(\omega t)$. $V = \frac{1}{2}m\omega^2 x^2$

$\frac{dV}{dx} = m\omega^2 x$; $\left\langle -\frac{dV}{dx}\right\rangle = -m\omega^2\langle x\rangle = -m\omega^2\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t) = -\sqrt{\frac{\hbar m\omega}{2}}\,\omega\cos(\omega t) = \frac{d\langle p\rangle}{dt}$. ✓

(e) $\Psi(x,t) = \frac{1}{\sqrt{2}} e^{-i\omega t/2}\left(\psi_0 + \psi_1 e^{-i\omega t}\right)$. At $t=0$, $|\Psi| = \frac{1}{\sqrt{2}}(\psi_0+\psi_1)$; at $t=\frac{\pi}{\omega}$, $|\Psi| = \frac{1}{\sqrt{2}}(\psi_0-\psi_1)$; at $t=\frac{2\pi}{\omega}$, $|\Psi| = \frac{1}{\sqrt{2}}(\psi_0+\psi_1)$; at $t=\frac{3\pi}{\omega}$, $|\Psi| = \frac{1}{\sqrt{2}}(\psi_0-\psi_1)$; at $t=\frac{4\pi}{\omega}$, $|\Psi| = \frac{1}{\sqrt{2}}(\psi_0+\psi_1)$; etc.



PROBLEM 2.18 (a) $\frac{d}{d\xi}\left(e^{-\xi^2}\right) = -2\xi e^{-\xi^2}$; $\left(\frac{d}{d\xi}\right)^2 e^{-\xi^2} = \frac{d}{d\xi}\left(-2\xi e^{-\xi^2}\right) = \left(-2+4\xi^2\right)e^{-\xi^2}$; $\left(\frac{d}{d\xi}\right)^3 e^{-\xi^2} = \frac{d}{d\xi}\left[\left(-2+4\xi^2\right)e^{-\xi^2}\right] = \left[8\xi + \left(-2+4\xi^2\right)(-2\xi)\right]e^{-\xi^2} = \left(12\xi - 8\xi^3\right)e^{-\xi^2}$; $\left(\frac{d}{d\xi}\right)^4 e^{-\xi^2} = \frac{d}{d\xi}\left[\left(12\xi-8\xi^3\right)e^{-\xi^2}\right] = \left[12-24\xi^2 + \left(12\xi-8\xi^3\right)(-2\xi)\right]e^{-\xi^2} = \left(12-48\xi^2+16\xi^4\right)e^{-\xi^2}$.

$H_3(\xi) = -e^{\xi^2}\left(\frac{d}{d\xi}\right)^3 e^{-\xi^2} = \boxed{-12\xi+8\xi^3}$; $H_4(\xi) = e^{\xi^2}\left(\frac{d}{d\xi}\right)^4 e^{-\xi^2} = \boxed{12-48\xi^2+16\xi^4}$.

(b) $H_5 = 2\xi H_4 - 8H_3 = 2\xi\left(12-48\xi^2+16\xi^4\right) - 8\left(-12\xi+8\xi^3\right) = \boxed{120\xi - 160\xi^3 + 32\xi^5}$.

$H_6 = 2\xi H_5 - 10H_4 = 2\xi\left(120\xi - 160\xi^3 + 32\xi^5\right) - 10\left(12-48\xi^2+16\xi^4\right) = \boxed{-120+720\xi^2-480\xi^4+64\xi^6}$.

(c) $\frac{dH_5}{d\xi} = 120 - 480\xi^2 + 160\xi^4 = 10\left(12-48\xi^2+16\xi^4\right) = (2)(5)H_4$.

$\frac{dH_6}{d\xi} = 1440\xi - 1920\xi^3 + 384\xi^5 = 12\left(120\xi - 160\xi^3 + 32\xi^5\right) = (2)(6)H_5$.

(d) $\frac{d}{dz}\left(e^{-z^2+2z\xi}\right) = (-2z+2\xi)e^{-z^2+2z\xi}$; putting in $z=0$, $\boxed{H_0(\xi) = 2\xi}$.

$\left(\frac{d}{dz}\right)^2\left(e^{-z^2+2z\xi}\right) = \frac{d}{dz}\left[(-2z+2\xi)e^{-z^2+2z\xi}\right] = \left[-2+(-2z+2\xi)^2\right]e^{-z^2+2z\xi}$; putting $z=0$, $\boxed{H_1 = -2+4\xi^2}$.

$\left(\frac{d}{dz}\right)^3\left(e^{-z^2+2z\xi}\right) = \frac{d}{dz}\left\{\left[-2+(-2z+2\xi)^2\right]e^{-z^2+2z\xi}\right\} = \left\{2(-2z+2\xi)(-2) + \left[-2+(-2z+2\xi)^2\right](-2z+2\xi)\right\}e^{-z^2+2z\xi}$;

putting $z=0$, $H_2 = -8\xi + \left(-2+4\xi^2\right)(2\xi) = \boxed{-12\xi+8\xi^3}$.

PROBLEM 2.19 $Ae^{ikx}+Be^{-ikx} = A(\cos kx + i\sin kx) + B(\cos kx - i\sin kx) = (A+B)\cos kx + i(A-B)\sin kx = C\cos kx + D\sin kx$,

with $\boxed{C=A+B;\ D=i(A-B)}$. [Note: for a real function, $B=A^*$.]

$F\cos(kx+\alpha) = F\left[e^{i(kx+\alpha)} + e^{-i(kx+\alpha)}\right]/2 = \left(\frac{F}{2}e^{i\alpha}\right)e^{ikx} + \left(\frac{F}{2}e^{-i\alpha}\right)e^{-ikx} = Ae^{ikx}+Be^{-ikx}$,

so $A = \frac{F}{2}e^{i\alpha}$, $B = \frac{F}{2}e^{-i\alpha}$; $|A|^2+|B|^2 = \frac{F^2}{4}+\frac{F^2}{4} = \frac{F^2}{2}$; $F = \boxed{\sqrt{2\left(|A|^2+|B|^2\right)} = 2|A|}$. $\frac{A}{B} = e^{2i\alpha}$

$\boxed{\alpha = \frac{1}{2}\tan^{-1}\left[\mathrm{Im}(A/B)/\mathrm{Re}(A/B)\right]}$. $G\sin(kx+\beta) = G\left[e^{i(kx+\beta)} - e^{-i(kx+\beta)}\right]/2i = \left(\frac{G}{2i}e^{i\beta}\right)e^{ikx} + \left(-\frac{G}{2i}e^{-i\beta}\right)e^{-ikx}$.

$A = \frac{G}{2i}e^{i\beta}$, $B = -\frac{G}{2i}e^{-i\beta}$. $\boxed{G = 2|A|}$; $\frac{A}{B} = -e^{2i\beta}$; $\boxed{\beta = \frac{1}{2}\tan^{-1}\left[\mathrm{Im}(A/B)/\mathrm{Re}(A/B)\right] - \pi/2}$.

PROBLEM 2.20 (a) $f(x) = b_0 + \sum_{n=1}^{\infty} \frac{a_n}{2i}\left(e^{in\pi x/a} - e^{-in\pi x/a}\right) + \sum_{n=1}^{\infty} \frac{b_n}{2}\left(e^{in\pi x/a} + e^{-in\pi x/a}\right)$

$$= b_0 + \sum_{n=1}^{\infty}\left(\frac{a_n}{2i} + \frac{b_n}{2}\right)e^{in\pi x/a} + \sum_{n=1}^{\infty}\left(-\frac{a_n}{2i} + \frac{b_n}{2}\right)e^{-in\pi x/a}.$$

Let $\boxed{c_0 \equiv b_0\ ;\ c_n \equiv \frac{1}{2}\left(\frac{a_n}{i} + b_n\right), \text{ for } n = 1,2,3,\dots;\ c_n \equiv \frac{1}{2}\left(-\frac{a_{-n}}{i} + b_{-n}\right), \text{ for } n = -1,-2,-3,\dots}$

Then $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/a}$. QED.

(b) $\int_{-a}^{a} f(x)\, e^{-im\pi x/a} dx = \sum_{n=-\infty}^{\infty} c_n \int_{-a}^{a} e^{i(n-m)\pi x/a} dx$

But for $n \neq m$, $\int_{-a}^{a} e^{i(n-m)\pi x/a} dx = \frac{e^{i(n-m)\pi x/a}}{i(n-m)\pi/a}\Big|_{-a}^{a} = \frac{e^{i(n-m)\pi} - e^{-i(n-m)\pi}}{i(n-m)\pi/a} = \frac{(-1)^{n-m} - (-1)^{n-m}}{i(n-m)\pi/a} = 0$,

whereas for $n = m$, $\int_{-a}^{a} e^{i(n-m)\pi x/a} dx = \int_{-a}^{a} dx = 2a$. $\left[\text{i.e. } \int_{-a}^{a} e^{i(n-m)\pi x/a} dx = 2a\,\delta_{nm}\right]$.

So all terms except $n = m$ are zero, and $\int_{-a}^{a} f(x)\, e^{-im\pi x/a} = 2a\, c_m$, so

$$c_n = \frac{1}{2a}\int_{-a}^{a} f(x)\, e^{-in\pi x/a} dx. \quad \text{QED.}$$

(c) $f(x) = \sum_{n=-\infty}^{\infty} \sqrt{\frac{\pi}{2}}\frac{1}{a} F(k) e^{ikx} = \frac{1}{\sqrt{2\pi}} \sum F(k) e^{ikx} \Delta k$, where $\boxed{\Delta k = \frac{\pi}{a}}$ is the increment in k from n to $(n+1)$. $F(k) = \sqrt{\frac{2}{\pi}}\, a\, \frac{1}{2a}\int_{-a}^{a} f(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} f(x) e^{-ikx} dx$.

(d) As $a \to \infty$, k becomes a continuous variable, $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k) e^{ikx} dk\ ;\ F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-ikx} dx$.

PROBLEM 2.21 (a) $1 = \int |\Psi|^2 dx = |A|^2 (2a) \Rightarrow \boxed{A = \frac{1}{\sqrt{2a}}}$.

(b) $\phi(k) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2a}}\int_{-a}^{a} e^{-ikx} dx = \frac{1}{2\sqrt{\pi a}}\left(\frac{e^{-ikx}}{-ik}\right)\Big|_{-a}^{a} = \frac{1}{\sqrt{\pi a}\,k}\left(\frac{e^{-ika} - e^{ika}}{-2i}\right) = \boxed{\frac{\sin(ka)}{\sqrt{\pi a}\ k}}$.

(c) Very small $a \Rightarrow$ highly localized in position; $\sin ka \approx ka$, so $\phi(k) \approx \sqrt{\frac{a}{\pi}}$ = constant. So a very broad range of k (hence $\lambda = 2\pi/k$) is represented. Well localized in position $\Rightarrow$ ill-defined wavelength. Very large $a \Rightarrow$ broad spread in position, $\phi(k)$ is sharply peaked about $k = 0$. Well-defined wavelength, non-localized position.

[The "width" of the $\phi(k)$ graph is given by $\sin(ka) = 0 \Rightarrow ka = \pm\pi$; $\Delta k = \frac{2\pi}{a}$, so the spread in k is inversely proportional to the spread in x.]

PROBLEM 2.22 (a) $1=|A|^2\int_{-\infty}^{\infty} e^{-2ax^2}dx = |A|^2\sqrt{\frac{\pi}{2a}}$; $\boxed{A=\left(\frac{2a}{\pi}\right)^{1/4}}$.

(b) $\phi(k)=\frac{1}{\sqrt{2\pi}}A\int_{-\infty}^{\infty}e^{-ax^2}e^{-ikx}dx$

$$\boxed{\int_{-\infty}^{\infty} e^{-(ax^2+bx)}dx = \int_{-\infty}^{\infty} e^{-y^2+(b^2/4a)}\frac{1}{\sqrt{a}}dy = \frac{1}{\sqrt{a}}e^{b^2/4a}\int_{-\infty}^{\infty}e^{-y^2}dy = \sqrt{\pi/a}\,e^{b^2/4a}.}$$

$$=\frac{1}{\sqrt{2\pi}}\left(\frac{2a}{\pi}\right)^{1/4}\sqrt{\frac{\pi}{a}}\,e^{-k^2/4a}$$

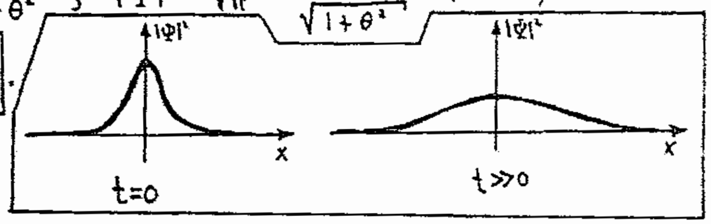
$$=\frac{1}{(2\pi a)^{1/4}}e^{-k^2/4a}.$$

$$\Psi(x,t)=\frac{1}{\sqrt{2\pi}}\frac{1}{(2\pi a)^{1/4}}\int_{-\infty}^{\infty}\underbrace{e^{-k^2/4a}\,e^{i(kx-\hbar k^2t/2m)}}_{e^{-[(\frac{1}{4a}+i\hbar t/2m)k^2-ixk]}}dk = \frac{1}{\sqrt{2\pi}(2\pi a)^{1/4}}\frac{\sqrt{\pi}}{\sqrt{\frac{1}{4a}+i\hbar t/2m}}\,e^{-x^2/4(\frac{1}{4a}+i\hbar t/2m)}$$

$$=\boxed{\left(\frac{2a}{\pi}\right)^{1/4}\frac{e^{-ax^2/(1+2i\hbar at/m)}}{\sqrt{1+2i\hbar at/m}}}.$$

(c) Let $\theta=2\hbar at/m$. Then $|\Psi|^2=\sqrt{\frac{2a}{\pi}}\dfrac{e^{-ax^2/(1+i\theta)}\,e^{-ax^2/(1-i\theta)}}{\sqrt{(1+i\theta)(1-i\theta)}}$. The exponent is

$-\frac{ax^2}{(1+i\theta)}-\frac{ax^2}{(1-i\theta)}=-ax^2\frac{(1-i\theta+1+i\theta)}{(1+i\theta)(1-i\theta)}=\frac{-2ax^2}{1+\theta^2}$; $|\Psi|^2=\sqrt{\frac{2a}{\pi}}\dfrac{e^{-2ax^2/(1+\theta^2)}}{\sqrt{1+\theta^2}}$. Or, with $w\equiv\sqrt{\frac{a}{1+\theta^2}}$, $\boxed{|\Psi|^2=\sqrt{\frac{2}{\pi}}\,we^{-2w^2x^2}}$.



As t increases, the graph of $|\Psi|^2$ flattens out and broadens.

(d) $\langle x\rangle=\int_{-\infty}^{\infty}x|\Psi|^2dx=\boxed{0}$ (odd integrand); $\langle p\rangle=m\frac{d\langle x\rangle}{dt}=\boxed{0}$. $\langle x^2\rangle=\sqrt{\frac{2}{\pi}}w\int_{-\infty}^{\infty}x^2e^{-2w^2x^2}dx=$

$=\sqrt{\frac{2}{\pi}}w\frac{1}{4w^2}\sqrt{\frac{\pi}{2w^2}}=\boxed{\frac{1}{4w^2}}$. $\langle p^2\rangle=-\hbar^2\int_{-\infty}^{\infty}\Psi^*\frac{d^2\Psi}{dx^2}dx$. Write

$\Psi=Be^{-bx^2}$, where $B\equiv\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+i\theta}}$ and $b\equiv\frac{a}{1+i\theta}$. $\frac{d^2\Psi}{dx^2}=B\frac{d}{dx}\left(-2bxe^{-bx^2}\right)$

$=-2bB(1-2bx^2)e^{-bx^2}$. $\Psi^*\frac{d^2\Psi}{dx^2}=-2b|B|^2(1-2bx^2)e^{-(b+b^*)x^2}$; $b+b^*=\frac{a}{1+i\theta}+\frac{a}{1-i\theta}=\frac{2a}{1+\theta^2}=2w^2$.

$|B|^2=\sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\theta^2}}=\sqrt{\frac{2}{\pi}}w$. So $\Psi^*\frac{d^2\Psi}{dx^2}=-2b\sqrt{\frac{2}{\pi}}w(1-2bx^2)e^{-2w^2x^2}$.

$\langle p^2\rangle=2b\hbar^2\sqrt{\frac{2}{\pi}}w\int_{-\infty}^{\infty}(1-2bx^2)e^{-2w^2x^2}dx=2b\hbar^2\sqrt{\frac{2}{\pi}}w\left\{\sqrt{\frac{\pi}{2w^2}}-2b\frac{1}{4w^2}\sqrt{\frac{\pi}{2w^2}}\right\}=2b\hbar^2\left(1-\frac{b}{2w^2}\right)$. But

$1-\frac{b}{2w^2}=1-\left(\frac{a}{1+i\theta}\right)\left(\frac{1+\theta^2}{2a}\right)=1-\frac{(1-i\theta)}{2}=\frac{1+i\theta}{2}=\frac{a}{2b}$, so $\langle p^2\rangle=2b\hbar^2\frac{a}{2b}=\boxed{\hbar^2a}$. $\boxed{\sigma_x=\frac{1}{2w}}$;

$\boxed{\sigma_p=\hbar\sqrt{a}}$. (e) $\sigma_x\sigma_p=\frac{1}{2w}\hbar\sqrt{a}=\frac{\hbar}{2}\sqrt{1+\theta^2}=\frac{\hbar}{2}\sqrt{1+(2\hbar at/m)^2}\geq\frac{\hbar}{2}$. ✓ Closest at $\boxed{t=0}$, at which time it is right at the uncertainty limit.

PROBLEM 2.23 (a) $(-2)^3 - 3(-2)^2 + 2(-2) - 1 = -8 - 12 - 4 - 1 = \boxed{-25}$.

(b) $\cos(3\pi) + 2 = -1 + 2 = \boxed{1}$. (c) $\boxed{0}$ ($x = 2$ is outside the domain of integration).

PROBLEM 2.24 (a) $\int_{-\infty}^{\infty} f(x)\,\delta(cx)\,dx$. Let $y \equiv cx$, so $dx = \frac{1}{c}dy$. $\left\{\begin{array}{l}\text{If } c>0,\ y: -\infty \to \infty. \\ \text{If } c<0,\ y: \infty \to -\infty.\end{array}\right\}$

$$\begin{cases} = \frac{1}{c}\int_{-\infty}^{\infty} f(y/c)\,\delta(y)\,dy = \frac{1}{c} f(0) \quad (c>0);\ \text{or} \\ = \frac{1}{c}\int_{\infty}^{-\infty} f(y/c)\,\delta(y)\,dy = -\frac{1}{c}\int_{-\infty}^{\infty} f(y/c)\,\delta(y)\,dy = -\frac{1}{c} f(0) \quad (c<0). \end{cases}$$

In either case, $\int_{-\infty}^{\infty} f(x)\,\delta(cx)\,dx = \frac{1}{|c|} f(0) = \int_{-\infty}^{\infty} f(x)\frac{1}{|c|}\delta(x)\,dx$. So $\delta(cx) = \frac{1}{|c|}\delta(x)$. ✓

(b) $\int_{-\infty}^{\infty} f(x)\frac{d\theta}{dx}\,dx = f\theta\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{df}{dx}\theta\,dx$ (integration by parts) $= f(\infty) - \int_{0}^{\infty}\frac{df}{dx}\,dx = f(\infty) - f(\infty) + f$

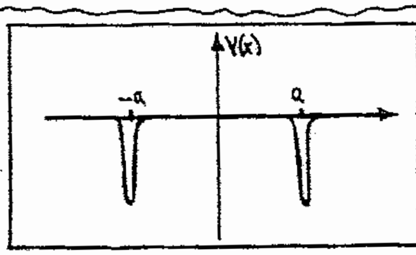
$= f(0) = \int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx$. So $\frac{d\theta}{dx} = \delta(x)$. ✓ [Makes sense: the θ function is cons

(so derivative is zero) except at $x=0$, where the derivative is infinite.]

PROBLEM 2.25 Put $f(x) = \delta(x)$ into eq. [2.85]: $F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)\,e^{-ikx}\,dx = \boxed{\frac{1}{\sqrt{2\pi}}}$.

$\therefore f(x) = \delta(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{ikx}\,dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}\,dk$. QED.

PROBLEM 2.26 (a)



(b) From Problem 2.1(c) the solutions are eve[n or] odd. Look first for even solutions:

$$\psi(x) = \begin{cases} Ae^{-\kappa x} & (x>a), \\ B(e^{\kappa x} + e^{-\kappa x}) & (-a<x<a), \\ Ae^{\kappa x} & (x<-a). \end{cases}$$

Continuity at a: $Ae^{-\kappa a} = B(e^{\kappa a} + e^{-\kappa a})$, or $\underline{A = B(e^{2\kappa a} + 1)}$. Discontinuous derivative at a: $\Delta\frac{d\psi}{dx} = -$

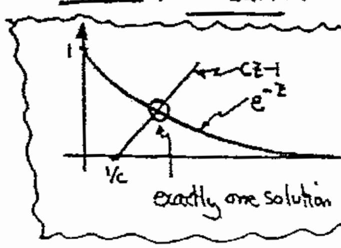
$\therefore -\kappa Ae^{-\kappa a} - [B(\kappa e^{\kappa a} - \kappa e^{-\kappa a})] = -\frac{2m\alpha}{\hbar^2}Ae^{-\kappa a} \Rightarrow A + B(e^{2\kappa a} - 1) = \frac{2m\alpha}{\hbar^2\kappa}A$; or

$B(e^{2\kappa a} - 1) = A\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) = B(e^{2\kappa a} + 1)\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) \Rightarrow e^{2\kappa a} - 1 = e^{2\kappa a}\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) + \frac{2m\alpha}{\hbar^2\kappa} - 1$.

$1 = \frac{2m\alpha}{\hbar^2\kappa} - 1 + \frac{2m\alpha}{\hbar^2\kappa}e^{-2\kappa a}$; $\frac{\hbar^2\kappa}{m\alpha} = 1 + e^{-2\kappa a}$, or $\boxed{e^{-2\kappa a} = \frac{\hbar^2\kappa}{m\alpha} - 1}$. This is a transce[ndental]

equation for κ (and hence for E). I'll solve it graphically: let $\underline{z \equiv 2\kappa a}$, $\underline{c \equiv \frac{\hbar^2}{2am\alpha}}$, so

$\underline{e^{-z} = cz - 1}$. Plot both sides and look for intersections:



From the graph, noting that c and z are both positive, we see that there is one (and only one) solution (for even ψ). If $\alpha = \hbar^2/2ma$, so $c = 1$, calculator gives $z = 1.278$, so

$\kappa^2 = -\frac{2mE}{\hbar^2} = \frac{z^2}{(2a)^2} \Rightarrow E = -\frac{(1.278)^2}{8}\left(\frac{\hbar^2}{ma^2}\right) = -0.204\left(\frac{\hbar^2}{ma^2}\right)$.

Odd solutions:

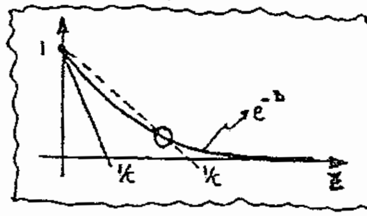
$$\psi(x) = \begin{cases} Ae^{-\kappa x} & (x>a) \\ B(e^{\kappa x}-e^{-\kappa x}) & (-a<x<a) \\ -Ae^{\kappa x} & (x<-a) \end{cases}$$

Continuity at a: $Ae^{-\kappa a} = B(e^{\kappa a}-e^{-\kappa a})$, or $\underline{A = B(e^{2\kappa a}-1)}$.

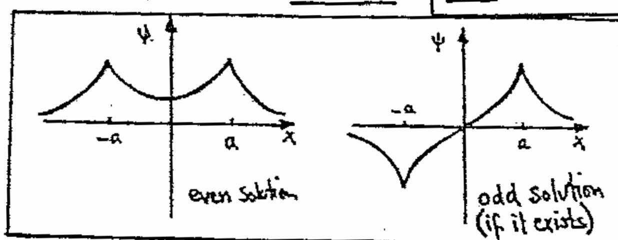
Discontinuity in ψ': $-A\kappa e^{-\kappa a} - B(\kappa e^{\kappa a}+\kappa e^{-\kappa a}) = -\frac{2m\alpha}{\hbar^2}Ae^{-\kappa a} \Rightarrow B(e^{2\kappa a}+1) = A\left(\frac{2m\alpha}{\hbar^2\kappa}-1\right)$, so

$e^{2\kappa a}+1 = (e^{2\kappa a}-1)\left(\frac{2m\alpha}{\hbar^2\kappa}-1\right) = e^{2\kappa a}\left(\frac{2m\alpha}{\hbar^2\kappa}-1\right) - \frac{2m\alpha}{\hbar^2\kappa}+1$, so $1 = \frac{2m\alpha}{\hbar^2\kappa}-1-\frac{2m\alpha}{\hbar^2\kappa}e^{-2\kappa a}$.

$\frac{\hbar^2\kappa}{m\alpha} = 1-e^{-2\kappa a}$, or $\boxed{e^{-2\kappa a} = 1-\frac{\hbar^2\kappa}{m\alpha}}$, or $\underline{e^{-z} = 1-cz}$.



This time there may or may not be a solution. Both graphs have their y-intercepts at 1, but if c is too large (α too small), there may be no intersection (solid line), whereas if c is smaller (dashed line) there will be. [$z=0 \Rightarrow \kappa=0$ is not a solution, since ψ is then non-normalizable.] The slope of e^{-z} (at $z=0$) is -1; the slope of $(1-cz)$ is $-c$. So there is an odd solution $\Leftrightarrow$ $\underline{c<1}$, or $\alpha > \hbar^2/2ma$. Conclusion: $\boxed{\text{One bound state if } \alpha \le \hbar^2/2ma;\ \text{two if } \alpha > \hbar^2/2ma.}$



$\alpha = \frac{\hbar^2}{ma} \Rightarrow c = \frac{1}{2}$. Even: $e^{-z} = \frac{1}{2}z-1 \Rightarrow z = 2.21772$

Odd: $e^{-z} = 1-\frac{1}{2}z \Rightarrow z = 1.59362$

$\boxed{E = -0.615(\hbar^2/ma^2);\ E = -0.317(\hbar^2/ma^2)}$.

$\alpha = \frac{\hbar^2}{4ma} \Rightarrow c = 2$. Only Even: $e^{-z} = 2z-1 \Rightarrow z = 0.738835$

$\boxed{E = -0.0682(\hbar^2/ma^2)}$

PROBLEM 2.27

$$\psi = \begin{cases} Ae^{ikx}+Be^{-ikx} & (x<-a) \\ Ce^{ikx}+De^{-ikx} & (-a<x<a) \\ Fe^{ikx} & (x>a) \end{cases}.$$

Impose boundary conditions:

(1) continuity at $-a$: $Ae^{-ika}+Be^{ika} = Ce^{-ika}+De^{ika}$, or $\underline{\beta A+B = \beta C+D}$. $\underline{\beta \equiv e^{-2ika}}$.

(2) continuity at $+a$: $Ce^{ika}+De^{-ika} = Fe^{ika}$, or $\underline{F = C+\beta D}$.

(3) disc. in ψ' at $-a$: $ik(Ce^{-ika}-De^{ika}) - ik(Ae^{-ika}-Be^{ika}) = -\frac{2m\alpha}{\hbar^2}(Ae^{-ika}+Be^{ika})$, or

$$\underline{\beta C - D = \beta(\gamma+1)A + B(\gamma-1)}. \qquad \underline{\gamma \equiv i2m\alpha/\hbar^2 k}.$$

(4) disc. in ψ' at $+a$: $ikFe^{ika} - ik(Ce^{ika}-De^{-ika}) = -\frac{2m\alpha}{\hbar^2}(Fe^{ika})$, or $\underline{C-\beta D = (1-\gamma)F}$.

To solve for C and D, $\begin{cases} \text{add (2) and (4): } 2C = F+(1-\gamma)F \Rightarrow \quad 2C = (2-\gamma)F. \\ \text{subtract (2) and (4): } 2\beta D = F-(1-\gamma)F \Rightarrow \quad 2D = (\gamma/\beta)F. \end{cases}$

$\begin{cases} \text{add (1) and (3): } 2\beta C = \beta A+B+\beta(\gamma+1)A+B(\gamma-1) \Rightarrow \quad 2C = (\gamma+2)A+(\gamma/\beta)B. \\ \text{subtract (1) and (3): } 2D = \beta A+B-\beta(\gamma+1)A-B(\gamma-1) \Rightarrow \quad 2D = -\gamma\beta A+(2-\gamma)B. \end{cases}$

Equate the two expressions for $2C$: $(2-\gamma)F = (\gamma+2)A+(\gamma/\beta)B$.
" " " " " $2D$: $(\gamma/\beta)F = -\gamma\beta A+(2-\gamma)B$.
Solve these for F and B, in terms of A.

Multiply first by $\beta(2-\gamma)$, second by γ, and subtract:

$$\beta(2-\gamma)^2 F = \beta(4-\gamma^2)A + \gamma(2-\gamma)B$$
$$(\gamma^2/\beta)F = -\beta\gamma^2 A + \gamma(2-\gamma)B$$

$$[\beta(2-\gamma)^2 - \gamma^2/\beta]F = \beta[4-\gamma^2+\gamma^2]A = 4\beta A \Rightarrow \frac{F}{A} = \frac{4}{(2-\gamma)^2 - \gamma^2/\beta^2}.$$

Let $g \equiv i/\gamma = \frac{\hbar^2 k}{2m\alpha}$; $\phi \equiv 4ka$, so $\gamma = \frac{i}{g}$, $\beta^2 = e^{-i\phi}$. Then: $\frac{F}{A} = \frac{4g^2}{(2g-i)^2 + e^{i\phi}}$.

Denominator: $4g^2 - 4ig - 1 + \cos\phi + i\sin\phi = (4g^2 - 1 + \cos\phi) + i(\sin\phi - 4g)$.

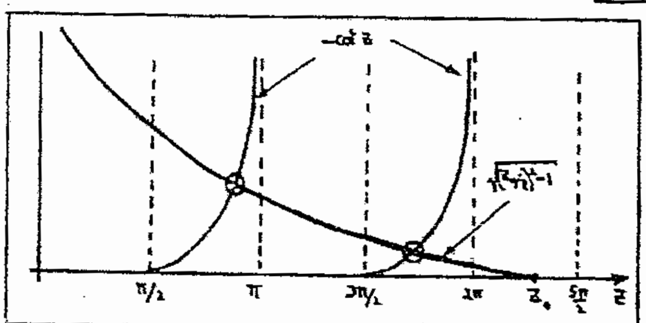
$$|\text{Denominator}|^2 = (4g^2-1+\cos\phi)^2 + (\sin\phi - 4g)^2 = 16g^4 + 1 + \cos^2\phi - 8g^2 - 2\cos\phi + 8g^2\cos\phi + \sin^2\phi - 8g\sin\phi$$
$$= 16g^4 + 8g^2 + 2 + (8g^2-2)\cos\phi - 8g\sin\phi$$

$$T = \left|\frac{F}{A}\right|^2 = \boxed{\frac{8g^4}{(8g^4+4g^2+1) + (4g^2-1)\cos\phi - 4g\sin\phi}, \text{ where } g \equiv \frac{\hbar^2 k}{2m\alpha} \text{ and } \phi \equiv 4ka.} \quad \boxed{R = 1 - T}.$$

PROBLEM 2.28 In place of [2.133], we have:
$$\psi(x) = \begin{cases} Fe^{-\kappa x} & (x>a) \\ D\sin(lx) & (0<x<a) \\ -\psi(-x) & (x<0) \end{cases}$$

Continuity of ψ: $Fe^{-\kappa a} = D\sin(la)$; continuity of ψ': $-F\kappa e^{-\kappa a} = Dl\cos(la)$. Divide: $-\kappa = l\cot$

$-\kappa a = la\cot(la) \Rightarrow \sqrt{z_0^2 - z^2} = -z\cot z$, or $\boxed{-\cot z = \sqrt{(z_0/z)^2 - 1}}$.



Wide, deep well: intersections are at $\pi, 2\pi, 3\pi,$ Same as [2.139], but now for n even. This in the rest of the states for the infinite sq

Shallow, narrow well: If $z_0 < \pi/2$ there is no bound state. The corresponding condition on

$$\boxed{V_0 < \frac{\pi^2\hbar^2}{8ma^2}}.$$

PROBLEM 2.29. $1 = 2\int_0^\infty |\psi|^2 dx = 2\left\{|D|^2\int_0^a \cos^2 lx\, dx + |F|^2\int_a^\infty e^{-2\kappa x}dx\right\}$

$$= 2\left\{|D|^2\left[\frac{x}{2} + \frac{1}{4l}\sin 2lx\right]_0^a + |F|^2\left[-\frac{1}{2\kappa}e^{-2\kappa x}\right]_a^\infty\right\} = 2\left\{|D|^2\left(\frac{a}{2} + \frac{\sin 2la}{4l}\right) + |F|^2\frac{e^{-2\kappa a}}{2\kappa}\right\}$$

But $F = De^{\kappa a}\cos la$ (eq. 2.134), so $1 = |D|^2\left[a + \frac{\sin(2la)}{2l} + \frac{\cos^2 la}{\kappa}\right]$. But $\kappa = l\tan(la)$ [eq.

So $1 = |D|^2\left[a + \frac{2\sin la\cos la}{2l} + \frac{\cos^3 la}{l\sin la}\right] = |D|^2\left[a + \frac{\cos la}{l\sin la}(\sin^2 la + \cos^2 la)\right] = |D|^2\left(a + \frac{1}{l\tan la}\right) = |D|$

$$\boxed{D = \frac{1}{\sqrt{a + 1/\kappa}}} \quad \boxed{F = \frac{e^{\kappa a}\cos la}{\sqrt{a+1/\kappa}}}.$$

PROBLEM 2.30 [2.137] $\Rightarrow z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$. We want α = area of potential $= 2aV_0$, held constant

$\therefore V_0 = \alpha/2a$; $z_0 = \frac{a}{\hbar}\sqrt{2m\frac{\alpha}{2a}} = \frac{1}{\hbar}\sqrt{m\alpha a} \to 0$. So z_0 is small.

So the intersection in Figure 2.13 occurs at very small z. Solve [2.138] for very small z, by expanding $\tan z$: $\tan z \cong z = \sqrt{(z_0/z)^2-1} \Rightarrow z^2 = \frac{z_0^2}{z^2} - 1 \Rightarrow z^4+z^2-z_0^2=0.$

$z^2 = \frac{-1 \pm \sqrt{1+4z_0^2}}{2} \cong \frac{1}{2}\left[-1+\left(1+2z_0^2 - \frac{1}{8}16z_0^4\right)\right] = \frac{1}{2}\left(2z_0^2-2z_0^4\right) = z_0^2 - z_0^4$. (Need + sign because z is small.) $z_0^2 - z^2 \cong z_0^4$. But $\kappa^2a^2 = z_0^2 - z^2$, so $\kappa a = z_0^2$. But we found that $z_0 \cong \frac{1}{\hbar}\sqrt{m\alpha a}$ here, so $\kappa a = \frac{1}{\hbar^2} m\alpha a$, or $\kappa = \frac{m\alpha}{\hbar^2}$. (At this point the a's have cancelled, and we can go to the limit $a \to 0$.) $\sqrt{-2mE}/\hbar = m\alpha/\hbar^2 \Rightarrow -2mE = m^2\alpha^2/\hbar^2$. $\boxed{E = -\frac{m\alpha^2}{2\hbar^2}}$ (which agrees with [2.111]).

In [2.151], $V_0 >> E \Rightarrow T^{-1} \cong 1 + \frac{V_0^2}{4EV_0}\sin^2\left(\frac{2a}{\hbar}\sqrt{2mV_0}\right)$. $V_0 = \frac{\alpha}{2a}$, so argument of $\sin$ is small, so replace $\sin x$ by x:

$$T^{-1} \cong 1 + \frac{V_0}{4E}\left(\frac{2a}{\hbar}\right)^2 2mV_0 = 1 + (2aV_0)^2\frac{m}{2\hbar^2 E}.$$ But $2aV_0 = \alpha$, so $T^{-1} = 1 + \frac{m\alpha^2}{2\hbar^2 E}$, in agreement with [2.123].

<u>PROBLEM 2.31</u> Multiply [2.147] by $\sin la$, [2.148] by $\frac{1}{l}\cos la$, and add:

$$\left.\begin{aligned} C\sin^2 la + D\sin la\cos la &= Fe^{ika}\sin la \\ C\cos^2 la - D\sin la\cos la &= \frac{ik}{l}Fe^{ika}\cos la \end{aligned}\right\} \quad C = Fe^{ika}\left[\sin la + \frac{ik}{l}\cos la\right].$$

Multiply [2.147] by $\cos la$, [2.148] by $\frac{1}{l}\sin la$, and subtract:

$$\left.\begin{aligned} C\sin la\cos la + D\cos^2 la &= Fe^{ika}\cos la \\ C\sin la\cos la - D\sin^2 la &= \frac{ik}{l}Fe^{ika}\sin la \end{aligned}\right\} \quad D = Fe^{ika}\left[\cos la - \frac{ik}{l}\sin la\right].$$

Put these into [2.145]:

$$(1)\quad Ae^{-ika} + Be^{ika} = -Fe^{ika}\left[\sin la + \frac{ik}{l}\cos la\right]\sin la + Fe^{ika}\left[\cos la - \frac{ik}{l}\sin la\right]\cos la$$
$$= Fe^{ika}\left[\cos^2 la - \frac{ik}{l}\sin la\cos la - \sin^2 la - \frac{ik}{l}\sin la\cos la\right]$$
$$= Fe^{ika}\left[\cos(2la) - \frac{ik}{l}\sin(2la)\right].$$ Likewise, from [2.146]:

$$(2)\quad Ae^{-ika} - Be^{ika} = -\frac{il}{k}Fe^{ika}\left[\left(\sin la + \frac{ik}{l}\cos la\right)\cos la + \left(\cos la - \frac{ik}{l}\sin la\right)\sin la\right]$$
$$= -\frac{il}{k}Fe^{ika}\left[\sin la\cos la + \frac{ik}{l}\cos^2 la + \sin la\cos la - \frac{ik}{l}\sin^2 la\right]$$
$$= -\frac{il}{k}Fe^{ika}\left[\sin(2la) + \frac{ik}{l}\cos(2la)\right] = Fe^{ika}\left[\cos(2la) - \frac{il}{k}\sin(2la)\right].$$

Add (1) and (2): $2Ae^{-ika} = Fe^{ika}\left[2\cos(2la) - i\left(\frac{k}{l}+\frac{l}{k}\right)\sin(2la)\right]$, or:

$$F = \frac{e^{-2ika}A}{\cos(2la) - i\frac{\sin(2la)}{2kl}(k^2+l^2)}$$ (confirming [2.150]); Subtract (2) from (1):

$$2Be^{ika} = Fe^{ika}\left[i\left(\frac{l}{k}-\frac{k}{l}\right)\sin(2la)\right] \Rightarrow B = i\,\frac{\sin(2la)}{2kl}\left(l^2-k^2\right)F \text{ (confirming [2.149]).}$$

$$T^{-1} = \left|\frac{A}{F}\right|^2 = \left|\cos(2la) - i\,\frac{\sin(2la)}{2kl}\left(k^2+l^2\right)\right|^2 = \cos^2(2la) + \frac{\sin^2(2la)}{(2kl)^2}\left(k^2+l^2\right)^2. \text{ But } \cos^2 2la = 1-\sin^2 \ldots$$

so
$$T^{-1} = 1 + \sin^2(2la)\underbrace{\left[\frac{(k^2+l^2)^2}{(2lk)^2} - 1\right]}_{\frac{1}{(2kl)^2}\left[k^4+2k^2l^2+l^4-4k^2l^2\right] = \frac{1}{(2kl)^2}\left[k^4-2k^2l^2+l^4\right] = \frac{(k^2-l^2)^2}{(2kl)^2}} = 1 + \frac{(k^2-l^2)^2}{(2kl)^2}\sin^2(2la).$$

But $k = \frac{\sqrt{2mE}}{\hbar}$; $l = \frac{\sqrt{2m(E+V_0)}}{\hbar}$; so $(2la) = \frac{2a}{\hbar}\sqrt{2m(E+V_0)}$; $k^2 - l^2 = \frac{1}{\hbar^2}2m(E-E-V_0) = -\frac{2mV_0}{\hbar^2}$

$$\frac{(k^2-l^2)^2}{(2kl)^2} = \frac{\left(\frac{2m}{\hbar^2}\right)^2 V_0^2}{4\left(\frac{2m}{\hbar^2}\right)^2 E(E+V_0)} = \frac{V_0^2}{4E(E+V_0)}. \quad \therefore T^{-1} = 1 + \frac{V_0^2}{4E(E+V_0)}\sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right), \text{ confirming [}\ldots$$

$$R = |B/A|^2 = \frac{|B|^2}{|F|^2}\frac{|F|^2}{|A|^2} = T\left|\frac{B}{F}\right|^2. \text{ From [2.149], } \left|\frac{B}{F}\right|^2 = \frac{\sin^2(2la)}{(2kl)^2}\left(l^2-k^2\right)^2 = \frac{V_0^2}{4E(E+V_0)}\sin^2\left(\frac{2a}{\hbar}\sqrt{2m\ldots}\right)$$

$$\therefore \boxed{R = \frac{\frac{V_0^2}{4E(E+V_0)}\sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)}{1 + \frac{V_0^2}{4E(E+V_0)}\sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)}}.$$

Let $\theta \equiv \frac{V_0^2}{4E(E+V_0)}\sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)$. Then

$T = \frac{1}{1+\theta}$, $R = \frac{\theta}{1+\theta}$, and $T + R = \frac{1+\theta}{1+\theta} = 1$. ✓

Problem 2.32 $E < V_0$.

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx} & (x<-a) \\ Ce^{\kappa x} + De^{-\kappa x} & (-a<x<a) \\ Fe^{ikx} & (x>a) \end{cases} \qquad k = \sqrt{2mE}/\hbar; \quad \kappa = \sqrt{2m(V_0-E)}/\hbar$$

(1) continuity of ψ at $-a$: $Ae^{-ika} + Be^{ika} = Ce^{-\kappa a} + De^{\kappa a}$

(2) continuity of ψ' at $-a$: $ik\left(Ae^{-ika} - Be^{ika}\right) = \kappa\left(Ce^{-\kappa a} - De^{\kappa a}\right)$

$\Rightarrow 2Ae^{-ika} = \left(1 - i\frac{\kappa}{k}\right)Ce^{-\kappa a} + \left(1 + i\frac{\kappa}{k}\right)De^{\kappa a}$

(3) continuity of ψ at $+a$: $Ce^{\kappa a} + De^{-\kappa a} = Fe^{ika}$

(4) continuity of ψ' at $+a$: $\kappa\left(Ce^{\kappa a} - De^{-\kappa a}\right) = ikFe^{ika}$

$\Rightarrow 2Ce^{\kappa a} = \left(1 + \frac{ik}{\kappa}\right)Fe^{ika}$; $2De^{-\kappa a} = \left(1 - \frac{ik}{\kappa}\right)Fe^{ika}$

$$2Ae^{-ika} = \left(1 - \frac{i\kappa}{k}\right)\left(1 + \frac{ik}{\kappa}\right)Fe^{ika}\frac{e^{-2\kappa a}}{2} + \left(1 + i\frac{\kappa}{k}\right)\left(1 - \frac{ik}{\kappa}\right)Fe^{ika}\frac{e^{2\kappa a}}{2}$$

$$= \frac{Fe^{ika}}{2}\left[\left(1 + i\left(\frac{k}{\kappa} - \frac{\kappa}{k}\right) + 1\right)e^{-2\kappa a} + \left(1 + i\left(\frac{\kappa}{k} - \frac{k}{\kappa}\right) + 1\right)e^{2\kappa a}\right]$$

$$= \frac{1}{2}Fe^{ika}\left[2\left(e^{-2\kappa a} + e^{2\kappa a}\right) + i\,\frac{(\kappa^2 - k^2)}{k\kappa}\left(e^{2\kappa a} - e^{-2\kappa a}\right)\right]. \text{ But } \sinh x = \frac{e^x - e^{-x}}{2},\ \cosh x = \frac{e^x + e^{-x}}{2} \text{ so}$$

$$= \frac{1}{2}Fe^{ika}\left[4\cosh(2\kappa a) + i\,\frac{(\kappa^2 - k^2)}{k\kappa}2\sinh(2\kappa a)\right] = 2Fe^{ika}\left[\cosh(2\kappa a) + i\,\frac{(\kappa^2 - k^2)}{2k\kappa}\sinh(2\kappa a)\right].$$

$T^{-1} = \left|\frac{A}{F}\right|^2 = \cosh^2(2\kappa a) + \frac{(\kappa^2 - k^2)^2}{(2\kappa k)^2}\sinh^2(2\kappa a)$. But $\cosh^2 = 1 + \sinh^2$, so

$$T^{-1} = 1 + \underbrace{\left[1 + \frac{(\kappa^2 - k^2)^2}{(2\kappa k)^2}\right]}_{\frac{4\kappa^2k^2 + k^4 + \kappa^4 - 2\kappa^2k^2}{(2\kappa k)^2} = \frac{(\kappa^2+k^2)^2}{(2\kappa k)^2} = \frac{\left(\frac{2mE}{\hbar^2} + \frac{2m(V_0-E)}{\hbar^2}\right)^2}{4\frac{2mE}{\hbar^2}\frac{2m(V_0-E)}{\hbar^2}} = \frac{V_0^2}{4E(V_0-E)}}\sinh^2(2\kappa a) = \boxed{1 + \frac{V_0^2}{4E(V_0-E)}\sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0-E)}\right)}.$$

You can get this from [… by switching the sign of … and using $\sin(i\theta) = i\sinh\ldots$

$\underline{E=V_0}$. $\psi = \begin{cases} Ae^{ikx}+Be^{-ikx} & (x<-a) \\ C+Dx & (-a<x<a) \\ Fe^{ikx} & (x>a) \end{cases}$ (In central region $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V_0\psi=E\psi \Rightarrow \frac{d^2\psi}{dx^2}=0$, so $\psi=C+Dx$.) $k=\frac{\sqrt{2mE}}{\hbar}$.

(1) Continuous ψ at $-a$: $Ae^{-ika}+Be^{ika}=C-Da$

(2) Continuous ψ at $+a$: $Fe^{ika}=C+Da$

$\Big\}\ 2Da=Fe^{ika}-Ae^{-ika}-Be^{+ika}$.

(3) Continuous ψ' at $-a$: $ik(Ae^{-ika}-Be^{ika})=D$

(4) Continuous ψ' at $+a$: $ikFe^{ika}=D$

$\Big\}\ Ae^{-2ika}-B=F$.

Put (4) into top equation: $Ae^{-2ika}+B=F-2aikF=(1-2iak)F$

$\Big\}\ 2Ae^{-2ika}=2F(1-ika)$

$$T^{-1}=\left|\frac{A}{F}\right|^2=1+(ka)^2=\boxed{1+\frac{2mE}{\hbar^2}a^2}.$$

You can get this from [2.151] by changing the sign of V_0 and taking the limit $E\to V_0$, using $\sin(\epsilon)\approx\epsilon$.

$\underline{E>V_0}$. This case is identical to the one in the book, only with $V_0\to -V_0$. So

$$\boxed{T^{-1}=1+\frac{V_0^2}{4E(E-V_0)}\sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E-V_0)}\right)}.$$

$\underline{\text{Problem 2.33}}$ (a) $\psi=\begin{cases} Ae^{ikx}+Be^{-ikx} & (x<0) \\ Fe^{-\kappa x} & (x>0) \end{cases}$ where $k=\sqrt{2mE}/\hbar$; $\kappa=\sqrt{2m(V_0-E)}/\hbar$.

(1) continuity of ψ: $A+B=F$

(2) continuity of ψ': $ik(A-B)=-\kappa F$

$\Big\}\ A+B=-\frac{ik}{\kappa}(A-B)\Rightarrow A\left(1+\frac{ik}{\kappa}\right)=-B\left(1-\frac{ik}{\kappa}\right)$.

$$R=\left|\frac{B}{A}\right|^2=\frac{|(1+ik/\kappa)|^2}{|(1-ik/\kappa)|^2}=\frac{1+(k/\kappa)^2}{1+(k/\kappa)^2}=\boxed{1}.$$

[Although the wave function penetrates into the barrier, it is eventually all reflected.

(b) $\psi=\begin{cases} Ae^{ikx}+Be^{-ikx} & (x<0) \\ Fe^{ilx} & (x>0) \end{cases}$ $k=\sqrt{2mE}/\hbar$; $l=\sqrt{2m(E-V_0)}/\hbar$.

(1) continuity of ψ: $A+B=F$

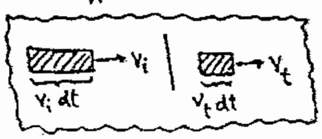
(2) continuity of ψ': $ik(A-B)=ilF$

$\Big\}\ A+B=\frac{k}{l}(A-B)$; $A\left(1-\frac{k}{l}\right)=-B\left(1+\frac{k}{l}\right)$.

$$R=\left|\frac{B}{A}\right|^2=\frac{(1-k/l)^2}{(1+k/l)^2}=\frac{(k-l)^2}{(k+l)^2}=\frac{(k-l)^4}{(k^2-l^2)^2}.$$

Now $k^2-l^2=\frac{2m}{\hbar^2}[E-E+V_0]=\left(\frac{2m}{\hbar^2}\right)V_0$;

$k-l=\frac{\sqrt{2m}}{\hbar}\left[\sqrt{E}-\sqrt{E-V_0}\right]$. So $\boxed{R=\frac{(\sqrt{E}-\sqrt{E-V_0})^4}{V_0^2}}$

(c) 

$T=\frac{P_t}{P_i}=\frac{|F|^2v_t}{|A|^2v_i}$ [Here P_i is the probability of finding particle in incident beam in box corresponding to time interval dt, and likewise for P_t.] But $\frac{v_t}{v_i}=\frac{\sqrt{E-V_0}}{\sqrt{E}}$ (from [2.81]).

So $T=\sqrt{\frac{E-V_0}{E}}\left|\frac{F}{A}\right|^2$. Alternatively, from Problem 1.9: $J_i=\frac{i\hbar}{2m}\left[Ae^{ikx}(-ikA^*e^{-ikx})-A^*e^{-ikx}(ikAe^{ikx})\right]$

$J_i = \frac{\hbar}{m}|A|^2 k$; $J_t = \frac{\hbar}{m}|F|^2 l$; $T = \frac{J_t}{J_i} = \left|\frac{F}{A}\right|^2 \frac{l}{k} = \left|\frac{F}{A}\right|^2 \sqrt{\frac{E-V_0}{E}}$. QED.

For $E < V_0$, $\boxed{T=0}$.

(d) For $E > V_0$, $F = A + B = A + A\frac{\left(\frac{k}{l}-1\right)}{\left(\frac{k}{l}+1\right)} = A\frac{2k/l}{\left(\frac{k}{l}+1\right)} = \frac{2k}{k+l} A$.

$$T = \left|\frac{F}{A}\right|^2 \frac{l}{k} = \left(\frac{2k}{k+l}\right)^2 \frac{l}{k} = \frac{4kl}{(k+l)^2} = \frac{4kl(k-l)^2}{(k^2-l^2)^2} = \boxed{\frac{4\sqrt{E}\sqrt{E-V_0}\left(\sqrt{E}-\sqrt{E-V_0}\right)^2}{V_0^2}}.$$

$$T+R = \frac{4kl}{(k+l)^2} + \frac{(k-l)^2}{(k+l)^2} = \frac{4kl + k^2 - 2kl + l^2}{(k+l)^2} = \frac{k^2+2kl+l^2}{(k+l)^2} = \frac{(k+l)^2}{(k+l)^2} = 1 \checkmark$$

Problem 2.34 (1) [2.115] $F + G = A + B$; (2) [2.117] $F - G = (1+2i\beta)A - (1-2i\beta)B$. $\beta = \frac{m\alpha}{\hbar^2 k}$.

Subtract: $2G = -2i\beta A + 2(1-i\beta)B \Rightarrow B = \frac{1}{1-i\beta}(i\beta A + G)$. Multiply (1) by $(1-2i\beta)$ and add:

$2(1-i\beta)F - 2i\beta G = 2A \Longrightarrow F = \frac{1}{1-i\beta}(A + i\beta G)$.

$$\boxed{S = \frac{1}{1-i\beta}\begin{pmatrix} i\beta & 1 \\ 1 & i\beta \end{pmatrix}}.$$

To get bound state energy, set $k \to i\kappa$. S-matrix blows up at: $1 = i\beta = i\frac{m\alpha}{\hbar^2 i\kappa} = \frac{m\alpha}{\hbar^2\kappa}$, or $\kappa = \frac{m\alpha}{\hbar^2}$. But [2.164] $\kappa = \frac{\sqrt{-2mE}}{\hbar}$, so $-2mE = \frac{m^2\alpha^2}{\hbar^2}$, or

$\boxed{E = -\frac{m\alpha^2}{2\hbar^2}}$, in agreement with [2.111].

Problem 2.35 For an even potential $(V(-x) = V(x))$ scattering from the right is the same as scattering from the left, with $x \longleftrightarrow -x$, $A \leftrightarrow G$, $B \leftrightarrow F$ (see Figure 2.15). Making this change in [2.158]:

$F = S_{11}G + S_{12}A$; $B = S_{21}G + S_{22}A$. Comparing [2.158] itself, we conclude: $\boxed{S_{11} = S_{22},\ S_{21} = S_{12}}$.

(Note that the delta-well S-matrix in Problem 2.34 has this property.) In the case of the finite square well [2.150] gives

$$S_{21} = \frac{e^{-2ika}}{\cos(2la) - i\frac{\sin(2la)}{2kl}(k^2+l^2)};$$

and [2.149], combined with [2.150], gives

$$S_{11} = \frac{i\frac{\sin(2la)}{2kl}(l^2-k^2)e^{-2ika}}{\cos(2la) - i\frac{\sin(2la)}{2kl}(k^2+l^2)}.$$

So

$$\boxed{S = \frac{e^{-2ika}}{\cos(2la) - i\frac{\sin(2la)}{2kl}(k^2+l^2)}\begin{bmatrix} i\frac{\sin(2la)}{2kl}(l^2-k^2) & 1 \\ 1 & i\frac{\sin(2la)}{2kl}(l^2-k^2)\end{bmatrix}}$$

Problem 2.36 Simplest method is to use the trig identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ to write $\sin^3(\pi x/a) = \frac{3}{4}\sin(\pi x/a) - \frac{1}{4}\sin(3\pi x/a)$. So ([2.24]) $\Psi(x,0) = \sqrt{\frac{a}{2}}A\left[\frac{3}{4}\psi_1(x) - \frac{1}{4}\psi_3(x)\right]$. Normalizing (using [2.34]): $\frac{a}{2}|A|^2\left(\frac{9}{16}+\frac{1}{16}\right) = \frac{5}{16}a|A|^2 = 1 \Rightarrow A = 4/\sqrt{5a}$. So $\Psi(x,0) = \frac{1}{\sqrt{5a}}\left(3\sin(\pi x/a) - \sin(3\pi x/a)\right)$.

$\therefore$ (eq. [2.32]): $\boxed{\Psi(x,t) = \frac{1}{\sqrt{5a}}\left[3\sin(\pi x/a)e^{-i\pi^2\hbar t/2ma^2} - \sin(3\pi x/a)e^{-9i\pi^2\hbar t/2ma^2}\right]}$.

Problem 2.37 $[2.41] \Rightarrow \left\{ \begin{array}{l} a_+ + a_- = \frac{2}{\sqrt{2m}}\hat{p} \Rightarrow \hat{p} = \sqrt{\frac{m}{2}}(a_+ + a_-) \\ a_+ - a_- = \frac{2}{\sqrt{2m}} i m\omega \hat{x} \Rightarrow \hat{x} = \frac{-i}{\sqrt{2m}\,\omega}(a_+ - a_-) \end{array} \right\}$

$\therefore \langle x \rangle = -\frac{i}{\sqrt{2m}\,\omega}\int \psi_n^*(a_+ - a_-)\psi_n\,dx.$ But $\left\{ \begin{array}{ll} a_+\psi_n = \sqrt{(n+1)\hbar\omega}\,\psi_{n+1} & [2.52] \\ a_-\psi_n = \sqrt{n\hbar\omega}\,\psi_{n-1} & [2.53] \end{array} \right\}$, so

$$= \frac{-i}{\sqrt{2m}\,\omega}\left\{\sqrt{(n+1)\hbar\omega}\int\psi_n^*\psi_{n+1}\,dx - \sqrt{n\hbar\omega}\int\psi_n^*\psi_{n-1}\,dx\right\} = \boxed{0}. \text{ (By orthogonality of } \{\psi_n\}\text{)}$$

$\langle p \rangle = m\frac{d\langle x\rangle}{dt} = \boxed{0}$. $\hat{x}^2 = \frac{-1}{2m\omega^2}(a_+ - a_-)(a_+ - a_-) = -\frac{1}{2m\omega^2}(a_+^2 - a_+a_- - a_-a_+ + a_-^2)$.

$\therefore \langle x^2 \rangle = -\frac{1}{2m\omega^2}\int\psi_n^*(a_+^2 - a_+a_- - a_-a_+ + a_-^2)\psi_n.$ But

$$\left\{ \begin{array}{l} a_+^2\psi_n = a_+\left(\sqrt{(n+1)\hbar\omega}\,\psi_{n+1}\right) = \sqrt{(n+1)\hbar\omega}\sqrt{(n+2)\hbar\omega}\,\psi_{n+2} = \sqrt{(n+1)(n+2)}\,\hbar\omega\,\psi_{n+2}. \\ a_+a_-\psi_n = a_+\left(\sqrt{n\hbar\omega}\,\psi_{n-1}\right) = \sqrt{n\hbar\omega}\sqrt{n\hbar\omega}\,\psi_n = n\hbar\omega\,\psi_n. \\ a_-a_+\psi_n = a_-\left(\sqrt{(n+1)\hbar\omega}\,\psi_{n+1}\right) = \sqrt{(n+1)\hbar\omega}\sqrt{(n+1)\hbar\omega}\,\psi_n = (n+1)\hbar\omega\,\psi_n. \\ a_-^2\psi_n = a_-\left(\sqrt{n\hbar\omega}\,\psi_{n-1}\right) = \sqrt{n\hbar\omega}\sqrt{(n-1)\hbar\omega}\,\psi_{n-2} = \sqrt{(n-1)n}\,\hbar\omega\,\psi_{n-2}. \end{array} \right.$$

$$\therefore \langle x^2 \rangle = -\frac{1}{2m\omega^2}\left\{0 - n\hbar\omega\int|\psi_n|^2dx - (n+1)\hbar\omega\int|\psi_n|^2dx + 0\right\} = \frac{1}{2m\omega^2}(2n+1)\hbar\omega = \boxed{\left(n+\tfrac{1}{2}\right)\frac{\hbar}{m\omega}}.$$

$$\hat{p}^2 = \frac{m}{2}(a_+ + a_-)(a_+ + a_-) = \frac{m}{2}\left(a_+^2 + a_+a_- + a_-a_+ + a_-^2\right) \Rightarrow$$

$$\langle p^2 \rangle = \frac{m}{2}\{0 + n\hbar\omega + (n+1)\hbar\omega + 0\} = \frac{m}{2}(2n+1)\hbar\omega = \boxed{\left(n+\tfrac{1}{2}\right)m\hbar\omega}.$$

$\langle T \rangle = \langle p^2/2m \rangle = \boxed{\frac{1}{2}\left(n+\frac{1}{2}\right)\hbar\omega}$; $\langle V \rangle = \langle \frac{1}{2}m\omega^2x^2 \rangle = \frac{1}{2}m\omega^2\left(n+\frac{1}{2}\right)\frac{\hbar}{m\omega} = \boxed{\frac{1}{2}\left(n+\frac{1}{2}\right)\hbar\omega}$.

(Note: $\langle T \rangle + \langle V \rangle = \left(n+\frac{1}{2}\right)\hbar\omega = E_n = \langle H \rangle$, as it should be.) $\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 \Rightarrow \sigma_x = \sqrt{n+1/2}\sqrt{\frac{\hbar}{m\omega}}$;

$\sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2 \Rightarrow \sigma_p = \sqrt{n+1/2}\sqrt{m\hbar\omega}$. So $\sigma_x\sigma_p = \underline{(n+1/2)\hbar} \geq \hbar/2$. ✓

Problem 2.38 Everything in Section 2.3.2 still applies, except that there is an additional boundary condition: $\psi(0) = 0$. This eliminates all the even solutions $(n = 0, 2, 4, \ldots)$, leaving only the odd solutions. So

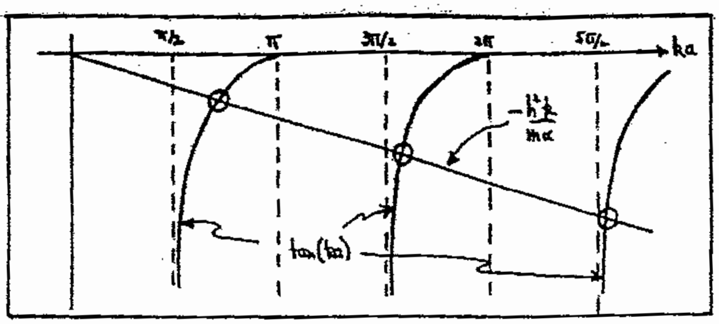
$$\boxed{E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega, \quad n = 1, 3, 5, \ldots}$$

Problem 2.39 $[2.18] \Rightarrow \psi(x) = A\sin kx + B\cos kx$, $0 \leq x \leq a$, with $k = \sqrt{2mE}/\hbar$.

Even solutions: $\psi(x) = \psi(-x) = A\sin(-kx) + B\cos(-kx) = -A\sin kx + B\cos kx \quad (-a \leq x \leq 0)$.

Boundary conditions: $\left\{ \begin{array}{l} \psi \text{ continuous at } 0:\ B = B \text{ (no new condition)}. \\ \psi' \text{ discontinuous ([2.107] with sign of } \alpha \text{ switched)}:\ Ak + Ak = \frac{2m\alpha}{\hbar^2}B \Rightarrow B = \frac{\hbar^2 k}{m\alpha}A. \\ \psi \to 0 \text{ at } x = a:\ A\sin(ka) + \frac{\hbar^2 k}{m\alpha}A\cos(ka) = 0 \Rightarrow \tan(ka) = -\frac{\hbar^2 k}{m\alpha}. \end{array} \right.$

$$\therefore \boxed{\psi = A\left[\sin kx + \frac{\hbar^2 k}{m\alpha}\cos kx\right]\ (0 \leq x \leq a);\ \psi(-x) = \psi(x)}.$$



From the graph, the allowed energies are slightly above

$$ka = \frac{n\pi}{2} \quad (n=1,3,5,\dots) \text{ —}$$

i.e. $$\boxed{E_n \gtrsim \frac{n^2\pi^2\hbar^2}{2m(2a)^2} \quad (n=1,3,5,\dots)}$$

These energies are somewhat higher than the corresponding energies for the infinite square well ([2.23], with $a \to 2a$). As $\alpha \to 0$, the straight line $(-\hbar^2 k/m\alpha)$ gets steeper and steeper, and the intersections get closer to $n\pi/2$; the energies then reduce to those of the ordinary infinite well. As $\alpha \to \infty$, the straight line approaches horizontal, and the intersections are at $n\pi$ $(n=1,2,3,\dots)$, so

$E_n \to \frac{n^2\pi^2\hbar^2}{2ma^2}$ — these are the allowed energies for the infinite square well of width a. At this point the barrier is impenetrable, and we have two isolated infinite square wells.

Odd solutions: $\psi(x) = -\psi(-x) = -A\sin(-kx) - B\cos(-kx) = A\sin(kx) - B\cos(kx)$, for $-a \le x \le 0$.

Boundary conditions:
$$\begin{cases} \psi \text{ continuous at } 0:\ B = -B \Rightarrow B = 0. \\ \psi' \text{ discontinuous}:\ Ak - Ak = \frac{2m\alpha}{\hbar^2}(0).\ \text{(No new condition.)} \\ \psi(a) = 0 \Rightarrow A\sin(ka) = 0 \Rightarrow ka = \frac{n\pi}{2} \quad (n=2,4,6,\dots). \end{cases}$$

$$\boxed{\psi(x) = A\sin(kx), \quad -a<x<a} \qquad \boxed{E_n = \frac{n^2\pi^2\hbar^2}{2m(2a)^2} \quad (n=2,4,6,\dots)}.$$

These are the exact (even-n) energies (and wave functions) for the infinite square well (of width $2a$). The point is that the odd solutions (even n) are zero at the origin, so they never "feel" the delta function at all.

Problem 2.40 (a) Normalization is same as before: $A = \left(\frac{2a}{\pi}\right)^{1/4}$.

(b) [2.86] $\Rightarrow$ $\phi(k) = \frac{1}{\sqrt{2\pi}}\left(\frac{2a}{\pi}\right)^{1/4}\int_{-\infty}^{\infty} e^{-ax^2} e^{ilx} e^{-ikx}\,dx$ ... same as before, only $k \to k-l$ ...

$$= \frac{1}{(2\pi a)^{1/4}} e^{-(k-l)^2/4a}. \qquad [2.83] \Rightarrow$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\frac{1}{(2\pi a)^{1/4}}\int_{-\infty}^{\infty} \underbrace{e^{-(k-l)^2/4a}\, e^{i(kx - \hbar k^2 t/2m)}}_{e^{-l^2/4a}\, e^{-\left[\left(\frac{1}{4a} + i\hbar t/2m\right)k^2 - (ix + l/2a)k\right]}}\,dk$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{(2\pi a)^{1/4}} e^{-l^2/4a}\sqrt{\frac{\pi}{\left(\frac{1}{4a} + i\hbar t/2m\right)}}\; e^{+(ix + \frac{l}{2a})^2/4\left(\frac{1}{4a} + i\hbar t/2m\right)}$$

$$= \boxed{\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1 + 2i\hbar a t/m}}\, e^{-l^2/4a}\, e^{+a(ix + l/2a)^2/(1 + 2ia\hbar t/m)}}$$

(c) Let $\theta \equiv 2\hbar a t/m$, as before: $|\Psi|^2 = \sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\theta^2}}\, e^{-\ell^2/2a}\, e^{a\left[\frac{(ix+\ell/2a)^2}{(1+i\theta)} + \frac{(-ix+\ell/2a)^2}{(1-i\theta)}\right]}$.

The term in square brackets:

$$[\;] = \frac{1}{1+\theta^2}\left[(1-i\theta)(ix+\ell/2a)^2 + (1+i\theta)(-ix+\ell/2a)^2\right] = \frac{1}{1+\theta^2}\Big[\left(-x^2+\frac{ix\ell}{a}+\frac{\ell^2}{4a^2}\right)+\left(-x^2-\frac{ix\ell}{a}+\frac{\ell^2}{4a^2}\right)$$

$$+\, i\theta\left(x^2-\frac{ix\ell}{a}-\frac{\ell^2}{4a^2}\right)+i\theta\left(-x^2-\frac{ix\ell}{a}+\frac{\ell^2}{4a^2}\right)\Big] = \frac{1}{1+\theta^2}\left[-2x^2+\frac{\ell^2}{2a^2}+2\theta\frac{x\ell}{a}\right]$$

$$= \frac{1}{(1+\theta^2)}\left[-2x^2+2\theta\frac{x\ell}{a}-\frac{\theta^2\ell^2}{2a^2}+\frac{\theta^2\ell^2}{2a^2}+\frac{\ell^2}{2a^2}\right] = \frac{-2}{(1+\theta^2)}\left(x-\frac{\theta\ell}{2a}\right)^2+\frac{\ell^2}{2a^2}.$$

$\therefore\ |\Psi|^2 = \sqrt{\frac{2}{\pi}}\sqrt{\frac{a}{1+\theta^2}}\; e^{-\ell^2/2a}\, e^{-\frac{2a}{1+\theta^2}(x-\theta\ell/2a)^2}\, e^{\ell^2/2a}$. Let $w \equiv \sqrt{\frac{a}{1+\theta^2}}$, as before:

$\boxed{|\Psi(x,t)|^2 = \sqrt{\frac{2}{\pi}}\, w\, e^{-2w^2(x-\theta\ell/2a)^2}}$. Same as before, except $x \to \left(x-\frac{\theta\ell}{2a}\right)$, so $|\Psi|^2$ has the same (flattening Gaussian) shape – only this time the center moves at constant speed $\underline{v = \hbar\ell/m}$.

(d) $\langle x\rangle = \int_{-\infty}^{\infty} x\,|\Psi(x,t)|^2 dx$. Let $y \equiv x-\theta\ell/2a = x-vt$, so $x = y+vt$.

$$= \int_{-\infty}^{\infty}(y+vt)\sqrt{\frac{2}{\pi}}\,w\,e^{-2w^2y^2}dy = vt.$$ (The first integral is trivially zero; the second is 1 by normalization)

$= \boxed{\frac{\hbar\ell}{m}t}$; $\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{\hbar\ell}$. $\langle x^2\rangle = \int_{-\infty}^{\infty}(y+vt)^2\sqrt{\frac{2}{\pi}}\,w\,e^{-2w^2y^2}dy = \frac{1}{4w^2}+0+(vt)^2$

(the first integral is same as before). $\boxed{\langle x^2\rangle = \frac{1}{4w^2}+(\hbar\ell t/m)^2}$

$\langle p^2\rangle = -\hbar^2\int_{-\infty}^{\infty}\Psi^*\frac{d^2\Psi}{dx^2}dx$. $\Psi = \left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+i\theta}}\,e^{-\ell^2/4a}\,e^{a(ix+\ell/2a)^2/(1+i\theta)}$, so

$$\frac{d\Psi}{dx} = \frac{2ia\left(ix+\frac{\ell}{2a}\right)}{(1+i\theta)}\Psi;\quad \frac{d^2\Psi}{dx^2} = \left[\frac{2ia(ix+\ell/2a)}{1+i\theta}\right]\frac{d\Psi}{dx}+\frac{2i^2a}{(1+i\theta)}\Psi = \left[\frac{-4a^2(ix+\ell/2a)^2}{(1+i\theta)^2}-\frac{2a}{(1+i\theta)}\right]\Psi$$

$$\langle p^2\rangle = \frac{4a^2\hbar^2}{(1+i\theta)^2}\int_{-\infty}^{\infty}\left[\left(ix+\frac{\ell}{2a}\right)^2+\frac{(1+i\theta)}{2a}\right]|\Psi|^2dx = \frac{4a^2\hbar^2}{(1+i\theta)^2}\int_{-\infty}^{\infty}\left[-\left(y+vt-\frac{i\ell}{2a}\right)^2+\frac{(1+i\theta)}{2a}\right]|\Psi|^2dy$$

$$= \frac{4a^2\hbar^2}{(1+i\theta)^2}\left\{-\int_{-\infty}^{\infty}y^2|\Psi|^2dy-2\left(vt-\frac{i\ell}{2a}\right)\int_{-\infty}^{\infty}y|\Psi|^2dy+\left[-\left(vt-\frac{i\ell}{2a}\right)^2+\frac{(1+i\theta)}{2a}\right]\int_{-\infty}^{\infty}|\Psi|^2dy\right\}$$

$$= \frac{4a^2\hbar^2}{(1+i\theta)^2}\left\{-\frac{1}{4w^2}+0-\left(vt-\frac{i\ell}{2a}\right)^2+\frac{(1+i\theta)}{2a}\right\} = \frac{4a^2\hbar^2}{(1+i\theta)^2}\left\{-\frac{1+\theta^2}{4a}-\left[\left(-\frac{i\ell}{2a}\right)(1+i\theta)\right]^2+\frac{(1+i\theta)}{2a}\right\}$$

$$= \frac{a\hbar^2}{(1+i\theta)}\left\{-(1-i\theta)+\frac{\ell^2}{a}(1+i\theta)+2\right\} = \frac{a\hbar^2}{1+i\theta}\left((1+i\theta)\left(1+\frac{\ell^2}{a}\right)\right) = \boxed{\hbar^2(a+\ell^2)}.$$

$\sigma_x^2 = \langle x^2\rangle-\langle x\rangle^2 = \frac{1}{4w^2}+\left(\frac{\hbar\ell t}{m}\right)^2-\left(\frac{\hbar\ell t}{m}\right)^2 = \frac{1}{4w^2} \Rightarrow \boxed{\sigma_x = \frac{1}{2w}}$; $\sigma_p^2 = \langle p^2\rangle-\langle p\rangle^2 = \hbar^2a+\hbar^2\ell^2-\hbar^2\ell^2 = \hbar^2 a$,

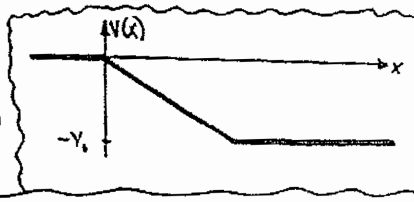
so $\boxed{\sigma_p = \hbar\sqrt{a}}$. (e) σ_x and σ_p are same as before, so uncertainty principle still holds.

PROBLEM 2.41 (a) $\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & (x<0) \\ Fe^{ilx} & (x>0) \end{cases}$ where $k \equiv \sqrt{2mE}/\hbar$, $l \equiv \sqrt{2m(E+V_0)}/\hbar$.

$$\left.\begin{array}{l}\text{continuity of } \psi \Rightarrow A+B=F \\ \text{continuity of } \psi' \Rightarrow ik(A-B) = ilF\end{array}\right\} A+B = \frac{k}{l}(A-B);\ A\left(1-\frac{k}{l}\right) = -B\left(1+\frac{k}{l}\right);\ \frac{B}{A} = -\left(\frac{1-k/l}{1+k/l}\right).$$

$$R = \left|\frac{B}{A}\right|^2 = \left(\frac{l-k}{l+k}\right)^2 = \left(\frac{\sqrt{E+V_0}-\sqrt{E}}{\sqrt{E+V_0}+\sqrt{E}}\right)^2 = \left(\frac{\sqrt{1+V_0/E}-1}{\sqrt{1+V_0/E}+1}\right)^2 = \left(\frac{\sqrt{1+3}-1}{\sqrt{1+3}+1}\right)^2 = \left(\frac{2-1}{2+1}\right)^2 = \boxed{\frac{1}{9}}.$$

(b) The cliff is a two-dimensional problem, and even if we pretend the car drops straight down, the potential as a function of distance along the (crooked, but now one-dimensional) path is $-mgx$ (with x the vertical coordinate), as shown.



PROBLEM 2.42

$$\left.\begin{array}{l} -\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + V\psi_1 = E\psi_1 \Rightarrow -\frac{\hbar^2}{2m}\psi_2\frac{d^2\psi_1}{dx^2} + V\psi_1\psi_2 = E\psi_1\psi_2 \\ -\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} + V\psi_2 = E\psi_2 \Rightarrow -\frac{\hbar^2}{2m}\psi_1\frac{d^2\psi_2}{dx^2} + V\psi_1\psi_2 = E\psi_1\psi_2 \end{array}\right\} \text{subtract} \Rightarrow -\frac{\hbar^2}{2m}\left[\psi_2\frac{d^2\psi_1}{dx^2} - \psi_1\frac{d^2\psi_2}{dx^2}\right] = 0.$$

But $\frac{d}{dx}\left[\psi_2\frac{d\psi_1}{dx} - \psi_1\frac{d\psi_2}{dx}\right] = \frac{d\psi_2}{dx}\frac{d\psi_1}{dx} + \psi_2\frac{d^2\psi_1}{dx^2} - \frac{d\psi_1}{dx}\frac{d\psi_2}{dx} - \psi_1\frac{d^2\psi_2}{dx^2} = \psi_2\frac{d^2\psi_1}{dx^2} - \psi_1\frac{d^2\psi_2}{dx^2}$. Since this is zero, it follows that $\psi_2\frac{d\psi_1}{dx} - \psi_1\frac{d\psi_2}{dx} = K$ (a constant). But $\psi \to 0$ at $\infty \Rightarrow$ the constant must be zero. Thus $\psi_2\frac{d\psi_1}{dx} = \psi_1\frac{d\psi_2}{dx}$, or $\frac{1}{\psi_1}\frac{d\psi_1}{dx} = \frac{1}{\psi_2}\frac{d\psi_2}{dx}$, so $\ln\psi_1 = \ln\psi_2 + \text{constant}$, or $\psi_1 = (\text{constant})\psi_2$. QED.

PROBLEM 2.43 $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$ (where x is measured around the circumference). $\frac{d^2\psi}{dx^2} = -k^2\psi$, with $k \equiv \sqrt{2mE}/\hbar$. $\psi = Ae^{ikx} + Be^{-ikx}$. But $\psi(x+a) = \psi(x)$, since $x+a$ is the same point as x, so $Ae^{ikx}e^{ika} + Be^{-ikx}e^{-ika} = Ae^{ikx} + Be^{-ikx}$. This is true for all x. In particular, for $x=0$:

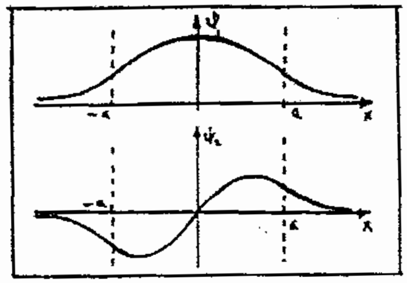
(1) $Ae^{ika} + Be^{-ika} = A+B$. And for $x = \frac{\pi}{2k}$: $Ae^{i\pi/2}e^{ika} + Be^{-i\pi/2}e^{-ika} = Ae^{i\pi/2} + Be^{-i\pi/2}$, or $iAe^{ika} + (-i)Be^{-ika} = iA + (-i)B$, or (2) $Ae^{ika} - Be^{-ika} = A - B$. Add (1) and (2):

$2Ae^{ika} = 2A$. Either $A = 0$, or else $e^{ika} = 1$, in which case $ka = 2n\pi$ $(n = 0, \pm1, \pm2, \dots)$.

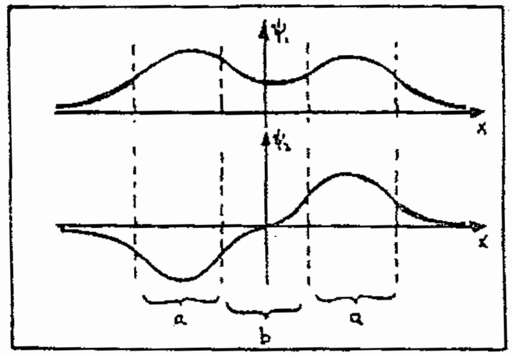
(But if $A = 0$, then $Be^{-ika} = B$, leading to the same conclusion.) So for every positive n there are two solutions: $\psi_n^+(x) = Ae^{i(2n\pi x/a)}$ and $\psi_n^-(x) = Be^{-i(2n\pi x/a)}$. ($n=0$ is OK too, but in that case there's just one solution.) Normalizing: $\int_0^a |\psi_\pm|^2 dx = 1 \Rightarrow A = B = \frac{1}{\sqrt{a}}$. (Any other solution with given energy is a linear combination of these.)

$$\boxed{\psi_n^\pm(x) = \frac{1}{\sqrt{a}}e^{\pm i(2n\pi x/a)};\quad E_n = \frac{2n^2\pi^2\hbar^2}{ma^2}\quad (n = 0, 1, 2, 3, \dots)}.$$

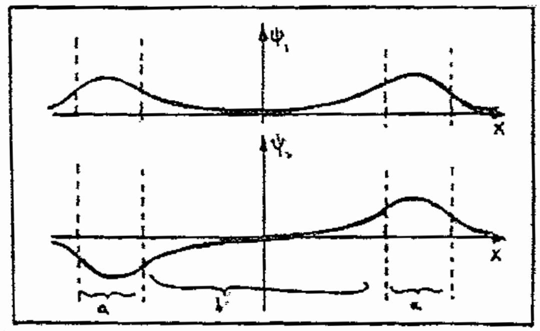
The theorem fails because here ψ does NOT go to zero at ∞; x is restricted to a finite range, and we are unable to determine the constant K.



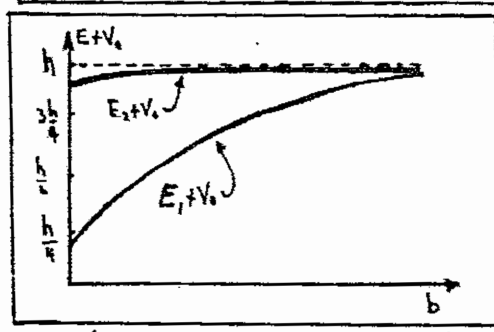
PROBLEM 2.44 (a) (i) $b=0 \Rightarrow$ ordinary finite square well. Exponential decay outside; sinusoidal inside (cos for ψ_1, sin for ψ_2). No nodes for ψ_1, one node for ψ_2.



(ii) Ground state is even. Exponential decay outside, sinusoidal inside the wells, hyperbolic cosine in barrier. First excited state is odd — hyperbolic sine in barrier. No nodes for ψ_1, one node for ψ_2.



(iii) For $b \gg a$, same as (ii), but wave function very small in barrier region. Essentially two isolated finite square wells. ψ_1 and ψ_2 are degenerate (in energy); they are even and odd linear combinations of the ground states of the two separate wells.



(Note: E_1, E_2 alone are negative.)

(b) From eq. [2.139] we know that for $b=0$ the energies fall slightly below

$$\left.\begin{aligned} E_1+V_0 &\cong \frac{\pi^2\hbar^2}{2m(2a)^2} = \frac{h}{4} \\ E_2+V_0 &\cong \frac{4\pi^2\hbar^2}{2m(2a)^2} = h \end{aligned}\right\} \text{ where } h \equiv \frac{\pi^2\hbar^2}{2ma^2}.$$

For $b \gg a$, the width of each (isolated) well is a, so

$$E_1+V_0 \approx E_2+V_0 \cong \frac{\pi^2\hbar^2}{2ma^2} = h$$

(again, slightly below this). Hence the graph shown.

[Incidentally, within each well, $\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(V_0+E)\psi$, so the more curved the wave function, the higher the energy. This is consistent with the graphs above.]

(c) The energy is lowest in configuration (i) — i.e. with $b \to 0$. So the electron tends to draw the nuclei $\boxed{\text{together}}$, promoting bonding of the atoms.

PROBLEM 2.45 $\quad \frac{\partial\Psi}{\partial t} = \Psi\left(-\frac{m\omega}{2\hbar}\right)\left[\frac{a^2}{2}\left(-2i\omega e^{-2i\omega t}\right) + \frac{i\hbar}{m} - 2ax(-i\omega)e^{-i\omega t}\right]$, so

$$i\hbar\frac{\partial\Psi}{\partial t}=\left[-\frac{1}{2}ma^2\omega^2e^{-2i\omega t}+\frac{1}{2}\hbar\omega+max\omega^2e^{-i\omega t}\right]\Psi.$$

$$\frac{\partial\Psi}{\partial x}=\Psi\left[\left(-\frac{m\omega}{2\hbar}\right)\left(2x-2ae^{-i\omega t}\right)\right]=-\frac{m\omega}{\hbar}\left(x-ae^{-i\omega t}\right)\Psi;\quad\frac{\partial^2\Psi}{\partial x^2}=-\frac{m\omega}{\hbar}\Psi-\frac{m\omega}{\hbar}\left(x-ae^{-i\omega t}\right)\frac{\partial\Psi}{\partial x}.$$

$$\frac{\partial^2\Psi}{\partial x^2}=-\frac{m\omega}{\hbar}+\left(\frac{m\omega}{\hbar}\right)^2\left(x-ae^{-i\omega t}\right)^2\Psi.$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+\frac{1}{2}m\omega^2x^2\Psi=-\frac{\hbar^2}{2m}\left[-\frac{m\omega}{\hbar}+\left(\frac{m\omega}{\hbar}\right)^2\left(x-ae^{-i\omega t}\right)^2\right]\Psi+\frac{1}{2}m\omega^2x^2\Psi$$

$$=\left[\frac{1}{2}\hbar\omega-\frac{1}{2}m\omega^2\left(x^2-2axe^{-i\omega t}+a^2e^{-2i\omega t}\right)+\frac{1}{2}m\omega^2x^2\right]\Psi$$

$$=\left[\frac{1}{2}\hbar\omega+max\omega^2e^{-i\omega t}-\frac{1}{2}m\omega^2a^2e^{-2i\omega t}\right]\Psi=i\hbar\frac{\partial\Psi}{\partial t}\quad\text{(comparing top line above).}$$

(b) $$|\Psi|^2=\sqrt{\frac{m\omega}{\pi\hbar}}\,e^{-\frac{m\omega}{2\hbar}\left[\left(x^2+\frac{a^2}{2}(1+e^{2i\omega t})-\frac{i\hbar t}{m}-2axe^{i\omega t}\right)+\left(x^2+\frac{a^2}{2}(1+e^{-2i\omega t})+\frac{i\hbar t}{m}-2axe^{-i\omega t}\right)\right]}$$

$$=\sqrt{\frac{m\omega}{\pi\hbar}}\,e^{-\frac{m\omega}{2\hbar}\left[2x^2+a^2+a^2\cos(2\omega t)-4ax\cos(\omega t)\right]}.\quad\text{But } a^2(1+\cos(2\omega t))=a^2(2\cos^2\omega t),\text{ so}$$

$$=\sqrt{\frac{m\omega}{\pi\hbar}}\,e^{-\frac{m\omega}{\hbar}\left(x^2-2ax\cos(\omega t)+a^2\cos^2(\omega t)\right)}=\boxed{\sqrt{\frac{m\omega}{\pi\hbar}}\,e^{-\frac{m\omega}{\hbar}(x-a\cos\omega t)^2}}.$$

The wave packet is a gaussian of fixed shape, whose center oscillates back and forth sinusoidally, with amplitude a and angular frequency ω.

(c) Note from Problem 2.12 (b) that this wave function is correctly normalized. Let $y\equiv x-a\cos\omega t$:

$$\langle x\rangle=\int x|\Psi|^2dx=\int(y+a\cos\omega t)|\Psi|^2dy=0+a\cos\omega t\int|\Psi|^2dy=\boxed{a\cos\omega t}.$$

$$\langle p\rangle=m\frac{d\langle x\rangle}{dt}=\boxed{-ma\omega\sin\omega t}.\quad\frac{d\langle p\rangle}{dt}=-ma\omega^2\cos\omega t.\quad V=\frac{1}{2}m\omega^2x^2\Rightarrow\frac{dV}{dx}=m\omega^2x.$$

$$\left\langle-\frac{dV}{dx}\right\rangle=-m\omega^2\langle x\rangle=-m\omega^2a\cos\omega t=\frac{d\langle p\rangle}{dt},\text{ so Ehrenfest's theorem \underline{is} satisfied.}$$

PROBLEM 2.46 (a) $\psi=\begin{cases}Ae^{ikx}+Be^{-ikx} & (0\le x\le a)\\ Fe^{ikx} & (x>a)\end{cases}$ where $k\equiv\sqrt{2mE}/\hbar$.

Boundary conditions: (1) $\psi(0)=0\Rightarrow B=-A$.

(2) $\psi(x)$ continuous at a: $A(e^{ika}-e^{-ika})=Fe^{ika}$.

(3) $\psi'(x)$ discontinuous: $ikFe^{ika}-ikA(e^{ika}+e^{-ika})=\frac{2m\alpha}{\hbar^2}\psi(a)=\frac{2m\alpha}{\hbar^2}A(e^{ika}-e^{-ika})$

$$\therefore A(e^{ika}-e^{-ika})-A(e^{ika}+e^{-ika})=-i\frac{2m\alpha}{\hbar^2k}A(e^{ika}-e^{-ika})\Rightarrow 2e^{-ika}=i\frac{2m\alpha}{\hbar^2k}(e^{ika}-e^{-ika})$$

$\boxed{1=i\frac{m\alpha}{\hbar^2k}\left(e^{2ika}-1\right)}$ This (implicitly) determines k, and $E=\hbar^2k^2/2m$.

(b) This wave function is not normalizable, and the Theorem in 2.1a applies only to normalized states.

(c) $\Psi(x,t) = \psi(x)e^{-iEt/\hbar} = \psi(x)e^{-iE_0t/\hbar}e^{\Gamma t/\hbar} \Rightarrow |\Psi|^2 = |\psi|^2 e^{+2\Gamma t/\hbar}$. Time for $|\Psi|^2$ to drop to $\frac{1}{e}$ of its initial value: $\boxed{\tau = -\frac{\hbar}{2\Gamma}}$. (Evidently Γ is negative.)

Problem 2.47 (a) $\frac{\partial\Psi}{\partial t} = \Psi\left[-\frac{m\alpha}{\hbar^2}\frac{\partial}{\partial t}|x-vt| - i\left(E + \frac{1}{2}mv^2\right)/\hbar\right]$. $\frac{\partial}{\partial t}|x-vt| = \begin{Bmatrix} -v, \text{ if } x-vt>0 \\ v, \text{ if } x-vt<0 \end{Bmatrix}$.

We can write this in terms of the θ-function [2.125]: $2\theta(z) - 1 = \begin{Bmatrix} 1, \text{ if } z>0 \\ -1, \text{ if } z<0 \end{Bmatrix}$,

so $\frac{\partial}{\partial t}|x-vt| = -v\left[2\theta(x-vt)-1\right]$.

$$[*] \quad i\hbar\frac{\partial\Psi}{\partial t} = \left\{ i\,\frac{m\alpha v}{\hbar}\left[2\theta(x-vt)-1\right] + E + \frac{1}{2}mv^2\right\}\Psi.$$

$\frac{\partial\Psi}{\partial x} = \Psi\left[-\frac{m\alpha}{\hbar^2}\frac{\partial}{\partial x}|x-vt| + \frac{imv}{\hbar}\right]$. $\quad \frac{\partial}{\partial x}|x-vt| = \begin{Bmatrix} 1, \text{ if } x>vt \\ -1, \text{ if } x<vt \end{Bmatrix} = 2\theta(x-vt)-1$.

$= \left\{-\frac{m\alpha}{\hbar^2}\left[2\theta(x-vt)-1\right] + \frac{imv}{\hbar}\right\}\Psi$. $\quad \frac{\partial^2\Psi}{\partial x^2} = \left\{-\frac{m\alpha}{\hbar^2}\left[2\theta(x-vt)-1\right] + \frac{imv}{\hbar}\right\}^2\Psi - \frac{2m\alpha}{\hbar^2}\frac{\partial}{\partial x}\theta(x-vt)\Psi$.

But (Problem 2.24 b) $\frac{\partial}{\partial x}\theta(x-vt) = \delta(x-vt)$, so

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} - \alpha\delta(x-vt)\Psi = \left\{-\frac{\hbar^2}{2m}\left[-\frac{m\alpha}{\hbar^2}\left[2\theta(x-vt)-1\right] + \frac{imv}{\hbar}\right]^2 + \alpha\delta(x-vt) - \alpha\delta(x-vt)\right\}\Psi$$

$$= -\frac{\hbar^2}{2m}\left\{\frac{m^2\alpha^2}{\hbar^4}\underbrace{\left(2\theta(x-vt)-1\right)^2}_{1} - \frac{m^2v^2}{\hbar^2} - 2i\frac{mv}{\hbar}\frac{m\alpha}{\hbar^2}\left[2\theta(x-vt)-1\right]\right\}\Psi$$

$$= \left\{-\frac{m\alpha^2}{2\hbar^2} + \frac{1}{2}mv^2 + i\,\frac{mv\alpha}{\hbar}\left[2\theta(x-vt)-1\right]\right\}\Psi = i\hbar\frac{\partial\Psi}{\partial t} \quad \text{(eq. [*] above). QED}$$

(b) $|\Psi|^2 = \frac{m\alpha}{\hbar^2}e^{-2m\alpha|y|/\hbar^2}$ $\quad (y \equiv x - vt)$. Check normalization: $2\frac{m\alpha}{\hbar^2}\int_0^\infty e^{-2m\alpha y/\hbar^2}dy = \frac{2m\alpha}{\hbar^2}\cdot\frac{\hbar^2}{2m\alpha} = 1$ ✓

$\langle H\rangle = \int_{-\infty}^{\infty}\Psi^* H\Psi\,dx$. $\quad$ But $H\Psi = i\hbar\frac{\partial\Psi}{\partial t}$, which we calculated above (*).

$$= \int\left\{\frac{im\alpha v}{\hbar}\left[2\theta(y)-1\right] + E + \frac{1}{2}mv^2\right\}|\Psi|^2dy = \boxed{E + \frac{1}{2}mv^2}. \text{ (Note that } [2\theta(y)-1] \text{ is odd in } y.)$$

Interpretation: the wave packet is dragged along (at speed v) with the delta-function. The total energy is the energy it would have in a stationary delta-function (E), plus kinetic energy due to the motion ($\frac{1}{2}mv^2$).

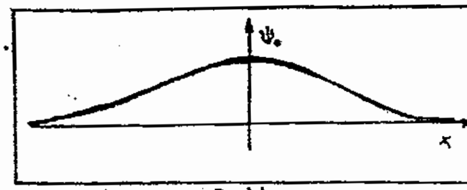
Problem 2.48 (a) $\psi_0 = A\,\text{sech}(ax)$; $\frac{d\psi_0}{dx} = -Aa\,\text{sech}(ax)\tanh(ax)$; $\frac{d^2\psi_0}{dx^2} = -Aa^2\left[-\text{sech}(ax)\tanh^2(ax) + \text{sech}(ax)\text{sech}^2(ax)\right]$

$H\psi_0 = -\frac{\hbar^2}{2m}\frac{d^2\psi_0}{dx^2} - \frac{\hbar^2a^2}{m}\text{sech}^2(ax)\psi_0 = \frac{\hbar^2}{2m}Aa^2\left[-\text{sech}(ax)\tanh^2(ax) + \text{sech}^3(ax)\right] - \frac{\hbar^2a^2}{m}A\,\text{sech}^3(ax)$

$= \frac{\hbar^2a^2A}{2m}\left[-\text{sech}(ax)\tanh^2(ax) + \text{sech}^3(ax) - 2\,\text{sech}^3(ax)\right] = -\frac{\hbar^2a^2}{2m}A\,\text{sech}(ax)\left[\quad \tanh^2(ax) + \text{sech}^2 ax\right]$.

But $(\tanh^2\theta + \operatorname{sech}^2\theta) = \frac{\sinh^2\theta}{\cosh^2\theta} + \frac{1}{\cosh^2\theta} = \frac{\sinh^2\theta + 1}{\cosh^2\theta} = 1$, so $H\psi_0 = -\frac{\hbar^2 a^2}{2m}\psi_0$. QED. $\boxed{E = -\frac{\hbar^2 a^2}{2m}}$.

$1 = |A|^2 \int_{-\infty}^{\infty} \operatorname{sech}^2(ax)\,dx = |A|^2 \frac{1}{a}\tanh(ax)\Big|_{-\infty}^{\infty} = \frac{2}{a}|A|^2 \Rightarrow \boxed{A = \sqrt{\frac{a}{2}}}$.



(b) $\frac{d\psi_k}{dx} = \frac{A}{ik+a}\left[(ik - a\tanh(ax))ik - a^2\operatorname{sech}^2 ax\right]e^{ikx}$

$$\frac{d^2\psi_k}{dx^2} = \frac{A}{ik+a}\left\{ik\left[(ik - a\tanh(ax))ik - a^2\operatorname{sech}^2 ax\right] - a^2 ik\operatorname{sech}^2 ax + a^3 2\operatorname{sech}^2 ax\tanh ax\right\}e^{ikx}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_k}{dx^2} + V\psi_k = \frac{A}{ik+a}\left\{-\frac{\hbar^2 ik}{2m}\left[-k^2 - iak\tanh ax - a^2\operatorname{sech}^2 ax\right] + \frac{\hbar^2}{2m}a^2 ik\operatorname{sech}^2 ax - \frac{\hbar^2 a^3}{m}\operatorname{sech}^2 ax\tanh ax - \frac{\hbar^2 a^2}{m}\operatorname{sech}^2 ax\,(ik - a\tanh ax)\right\}e^{ikx}$$

$$= \frac{Ae^{ikx}}{ik+a}\cdot\frac{\hbar^2}{2m}\left\{ik^3 - ak^2\tanh ax + ia^2k\operatorname{sech}^2 ax + ia^2k\operatorname{sech}^2 ax - 2a^3\operatorname{sech}^2 ax\tanh ax - 2ia^2k\operatorname{sech}^2 ax + 2a^3\operatorname{sech}^2 ax\tanh ax\right\} = \frac{Ae^{ikx}}{ik+a}\cdot\frac{\hbar^2}{2m}\cdot k^2(ik - a\tanh ax) = \frac{\hbar^2 k^2}{2m}\psi_k$$

$= E\psi_k$. QED.

As $x \to +\infty$, $\tanh(ax) \to +1$, so $\boxed{\psi_k(x) \to A\left(\frac{ik-a}{ik+a}\right)e^{ikx}}$, which represents a transmitted wave. $\boxed{R = 0}$. $T = \left|\frac{ik-a}{ik+a}\right|^2 = \left(\frac{-ik-a}{-ik+a}\right)\left(\frac{ik-a}{ik+a}\right) = \boxed{1}$.

(c) For scattering from the left $(G = 0)$, $\psi(x) = \begin{cases} Ae^{ikx} + B(0) & \\ A\left(\frac{ik-a}{ik+a}\right)e^{ikx} & x \to \infty \end{cases}$, so $S_{11} = 0$, $S_{21} = \frac{ik-a}{ik+a}$.

From Problem 2.35 we immediately conclude: $S_{22} = 0$, $S_{21} = S_{12}$.

$\boxed{S = \left(\frac{ik-a}{ik+a}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$. For bound states, set $k \to i\kappa$. Then $S = \left(\frac{-\kappa-a}{-\kappa+a}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which blows up at $\kappa = a$, or $E = -\frac{\hbar^2\kappa^2}{2m} = \boxed{-\frac{\hbar^2 a^2}{2m}}$. This agrees with (a). Evidently there is just one bound state.

PROBLEM 2.49 (a) $B = S_{11}A + S_{12}G \Rightarrow G = \frac{1}{S_{12}}(B - S_{11}A) = M_{21}A + M_{22}B \Rightarrow M_{21} = -\frac{S_{11}}{S_{12}}$, $M_{22} = \frac{1}{S_{12}}$.

$F = S_{21}A + S_{22}B = S_{21}A + \frac{S_{22}}{S_{12}}(B - S_{11}A) = -\frac{(S_{11}S_{22} - S_{12}S_{21})}{S_{12}}A + \frac{S_{22}}{S_{12}}B = M_{11}A + M_{12}B \Rightarrow M_{11} = -\frac{\det S}{S_{12}}$, $M_{12} = \frac{S_{22}}{S_{12}}$.

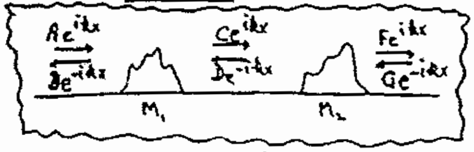
$\boxed{M = \frac{1}{S_{12}}\begin{pmatrix} -\det S & S_{22} \\ -S_{11} & 1 \end{pmatrix}}$. Conversely: $G = M_{21}A + M_{22}B \Rightarrow B = \frac{1}{M_{22}}(G - M_{21}A) = S_{11}A + S_{12}G \Rightarrow$

$S_{11} = -\frac{M_{21}}{M_{22}}$; $S_{12} = \frac{1}{M_{22}}$. $F = M_{11}A + M_{12}B = M_{11}A + \frac{M_{12}}{M_{22}}(G - M_{21}A)$

$F = \frac{(M_{11}M_{22} - M_{12}M_{21})}{M_{22}}A + \frac{M_{12}}{M_{22}}G = S_{21}A + S_{22}G \Rightarrow S_{21} = \frac{\det M}{M_{22}}$; $S_{22} = \frac{M_{12}}{M_{22}}$. $\boxed{S = \frac{1}{M_{22}}\begin{pmatrix} -M_{21} & 1 \\ \det M & M_{12} \end{pmatrix}}$.

It happens that the time-reversal invariance of the Schrödinger equation, plus conservation of probability, requires $M_{22} = M_{11}^*$, $M_{21} = M_{12}^*$, and $\det M = 1$; but I won't use this here. See Merzbacher's Quantum Mechanics. Similarly, for even potentials $S_{11} = S_{22}$, $S_{12} = S_{21}$ (2.35)

$R_l = |S_{11}|^2 = \boxed{\left|\frac{M_{21}}{M_{22}}\right|^2}$; $T_l = |S_{21}|^2 = \boxed{\left|\frac{\det M}{M_{22}}\right|^2}$; $R_r = |S_{22}|^2 = \boxed{\left|\frac{M_{12}}{M_{22}}\right|^2}$; $T_r = |S_{12}|^2 = \boxed{\frac{1}{|M_{22}|^2}}$.

(b) 

$\begin{pmatrix}F\\G\end{pmatrix} = M_2 \begin{pmatrix}C\\D\end{pmatrix}$; $\begin{pmatrix}C\\D\end{pmatrix} = M_1 \begin{pmatrix}A\\B\end{pmatrix}$, so

$\begin{pmatrix}F\\G\end{pmatrix} = M_2 M_1 \begin{pmatrix}A\\B\end{pmatrix} = M \begin{pmatrix}A\\B\end{pmatrix}$. $\therefore M = M_2 M_1$. QED.

(c) $\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & (x<a) \\ Fe^{ikx} + Ge^{-ikx} & (x>a) \end{cases}$. $\begin{cases}\text{Continuity of } \psi: \; Ae^{ika} + Be^{-ika} = Fe^{ika} + Ge^{-ika} \\ \text{Discontinuity of } \psi': \; ik(Fe^{ika} - Ge^{-ika}) - ik(Ae^{ika} - Be^{-ika}) = -\frac{2m\alpha}{\hbar^2}\psi(a) \end{cases}$

$$= -\frac{2m\alpha}{\hbar^2}(Ae^{ika} + Be^{-ika}).$$

$$Fe^{2ika} + G = Ae^{2ika} + B$$

$$Fe^{2ika} - G = Ae^{2ika} - B + i\frac{2m\alpha}{\hbar^2 k}(Ae^{2ika} + B)$$

Add: $2Fe^{2ika} = 2Ae^{2ika} + i\frac{2m\alpha}{\hbar^2 k}(Ae^{2ika} + B) \Rightarrow F = \left[1 + i\frac{m\alpha}{\hbar^2 k}\right]A + i\frac{m\alpha}{\hbar^2 k}e^{-2ika}B = M_{11}A + M_{12}B$.

So $M_{11} = (1+i\beta)$; $M_{12} = i\beta e^{-2ika}$; $\beta \equiv \frac{m\alpha}{\hbar^2 k}$.

Subtract: $2G = 2B - 2i\beta e^{2ika}A - 2i\beta B \Rightarrow G = (1-i\beta)B - i\beta e^{2ika}A = M_{21}A + M_{22}B$, so

$M_{21} = -i\beta e^{2ika}$; $M_{22} = (1-i\beta)$.

$$\therefore \boxed{M = \begin{pmatrix} (1+i\beta) & i\beta e^{-2ika} \\ -i\beta e^{2ika} & (1-i\beta) \end{pmatrix}}.$$

(d) $M_1 = \begin{pmatrix} (1+i\beta) & i\beta e^{-2ika} \\ -i\beta e^{2ika} & (1-i\beta) \end{pmatrix}$; to get M_2, just switch the sign of a: $M_2 = \begin{pmatrix} (1+i\beta) & i\beta e^{2ika} \\ -i\beta e^{-2ika} & (1-i\beta) \end{pmatrix}$.

$$M = M_2 M_1 = \boxed{\begin{bmatrix} \left[1 + 2i\beta + \beta^2(e^{4ika} - 1)\right] & 2i\beta\left[\cos(2ka) + \beta\sin(2ka)\right] \\ -2i\beta\left[\cos(2ka) + \beta\sin(2ka)\right] & \left[1 - 2i\beta + \beta^2(e^{-4ika} - 1)\right] \end{bmatrix}}$$

$$T = T_l = T_r = \frac{1}{|M_{22}|^2} \Rightarrow T^{-1} = \left[1 + 2i\beta + \beta^2(e^{4ika} - 1)\right]\left[1 - 2i\beta + \beta^2(e^{-4ika} - 1)\right]$$

$$= 1 - 2i\beta + \beta^2 e^{-4ika} - \beta^2 + 2i\beta + 4\beta^2 + 2i\beta^3 e^{-4ika} - 2i\beta^3 + \beta^2 e^{4ika} - \beta^2 - 2i\beta^3 e^{4ika} + 2i\beta^3$$

$$+ \beta^4(1 - e^{4ika} - e^{-4ika} + 1) = 1 + 2\beta^2 + \beta^2(e^{4ika} + e^{-4ika}) - 2i\beta^3(e^{4ika} - e^{-4ika}) + 2\beta^4$$

$$-\beta^4(e^{4ika} + e^{-4ika}) = 1 + 2\beta^2 + 2\beta^2\cos(4ka) - 2i\beta^3 2i\sin(4ka) + 2\beta^4 - 2\beta^4\cos(4ka)$$

$$= 1 + 2\beta^2(1 + \cos(4ka)) + 4\beta^3\sin(4ka) + 2\beta^4(1 - \cos(4ka))$$

$$= 1 + 4\beta^2\cos^2(2ka) + 8\beta^3\sin(2ka)\cos(2ka) + 4\beta^4\sin^2(2ka)$$

$$\boxed{T^{-1} = 1 + 4\beta^2\left(\cos(2ka) + \beta\sin(2ka)\right)^2}$$

Chapter 3

Problem 3.1 (a) $\boxed{\text{Yes}}$; $\boxed{\text{Two-dimensional}}$. (b) $\boxed{\text{No}}$ – the sum of two such vectors has $a_z = 2$, and is not in the set. (c) $\boxed{\text{Yes}}$; $\boxed{\text{One-dimensional}}$.

Problem 3.2 (a) $\boxed{\text{Yes}}$; $\boxed{1, x, x^2, \dots, x^{N-1}}$ is a convenient basis. Dimension: $\boxed{N}$.

(b) $\boxed{\text{Yes}}$; $\boxed{1, x^2, x^4, \dots}$. Dimension $\boxed{N/2}$ (if N is even) or $\boxed{(N-1)/2}$ (if N is odd).

(c) $\boxed{\text{No}}$ – sum of two such "vectors" is not in the space.

(d) $\boxed{\text{Yes}}$; $\boxed{(x-1), (x-1)^2, (x-1)^3, \dots, (x-1)^{N-1}}$. Dimension: $\boxed{N-1}$.

(e) $\boxed{\text{No}}$ – sum of two such "vectors" would have value 2 at $x=0$.

Problem 3.3 Suppose $|\alpha\rangle = a_1|e_1\rangle + a_2|e_2\rangle + \cdots + a_n|e_n\rangle$ and $|\alpha\rangle = b_1|e_1\rangle + b_2|e_2\rangle + \cdots + b_n|e_n\rangle$.

Subtract: $0 = (a_1-b_1)|e_1\rangle + (a_2-b_2)|e_2\rangle + \cdots + (a_n-b_n)|e_n\rangle$. Suppose $a_j \neq b_j$ for some j. Then we can divide by (a_j-b_j) to get: $|e_j\rangle = -\frac{(a_1-b_1)}{(a_j-b_j)}|e_1\rangle - \frac{(a_2-b_2)}{(a_j-b_j)}|e_2\rangle - \cdots - 0|e_j\rangle - \cdots - \frac{(a_n-b_n)}{(a_j-b_j)}|e_n\rangle$, so $|e_j\rangle$ is linearly dependent on the others, and hence $\{|e_j\rangle\}$ is not a basis. If $\{|e_j\rangle\}$ is a basis, therefore, the components must all be equal ($a_1=b_1, a_2=b_2, \dots, a_n=b_n$). QED.

Problem 3.4 (i) $\langle e_1|e_1\rangle = |1+i|^2 + 1 + |i|^2 = (1+i)(1-i) + 1 + (i)(-i) = 1+1+1+1 = 4$. $\|e_1\| = 2$

$$\boxed{|e_1'\rangle = \tfrac{1}{2}(1+i)\,\hat{\imath} + \tfrac{1}{2}\hat{\jmath} + \tfrac{i}{2}\hat{k}}.$$

(ii) $\langle e_1'|e_2\rangle = \frac{1}{2}(1-i)(i) + \frac{1}{2}(3) + (\frac{-i}{2})1 = \frac{1}{2}(i+1+3-i) = 2.$

$|e_2''\rangle \equiv |e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle = (i-1-i)\hat{\imath} + (3-1)\hat{\jmath} + (1-i)\hat{k} = (-1)\hat{\imath} + (2)\hat{\jmath} + (1-i)\hat{k}$.

$\langle e_2''|e_2''\rangle = 1+4+2 = 7$. $\boxed{|e_2'\rangle = \frac{1}{\sqrt{7}}\left(-\hat{\imath} + 2\hat{\jmath} + (1-i)\hat{k}\right)}$.

(iii) $\langle e_1'|e_3\rangle = \frac{1}{2}\cdot 28 = 14$; $\langle e_2'|e_3\rangle = \frac{2}{\sqrt{7}}28 = 8\sqrt{7}$

$|e_3''\rangle = |e_3\rangle - \langle e_1'|e_3\rangle|e_1'\rangle - \langle e_2'|e_3\rangle|e_2'\rangle = |e_3\rangle - 7|e_1\rangle - 8|e_2''\rangle$

$= (0-7-7i+8)\hat{\imath} + (28-7-16)\hat{\jmath} + (0-7i-8+8i)\hat{k} = (1-7i)\hat{\imath} + 5\hat{\jmath} + (-8+i)\hat{k}$.

$\|e_3''\|^2 = 1+49+25+64+1 = 140$. $\boxed{|e_3'\rangle = \frac{1}{2\sqrt{35}}\left[(1-7i)\hat{\imath} + 5\hat{\jmath} + (-8+i)\hat{k}\right]}$.

Problem 3.5 From [3.21]: $\langle\gamma|\gamma\rangle = \langle\gamma|\left(|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}|\alpha\rangle\right) = \langle\gamma|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}\langle\gamma|\alpha\rangle$. From [3.19]:

$\langle\gamma|\beta\rangle^* = \langle\beta|\gamma\rangle = \langle\beta|\left(|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}|\alpha\rangle\right) = \langle\beta|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}\langle\beta|\alpha\rangle = \langle\beta|\beta\rangle - \frac{|\langle\alpha|\beta\rangle|^2}{\langle\alpha|\alpha\rangle}$, which is real.

$\langle\gamma|\alpha\rangle^* = \langle\alpha|\gamma\rangle = \langle\alpha|\left(|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}|\alpha\rangle\right) = \langle\alpha|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}\langle\alpha|\alpha\rangle = 0$. $\therefore$ $\langle\gamma|\alpha\rangle = 0$.

$\therefore \langle\gamma|\gamma\rangle = \langle\beta|\beta\rangle - \frac{|\langle\alpha|\beta\rangle|^2}{\langle\alpha|\alpha\rangle} \geq 0$, and hence $|\langle\alpha|\beta\rangle|^2 \leq \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle$. QED.

Problem 3.6 $\langle\alpha|\beta\rangle = (1-i)(4-i) + (i)(0) + (-i)(2-2i) = 4 - 5i - 1 - 2i - 2 = 1 - 7i$; $\langle\beta|\alpha\rangle = 1 + 7i$

$\langle\alpha|\alpha\rangle = 1+1+1+1 = 4$; $\langle\beta|\beta\rangle = 16+1+4+4 = 25$. $\cos\theta = \sqrt{\dfrac{1+49}{4\cdot 25}} = \dfrac{1}{\sqrt{2}}$. $\boxed{\theta = 45^\circ}$

Problem 3.7 Let $|\gamma\rangle \equiv |\alpha\rangle + |\beta\rangle$; $\langle\gamma|\gamma\rangle = \langle\gamma|\alpha\rangle + \langle\gamma|\beta\rangle$.

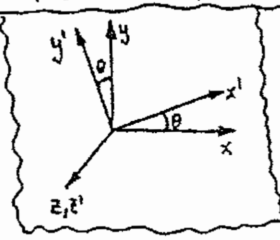
$\langle\gamma|\alpha\rangle^* = \langle\alpha|\gamma\rangle = \langle\alpha|\alpha\rangle + \langle\alpha|\beta\rangle \Rightarrow \langle\gamma|\alpha\rangle = \langle\alpha|\alpha\rangle + \langle\beta|\alpha\rangle$.

$\langle\gamma|\beta\rangle^* = \langle\beta|\gamma\rangle = \langle\beta|\alpha\rangle + \langle\beta|\beta\rangle \Rightarrow \langle\gamma|\beta\rangle = \langle\alpha|\beta\rangle + \langle\beta|\beta\rangle$.

$$\therefore \|(|\alpha\rangle + |\beta\rangle)\|^2 = \langle\gamma|\gamma\rangle = \langle\alpha|\alpha\rangle + \langle\beta|\beta\rangle + \underbrace{\langle\alpha|\beta\rangle + \langle\beta|\alpha\rangle}_{2\,\mathrm{Re}(\langle\alpha|\beta\rangle) \le 2|\langle\alpha|\beta\rangle| \le 2\sqrt{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle}\ \text{(by Schwarz inequality)}}$$

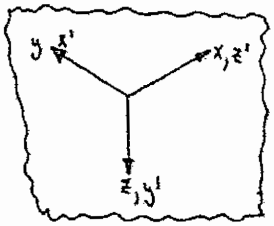
$\le \|\alpha\|^2 + \|\beta\|^2 + 2\|\alpha\|\,\|\beta\| = (\|\alpha\| + \|\beta\|)^2$. So $\|(|\alpha\rangle + |\beta\rangle)\| \le \|\alpha\| + \|\beta\|$. QED.

Problem 3.8 (a) $\hat{i}' = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$; $\hat{j}' = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$; $\hat{k}' = \hat{k}$.



$$T = \boxed{\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}}.$$

(b)



$\hat{i}' = \hat{j}$, $\hat{j}' = \hat{k}$, $\hat{k}' = \hat{i}$. $\boxed{T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}$.

(c) $\hat{i}' = \hat{i}$, $\hat{j}' = \hat{j}$, $\hat{k}' = -\hat{k} \Rightarrow$ $T = \boxed{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$.

(d) $\boxed{\text{No}}$. For a linear transformation [3.29], $\hat{T}|0\rangle = |0\rangle$, where $|0\rangle$ is the zero vector $(0,0,0)$. But under translation the zero vector goes to (x_0, y_0, z_0).

Problem 3.9 (a) $\boxed{\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ 3i & (3-2i) & 4 \end{pmatrix}}$; (b) $\begin{pmatrix} (-2+0-1) & (0+1+3i) & (i+0+2i) \\ (4+0+3i) & (0+0+9) & (-2i+0+6) \\ (4i+0+2i) & (0-2i+6) & (2+0+4) \end{pmatrix} = \boxed{\begin{pmatrix} -3 & (1+3i) & 3i \\ (4+3i) & 9 & (6-2i) \\ 6i & (6-2i) & 6 \end{pmatrix}}$

(c) $BA = \begin{pmatrix} (-2+0+2) & (2+0-2) & (2i+0-2i) \\ (0+2+0) & (0+0+0) & (0+3+0) \\ (-i+6+4i) & (i+0-4i) & (-1+9+4) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 3 \\ (6+3i) & -3i & 12 \end{pmatrix}$; $[A,B] = AB - BA = \boxed{\begin{pmatrix} -3 & (1+3i) & 3i \\ (2+3i) & 9 & (3-2i) \\ (-6+3i) & (6+i) & -6 \end{pmatrix}}$.

(d) $\boxed{\begin{pmatrix} -1 & 2 & 2i \\ 1 & 0 & -2i \\ i & 3 & 2 \end{pmatrix}}$. (e) $\boxed{\begin{pmatrix} -1 & 1 & -i \\ 2 & 0 & 3 \\ -2i & 2i & 2 \end{pmatrix}}$. (f) $\boxed{\begin{pmatrix} -1 & 2 & -2i \\ 1 & 0 & 2i \\ -i & 3 & 2 \end{pmatrix}}$. (g) $2+1+2 = \boxed{5}$

(h) $4+0+0-1-0-0 = \boxed{3}$. $B^{-1} = \frac{1}{3}\tilde{C}$; $C = \begin{pmatrix} \begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ i & 2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ i & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & -i \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & -i \\ i & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ i & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & -i \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & -i \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 2 & 0 & -i \\ -3i & 3 & -6 \\ i & 0 & 2 \end{pmatrix}$

$$\boxed{B^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -3i & i \\ 0 & 3 & 0 \\ -i & -6 & 2 \end{pmatrix}}.$$

$$BB^{-1} = \frac{1}{3}\begin{pmatrix} (4+0-1) & (-6i+0+6i) & (2i+0-2i) \\ (0+0+0) & (0+3+0) & (0+0+0) \\ (2i+0-2i) & (3+9-12) & (-1+0+4) \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

$\det A = 0 + 6i + 4 - 0 - 6i - 4 = 0$. $\boxed{\text{No}}$ — A does not have an inverse.

Problem 3.10 (a) $\begin{pmatrix} -i+2i+2i \\ 2i+0+6 \\ -2+4+4 \end{pmatrix} = \boxed{\begin{pmatrix} 3i \\ 6+2i \\ 6 \end{pmatrix}}$; (b) $\begin{pmatrix} -i & -2i & 2 \end{pmatrix}\begin{pmatrix} 2 \\ (1-i) \\ 0 \end{pmatrix} = -2i - 2i(1-i) + 0 = \boxed{-2-4i}$;

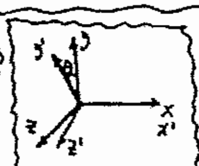
(c) $\begin{pmatrix} i & 2i & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 & -i \\ 0 & 1 & 0 \\ i & 3 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ (1-i) \\ 0 \end{pmatrix} = \begin{pmatrix} i & 2i & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 1-i \\ 3-i \end{pmatrix} = 4i + 2i(1-i) + 2(3-i) = \boxed{8+4i}$;

(d) $\begin{pmatrix} i \\ 2i \\ 2 \end{pmatrix}\begin{pmatrix} 2 & (1+i) & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 2i & (-1+i) & 0 \\ 4i & (-2+2i) & 0 \\ 4 & (2+2i) & 0 \end{pmatrix}}$.

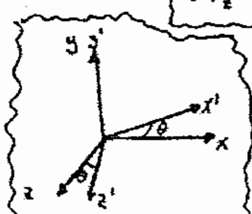
Problem 3.11 (a) $\boxed{S = \frac{1}{2}(T+\tilde{T})}$; $\boxed{A = \frac{1}{2}(T-\tilde{T})}$; (b) $\boxed{R = \frac{1}{2}(T+T^*)}$; $\boxed{I = \frac{1}{2}(T-T^*)}$;

(c) $\boxed{H = \frac{1}{2}(T+T^\dagger)}$; $\boxed{K = \frac{1}{2}(T-T^\dagger)}$.

Problem 3.12 $(\widetilde{ST})_{ki} = (ST)_{ik} = \sum_{j=1}^{n} S_{ij}T_{jk} = \sum_{j=1}^{n} \tilde{T}_{kj}\tilde{S}_{ji} = (\tilde{T}\tilde{S})_{ki}$. So $\widetilde{ST} = \tilde{T}\tilde{S}$. QED.
$(ST)^\dagger = (\widetilde{ST})^* = (\tilde{T}\tilde{S})^* = \tilde{T}^*\tilde{S}^* = T^\dagger S^\dagger$. QED. Check [3.58]: $(T^{-1}S^{-1})(ST) = T^{-1}(S^{-1}S)T = T^{-1}T = 1$, so $(ST)^{-1} = T^{-1}S^{-1}$. QED. Suppose $U^\dagger = U^{-1}$ and $W^\dagger = W^{-1}$ (i.e. U and W are unitary). Then $(WU)^\dagger = U^\dagger W^\dagger = U^{-1}W^{-1} = (WU)^{-1}$, so WU is unitary. Suppose $H = H^\dagger$ and $J = J^\dagger$ (i.e. H and J are Hermitian). Then $(HJ)^\dagger = J^\dagger H^\dagger = JH$; the product is Hermitian $\iff$ this is HJ — i.e. $\iff$ $\boxed{[H,J]=0}$ (they commute). $(U+W)^\dagger = U^\dagger + W^\dagger = U^{-1} + W^{-1} \stackrel{?}{=} (U+W)^{-1}$. $\boxed{\text{No}}$ — the sum of two unitary matrices is *not* unitary. $(H+J)^\dagger = H^\dagger + J^\dagger = H + J$. $\boxed{\text{YES}}$ — The sum of two Hermitian matrices *is* Hermitian.

Problem 3.13 

$\hat{i}' = \hat{i}$, $\hat{j}' = \cos\theta\,\hat{j} + \sin\theta\,\hat{k}$, $\hat{k}' = \cos\theta\,\hat{k} - \sin\theta\,\hat{j}$. $\boxed{T_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}}$.



$$\left.\begin{aligned} \hat{i}' &= \cos\theta\,\hat{i} - \sin\theta\,\hat{k} \\ \hat{j}' &= \hat{j} \\ \hat{k}' &= \cos\theta\,\hat{k} + \sin\theta\,\hat{i}. \end{aligned}\right\} \quad \boxed{T_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}}.$$

$$\left.\begin{aligned} \hat{i}' &= \hat{j} \\ \hat{j}' &= -\hat{i} \\ \hat{k}' &= \hat{k} \end{aligned}\right\} \quad \boxed{S = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \quad S^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$ST_xS^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -\cos\theta & 0 & -\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} = T_y.$$

$$ST_yS^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \cos\theta & \sin\theta \\ -1 & 0 & 0 \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} = T_x(-\theta).$$

Is this what we would expect? Yes: for rotation about the x axis now means rotation about the y axis, and rotation about the y axis has become rotation about the $-x$ axis — which is to say, rotation in the opposite direction about the $+x$ axis.

Problem 3.14 From [3.63] we have $A^f B^f = SA^eS^{-1}SB^eS^{-1} = S(A^eB^e)S^{-1} = SC^eS^{-1} = C^f$. ✓

Suppose $S^\dagger = S^{-1}$ and $H^e = H^{e\dagger}$ (S unitary, H^e Hermitian). Then

$$H^{f\dagger} = (SH^eS^{-1})^\dagger = (S^{-1})^\dagger H^{e\dagger} S^\dagger = SH^eS^{-1} = H^f, \text{ so } H^f \text{ is Hermitian.}$$ ✓

In an orthonormal basis, $\langle\alpha|\beta\rangle = a^\dagger b$ (eq. [3.50]). So if $\{|f_i\rangle\}$ is orthonormal, $\langle\alpha|\beta\rangle = a^{f\dagger} b^f$. But $b^f = Sb^e$ [3.62], and also $a^{f\dagger} = a^{e\dagger}S^\dagger$. So $\langle\alpha|\beta\rangle = a^{e\dagger}S^\dagger S b^e$. This is equal to $a^{e\dagger}b^e$ (and hence $\{|e_i\rangle\}$ is also orthonormal), for all vectors $|\alpha\rangle$ and $|\beta\rangle \Longleftrightarrow S^\dagger S = 1$ — i.e. S is unitary.

Problem 3.15 $\mathrm{Tr}(T_1T_2) = \sum_{i=1}^n (T_1T_2)_{ii} = \sum_{i=1}^n\sum_{j=1}^n (T_1)_{ij}(T_2)_{ji} = \sum_{j=1}^n\sum_{i=1}^n (T_2)_{ji}(T_1)_{ij} = \sum_{j=1}^n (T_2T_1)_{jj} = \mathrm{Tr}(T_2T_1)$.

$\boxed{\text{No.}}$ Counterexample: $T_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $T_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $T_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

$T_1T_2T_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \mathrm{Tr}(T_1T_2T_3) = 1.$

$T_2T_1T_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \mathrm{Tr}(T_2T_1T_3) = 0.$

Problem 3.16 $U^\dagger U = 1 \Rightarrow (U^\dagger U)_{ik} = \delta_{ik} \Rightarrow \sum_{j=1}^n (U^\dagger)_{ij}(U)_{jk} = \sum_{j=1}^n (U)^*_{ji}(U)_{jk} = \delta_{ik}$.

Construct the set of n vectors $U_{ij} \equiv a^{(j)}_i$ (i.e. the i^{th} element of the j^{th} vector is U_{ij}). $a^{(i)\dagger}a^{(j)} = \delta_{ij}$, so these vectors are orthonormal. Here $a^{(j)}$ is the j^{th} column of U.

Similarly, $UU^\dagger = 1 \Rightarrow (UU^\dagger)_{ik} = \delta_{ik} \Rightarrow \sum_{j=1}^n U_{ij}U^\dagger_{jk} = \sum_{j=1}^n U^*_{kj}U_{ij} = \delta_{ki}$. This time let the vectors $b^{(j)}$ be the rows of U: $b^{(i)}_j \equiv U_{ji}$. $b^{(k)\dagger}b^{(i)} = \sum_{j=1}^n b^{(k)*}_j b^{(i)}_j = \sum_{j=1}^n U^*_{kj}U_{ij} = \delta_{ki}$, so the rows are also orthonormal.

Problem 3.17
$$\begin{vmatrix} (\cos\theta-\lambda) & -\sin\theta \\ \sin\theta & (\cos\theta-\lambda) \end{vmatrix} = (\cos\theta-\lambda)^2 + \sin^2\theta = \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta = 0, \text{ or}$$
$$\lambda^2 - 2\lambda\cos\theta + 1 = 0.$$

$\therefore \lambda = \dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \sqrt{-\sin^2\theta} = \cos\theta \pm i\sin\theta = \boxed{e^{\pm i\theta}}$. So there are two eigenvalues, both of them complex. Only if $\sin\theta = 0$ does this matrix possess real eigenvalues — i.e. only if $\boxed{\theta = 0 \text{ or } \pi}$.

Eigenvectors: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = e^{\pm i\theta}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \cos\theta\,\alpha - \sin\theta\,\beta = (\cos\theta \pm i\sin\theta)\alpha \Rightarrow \beta = \mp i.$

Normalizing, for convenience:
$$\boxed{a^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}; \; a^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}}.$$

$(S^{-1})_{11} = a_1^{(1)} = \frac{1}{\sqrt{2}}$; $(S^{-1})_{12} = a_1^{(2)} = \frac{1}{\sqrt{2}}$; $(S^{-1})_{21} = a_2^{(1)} = -i/\sqrt{2}$; $(S^{-1})_{22} = a_2^{(2)} = i/\sqrt{2}$.

$S^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$; $S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$. $S\,TS^{-1} = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} (\cos\theta + i\sin\theta) & (\cos\theta - i\sin\theta) \\ (\sin\theta - i\cos\theta) & (\sin\theta + i\cos\theta) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} e^{i\theta} & e^{-i\theta} \\ -ie^{i\theta} & ie^{-i\theta} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2e^{i\theta} & 0 \\ 0 & 2e^{-i\theta} \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$. ✓

PROBLEM 3.18 $\begin{vmatrix} (1-\lambda) & 1 \\ 0 & (1-\lambda) \end{vmatrix} = (1-\lambda)^2 = 0 \Rightarrow \boxed{\lambda = 1}$ (only one eigenvalue).

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \alpha + \beta = \alpha \Rightarrow \beta = 0$. $\therefore$ $\boxed{a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ (only one eigenvector – up to an arbitrary constant factor).

Since the eigenvectors do not span the space, this matrix $\boxed{\text{cannot be diagonalized}}$. [If it could be diagonalized the diagonal form would have to be $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, since the only eigenvalue is 1. But in that case $1 = SMS^{-1}$. Multiplying from the left by S^{-1} and from the right by S: $S^{-1}1S = S^{-1}SMS^{-1}S = M$. But $S^{-1}1S = S^{-1}S = 1$. So $1 = M$, which is false.]

PROBLEM 3.19 Expand the determinant [3.71] by minors, using the first column:

$$\det(T - \lambda 1) = (T_{11} - \lambda)\begin{vmatrix} (T_{22}-\lambda) & \cdots & \\ \vdots & \ddots & \\ \vdots & & (T_{nn}-\lambda) \end{vmatrix} + \sum_{j=2}^{n} T_{j1}\,\text{cofactor}(T_{j1}).$$

But the cofactor of T_{j1} (for $j>1$) is missing two of the original diagonal elements: $(T_{11}-\lambda)$ (from the first column), and $(T_{jj}-\lambda)$ (from the j^{th} row). So its highest power of λ will be $(n-2)$. Thus terms in λ^n and λ^{n-1} come exclusively from the first term above. Indeed, the same argument applied now to the cofactor of $(T_{11}-\lambda)$ — and repeated as we expand that determinant — shows that only the product of the diagonal elements contributes to λ^n and λ^{n-1}:

$$(T_{11}-\lambda)(T_{22}-\lambda)\cdots(T_{nn}-\lambda) = (-\lambda)^n + (-\lambda)^{n-1}(T_{11} + T_{22} + \cdots + T_{nn}) + \cdots$$

Evidently, then, $C_n = (-1)^n$, and $C_{n-1} = (-1)^{n-1}\text{Tr}(T)$. To get C_0 – the term with no factors of λ – we simply set $\lambda = 0$. Thus $C_0 = \det(T)$.

For a 3x3 matrix, $\begin{vmatrix} (T_{11}-\lambda) & T_{12} & T_{13} \\ T_{21} & (T_{22}-\lambda) & T_{23} \\ T_{31} & T_{32} & (T_{33}-\lambda) \end{vmatrix} = (T_{11}-\lambda)(T_{22}-\lambda)(T_{33}-\lambda) + T_{12}T_{23}T_{31} + T_{13}T_{21}T_{32} - T_{31}T_{13}(T_{22}-\lambda) - T_{32}T_{23}(T_{11}-\lambda) - T_{12}T_{21}(T_{33}-\lambda)$

$= -\lambda^3 + \lambda^2(T_{11}+T_{22}+T_{33}) - \lambda(T_{11}T_{22} + T_{11}T_{33} + T_{22}T_{33}) + \lambda(T_{13}T_{31} + T_{23}T_{32} + T_{12}T_{21}) + T_{11}T_{22}T_{33} + T_{12}T_{23}T_{31} + T_{13}T_{21}T_{32}$
$- T_{31}T_{13}T_{22} - T_{32}T_{23}T_{11} - T_{12}T_{21}T_{33} = -\lambda^3 + \lambda^2\,\text{Tr}(T) + \lambda C_1 + \det(T)$, with

$$\boxed{C_1 = (T_{13}T_{31} + T_{23}T_{32} + T_{12}T_{21}) - (T_{11}T_{22} + T_{11}T_{33} + T_{22}T_{33})}.$$

PROBLEM 3.20 The characteristic equation is an n^{th} order polynomial, which can be factored in terms of its n (complex) roots: $(\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda) = (-\lambda)^n + (-\lambda)^{n-1}(\lambda_1 + \lambda_2 + \cdots + \lambda_n) + \cdots + (\lambda_1\lambda_2\cdots\lambda_n) = 0$

Comparing [3.81], it follows that $\text{Tr}(T) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ and $\det(T) = \lambda_1\lambda_2\cdots\lambda_n$. QED

PROBLEM 3.21 Let $|\gamma\rangle = |\alpha\rangle + c|\beta\rangle$. Then $\langle\gamma|\hat{T}\gamma\rangle = \langle\alpha|\hat{T}\alpha\rangle + c\langle\alpha|\hat{T}\beta\rangle + c^*\langle\beta|\hat{T}\alpha\rangle + |c|^2\langle\beta|\hat{T}\beta\rangle$ and $\langle\hat{T}\gamma|\gamma\rangle = \langle\hat{T}\alpha|\alpha\rangle + c^*\langle\hat{T}\beta|\alpha\rangle + c\langle\hat{T}\alpha|\beta\rangle + |c|^2\langle\hat{T}\beta|\beta\rangle$. Suppose $\langle\hat{T}\gamma|\gamma\rangle = \langle\gamma|\hat{T}\gamma\rangle$ for all vectors (in particular, then, $\langle\hat{T}\alpha|\alpha\rangle = \langle\alpha|\hat{T}\alpha\rangle$ and $\langle\hat{T}\beta|\beta\rangle = \langle\beta|\hat{T}\beta\rangle$). Then

$$c\langle\alpha|\hat{T}\beta\rangle + c^*\langle\beta|\hat{T}\alpha\rangle = c\langle\hat{T}\alpha|\beta\rangle + c^*\langle\hat{T}\beta|\alpha\rangle \text{ — for any complex number } c.$$

In particular, for $c=1$: $\quad \langle\alpha|\hat{T}\beta\rangle + \langle\beta|\hat{T}\alpha\rangle = \langle\hat{T}\alpha|\beta\rangle + \langle\hat{T}\beta|\alpha\rangle$,

while, for $c=i$: $\quad \langle\alpha|\hat{T}\beta\rangle - \langle\beta|\hat{T}\alpha\rangle = \langle\hat{T}\alpha|\beta\rangle - \langle\hat{T}\beta|\alpha\rangle$ (I cancelled the i's).

Adding: $\langle\alpha|\hat{T}\beta\rangle = \langle\hat{T}\alpha|\beta\rangle$. QED. $\boxed{\text{No}}$ — it is not sufficient that $\langle e_n|\hat{T}e_n\rangle = \langle\hat{T}e_n|e_n\rangle$ for every member of an orthonormal basis — you would need $\langle e_n|\hat{T}e_m\rangle = \langle\hat{T}e_n|e_m\rangle$ for all n and all m.
Counterexample: Let $\hat{T}|e_1\rangle = |e_2\rangle$, $\hat{T}|e_2\rangle = 0$. Then $\langle e_1|\hat{T}e_1\rangle = 0 = \langle\hat{T}e_1|e_1\rangle$, $\langle e_2|\hat{T}e_2\rangle = 0 = \langle\hat{T}e_2|e_2\rangle$, but $\langle e_2|\hat{T}e_1\rangle = 1$, whereas $\langle\hat{T}e_2|e_1\rangle = 0$.

PROBLEM 3.22 (a) $\mathsf{T}^\dagger = \tilde{\mathsf{T}}^* = \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix} = \mathsf{T}$. ✓

(b) $\begin{vmatrix} (1-\lambda) & (1-i) \\ (1+i) & (0-\lambda) \end{vmatrix} = -(1-\lambda)\lambda - 1 - 1 = 0;$

$$\lambda^2 - \lambda - 2 = 0; \quad \lambda = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}. \quad \boxed{\lambda_1 = 2,\ \lambda_2 = -1}.$$

(c) $\begin{pmatrix} 1 & (1-i) \\ (1+i) & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 2\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \alpha + (1-i)\beta = 2\alpha \Rightarrow \alpha = (1-i)\beta$. $|\alpha|^2 + |\beta|^2 = 1 \Rightarrow 2|\beta|^2 + |\beta|^2 = 1 \Rightarrow \beta = 1/\sqrt{3}$.

$\boxed{\mathbf{a}^{(1)} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1-i \\ 1 \end{pmatrix}}$. $\begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \alpha + (1-i)\beta = -\alpha$; $\alpha = -\frac{1}{2}(1-i)\beta$. $\frac{1}{4}\cdot 2|\beta|^2 + |\beta|^2 = 1 \Rightarrow$

$\frac{3}{2}|\beta|^2 = 1$; $\beta = \sqrt{\frac{2}{3}}$. $\quad \boxed{\mathbf{a}^{(2)} = \frac{1}{\sqrt{6}}\begin{pmatrix} i-1 \\ 2 \end{pmatrix}}$. $\left[\mathbf{a}^{(1)\dagger}\mathbf{a}^{(2)} = \frac{1}{3\sqrt{2}}\left((1+i)\ \ 1\right)\begin{pmatrix} i-1 \\ 2 \end{pmatrix} = \frac{1}{3\sqrt{2}}\left[i-1-1-i+2\right] = 0 \checkmark\right]$

(d) $[3.80] \Rightarrow (S^{-1})_{11} = a^{(1)}_1 = \frac{1}{\sqrt{3}}(1-i)$; $(S^{-1})_{12} = a^{(2)}_1 = \frac{1}{\sqrt{6}}(i-1)$; $(S^{-1})_{21} = a^{(1)}_2 = \frac{1}{\sqrt{3}}$; $(S^{-1})_{22} = a^{(2)}_2 = \frac{2}{\sqrt{6}}$.

$$S^{-1} = \frac{1}{\sqrt{3}}\begin{pmatrix} (1-i) & (i-1)/\sqrt{2} \\ 1 & \sqrt{2} \end{pmatrix}; \quad S = (S^{-1})^\dagger = \frac{1}{\sqrt{3}}\begin{pmatrix} (1+i) & 1 \\ \frac{(-i-1)}{\sqrt{2}} & \sqrt{2} \end{pmatrix}.$$

$$STS^{-1} = \frac{1}{3}\begin{pmatrix} (1+i) & 1 \\ -\frac{(1+i)}{\sqrt{2}} & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & (1-i) \\ (1+i) & 0 \end{pmatrix}\begin{pmatrix} (1-i) & (i-1)/\sqrt{2} \\ 1 & \sqrt{2} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} (1+i) & 1 \\ -\frac{(1+i)}{\sqrt{2}} & \sqrt{2} \end{pmatrix}\begin{pmatrix} 2(1-i) & (1-i)/\sqrt{2} \\ 2 & -\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 6 & 0 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}. \checkmark$$

(e) $\boxed{\mathrm{Tr}(\mathsf{T}) = 1}$; $\det(\mathsf{T}) = 0 - (1+i)(1-i) = \boxed{-2}$. $\mathrm{Tr}(STS^{-1}) = 2 - 1 = 1$ ✓; $\det(STS^{-1}) = -2$ ✓

PROBLEM 3.23 (a) $\det(\mathsf{T}) = 8 - 1 - 1 - 2 - 2 - 2 = \boxed{0}$. $\mathrm{Tr}(\mathsf{T}) = \boxed{6}$.

(b) $\begin{vmatrix} (2-\lambda) & i & 1 \\ -i & (2-\lambda) & i \\ 1 & -i & (2-\lambda) \end{vmatrix} = (2-\lambda)^3 - 1 - 1 - (2-\lambda) - (2-\lambda) - (2-\lambda) = 8 - 12\lambda + 6\lambda^2 - \lambda^3 - 8 + 3\lambda = 0 \Rightarrow$

$$-\lambda^3 + 6\lambda^2 - 9\lambda = 0, \text{ or } -\lambda(\lambda^2 - 6\lambda + 9) = 0. \quad \boxed{\lambda_1 = 0,\ \lambda_2 = \lambda_3 = 3}$$

$\lambda_1+\lambda_2+\lambda_3 = 6 = \mathrm{Tr}(\mathsf{T})$. ✓ $\lambda_1\lambda_2\lambda_3 = 0 = \det(\mathsf{T})$. ✓ Diagonal form: $\boxed{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}}$.

(c) $\begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0 \Rightarrow \begin{cases} 2\alpha + i\beta + \gamma = 0 \\ -i\alpha + 2\beta + i\gamma = 0 \Rightarrow \alpha + 2i\beta - \gamma = 0 \end{cases}$ add $\Rightarrow 3\alpha + 3i\beta = 0 \Rightarrow \beta = i\alpha$. $2\alpha - \alpha + \gamma = 0 \Rightarrow \gamma = -\alpha$.

$\mathsf{a}^{(1)} = \begin{pmatrix} \alpha \\ i\alpha \\ -\alpha \end{pmatrix}$. Normalizing: $|\alpha|^2 + |\alpha|^2 + |\alpha|^2 = 1 \Rightarrow \alpha = \frac{1}{\sqrt{3}}$. $\boxed{\mathsf{a}^{(1)} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix}}$.

$\begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 3\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \Rightarrow \begin{cases} 2\alpha + i\beta + \gamma = 3\alpha \Rightarrow -\alpha + i\beta + \gamma = 0 \\ -i\alpha + 2\beta + i\gamma = 3\beta \Rightarrow \alpha - i\beta - \gamma = 0 \\ \alpha - i\beta + 2\gamma = 3\gamma \Rightarrow \alpha - i\beta - \gamma = 0 \end{cases}$

The three equations are redundant – there is only one condition here: $\alpha - i\beta - \gamma = 0$. We could pick $\gamma = 0$, $\beta = -i\alpha$; or $\beta = 0$, $\gamma = \alpha$. Then $\mathsf{a}_0^{(2)} = \begin{pmatrix} \alpha \\ -i\alpha \\ 0 \end{pmatrix}$; $\mathsf{a}_0^{(3)} = \begin{pmatrix} \alpha \\ 0 \\ \alpha \end{pmatrix}$.

But these are not orthogonal, so we use the Gram-Schmidt procedure (Problem 3.4): first normalize $\boxed{\mathsf{a}^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}}$. $\mathsf{a}^{(2)\dagger}\mathsf{a}_0^{(3)} = \frac{\alpha}{\sqrt{2}}(1\ i\ 0)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{\alpha}{\sqrt{2}}$. $\mathsf{a}_0^{(3)} - (\mathsf{a}^{(2)\dagger}\mathsf{a}_0^{(3)})\mathsf{a}^{(2)} = \alpha\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{\alpha}{2}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} = \alpha\begin{pmatrix} 1/2 \\ i/2 \\ 1 \end{pmatrix}$.

Normalize: $|\alpha|^2\left(\frac{1}{4} + \frac{1}{4} + 1\right) = \frac{3}{2}|\alpha|^2 \Rightarrow \alpha = \sqrt{\frac{2}{3}}$. $\boxed{\mathsf{a}^{(3)} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ i \\ 2 \end{pmatrix}}$.

$\mathsf{a}^{(1)\dagger}\mathsf{a}^{(2)} = \frac{1}{\sqrt{6}}(1\ {-i}\ {-1})\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} = \frac{1}{\sqrt{6}}(1-1+0) = 0$ ✓ $\mathsf{a}^{(1)\dagger}\mathsf{a}^{(3)} = \frac{1}{3\sqrt{2}}(1\ {-i}\ {-1})\begin{pmatrix} 1 \\ i \\ 2 \end{pmatrix} = \frac{1}{3\sqrt{2}}(1+1-2) = 0$. ✓

(d) S^{-1} is the matrix whose columns are the eigenvectors of T (eq. [3.80]):

$$\mathsf{S}^{-1} = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2}i & -\sqrt{3}i & i \\ -\sqrt{2} & 0 & 2 \end{pmatrix}; \quad \mathsf{S} = (\mathsf{S}^{-1})^\dagger = \boxed{\frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & -\sqrt{2}i & -\sqrt{2} \\ \sqrt{3} & \sqrt{3}i & 0 \\ 1 & -i & 2 \end{pmatrix}}.$$

$$\mathsf{S}\mathsf{T}\mathsf{S}^{-1} = \frac{1}{6}\begin{pmatrix} \sqrt{2} & -\sqrt{2}i & -\sqrt{2} \\ \sqrt{3} & \sqrt{3}i & 0 \\ 1 & -i & 2 \end{pmatrix}\underbrace{\begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix}\begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2}i & -\sqrt{3}i & i \\ -\sqrt{2} & 0 & 2 \end{pmatrix}}_{\begin{pmatrix} 0 & 3\sqrt{3} & 3 \\ 0 & -3\sqrt{3}i & 3i \\ 0 & 0 & 6 \end{pmatrix}} = \frac{1}{6}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}. \checkmark$$

PROBLEM 3.24 (a) $\langle \hat{U}\alpha|\hat{U}\beta\rangle = \langle \hat{U}^\dagger\hat{U}\alpha|\beta\rangle = \langle\alpha|\beta\rangle$. (b) $\hat{U}|\alpha\rangle = \lambda|\alpha\rangle \Rightarrow \langle\hat{U}\alpha|\hat{U}\alpha\rangle = |\lambda|^2\langle\alpha|\alpha\rangle$. But from (a) this is also $\langle\alpha|\alpha\rangle$. So $|\lambda| = 1$. (c) $\hat{U}|\alpha\rangle = \lambda|\alpha\rangle$, $\hat{U}|\beta\rangle = \mu|\beta\rangle \Rightarrow |\beta\rangle = \mu\hat{U}^{-1}|\beta\rangle$, so $\hat{U}^\dagger|\beta\rangle = \frac{1}{\mu}|\beta\rangle = \mu^*|\beta\rangle$ (from (b)). So $\langle\beta|\hat{U}\alpha\rangle = \lambda\langle\beta|\alpha\rangle$ $= \langle\hat{U}^\dagger\beta|\alpha\rangle = \mu\langle\beta|\alpha\rangle$, or $(\lambda-\mu)\langle\beta|\alpha\rangle = 0$. So if $\lambda \neq \mu$, then $\langle\beta|\alpha\rangle = 0$. QED.

Problem 3.25 $|e_1\rangle = 1$; $\langle e_1|e_1\rangle = \int_{-1}^{1} 1\,dx = 2$; $\boxed{|e_1'\rangle = \frac{1}{\sqrt{2}}}$. $|e_2\rangle = x$; $\langle e_1'|e_2\rangle = \frac{1}{\sqrt{2}}\int_{-1}^{1} x\,dx = 0$;

$\langle e_2|e_2\rangle = \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$. So $\boxed{|e_2'\rangle = \sqrt{\frac{3}{2}}\,x}$. $|e_3\rangle = x^2$; $\langle e_1'|e_3\rangle = \frac{1}{\sqrt{2}}\int_{-1}^{1} x^2 dx = \frac{1}{\sqrt{2}}\frac{2}{3}$.

$\langle e_2'|e_3\rangle = \sqrt{\frac{3}{2}}\int_{-1}^{1} x^3 dx = 0$. So (Problem 3.4) $|e_3''\rangle = |e_3\rangle - \frac{1}{\sqrt{2}}\frac{2}{3}|e_1'\rangle = x^2 - \frac{1}{3}$. $\langle e_3''|e_3''\rangle = \int_{-1}^{1}(x^2-\frac{1}{3})^2 dx$

$= \left[\frac{x^5}{5} - \frac{2}{3}\frac{x^3}{3} + \frac{1}{9}x\right]\Big|_{-1}^{1} = \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{18-10}{45} = \frac{8}{45}$. $|e_3'\rangle = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) = \boxed{\sqrt{\frac{5}{2}}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)}$. $|e_4\rangle = x^3$.

$\langle e_1'|e_4\rangle = \frac{1}{\sqrt{2}}\int_{-1}^{1} x^3 dx = 0$; $\langle e_2'|e_4\rangle = \sqrt{\frac{3}{2}}\int_{-1}^{1} x^4 dx = \sqrt{\frac{3}{2}}\cdot\frac{2}{5}$; $\langle e_3'|e_4\rangle = \sqrt{\frac{5}{2}}\int_{-1}^{1}\left(\frac{3}{2}x^5 - \frac{1}{2}x^3\right)dx = 0$.

$\therefore |e_4''\rangle = |e_4\rangle - \langle e_2'|e_4\rangle|e_2'\rangle = x^3 - \sqrt{\frac{3}{2}}\frac{2}{5}\sqrt{\frac{3}{2}}x = x^3 - \frac{3}{5}x$. $\langle e_4''|e_4''\rangle = \int_{-1}^{1}(x^3 - \frac{3}{5}x)^2 dx$

$= \left[\frac{x^7}{7} - \frac{2\cdot 3}{5}\frac{x^5}{5} + \frac{9}{25}\frac{x^3}{3}\right]\Big|_{-1}^{1} = \frac{2}{7} - \frac{12}{25} + \frac{18}{75} = \frac{150-252+126}{3\cdot 7\cdot 25} = \frac{8}{7\cdot 25}$. $|e_4'\rangle = \frac{5}{2}\sqrt{\frac{7}{2}}\left(x^3 - \frac{3}{5}x\right)$

$|e_4'\rangle = \boxed{\sqrt{\frac{7}{2}}\left(\frac{5}{2}x^3 - \frac{3}{2}x\right)}$. $P_0(x) = \sqrt{2}\,|e_1'\rangle = \boxed{1}$; $P_1(x) = \sqrt{\frac{2}{3}}\,|e_2'\rangle = \boxed{x}$; $P_2(x) = \sqrt{\frac{2}{5}}\,|e_3'\rangle$

$\boxed{P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}}$; $P_3(x) = \sqrt{\frac{2}{7}}\,|e_4'\rangle = \boxed{\frac{5}{2}x^3 - \frac{3}{2}x}$.

Problem 3.26 $f(x) = \sum_{n=0}^{N-1}\left[\left(\frac{a_n}{2i}\right)\left(e^{in\pi x} - e^{-in\pi x}\right) + \left(\frac{b_n}{2}\right)\left(e^{in\pi x} + e^{-in\pi x}\right)\right]$

$$= \sum_{n=0}^{N-1}\left[\left(\frac{a_n}{2i} + \frac{b_n}{2}\right)e^{in\pi x} + \left(\frac{-a_n}{2i} + \frac{b_n}{2}\right)e^{-in\pi x}\right] = \sum_{n=-(N-1)}^{(N-1)} c_n e^{in\pi x},$$

where $\left\{\begin{array}{l} c_n = \frac{1}{2}(-ia_n + b_n),\ \text{for } n = 1,2,3,\dots,N-1 \\ c_0 = b_0 \\ c_n = \frac{1}{2}(ia_n + b_n),\ \text{for } n = -1,-2,-3,\dots,-(N-1) \end{array}\right\}$. So the set does span the space. Is it orthonormal?

$$\langle e_m|e_n\rangle = \frac{1}{2}\int_{-1}^{1} e^{-im\pi x}e^{in\pi x}dx = \left\{\begin{array}{ll} \frac{1}{2}\,\frac{e^{-i(m-n)\pi x}}{-i(m-n)\pi}\Big|_{-1}^{1} = 0, & \text{for } m \neq n \\ \frac{1}{2}\int_{-1}^{1} dx = 1, & \text{for } m = n \end{array}\right\} = \delta_{mn}.$$ Yes — orthonormal.

So it's also a basis (i.e. no "extra" functions included), since orthogonal vectors are necessarily linearly independent. Dimension: $2(N-1) + 1 = \boxed{2N-1}$.

Problem 3.27 The question is whether $\int_{-\infty}^{\infty} p(x)q(x)e^{-x^2}dx$ is necessarily finite. The answer is yes, for $\int_{-\infty}^{\infty} x^n e^{-x^2}dx$ is finite for any power n. $\langle e_1|e_1\rangle = \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$. So $\boxed{|e_1'\rangle = (\pi)^{-1/4}e^{-x^2/2}}$.

$\langle e_1'|e_2\rangle = (\pi)^{-1/4}\int_{-\infty}^{\infty} x e^{-x^2}dx = 0$. $\langle e_2|e_2\rangle = \int_{-\infty}^{\infty} x^2 e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$. So $\boxed{|e_2'\rangle = (\pi)^{-1/4}\sqrt{2}\,x\,e^{-x^2/2}}$.

$\langle e_1'|e_3\rangle = \pi^{-1/4}\int_{-\infty}^{\infty} x^2 e^{-x^2}dx = \pi^{-1/4}\frac{\sqrt{\pi}}{2}$; $\langle e_2'|e_3\rangle = \pi^{-1/4}\sqrt{2}\int_{-\infty}^{\infty} x^3 e^{-x^2}dx = 0$. $|e_3''\rangle = |e_3\rangle - \pi^{-1/4}\frac{\sqrt{\pi}}{2}|e_1'\rangle$

$|e_3''\rangle = x^2 e^{-x^2/2} - \frac{1}{2}e^{-x^2/2} = (x^2 - \frac{1}{2})e^{-x^2/2}$. $\langle e_3''|e_3''\rangle = \int_{-\infty}^{\infty}(x^2-\frac{1}{2})^2 e^{-x^2}dx = \int_{-\infty}^{\infty}(x^4 - x^2 + \frac{1}{4})e^{-x^2}dx$.

$\langle e_3''|e_3''\rangle = \frac{3}{4}\sqrt{\pi} - \frac{1}{2}\sqrt{\pi} + \frac{1}{4}\sqrt{\pi} = \frac{1}{2}\sqrt{\pi}$; $\boxed{|e_3'\rangle = \pi^{-1/4}\sqrt{2}\,(x^2-\tfrac{1}{2})e^{-x^2/2}}$

$\langle e_1'|e_4\rangle = \pi^{-1/4}\int_{-\infty}^{\infty} x^3 e^{-x^2}dx = 0$; $\langle e_2'|e_4\rangle = \pi^{-1/4}\sqrt{2}\int_{-\infty}^{\infty} x^4 e^{-x^2}dx = \pi^{-1/4}\sqrt{2}\,\frac{3}{4}\sqrt{\pi}$;

$\langle e_3'|e_4\rangle = \pi^{-1/4}\sqrt{2}\int_{-\infty}^{\infty}(x^5 - \frac{x^3}{2})e^{-x^2}dx = 0.$ ∴ $|e_4''\rangle = |e_4\rangle - \pi^{-1/4}\sqrt{2}\,\frac{3}{4}\sqrt{\pi}\,|e_2'\rangle$

$|e_4''\rangle = x^3e^{-x^2/2} - \frac{3}{2}\,x\,e^{-x^2/2} = (x^3-\frac{3}{2}x)e^{-x^2/2}$. $\langle e_4''|e_4''\rangle = \int_{-\infty}^{\infty}(x^6-3x^4+\frac{9}{4}x^2)e^{-x^2}dx$.

$\langle e_4''|e_4''\rangle = \frac{15}{8}\sqrt{\pi} - 3\cdot\frac{3}{4}\sqrt{\pi} + \frac{9}{4}\cdot\frac{1}{2}\sqrt{\pi} = \frac{3}{4}\sqrt{\pi}$. $\boxed{|e_4'\rangle = \frac{2}{\sqrt{3}}\pi^{-1/4}(x^3-\tfrac{3}{2}x)e^{-x^2/2}}$.

Comment: Except for the common factor $\pi^{-1/4}e^{-x^2/2}$, we are generating a sequence of polynomials that are reminiscent of the Hermite polynomials (Table 2.1). In fact, $|e_{n+1}\rangle = \frac{\pi^{-1/4}}{\sqrt{2^n n!}}H_n(x)$.

PROBLEM 3.28 $\left(\frac{d^2}{dx^2}-x^2\right)e^{-x^2/2} = \frac{d}{dx}\left(-xe^{-x^2/2}\right) - x^2e^{-x^2/2} = -e^{-x^2/2} + x^2e^{-x^2/2} - x^2e^{-x^2/2} = -e^{-x^2/2}$, so it is an eigenfunction, and the eigenvalue is $\boxed{-1}$.

PROBLEM 3.29 (a) $\hat{D}|e_n\rangle = \frac{d}{dx}x^{n-1} = (n-1)x^{n-2} = (n-1)|e_{n-1}\rangle$.
The nonzero matrix elements are $1, 2, 3, \dots, (N-1)$, on the subdiagonal just above the main diagonal.

(a)

$$D = \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 2 & \cdots & & 0 \\ \cdots & & & 3 & & \vdots \\ & & & & 4 & \vdots \\ & & & & \ddots & 0 \\ & & & & & (N-1) \\ 0 & 0 & & \cdots & & 0 \end{bmatrix}$$

(b) $\hat{D}|e_n\rangle = in\pi\frac{1}{\sqrt{2}}e^{in\pi x} = in\pi|e_n\rangle$. The nonzero matrix elements are on the main diagonal: $in\pi$.

(b)

$$D = i\pi\begin{bmatrix} \ddots & & & & & \\ & 2 & & & 0 & \\ & & 1 & & & \\ & & & 0 & & \\ & 0 & & & -1 & \\ & & & & & -2 \\ & & & & & & \ddots \end{bmatrix}$$

(c) $\hat{X}|e_n\rangle = x^n = |e_{n+1}\rangle$. The nonzero matrix elements are on the subdiagonal just below the main diagonal, and they are all 1's. Only with respect to an orthonormal basis is a Hermitian operator represented by a matrix equal to its transpose conjugate.

(c)

$$X = \begin{bmatrix} 0 & 0 & & & & \\ 1 & 0 & & & 0 & \\ 0 & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & 0 & & & \ddots & 0 \quad 0 \\ & & & & & 1 \quad 0 \end{bmatrix}$$

PROBLEM 3.30 $\hat{X}|e_n\rangle = \sqrt{n-\frac{1}{2}}\;x\,P_{n-1} = \sqrt{n-\frac{1}{2}}\;\frac{1}{2n-1}\left[nP_n + (n-1)P_{n-2}\right] = \frac{1}{2\sqrt{n-\frac{1}{2}}}\left\{\frac{n}{\sqrt{n+\frac{1}{2}}}|e_{n+1}\rangle + \frac{(n-1)}{\sqrt{n-\frac{3}{2}}}|e_{n-1}\rangle\right\}$. ∴ $\langle e_m|\hat{X}|e_n\rangle = X_{mn} = \frac{1}{2}\left\{\frac{n}{\sqrt{(n-\frac{1}{2})(n+\frac{1}{2})}}\delta_{m,n+1} + \frac{(n-1)}{\sqrt{n-\frac{1}{2}}\sqrt{n-\frac{3}{2}}}\delta_{m,n-1}\right\}$, or:

$$\underline{\underline{X_{mn} = \frac{n}{\sqrt{(2n-1)(2n+1)}}\,\delta_{m,n+1} + \frac{m}{\sqrt{(2m-1)(2m+1)}}\,\delta_{m,n-1}}}$$

The first term gives subdiagonal below main: $\frac{1}{\sqrt{1\cdot 3}}, \frac{2}{\sqrt{3\cdot 5}}, \frac{3}{\sqrt{5\cdot 7}}, \dots$; the second gives the same above the main diagonal.

$$X = \begin{bmatrix} 0 & 1/\sqrt{1\cdot3} & 0 & 0 & \\ 1/\sqrt{1\cdot3} & 0 & 2/\sqrt{3\cdot5} & 0 & \vdots \\ 0 & 2/\sqrt{3\cdot5} & 0 & 3/\sqrt{5\cdot7} & \vdots \\ 0 & 0 & 3/\sqrt{5\cdot7} & 0 & \ddots \\ & \cdots & & & \end{bmatrix}$$

$$D|e_n\rangle = \sqrt{n-\tfrac12}\,\frac{d}{dx}P_{n-1} = \sqrt{n-\tfrac12}\sum_k (2n-4k-3)\,P_{n-2k-2}$$
$$= \sqrt{n-\tfrac12}\sum_k \frac{(2n-4k-3)}{\sqrt{n-2k-3/2}}\,|e_{n-2k-1}\rangle$$
$$= \sqrt{2n-1}\sum_k \sqrt{2n-4k-3}\;|e_{n-2k-1}\rangle .$$

$\therefore D_{mn} = \langle e_m|D|e_n\rangle = \sqrt{2n-1}\sum_k \sqrt{2n-4k-3}\;\delta_{m,n-2k-1}$. $\quad m = n-2k-1 \Rightarrow 2k = n-m-1;\ k=\frac12(n-m-1)$.

So n and m must be of opposite parity. $2n-4k = 2n-2(n-m-1) = 2n-2n+2m+2 = 2m+2$, so $2n-4k-3 = 2m-1$. $\therefore$ $\underline{D_{mn} = \sqrt{2n-1}\sqrt{2m-1}}$, with $n>m$ (else k is negative) and of opposite parity.

$$D = \begin{bmatrix} 0 & \sqrt{1\cdot3} & 0 & \sqrt{1\cdot7} & 0 & \sqrt{1\cdot11} & \cdots \\ 0 & 0 & \sqrt{3\cdot5} & 0 & \sqrt{3\cdot9} & 0 & \sqrt{3\cdot13} & \cdots \\ 0 & 0 & 0 & \sqrt{5\cdot7} & 0 & \sqrt{5\cdot11} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{7\cdot9} & 0 & \sqrt{7\cdot13} & \cdots \\ & & \cdots & & & & \end{bmatrix}$$

By inspection, $X = X^\dagger$, but $iD \neq (iD)^\dagger$.

Problem 3.31 Integrate by parts twice: $\langle f|\hat{D}^2 g\rangle = \int_a^b f^* \frac{d^2 g}{dx^2}dx = f^*\frac{dg}{dx}\Big|_a^b - \int_a^b \frac{df^*}{dx}\frac{dg}{dx}dx$

$= f^*\frac{dg}{dx}\Big|_a^b - \frac{df^*}{dx}g\Big|_a^b + \int_a^b \frac{d^2f^*}{dx^2}g\,dx = \left(f^*\frac{dg}{dx} - \frac{df^*}{dx}g\right)\Big|_a^b + \langle \hat{D}^2 f|g\rangle$. So $\hat{D}^2$ is Hermitian for functions that satisfy $\boxed{\left(f^*\frac{dg}{dx} - \frac{df^*}{dx}g\right)\Big|_a^b = 0}$. $\hat{D}^2|e_n\rangle = \frac{d^2}{dx^2}(x^{n-1}) = (n-1)(n-2)x^{n-3} = (n-1)(n-2)|e_{n-2}\rangle$.

$D_{mn} = \langle e_m|\hat{D}^2|e_n\rangle = (n-1)(n-2)\delta_{m,n-2}$. Nonzero matrix elements occur on the diagonal two above the main diagonal.

$$D = \begin{bmatrix} 0 & 0 & 1\cdot2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 2\cdot3 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 3\cdot4 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 4\cdot5 & \ddots \\ & & & \cdots & & & \end{bmatrix}$$

This is the square of $\hat{D}$ in 3.29a:

$$D\cdot D = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \vdots \\ 0 & 0 & 2 & 0 & 0 & \vdots \\ 0 & 0 & 0 & 3 & 0 & \\ 0 & 0 & 0 & 0 & 4 & \\ & & \cdots & & & \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \vdots \\ 0 & 0 & 2 & 0 & 0 & \vdots \\ 0 & 0 & 0 & 3 & 0 & \vdots \\ 0 & 0 & 0 & 0 & 4 & \\ & & \cdots & & & \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1\cdot2 & 0 & 0 & \vdots \\ 0 & 0 & 0 & 2\cdot3 & 0 & \vdots \\ 0 & 0 & 0 & 0 & 3\cdot4 & \vdots \\ & & \cdots & & & \end{pmatrix}. \checkmark$$

Problem 3.32 (a) $\left.\begin{aligned} f(1) &= \sum\left[a_n\sin(n\pi) + b_n\cos(n\pi)\right] = \sum(-1)^n b_n \\ f(-1) &= \sum\left[a_n\sin(-n\pi) + b_n\cos(-n\pi)\right] = \sum(-1)^n b_n \end{aligned}\right\}$ $\therefore f(-1) = f(1)$ for all functions in this space.

So the boundary term in [3.97] vanishes, and hence $i\hat{D}$ is Hermitian.

(b) $i\hat{D}f = i\frac{df}{dx} = \lambda f \Rightarrow \frac{df}{dx} = -i\lambda f \Rightarrow f = Ae^{-i\lambda x}$. But $f(-1)=f(1) \Rightarrow Ae^{i\lambda} = Ae^{-i\lambda} \Rightarrow e^{2i\lambda} = 1$.

So $\boxed{\lambda = n\pi \quad (n=0,\pm1,\pm2,\dots)}$. Normalizing: $1 = |A|^2\int_{-1}^{1} e^{i\lambda x}e^{-i\lambda x}\,dx = 2|A|^2 \Rightarrow A = \frac{1}{\sqrt{2}}$. $\boxed{f(x) = \frac{1}{\sqrt{2}}e^{-in\pi x}}$

(c) The eigenvalues are real. ✓ For $n \neq m$, $\langle f_m | f_n\rangle = \frac{1}{2}\int_{-1}^{1} e^{im\pi x}e^{-in\pi x}\,dx = \frac{1}{2}\left.\frac{e^{i(m-n)\pi x}}{i(m-n)\pi}\right|_{-1}^{1} = 0$,
so they are orthogonal. ✓ And they are complete, because they are precisely the basis [3.93].

(d) This is Dirichlet's Theorem, in the form of Problem 2.20 (a).

PROBLEM 3.33 (a) $\langle(\alpha f+\beta g)|(\alpha f+\beta g)\rangle = |\alpha|^2\langle f|f\rangle + \alpha^*\beta\langle f|g\rangle + \beta^*\alpha\langle g|f\rangle + |\beta|^2\langle g|g\rangle$. By assumption $\langle f|f\rangle$ and $\langle g|g\rangle$ are finite; by the Schwarz inequality (but see footnote 15) $\langle f|g\rangle$ and $\langle g|f\rangle$ are also finite.

(b) $\langle f|f\rangle = 2\int_0^1 x^{2\nu}dx = 2\left.\frac{x^{2\nu+1}}{2\nu+1}\right|_0^1 = \frac{1}{\nu-\frac{1}{2}}\left(1-0^{2\nu+1}\right)$. This is finite provided the power of zero is positive: $2\nu+1>0$, so $\nu > -\frac{1}{2}$. [If $\nu = -\frac{1}{2}$, $\int_0^1 x^{2\nu}dx = \int_0^1 \frac{1}{x}dx = \ln x\Big|_0^1 = \infty$, so $\nu = -\frac{1}{2}$ itself is no good.] $\boxed{\nu > -\frac{1}{2}}$.

(c) $f(x) = \frac{1}{1+x}$, convergent for $|x|<1$. $\langle f|f\rangle = \int_{-a}^{a}\frac{1}{(1+x)^2}dx = \left.\frac{-1}{(1+x)}\right|_{-a}^{a} = \frac{-1}{1+a} + \frac{1}{1-a} = \frac{2a}{1-a^2}$.
$\boxed{0<a<1}$ (above 1 the function $f(x)$ is undefined).

(d) $\langle f|f\rangle = 2\int_0^\infty e^{-2x}dx = 1$, so it is in L_2. $[3.111] \Rightarrow a_\lambda = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\lambda x}e^{-|x|}dx$

$$a_\lambda = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{i\lambda x}e^{-x}dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}e^{i\lambda x}e^{x}dx = \frac{1}{\sqrt{2\pi}}\left\{\int_0^\infty e^{(i\lambda-1)x}dx + \int_0^\infty e^{(-i\lambda-1)x}dx\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{e^{(i\lambda-1)x}}{(i\lambda-1)} + \frac{e^{-(i\lambda+1)x}}{-(i\lambda+1)}\right\}\Bigg|_0^\infty = \frac{1}{\sqrt{2\pi}}\left\{\frac{1}{1-i\lambda}+\frac{1}{1+i\lambda}\right\} = \boxed{\sqrt{\frac{2}{\pi}}\,\frac{1}{(1+\lambda^2)}}.$$

(e) $\hat{D}^2_{\lambda\mu} = \langle f_\lambda|\hat{D}^2|f_\mu\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\lambda x}\left(\frac{d^2}{dx^2}\right)e^{-i\mu x}dx = -\mu^2\,\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(\lambda-\mu)x}dx = \boxed{-\mu^2\,\delta(\lambda-\mu)}$.

PROBLEM 3.34 When we say a Taylor series converges to a function, we mean that the sequence of partial sums:
$$f_N \equiv \sum_{i=0}^{N} a_i x^i$$
Converges to f (as $N\to\infty$) — but the coefficients a_i are fixed numbers (to wit: $\frac{1}{i!}\left(\frac{d}{dx}\right)^i f(x)\Big|_{x=0}$) – you're not allowed to change a_1 (say) when you go from $N=400$ to $N=401$. If you are free to alter the coefficients as you increase N, then you can fit non-differentiable, discontinuous functions. When you expand in terms of Legendre polynomials this is precisely what happens, for when you tack on a higher P_n, this brings in powers less than n that are contained in P_n. [Incidentally, when you expand in orthogonal functions (such as P_n, but not simple powers) the coefficients do not change – the "best fitting" partial sum has coefficients given by Fourier's trick at each order.]

Problem 3.35 (a) $1 = \langle\Psi|\Psi\rangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_n^* c_m \langle e_n|e_m\rangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_n^* c_m \delta_{nm} = \sum_{n=1}^{\infty} |c_n|^2$. QED.

(b) $1 = \langle\Psi|\Psi\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c_k^* c_l \langle e_k|e_l\rangle \, dl\, dk = \iint_{-\infty}^{\infty} c_k^* c_l \, \delta(k-l)\, dl\, dk = \int_{-\infty}^{\infty} |c_k|^2 dk$. QED.

(c) From [3.120] and postulate 2:

$$\langle Q\rangle = \langle\Psi|\hat{Q}\Psi\rangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_n^* c_m \langle e_n|\hat{Q}e_m\rangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_n^* c_m \lambda_m \langle e_n|e_m\rangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_n^* c_m \lambda_m \delta_{nm} = \sum_{n=1}^{\infty} \lambda_n |c_n|^2.$$

From [3.125] and postulate 2:

$$\langle Q\rangle = \langle\Psi|\hat{Q}\Psi\rangle = \iint_{-\infty}^{\infty} c_k^* c_l \langle e_k|\hat{Q}e_l\rangle \, dl\, dk = \iint_{-\infty}^{\infty} c_k^* c_l \lambda_l \langle e_k|e_l\rangle \, dl\, dk = \iint_{-\infty}^{\infty} c_k^* c_l \lambda_l \, \delta(k-l)\, dl\, dk$$

$$= \int_{-\infty}^{\infty} \lambda_k |c_k|^2 dk.$$

Problem 3.36 (a) $\Psi(x,0) = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2) \Rightarrow c_1 = \frac{1}{\sqrt{2}}, \; c_2 = \frac{1}{\sqrt{2}}$, all others zero. $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$.

Possibilities are: $\boxed{E_1 = \frac{\pi^2\hbar^2}{2ma^2}, \text{ probability } \frac{1}{2}, \text{ and } E_2 = \frac{2\pi^2\hbar^2}{ma^2}, \text{ probability } \frac{1}{2}.}$

$\langle H\rangle = \frac{1}{2}\left(\frac{\pi^2\hbar^2}{2ma^2}\right) + \frac{1}{2}\left(\frac{2\pi^2\hbar^2}{ma^2}\right) = \boxed{\frac{5}{4}\frac{\pi^2\hbar^2}{ma^2}}$. (Same as before — Problem 2.6(c).)

(b) $\Psi(x,0) = \sqrt{\frac{30}{a^5}}\, x(a-x)$. $c_n = \langle\psi_n|\Psi\rangle = \sqrt{\frac{30}{a^5}}\sqrt{\frac{2}{a}}\int_0^a (ax - x^2)\sin\left(\frac{n\pi}{a}x\right)dx$

$$c_n = \frac{2\sqrt{15}}{a^3}\left\{ a\left[\frac{a^2}{n^2\pi^2}\sin\left(\frac{n\pi}{a}x\right) - \frac{ax}{n\pi}\cos\left(\frac{n\pi}{a}x\right)\right] - \left[\frac{2a^2x}{n^2\pi^2}\sin\left(\frac{n\pi}{a}x\right) - \left(\frac{a}{n\pi}\right)^3\left(\left(\frac{n\pi x}{a}\right)^2 - 2\right)\cos\left(\frac{n\pi}{a}x\right)\right]\right\}\Bigg|_0^a$$

$$= \frac{2\sqrt{15}}{a^3}\left\{-\frac{a^3}{n\pi}\cos(n\pi) + \left(\frac{a}{n\pi}\right)^3\left[(n^2\pi^2 - 2)\cos(n\pi) + 2\right]\right\} = \frac{2\sqrt{15}}{n\pi}\left[-\cos(n\pi) + \cos(n\pi) + \frac{2}{(n\pi)^2}(1 - \cos(n\pi))\right]$$

$$= \frac{4\sqrt{15}}{(n\pi)^3}\left[1 - (-1)^n\right] = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{8\sqrt{15}}{(n\pi)^3}, & \text{if } n \text{ is odd} \end{cases}.$$

Possibilities are: $\boxed{E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}, \text{ probability } \frac{64\times 15}{(n\pi)^6} \quad (n = 1, 3, 5, \ldots).}$

$\langle H\rangle = \sum_{n=1,3,5,\ldots} \frac{64\times 15}{\pi^6 n^6}\frac{n^2\pi^2\hbar^2}{2ma^2} = \frac{32\times 15\hbar^2}{\pi^4 ma^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots\right]$. Math table gives $\frac{1}{6}\left(\frac{\pi}{2}\right)^4$ for this sum. So $\langle H\rangle = \frac{32\times 15\hbar^2}{\pi^4 ma^2}\frac{1}{6}\frac{\pi^4}{16} = \boxed{\frac{5\hbar^2}{ma^2}}$. (Same as before — Problem 2.8(b)).

Problem 3.37 $\frac{\hbar}{i}\frac{d}{dx}e_p = \frac{\hbar}{i}\frac{1}{\sqrt{2\pi\hbar}}\frac{ip}{\hbar}e^{ipx/\hbar} = p\, e_p$. So it *is* an eigenfunction, with eigenvalue p.

$\langle e_p|e_q\rangle = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-ipx/\hbar} e^{iqx/\hbar} dx = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{i(q-p)y}\hbar\, dy$ (I let $y \equiv x/\hbar$).

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(q-p)y} dy = \delta(q-p)$ [2.126]. So they are "orthonormal", in the sense [3.124].

Problem 3.38. $\Psi_0(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} e^{-i\omega t/2}$. $\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-i\omega t/2}\int_{-\infty}^{\infty} e^{-ipx/\hbar} e^{-\frac{m\omega}{2\hbar}x^2}dx.$

From Problem 2.22(b):

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-i\omega t/2}\sqrt{\frac{2\pi\hbar}{m\omega}}\; e^{-p^2/2m\omega\hbar} = \boxed{\frac{1}{(\pi m\omega\hbar)^{1/4}} e^{-p^2/2m\omega\hbar} e^{-i\omega t/2}}.$$

$|\Phi(p,t)|^2 = \frac{1}{\sqrt{\pi m\omega\hbar}} e^{-p^2/m\omega\hbar}$. Maximum classical momentum: $\frac{1}{2m}p^2 = E = \frac{1}{2}\hbar\omega \Rightarrow p = \sqrt{m\omega\hbar}$.

So probability outside classical range is: $P = \int_{-\infty}^{-\sqrt{m\omega\hbar}} |\Phi|^2 dp + \int_{\sqrt{m\omega\hbar}}^{\infty} |\Phi|^2 dp = 1 - 2\int_0^{\sqrt{m\omega\hbar}} |\Phi|^2 dp$

$\int_0^{\sqrt{m\omega\hbar}} |\Phi|^2 dp = \frac{1}{\sqrt{\pi m\omega\hbar}}\int_0^{\sqrt{m\omega\hbar}} e^{-p^2/m\omega\hbar} dp$. Let $z \equiv (p/\sqrt{m\omega\hbar})\sqrt{2}$, so $dp = \sqrt{\frac{m\omega\hbar}{2}}\,dz$.

$$= \frac{1}{\sqrt{2\pi}}\int_0^{\sqrt{2}} e^{-z^2/2}dz = F(\sqrt{2}) - \tfrac{1}{2}, \text{ in CRC Table notation.}$$

$P = 1 - 2\left(F(\sqrt{2}) - \frac{1}{2}\right) = 1 - 2F(\sqrt{2}) + 1 = 2\left(1 - F(\sqrt{2})\right) = 0.157$. To two digits: $\boxed{0.16}$. (Same as Problem 2.15.)

Problem 3.39 $\left[x, \frac{p^2}{2m} + V\right] = \frac{1}{2m}[x, p^2] + [x, V]$. $[x, p^2] = xp^2 - p^2x = xp^2 - pxp + pxp - p^2x = [x,p]p + p[x,p]$.

From [3.140]: $[x, p^2] = i\hbar p + pi\hbar = 2i\hbar p$; $[x, V] = 0$. So $\left[x, \frac{p^2}{2m} + V\right] = \frac{1}{2m}2i\hbar p = i\hbar p/m$.

$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\frac{i\hbar}{m}\langle p\rangle\right)^2 = \left(\frac{\hbar}{2m}\langle p\rangle\right)^2 \Rightarrow \quad \sigma_x\sigma_H \geq \frac{\hbar}{2m}|\langle p\rangle|$. QED. For stationary states $\sigma_H = 0$ and $\langle p\rangle = 0$, so it just says $0 \geq 0$.

Problem 3.40. (a) $[A', B'] = A'B' - B'A' = SAS^{-1}SBS^{-1} - SBS^{-1}SAS^{-1} = SABS^{-1} - SBAS^{-1} = S[A,B]S^{-1} = 0$. ✓

(b) $$D_1D_2 = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \end{pmatrix}\begin{pmatrix} e_1 & & 0 \\ & e_2 & \\ 0 & & \ddots \end{pmatrix} = \begin{pmatrix} (d_1e_1) & & 0 \\ & (d_2e_2) & \\ 0 & & \ddots \end{pmatrix} = \begin{pmatrix} (e_1d_1) & & 0 \\ & (e_2d_2) & \\ 0 & & \ddots \end{pmatrix} = D_2D_1.$$ ✓

(c) $[A,B] = 0$; $\quad A|\alpha\rangle = \lambda|\alpha\rangle$. $AB|\alpha\rangle = BA|\alpha\rangle = \lambda B|\alpha\rangle$. So $B|\alpha\rangle$ is also an eigenvector of A, with the same eigenvalue λ. But the spectrum of A is nondegenerate, so $B|\alpha\rangle$ must be a multiple of $|\alpha\rangle$ itself: $B|\alpha\rangle = \mu|\alpha\rangle$. So $|\alpha\rangle$ is an eigenvector of B. ($B|\alpha\rangle$ could be zero — in which case it is not actually an eigenvector of A, but in that case $|\alpha\rangle$ is an eigenvector of B with eigenvalue 0.)

Problem 3.41 (a) $[AB, C] = ABC - CAB = ABC - ACB + ACB - CAB = A[B,C] + [A,C]B$. ✓

(b) $[x,p] = i\hbar$; $[x^2, p] = x[x,p] + [x,p]x = x(i\hbar) + (i\hbar)x = 2i\hbar x$; $[x^3, p] = x[x^2,p] + [x,p]x^2 = 3i\hbar x^2$.

Induction: assume $[x^n, p] = i\hbar n x^{n-1}$, then $[x^{n+1}, p] = x[x^n, p] + [x,p]x^n = i\hbar n x^n + i\hbar x^n = i\hbar(n+1)x^n$.

QED.

(c) $f(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow [f, p] = \sum_{n=0}^{\infty} a_n [x^n, p] = \sum_{n=0}^{\infty} a_n i\hbar n x^{n-1} = i\hbar\sum_{n=0}^{\infty} a_n n x^{n-1} = i\hbar\frac{df}{dx}$. ✓

Problem 3.42 $\frac{d\Psi}{dx} = \frac{i}{\hbar}\left[iax - ia\langle x\rangle + \langle p\rangle\right]\Psi = \frac{a}{\hbar}\left[-x + \langle x\rangle + \frac{i}{a}\langle p\rangle\right]\Psi$

$$\frac{d\Psi}{\Psi} = \frac{a}{\hbar}\left[-x + \langle x\rangle + i\langle p\rangle/a\right]dx \Rightarrow \ln\Psi = \frac{a}{\hbar}\left[-\frac{x^2}{2} + \langle x\rangle x + i\langle p\rangle x/a\right] + \text{constant}.$$

Let constant $= -\frac{\langle x\rangle^2 a}{2\hbar} + B$ (B a new constant). Then $\ln\Psi = -\frac{a}{2\hbar}(x-\langle x\rangle)^2 + i\langle p\rangle x/\hbar + B$.

$$\Psi = e^{-\frac{a}{2\hbar}(x-\langle x\rangle)^2} e^{i\langle p\rangle x/\hbar} e^{B} = A\, e^{-\frac{a}{2\hbar}(x-\langle x\rangle)^2} e^{i\langle p\rangle x/\hbar}, \text{ where } A \equiv e^{B}.$$

Problem 3.43 (a) 1 commutes with everything, so $\boxed{\frac{d}{dt}\langle\Psi|\Psi\rangle = 0}$ (this is the conservation of normalization, which we originally proved in eq. [1.27]).

(b) Anything commutes with itself, so $[H,H]=0$, and hence $\boxed{\frac{d}{dt}\langle H\rangle = 0}$ (assuming H has no explicit time dependence) — this is conservation of energy, in the sense of the comment following eq. [2.35].

(c) $[H,x] = -i\hbar p/m$ (see Problem 3.39). So $\frac{d\langle x\rangle}{dt} = \frac{i}{\hbar}\left(-\frac{i\hbar\langle p\rangle}{m}\right) = \boxed{\langle p\rangle/m}$ (eq. [1.33]).

(d) $[H,p] = \left[\frac{p^2}{2m} + V, p\right] = [V,p] = i\hbar\frac{dV}{dx}$ (Problem 3.41(c)). So $\frac{d\langle p\rangle}{dt} = \frac{i}{\hbar}\left(i\hbar\left\langle\frac{dV}{dx}\right\rangle\right) = \boxed{\left\langle -\frac{\partial V}{\partial x}\right\rangle}$.

This is Ehrenfest's theorem (eq. [1.38]).

Problem 3.44 $\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_1 e^{-iE_1t/\hbar} + \psi_2 e^{-iE_2t/\hbar}\right)$. $H^2\Psi = \frac{1}{\sqrt{2}}\left[(H^2\psi_1)e^{-iE_1t/\hbar} + (H^2\psi_2)e^{-iE_2t/\hbar}\right]$.

$H\psi_1 = E_1\psi_1 \Rightarrow H^2\psi_1 = E_1H\psi_1 = E_1^2\psi_1$. So

$$\langle H^2\rangle = \frac{1}{2}\left\langle\left(\psi_1 e^{-iE_1t/\hbar} + \psi_2 e^{-iE_2t/\hbar}\right)\middle|\left(E_1^2\psi_1 e^{-iE_1t/\hbar} + E_2^2\psi_2 e^{-iE_2t/\hbar}\right)\right\rangle$$

$$= \frac{1}{2}\left\{\langle\psi_1|\psi_1\rangle e^{iE_1t/\hbar}E_1^2 e^{-iE_1t/\hbar} + \langle\psi_1|\psi_2\rangle e^{iE_1t/\hbar}E_2^2 e^{-iE_2t/\hbar} + \langle\psi_2|\psi_1\rangle e^{iE_2t/\hbar}E_1^2\, e^{-iE_1t/\hbar}\right.$$

$$\left. + \langle\psi_2|\psi_2\rangle e^{iE_2t/\hbar}E_2^2 e^{-iE_2t/\hbar}\right\} = \frac{1}{2}\left(E_1^2 + 0 + 0 + E_2^2\right) = \frac{1}{2}\left(E_1^2 + E_2^2\right).$$

$\langle H\rangle = \frac{1}{2}(E_1 + E_2)$ (Problem 2.6(e)). $\sigma_H^2 = \langle H^2\rangle - \langle H\rangle^2 = \frac{1}{2}(E_1^2 + E_2^2) - \frac{1}{4}(E_1 + E_2)^2$

$$\sigma_H^2 = \frac{1}{4}\left(2E_1^2 + 2E_2^2 - E_1^2 - 2E_1E_2 - E_2^2\right) = \frac{1}{4}\left(E_1^2 - 2E_1E_2 + E_2^2\right) = \frac{1}{4}(E_2 - E_1)^2. \quad \boxed{\sigma_H = \frac{1}{2}(E_2 - E_1)}.$$

$$\langle x^2\rangle = \frac{1}{2}\left\{\langle\psi_1|x^2|\psi_1\rangle + \langle\psi_2|x^2|\psi_2\rangle + \langle\psi_1|x^2|\psi_2\rangle e^{i(E_1-E_2)t/\hbar} + \langle\psi_2|x^2|\psi_1\rangle e^{i(E_2-E_1)t/\hbar}\right\}$$

$$\langle\psi_n|x^2|\psi_m\rangle = \frac{2}{a}\int_0^a x^2\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)dx = \frac{1}{a}\int_0^a x^2\left[\cos\left(\frac{n-m}{a}\pi x\right) - \cos\left(\frac{n+m}{a}\pi x\right)\right]dx.$$

Now $\int_0^a x^2\cos\left(\frac{k}{a}\pi x\right)dx = \left\{\frac{2a^2x}{k^2\pi^2}\cos\left(\frac{k}{a}\pi x\right) + \left(\frac{a}{k\pi}\right)^3\left[\left(\frac{k\pi x}{a}\right)^2 - 2\right]\sin\left(\frac{k}{a}\pi x\right)\right\}\Big|_0^a$

$$= \frac{2a^3}{k^2\pi^2}\cos(k\pi) = \frac{2a^3}{k^2\pi^2}(-1)^k \quad (\text{for } k = \text{nonzero integer}).$$

$$\therefore \langle\psi_n|x^2|\psi_m\rangle = \frac{2a^2}{\pi^2}\left[\frac{(-1)^{n-m}}{(n-m)^2} - \frac{(-1)^{n+m}}{(n+m)^2}\right] = \frac{2a^2}{\pi^2}(-1)^{n+m}\frac{4nm}{(n^2-m^2)^2}.$$

So $\langle\psi_1|x^2|\psi_2\rangle = \langle\psi_2|x^2|\psi_1\rangle = -\frac{16a^2}{9\pi^2}$. Meanwhile, from Problem 2.5: $\langle\psi_n|x^2|\psi_n\rangle = a^2\left[\frac{1}{3} - \frac{1}{2(n\pi)^2}\right]$.

So $\langle x^2\rangle = \frac{1}{2}\left\{a^2\left[\frac{1}{3}-\frac{1}{2\pi^2}\right] + a^2\left[\frac{1}{3}-\frac{1}{8\pi^2}\right] - \frac{16a^2}{9\pi^2}\left[\underbrace{e^{i(E_2-E_1)t/\hbar} + e^{-i(E_2-E_1)t/\hbar}}_{2\cos\left(\frac{E_2-E_1}{\hbar}t\right).}\right]\right\}$

$$\frac{E_2-E_1}{\hbar} = \frac{(4-1)\pi^2\hbar^2}{\hbar\,2ma^2} = \frac{3\pi^2\hbar}{2ma^2} = 3\omega \text{ (in the notation of Problem 2.6 (b)).}$$

$\langle x^2\rangle = \frac{a^2}{2}\left\{\frac{2}{3} - \frac{5}{8\pi^2} - \frac{32}{9\pi^2}\cos(3\omega t)\right\}$. From Problem 2.6 (c), $\langle x\rangle = \frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t)\right]$.

So $\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{a^2}{4}\left\{\frac{4}{3} - \frac{5}{4\pi^2} - \frac{64}{9\pi^2}\cos(3\omega t) - 1 + \frac{64}{9\pi^2}\cos(3\omega t) - \left(\frac{32}{9\pi^2}\right)^2\cos^2(3\omega t)\right\}$

$\boxed{\sigma_x^2 = \frac{a^2}{4}\left[\frac{1}{3} - \frac{5}{4\pi^2} - \left(\frac{32}{9\pi^2}\right)^2\cos^2(3\omega t)\right]}$. And, from Problem 2.6(d): $\boxed{\frac{d\langle x\rangle}{dt} = \frac{8\hbar}{3ma}\sin(3\omega t).}$

Energy-time uncertainty principle [3.149] says $\sigma_H^2\sigma_x^2 \geq \frac{\hbar^2}{4}\left(\frac{d\langle x\rangle}{dt}\right)^2$. Here

$$\sigma_H^2\sigma_x^2 = \frac{1}{4}(3\hbar\omega)^2\frac{a^2}{4}\left[\frac{1}{3} - \frac{5}{4\pi^2} - \left(\frac{32}{9\pi^2}\right)^2\cos^2(3\omega t)\right] = (\hbar\omega a)^2\left(\frac{3}{4}\right)^2\left[\frac{1}{3} - \frac{5}{4\pi^2} - \left(\frac{32}{9\pi^2}\right)^2\cos^2(3\omega t)\right].$$

$$\frac{\hbar^2}{4}\left(\frac{d\langle x\rangle}{dt}\right)^2 = \left(\frac{\hbar}{2}\cdot\frac{8\hbar}{3ma}\right)^2\sin^2(3\omega t) \qquad \left(\text{since } \frac{\hbar}{ma} = \frac{2a\omega}{\pi}\right)$$

$$= \left(\frac{8}{3\pi^2}\right)^2(\hbar\omega a)^2\sin^2(3\omega t).$$

So the uncertainty principle holds if: $\left(\frac{3}{4}\right)^2\left[\frac{1}{3} - \frac{5}{4\pi^2} - \left(\frac{32}{9\pi^2}\right)^2\cos^2(3\omega t)\right] \geq \left(\frac{8}{3\pi^2}\right)^2\sin^2(3\omega t)$, which is to say, if

$$\frac{1}{3} - \frac{5}{4\pi^2} \geq \left(\frac{32}{9\pi^2}\right)^2\cos^2(3\omega t) + \left(\frac{4}{3}\cdot\frac{8}{3\pi^2}\right)^2\sin^2(3\omega t) = \left(\frac{32}{9\pi^2}\right)^2.$$

Evaluating both sides: $\frac{1}{3} - \frac{5}{4\pi^2} = 0.20668$; $\left(\frac{32}{9\pi^2}\right)^2 = 0.12978$. So it holds. (whew!)

PROBLEM 3.45 From Problem 2.40, we have: $\langle x\rangle = \frac{\hbar\ell}{m}t$, so $\boxed{\frac{d\langle x\rangle}{dt} = \frac{\hbar\ell}{m}}$, $\sigma_x^2 = \frac{1}{4w^2} = \boxed{\frac{1+\theta^2}{4a}}$, where $\theta = 2\hbar at/m$. And $\langle H\rangle = \frac{1}{2m}\langle p^2\rangle = \frac{1}{2m}\hbar^2(a+\ell^2)$. We need $\langle H^2\rangle$ (to get σ_H). Now $H = \frac{p^2}{2m}$, so $\langle H^2\rangle = \frac{1}{4m^2}\langle p^4\rangle = \frac{1}{4m^2}\int_{-\infty}^{\infty} p^4|\Phi(p,t)|^2 dp$ (3.133), where (3.132):

$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\Psi(x,t)\,dx$. From Problem 2.40: $\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+i\theta}}e^{-\ell^2/4a}e^{a(ix+\frac{\ell}{2a})^2/(1+i\theta)}$

So $\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+i\theta}}e^{-\ell^2/4a}\int_{-\infty}^{\infty} e^{-ipx/\hbar}e^{a(ix+\frac{\ell}{2a})^2/(1+i\theta)}dx$. Let $y \equiv x - \frac{i\ell}{2a}$.

$$= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+i\theta}}e^{-\ell^2/4a}e^{p\ell/2a\hbar}\int_{-\infty}^{\infty} e^{-ipy/\hbar}e^{-ay^2/(1+i\theta)}dy.$$ See Problem 2.22a for integral.

$$= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+i\theta}}e^{-\ell^2/4a}e^{p\ell/2a\hbar}\sqrt{\frac{\pi(1+i\theta)}{a}}\,e^{-\frac{p^2(1+i\theta)}{4a\hbar^2}} = \frac{1}{\sqrt{\hbar}}\left(\frac{1}{2a\pi}\right)^{1/4}e^{-\frac{\ell^2}{4a}}e^{p\ell/2a\hbar}e^{-p^2(1+i\theta)/4a\hbar^2}.$$

$$|\Phi(p,t)|^2 = \frac{1}{\sqrt{2a\pi}}\frac{1}{\hbar}e^{-\frac{\ell^2}{2a}}e^{p\ell/a\hbar}e^{-p^2/2a\hbar^2} = \frac{1}{\hbar\sqrt{2a\pi}}e^{-\frac{1}{2a}(\ell^2-2p\ell/\hbar+p^2/\hbar^2)}$$

$$= \frac{1}{\hbar\sqrt{2a\pi}}e^{-(\ell-p/\hbar)^2/2a}. \quad \langle p^4\rangle = \frac{1}{\hbar\sqrt{2a\pi}}\int_{-\infty}^{\infty}p^4e^{-(\ell-p/\hbar)^2/2a}dp.$$ Let $\frac{p}{\hbar}-\ell\equiv z$, so

$p=\hbar(z+\ell)$. $\langle p^4\rangle = \frac{1}{\hbar\sqrt{2a\pi}}\hbar^4\hbar\int_{-\infty}^{\infty}(z+\ell)^4e^{-z^2/2a}dz$. Only even powers of z will survive:

$$= \frac{\hbar^4}{\sqrt{2a\pi}}\int_{-\infty}^{\infty}(z^4+6z^2\ell^2+\ell^4)e^{-z^2/2a}dz = \frac{\hbar^4}{\sqrt{2a\pi}}\left\{\frac{3(2a)^2}{4}\sqrt{2a\pi}+6\ell^2\frac{(2a)}{2}\sqrt{2a\pi}+\ell^4\sqrt{2a\pi}\right\}$$

$$= \hbar^4(3a^2+6a\ell^2+\ell^4). \quad \therefore \langle H^2\rangle = \frac{\hbar^4}{4m^2}(3a^2+6a\ell^2+\ell^4).$$

$$\sigma_H^2 = \langle H^2\rangle - \langle H\rangle^2 = \frac{\hbar^4}{4m^2}\left[3a^2+6a\ell^2+\ell^4-a^2-2a\ell^2-\ell^4\right] = \frac{\hbar^4}{4m^2}(2a^2+4a\ell^2) = \boxed{\frac{\hbar^4 a}{2m^2}(a+2\ell^2)}$$

$$\sigma_H^2\sigma_x^2 = \frac{\hbar^4 a}{2m^2}(a+2\ell^2)\frac{1}{4a}\left(1+(2\hbar at/m)^2\right) = \frac{\hbar^4\ell^2}{4m^2}\left(1+\frac{a}{2\ell^2}\right)\left(1+\left(\frac{2\hbar at}{m}\right)^2\right) \geq \frac{\hbar^4\ell^2}{4m^2} = \frac{\hbar^2}{4}\left(\frac{\hbar\ell}{m}\right)^2 = \frac{\hbar^2}{4}\left(\frac{d\langle x\rangle}{dt}\right)^2.$$

So it works.

Problem 3.46 For $Q=x$, [3.149] says $\sigma_H\sigma_x \geq \frac{\hbar}{2}\left|\frac{d\langle x\rangle}{dt}\right|$. But $\langle p\rangle = m\frac{d\langle x\rangle}{dt}$, so

$\sigma_x\sigma_H \geq \frac{\hbar}{2m}|\langle p\rangle|$, which is the Griffiths uncertainty principle of Problem 3.39.

Problem 3.47 (a) (i) $M^2 = \begin{pmatrix}0&0&4\\0&0&0\\0&0&0\end{pmatrix}$; $M^3 = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$, so $e^M = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\begin{pmatrix}0&1&3\\0&0&4\\0&0&0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}0&0&4\\0&0&0\\0&0&0\end{pmatrix}$

$e^M = \boxed{\begin{pmatrix}1&1&5\\0&1&4\\0&0&1\end{pmatrix}}$. (ii) $M^2 = \begin{pmatrix}-\theta^2&0\\0&-\theta^2\end{pmatrix} = -\theta^2 1$; $M^3 = -\theta^2 M$; $M^4 = \theta^4 1$; etc.

$$\therefore e^M = 1+\theta\begin{pmatrix}0&1\\-1&0\end{pmatrix}-\frac{1}{2}\theta^2 1-\frac{\theta^3}{3!}\begin{pmatrix}0&1\\-1&0\end{pmatrix}+\frac{\theta^4}{4!}1+\cdots = \left(1-\frac{\theta^2}{2}+\frac{\theta^4}{4!}-\cdots\right)1+\begin{pmatrix}0&1\\-1&0\end{pmatrix}\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots\right)$$

$$= \cos\theta\begin{pmatrix}1&0\\0&1\end{pmatrix}+\sin\theta\begin{pmatrix}0&1\\-1&0\end{pmatrix} = \boxed{\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}}.$$

(b) $SMS^{-1} = D = \begin{pmatrix}d_1&&0\\&d_2&\\0&&\ddots d_n\end{pmatrix}$ for some S. $Se^MS^{-1} = S(1+M+\frac{1}{2}M^2+\frac{1}{3!}M^3+\cdots)S^{-1}$. Insert $SS^{-1}=1$:

$$Se^MS^{-1} = 1+SMS^{-1}+\frac{1}{2}SMS^{-1}SMS^{-1}+\frac{1}{3!}SMS^{-1}SMS^{-1}SMS^{-1}+\cdots = 1+D+\frac{1}{2}D^2+\frac{1}{3!}D^3+\cdots = e^D.$$

$\therefore \det(e^D) = \det(Se^MS^{-1}) = \det(S)\det(e^M)\det(S^{-1}) = \det(e^M)$. But

$$D = \begin{pmatrix}d_1&&0\\&d_2&\\0&&\ddots d_n\end{pmatrix} \Rightarrow D^2 = \begin{pmatrix}d_1^2&&0\\&d_2^2&\\0&&\ddots d_n^2\end{pmatrix},\text{ etc, so } e^D = 1+\begin{pmatrix}d_1&&\\&d_2&\\&&\ddots d_n\end{pmatrix}+\frac{1}{2}\begin{pmatrix}d_1^2&&\\&d_2^2&\\&&\ddots d_n^2\end{pmatrix}+\frac{1}{3!}\begin{pmatrix}d_1^3&&\\&d_2^3&\\&&\ddots d_n^3\end{pmatrix}+\cdots$$

$$e^D = \begin{pmatrix}e^{d_1}&&0\\&e^{d_2}&\\0&&\ddots e^{d_n}\end{pmatrix}. \quad \therefore \det(e^D) = e^{d_1}e^{d_2}\cdots e^{d_n} = e^{(d_1+d_2+\cdots+d_n)} = e^{\mathrm{Tr}\,D} = e^{\mathrm{Tr}(M)} \quad (\text{eq. [3.67]}).$$

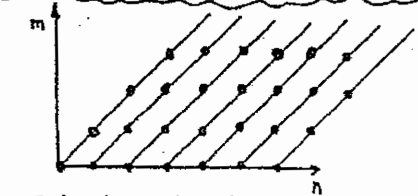
So $\det(e^M) = e^{\mathrm{Tr}M}$. QED.

(c) Matrices that commute obey the same algebraic rules as ordinary numbers, so the standard proofs of $e^{x+y} = e^x e^y$ will do the job. Here are two:

(i) Combinatorial argument: $e^{M+N} = \sum_{n=0}^{\infty} \frac{1}{n!}(M+N)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} M^m N^{n-m}$ (binomial theorem – invalid if order matters)

$$e^{M+N} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{m!(n-m)!} M^m N^{n-m}.$$

Instead of summing vertically first, for fixed n ($m: 0 \to n$), sum horizontally first, for fixed m ($n: m \to \infty$, or $k \equiv n-m: 0 \to \infty$) – see diagram.



Each dot is a term in the double sum.

$$e^{M+N} = \sum_{m=0}^{\infty} \frac{1}{m!} M^m \sum_{k=0}^{\infty} \frac{1}{k!} N^k = e^M e^N. \quad \text{QED.}$$

(ii) Analytic method: Let $S(\lambda) = e^{\lambda M} e^{\lambda N}$. $\frac{dS}{d\lambda} = Me^{\lambda M}e^{\lambda N} + e^{\lambda M} N e^{\lambda N} = (M+N) e^{\lambda M} e^{\lambda N} = (M+N)S$.

(The second equality, in which we pull N through $e^{\lambda M}$, would not hold if M and N did not commute.)

$\therefore S(\lambda) = e^{(M+N)\lambda} A$, for some constant A. But $S(0) = 1$, so $A = 1$, so $e^{\lambda M} e^{\lambda N} = e^{\lambda(M+N)}$, and (setting $\lambda = 1$) $e^M e^N = e^{(M+N)}$. [This latter method is the one that generalizes most easily when M and N do not commute – leading to the Baker–Campbell–Hausdorff Lemma.]

As a counterexample when $[M,N] \neq 0$, let $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $N = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$. Then $M^2 = 0$, $N^2 = 0$, so

$e^M = 1 + M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $e^N = 1 + N = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$; $e^M e^N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. But $(M+N) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

so (from a(ii)): $e^{M+N} = \begin{pmatrix} \cos(1) & \sin(1) \\ -\sin(1) & \cos(1) \end{pmatrix}$. The two are clearly unequal.

(d) $e^{iH} = \sum_{n=0}^{\infty} \frac{1}{n!} i^n H^n \Rightarrow (e^{iH})^\dagger = \sum_{n=0}^{\infty} \frac{1}{n!} (-i)^n (H^\dagger)^n = \sum_{n=0}^{\infty} \frac{1}{n!} (-i)^n H^n = e^{-iH}$ (for H Hermitian).

$\therefore (e^{iH})^\dagger (e^{iH}) = e^{-iH} e^{iH} = e^{i(H-H)} = 1$ (using (c)). So e^{iH} is unitary. ✓

Problem 3.48 (a) New allowed energies: $E_n = \frac{n^2 \pi^2 \hbar^2}{2m(2a)^2}$. $\Psi = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right)$, $\psi_n = \sqrt{\frac{2}{2a}} \sin\left(\frac{n\pi}{2a}x\right)$.

$$\langle \psi_n | \Psi \rangle = \frac{\sqrt{2}}{a} \int_0^a \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{n\pi}{2a}x\right) dx = \frac{\sqrt{2}}{a} \frac{1}{2} \int_0^a \left[\cos\left(\left(\frac{n}{2}-1\right)\frac{\pi x}{a}\right) - \cos\left(\left(\frac{n}{2}+1\right)\frac{\pi x}{a}\right)\right] dx. \quad \text{For } n \neq 2,$$

$$= \frac{1}{\sqrt{2}a} \left\{ \frac{\sin\left(\left(\frac{n}{2}-1\right)\frac{\pi x}{a}\right)}{\left(\frac{n}{2}-1\right)\frac{\pi}{a}} - \frac{\sin\left(\left(\frac{n}{2}+1\right)\frac{\pi x}{a}\right)}{\left(\frac{n}{2}+1\right)\frac{\pi}{a}} \right\}\Bigg|_0^a = \frac{1}{\sqrt{2}\pi} \left\{ \frac{\sin\left(\left(\frac{n}{2}-1\right)\pi\right)}{\left(\frac{n}{2}-1\right)} - \frac{\sin\left(\left(\frac{n}{2}+1\right)\pi\right)}{\left(\frac{n}{2}+1\right)} \right\}$$

$$= \frac{\sin\left[\left(\frac{n}{2}+1\right)\pi\right]}{\sqrt{2}\pi} \left[\frac{1}{\left(\frac{n}{2}-1\right)} - \frac{1}{\left(\frac{n}{2}+1\right)} \right] = \frac{4\sqrt{2}}{\pi} \frac{\sin\left[\left(\frac{n}{2}+1\right)\pi\right]}{(n^2-4)} = \begin{cases} 0, & \text{if } n \text{ is even} \\ \pm \frac{4\sqrt{2}}{\pi(n^2-4)}, & \text{if } n \text{ is odd} \end{cases}.$$

$\langle \psi_2 | \Psi \rangle = \frac{\sqrt{2}}{a} \int_0^a \sin^2\left(\frac{\pi}{a}x\right) dx = \frac{\sqrt{2}}{a} \int_0^a \frac{1}{2} dx = \frac{1}{\sqrt{2}}$. So probability of getting E_n is

$$P_n = |\langle \psi_n | \Psi \rangle|^2 = \begin{cases} \frac{1}{2}, & \text{if } n = 2 \\ \frac{32}{\pi^2(n^2-4)^2}, & \text{if } n \text{ is odd} \\ 0, & \text{otherwise} \end{cases}.$$

Most Probable: $E_2 = \boxed{\frac{\pi^2 \hbar^2}{2ma^2}}$ (same as before)

Probability: $P_2 = \boxed{1/2}$.

(b) Next most probable is $E_1 = \boxed{\frac{\pi^2\hbar^2}{8ma^2}}$, with probability $P_1 = \boxed{\frac{32}{9\pi^2} = 0.36025}$.

(c) $\langle H\rangle = \langle\Psi|H|\Psi\rangle = \frac{2}{a}\int_0^a \sin\left(\frac{\pi}{a}x\right)\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right]\sin\left(\frac{\pi}{a}x\right)dx\ldots$ but this is exactly the same as before the wall moved — for which we know the answer: $\langle H\rangle = \boxed{\frac{\pi^2\hbar^2}{2ma^2}}$.

PROBLEM 3.49 Evidently $\Psi(x,t) = c_0\psi_0(x)e^{-iE_0t/\hbar} + c_1\psi_1(x)e^{-iE_1t/\hbar}$, with $|c_0|^2 = |c_1|^2 = \frac{1}{2}$, so $c_0 = \frac{1}{\sqrt{2}}e^{i\theta_0}$, $c_1 = \frac{1}{\sqrt{2}}e^{i\theta_1}$, for some θ_0, θ_1.

$$\langle x\rangle = |c_0|^2\langle\psi_0|x|\psi_0\rangle + c_0^*c_1\langle\psi_0|x|\psi_1\rangle e^{i(E_0-E_1)t/\hbar} + c_1^*c_0\langle\psi_1|x|\psi_0\rangle e^{i(E_1-E_0)t/\hbar} + |c_1|^2\langle\psi_1|x|\psi_1\rangle.$$

But (see Problem 2.37): $\langle\psi_0|x|\psi_0\rangle = \langle\psi_1|x|\psi_1\rangle = 0$, and

$$\langle\psi_0|x|\psi_1\rangle = \frac{-i}{\sqrt{2m}\,\omega}\langle\psi_0|(a_+-a_-)|\psi_1\rangle = \frac{i}{\sqrt{2m}\,\omega}\langle\psi_0|a_-|\psi_1\rangle = \frac{i}{\sqrt{2m}\,\omega}\langle\psi_0|\sqrt{\hbar\omega}|\psi_0\rangle = i\sqrt{\frac{\hbar}{2m\omega}}.$$

$$\langle\psi_1|x|\psi_0\rangle = \frac{-i}{\sqrt{2m}\,\omega}\langle\psi_1|(a_+-a_-)|\psi_0\rangle = \frac{-i}{\sqrt{2m}\,\omega}\langle\psi_1|a_+|\psi_0\rangle = \frac{-i}{\sqrt{2m}\,\omega}\langle\psi_1|\sqrt{\hbar\omega}|\psi_1\rangle = -i\sqrt{\frac{\hbar}{2m\omega}}.$$

$$\therefore \langle x\rangle = \frac{1}{2}e^{-i\theta_0}e^{i\theta_1}\,i\sqrt{\frac{\hbar}{2m\omega}}\,e^{-i(E_1-E_0)t/\hbar} + \frac{1}{2}e^{i\theta_0}e^{-i\theta_1}e^{i(E_1-E_0)t/\hbar}\left(-i\sqrt{\frac{\hbar}{2m\omega}}\right).$$ But $(E_1-E_0) = \left(\frac{3}{2}\hbar\omega - \frac{1}{2}\hbar\omega\right) = \hbar\omega$.

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\frac{1}{2i}\left[e^{i(\omega t+\theta_0-\theta_1)} - e^{-i(\omega t+\theta_0-\theta_1)}\right] = \sqrt{\frac{\hbar}{2m\omega}}\sin(\omega t+\theta_0-\theta_1).$$ Maximum: $\boxed{\sqrt{\frac{\hbar}{2m\omega}}}$.

If it assumes this value at $t=0$, then $\sin(\theta_0-\theta_1) = 1$, so $\theta_0 - \theta_1 = \frac{\pi}{2}$; $\theta_1 = \theta_0 - \frac{\pi}{2}$;

$e^{i\theta_1} = e^{i\theta_0}e^{-i\pi/2} = -i\,e^{i\theta_0}$. $\therefore$ $\Psi(x,t) = \boxed{\frac{e^{i\theta_0}}{\sqrt{2}}\left(\psi_0 e^{-iE_0t/\hbar} - i\psi_1 e^{-iE_1t/\hbar}\right)}$,

or, explicitly: $\Psi(x,t) = \boxed{\frac{e^{i(\theta_0-\omega t/2)}}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[1 - i\sqrt{\frac{2m\omega}{\hbar}}\,x\,e^{-i\omega t}\right]e^{-m\omega x^2/2\hbar}}$.

PROBLEM 3.50. $x = -\frac{i}{\sqrt{2m}\,\omega}(a_+ - a_-)$ (Problem 2.37). $\left\{\begin{array}{l} a_+|n\rangle = \sqrt{(n+1)\hbar\omega}\,|n+1\rangle \\ a_-|n\rangle = \sqrt{n\hbar\omega}\,|n-1\rangle\end{array}\right\}$

$$\langle n|x|n'\rangle = -\frac{i}{\sqrt{2m}\,\omega}\langle n|(a_+-a_-)|n'\rangle = \frac{-i}{\sqrt{2m}\,\omega}\left\{\sqrt{(n'+1)\hbar\omega}\,\langle n|n'+1\rangle - \sqrt{n'\hbar\omega}\,\langle n|n'-1\rangle\right\}$$

$$= -i\sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n'+1}\,\delta_{n,n'+1} - \sqrt{n'}\,\delta_{n,n'-1}\right] = \boxed{i\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n'}\,\delta_{n,n'-1} - \sqrt{n}\,\delta_{n',n-1}\right)}.$$

$p = \sqrt{\frac{m}{2}}(a_+ + a_-) \Rightarrow \langle n|p|n'\rangle = \boxed{\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{n'}\,\delta_{n,n'-1} + \sqrt{n}\,\delta_{n',n-1}\right)}$. Noting that n, n': $0 \to \infty$,

the matrices are:

$$\boxed{X = i\sqrt{\frac{\hbar}{2m\omega}}\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & 0 & \\ -\sqrt{1} & 0 & \sqrt{2} & 0 & 0 & 0 & \\ 0 & -\sqrt{2} & 0 & \sqrt{3} & 0 & 0 & \cdots \\ 0 & 0 & -\sqrt{3} & 0 & \sqrt{4} & 0 & \\ 0 & 0 & 0 & -\sqrt{4} & 0 & \sqrt{5} & \\ & & & \cdots & & & \end{pmatrix}};\quad \boxed{P = \sqrt{\frac{m\hbar\omega}{2}}\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & \\ & & \cdots & & & \end{pmatrix}}$$

$$X^2 = -\frac{\hbar}{2m\omega}\begin{pmatrix} -1 & 0 & \sqrt{1\cdot 2} & 0 & 0 & 0 & \\ 0 & -3 & 0 & \sqrt{2\cdot 3} & 0 & 0 & \\ \sqrt{1\cdot 2} & 0 & -5 & 0 & \sqrt{3\cdot 4} & 0 & \cdots \\ 0 & \sqrt{2\cdot 3} & 0 & -7 & 0 & \sqrt{4\cdot 5} & \\ & & & \cdots & & & \end{pmatrix}; \quad P^2 = \frac{m\hbar\omega}{2}\begin{pmatrix} 1 & 0 & \sqrt{1\cdot 2} & 0 & 0 & \\ 0 & 3 & 0 & \sqrt{2\cdot 3} & 0 & \\ \sqrt{1\cdot 2} & 0 & 5 & 0 & \sqrt{3\cdot 4} & \cdots \\ 0 & \sqrt{2\cdot 3} & 0 & 7 & 0 & \\ & & & \cdots & & \end{pmatrix},$$

So
$$H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 X^2 = \frac{\hbar\omega}{4}\begin{pmatrix} 1 & 0 & \sqrt{1\cdot 2} & 0 & 0 & \\ 0 & 3 & 0 & \sqrt{2\cdot 3} & 0 & \\ \sqrt{1\cdot 2} & 0 & 5 & 0 & \sqrt{3\cdot 4} & \cdots \\ 0 & \sqrt{2\cdot 3} & 0 & 7 & 0 & \\ & & \cdots & & & \end{pmatrix} - \frac{\hbar\omega}{4}\begin{pmatrix} -1 & 0 & \sqrt{1\cdot 2} & 0 & 0 \\ 0 & -3 & 0 & \sqrt{2\cdot 3} & 0 \\ \sqrt{1\cdot 2} & 0 & -5 & 0 & \sqrt{4\cdot 3} \\ 0 & \sqrt{2\cdot 3} & 0 & -7 & 0 \\ & & \cdots & & \end{pmatrix}$$

$$= \frac{\hbar\omega}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & \\ 0 & 3 & 0 & 0 & \\ 0 & 0 & 5 & 0 & \cdots \\ 0 & 0 & 0 & 7 & \\ & & \cdots & & \end{pmatrix}.$$ The diagonal elements are: $H_{nn} = (n+\frac{1}{2})\hbar\omega$, as expected.

Problem 3.51 From eq. [3.132]: $\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\,\Psi(x,t)\,dx$. So, by Plancherel's theorem [eq. 2.85]
$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{ipx/\hbar}\,\Phi(p,t)\,dp. \qquad (\text{Let } p = \hbar k.)$$

$$\langle x\rangle = \int \Psi^* x \Psi\,dx = \int\left\{\frac{1}{\sqrt{2\pi\hbar}}\int e^{-ip'x/\hbar}\Phi^*(p',t)\,dp'\right\} x \left\{\frac{1}{\sqrt{2\pi\hbar}}\int e^{+ipx/\hbar}\Phi(p,t)\,dp\right\}dx.$$

But $x e^{ipx/\hbar} = -i\hbar\frac{d}{dp}\left(e^{ipx/\hbar}\right)$, so (integrating by parts):
$$x\int e^{ipx/\hbar}\Phi\,dp = \int \frac{\hbar}{i}\frac{d}{dp}\left(e^{ipx/\hbar}\right)\Phi\,dp = \int e^{ipx/\hbar}\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\Phi(p,t)\right)dp. \text{ So}$$

$$\langle x\rangle = \frac{1}{2\pi\hbar}\iiint\left[e^{-ip'x/\hbar}\Phi^*(p',t)e^{ipx/\hbar}\left(-\frac{\hbar\partial}{i\partial p}\Phi(p,t)\right)\right]dp'\,dp\,dx.$$ Do the x integral first, letting $y = \frac{x}{\hbar}$:
$$\frac{1}{2\pi\hbar}\int e^{-ip'x/\hbar}e^{ipx/\hbar}dx = \frac{1}{2\pi}\int e^{i(p-p')y}\,dy = \delta(p-p'), \quad (\text{eq } [2.126]), \text{ so}$$
$$\langle x\rangle = \iint \Phi^*(p',t)\,\delta(p-p')\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\Phi(p,t)\right)dp'\,dp = \int \Phi^*(p,t)\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\Phi(p,t)\right)dp. \quad \text{QED.}$$

Problem 3.52 $\Psi_n(x,t) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)e^{-iE_nt/\hbar}$. $\qquad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$.

$$\Phi_n(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\,\Psi_n(x,t)\,dx = \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2}{a}}\,e^{-iE_nt/\hbar}\int_0^a e^{-ipx/\hbar}\sin\left(\frac{n\pi}{a}x\right)dx.$$

Write $\sin\left(\frac{n\pi}{a}x\right) = \frac{1}{2i}\left(e^{in\pi x/a} - e^{-in\pi x/a}\right)$. Then
$$\Phi_n(p,t) = \frac{1}{\sqrt{\pi\hbar a}}\,\frac{1}{2i}\,e^{-iE_nt/\hbar}\int_0^a\left[e^{i(n\pi/a - p/\hbar)x} - e^{-i(n\pi/a + p/\hbar)x}\right]dx$$
$$= \frac{1}{\sqrt{\pi\hbar a}}\,\frac{1}{2i}\,e^{-iE_nt/\hbar}\left\{\frac{e^{i(n\pi/a - p/\hbar)x}}{i(n\pi/a - p/\hbar)} + \frac{e^{-i(n\pi/a + p/\hbar)x}}{i(n\pi/a + p/\hbar)}\right\}\Bigg|_0^a$$
$$= \frac{-1}{2\sqrt{\hbar\pi a}}\,e^{-iE_nt/\hbar}\left\{\frac{e^{i(n\pi - pa/\hbar)} - 1}{(n\pi/a - p/\hbar)} + \frac{e^{-i(n\pi + pa/\hbar)} - 1}{(n\pi/a + p/\hbar)}\right\}.$$

$$\Phi_n(p,t) = \frac{-1}{2\sqrt{\pi\hbar a}}\, e^{-iE_n t/\hbar}\left\{ \frac{(-1)^n e^{-ipa/\hbar}-1}{(n\pi - ap/\hbar)}\,a + \frac{(-1)^n e^{-ipa/\hbar}-1}{(n\pi + ap/\hbar)}\,a \right\}$$

$$= -\frac{1}{2}\sqrt{\frac{a}{\pi\hbar}}\, e^{-iE_n t/\hbar}\,\frac{2n\pi}{(n\pi)^2-(ap/\hbar)^2}\left[(-1)^n e^{-ipa/\hbar}-1\right] = \boxed{\sqrt{\frac{a\pi}{\hbar}}\,\frac{n e^{-iE_n t/\hbar}}{(n\pi)^2-(ap/\hbar)^2}\left[1-(-1)^n e^{-ipa/\hbar}\right]}.$$

<u>For even n</u>, $[\,] = 1-e^{-ipa/\hbar} = e^{-ipa/2\hbar}\left(e^{ipa/2\hbar}-e^{-ipa/2\hbar}\right) = 2ie^{-ipa/2\hbar}\sin(pa/2\hbar)$.

<u>For odd n</u>, $[\,] = 1+e^{-ipa/\hbar} = e^{-ipa/2\hbar}\left(e^{ipa/2\hbar}+e^{-ipa/2\hbar}\right) = 2\,e^{-ipa/2\hbar}\cos(pa/2\hbar)$.

$$\therefore |\Phi_n(p,t)|^2 = \boxed{4\frac{a\pi}{\hbar}\,\frac{n^2}{[(n\pi)^2-(ap/\hbar)^2]^2}\begin{Bmatrix}\sin^2(pa/2\hbar), \text{ if } n \text{ is even}\\ \cos^2(pa/2\hbar), \text{ if } n \text{ is odd}\end{Bmatrix}}.$$

[<u>Note</u>: the apparent singularity at $(n\pi)^2 = (ap/\hbar)^2$ is not a problem, for suppose $\frac{ap}{\hbar} = \pm(n\pi+\epsilon)$, for small ϵ. Then $[(n\pi)^2-(ap/\hbar)^2] = (n\pi)^2-(n\pi+\epsilon)^2 = (n\pi)^2-(n\pi)^2-2\epsilon(n\pi)-\epsilon^2 \cong -2\epsilon n\pi$, so

$\frac{n^2}{[(n\pi)^2-(ap/\hbar)^2]^2} \cong \frac{n^2}{4\epsilon^2 n^2\pi^2} = \frac{1}{4\pi^2\epsilon^2}$. But the $\sin^2$ (or $\cos^2$) in the numerator kills the ϵ^2:

$\sin^2(pa/2\hbar) = \sin^2(\pm(n\pi+\epsilon)/2) = \sin^2(\frac{n\pi}{2}+\frac{\epsilon}{2}) = \sin^2(\epsilon/2) \cong \epsilon^2/4$ (n even),

$\cos^2(pa/2\hbar) = \cos^2(\pm(n\pi+\epsilon)/2) = \cos^2(\frac{n\pi}{2}+\frac{\epsilon}{2}) = \sin^2(\epsilon/2) \cong \epsilon^2/4$ (n odd).

So $|\Phi_n|^2$ is finite <u>even</u> at such points.]

<u>Problem 3.53</u>. $[3.148] \Rightarrow \frac{d}{dt}\langle xp\rangle = \frac{i}{\hbar}\langle [H,xp]\rangle$. Eq. $[3.142] \Rightarrow [H,xp] = [H,x]p + x[H,p]$.

Problem 3.39 $\Rightarrow [H,x] = -i\hbar p/m$; Problem 3.43(d) $\Rightarrow [H,p] = i\hbar\frac{dV}{dx}$. So

$\frac{d}{dt}\langle xp\rangle = \frac{i}{\hbar}\left[-\frac{i\hbar}{m}\langle p^2\rangle + i\hbar\langle x\frac{dV}{dx}\rangle\right] = 2\langle\frac{p^2}{2m}\rangle - \langle x\frac{dV}{dx}\rangle = 2\langle T\rangle - \langle x\frac{dV}{dx}\rangle$. QED.

In a stationary state all expectation values (at least, for operators that do not depend explicitly on t) are time-independent (see item 1 on page 22). For the harmonic oscillator, $V = \frac{1}{2}m\omega^2x^2$, so

$\frac{dV}{dx} = m\omega^2 x$, so $x\frac{dV}{dx} = m\omega^2x^2 = 2V$. $\therefore [3.159] \Rightarrow 2\langle T\rangle = 2\langle V\rangle$, or $\langle T\rangle = \langle V\rangle$. QED.

In Problem 2.14 (c) we found $\langle T\rangle = \langle V\rangle = \frac{1}{4}\hbar\omega$ (for $n=0$); $\langle T\rangle = \langle V\rangle = \frac{3}{4}\hbar\omega$ (for $n=1$). ✓

In Problem 2.37 we found $\langle T\rangle = \langle V\rangle = \frac{1}{2}(n+\frac{1}{2})\hbar\omega$, for <u>all</u> stationary states. ✓

Problem 3.54 If $[H,Q]=0$, then H and Q are compatible observables, which means they have a complete set of simultaneous eigenfunctions. (See Footnote 26) $H|e_n\rangle = E_n|e_n\rangle$, $Q|e_n\rangle = \lambda_n|e_n\rangle$.
The solution to the (time-dependent) Schrödinger equation:

$$\Psi(x,t) = \sum_{n=1}^{\infty} a_n e^{-iE_nt/\hbar}|e_n\rangle$$ is also an expansion in eigenfunctions of Q, with the coefficients $c_n = a_n e^{-iE_nt/\hbar}$. Thus $|c_n|^2 = |a_n|^2$, and it's independent of time. QED.

Problem 3.55 (a) Expanding in a Taylor series: $f(x+x_0) = \sum_{n=0}^{\infty} \frac{1}{n!} x_0^n \left(\frac{d}{dx}\right)^n f(x)$. But $p = \frac{\hbar}{i}\frac{d}{dx}$, so $\frac{d}{dx} = ip/\hbar$. $\therefore f(x+x_0) = \sum_{n=0}^{\infty} \frac{1}{n!} x_0^n \left(\frac{ip}{\hbar}\right)^n f(x) = e^{ipx_0/\hbar} f(x)$.

(b) $\Psi(x,t+t_0) = \sum_{n=0}^{\infty} \frac{1}{n!} t_0^n \left(\frac{\partial}{\partial t}\right)^n \Psi(x,t)$. But $i\hbar\frac{\partial\Psi}{\partial t} = H\Psi$. [Note: it is emphatically NOT the case that $i\hbar\frac{\partial}{\partial t} = H$ — these two operators have the same effect only when acting on solutions to the (time-dependent) Schrödinger equation — as here.] Also: $\left(i\hbar\frac{\partial}{\partial t}\right)^2\Psi = i\hbar\frac{\partial}{\partial t}(H\Psi) = H\left(i\hbar\frac{\partial\Psi}{\partial t}\right) = H^2\Psi$, provided H is not explicitly dependent on t. And so on. So

$$\Psi(x,t+t_0) = \sum_{n=0}^{\infty} \frac{1}{n!} t_0^n \left(-\frac{i}{\hbar}H\right)^n \Psi = e^{-iHt_0/\hbar}\Psi(x,t).$$

(c) $\langle Q\rangle_{t+t_0} = \langle \Psi(x,t+t_0)|Q(x,p,t+t_0)|\Psi(x,t+t_0)\rangle$. But $\Psi(x,t+t_0) = e^{-iHt_0/\hbar}\Psi(x,t)$, so—
Using the Hermiticity of H to write $\left(e^{-iHt_0/\hbar}\right)^\dagger = e^{iHt_0/\hbar}$:

$\langle Q\rangle_{t+t_0} = \langle \Psi(x,t)|e^{iHt_0/\hbar}Q(x,p,t+t_0)e^{-iHt_0/\hbar}|\Psi(x,t)\rangle$. If $t_0 = dt$ is very small, expanding to first order, we have:

$$\langle Q\rangle_t + \frac{d\langle Q\rangle}{dt}dt = \langle\Psi(x,t)|\underbrace{\left(1+iH/\hbar\, dt\right)\left(Q(x,p,t)+\frac{\partial Q}{\partial t}dt\right)\left(1-iH/\hbar\, dt\right)}_{Q(x,p,t)+i\frac{H}{\hbar}dt\,Q - Q\left(i\frac{H}{\hbar}dt\right)+\frac{\partial Q}{\partial t}dt = Q+\frac{i}{\hbar}[H,Q]dt+\frac{\partial Q}{\partial t}dt}|\Psi(x,t)\rangle$$

$$= \langle Q\rangle_t + \frac{i}{\hbar}\langle[H,Q]\rangle dt + \left\langle\frac{\partial Q}{\partial t}\right\rangle dt$$

$$\therefore \frac{d\langle Q\rangle}{dt} = \frac{i}{\hbar}\langle[H,Q]\rangle + \left\langle\frac{\partial Q}{\partial t}\right\rangle.$$

Problem 3.56 $\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_1 e^{-iE_1t/\hbar} + \psi_2 e^{-iE_2t/\hbar}\right)$. $\langle\Psi(x,t)|\Psi(x,0)\rangle = 0 \Rightarrow$

$\frac{1}{2}\left\{e^{+iE_1t/\hbar}\langle\psi_1|\psi_1\rangle + e^{+iE_1t/\hbar}\langle\psi_1|\psi_2\rangle + e^{+iE_2t/\hbar}\langle\psi_2|\psi_1\rangle + e^{+iE_2t/\hbar}\langle\psi_2|\psi_2\rangle\right\} = \frac{1}{2}\left[e^{iE_1t/\hbar} + e^{iE_2t/\hbar}\right] = 0$,
or $e^{iE_2t/\hbar} = -e^{iE_1t/\hbar}$, so $e^{i(E_2-E_1)t/\hbar} = -1 = e^{i\pi} \Rightarrow \frac{(E_2-E_1)t}{\hbar} = \pi$ (orthogonality also at $3\pi, 5\pi$, etc, but this is the first occurrence). $\therefore \Delta t \equiv t/\pi = \frac{\hbar}{E_2-E_1}$. But $\Delta E = \sigma_H = \frac{1}{2}(E_2-E_1)$. (Problem 3.44). So $\Delta t\,\Delta E = \frac{\hbar}{2}$.

PROBLEM 3.57 (a) $P^2|\beta\rangle = P(P|\beta\rangle) = P(\langle\alpha|\beta\rangle|\alpha\rangle) = \langle\alpha|\beta\rangle(P|\alpha\rangle) = \langle\alpha|\beta\rangle\underbrace{\langle\alpha|\alpha\rangle}_{1}|\alpha\rangle = \langle\alpha|\beta\rangle|\alpha\rangle = P|\beta\rangle$.

Since $P^2|\beta\rangle = P|\beta\rangle$ for any vector $|\beta\rangle$, $P^2 = P$. QED. [Note: to say two operators are equal means that they have the same effect on all vectors.]

(b) Since $\{|e_j\rangle\}$ is a basis, any vector $|\beta\rangle$ can be written as a linear combination:

$$|\beta\rangle = \sum_{j=1}^{n} b_j|e_j\rangle.$$ By Fourier's trick, since $\{|e_j\rangle\}$ are orthonormal, $\langle e_j|\beta\rangle = b_j$.

So $|\beta\rangle = \sum_{j=1}^{n}|e_j\rangle\langle e_j|\beta\rangle = \left(\sum_{j=1}^{n}|e_j\rangle\langle e_j|\right)|\beta\rangle$. Since this is true for any $|\beta\rangle$, it follows that

$$\sum_{j=1}^{n}|e_j\rangle\langle e_j| = 1.$$ QED.

(c) $|\alpha\rangle = \sum_{j=1}^{n} a_j|e_j\rangle \Rightarrow Q|\alpha\rangle = \sum_{j=1}^{n} a_j Q|e_j\rangle = \sum_{j=1}^{n}\langle e_j|\alpha\rangle\lambda_j|e_j\rangle = \left(\sum_{j=1}^{n}\lambda_j|e_j\rangle\langle e_j|\right)|\alpha\rangle.$

So $Q = \sum_{j=1}^{n}\lambda_j|e_j\rangle\langle e_j|$. QED.

PROBLEM 3.58 (a) $\det\begin{pmatrix}(h-\lambda) & g\\ g & (h-\lambda)\end{pmatrix} = (h-\lambda)^2 - g^2 = 0 \Rightarrow h-\lambda = \pm g \Rightarrow \boxed{\lambda = h\mp g}.$

$\begin{pmatrix}h & g\\ g & h\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = (h\mp g)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \Rightarrow h\alpha + g\beta = (h\mp g)\alpha = h\alpha \mp g\alpha \Rightarrow \beta = \mp\alpha.$ Upper sign $\Rightarrow \beta = -\alpha$:

So $\boxed{|u\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}}$; lower sign $\beta = \alpha \Rightarrow \boxed{|l\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}}$.

(b) $$|\Psi(0)\rangle = \begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{\sqrt{2}}(|u\rangle + |l\rangle) \Rightarrow |\Psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i(h-g)t/\hbar}|u\rangle + e^{-i(h+g)t/\hbar}|l\rangle\right]$$

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-iht/\hbar}\left[e^{igt/\hbar}|u\rangle + e^{-igt/\hbar}|l\rangle\right] = \frac{1}{2}e^{-iht/\hbar}\left[\begin{pmatrix}e^{igt/\hbar}\\ -e^{igt/\hbar}\end{pmatrix} + \begin{pmatrix}e^{-igt/\hbar}\\ e^{-igt/\hbar}\end{pmatrix}\right]$$

$$= e^{-iht/\hbar}\begin{pmatrix}\frac{1}{2}\left(e^{igt/\hbar} + e^{-igt/\hbar}\right)\\ -\frac{1}{2}\left(e^{igt/\hbar} - e^{-igt/\hbar}\right)\end{pmatrix} = \boxed{e^{-iht/\hbar}\begin{pmatrix}\cos(gt/\hbar)\\ -i\sin(gt/\hbar)\end{pmatrix}}.$$

CHAPTER 4

Problem 4.1 (a) $[x,y]=xy-yx=0$, etc, so $\boxed{[r_i,r_j]=0}$. $[p_x,p_y]f=\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\left(\frac{\hbar}{i}\frac{\partial f}{\partial y}\right)-\left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right)\left(\frac{\hbar}{i}\frac{\partial f}{\partial x}\right)$

$$=-\hbar^2\left(\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x}\right)=0$$

(by equality of cross-derivatives), so $\boxed{[p_i,p_j]=0}$.

$[x,p_x]f=\frac{\hbar}{i}\left(x\frac{\partial f}{\partial x}-\frac{\partial}{\partial x}(xf)\right)=\frac{\hbar}{i}\left(x\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial x}-f\right)=i\hbar f$, so $[x,p_x]=i\hbar$ — likewise $[y,p_y]$ and $[z,p_z]$.

$[y,p_x]f=\frac{\hbar}{i}\left(y\frac{\partial f}{\partial x}-\frac{\partial}{\partial x}(yf)\right)=\frac{\hbar}{i}\left(y\frac{\partial f}{\partial x}-y\frac{\partial f}{\partial x}\right)=0$ (since $\frac{\partial y}{\partial x}=0$). So $[y,p_x]=0$, and same goes for the other "mixed" commutators. Thus $\boxed{[r_i,p_j]=-[p_j,r_i]=i\hbar\delta_{ij}}$.

(b) Use [3.148]: $\frac{d\langle x\rangle}{dt}=\frac{i}{\hbar}\langle[H,x]\rangle$; $[H,x]=\left[\frac{p^2}{2m}+V,x\right]=\frac{1}{2m}[p_x^2+p_y^2+p_z^2,x]=\frac{1}{2m}[p_x^2,x]$

$$=\frac{1}{2m}\{p_x[p_x,x]+[p_x,x]p_x\}=\frac{1}{2m}\left((-i\hbar)p_x+(-i\hbar)p_x\right)$$

$$=-i\frac{\hbar}{m}p_x.$$

$\therefore\ \frac{d\langle x\rangle}{dt}=\frac{i}{\hbar}\left(-i\frac{\hbar}{m}\langle p_x\rangle\right)=\frac{1}{m}\langle p_x\rangle$. Same for y and z, so: $\boxed{\frac{d\langle\vec r\rangle}{dt}=\frac{1}{m}\langle\vec p\rangle}$.

$\frac{d\langle p_x\rangle}{dt}=\frac{i}{\hbar}\langle[H,p_x]\rangle$; $[H,p_x]=\left[\frac{p^2}{2m}+V,p_x\right]=[V,p_x]=i\hbar\frac{\partial V}{\partial x}$ (Problem 3.43 d).

$=\frac{i}{\hbar}(i\hbar)\left\langle\frac{\partial V}{\partial x}\right\rangle=\left\langle-\frac{\partial V}{\partial x}\right\rangle$. Same for y and z, so: $\boxed{\frac{d\langle\vec p\rangle}{dt}=\langle-\vec\nabla V\rangle}$.

(This is Ehrenfest's Theorem in 3 dimensions.)

(c) From [3.139]: $\sigma_x\sigma_{p_x}\geq\left|\frac{1}{2i}\langle[x,p_x]\rangle\right|=\left|\frac{1}{2i}i\hbar\right|=\frac{\hbar}{2}$. So $\boxed{\sigma_{r_i}\sigma_{p_j}\geq\frac{\hbar}{2}\delta_{ij}}$.

Problem 4.2 (a) $[4.8]\Rightarrow\ -\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)=E\psi$ (inside the box).

Separable solutions: $\psi(x,y,z)=X(x)Y(y)Z(z)$. Put this in, and divide by XYZ:

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=-\frac{2m}{\hbar^2}E=-(k_x^2+k_y^2+k_z^2),\quad(\star)$$

where $E\equiv\frac{(k_x^2+k_y^2+k_z^2)\hbar^2}{2m}$, and k_x,k_y,k_z are three constants. $\frac{d^2X}{dx^2}=-k_x^2X$; $\frac{d^2Y}{dy^2}=-k_y^2Y$; $\frac{d^2Z}{dz^2}=-k_z^2Z$.

[The three terms on the left of $\star$ are functions of x, y, and z respectively, so each must be a constant. I have called the three separation constants k_x^2, k_y^2, and k_z^2 — as we'll soon see, they must be positive.]

Solution: $X(x)=A_x\sin k_xx+B_x\cos k_xx$; $Y(y)=A_y\sin k_yy+B_y\cos k_yy$; $Z(z)=A_z\sin k_zz+B_z\cos k_zz$.

But $X(0)=0$, so $B_x=0$; $Y(0)=0$, so $B_y=0$; $Z(0)=0$, so $B_z=0$. And $X(a)=0\Rightarrow\sin(k_xa)=0\Rightarrow$ $k_x=\frac{n_x\pi}{a}$ $(n_x=1,2,3,\dots)$. [As before — p. 26 — $n_x\neq0$, and negative values are redundant.]

Likewise $k_y=\frac{n_y\pi}{a}$ and $k_z=\frac{n_z\pi}{a}$. So $E=\frac{\hbar^2}{2m}\frac{\pi^2}{a^2}(n_x^2+n_y^2+n_z^2)$, and

$\psi(x,y,z) = A_x A_y A_z \sin\left(\frac{n_x\pi}{a}x\right)\sin\left(\frac{n_y\pi}{a}y\right)\sin\left(\frac{n_z\pi}{a}z\right)$. We may as well normalize X, Y, and Z separately — then (p. 26) $A_x = A_y = A_z = \sqrt{\frac{2}{a}}$. Conclusion:

$$\boxed{\psi(x,y,z) = \left(\frac{2}{a}\right)^{3/2}\sin\left(\frac{n_x\pi}{a}x\right)\sin\left(\frac{n_y\pi}{a}y\right)\sin\left(\frac{n_z\pi}{a}z\right);\ E = \frac{\pi^2\hbar^2}{2ma^2}(n_x^2+n_y^2+n_z^2);\ n_x, n_y, n_z = 1,2,3,\ldots}$$

(b)

n_x	n_y	n_z	$(n_x^2+n_y^2+n_z^2)$
1	1	1	3
1	1	2	6
1	2	1	
2	1	1	
1	2	2	9
2	1	2	
2	2	1	
1	1	3	11
1	3	1	
3	1	1	
2	2	2	12
1	2	3	14
1	3	2	
2	1	3	
2	3	1	
3	1	2	
3	2	1	

$E_1 = 3\frac{\pi^2\hbar^2}{2ma^2}$; degeneracy: 1.

$E_2 = 6\frac{\pi^2\hbar^2}{2ma^2}$; $d = 3$.

$E_3 = 9\frac{\pi^2\hbar^2}{2ma^2}$; $d = 3$.

$E_4 = 11\frac{\pi^2\hbar^2}{2ma^2}$; $d = 3$.

$E_5 = 12\frac{\pi^2\hbar^2}{2ma^2}$; $d = 1$.

$E_6 = 14\frac{\pi^2\hbar^2}{2ma^2}$; $d = 6$.

(c) The next combinations are: E_7 (322), E_8 (411), E_9 (331), E_{10} (421), E_{11} (332), E_{12} (422), E_{13} (431), and E_{14} (333 and 511). Degeneracy: $\boxed{4}$. Simple combinatorics suggests degeneracies of 1 ($n_x = n_y = n_z$), 3 (two the same, one different), or 6 (all three different). But in the case of E_{14} there is a numerical "accident": $3^2+3^2+3^2 = 27$, but $5^2+1^2+1^2$ is also 27, so the degeneracy is greater than combinatorial reasoning alone would suggest.

PROBLEM 4.3 [4.32] $\Rightarrow Y_0^0 = \frac{1}{\sqrt{4\pi}} P_0^0(\cos\theta)$; [4.27] $\Rightarrow P_0^0(x) = P_0(x)$; [4.28] $\Rightarrow P_0(x) = 1$.

So $\boxed{Y_0^0 = \frac{1}{\sqrt{4\pi}}}$. $Y_2^1 = -\sqrt{\frac{5}{4\pi}\frac{1}{3\cdot 2}}\, e^{i\phi} P_2^1(\cos\theta)$; $P_2^1(x) = \sqrt{1-x^2}\frac{d}{dx}P_2(x)$;

$P_2(x) = \frac{1}{4\cdot 2}\left(\frac{d}{dx}\right)^2(x^2-1)^2 = \frac{1}{8}\frac{d}{dx}\left[2(x^2-1)2x\right] = \frac{1}{2}\left[x^2-1+x(2x)\right] = \frac{1}{2}(3x^2-1)$; so

$P_2^1(x) = \sqrt{1-x^2}\frac{d}{dx}\left[\frac{3}{2}x^2-\frac{1}{2}\right] = \sqrt{1-x^2}\,3x$; $P_2^1(\cos\theta) = 3\cos\theta\sin\theta$. $\boxed{Y_2^1 = -\sqrt{\frac{15}{8\pi}}\, e^{i\phi}\sin\theta\cos\theta}$.

Normalization: $\iint |Y_0^0|^2 \sin\theta\, d\theta\, d\phi = \frac{1}{4\pi}\left[\int_0^\pi \sin\theta\, d\theta\right]\left[\int_0^{2\pi} d\phi\right] = \frac{1}{4\pi}(2)(2\pi) = 1.$ ✓

$\iint |Y_2^1|^2 \sin\theta\, d\theta\, d\phi = \frac{15}{8\pi}\int_0^\pi \sin^2\theta\cos^2\theta\sin\theta\, d\theta \int_0^{2\pi} d\phi = \frac{15}{4}\int_0^\pi \cos^2\theta(1-\cos^2\theta)\sin\theta\, d\theta = \frac{15}{4}\left[-\frac{\cos^3\theta}{3}+\frac{\cos^5\theta}{5}\right]\Big|_0^\pi$

$$= \frac{15}{4}\left[\frac{2}{3}-\frac{2}{5}\right] = \frac{5}{2}-\frac{3}{2} = 1\checkmark.$$ <u>Orthogonality</u>: $\iint Y_0^{1*}\, Y_2^1 \sin\theta\, d\theta\, d\phi =$

$$= -\frac{1}{\sqrt{4\pi}}\sqrt{\frac{15}{8\pi}}\,\underbrace{\left[\int_0^\pi \sin\theta\cos\theta\sin\theta\, d\theta\right]}_{\left(\frac{\sin^3\theta}{3}\right)\Big|_0^\pi = 0}\underbrace{\left[\int_0^{2\pi} e^{i\phi}\, d\phi\right]}_{\left(\frac{e^{i\phi}}{i}\right)\Big|_0^{2\pi} = 0} = 0.\ \checkmark$$

<u>PROBLEM 4.4</u> $\frac{d\Theta}{d\theta} = \frac{A}{\tan(\theta/2)}\frac{1}{2}\sec^2(\theta/2) = \frac{A}{2}\frac{1}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{A}{\sin\theta}$. $\therefore \frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = \frac{d}{d\theta}(A) = 0.$

With $l=m=0$, [4.25] reads: $\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0$. So $A\ln(\tan\theta/2)$ <u>does</u> satisfy [4.25].

$\Theta(0) = A\ln(0) = A(-\infty)$. $\boxed{\text{Blows up at } \theta=0}$ $\Theta(\pi) = A\ln(\tan(\pi/2)) = A\ln(\infty) = A(\infty)$. $\boxed{\text{Blows up at } \theta=\pi.}$

<u>PROBLEM 4.5</u> $Y_l^l = (-1)^l\sqrt{\frac{(2l+1)}{4\pi}\frac{1}{(2l)!}}\, e^{il\phi}P_l^l(\cos\theta)$. $P_l^l(x) = (1-x^2)^{l/2}\left(\frac{d}{dx}\right)^l P_l(x)$.

$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$, so $P_l^l(x) = \frac{1}{2^l l!}(1-x^2)^{l/2}\left(\frac{d}{dx}\right)^{2l}(x^2-1)^l$. Now $(x^2-1)^l = x^{2l}+\dots$, where

all the other terms involve powers of x <u>less</u> than $2l$, and hence give zero when differentiated $2l$ times.

So $P_l^l(x) = \frac{1}{2^l l!}(1-x^2)^{l/2}\left(\frac{d}{dx}\right)^{2l}x^{2l}$. But $\left(\frac{d}{dx}\right)^n x^n = n!$, so $P_l^l = \frac{(2l)!}{2^l l!}(1-x^2)^{l/2}$.

$$\therefore Y_l^l = (-1)^l\sqrt{\frac{(2l+1)}{4\pi(2l)!}}\, e^{il\phi}\frac{(2l)!}{2^l l!}(\sin\theta)^l = \boxed{\frac{1}{l!}\sqrt{\frac{(2l+1)!}{4\pi}}\left(-\frac{1}{2}e^{i\phi}\sin\theta\right)^l}.$$

$Y_3^2 = \sqrt{\frac{7}{4\pi}\cdot\frac{1}{5!}}\, e^{2i\phi}P_3^2(\cos\theta)$; $P_3^2(x) = (1-x^2)\left(\frac{d}{dx}\right)^2 P_3(x)$; $P_3(x) = \frac{1}{8\cdot 3!}\left(\frac{d}{dx}\right)^3(x^2-1)^3$

$P_3 = \frac{1}{8\cdot3\cdot2}\left(\frac{d}{dx}\right)^2\left[6x(x^2-1)^2\right] = \frac{1}{8}\frac{d}{dx}\left[(x^2-1)^2+4x^2(x^2-1)\right] = \frac{1}{8}\left(4x(x^2-1)+8x(x^2-1)+4x^2\cdot2x\right)$

$= \frac{1}{2}(x^3-x+2x^3-2x+2x^3) = \frac{1}{2}(5x^3-3x)$. $P_3^2(x) = \frac{1}{2}(1-x^2)\left(\frac{d}{dx}\right)^2(5x^3-3x) = \frac{1}{2}(1-x^2)\frac{d}{dx}(15x^2-3)$

$P_3^2(x) = \frac{1}{2}(1-x^2)30x = 15x(1-x^2)$. $Y_3^2 = \sqrt{\frac{7}{4\pi}\frac{1}{5!}}\,15e^{2i\phi}\cos\theta\sin^2\theta = \boxed{\frac{1}{4}\sqrt{\frac{105}{2\pi}}\, e^{2i\phi}\sin^2\theta\cos\theta}.$

<u>Check that Y_l^l satisfies [4.18]</u>: Let $\frac{1}{l!}\sqrt{\frac{(2l+1)!}{4\pi}}\left(-\frac{1}{2}\right)^l = A$. $Y_l^l = A(e^{i\phi}\sin\theta)^l$.

$\frac{\partial Y_l^l}{\partial\theta} = Ae^{il\phi}l(\sin\theta)^{l-1}\cos\theta$; $\sin\theta\frac{\partial Y_l^l}{\partial\theta} = l\cos\theta\, Y_l^l$; $\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_l^l}{\partial\theta}\right) = l\cos\theta\left(\sin\theta\frac{\partial Y_l^l}{\partial\theta}\right) - l\sin^2\theta\, Y_l^l$
$= (l^2\cos^2\theta - l\sin^2\theta)Y_l^l$.

$\frac{\partial^2 Y_l^l}{\partial\phi^2} = -l^2 Y_l^l$. So left side of [4.18] is $\left[l^2(1-\sin^2\theta) - l\sin^2\theta - l^2\right]Y_l^l = -l(l+1)\sin^2\theta\, Y_l^l$, which

matches the right side. Now <u>check that Y_3^2 satisfies [4.18]</u>: $Y_3^2 = Ae^{2i\phi}\sin^2\theta\cos\theta$, so

$\frac{\partial Y}{\partial \theta} = Ae^{2i\phi}(2\sin\theta\cos^2\theta - \sin^3\theta)$; $\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) = Ae^{2i\phi}\sin\theta\frac{\partial}{\partial\theta}(2\sin^2\theta\cos^2\theta - \sin^4\theta) =$

$= Ae^{2i\phi}\sin\theta(4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta - 4\sin^3\theta\cos\theta) = 4Ae^{2i\phi}\sin^2\theta\cos\theta(\cos^2\theta - 2\sin^2\theta)$

$= 4(\cos^2\theta - 2\sin^2\theta)Y.$ $\quad \frac{\partial^2 Y}{\partial\phi^2} = -4Y.$ So left side of [4.18] is

$4(\cos^2\theta - 2\sin^2\theta - 1)Y = 4(-3\sin^2\theta)Y = -\ell(\ell+1)\sin^2\theta Y$, where $\ell = 3$ — so it fits right side of [4.18].

Problem 4.6 $\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)dx = \frac{1}{2^\ell \ell!}\frac{1}{2^{\ell'}\ell'!}\int_{-1}^{1}\left[\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell\right]\left[\left(\frac{d}{dx}\right)^{\ell'}(x^2-1)^{\ell'}\right]dx$. If $\ell \neq \ell'$, we may as well let ℓ be the larger of the two ($\ell > \ell'$). Integrate by parts, pulling successively each derivative off the first term onto the second:

$$2^\ell \ell! \, 2^{\ell'}\ell'! \int_{-1}^{1} P_\ell(x)P_{\ell'}(x)dx = \left[\left(\frac{d}{dx}\right)^{\ell-1}(x^2-1)^\ell\right]\left[\left(\frac{d}{dx}\right)^{\ell'}(x^2-1)^{\ell'}\right]\Bigg|_{-1}^{1} - \int_{-1}^{1}\left[\left(\frac{d}{dx}\right)^{\ell-1}(x^2-1)^\ell\right]\left[\left(\frac{d}{dx}\right)^{\ell'+1}(x^2-1)^{\ell'}\right]dx$$

$$= \cdots (\text{boundary terms}) \cdots + (-1)^\ell \int_{-1}^{1}(x^2-1)^\ell \left(\frac{d}{dx}\right)^{\ell'+\ell}(x^2-1)^{\ell'}dx.$$

But the remaining integral is zero, because $\left(\frac{d}{dx}\right)^{\ell'+\ell}(x^2-1)^{\ell'} = 0$. ($\ell'+\ell > 2\ell'$, and $(x^2-1)^{\ell'}$ is a polynomial whose highest power is 2ℓ — any more than 2ℓ derivatives will kill it.) Now, the boundary terms are of the form

$$\left[\left(\frac{d}{dx}\right)^{\ell-n}(x^2-1)^\ell\right]\left[\left(\frac{d}{dx}\right)^{\ell'+n-1}(x^2-1)^{\ell'}\right]\Bigg|_{-1}^{+1}, \quad n = 1, 2, 3, \ldots, \ell.$$

Look at the first term: $(x^2-1)^\ell = \underbrace{(x^2-1)(x^2-1)\ldots(x^2-1)}_{\ell \text{ factors}}$. $0, 1, 2, \ldots, \ell-1$ derivatives of this will still leave at least one overall factor of (x^2-1). [Zero derivatives leaves all ℓ factors; 1 derivative leaves $\ell-1$: $\frac{d}{dx}(x^2-1)^\ell = 2\ell x(x^2-1)^{\ell-1}$; 2 derivatives leaves $\ell-2$: $\frac{d^2}{dx^2}(x^2-1)^\ell = 2\ell(x^2-1)^{\ell-1} + 2\ell(\ell-1)2x^2(x^2-1)^{\ell-2}$, and so on.] So the boundary terms are all zero, and hence $\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)dx = 0$. This leaves only the case $\ell = \ell'$. Again the boundary terms vanish, but this time the remaining integral does not:

$$(2^\ell \ell!)^2 \int_{-1}^{1}(P_\ell(x))^2 dx = (-1)^\ell \int_{-1}^{1}(x^2-1)^\ell \underbrace{\left(\frac{d}{dx}\right)^{2\ell}(x^2-1)^\ell}_{\left(\frac{d}{dx}\right)^{2\ell}(x^{2\ell}) = (2\ell)!} dx = (-1)^\ell (2\ell)! \int_{-1}^{1}(x^2-1)^\ell dx = 2(2\ell)!\int_0^1 (1-x^2)^\ell dx$$

Let $x = \cos\theta$, so $dx = -\sin\theta\, d\theta$, $(1-x^2) = \sin^2\theta$, $\theta: \pi/2 \to 0$. So $\int_0^1 (1-x^2)^\ell dx = \int_{\pi/2}^{0}(\sin\theta)^{2\ell}(-\sin\theta)d\theta$

$$= \int_0^{\pi/2}(\sin\theta)^{2\ell+1}d\theta = \frac{(2)(4)\cdots(2\ell)}{(1)(3)(5)\cdots(2\ell+1)} = \frac{(2^\ell \ell!)^2}{1\cdot2\cdot3\cdots(2\ell+1)} = \frac{(2^\ell \ell!)^2}{(2\ell+1)!}.$$

$$\therefore \int_{-1}^{1}(P_\ell(x))^2 dx = \frac{1}{(2^\ell \ell!)^2}2(2\ell)!\frac{(2^\ell \ell!)^2}{(2\ell+1)!} = \frac{2}{2\ell+1}. \quad \text{So } \int_{-1}^{1}P_\ell(x)P_{\ell'}(x)dx = \frac{2}{2\ell+1}\delta_{\ell\ell'}. \text{ QED.}$$

Problem 4.7 (a) $j_2(x) = (-x)^2\left(\frac{1}{x}\frac{d}{dx}\right)^2\frac{\sin x}{x} = x^2\left(\frac{1}{x}\frac{d}{dx}\right)\left(\frac{1}{x}\frac{d}{dx}\left(\frac{\sin x}{x}\right)\right) = x\frac{d}{dx}\left(\frac{1}{x}\cdot\frac{x\cos x - \sin x}{x^2}\right)$

$$= x\frac{d}{dx}\left(\frac{\cos x}{x^2} - \frac{\sin x}{x^3}\right) = x\left[\frac{-x^2\sin x - 2x\cos x}{x^4} - \frac{x^3\cos x - 3x^2\sin x}{x^6}\right] = -\frac{\sin x}{x} - 2\frac{\cos x}{x^2} - \frac{\cos x}{x^2} + 3\frac{\sin x}{x^3}.$$

$\therefore \boxed{j_2(x) = \left(\frac{3}{x^3}-\frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x}$ $\quad n_2(x) = -(-x)^2\left(\frac{1}{x}\frac{d}{dx}\right)^2\frac{\cos x}{x} = -x^2\left(\frac{1}{x}\frac{d}{dx}\right)\left(\frac{1}{x}\frac{d}{dx}\left(\frac{\cos x}{x}\right)\right)$

$$= -x\frac{d}{dx}\left[\frac{1}{x}\cdot\frac{-x\sin x-\cos x}{x^2}\right] = x\frac{d}{dx}\left(\frac{\sin x}{x^2}+\frac{\cos x}{x^3}\right) = x\left[\frac{x^2\cos x - 2x\sin x}{x^4} + \frac{-x^3\sin x - 3x^2\cos x}{x^6}\right]$$

$$= \frac{\cos x}{x} - 2\frac{\sin x}{x^2} - \frac{\sin x}{x^2} - 3\frac{\cos x}{x^3} = \boxed{-\left(\frac{3}{x^3}-\frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x}.$$

(b) Keeping terms to order x^2: $j_2(x) \cong \left(\frac{3}{x^3}-\frac{1}{x}\right)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\right) - \frac{3}{x^2}\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\right)$

$$\cong \frac{3}{x^2}-\frac12+\frac{x^2}{40}-1+\frac{x^2}{6}-\frac{3}{x^2}+\frac32-\frac{x^2}{8} = \frac{1}{x^2}(3-3)+\left(-\frac12-1+\frac32\right)+x^2\left(\frac1{40}+\frac16-\frac18\right) = x^2\,\frac{3+20-15}{120}$$

$=\frac{8}{120}x^2 = \boxed{\frac{1}{15}x^2}$. (Finite — in fact, zero, as $x\to 0$.) For n_2 no such cancellations occur — the dominant term as $x\to 0$ is $\frac{1}{x^3}$: $n_2(x)\cong -\left(\frac{3}{x^3}-\frac1x\right)(1)-\frac{3}{x^2}(x)\cong \boxed{-\frac{3}{x^3}}$, which blows up as $x\to 0$.

PROBLEM 4.8 (a) $u = Arj_1(kr) = A\left[\frac{\sin(kr)}{k^2r}-\frac{\cos(kr)}{k}\right] = \frac{A}{k}\left[\frac{\sin(kr)}{(kr)}-\cos(kr)\right]$.

$$\frac{d}{dr}u = \frac{A}{k}\left[\frac{k^2r\cos(kr)-k\sin(kr)}{(kr)^2}+k\sin(kr)\right] = A\left[\frac{\cos(kr)}{kr}-\frac{\sin(kr)}{(kr)^2}+\sin(kr)\right]$$

$$\frac{d^2u}{dr^2} = A\left\{\frac{-k^2r\sin(kr)-k\cos(kr)}{(kr)^2} - \frac{k^3r^2\cos(kr)-2k^2r\sin(kr)}{(kr)^4}+k\cos(kr)\right\}$$

$$= Ak\left[-\frac{\sin(kr)}{kr}-\frac{\cos(kr)}{(kr)^2}-\frac{\cos(kr)}{(kr)^2}+2\frac{\sin(kr)}{(kr)^3}+\cos(kr)\right]$$

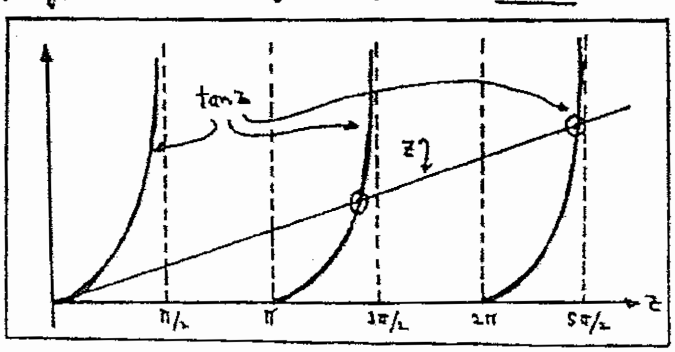
$$= Ak\left[\left(1-\frac{2}{(kr)^2}\right)\cos(kr)+\left(\frac{2}{(kr)^3}-\frac{1}{(kr)}\right)\sin(kr)\right].$$ With $V=0$ and $l=1$, [4.37] reads:

$\frac{d^2u}{dr^2}-\frac{2}{r^2}u = -\frac{2mE}{\hbar^2}u = -k^2u$. In this case the left side is

$$Ak\left\{\left(1-\frac{2}{(kr)^2}\right)\cos(kr)+\left(\frac{2}{(kr)^3}-\frac{1}{(kr)}\right)\sin(kr)-\frac{2}{(kr)^2}\left(\frac{\sin(kr)}{kr}-\cos(kr)\right)\right\}$$

$$= Ak\left\{\cos(kr)-\frac{\sin(kr)}{kr}\right\} = -k^2u.$$ So this u does satisfy [4.37].

(b) Equation [4.48] $\Rightarrow j_1(z)=0$, where $z\equiv ka$. Thus $\frac{\sin z}{z^2}-\frac{\cos z}{z}=0$, or $\boxed{\tan z = z}$.



For high z (large n, if $n=1,2,3,\ldots$ counts the allowed energies in increasing order), the intersections occur slightly below $z=(n+\frac12)\pi$.

$\therefore E = \frac{\hbar^2k^2}{2m}=\frac{\hbar^2z^2}{2ma^2}=\frac{\hbar^2\pi^2}{2ma^2}\left(n+\frac12\right)^2$. QED.

Problem 4.9 For $r \le a$, we have $u(r) = A\sin(kr)$, with $k \equiv \sqrt{2mE}/\hbar$ (same as p. 130).
For $r \ge a$, [4.37] with $l=0$, $V=V_0 \Rightarrow$

$$\frac{d^2u}{dr^2} = \frac{2m}{\hbar^2}(V_0 - E)u = \lambda^2 u, \text{ with } \lambda \equiv \sqrt{2m(V_0-E)}/\hbar.$$

(Note: for bound state, $E < V_0$).

$\therefore u(r) = Ce^{\lambda r} + De^{-\lambda r}$. But C term blows up, so $u(r) = De^{-\lambda r}$.

Continuity at $r=a$: $A\sin(ka) = De^{-\lambda a}$
continuity of u' at $r=a$: $Ak\cos(ka) = -D\lambda e^{-\lambda a}$

Divide: $\frac{1}{k}\tan(ka) = -\frac{1}{\lambda}$, or $-\tan ka = \frac{k}{\lambda}$.

Let $ka \equiv z$; $\frac{k}{\lambda} = \frac{z}{\sqrt{\frac{2mV_0a^2}{\hbar^2} - z^2}}$. Let $z_0 \equiv \frac{\sqrt{2mV_0}\,a}{\hbar}$. $\boxed{-\tan z = \frac{z}{\sqrt{z_0^2 - z^2}}}$.

This is exactly the same transcendental equation we encountered in Problem 2.28 – see graph there.
There is no solution if $z_0 < \pi/2$, which is to say, if $\frac{2mV_0a^2}{\hbar^2} < \frac{\pi^2}{4}$, or $V_0a^2 < \frac{\pi^2\hbar^2}{8m}$.
Otherwise, the ground state energy is somewhere between $z = \pi/2$ and $z = \pi$:

$$E = \frac{\hbar^2k^2a^2}{2ma^2} = \frac{\hbar^2}{2ma^2}z^2, \text{ so } \boxed{\frac{\hbar^2\pi^2}{8ma^2} < E_0 < \frac{\hbar^2\pi^2}{2ma^2}}$$

(Precise location, of course, depends on V_0.)

Problem 4.10 R_{30}. $(n=3, l=0)$. [4.62] $\Rightarrow v(\rho) = \sum_{j=0} a_j \rho^j$.

[4.76] $\Rightarrow a_1 = \frac{2(1-3)}{(1)(2)}a_0 = -2a_0$; $a_2 = \frac{2(2-3)}{(2)(3)}a_1 = -\frac{1}{3}a_1 = \frac{2}{3}a_0$; $a_3 = \frac{2(3-3)}{(3)(4)}a_2 = 0$; etc.

[4.73] $\Rightarrow \rho = \frac{r}{3a}$; [4.75] $\Rightarrow R_{30} = \frac{1}{r}\rho e^{-\rho}v(\rho) = \frac{1}{r}\frac{r}{3a}e^{-r/3a}\left(a_0 - 2a_0\frac{r}{3a} + \frac{2}{3}a_0\left(\frac{r}{3a}\right)^2\right)$

$$= \boxed{\left(\frac{a_0}{3a}\right)\left(1 - \frac{2}{3}\left(\frac{r}{a}\right) + \frac{2}{27}\left(\frac{r}{a}\right)^2\right)e^{-r/3a}}.$$

R_{31}. $(n=3, l=1)$. $a_1 = \frac{2(2-3)}{(1)(4)}a_0 = -\frac{1}{2}a_0$; $a_2 = \frac{2(3-3)}{(2)(5)}a_1 = 0$; etc.

$$R_{31} = \frac{1}{r}\left(\frac{r}{3a}\right)^2 e^{-r/3a}\left(a_0 - \frac{1}{2}a_0\frac{r}{3a}\right) = \boxed{\left(\frac{a_0}{9a^2}\right)r\left(1 - \frac{1}{6}\left(\frac{r}{a}\right)\right)e^{-r/3a}}.$$

R_{32}. $(n=3, l=2)$. $a_1 = \frac{2(3-3)}{(1)(6)}a_0 = 0$; etc. $R_{32} = \frac{1}{r}\left(\frac{r}{3a}\right)^3 e^{-r/3a}(a_0) = \boxed{\left(\frac{a_0}{27a^3}\right)r^2e^{-r/3a}}$.

Problem 4.11 (a) [4.31] $\Rightarrow \int_0^\infty |R|^2r^2dr = 1$. [4.82] $\Rightarrow R_{20} = \left(\frac{a_0}{2a}\right)\left(1 - \frac{r}{2a}\right)e^{-r/2a}$. Let $\frac{r}{a} \equiv z$.

$$1 = \left(\frac{a_0}{2a}\right)^2 a^3\int_0^\infty\left(1 - \frac{z}{2}\right)^2e^{-z}z^2dz = \frac{a_0^2a}{4}\int_0^\infty\left(z^2 - z^3 + \frac{1}{4}z^4\right)e^{-z}dz = \frac{a_0^2a}{4}\left[2 - 6 + \frac{1}{4}\cdot 24\right] = \frac{a}{2}a_0^2.$$

$\therefore \boxed{a_0 = \sqrt{\frac{2}{a}}}$. [4.15] $\Rightarrow \psi_{200} = R_{20}Y_0^0$. Table 4.2 $\Rightarrow Y_0^0 = \frac{1}{\sqrt{4\pi}}$. $\therefore \psi_{200} = \frac{1}{\sqrt{4\pi}}\sqrt{\frac{2}{a}}\frac{1}{2a}\left(1 - \frac{r}{2a}\right)e^{-r/2a}$

$$\boxed{\psi_{200} = \frac{1}{\sqrt{2\pi a}}\frac{1}{2a}\left(1 - \frac{r}{2a}\right)e^{-r/2a}}.$$

(b) $R_{21} = \frac{a_0}{4a^2} r e^{-r/2a}$. $\quad 1 = \left(\frac{a_0}{4a^2}\right)^2 a^5 \int_0^\infty z^4 e^{-z} dz = \frac{a_0^2 a}{16} \cdot 24 = \frac{3}{2} a\, a_0^2$, so $\boxed{a_0 = \sqrt{\frac{2}{3a}}}$.

$R_{21} = \frac{1}{\sqrt{6a}} \frac{1}{2a^2} r e^{-r/2a}$. $\quad \psi_{21\pm1} = \frac{1}{\sqrt{6a}} \frac{1}{2a^2} r e^{-r/2a} \left(\mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}\right)$

$\boxed{\psi_{21\pm1} = \mp \frac{1}{\sqrt{\pi a}} \frac{1}{8a^2} r e^{-r/2a} \sin\theta e^{\pm i\phi}}$. $\quad \psi_{210} = \frac{1}{\sqrt{6a}} \frac{1}{2a^2} r e^{-r/2a} \left(\sqrt{\frac{3}{4\pi}} \cos\theta\right)$

$$\boxed{\psi_{210} = \frac{1}{\sqrt{2\pi a}} \frac{1}{4a^2} r e^{-r/2a} \cos\theta}.$$

Problem 4.12 (a) $L_0 = e^x e^{-x} = \boxed{1}$. $\quad L_1 = e^x \frac{d}{dx}(e^{-x}x) = e^x\left[e^{-x} - e^{-x}x\right] = \boxed{1-x}$.

$L_2 = e^x\left(\frac{d}{dx}\right)^2(e^{-x}x^2) = e^x \frac{d}{dx}\left(2xe^{-x} - e^{-x}x^2\right) = e^x\left(2e^{-x} - 2xe^{-x} + e^{-x}x^2 - 2xe^{-x}\right) = \boxed{2-4x+x^2}$.

$L_3 = e^x\left(\frac{d}{dx}\right)^3(e^{-x}x^3) = e^x\left(\frac{d}{dx}\right)^2\left(-e^{-x}x^3 + 3x^2e^{-x}\right) = e^x \frac{d}{dx}\left(e^{-x}x^3 - 3x^2e^{-x} - 3x^2e^{-x} + 6xe^{-x}\right)$

$= e^x\left(-e^{-x}x^3 + 3x^2e^{-x} + 6x^2e^{-x} - 12xe^{-x} - 6xe^{-x} + 6e^{-x}\right) = \boxed{6-18x+9x^2-x^3}$.

(b) $v(\rho) = L_2^5(2\rho)$; $\quad L_2^5(x) = L_{7-5}^5(x) = (-1)^5\left(\frac{d}{dx}\right)^5 L_7(x)$.

$L_7(x) = e^x\left(\frac{d}{dx}\right)^7(x^7e^{-x}) = e^x\left(\frac{d}{dx}\right)^6\left(7x^6e^{-x} - x^7e^{-x}\right) = e^x\left(\frac{d}{dx}\right)^5\left(42x^5e^{-x} - 7x^6e^{-x} - 7x^6e^{-x} + x^7e^{-x}\right)$

$= e^x\left(\frac{d}{dx}\right)^4\left(210x^4e^{-x} - 42x^5e^{-x} - 84x^5e^{-x} + 14x^6e^{-x} + 7x^6e^{-x} - x^7e^{-x}\right)$

$= e^x\left(\frac{d}{dx}\right)^3\left(840x^3e^{-x} - 210x^4e^{-x} - 630x^4e^{-x} + 126x^5e^{-x} + 126x^5e^{-x} - 21x^6e^{-x} - 7x^6e^{-x} + x^7e^{-x}\right)$

$= e^x\left(\frac{d}{dx}\right)^2\left(2520x^2e^{-x} - 840x^3e^{-x} - 3360x^3e^{-x} + 840x^4e^{-x} + 1260x^4e^{-x} - 252x^5e^{-x} - 168x^5e^{-x}\right.$
$\left. + 28x^6e^{-x} + 7x^6e^{-x} - x^7e^{-x}\right)$

$= e^x\left(\frac{d}{dx}\right)\left(5040xe^{-x} - 2520x^2e^{-x} - 12600x^2e^{-x} + 4200x^3e^{-x} + 8400x^3e^{-x} - 2100x^4e^{-x}\right.$
$\left. - 2100x^4e^{-x} + 420x^5e^{-x} + 210x^5e^{-x} - 35x^6e^{-x} - 7x^6e^{-x} + x^7e^{-x}\right)$

$= e^x\left(5040e^{-x} - 5040xe^{-x} - 30240xe^{-x} + 15120x^2e^{-x} + 37800x^2e^{-x} - 12600x^3e^{-x} - 8400x^3e^{-x}\right.$
$+ 2100x^4e^{-x} - 8400x^3e^{-x} + 2100x^4e^{-x} + 3150x^4e^{-x} - 630x^5e^{-x} - 252x^5e^{-x} + 42x^6e^{-x}$
$\left. + 7x^6e^{-x} - x^7e^{-x}\right)$

$= 5040 - 35280x + 52920x^2 - 29400x^3 + 7350x^4 - 882x^5 + 49x^6 - x^7$

$L_2^5 = -\left(\frac{d}{dx}\right)^5\left(-882x^5 + 49x^6 - x^7\right) = \quad -\left[-882(5\cdot4\cdot3\cdot2) + 49(6\cdot5\cdot4\cdot3\cdot2)x - 7\cdot6\cdot5\cdot4\cdot3\, x^2\right]$

$= 60\left((882\times2) - (49\times12)x + 42x^2\right) = 420\left(252 - 84x + 6x^2\right) = 2520\left(42 - 14x + x^2\right)$.

$v(\rho) = 2520\left(42 - 28\rho + 4\rho^2\right) \quad = \boxed{5040\left(21 - 14\rho + 2\rho^2\right)}$.

(c) [4.62] $\Rightarrow v(\rho) = \sum_{j=0}^{\infty} a_j \rho^j$. [4.76] $\Rightarrow a_1 = \frac{2(3-5)}{(1)(6)} a_0 = -\frac{2}{3} a_0$

$a_2 = \frac{2(4-5)}{(2)(7)} a_1 = -\frac{1}{7} a_1 = \frac{2}{21} a_0$; $a_3 = \frac{2(5-5)}{(3)(8)} a_2 = 0$; etc.

$$v(\rho) = a_0 - \frac{2}{3} a_0 \rho + \frac{2}{21} a_0 \rho^2 = \boxed{\frac{a_0}{21}\left(21 - 14\rho + 2\rho^2\right)}.$$

Problem 4.13 (a) $\psi = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$, so $\langle r^n \rangle = \frac{1}{\pi a^3} \int r^n e^{-2r/a} (r^2 \sin\theta\, dr\, d\theta\, d\phi)$

$$\langle r^n \rangle = \frac{4\pi}{\pi a^3} \int_0^\infty r^{n+2} e^{-2r/a} dr. \quad \langle r \rangle = \frac{4}{a^3} \int_0^\infty r^3 e^{-2r/a} dr = \frac{4}{a^3}\, 3! \left(\frac{a}{2}\right)^4 = \boxed{\frac{3}{2} a}.$$

$$\langle r^2 \rangle = \frac{4}{a^3} \int_0^\infty r^4 e^{-2r/a} dr = \frac{4}{a^3}\, 4! \left(\frac{a}{2}\right)^5 = \boxed{3a^2}.$$

(b) $\boxed{\langle x \rangle = 0}$; $\langle x^2 \rangle = \frac{1}{3} \langle r^2 \rangle = \boxed{a^2}$.

(c) $\psi_{211} = R_{21} Y_1^1 = -\frac{1}{\sqrt{\pi a}} \frac{1}{8a^2} r e^{-r/2a} \sin\theta\, e^{i\phi}$. (Problem 4.11(b))

$$\langle x^2 \rangle = \frac{1}{\pi a} \frac{1}{(8a^2)^2} \int \left(r^2 e^{-r/a} \sin^2\theta\right)\left(r^2 \sin^2\theta \cos^2\phi\right)\left(r^2 \sin\theta\, dr\, d\theta\, d\phi\right)$$

$$= \frac{1}{64\pi a^5} \int_0^\infty r^6 e^{-r/a} dr \int_0^\pi \sin^5\theta\, d\theta \int_0^{2\pi} \cos^2\phi\, d\phi = \frac{1}{64\pi a^5}\left(6!\, a^7\right)\left(2 \cdot \frac{2\cdot 4}{1\cdot 3\cdot 5}\right)\left(\frac{1}{2}\cdot 2\pi\right)$$

$$= \boxed{12a^2}.$$

Problem 4.14 (a) $P = \int |\Psi|^2 d^3r = \frac{4\pi}{\pi a^3} \int_0^b e^{-2r/a} r^2 dr = \frac{4}{a^3}\left\{-\frac{a}{2} r^2 e^{-2r/a} + \frac{a^3}{4} e^{-2r/a}\left(-\frac{2r}{a} - 1\right)\right\}\Big|_0^b$

$$= -\left(1 + \frac{2r}{a} + \frac{2r^2}{a^2}\right) e^{-2r/a}\Big|_0^b = \boxed{1 - \left(1 + \frac{2b}{a} + 2\frac{b^2}{a^2}\right) e^{-2b/a}}.$$

(b) $P = 1 - \left(1 + \epsilon + \frac{1}{2}\epsilon^2\right) e^{-\epsilon} \cong 1 - \left(1 + \epsilon + \frac{1}{2}\epsilon^2\right)\left(1 - \epsilon + \frac{\epsilon^2}{2} - \frac{\epsilon^3}{3!}\right) \cong 1 - 1 + \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{6} - \epsilon + \epsilon^2 - \frac{\epsilon^3}{2}$

$$-\frac{1}{2}\epsilon^2 + \frac{1}{2}\epsilon^3 = \epsilon^3\left(\frac{1}{6} - \frac{1}{2} + \frac{1}{2}\right) = \frac{1}{6}\epsilon^3 = \frac{1}{6}\left(\frac{2b}{a}\right)^3 = \boxed{\frac{4}{3}\left(\frac{b}{a}\right)^3}.$$

(c) $|\psi(0)|^2 = \frac{1}{\pi a^3} \Rightarrow P \simeq \frac{4}{3}\pi b^3 \frac{1}{\pi a^3} = \frac{4}{3}\left(\frac{b}{a}\right)^3$. ✓

(d) $P = \frac{4}{3}\left(\frac{10^{-15}}{.5 \times 10^{-10}}\right)^3 = \frac{4}{3}\left(2 \times 10^{-5}\right)^3 = \frac{4}{3} \cdot 8 \times 10^{-15} = \frac{32}{3} \times 10^{-15} = \boxed{1.07 \times 10^{-14}}$.

Problem 4.15 (a) [4.75] $\Rightarrow R_{n(n-1)} = \frac{1}{r} \rho^n e^{-\rho} v(\rho)$. $\rho = \frac{r}{na}$.

[4.76] $\Rightarrow a_1 = \frac{2(n-n)}{(1)(2n)} a_0 = 0$, etc. $\therefore v(\rho) = a_0$. $R_{n(n-1)} = N_n r^{n-1} e^{-r/na}$, where $\frac{a_0}{(na)^n}$.

$$1 = \int_0^\infty |R|^2 r^2 dr = (N_n)^2 \int_0^\infty r^{2n} e^{-2r/na} dr = (N_n)^2 (2n)! \left(\frac{na}{2}\right)^{2n+1}; \quad \boxed{N_n = \left(\frac{2}{na}\right)^n \sqrt{\frac{2}{na(2n)!}}}.$$

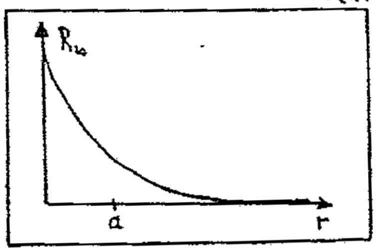
(b) $\langle r^\ell\rangle = \int_0^\infty |R|^2 r^{\ell+2}\,dr = N_n^2 \int_0^\infty r^{2n+\ell} e^{-2r/na}\,dr.$ $\quad \langle r\rangle = \left(\frac{2}{na}\right)^{2n+1}\frac{1}{(2n)!}(2n+1)!\left(\frac{na}{2}\right)^{2n+2}.$

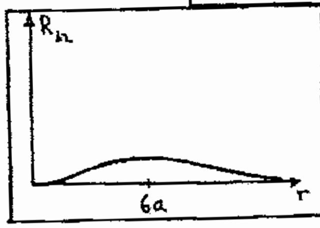
$\boxed{\langle r\rangle = (n+\tfrac{1}{2})na}.$ $\quad \langle r^2\rangle = \left(\frac{2}{na}\right)^{2n+1}\frac{1}{(2n)!}(2n+2)!\left(\frac{na}{2}\right)^{2n+3} = (2n+2)(2n+1)\left(\frac{na}{2}\right)^2.$

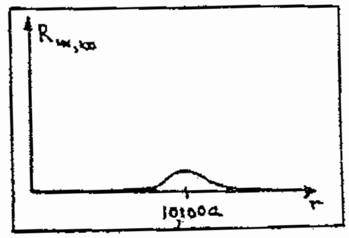
$\boxed{\langle r^2\rangle = (n+\tfrac{1}{2})(n+1)(na)^2}.$

(c) $\sigma_r^2 = \langle r^2\rangle - \langle r\rangle^2 = \left[(n+\tfrac{1}{2})(n+1)(na)^2 - (n+\tfrac{1}{2})^2(na)^2\right] = (n+\tfrac{1}{2})(na)^2(n+1-n-\tfrac{1}{2})$

$= \frac{1}{2}(n+\tfrac{1}{2})(na)^2 = \frac{1}{2(n+\frac{1}{2})}\langle r\rangle^2.$ So $\boxed{\sigma_r = \frac{\langle r\rangle}{\sqrt{2n+1}}}.$







Maxima at: $\frac{dR_{n,n-1}}{dr} = 0 \Rightarrow (n-1)r^{n-2}e^{-r/na} - \frac{1}{na}r^{n-1}e^{-r/na} = 0 \Rightarrow r = na(n-1).$

Problem 4.16 (a) $\boxed{V(r) = -G\frac{Mm}{r}}$. [So $\frac{e^2}{4\pi\epsilon_0} \to GMm$ to translate hydrogen results.]

(b) [4.72] $\Rightarrow a = \left(\frac{4\pi\epsilon_0}{e^2}\right)\frac{\hbar^2}{m} \to \boxed{\frac{\hbar^2}{GMm^2}} = \frac{(1.0546\times10^{-34}\,\mathrm{J\,s})^2}{\left(6.6726\times10^{-11}\frac{\mathrm{m}^3}{\mathrm{kg\,s}^2}\right)(1.9892\times10^{30}\,\mathrm{kg})(5.98\times10^{24}\,\mathrm{kg})^2}$

$= \boxed{2.34\times10^{-138}\,\mathrm{m}}$

(c) [4.70] $\Rightarrow \boxed{E_n = -\left[\frac{m}{2\hbar^2}(GMm)^2\right]\frac{1}{n^2}}.$ $\quad E_c = \frac{1}{2}mv^2 - G\frac{Mm}{r_0}.$ $\quad$ But $G\frac{Mm}{r_0^2} = \frac{mv^2}{r_0} \Rightarrow \frac{1}{2}mv^2 = \frac{GMm}{2r_0}$

So $E_c = -\frac{GMm}{2r_0} = -\left[\frac{m}{2\hbar^2}(GMm)^2\right]\frac{1}{n^2} \Rightarrow n^2 = \frac{GMm^2}{\hbar^2}r_0 = \frac{r_0}{a} \Rightarrow \boxed{n = \sqrt{\frac{r_0}{a}}}.$

$r_0 =$ earth-sun distance $= 1.496\times10^{11}\,\mathrm{m} \Rightarrow n = \sqrt{\frac{1.496\times10^{11}}{2.34\times10^{-138}}} = \sqrt{6.39\times10^{148}} = \boxed{2.53\times10^{74}}$

(d) $\Delta E = -\left[\frac{G^2M^2m^3}{2\hbar^2}\right]\left[\frac{1}{(n+1)^2} - \frac{1}{n^2}\right].$ $\quad \frac{1}{(n+1)^2} = \frac{1}{n^2(1+\frac{1}{n})^2} \cong \frac{1}{n^2}\left(1-\frac{2}{n}\right).$

So $\left[\frac{1}{(n+1)^2} - \frac{1}{n^2}\right] \cong \frac{1}{n^2}\left(1-\frac{2}{n}-1\right) = \frac{-2}{n^3}.$ $\quad \therefore \Delta E = \frac{G^2M^2m^3}{\hbar^2 n^3}$

$\Delta E = \frac{(6.67\times10^{-11})^2(1.99\times10^{30})^2(5.98\times10^{24})^3}{(1.055\times10^{-34})^2(2.53\times10^{74})^3} = \boxed{2.09\times10^{-41}\,\mathrm{J}}.$ $\quad \nu = \frac{E}{h};$ $\quad \lambda = \frac{c}{\nu} = \frac{ch}{E}.$

$\lambda = (3\times10^8)(6.63\times10^{-34})/(2.09\times10^{-41}) = \boxed{9.52\times10^{15}\,\mathrm{m}}.$

PROBLEM 4.17 The potential [4.52] is replaced by $V(r) = -\frac{Ze^2}{4\pi\epsilon_0}\frac{1}{r}$. So $e^2 \to Ze^2$.

$\boxed{E_n(Z) = Z^2 E_n}$ $\boxed{E_1(Z) = Z^2 E_1}$ $\boxed{a(Z) = \frac{1}{Z}a}$ $\boxed{R(Z) = Z^2 R}$. Lyman lines range from $n_i = 2$ to $n_i = \infty$: $\frac{1}{\lambda} = R\left(1-\frac{1}{4}\right) = \frac{3}{4}R \Rightarrow \lambda = \frac{4}{3R}$; $\frac{1}{\lambda} = R\left(1-\frac{1}{\infty}\right) = R \Rightarrow \lambda = \frac{1}{R}$.

So for $Z=2$: $\lambda = \frac{1}{4R} = \frac{1}{4(1.097\times 10^7)} = \boxed{2.28\times 10^{-8}\,\text{m}}$ to $\lambda = \frac{1}{3R} = \boxed{3.04\times 10^{-8}\,\text{m}}$, $\boxed{\text{ultraviolet}}$;

for $Z=3$: $\lambda = \frac{1}{9R} = \boxed{1.01\times 10^{-8}\,\text{m}}$ to $\lambda = \frac{4}{27R} = \boxed{1.35\times 10^{-8}\,\text{m}}$ – also $\boxed{\text{ultraviolet}}$.

PROBLEM 4.18 (a) $L^2 f = \lambda f$, $L_z f = \mu f$. In this state $\langle L^2\rangle = \lambda = \langle L_x^2 + L_y^2 + L_z^2\rangle = \langle L_x^2\rangle + \langle L_y^2\rangle + \mu^2$. But L_x is Hermitian, so $\langle L_x^2\rangle = \langle f|L_x^2 f\rangle = \langle L_x f|L_x f\rangle = \|L_x f\|^2 \geq 0$, and likewise $\langle L_y^2\rangle \geq 0$. So $\lambda \geq \mu^2$. QED.

(b) Implication: There is no state in which the angular momentum points in a specific direction. If there were, then we would simultaneously know L_x, L_y, and L_z — but these are incompatible observables — if you know L_x precisely you cannot know L_y and L_z precisely. The uncertainty principle [4.100] says that L_x, L_y and L_z cannot be simultaneously determined — except in the special case $\langle L_x\rangle = \langle L_y\rangle = \langle L_z\rangle = 0$ — which is to say, when $\ell = 0$ and $m = 0$.

PROBLEM 4.19 $\langle f|L_\pm g\rangle = \langle f|L_x g\rangle \pm i\langle f|L_y g\rangle = \langle L_x f|g\rangle \pm i\langle L_y f|g\rangle = \langle (L_x \mp iL_y)f|g\rangle = \langle L_\mp f|g\rangle$, So $L_\pm^\dagger = L_\mp$. Now, using [4.112]b in the form $L_\mp L_\pm = L^2 - L_z^2 \mp \hbar L_z$:

$$\langle f_\ell^m|L_\mp L_\pm f_\ell^m\rangle = \langle f_\ell^m|(L^2 - L_z^2 \mp \hbar L_z)f_\ell^m\rangle = \langle f_\ell^m|[\hbar^2\ell(\ell+1) - \hbar^2 m^2 \mp \hbar^2 m]f_\ell^m\rangle$$

$$= \hbar^2[\ell(\ell+1) - m(m\pm 1)]\langle f_\ell^m|f_\ell^m\rangle = \hbar^2[\ell(\ell+1) - m(m\pm 1)].$$

$$= \langle L_\pm f_\ell^m|L_\pm f_\ell^m\rangle = \langle A_\ell^m f_\ell^{m\pm 1}|A_\ell^m f_\ell^{m\pm 1}\rangle = |A_\ell^m|^2\langle f_\ell^{m\pm 1}|f_\ell^{m\pm 1}\rangle = |A_\ell^m|^2.$$

So $\boxed{A_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)}}$.

PROBLEM 4.20 (a) $[L_z, x] = [xp_y - yp_x, x] = [xp_y, x] - [yp_x, x] = 0 - y[p_x, x] = i\hbar y$. ✓

$[L_z, y] = [xp_y - yp_x, y] = [xp_y, y] - [yp_x, y] = x[p_y, y] - 0 = -i\hbar x$. ✓

$[L_z, z] = [xp_y - yp_x, z] = [xp_y, z] - [yp_x, z] = 0 - 0 = 0$. ✓

$[L_z, p_x] = [xp_y - yp_x, p_x] = [xp_y, p_x] - [yp_x, p_x] = p_y[x, p_x] = i\hbar p_y$. ✓

$[L_z, p_y] = [xp_y - yp_x, p_y] = [xp_y, p_y] - [yp_x, p_y] = 0 - p_x[y, p_y] = -i\hbar p_x$. ✓

$[L_z, p_z] = [xp_y - yp_x, p_z] = [xp_y, p_z] - [yp_x, p_z] = 0 - 0 = 0$. ✓

(b) $[L_z, L_x] = [L_z, yp_z - zp_y] = [L_z, yp_z] - [L_z, zp_y] = [L_z, y]p_z - [L_z, p_y]z$
$= -i\hbar x p_z + i\hbar p_x z = i\hbar(zp_x - xp_z) = i\hbar L_y$. (So, by cyclic permutation of the indices, $[L_x, L_y] = i\hbar L_z$.)

(c) $[L_z, r^2] = [L_z, x^2] + [L_z, y^2] + [L_z, z^2] = [L_z, x]x + x[L_z, x] + [L_z, y]y + y[L_z, y] + 0$
$= i\hbar y x + x i\hbar y + (-i\hbar x)y + y(-i\hbar x) = \boxed{0}$.

$[L_z, p^2] = [L_z, p_x^2] + [L_z, p_y^2] + [L_z, p_z^2] = [L_z, p_x]p_x + p_x[L_z, p_x] + [L_z, p_y]p_y + p_y[L_z, p_y] + 0$
$= i\hbar p_y p_x + p_x i\hbar p_y + (-i\hbar p_x)p_y + p_y(-i\hbar p_x) = \boxed{0}$.

(d) It follows from (c) that all three components of $\vec{L}$ commute with r^2 and p^2, and hence with $H = \frac{1}{2m}p^2 + V(\sqrt{r^2})$. QED

Problem 4.21 (a) [3.148] $\Rightarrow$ $\frac{d\langle L_x\rangle}{dt} = \frac{i}{\hbar}\langle [H, L_x]\rangle$. $[H, L_x] = \frac{1}{2m}[p^2, L_x] + [V, L_x]$.

The first term is zero (Problem 4.20(c)) — the second would be too if V were a function only of $r = |\vec{r}|$.
$\therefore$ $[H, L_x] = [V, yp_z - zp_y] = y[V, p_z] - z[V, p_y]$. But (Problem 3.41(c)): $[V, p_z] = i\hbar\frac{\partial V}{\partial z}$, and $[V, p_y] = i\hbar\frac{\partial V}{\partial y}$. So $[H, L_x] = y\, i\hbar\frac{\partial V}{\partial z} - z\, i\hbar\frac{\partial V}{\partial y} = i\hbar[\vec{r}\times(\vec{\nabla}V)]_x$. So

$$\frac{d\langle L_x\rangle}{dt} = -\langle[\vec{r}\times(\vec{\nabla}V)]_x\rangle.$$ Same goes for the other two components, so

$$\frac{d\langle\vec{L}\rangle}{dt} = \langle(\vec{r}\times(-\vec{\nabla}V))\rangle = \langle\vec{N}\rangle. \quad \text{QED.}$$

(b) If $V(\vec{r}) = V(r)$, then $\vec{\nabla}V = \frac{\partial V}{\partial r}\hat{r}$, and $\vec{r}\times\hat{r} = 0$, so $\frac{d\langle\vec{L}\rangle}{dt} = 0$. QED.

Problem 4.22 (a) $\boxed{L_+ Y_l^l = 0}$ (top of the ladder).

(b) $L_z Y_l^l = \hbar l Y_l^l \Rightarrow \frac{\hbar}{i}\frac{\partial}{\partial\phi}Y_l^l = \hbar l Y_l^l$, so $\frac{\partial Y_l^l}{\partial\phi} = il\,Y_l^l$, and hence $Y_l^l = f(\theta)e^{il\phi}$.

[Note: $f(\theta)$ is the "constant" here — it's constant with respect to ϕ ... but still can depend on θ.]

$L_+ Y_l^l = 0 \Rightarrow \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left[f(\theta)e^{il\phi}\right] = 0$, or $\frac{df}{d\theta}e^{il\phi} + if\cot\theta\, il\, e^{il\phi} = 0$, so

$\frac{df}{d\theta} = l\cot\theta f$; $\frac{df}{f} = l\cot\theta\, d\theta \Rightarrow \int\frac{df}{f} = l\int\frac{\cos\theta}{\sin\theta}d\theta \Rightarrow \ln f = l\ln(\sin\theta) + \text{constant}$.

$\ln f = \ln(\sin^l\theta) + K \Rightarrow \ln\left(\frac{f}{\sin^l\theta}\right) = K \Rightarrow \frac{f}{\sin^l\theta} = \text{constant} \Rightarrow f(\theta) = A\sin^l\theta$.

$\therefore \boxed{Y_l^l(\theta,\phi) = A\left(e^{i\phi}\sin\theta\right)^l}$

(c) $1 = A^2\int\sin^{2l}\theta\,\sin\theta\, d\theta\, d\phi = 2\pi A^2\int_0^\pi \sin^{(2l+1)}\theta\, d\theta = 2\pi A^2\, 2\,\frac{(2\cdot4\cdot6\cdot\cdots\cdot(2l))}{1\cdot3\cdot5\cdot\cdots\cdot(2l+1)}$

$= 4\pi A^2\frac{(2\cdot4\cdot6\cdot\cdots\cdot2l)^2}{1\cdot2\cdot3\cdot4\cdot5\cdot\cdots\cdot(2l+1)} = 4\pi A^2\frac{(2^l l!)^2}{(2l+1)!}$. $\therefore$ $\boxed{A = \frac{1}{2^{l+1}l!}\sqrt{\frac{(2l+1)!}{\pi}}}$ Same as Problem 4.5, except for $(-1)^l$, which is arbitrary anyway.

PROBLEM 4.23 $L_+Y_2^1 = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left[-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi}\right]$

$$= -\sqrt{\frac{15}{8\pi}}\hbar e^{i\phi}\left\{e^{i\phi}(\cos^2\theta - \sin^2\theta) + i\frac{\cos\theta}{\sin\theta}\sin\theta\cos\theta\, i e^{i\phi}\right\} = -\sqrt{\frac{15}{8\pi}}\hbar e^{2i\phi}(\cos^2\theta - \sin^2\theta - \cos^2\theta)$$

$$= \sqrt{\frac{15}{8\pi}}\hbar\left(e^{i\phi}\sin\theta\right)^2 = \hbar\sqrt{2\cdot3 - 1\cdot2}\,Y_2^2 = 2\hbar Y_2^2. \quad \therefore \boxed{Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\left(e^{i\phi}\sin\theta\right)^2}.$$

PROBLEM 4.24 (a) Eigenfunctions of a Hermitian operator belonging to different eigenvalues are orthogonal. (Theorem 2, p. 93) Spherical harmonics are eigenfunctions of the Hermitian operator L^2, so those with different l are orthogonal. They are also eigenfunctions of L_z, so those with the same l are orthogonal if they have different m. Since there is only one Y_l^m for given l and m, they're all orthogonal.

(b) The same goes for ψ_{nlm}, but these are also eigenfunctions of the Hermitian operator H, so those with different E (which is to say, different n) are also orthogonal to one another.

[Note: this only proves the orthogonality of Y_l^m and ψ_{nlm}, not normalization, of course.]

PROBLEM 4.25 (a) $H = 2\left(\frac{1}{2}mv^2\right) = mv^2$; $|\vec{L}| = 2\frac{a}{2}mv = amv$, so $L^2 = a^2m^2v^2$. $\therefore H = \frac{L^2}{ma^2}$.

But we know the eigenvalues of L^2: $\hbar^2 l(l+1)$ — or, since we usually label energies with n:

$$\boxed{E_n = \frac{\hbar^2 n(n+1)}{ma^2} \quad (n = 0, 1, 2, \dots)}.$$

(b) $\boxed{\psi_{nm}(\theta,\phi) = Y_n^m(\theta,\phi)}$ — the ordinary spherical harmonics. The degeneracy of the n^{th} energy level is the number of m-values for given n: $\boxed{2n+1}$.

PROBLEM 4.26 $r_c = \dfrac{(1.6\times10^{-19})^2}{4\pi(8.85\times10^{-12})(9.11\times10^{-31})(3.0\times10^8)^2} = 2.81\times10^{-15}\,\text{m}.$

$$L = \tfrac{1}{2}\hbar = I\omega = \left(\tfrac{2}{5}mr^2\right)\left(\tfrac{v}{r}\right) = \tfrac{2}{5}mrv \Rightarrow v = \frac{5\hbar}{4mr} = \frac{(5)(1.055\times10^{-34})}{(4)(9.11\times10^{-31})(2.81\times10^{-15})} = \boxed{5.15\times10^{10}\,\text{m/s}}.$$

Since the speed of light is 3×10^8 m/s, a point on the equator would be going more than 100 times the speed of light. Nope: doesn't look like a very good model for spin.

PROBLEM 4.27

(a) $[S_x, S_y] = S_xS_y - S_yS_x = \frac{\hbar^2}{4}\left\{\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} - \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right\} = \frac{\hbar^2}{4}\left\{\begin{pmatrix}i&0\\0&-i\end{pmatrix} - \begin{pmatrix}-i&0\\0&i\end{pmatrix}\right\}$

$$= \frac{\hbar^2}{4}\begin{pmatrix}2i&0\\0&-2i\end{pmatrix} = i\hbar\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = i\hbar S_z. \checkmark$$

(b) $\sigma_x\sigma_x = \begin{pmatrix}1&0\\0&1\end{pmatrix} = 1 = \sigma_y\sigma_y = \sigma_z\sigma_z$, so $\sigma_j\sigma_j = 1$ for $j = x, y,$ or z.

$\sigma_x\sigma_y = \begin{pmatrix}i&0\\0&-i\end{pmatrix} = i\sigma_z$; $\sigma_y\sigma_z = \begin{pmatrix}0&i\\i&0\end{pmatrix} = i\sigma_x$; $\sigma_z\sigma_x = \begin{pmatrix}0&1\\-1&0\end{pmatrix} = i\sigma_y$; similarly $\sigma_y\sigma_x = \begin{pmatrix}-i&0\\0&i\end{pmatrix} = -i\sigma_z$;

$\sigma_z\sigma_y = \begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix} = -i\sigma_x$; $\sigma_x\sigma_z = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = -i\sigma_y$. Equation [4.153] puts all this into a single equation. [Note: $\epsilon_{jkl} = \begin{cases}1 \text{ if } jkl = xyz, yzx, \text{or } zxy\\ -1 \text{ if } jkl = xzy, yxz, \text{or } zyx\end{cases}$ and zero otherwise.]

Problem 4.28 (a) $\chi^\dagger\chi = |A|^2(9+16) = 25|A|^2 = 1 \Rightarrow \boxed{A = 1/5}$.

(b) $\langle S_x\rangle = \chi^\dagger S_x\chi = \frac{1}{25}\frac{\hbar}{2}(-3i\ \ 4)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}3i\\ 4\end{pmatrix} = \frac{\hbar}{50}(-3i\ \ 4)\begin{pmatrix}4\\ 3i\end{pmatrix} = \frac{\hbar}{50}(-12i+12i) = \boxed{0}$.

$\langle S_y\rangle = \chi^\dagger S_y\chi = \frac{1}{25}\frac{\hbar}{2}(-3i\ \ 4)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}3i\\ 4\end{pmatrix} = \frac{\hbar}{50}(-3i\ \ 4)\begin{pmatrix}-4i\\ -3\end{pmatrix} = \frac{\hbar}{50}(-12-12) = -\frac{24}{50}\hbar = \boxed{-\frac{12}{25}\hbar}$.

$\langle S_z\rangle = \chi^\dagger S_z\chi = \frac{1}{25}\frac{\hbar}{2}(-3i\ \ 4)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}3i\\ 4\end{pmatrix} = \frac{\hbar}{50}(-3i\ \ 4)\begin{pmatrix}3i\\ -4\end{pmatrix} = \frac{\hbar}{50}(9-16) = \boxed{-\frac{7}{50}\hbar}$.

(c) $\langle S_x^2\rangle = \langle S_y^2\rangle = \langle S_z^2\rangle = \frac{\hbar^2}{4}$ (always, for spin 1/2), so $\sigma_{S_x}^2 = \langle S_x^2\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4} - 0$, so $\boxed{\sigma_{S_x} = \frac{\hbar}{2}}$.

$\sigma_{S_y}^2 = \langle S_y^2\rangle - \langle S_y\rangle^2 = \frac{\hbar^2}{4} - \left(\frac{12}{25}\right)^2\hbar^2 = \frac{\hbar^2}{2500}(625-576) = \frac{49}{2500}\hbar^2 \Rightarrow \boxed{\sigma_{S_y} = \frac{7}{50}\hbar}$.

$\sigma_{S_z}^2 = \langle S_z^2\rangle - \langle S_z\rangle^2 = \frac{\hbar^2}{4} - \left(\frac{7}{50}\right)^2\hbar^2 = \frac{\hbar^2}{2500}(625-49) = \frac{576}{2500}\hbar^2 \Rightarrow \boxed{\sigma_{S_z} = \frac{12}{25}\hbar}$.

(d) $\sigma_{S_x}\sigma_{S_y} = \frac{\hbar}{2}\cdot\frac{7}{50}\hbar \overset{?}{\geq} \frac{\hbar}{2}|\langle S_z\rangle| = \frac{\hbar}{2}\cdot\frac{7}{50}\hbar$; ✓ (right at the uncertainty limit).

$\sigma_{S_y}\sigma_{S_z} = \frac{7}{50}\hbar\cdot\frac{12}{25}\hbar \overset{?}{\geq} \frac{\hbar}{2}|\langle S_x\rangle| = 0$; ✓ (trivial).

$\sigma_{S_z}\sigma_{S_x} = \frac{12}{25}\hbar\cdot\frac{\hbar}{2} \overset{?}{\geq} \frac{\hbar}{2}|\langle S_y\rangle| = \frac{\hbar}{2}\cdot\frac{12}{25}\hbar$; ✓ (right at the uncertainty limit).

Problem 4.29 $\langle S_x\rangle = \frac{\hbar}{2}(a^*\ \ b^*)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \frac{\hbar}{2}(a^*\ \ b^*)\begin{pmatrix}b\\ a\end{pmatrix} = \boxed{\frac{\hbar}{2}(a^*b + b^*a)} = \hbar\,\mathrm{Re}(ab^*)$.

$\langle S_y\rangle = \frac{\hbar}{2}(a^*\ \ b^*)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \frac{\hbar}{2}(a^*\ \ b^*)\begin{pmatrix}-ib\\ ia\end{pmatrix} = \frac{\hbar}{2}(-ia^*b + iab^*) = \boxed{\frac{\hbar}{2}i(ab^* - a^*b)} = -\hbar\,\mathrm{Im}(ab^*)$.

$\langle S_z\rangle = \frac{\hbar}{2}(a^*\ \ b^*)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \frac{\hbar}{2}(a^*\ \ b^*)\begin{pmatrix}a\\ -b\end{pmatrix} = \frac{\hbar}{2}(a^*a - b^*b) = \boxed{\frac{\hbar}{2}(|a|^2 - |b|^2)}$.

$S_x^2 = \frac{\hbar^2}{4}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{\hbar^2}{4}$; $S_y^2 = \frac{\hbar^2}{4}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \frac{\hbar^2}{4}$; $S_z^2 = \frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \frac{\hbar^2}{4}$; so

$\boxed{\langle S_x^2\rangle = \langle S_y^2\rangle = \langle S_z^2\rangle = \hbar^2/4}$ $\quad \langle S_x^2\rangle + \langle S_y^2\rangle + \langle S_z^2\rangle = \frac{3}{4}\hbar^2 \overset{?}{=} s(s+1)\hbar^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 = \frac{3}{4}\hbar^2 = \langle S^2\rangle$. ✓

Problem 4.30 (a) $S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$. $\det\begin{pmatrix}-\lambda & -i\hbar/2\\ i\hbar/2 & -\lambda\end{pmatrix} = \lambda^2 - \frac{\hbar^2}{4} \Rightarrow \boxed{\lambda = \pm\frac{\hbar}{2}}$ (of course).

$\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \Rightarrow -i\beta = \pm\alpha$; $|\alpha|^2 + |\beta|^2 = 1 \Rightarrow |\alpha|^2 + |\alpha|^2 = 1 \Rightarrow \alpha = 1/\sqrt{2}$.

$$\boxed{\chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix};\quad \chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}}.$$

(b) $c_+ = \chi_+^{(y)\dagger}\chi = \frac{1}{\sqrt{2}}(1\ \ -i)\begin{pmatrix}a\\ b\end{pmatrix} = \frac{1}{\sqrt{2}}(a - ib)$. $\boxed{+\frac{\hbar}{2}\text{, with probability } \frac{1}{2}|a-ib|^2}$.

$c_- = \chi_-^{(y)\dagger}\chi = \frac{1}{\sqrt{2}}(1\ \ i)\begin{pmatrix}a\\ b\end{pmatrix} = \frac{1}{\sqrt{2}}(a + ib)$. $\boxed{-\frac{\hbar}{2}\text{, with probability } \frac{1}{2}|a+ib|^2}$.

$P_+ + P_- = \frac{1}{2}\left[(a^* + ib^*)(a - ib) + (a^* - ib^*)(a + ib)\right] = \frac{1}{2}\left[|a|^2 - ia^*b + iab^* + |b|^2 + |a|^2 + ia^*b - iab^* + |b|^2\right]$
$= |a|^2 + |b|^2 = 1$. ✓ $\quad$ (c) $\boxed{\hbar^2/4}$, probability $\boxed{1}$.

Problem 4.31 $S_r = \vec{S}\cdot\hat{r} = S_x \sin\theta\cos\phi + S_y \sin\theta\sin\phi + S_z\cos\theta$

$$= \frac{\hbar}{2}\left\{\begin{pmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0\end{pmatrix} + \begin{pmatrix} 0 & -i\sin\theta\sin\phi \\ i\sin\theta\sin\phi & 0\end{pmatrix} + \begin{pmatrix}\cos\theta & 0 \\ 0 & -\cos\theta\end{pmatrix}\right\} = \frac{\hbar}{2}\begin{pmatrix}\cos\theta & (\sin\theta)(\cos\phi - i\sin\phi) \\ (\sin\theta)(\cos\phi + i\sin\phi) & -\cos\theta\end{pmatrix}$$

$$= \boxed{\frac{\hbar}{2}\begin{pmatrix}\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta\end{pmatrix}}. \quad \det\begin{pmatrix}(\frac{\hbar}{2}\cos\theta - \lambda) & \frac{\hbar}{2}e^{-i\phi}\sin\theta \\ \frac{\hbar}{2}e^{i\phi}\sin\theta & (-\frac{\hbar}{2}\cos\theta - \lambda)\end{pmatrix} = -\frac{\hbar^2}{4}\cos^2\theta + \lambda^2 - \frac{\hbar^2}{4}\sin^2\theta = 0 \Rightarrow$$

$$\lambda^2 = \frac{\hbar^2}{4}(\sin^2\theta + \cos^2\theta) = \frac{\hbar^2}{4} \Rightarrow \boxed{\lambda = \pm\hbar/2} \text{ (of course).}$$

$$\frac{\hbar}{2}\begin{pmatrix}\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \Rightarrow \quad \alpha\cos\theta + \beta e^{-i\phi}\sin\theta = \pm\alpha; \; \beta = e^{i\phi}\frac{(\pm 1 - \cos\theta)}{\sin\theta}\alpha.$$

Upper sign: use $\quad 1-\cos\theta = 2\sin^2\theta/2$, $\sin\theta = 2\sin\theta/2\cos\theta/2$: $\quad \beta = e^{i\phi}\frac{\sin\theta/2}{\cos\theta/2}\alpha$.

$$|\alpha|^2 + |\beta|^2 = 1 \Rightarrow |\alpha|^2 + \frac{\sin^2\theta/2}{\cos^2\theta/2}|\alpha|^2 = |\alpha|^2\frac{1}{\cos^2\theta/2} = 1 \Rightarrow \alpha = \cos\theta/2; \; \beta = e^{i\phi}\sin\theta/2.$$

$$\therefore \boxed{\chi_+^{(r)} = \begin{pmatrix}\cos\theta/2 \\ e^{i\phi}\sin\theta/2\end{pmatrix}}.$$

Lower sign: Use $1+\cos\theta = 2\cos^2\theta/2$: $\beta = -e^{i\phi}\frac{\cos\theta/2}{\sin\theta/2}\alpha$; $1 = |\alpha|^2\left(1 + \frac{\cos^2\theta/2}{\sin^2\theta/2}\right) = |\alpha|^2\frac{1}{\sin^2\theta/2}$.

$\therefore \alpha = \sin\theta/2$, $\beta = -e^{i\phi}\cos\theta/2$. $\qquad \boxed{\chi_-^{(r)} = \begin{pmatrix}\sin\theta/2 \\ -e^{i\phi}\cos\theta/2\end{pmatrix}}.$

Problem 4.32 There are 3 states: $\chi_+ = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\chi_0 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\chi_- = \begin{pmatrix}0\\0\\1\end{pmatrix}$. $S_z\chi_+ = \hbar\chi_+$, $S_z\chi_0 = 0$, $S_z\chi_- = -\hbar\chi_-$.

So $\boxed{S_z = \hbar\begin{pmatrix}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1\end{pmatrix}}$ $\quad \left.\begin{array}{l} S_+\chi_+ = 0, \; S_+\chi_0 = \hbar\sqrt{2}\chi_+, \; S_+\chi_- = \hbar\sqrt{2}\chi_0 \\ S_-\chi_+ = \hbar\sqrt{2}\chi_0, \; S_-\chi_0 = \hbar\sqrt{2}\chi_-, \; S_-\chi_- = 0\end{array}\right\}$ from [4.136].

$$\therefore S_+ = \sqrt{2}\hbar\begin{pmatrix}0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0\end{pmatrix}, \; S_- = \sqrt{2}\hbar\begin{pmatrix}0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{pmatrix}. \quad S_x = \frac{1}{2}(S_+ + S_-) = \boxed{\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0\end{pmatrix}}.$$

$$S_y = \frac{1}{2i}(S_+ - S_-) = \boxed{\frac{i\hbar}{\sqrt{2}}\begin{pmatrix}0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0\end{pmatrix}}$$

Problem 4.33 (a) Using [4.151] and [4.163]:

$$c_+^{(x)} = \chi_+^{(x)\dagger}\chi = \frac{1}{\sqrt{2}}(1\;1)\begin{pmatrix}\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2} \\ \sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\end{pmatrix} = \frac{1}{\sqrt{2}}\left[\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2} + \sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\right].$$

$$P_+^{(x)}(t) = |c_+^{(x)}|^2 = \frac{1}{2}\left[\cos\frac{\alpha}{2}e^{-i\gamma B_0 t/2} + \sin\frac{\alpha}{2}e^{i\gamma B_0 t/2}\right]\left[\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2} + \sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\right]$$

$$= \frac{1}{2}\left\{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} + \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\left(e^{i\gamma B_0 t} + e^{-i\gamma B_0 t}\right)\right\} = \frac{1}{2}\left(1 + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\cos(\gamma B_0 t)\right)$$

$$= \boxed{\frac{1}{2}\left(1 + \sin\alpha\cos(\gamma B_0 t)\right)}.$$

(b) From Problem 4.30 (a): $\chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$. $c_+^{(y)} = \chi_+^{(y)\dagger}\chi = \frac{1}{\sqrt{2}}(1\ -i)\begin{pmatrix}\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2}\\ \sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\end{pmatrix}$

$$c_+^{(y)} = \frac{1}{\sqrt{2}}\left[\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2} - i\sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\right];$$

$$P_+^{(y)}(t) = |c_+^{(y)}|^2 = \frac{1}{2}\left[\cos\frac{\alpha}{2}e^{-i\gamma B_0 t/2} + i\sin\frac{\alpha}{2}e^{i\gamma B_0 t/2}\right]\left[\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2} - i\sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\right]$$

$$= \frac{1}{2}\left\{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\left(e^{i\gamma B_0 t} - e^{-i\gamma B_0 t}\right)\right\} = \frac{1}{2}\left(1 - 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sin(\gamma B_0 t)\right)$$

$$= \boxed{\frac{1}{2}\left(1 - \sin\alpha\sin(\gamma B_0 t)\right)}.$$

(c) $\chi_+^{(z)} = \begin{pmatrix}1\\0\end{pmatrix}$. $c_+^{(z)} = (1\ 0)\begin{pmatrix}\cos\frac{\alpha}{2}e^{i\gamma B_0 t/2}\\ \sin\frac{\alpha}{2}e^{-i\gamma B_0 t/2}\end{pmatrix} = \cos\frac{\alpha}{2}e^{i\gamma B_0 t/2}$. $P_+^{(z)}(t) = |c_+^{(z)}|^2 = \boxed{\cos^2\frac{\alpha}{2}}$.

Problem 4.34 (a) $H = -\gamma\vec{B}\cdot\vec{S} = -\gamma B_0\cos\omega t\, S_z = \boxed{-\frac{\gamma B_0\hbar}{2}\cos\omega t\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}$.

(b) $\chi(t) = \begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix}$, with $\alpha(0) = \beta(0) = 1/\sqrt{2}$.

$$i\hbar\frac{\partial\chi}{\partial t} = i\hbar\begin{pmatrix}\dot\alpha\\ \dot\beta\end{pmatrix} = H\chi = -\frac{\gamma B_0\hbar}{2}\cos\omega t\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = -\frac{\gamma B_0\hbar}{2}\cos\omega t\begin{pmatrix}\alpha\\ -\beta\end{pmatrix}.$$

$$\dot\alpha = i\left(\frac{\gamma B_0}{2}\right)\cos\omega t\,\alpha \Rightarrow \frac{d\alpha}{\alpha} = i\left(\frac{\gamma B_0}{2}\right)\cos\omega t\,dt \Rightarrow \ln\alpha = \frac{i\gamma B_0}{2}\frac{\sin\omega t}{\omega} + \text{constant}.$$

$$\alpha(t) = Ae^{i(\gamma B_0/2\omega)\sin\omega t}. \quad \alpha(0) = A = \frac{1}{\sqrt{2}},\text{ so } \alpha(t) = \frac{1}{\sqrt{2}}e^{i(\gamma B_0/2\omega)\sin\omega t}.$$

$$\dot\beta = -i\left(\frac{\gamma B_0}{2}\right)\cos\omega t\,\beta \Rightarrow \beta(t) = \frac{1}{\sqrt{2}}e^{-i(\gamma B_0/2\omega)\sin\omega t}$$

$$\boxed{\chi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{i(\gamma B_0/2\omega)\sin\omega t}\\ e^{-i(\gamma B_0/2\omega)\sin\omega t}\end{pmatrix}}.$$

(c) $$c_-^{(x)} = \chi_-^{(x)\dagger}\chi = \frac{1}{2}(1\ -1)\begin{pmatrix}e^{i(\gamma B_0/2\omega)\sin\omega t}\\ e^{-i(\gamma B_0/2\omega)\sin\omega t}\end{pmatrix} = \frac{1}{2}\left[e^{i(\gamma B_0/2\omega)\sin\omega t} - e^{-i(\gamma B_0/2\omega)\sin\omega t}\right]$$

$$= i\sin\left[\frac{\gamma B_0}{2\omega}\sin(\omega t)\right]. \qquad P_-^{(x)}(t) = |c_-^{(x)}|^2 = \boxed{\sin^2\left[\frac{\gamma B_0}{2\omega}\sin(\omega t)\right]}.$$

(d) The argument of $\sin^2$ must reach $\pi/2$ (so $P=1$). $\therefore \frac{\gamma B_0}{2\omega} = \frac{\pi}{2}$, or $\boxed{B_0 = \frac{\pi\omega}{\gamma}}$.

Problem 4.35 (a) $S_-|10\rangle = (S_-^{(1)} + S_-^{(2)})\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) = \frac{1}{\sqrt{2}}\left[(S_-\uparrow)\downarrow + (S_-\downarrow)\uparrow + \uparrow(S_-\downarrow) + \downarrow(S_-\uparrow)\right]$.

But $S_-\uparrow = \hbar\downarrow$, $S_-\downarrow = 0$ (eq. [4.143]), so $S_-|10\rangle = \frac{1}{\sqrt{2}}\left[\hbar\downarrow\downarrow + 0 + 0 + \hbar\downarrow\downarrow\right] = \sqrt{2}\hbar\downarrow\downarrow = \sqrt{2}\hbar|1\,-1\rangle$. ✓

(b) $S_\pm|00\rangle = (S_\pm^{(1)} + S_\pm^{(2)})\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) = \frac{1}{\sqrt{2}}\left[(S_\pm\uparrow)\downarrow - (S_\pm\downarrow)\uparrow + \uparrow(S_\pm\downarrow) - \downarrow(S_\pm\uparrow)\right]$

$S_+|00\rangle = \frac{1}{\sqrt{2}}(0 - \hbar\uparrow\uparrow + \hbar\uparrow\uparrow - 0) = 0$; $S_-|00\rangle = \frac{1}{\sqrt{2}}(\hbar\downarrow\downarrow - 0 + 0 - \hbar\downarrow\downarrow) = 0$. ✓

(c) $$S^2|11\rangle = \left[(S^{(1)})^2 + (S^{(2)})^2 + 2\vec{S}^{(1)}\cdot\vec{S}^{(2)}\right]\uparrow\uparrow = (S^2\uparrow)\uparrow + \uparrow(S^2\uparrow) + 2\left[(S_x\uparrow)(S_x\uparrow) + (S_y\uparrow)(S_y\uparrow) + (S_z\uparrow)(S_z\uparrow)\right]$$

$$= \frac{3}{4}\hbar^2\uparrow\uparrow + \frac{3}{4}\hbar^2\uparrow\uparrow + 2\left[\frac{\hbar}{2}\downarrow\frac{\hbar}{2}\downarrow + \frac{i\hbar}{2}\downarrow\frac{i\hbar}{2}\downarrow + \frac{\hbar}{2}\uparrow\frac{\hbar}{2}\uparrow\right] = \frac{3}{2}\hbar^2\uparrow\uparrow + 2\left[\frac{\hbar^2}{4}\uparrow\uparrow\right] = 2\hbar^2\uparrow\uparrow$$

$= 2\hbar^2|11\rangle = (1)(1+1)\hbar^2|11\rangle$, as it should be.

$$S^2|1\,-1\rangle = \left[(S^{(1)})^2 + (S^{(2)})^2 + 2\vec{S}^{(1)}\cdot\vec{S}^{(2)}\right]\downarrow\downarrow = \frac{3\hbar^2}{4}\downarrow\downarrow + \frac{3\hbar^2}{4}\downarrow\downarrow + 2\left[(S_x\downarrow)(S_x\downarrow) + (S_y\downarrow)(S_y\downarrow) + (S_z\downarrow)(S_z\downarrow)\right]$$

$$= \frac{3}{2}\hbar^2\downarrow\downarrow + 2\left[\left(\frac{\hbar}{2}\uparrow\right)\left(\frac{\hbar}{2}\uparrow\right) + \left(-\frac{i\hbar}{2}\uparrow\right)\left(-\frac{i\hbar}{2}\uparrow\right) + \left(-\frac{\hbar}{2}\downarrow\right)\left(-\frac{\hbar}{2}\downarrow\right)\right] = \frac{3}{2}\hbar^2\downarrow\downarrow + 2\frac{\hbar^2}{4}\downarrow\downarrow = 2\hbar^2\downarrow\downarrow = 2\hbar^2|1\,-1\rangle. ✓$$

Problem 4.36 (a) $\frac{1}{2}$ and $\frac{1}{2}$ gives 1 or zero; $\frac{1}{2}$ and 1 gives $3/2$ or $1/2$, $\frac{1}{2}$ and 0 gives $1/2$ only. So baryons can have $\boxed{\text{spin } 3/2 \text{ or spin } 1/2}$ (and the latter can be achieved in two distinct ways).

[Incidentally, the lightest baryons do carry spin $1/2$ (proton, neutron, etc) or $3/2$ (Δ, Ω^-, etc) — heavier baryons can have higher total spin, but this is because the quarks have orbital angular momentum as well.]

(b) $\frac{1}{2}$ and $\frac{1}{2}$ gives $\boxed{\text{one or zero}}$. [Again, these are the observed spins for the lightest mesons — π's and K's have spin 0, ρ's and ω's have spin 1.]

Problem 4.37 (a) From the 2×1 Clebsch-Gordan table we get

$|3\,1\rangle = \sqrt{\frac{1}{15}}|2\,2\rangle|1\,-1\rangle + \sqrt{\frac{8}{15}}|2\,1\rangle|1\,0\rangle + \sqrt{\frac{6}{15}}|2\,0\rangle|1\,1\rangle$, so you might get

$$\boxed{2\hbar\ (\text{probability } \tfrac{1}{15}),\ \hbar\ (\text{probability } \tfrac{8}{15}),\ \text{or } 0\ (\text{probability } \tfrac{6}{15})}.$$

(b) From the $1\times\frac{1}{2}$ table: $|1\,0\rangle|\frac{1}{2}\,-\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}\,|\frac{3}{2}\,-\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}\,|\frac{1}{2}\,-\frac{1}{2}\rangle$. So total is $\frac{3}{2}$ or $\frac{1}{2}$, with $l(l+1)\hbar^2 = \frac{15}{4}\hbar^2$ and $\frac{3}{4}\hbar^2$, respectively. Thus you could get

$$\boxed{\tfrac{15}{4}\hbar^2\ (\text{probability } 2/3),\ \text{or } \tfrac{3}{4}\hbar^2\ (\text{probability } 1/3)}.$$

Problem 4.38 Using [4.179]: $[S^2, S_z^{(1)}] = [S^{(1)2}, S_z^{(1)}] + [S^{(2)2}, S_z^{(1)}] + 2[\vec{S}^{(1)}\cdot\vec{S}^{(2)}, S_z^{(1)}]$.

But $[S^2, S_z] = 0$ [4.102], and anything with superscript (2) commutes with anything with superscript (1). So $[S^2, S_z^{(1)}] = 2\{S_x^{(2)}[S_x^{(1)}, S_z^{(1)}] + S_y^{(2)}[S_y^{(1)}, S_z^{(1)}] + S_z^{(2)}[S_z^{(1)}, S_z^{(1)}]\}$

$$= 2\{-i\hbar\, S_y^{(1)}S_x^{(2)} + i\hbar\, S_x^{(1)}S_y^{(2)}\} = 2i\hbar\,(\vec{S}^{(1)}\times\vec{S}^{(2)})_z.$$

$\boxed{[S^2, S_z^{(1)}] = 2i\hbar\left(S_x^{(1)}S_y^{(2)} - S_y^{(1)}S_x^{(2)}\right)}$, and $[S^2, \vec{S}^{(1)}] = 2i\hbar\,(\vec{S}^{(1)}\times\vec{S}^{(2)})$.

[Note that $[S^2, \vec{S}^{(2)}] = 2i\hbar\,(\vec{S}^{(2)}\times\vec{S}^{(1)}) = -2i\hbar\,(\vec{S}^{(1)}\times\vec{S}^{(2)})$, so $[S^2, (\vec{S}^{(1)}+\vec{S}^{(2)})] = 0$.]

Problem 4.39 (a) $-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + \frac{1}{2}m\omega^2(x^2+y^2+z^2)\psi = E\psi$.

Let $\psi(x,y,z) = X(x)Y(y)Z(z)$; plug in, divide by XYZ, and collect terms:

$$\left(-\frac{\hbar^2}{2m}\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{2}m\omega^2x^2\right) + \left(-\frac{\hbar^2}{2m}\frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{2}m\omega^2y^2\right) + \left(-\frac{\hbar^2}{2m}\frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{1}{2}m\omega^2z^2\right) = E$$

The first term is a function only of x, the second only of y, and the third only of z. So each is a constant (call them E_x, E_y, E_z, with $E_x + E_y + E_z = E$).

Thus $-\frac{\hbar^2}{2m}\frac{d^2X}{dx^2}+\frac{1}{2}m\omega^2x^2X=E_xX$; $-\frac{\hbar^2}{2m}\frac{d^2Y}{dy^2}+\frac{1}{2}m\omega^2y^2Y=E_yY$; $-\frac{\hbar^2}{2m}\frac{d^2Z}{dz^2}+\frac{1}{2}m\omega^2z^2Z=E_zZ$.

Each of these is simply the one-dimensional Harmonic Oscillator (eq. [2.39]). We know the allowed energies [2.50]:

$$E_x=(n_x+\tfrac{1}{2})\hbar\omega;\quad E_y=(n_y+\tfrac{1}{2})\hbar\omega;\quad E_z=(n_z+\tfrac{1}{2})\hbar\omega,\ \text{where } n_x,n_y,n_z=0,1,2,3,\dots$$

So $E=(n_x+n_y+n_z+\frac{3}{2})\hbar\omega=\boxed{(n+\frac{3}{2})\hbar\omega}$, with $n\equiv n_x+n_y+n_z$.

(b) The question is: "how many ways can we add three non-negative integers to get sum n?"

If $n_x=n$, then $n_y=n_z=0$. One way.

If $n_x=n-1$, then $n_y=0,n_z=1$, or else $n_y=1,n_z=0$. Two ways.

If $n_x=n-2$, then $n_y=0,n_z=2$, or else $n_y=1,n_z=1$, or else $n_y=2,n_z=0$. Three ways.

etc. Evidently $d(n)=1+2+3+\cdots+(n+1)=\boxed{\frac{(n+1)(n+2)}{2}}$.

PROBLEM 4.40 [4.37]: $-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+\left[\frac{1}{2}m\omega^2r^2+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u=Eu$.

Following [2.55], let $\xi\equiv\sqrt{\frac{m\omega}{\hbar}}\,r$. Then: $-\frac{\hbar^2}{2m}\frac{m\omega}{\hbar}\frac{d^2u}{d\xi^2}+\left[\frac{1}{2}m\omega^2\frac{\hbar}{m\omega}\xi^2+\frac{\hbar^2}{2m}\frac{m\omega}{\hbar}\frac{l(l+1)}{\xi^2}\right]u=Eu$,

or $\frac{d^2u}{d\xi^2}=\left[\xi^2+\frac{l(l+1)}{\xi^2}-K\right]u$, where $K\equiv\frac{2E}{\hbar\omega}$ (as in [2.57]).

At large ξ, $\frac{d^2u}{d\xi^2}\cong\xi^2u$, and $u\sim(\ )e^{-\xi^2/2}$ (see [2.61]).

At small ξ, $\frac{d^2u}{d\xi^2}\cong\frac{l(l+1)}{\xi^2}u$, and $u\sim(\ )\xi^{l+1}$ (see [4.59]).

So let $u(\xi)=\xi^{l+1}e^{-\xi^2/2}v(\xi)$. (This simply defines a new function $v(\xi)$.)

$$\frac{du}{d\xi}=(l+1)\xi^le^{-\xi^2/2}v-\xi^{l+2}e^{-\xi^2/2}v+\xi^{l+1}e^{-\xi^2/2}v'.$$

$$\begin{aligned}\frac{d^2u}{d\xi^2}&=l(l+1)\xi^{l-1}e^{-\xi^2/2}v-(l+1)\xi^{l+1}e^{-\xi^2/2}v+(l+1)\xi^le^{-\xi^2/2}v'\\&\quad-(l+2)\xi^{l+1}e^{-\xi^2/2}v+\xi^{l+3}e^{-\xi^2/2}v-\xi^{l+2}e^{-\xi^2/2}v'\\&\quad+(l+1)\xi^le^{-\xi^2/2}v'-\xi^{l+2}e^{-\xi^2/2}v'+\xi^{l+1}e^{-\xi^2/2}v''\end{aligned}$$

$$\begin{aligned}&=\cancel{l(l+1)\xi^{l-1}e^{-\xi^2/2}v}-(2l+3)\xi^{l+1}e^{-\xi^2/2}v+\cancel{\xi^{l+3}e^{-\xi^2/2}v}+2(l+1)\xi^le^{-\xi^2/2}v'\\&\quad-2\xi^{l+2}e^{-\xi^2/2}v'+\xi^{l+1}e^{-\xi^2/2}v''=\cancel{\xi^{l+3}e^{-\xi^2/2}v}+\cancel{l(l+1)\xi^{l-1}e^{-\xi^2/2}v}-K\xi^{l+1}e^{-\xi^2/2}v.\end{aligned}$$

Cancelling the indicated terms, and dividing off $\xi^{l+1}e^{-\xi^2/2}$, we have:

$$v''+2v'\left(\frac{l+1}{\xi}-\xi\right)+(K-2l-3)v=0.\quad \text{Let } v(\xi)=\sum_{j=0}^{\infty}a_j\xi^j,\ \text{so}$$

$$v'=\sum_{j=1}^{\infty}ja_j\xi^{j-1};\quad v''=\sum_{j=2}^{\infty}j(j-1)a_j\xi^{j-2}.\quad \text{So}$$

$$\sum_{j=2}^{\infty} j(j-1)a_j \xi^{j-2} + 2(\ell+1)\sum_{j=1}^{\infty} j a_j \xi^{j-2} - 2\sum_{j=1}^{\infty} j a_j \xi^j + (K-2\ell-3)\sum_{j=0}^{\infty} a_j \xi^j = 0.$$

In the first two sums, let $j \to j+2$ (rename the dummy variable):

$$\sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2}\xi^j + 2(\ell+1)\sum_{j=0}^{\infty}(j+2)a_{j+2}\xi^j - 2\sum_{j=0}^{\infty} j a_j \xi^j + (K-2\ell-3)\sum_{j=0}^{\infty} a_j \xi^j = 0.$$

<u>Note</u>: The second sum should start at $j=-1$ — to eliminate this term (there is no compensating one in ξ^{-1}) we must take $\underline{\underline{a_1 = 0}}$. Combining the terms:

$$\sum_{j=0}^{\infty}\left\{(j+2)\ (j+2\ell+3)a_{j+2} + (K-2j-2\ell-3)a_j\right\}\xi^j = 0, \text{ so}$$

$$\boxed{a_{j+2} = \frac{(2j+2\ell+3-K)}{(j+2)(j+2\ell+3)}\,a_j}.$$

Since $a_1 = 0$, this gives us a single sequence: $a_0, a_2, a_4, \ldots$. But the series must terminate (else we get the wrong behavior as $\xi \to \infty$), so there occurs some maximal (even) number j_{max} such that $a_{j_{max}+2} = 0$. Thus:

$$K = 2j_{max} + 2\ell + 3.$$

But $E = \frac{1}{2}\hbar\omega K$, so $E = (j_{max} + \ell + \frac{3}{2})\hbar\omega$. Or, letting $j_{max} + \ell = n$,

$\boxed{E_n = (n+\frac{3}{2})\hbar\omega}$, and n can be any nonnegative integer.

[Incidentally, we can also determine the degeneracy of E_n. Suppose n is <u>even</u> — then (since j_{max} is even) $\ell = 0, 2, 4, \ldots, n$. For each ℓ there are $(2\ell+1)$ values for m. So

$d(n) = \sum_{\ell=0,2,4,\ldots}^{n} (2\ell+1)$. Let $j = \frac{\ell}{2}$; then $d(n) = \sum_{j=0}^{n/2}(4j+1) = 4\sum_{j=0}^{n/2} j + \sum_{j=0}^{n/2} 1$

$$= 4\,\frac{(\frac{n}{2})(\frac{n}{2}+1)}{2} + (\frac{n}{2}+1) = (\frac{n}{2}+1)(n+1) = \frac{(n+1)(n+2)}{2},$$

as before (Problem 4.39 b).]

<u>PROBLEM 4.41</u> (a) $\frac{d}{dt}\langle \vec{r}\cdot\vec{p}\rangle = \frac{i}{\hbar}\langle[H, \vec{r}\cdot\vec{p}]\rangle$.

$$[H, \vec{r}\cdot\vec{p}] = \sum_{i=1}^{3}[H, r_i p_i] = \sum_{i=1}^{3}\left([H, r_i]p_i + r_i[H, p_i]\right)$$

$$= \sum_{i=1}^{3}\left(\frac{1}{2m}[p^2, r_i]p_i + r_i[V, p_i]\right). \quad [p^2, r_i] = \sum_{j=1}^{3}[p_j p_j, r_i] = \sum_{j=1}^{3}\left(p_j[p_j, r_i] + [p_j, r_i]p_j\right)$$

$$= \sum_{j=1}^{3}\left(p_j(-i\hbar\delta_{ij}) + (-i\hbar\delta_{ij})p_j\right) = -2i\hbar p_i.$$

$[V, p_i] = i\hbar\frac{\partial V}{\partial r_i}$ (Problem 3.41(c)). $[H, \vec{r}\cdot\vec{p}] = \sum_{i=1}^{3}\left(\frac{1}{2m}(-2i\hbar)p_i p_i + r_i\left(i\hbar\frac{\partial V}{\partial r_i}\right)\right) = i\hbar\left(-\frac{p^2}{m} + \vec{r}\cdot\vec{\nabla}V\right).$

$\frac{d}{dt}\langle\vec{r}\cdot\vec{p}\rangle = \langle\frac{p^2}{m} - \vec{r}\cdot\vec{\nabla}V\rangle = 2\langle T\rangle - \langle\vec{r}\cdot\vec{\nabla}V\rangle$. For stationary states $\frac{d}{dt}\langle\vec{r}\cdot\vec{p}\rangle = 0$, so

$2\langle T\rangle = \langle \vec{r}\cdot\vec{\nabla}V\rangle$. QED

(b) $V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} \Rightarrow \vec{\nabla}V = \frac{e^2}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r} \Rightarrow \vec{r}\cdot\vec{\nabla}V = \frac{e^2}{4\pi\epsilon_0}\frac{1}{r} = -V$. So $2\langle T\rangle = -\langle V\rangle$.

But $\langle T\rangle + \langle V\rangle = E_n$, so $\langle T\rangle - 2\langle T\rangle = E_n$, or $\langle T\rangle = -E_n$; $\langle V\rangle = 2E_n$. QED

(c) $V = \frac{1}{2}m\omega^2 r^2 \Rightarrow \vec{\nabla}V = m\omega^2 r\hat{r} \Rightarrow \vec{r}\cdot\vec{\nabla}V = m\omega^2 r^2 = 2V$. So $2\langle T\rangle = 2\langle V\rangle$, or $\langle T\rangle = \langle V\rangle$.

But $\langle T\rangle + \langle V\rangle = E_n$, so $\langle T\rangle = \langle V\rangle = \frac{1}{2}E_n$. QED

PROBLEM 4.42 (a) $\psi = \frac{1}{\sqrt{\pi a^3}}e^{-r/a} \Rightarrow \phi(\vec{p}) = \frac{1}{(2\pi\hbar)^{3/2}}\frac{1}{\sqrt{\pi a^3}}\int e^{-i\vec{p}\cdot\vec{r}/\hbar}e^{-r/a}r^2\sin\theta\,dr\,d\theta\,d\phi$.

With axes as suggested, $\vec{p}\cdot\vec{r} = pr\cos\theta$. Doing the (trivial) ϕ integral:

$$\phi(\vec{p}) = \frac{2\pi}{(2\pi a\hbar)^{3/2}}\frac{1}{\sqrt{\pi}}\int_0^\infty r^2 e^{-r/a}\left\{\int_0^\pi e^{-ipr\cos\theta/\hbar}\sin\theta\,d\theta\right\}dr.$$

$$\int_0^\pi e^{-ipr\cos\theta/\hbar}\sin\theta\,d\theta = \frac{\hbar}{ipr}e^{-ipr\cos\theta/\hbar}\Big|_0^\pi = \frac{\hbar}{ipr}\left(e^{ipr/\hbar} - e^{-ipr/\hbar}\right) = \frac{2\hbar}{pr}\sin(pr/\hbar).$$

$$\phi(\vec{p}) = \frac{1}{\pi\sqrt{2}}\frac{1}{(a\hbar)^{3/2}}\frac{2\hbar}{p}\int_0^\infty r e^{-r/a}\sin(pr/\hbar)\,dr.$$

$$\int_0^\infty r e^{-r/a}\sin(pr/\hbar)\,dr = \frac{1}{2i}\left\{\int_0^\infty r e^{-r/a}e^{ipr/\hbar}dr - \int_0^\infty r e^{-r/a}e^{-ipr/\hbar}dr\right\}$$

$$= \frac{1}{2i}\left\{\frac{1}{\left(\frac{1}{a}-\frac{ip}{\hbar}\right)^2} - \frac{1}{\left(\frac{1}{a}+\frac{ip}{\hbar}\right)^2}\right\} = \frac{1}{2i}\frac{(2ip/a\hbar)2}{\left[\left(\frac{1}{a}\right)^2 + (p/\hbar)^2\right]^2} = \frac{(2p/\hbar)a^3}{\left[1+(ap/\hbar)^2\right]^2}$$

$$\phi(\vec{p}) = \sqrt{\frac{2}{\hbar}}\frac{1}{(a)^{3/2}}\frac{1}{\pi p}\frac{2pa^3}{\hbar}\frac{1}{\left[1+(ap/\hbar)^2\right]^2} = \boxed{\frac{1}{\pi}\left(\frac{2a}{\hbar}\right)^{3/2}\frac{1}{\left[1+(ap/\hbar)^2\right]^2}}$$

(b) $$\int|\phi|^2 d^3p = 4\pi\int_0^\infty p^2|\phi|^2 dp = 4\pi\cdot\frac{1}{\pi^2}\left(\frac{2a}{\hbar}\right)^3\int_0^\infty\frac{p^2}{\left[1+(ap/\hbar)^2\right]^4}dp.$$

From math tables: $\int_0^\infty \frac{x^2}{(m+x^2)^4}dx = \frac{\pi}{32}m^{-5/2}$, so $\int_0^\infty\frac{p^2}{\left[1+(ap/\hbar)^2\right]^4}dp = \left(\frac{\hbar}{a}\right)^8\frac{\pi}{32}\left(\frac{\hbar}{a}\right)^{-5} = \frac{\pi}{32}\left(\frac{\hbar}{a}\right)^3$.

So $\int|\phi|^2 d^3p = \frac{32}{\pi}\left(\frac{a}{\hbar}\right)^3\cdot\frac{\pi}{32}\left(\frac{\hbar}{a}\right)^3 = 1.$ ✓

(c) $\langle p^2\rangle = \int p^2|\phi|^2 d^3p = \frac{1}{\pi^2}\left(\frac{2a}{\hbar}\right)^3 4\pi\int_0^\infty\frac{p^4}{\left[1+(ap/\hbar)^2\right]^4}dp$. From math tables:

$$\int_0^\infty\frac{x^4}{[m+x^2]^4}dx = \left(\frac{\pi}{32}\right)m^{-3/2}. \quad \therefore \quad \langle p^2\rangle = \frac{4}{\pi}\left(\frac{2a}{\hbar}\right)^3\left(\frac{\hbar}{a}\right)^8\frac{\pi}{32}\left(\frac{\hbar}{a}\right)^{-3} = \boxed{\frac{\hbar^2}{a^2}}$$

(d) $\langle T\rangle = \frac{1}{2m}\langle p^2\rangle = \frac{1}{2m}\frac{\hbar^2}{a^2} = \frac{\hbar^2}{2m}\frac{m^2}{\hbar^4}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 = \frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 = \boxed{-E_1}$, which is consistent with [4.191].

PROBLEM 4.43 (a) From Tables 4.2 and 4.6, $\psi_{321} = R_{32} Y_2^1 = \frac{4}{81\sqrt{30}} \frac{1}{a^{3/2}}\left(\frac{r}{a}\right)^2 e^{-r/3a}\left[-\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta e^{i\phi}\right]$

$$= \boxed{-\frac{1}{\sqrt{\pi}} \frac{1}{81 a^{7/2}} r^2 e^{-r/3a} \sin\theta\cos\theta\, e^{i\phi}}$$

(b) $\int |\psi|^2 d^3r = \frac{1}{\pi}\frac{1}{(81)^2 a^7}\int r^4 e^{-2r/3a}\sin^2\theta\cos^2\theta\, r^2\sin\theta\, dr\, d\theta\, d\phi$

$$= \frac{1}{\pi(81)^2 a^7} 2\pi \int_0^\infty r^6 e^{-2r/3a} dr \int_0^\pi (1-\cos^2\theta)\cos^2\theta\sin\theta\, d\theta$$

$$= \frac{2}{(81)^2 a^7}\left[6!\left(\frac{3a}{2}\right)^7\right]\left[-\frac{\cos^3\theta}{3} + \frac{\cos^5\theta}{5}\right]\Big|_0^\pi = \frac{2}{3^8 a^7} 6\cdot5\cdot4\cdot3\cdot2\, \frac{3^7 a^7}{2^7}\left[\frac{2}{3}-\frac{2}{5}\right]$$

$$= \frac{3\cdot5}{4}\cdot\frac{4}{15} = 1. \checkmark$$

(c) $\langle r^s\rangle = \int_0^\infty r^s |R_{32}|^2 r^2 dr = \left(\frac{4}{81}\right)^2 \frac{1}{30}\frac{1}{a^7}\int_0^\infty r^{s+6} e^{-2r/3a} dr = \frac{8}{15(81)^2 a^7}(s+6)!\left(\frac{3a}{2}\right)^{s+7}$

$$= \boxed{(s+6)!\left(\frac{3a}{2}\right)^s \frac{1}{720}} = \frac{(s+6)!}{6!}\left(\frac{3a}{2}\right)^s. \text{ Finite for } \boxed{s>-7}.$$

PROBLEM 4.44 We may as well choose axes so that $\hat{a}$ lies along the z axis and $\hat{b}$ is in the xz plane. Then $S_a^{(1)} = S_z^{(1)}$, and $S_b^{(2)} = \cos\theta\, S_z^{(2)} + \sin\theta\, S_x^{(2)}$. $\langle 00|S_a^{(1)} S_b^{(2)}|00\rangle$ is to be calculated.

$S_a^{(1)}S_b^{(2)}|00\rangle = \frac{1}{\sqrt{2}}\left(S_z^{(1)}(\cos\theta S_z^{(2)} + \sin\theta S_x^{(2)})\right)(\uparrow\downarrow - \downarrow\uparrow) = \frac{1}{\sqrt{2}}\left\{(S_z\uparrow)(\cos\theta\, S_z\downarrow + \sin\theta\, S_x\downarrow) - (S_z\downarrow)(\cos\theta\, S_z\uparrow + \sin\theta\, S_x\uparrow)\right\}$

$$= \frac{1}{\sqrt{2}}\left\{\frac{\hbar}{2}\uparrow\left(\cos\theta(-\tfrac{\hbar}{2}\downarrow) + \sin\theta(\tfrac{\hbar}{2}\uparrow)\right) - \left(-\tfrac{\hbar}{2}\downarrow\right)\left(\cos\theta(\tfrac{\hbar}{2}\uparrow) + \sin\theta(\tfrac{\hbar}{2}\downarrow)\right)\right\} \quad (\text{using } [4.145])$$

$$= \frac{\hbar^2}{4}\left\{\cos\theta\frac{1}{\sqrt{2}}(-\uparrow\downarrow + \downarrow\uparrow) + \sin\theta(\uparrow\uparrow + \downarrow\downarrow)\right\} = \frac{\hbar^2}{4}\left\{-\cos\theta|00\rangle + \sin\theta(|11\rangle + |1\,-1\rangle)\right\}.$$

So $\langle S_a^{(1)} S_b^{(2)}\rangle = \langle 00|S_a^{(1)} S_b^{(2)}|00\rangle = \frac{\hbar^2}{4}\langle 00|\left[-\cos\theta|00\rangle + \sin\theta(|11\rangle + |1\,-1\rangle)\right] = -\frac{\hbar^2}{4}\cos\theta\langle 00|00\rangle + 0$

(by orthogonality), so $\langle S_a^{(1)} S_b^{(2)}\rangle = -\frac{\hbar^2}{4}\cos\theta$. QED.

PROBLEM 4.45 First note from [4.147] and [4.136] that

$$S_x|s\ m\rangle = \frac{1}{2}\{S_+|s\,m\rangle + S_-|s\,m\rangle\} = \frac{\hbar}{2}\left\{\sqrt{s(s+1)-m(m+1)}\,|s\ m+1\rangle + \sqrt{s(s+1)-m(m-1)}\,|s\ m-1\rangle\right\}$$

$$S_y|s\ m\rangle = \frac{1}{2i}\{S_+|s\,m\rangle - S_-|s\,m\rangle\} = \frac{\hbar}{2i}\left\{\sqrt{s(s+1)-m(m+1)}\,|s\ m+1\rangle - \sqrt{s(s+1)-m(m-1)}\,|s\ m-1\rangle\right\}$$

Now, using [4.179] and [4.145]:

$$S^2|s\ m\rangle = \left[(S^{(1)})^2 + (S^{(2)})^2 + 2\left(S_x^{(1)}S_x^{(2)} + S_y^{(1)}S_y^{(2)} + S_z^{(1)}S_z^{(2)}\right)\right]\left[A|\tfrac{1}{2}\tfrac{1}{2}\rangle|s_2\ m-\tfrac{1}{2}\rangle + B|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|s_2\ m+\tfrac{1}{2}\rangle\right]$$

$$= A\Big\{(S^2|\tfrac12\tfrac12\rangle)|s_2\ m-\tfrac12\rangle + |\tfrac12\tfrac12\rangle(S^2|s_2\ m-\tfrac12\rangle) + 2\Big[(S_x|\tfrac12\tfrac12\rangle)(S_x|s_2\ m-\tfrac12\rangle) + (S_y|\tfrac12\tfrac12\rangle)(S_y|s_2\ m-\tfrac12\rangle)$$
$$+ (S_z|\tfrac12\tfrac12\rangle)(S_z|s_2\ m-\tfrac12\rangle)\Big]\Big\}$$

$$+ B\Big\{(S^2|\tfrac12\,-\tfrac12\rangle)|s_2\ m+\tfrac12\rangle + |\tfrac12\,-\tfrac12\rangle(S^2|s_2\ m+\tfrac12\rangle) + 2\Big[(S_x|\tfrac12\,-\tfrac12\rangle)(S_x|s_2\ m+\tfrac12\rangle) + (S_y|\tfrac12\,-\tfrac12\rangle)(S_y|s_2\ m+\tfrac12\rangle)$$
$$+ (S_z|\tfrac12\,-\tfrac12\rangle)(S_z|s_2\ m+\tfrac12\rangle)\Big]\Big\}$$

$$S^2|s\,m\rangle = A\Big\{\tfrac{3}{4}\hbar^2|\tfrac12\,\tfrac12\rangle|s_2\,m-\tfrac12\rangle + \hbar^2 s_2(s_2+1)|\tfrac12\,\tfrac12\rangle|s_2\,m-\tfrac12\rangle$$

$$+2\Big[\tfrac{\hbar}{2}|\tfrac12\,-\tfrac12\rangle\tfrac{\hbar}{2}\Big(\sqrt{s_2(s_2+1)-(m-\tfrac12)(m+\tfrac12)}\,|s_2\,m+\tfrac12\rangle + \sqrt{s_2(s_2+1)-(m-\tfrac12)(m-\tfrac12)}\,|s_2\,m-\tfrac32\rangle\Big)$$

$$+\left(\tfrac{i\hbar}{2}\right)|\tfrac12\,-\tfrac12\rangle\tfrac{\hbar}{2i}\Big(\sqrt{s_2(s_2+1)-(m-\tfrac12)(m+\tfrac{8}{2})}\,|s_2\,m+\tfrac12\rangle - \sqrt{s_2(s_2+1)-(m-\tfrac12)(m-\tfrac32)}\,|s_2\,m-\tfrac32\rangle\Big)$$

$$+\tfrac{\hbar}{2}|\tfrac12\,\tfrac12\rangle\hbar(m-\tfrac12)|s_2\,m-\tfrac12\rangle\Big]\Big\} + B\Big\{\tfrac34\hbar^2|\tfrac12\,-\tfrac12\rangle|s_2\,m+\tfrac12\rangle + \hbar^2 s_2(s_2+1)|\tfrac12\,-\tfrac12\rangle|s_2\,m+\tfrac12\rangle$$

$$+2\Big[\tfrac{\hbar}{2}|\tfrac12\,\tfrac12\rangle\tfrac{\hbar}{2}\Big(\sqrt{s_2(s_2+1)-(m+\tfrac12)(m+\tfrac32)}\,|s_2\,m+\tfrac32\rangle + \sqrt{s_2(s_2+1)-(m+\tfrac12)(m-\tfrac12)}\,|s_2\,m-\tfrac12\rangle\Big)$$

$$+\left(-\tfrac{i\hbar}{2}\right)|\tfrac12\,\tfrac12\rangle\tfrac{\hbar}{2i}\Big(\sqrt{s_2(s_2+1)-(m+\tfrac12)(m+\tfrac32)}\,|s_2\,m+\tfrac32\rangle - \sqrt{s_2(s_2+1)-(m+\tfrac12)(m-\tfrac12)}\,|s_2\,m-\tfrac12\rangle\Big)$$

$$+\left(-\tfrac{\hbar}{2}\right)|\tfrac12\,-\tfrac12\rangle\hbar(m+\tfrac12)|s_2\,m+\tfrac12\rangle\Big]\Big\}$$

$$= \hbar^2\Big\{A\Big[\tfrac34 + s_2(s_2+1) + m - \tfrac12\Big] + B\sqrt{s_2(s_2+1)-m^2+\tfrac14}\Big\}|\tfrac12\,\tfrac12\rangle|s_2\,m-\tfrac12\rangle$$

$$+\hbar^2\Big\{B\Big[\tfrac34 + s_2(s_2+1) - m - \tfrac12\Big] + A\sqrt{s_2(s_2+1)-m^2+\tfrac14}\Big\}|\tfrac12\,-\tfrac12\rangle|s_2\,m+\tfrac12\rangle = \hbar^2 s(s+1)|s\,m\rangle$$

$$= \hbar^2 s(s+1)\Big[A|\tfrac12\,\tfrac12\rangle|s_2\,m-\tfrac12\rangle + B|\tfrac12\,-\tfrac12\rangle|s_2\,m+\tfrac12\rangle\Big].$$

$$\therefore \begin{cases} A\big[s_2(s_2+1)+\tfrac14+m\big] + B\sqrt{s_2(s_2+1)-m^2+\tfrac14} = s(s+1)A \\ B\big[s_2(s_2+1)+\tfrac14-m\big] + A\sqrt{s_2(s_2+1)-m^2+\tfrac14} = s(s+1)B\,. \end{cases}$$

$$\text{or:}\left\{\begin{aligned} A\big[s_2(s_2+1)-s(s+1)+\tfrac14+m\big] + B\sqrt{s_2(s_2+1)-m^2+\tfrac14} &= 0 \\ B\big[s_2(s_2+1)-s(s+1)+\tfrac14-m\big] + A\sqrt{s_2(s_2+1)-m^2+\tfrac14} &= 0 \end{aligned}\right\} \text{or:}\left\{\begin{aligned} A(a+m)+Bb&=0 \\ B(a-m)+Ab&=0 \end{aligned}\right\},$$

where $a \equiv s_2(s_2+1)-s(s+1)+\frac14$, $b \equiv \sqrt{s_2(s_2+1)-m^2+\frac14}$. Multiply by $(a-b)$ and b — then subtract:

$A(a^2-m^2)+Bb(a-m)=0$; $Bb(a-m)+Ab^2=0 \Rightarrow A(a^2-m^2-b^2)=0 \Rightarrow a^2-b^2=m^2$, or:

$$\big[s_2(s_2+1)-s(s+1)+\tfrac14\big]^2 - s_2(s_2+1)+m^2-\tfrac14 = m^2 \Rightarrow \big[s_2(s_2+1)-s(s+1)+\tfrac14\big]^2 = s_2^2+s_2+\tfrac14 = (s_2+\tfrac12)^2$$

$\therefore\ s_2(s_2+1)-s(s+1)+\frac14 = \pm(s_2+\frac12)$; $\quad s(s+1) = s_2(s_2+1)\mp(s_2+\frac12)+\frac14$. Add $\frac14$ to both sides:

$$s^2+s+\tfrac14 = (s+\tfrac12)^2 = s_2(s_2+1)\mp(s_2+\tfrac12)+\tfrac12 = \begin{cases} s_2^2+s_2-s_2-\tfrac12+\tfrac12 = s_2^2 \\ s_2^2+s_2+s_2+\tfrac12+\tfrac12 = (s_2+1)^2 \end{cases}$$

So $\begin{cases} s+\frac12 = \pm s_2 \Rightarrow s = \pm s_2 - \frac12 = \begin{cases} s_2-\frac12 \\ -s_2-\frac12 \end{cases} \\ s+\frac12 = \pm(s_2+1) \Rightarrow s = \pm(s_2+1)-\frac12 = \begin{cases} s_2+\frac12 \\ -s_2-\frac32 \end{cases} \end{cases}$. But $s \geq 0$, so the possibilities are $\boxed{s = s_2 \pm 1/2}$

Then: $a = s_2^2+s_2-(s_2\pm\frac12)(s_2\pm\frac12+1)+\frac14 = s_2^2+s_2-s_2^2\mp\frac12 s_2 - s_2 \mp \frac12 s_2 - \frac14 \mp \frac12 + \frac14 = \mp s_2 \mp \frac12$

$= \mp(s_2+\frac12)$. $\quad b = \sqrt{(s_2^2+s_2+\frac14)-m^2} = \sqrt{(s_2+\frac12)^2-m^2} = \sqrt{(s_2+\frac12+m)(s_2+\frac12-m)}$.

$$\therefore\ A\big(\mp(s_2+\tfrac12)+m\big) = \mp A(s_2+\tfrac12\mp m) = -Bb = -B\sqrt{(s_2+\tfrac12+m)(s_2+\tfrac12-m)} \Rightarrow$$

$$A\sqrt{s_2+\tfrac12\mp m} = \pm B\sqrt{s_2+\tfrac12\pm m}.$$

But $|A|^2+|B|^2=1 \Rightarrow |A|^2+|A|^2\left(\frac{s_2+\frac{1}{2}\mp m}{s_2+\frac{1}{2}\pm m}\right)=\frac{|A|^2}{(s_2+\frac{1}{2}\pm m)}\left[s_2+\frac{1}{2}\pm m+s_2+\frac{1}{2}\mp m\right]=\frac{(2s_2+1)}{(s_2+\frac{1}{2}\pm m)}|A|^2$

$$\boxed{A=\sqrt{\frac{s_2\pm m+\frac{1}{2}}{2s_2+1}}}.\quad B=\pm A\frac{\sqrt{s_2+\frac{1}{2}\mp m}}{\sqrt{s_2+\frac{1}{2}\pm m}}=\boxed{\pm\sqrt{\frac{s_2\mp m+\frac{1}{2}}{2s_2+1}}}.$$

If $s_2=1$, $A=\frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}\pm m}=\sqrt{\frac{1}{2}\pm\frac{m}{3}}$; $B=\pm\frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}\mp m}=\pm\sqrt{\frac{1}{2}\mp\frac{m}{3}}$. $s=1\pm\frac{1}{2}$.

Plus sign:
$$\begin{cases}|\frac{3}{2}\,\frac{3}{2}\rangle=|\frac{1}{2}\,\frac{1}{2}\rangle|1\,1\rangle & (m=3/2)\\ |\frac{3}{2}\,\frac{1}{2}\rangle=\sqrt{\frac{2}{3}}|\frac{1}{2}\,\frac{1}{2}\rangle|1\,0\rangle+\sqrt{\frac{1}{3}}|\frac{1}{2}\,-\frac{1}{2}\rangle|1\,1\rangle & (m=1/2)\\ |\frac{3}{2}\,-\frac{1}{2}\rangle=\sqrt{\frac{1}{3}}|\frac{1}{2}\,\frac{1}{2}\rangle|1\,-1\rangle+\sqrt{\frac{2}{3}}|\frac{1}{2}\,-\frac{1}{2}\rangle|1\,0\rangle & (m=-1/2)\\ |\frac{3}{2}\,-\frac{3}{2}\rangle=|\frac{1}{2}\,-\frac{1}{2}\rangle|1\,-1\rangle & (m=-3/2)\end{cases}$$

Lower sign:
$$\begin{cases}|\frac{1}{2}\,\frac{1}{2}\rangle=\sqrt{\frac{1}{3}}|\frac{1}{2}\,\frac{1}{2}\rangle|1\,0\rangle-\sqrt{\frac{2}{3}}|\frac{1}{2}\,-\frac{1}{2}\rangle|1\,1\rangle & (m=1/2)\\ |\frac{1}{2}\,-\frac{1}{2}\rangle=\sqrt{\frac{2}{3}}|\frac{1}{2}\,\frac{1}{2}\rangle|1\,-1\rangle-\sqrt{\frac{1}{3}}|\frac{1}{2}\,-\frac{1}{2}\rangle|1\,0\rangle & (m=-1/2)\end{cases}$$

(Comparing the Clebsch-Gordan table I see that the conventional signs are reversed for the $s=1/2$ states.)

Problem 4.46 $|\frac{3}{2}\,\frac{3}{2}\rangle=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$; $|\frac{3}{2}\,\frac{1}{2}\rangle=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$; $|\frac{3}{2}\,-\frac{1}{2}\rangle=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$; $|\frac{3}{2}\,-\frac{3}{2}\rangle=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$.

$$[4.136]\Rightarrow\begin{cases}S_+|\frac{3}{2}\,\frac{3}{2}\rangle=0,\ S_+|\frac{3}{2}\,\frac{1}{2}\rangle=\sqrt{3}\hbar|\frac{3}{2}\,\frac{3}{2}\rangle,\ S_+|\frac{3}{2}\,-\frac{1}{2}\rangle=2\hbar|\frac{3}{2}\,\frac{1}{2}\rangle,\ S_+|\frac{3}{2}\,-\frac{3}{2}\rangle=\sqrt{3}\hbar|\frac{3}{2}\,-\frac{1}{2}\rangle.\\ S_-|\frac{3}{2}\,\frac{3}{2}\rangle=\sqrt{3}\hbar|\frac{3}{2}\,\frac{1}{2}\rangle,\ S_-|\frac{3}{2}\,\frac{1}{2}\rangle=2\hbar|\frac{3}{2}\,-\frac{1}{2}\rangle,\ S_-|\frac{3}{2}\,-\frac{1}{2}\rangle=\sqrt{3}\hbar|\frac{3}{2}\,-\frac{3}{2}\rangle,\ S_-|\frac{3}{2}\,-\frac{3}{2}\rangle=0.\end{cases}$$

So: $S_+=\hbar\begin{pmatrix}0&\sqrt{3}&0&0\\0&0&2&0\\0&0&0&\sqrt{3}\\0&0&0&0\end{pmatrix}$; $S_-=\hbar\begin{pmatrix}0&0&0&0\\\sqrt{3}&0&0&0\\0&2&0&0\\0&0&\sqrt{3}&0\end{pmatrix}$; $S_x=\frac{1}{2}(S_++S_-)=\boxed{\frac{\hbar}{2}\begin{pmatrix}0&\sqrt{3}&0&0\\\sqrt{3}&0&2&0\\0&2&0&\sqrt{3}\\0&0&\sqrt{3}&0\end{pmatrix}}$

$$\det\begin{pmatrix}-\lambda&\sqrt{3}&0&0\\\sqrt{3}&-\lambda&2&0\\0&2&-\lambda&\sqrt{3}\\0&0&\sqrt{3}&-\lambda\end{pmatrix}=-\lambda\begin{vmatrix}-\lambda&2&0\\2&-\lambda&\sqrt{3}\\0&\sqrt{3}&-\lambda\end{vmatrix}-\sqrt{3}\begin{vmatrix}\sqrt{3}&2&0\\0&-\lambda&\sqrt{3}\\0&\sqrt{3}&-\lambda\end{vmatrix}$$

$$=-\lambda\left[-\lambda^3+3\lambda+4\lambda\right]-\sqrt{3}\left[\sqrt{3}\lambda^2-3\sqrt{3}\right]=\lambda^4-7\lambda^2-3\lambda^2+9=0,\text{ or }\lambda^4-10\lambda^2+9=0;$$

$(\lambda^2-9)(\lambda^2-1)=0$; $\lambda=\pm3,\pm1$. So the eigenvalues of S_x are $\boxed{\frac{3}{2}\hbar,\ \frac{1}{2}\hbar,\ -\frac{1}{2}\hbar,\ -\frac{3}{2}\hbar}$.

Problem 4.47 $L_+Y_l^m=\hbar\sqrt{l(l+1)-m(m+1)}\,Y_l^{m+1}$ [4.120, 4.121].

$$\hbar e^{i\phi}\left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)B_l^m e^{im\phi}P_l^m(\cos\theta)=\hbar\sqrt{l(l+1)-m(m+1)}\,B_l^{m+1}e^{i(m+1)\phi}P_l^{m+1}(\cos\theta)\qquad[4.130]$$

$$B_l^m\left(\frac{d}{d\theta}-m\cot\theta\right)P_l^m(\cos\theta)=\sqrt{l(l+1)-m(m+1)}\,B_l^{m+1}P_l^{m+1}(\cos\theta)$$

Let $x \equiv \cos\theta$; $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{x}{\sqrt{1-x^2}}$; $\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = -\sin\theta\frac{d}{dx} = -\sqrt{1-x^2}\frac{d}{dx}$.

$$B_\ell^m\left[-\sqrt{1-x^2}\frac{d}{dx} - m\frac{x}{\sqrt{1-x^2}}\right]P_\ell^m(x) = -B_\ell^m\frac{1}{\sqrt{1-x^2}}\underbrace{\left[(1-x^2)\frac{dP_\ell^m}{dx} + mxP_\ell^m\right]}_{\sqrt{1-x^2}\,P_\ell^{m+1}\ [4.196]} = \sqrt{\ell(\ell+1)-m(m+1)}\,B_\ell^{m+1}P_\ell^{m+1}(x).$$

$\therefore \boxed{B_\ell^{m+1} = \frac{-1}{\sqrt{\ell(\ell+1)-m(m+1)}}B_\ell^m}$. Now $\ell(\ell+1)-m(m+1) = (\ell-m)(\ell+m+1)$, so

$$B_\ell^{m+1} = \frac{-1}{\sqrt{\ell-m}\sqrt{\ell+1+m}}B_\ell^m. \quad \therefore\ B_\ell^1 = \frac{-1}{\sqrt{\ell}\sqrt{\ell+1}}B_\ell^0;\ B_\ell^2 = \frac{-1}{\sqrt{\ell-1}\sqrt{\ell+2}}B_\ell^1 = \frac{1}{\sqrt{\ell(\ell-1)}\sqrt{(\ell+1)(\ell+2)}}B_\ell^0$$

$B_\ell^3 = \frac{-1}{\sqrt{\ell-2}\sqrt{\ell+3}}B_\ell^2 = \frac{-1}{\sqrt{(\ell+3)(\ell+2)(\ell+1)\ell(\ell-1)(\ell-2)}}B_\ell^0$, etc. Evidently there is an overall sign factor $(-1)^m$, and inside the square root the quantity is $(\ell+m)!/(\ell-m)!$ Thus: $\boxed{B_\ell^m = (-1)^m\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\,C(\ell)}$ (where $C(\ell) \equiv B_\ell^0$), for $m \geq 0$. For $m<0$, we have $B_\ell^{-1} = \frac{-B_\ell^0}{\sqrt{(\ell+1)\ell}}$; $B_\ell^{-2} = \frac{-1}{\sqrt{(\ell+2)(\ell-1)}}B_\ell^{-1} = \frac{1}{\sqrt{(\ell+2)(\ell+1)\ell(\ell-1)}}B_\ell^0$, etc.

Evidently $B_\ell^{-m} = B_\ell^m$, so in general: $B_\ell^m = (-1)^m\sqrt{\frac{(\ell-|m|)!}{(\ell+|m|)!}}\,C(\ell)$.

Now, Problem 4.22 says: $Y_\ell^\ell = \frac{1}{2^\ell \ell!}\sqrt{\frac{(2\ell+1)!}{\pi}}\left(e^{i\phi}\sin\theta\right)^\ell = B_\ell^\ell e^{i\ell\phi}P_\ell^\ell(\cos\theta)$.

But $P_\ell^\ell(x) = (1-x^2)^{\ell/2}\left(\frac{d}{dx}\right)^\ell\frac{1}{2^\ell\ell!}\left(\frac{d}{dx}\right)^\ell(x^2-1)^\ell = \frac{(1-x^2)^{\ell/2}}{2^\ell\ell!}\underbrace{\left(\frac{d}{dx}\right)^{2\ell}\left(x^{2\ell}-\dots\right)}_{(2\ell)!} = \frac{(2\ell)!}{2^\ell\ell!}(1-x^2)^{\ell/2}$, so

$$P_\ell^\ell(\cos\theta) = \frac{(2\ell)!}{2^\ell\ell!}(\sin\theta)^\ell.$$

So: $\frac{1}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{\pi}}\left(e^{i\phi}\sin\theta\right)^\ell = B_\ell^\ell e^{i\ell\phi}\frac{(2\ell)!}{2^\ell\ell!}(\sin\theta)^\ell \Rightarrow B_\ell^\ell = \frac{1}{(2\ell)!}\sqrt{\frac{(2\ell+1)!}{\pi}} = \sqrt{\frac{(2\ell+1)}{\pi(2\ell)!}}$.

But $B_\ell^\ell = (-1)^\ell\sqrt{\frac{1}{(2\ell)!}}\,C(\ell)$, so $\boxed{C(\ell) = (-1)^\ell\sqrt{\frac{2\ell+1}{\pi}}}$, and I conclude:

$\boxed{B_\ell^m = (-1)^{\ell+m}\sqrt{\frac{(2\ell+1)}{\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}}$. (This agrees with [4.32] except for the overall sign, which of course is purely conventional.)

Problem 4.48 (a) For both terms, $l=1$, so $\hbar^2(1)(2) = \boxed{2\hbar^2}$, $\boxed{P=1}$.

(b) $\boxed{0, P=1/3}$, or $\boxed{\hbar, P=2/3}$. (c) $\boxed{\frac{3}{4}\hbar^2, P=1}$. (d) $\boxed{\hbar/2, P=1/3}$, or $\boxed{-\hbar/2, P=2/3}$.

(e) From the $1\times\frac{1}{2}$ Clebsch-Gordan table (or Problem 4.45):

$$\frac{1}{\sqrt3}\left|\tfrac12\,\tfrac12\right\rangle|1\,0\rangle + \sqrt{\tfrac23}\left|\tfrac12\,-\tfrac12\right\rangle|1\,1\rangle = \frac{1}{\sqrt3}\left[\sqrt{\tfrac23}\left|\tfrac32\,\tfrac12\right\rangle - \frac{1}{\sqrt3}\left|\tfrac12\,\tfrac12\right\rangle\right] + \sqrt{\tfrac23}\left[\frac{1}{\sqrt3}\left|\tfrac32\,\tfrac12\right\rangle + \sqrt{\tfrac23}\left|\tfrac12\,\tfrac12\right\rangle\right]$$

$$= \left(\frac{2\sqrt2}{3}\right)\left|\tfrac32\,\tfrac12\right\rangle + \left(\frac13\right)\left|\tfrac12\,\tfrac12\right\rangle.$$ So $s = \frac32$ or $\frac12$.

$\boxed{\frac{15}{4}\hbar^2,\ P=\frac89}$, or $\boxed{\frac34\hbar^2,\ P=\frac19}$. (f) $\boxed{\frac12\hbar,\ P=1}$.

(g) $$|\Psi|^2 = |R_{21}|^2\left\{\frac13|Y_1^0|^2\underbrace{(\chi_+^\dagger\chi_+)}_{1} + \frac{\sqrt2}{3}\left(Y_1^{0*}Y_1^1\underbrace{(\chi_+^\dagger\chi_-)}_{0} + Y_1^{1*}Y_1^0\underbrace{(\chi_-^\dagger\chi_+)}_{0}\right) + \frac23|Y_1^1|^2\underbrace{(\chi_-^\dagger\chi_-)}_{1}\right\}$$

[Tables 4.6, 4.2] $$= \frac13|R_{21}|^2\left(|Y_1^0|^2 + 2|Y_1^1|^2\right) = \frac13\cdot\frac{1}{24}\cdot\frac{1}{a^3}\cdot\frac{r^2}{a^2}e^{-r/a}\left[\frac{3}{4\pi}\cos^2\theta + 2\cdot\frac{3}{8\pi}\sin^2\theta\right]$$

$$= \frac{1}{3\cdot24\cdot a^5}r^2e^{-r/a}\cdot\frac{3}{4\pi}(\cos^2\theta+\sin^2\theta) = \boxed{\frac{1}{96\pi a^5}r^2e^{-r/a}}.$$

(h) $$\frac13|R_{21}|^2\int|Y_1^0|^2\sin\theta\,d\theta\,d\phi = \frac13|R_{21}|^2 = \frac13\cdot\frac{1}{24a^3}r^2e^{-r/a} = \boxed{\frac{1}{72a^5}r^2e^{-r/a}}.$$

<u>Problem 4.49</u> (a) [4.129] says $L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$, so this problem is <u>identical</u> to Problem 3.55, with $\hat p \to L_z$ and $x \to \phi$.

(b) First note that if M is a matrix such that $M^2=1$, then

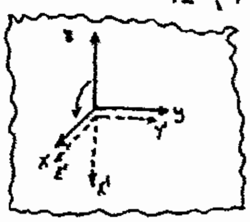
$$e^{iM\phi} = 1 + iM\phi + \tfrac12(iM\phi)^2 + \tfrac{1}{3!}(iM\phi)^3 + \cdots = 1 + i\phi M - \tfrac12\phi^2 - iM\tfrac{1}{3!}\phi^3 + \cdots$$
$$= \left(1 - \tfrac12\phi^2 + \tfrac{1}{4!}\phi^4 - \cdots\right) + iM\left(\phi - \tfrac{1}{3!}\phi^3 + \tfrac{1}{5!}\phi^5 - \cdots\right) = \cos\phi + iM\sin\phi.$$

So $R = e^{i\pi\sigma_x/2} = \cos\frac{\pi}{2} + i\sigma_x\sin\frac{\pi}{2}$ (because $\sigma_x^2 = 1$ — see Problem 4.27)

$= i\sigma_x = \boxed{i\begin{pmatrix}0&1\\1&0\end{pmatrix}}$. $\therefore\ R\chi_+ = i\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = i\begin{pmatrix}0\\1\end{pmatrix} = i\chi_-$. So it <u>does</u> convert spin up into spin down (with a factor of i).

(c) $R = e^{i\pi\sigma_y/4} = \cos\frac{\pi}{4} + i\sigma_y\sin\frac{\pi}{4} = \frac{1}{\sqrt2}(1+i\sigma_y) = \frac{1}{\sqrt2}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix} + i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right) = \boxed{\frac{1}{\sqrt2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}}$.

$R\chi_+ = \frac{1}{\sqrt2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt2}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt2}(\chi_+ - \chi_-) = \chi_-^{(x)}$ (eq. [4.151]). What <u>had</u> been spin <u>up</u> along z is now spin <u>down</u> along x' (see figure).



(d) $R = e^{i\pi\sigma_z} = \cos\pi + i\sigma_z\sin\pi = \boxed{-1}$ — rotation by 360° changes the <u>sign</u> of the spinor. But since the sign of χ is arbitrary, it doesn't <u>matter</u>.

(e) $(\vec\sigma\cdot\hat n)^2 = (\sigma_x n_x + \sigma_y n_y + \sigma_z n_z)(\sigma_x n_x + \sigma_y n_y + \sigma_z n_z) = \sigma_x^2 n_x^2 + \sigma_y^2 n_y^2 + \sigma_z^2 n_z^2 + n_x n_y(\sigma_x\sigma_y + \sigma_y\sigma_x) + n_x n_z(\sigma_x\sigma_z + \sigma_z\sigma_x) + n_y n_z(\sigma_y\sigma_z - \sigma_z\sigma_y)$.

But $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$, and $\sigma_x\sigma_y + \sigma_y\sigma_x = \sigma_x\sigma_z + \sigma_z\sigma_x = \sigma_y\sigma_z + \sigma_z\sigma_y = 0$ (Problem 4.27), so $(\vec\sigma\cdot\hat n)^2 = n_x^2 + n_y^2 + n_z^2 = 1$. So $e^{i(\vec\sigma\cdot\hat n)\varphi/2} = \cos(\varphi/2) + i(\vec\sigma\cdot\hat n)\sin(\varphi/2)$. QED.

Problem 4.50 (a) $[q_1,q_2]=\frac{1}{2}\left[x+\left(\frac{a^2}{\hbar}\right)p_y, x-\left(\frac{a^2}{\hbar}\right)p_y\right]=0$, because $[x,p_y]=[x,x]=[p_y,p_y]=0$.

$[p_1,p_2]=\frac{1}{2}\left[p_x-\left(\frac{\hbar}{a^2}\right)y, p_x+\left(\frac{\hbar}{a^2}\right)y\right]=0$, because $[y,p_x]=[y,y]=[p_x,p_x]=0$.

$[q_1,p_1]=\frac{1}{2}\left[x+\left(\frac{a^2}{\hbar}\right)p_y, p_x-\left(\frac{\hbar}{a^2}\right)y\right]=\frac{1}{2}\left\{[x,p_x]-[p_y,y]\right\}=\frac{1}{2}(i\hbar-(-i\hbar))=i\hbar$.

$[q_2,p_2]=\frac{1}{2}\left[x-\left(\frac{a^2}{\hbar}\right)p_y, p_x+\left(\frac{\hbar}{a^2}\right)y\right]=\frac{1}{2}\left\{[x,p_x]-[p_y,y]\right\}=i\hbar$.

[See eq. [4.10] for the canonical commutators.]

(b) $q_1^2-q_2^2=\frac{1}{2}\left\{x^2+\frac{a^2}{\hbar}(xp_y+p_yx)+\left(\frac{a^2}{\hbar}\right)^2p_y^2-x^2+\frac{a^2}{\hbar}(xp_y+p_yx)-\left(\frac{a^2}{\hbar}\right)^2p_y^2\right\}=\frac{2a^2}{\hbar}xp_y$.

$p_1^2-p_2^2=\frac{1}{2}\left\{p_x^2-\frac{\hbar}{a^2}(p_xy+yp_x)+\left(\frac{\hbar}{a^2}\right)^2y^2-p_x^2-\frac{\hbar}{a^2}(p_xy+yp_x)-\left(\frac{\hbar}{a^2}\right)^2y^2\right\}=-\frac{2\hbar}{a^2}yp_x$.

So $\frac{\hbar}{2a^2}(q_1^2-q_2^2)+\frac{a^2}{2\hbar}(p_1^2-p_2^2)=xp_y-yp_x=L_z$.

(c) $H=\frac{1}{2m}p^2+\frac{1}{2}m\omega^2x^2=\frac{a^2}{2\hbar}p^2+\frac{\hbar}{2a^2}x^2= \quad H(x,p)$.

Then $H(q_1,p_1)=\frac{a^2}{2\hbar}p_1^2+\frac{\hbar}{2a^2}q_1^2\equiv H_1$, $H(q_2,p_2)=\frac{a^2}{2\hbar}p_2^2+\frac{\hbar}{2a^2}q_2^2\equiv H_2$.

$\therefore L_z=H_1-H_2$.

(d) The eigenvalues of H_1 are $(n_1+\frac{1}{2})\hbar$, and those of H_2 are $(n_2+\frac{1}{2})\hbar$, so the eigenvalues of L_z are $(n_1+\frac{1}{2})\hbar-(n_2+\frac{1}{2})\hbar=(n_1-n_2)\hbar=m\hbar$, and m is an integer, because n_1 and n_2 are.

Problem 4.51 (a) $[3.148]\Rightarrow\frac{d\langle\vec{r}\rangle}{dt}=\frac{i}{\hbar}\langle[H,\vec{r}]\rangle$.

$H=\frac{1}{2m}(\vec{p}-q\vec{A})\cdot(\vec{p}-q\vec{A})+q\varphi=\frac{1}{2m}\left[p^2-q(\vec{p}\cdot\vec{A}+\vec{A}\cdot\vec{p})+q^2A^2\right]+q\varphi$.

$[H,x]=\frac{1}{2m}[p^2,x]-\frac{q}{2m}\left[(\vec{p}\cdot\vec{A}+\vec{A}\cdot\vec{p}),x\right]$.

$[p^2,x]=[(p_x^2+p_y^2+p_z^2),x]=[p_x^2,x]=p_x[p_x,x]+[p_x,x]p_x=p_x(-i\hbar)+(-i\hbar)p_x=-2i\hbar p_x$.

$[\vec{p}\cdot\vec{A},x]=[(p_xA_x+p_yA_y+p_zA_z),x]=[p_xA_x,x]=p_x[A_x,x]+[p_x,x]A_x=0+(-i\hbar)A_x=-i\hbar A_x$.

$[\vec{A}\cdot\vec{p},x]=[(A_xp_x+A_yp_y+A_zp_z),x]=[A_xp_x,x]=A_x[p_x,x]+[A_x,x]p_x=A_x(-i\hbar)+0=-i\hbar A_x$.

$[H,x]=\frac{1}{2m}(-2i\hbar p_x)-\frac{q}{2m}(-2i\hbar A_x)=-\frac{i\hbar}{m}(p_x-qA_x)$; $\therefore [H,\vec{r}]=-\frac{i\hbar}{m}(\vec{p}-q\vec{A})$.

$$\frac{d\langle\vec{r}\rangle}{dt}=\frac{1}{m}\langle(\vec{p}-q\vec{A})\rangle.\quad \text{QED.}$$

(b) We define the operator $\vec{v}\equiv\frac{1}{m}(\vec{p}-q\vec{A})$; $\frac{d\langle\vec{v}\rangle}{dt}=\frac{i}{\hbar}\langle[H,\vec{v}]\rangle+\langle\frac{\partial\vec{v}}{\partial t}\rangle$. $\frac{\partial\vec{v}}{\partial t}=-\frac{q}{m}\frac{\partial\vec{A}}{\partial t}$.

$H=\frac{1}{2}mv^2+q\varphi\Rightarrow[H,\vec{v}]=\frac{m}{2}[v^2,\vec{v}]+q[\varphi,\vec{v}]$. $[\varphi,\vec{v}]=\frac{1}{m}[\varphi,\vec{p}]$.

$[\varphi,p_x]=i\hbar\frac{\partial\varphi}{\partial x}$ (Problem 3.41), so $[\varphi,\vec{p}]=i\hbar\vec{\nabla}\varphi$, and $[\varphi,\vec{v}]=\frac{i\hbar}{m}\vec{\nabla}\varphi$.

$[v^2,v_x]=[(v_x^2+v_y^2+v_z^2),v_x]=[v_y^2,v_x]+[v_z^2,v_x]=v_y[v_y,v_x]+[v_y,v_x]v_y+v_z[v_z,v_x]+[v_z,v_x]v_z$.

$$[v_y, v_x] = \frac{1}{m^2}\left[(p_y - qA_y),(p_x - qA_x)\right] = -\frac{q}{m^2}\left\{[A_y,p_x]+[p_y,A_x]\right\} = -\frac{q}{m^2}\left\{i\hbar\frac{\partial A_y}{\partial x} - i\hbar\frac{\partial A_x}{\partial y}\right\}$$

$$= -\frac{i\hbar q}{m^2}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right) = -\frac{i\hbar q}{m^2}(\vec\nabla\times\vec A)_z = -\frac{i\hbar q}{m^2}B_z .$$

$$[v_z, v_x] = \frac{1}{m^2}\left[(p_z - qA_z),(p_x - qA_x)\right] = -\frac{q}{m^2}\left\{[A_z,p_x]+[p_z,A_x]\right\} = -\frac{q}{m^2}\left\{i\hbar\frac{\partial A_z}{\partial x} - i\hbar\frac{\partial A_x}{\partial z}\right\}$$

$$= -\frac{i\hbar q}{m^2}\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right) = \frac{i\hbar q}{m^2}(\vec\nabla\times\vec A)_y = \frac{i\hbar q}{m^2}B_y .$$

$$\therefore [v^2, v_x] = \frac{i\hbar q}{m^2}\left\{-v_yB_z - B_zv_y + v_zB_y + B_yv_z\right\} = \frac{i\hbar q}{m^2}\left[-(\vec v\times\vec B)_x + (\vec B\times\vec v)_x\right]$$

$\therefore [v^2,\vec v] = \frac{i\hbar q}{m^2}\left\{(\vec B\times\vec v) - (\vec v\times\vec B)\right\}$. Putting all this together:

$$\frac{d\langle\vec v\rangle}{dt} = \frac{i}{\hbar}\left\langle\left\{\frac{m}{2}\frac{i\hbar q}{m^2}\left[\vec B\times\vec v - \vec v\times\vec B\right] + q\frac{i\hbar}{m}\vec\nabla\varphi\right\}\right\rangle - \frac{q}{m}\left\langle\frac{\partial\vec A}{\partial t}\right\rangle$$

✗ $$m\frac{d\langle\vec v\rangle}{dt} = \frac{q}{2}\langle(\vec v\times\vec B)-(\vec B\times\vec v)\rangle + q\left\langle-\vec\nabla\varphi - \frac{\partial\vec A}{\partial t}\right\rangle = \frac{q}{2}\langle(\vec v\times\vec B - \vec B\times\vec v)\rangle + q\langle\vec E\rangle .$$

Or, since $\vec v\times\vec B - \vec B\times\vec v = \frac{1}{m}\left[(\vec p - q\vec A)\times\vec B - \vec B\times(\vec p - q\vec A)\right] = \frac{1}{m}\left[\vec p\times\vec B - \vec B\times\vec p\right] - \frac{q}{m}\left[\vec A\times\vec B - \vec B\times\vec A\right]$.

[Note: $\vec p$ does not commute with $\vec B$, so the order matters in the first term. But $\vec A$ does commute with $\vec B$, so $\vec B\times\vec A = -\vec A\times\vec B$ in the second.]

$$m\frac{d\langle\vec v\rangle}{dt} = q\langle\vec E\rangle + \frac{q}{2m}\langle\vec p\times\vec B - \vec B\times\vec p\rangle - \frac{q^2}{m}\langle\vec A\times\vec B\rangle . \quad \text{QED.}$$

(c) Go back to eq. ✗, and use $\langle\vec E\rangle = \vec E$, $\langle\vec v\times\vec B\rangle = \langle\vec v\rangle\times\vec B$; $\langle\vec B\times\vec v\rangle = \vec B\times\langle\vec v\rangle = -\langle\vec v\rangle\times\vec B$.

Then $m\frac{d\langle\vec v\rangle}{dt} = q\langle\vec v\rangle\times\vec B + q\vec E$. QED

Problem 4.52 (a) $\vec E = -\vec\nabla\varphi = \boxed{-2Kz\,\hat k}$. $\vec B = \vec\nabla\times\vec A = \begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ -\frac{B_0 y}{2} & \frac{B_0 x}{2} & 0\end{vmatrix} = \boxed{B_0\,\hat k}$.

(b) Time-independent version of [4.202]:

$$\frac{1}{2m}\left(\frac{\hbar}{i}\vec\nabla - q\vec A\right)\cdot\left(\frac{\hbar}{i}\vec\nabla - q\vec A\right)\psi + q\varphi\psi = E\psi \Rightarrow$$

$-\frac{\hbar^2}{2m}\nabla^2\psi + \frac{i\hbar q}{2m}\left[\vec\nabla\cdot(\vec A\psi) + \vec A\cdot(\vec\nabla\psi)\right] + \frac{q^2}{2m}A^2 + q\varphi\psi = E\psi$. But $\vec\nabla\cdot(\vec A\psi) = (\vec\nabla\cdot\vec A)\psi + \vec A\cdot(\vec\nabla\psi)$, so

$$\boxed{-\frac{\hbar^2}{2m}\nabla^2\psi + \frac{i\hbar q}{m}\left[\vec A\cdot(\vec\nabla\psi) + \frac{1}{2}(\vec\nabla\cdot\vec A)\psi\right] + \left(\frac{q^2}{2m}A^2 + q\varphi\right)\psi = E\psi}$$ (OK when A, φ are independent of t).

In the present case, $\vec\nabla\cdot\vec A = 0$; $\vec A\cdot(\vec\nabla\psi) = \frac{B_0}{2}\left(-y\frac{\partial\psi}{\partial x} + x\frac{\partial\psi}{\partial y}\right)$; $A^2 = \frac{B_0^2}{4}(x^2+y^2)$; $\varphi = Kz^2$.

Now $L_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$, so $-\frac{\hbar^2}{2m}\nabla^2\psi - \frac{qB_0}{2m}L_z\psi + \left(\frac{q^2B_0^2}{8m}(x^2+y^2) + qKz^2\right)\psi = E\psi$.

Since L_z commutes with H, we may as well pick simultaneous eigenfunctions of both: $L_z\psi = \bar m\hbar\psi$, where $\bar{m} = 0, \pm 1, \pm 2, \dots$. Then $\left\{-\frac{\hbar^2}{2m}\nabla^2 + \frac{(qB_0)^2}{8m}(x^2+y^2) + qKz^2\right\}\psi = \left(E + \frac{qB_0\hbar}{2m}\bar{m}\right)\psi$.

Let $\underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{\omega_1 \equiv qB_0/2m, \ \omega_2 \equiv \sqrt{\frac{2Kq}{m}}}$, and go to cylindrical coordinates (r, ϕ, z):

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}\right] + \left[\frac{1}{2}m\omega_1^2(x^2+y^2) + \frac{1}{2}m\omega_2^2 z^2\right]\psi = (E + \hbar\omega_1\bar{m})\psi .$$

But $L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$ [4.129], so $\frac{\partial^2\psi}{\partial\phi^2} = -\frac{1}{\hbar^2}L_z^2\psi = -\frac{1}{\hbar^2}\bar{m}^2\hbar^2\psi = -\bar{m}^2\psi$. Use separation of variables:

$\psi(r,\phi,z) = R(r)\,\Phi(\phi)\,Z(z)$:

$$-\frac{\hbar^2}{2m}\left[\Phi Z\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{\bar{m}^2}{r^2}R\Phi Z + R\Phi\frac{d^2Z}{dz^2}\right] + \left(\frac{1}{2}m\omega_1^2 r^2 + \frac{1}{2}m\omega_2^2 z^2\right)R\Phi Z = (E + \bar{m}\hbar\omega_1)R\Phi Z.$$

Divide by $R\Phi Z$ and collect terms:

$$\left\{-\frac{\hbar^2}{2m}\left[\frac{1}{Rr}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{\bar{m}^2}{r^2}\right] + \frac{r^2}{2}m\omega_1^2\right\} + \left\{-\frac{\hbar^2}{2m}\frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{1}{2}m\omega_2^2 z^2\right\} = (E + \bar{m}\hbar\omega_1)$$

The first term depends only on r, the second only on z, so they're both constants — E_r and E_z.

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{\bar{m}^2}{r^2}R\right] + \frac{1}{2}m\omega_1^2 r^2 R = E_r R \ ; \quad -\frac{\hbar^2}{2m}\frac{d^2Z}{dz^2} + \frac{1}{2}m\omega_2^2 z^2 Z = E_z Z \ ; \quad \underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{E = E_r + E_z - \bar{m}\hbar\omega_1}.$$

These are 2-dimensional and one-dimensional harmonic oscillators, respectively — we can read off $\underline{E_z = (n_z + \frac{1}{2})\hbar\omega_2}$ $(n_z = 0, 1, 2, \dots)$ immediately. To get E_r, let $u(r) = r^{1/2}R$, and follow the method of page 129: $R = u/\sqrt{r} \Rightarrow \frac{dR}{dr} = \frac{u'}{\sqrt{r}} - \frac{1}{2}\frac{u}{r^{3/2}}$; $r\frac{dR}{dr} = \sqrt{r}\,u' - \frac{1}{2}\frac{u}{\sqrt{r}}$; $\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \sqrt{r}\,u'' + \frac{1}{4}\frac{u}{r^{3/2}}$

$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{du}\right) = \frac{u''}{\sqrt{r}} + \frac{1}{4}\frac{u}{r^{5/2}}$. $\therefore$ $-\frac{\hbar^2}{2m}\left[\frac{u''}{\sqrt{r}} + \frac{1}{4}\frac{u}{r^2}\frac{1}{\sqrt{r}} - \frac{\bar{m}^2}{r^2}\frac{u}{\sqrt{r}}\right] + \frac{1}{2}m\omega_1^2 r^2\frac{u}{\sqrt{r}} = E_r\frac{u}{\sqrt{r}}$; or

$-\frac{\hbar^2}{2m}\left[\frac{d^2u}{dr^2} + \left(\frac{1}{4} - \bar{m}^2\right)\frac{u}{r^2}\right] + \frac{1}{2}m\omega_1^2 r^2 u = E_r u$. Now, this is identical to the equation we encountered in Problem 4.40 (for the 3-D harmonic oscillator), only with $l(l+1) \to \bar{m}^2 - \frac{1}{4}$ — which is to say, $l^2 + l + \frac{1}{4} = \bar{m}^2$, or $(l + \frac{1}{2})^2 = \bar{m}^2$, or $l = |\bar{m}| - 1/2$. [Note: our present equation depends only on $\bar{m}^2$, and hence is the same for either sign; but the solution in 4.40 assumed $l + 1/2 \geq 0$ (else u is not normalizable) so we need $|\bar{m}|$ here.] The solution (from 4.40) is

$$E_r = (j_{max} + l + 3/2)\hbar\omega \longrightarrow (j_{max} + |\bar{m}| + 1)\hbar\omega_1 \text{, where } j_{max} = 0, 2, 4, 6, \dots$$

(This makes sense: the eigenvalues for a two-dimensional oscillator are $(n+1)\hbar\omega$, with $n = j_{max} + |\bar{m}|$.)

$$\therefore \ E = (j_{max} + |\bar{m}| + 1)\hbar\omega_1 + (n_z + \tfrac{1}{2})\hbar\omega_2 - \bar{m}\hbar\omega_1 = \boxed{(n_r + 1)\hbar\omega_1 + (n_z + \tfrac{1}{2})\hbar\omega_2}\text{, where } \omega_1 = qB_0/2m,$$

$\omega_2 = \sqrt{\frac{2Kq}{m}}$, $n_z = 0, 1, 2, 3, \dots$, and n_r is an $\underline{\text{even}}$ integer (if $\bar{m} \geq 0$, $n_r = j_{max}$; if $\bar{m} < 0$, $n_r = j_{max} - 2\bar{m}$):

$n_r = 0, 2, 4, 6, \dots$.

Problem 4.53 (a) $\vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times(\vec{\nabla}\lambda) = \vec{\nabla}\times\vec{A} = \vec{B}$. ($\vec{\nabla}\times\vec{\nabla}\lambda = 0$, by equality of cross-derivatives: $(\vec{\nabla}\times\vec{\nabla}\lambda)_x = \frac{\partial}{\partial y}\left(\frac{\partial\lambda}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\lambda}{\partial y}\right) = 0$, etc.)

$\vec{E}' = -\vec{\nabla}\varphi' - \frac{\partial\vec{A}'}{\partial t} = -\vec{\nabla}\varphi + \vec{\nabla}\left(\frac{\partial\Lambda}{\partial t}\right) - \frac{\partial\vec{A}}{\partial t} - \frac{\partial}{\partial t}(\vec{\nabla}\Lambda) = -\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t} = \vec{E}$. (Again: $\vec{\nabla}\left(\frac{\partial\Lambda}{\partial t}\right) = \frac{\partial}{\partial t}(\vec{\nabla}\Lambda)$ by equality of cross-derivatives.)

(b) $\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A} - q(\vec{\nabla}\Lambda)\right)e^{iq\Lambda/\hbar}\Psi = q(\vec{\nabla}\Lambda)e^{iq\Lambda/\hbar}\Psi + \frac{\hbar}{i}e^{iq\Lambda/\hbar}\vec{\nabla}\Psi - q\vec{A}e^{iq\Lambda/\hbar}\Psi - q(\vec{\nabla}\Lambda)e^{iq\Lambda/\hbar}\Psi$

$$= \frac{\hbar}{i}e^{iq\Lambda/\hbar}\vec{\nabla}\Psi - q\vec{A}e^{iq\Lambda/\hbar}\Psi.$$

$$\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A} - q(\vec{\nabla}\Lambda)\right)^2 e^{iq\Lambda/\hbar}\Psi = \left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A} - q(\vec{\nabla}\Lambda)\right)\left[\frac{\hbar}{i}e^{iq\Lambda/\hbar}\vec{\nabla}\Psi - q\vec{A}e^{iq\Lambda/\hbar}\Psi\right]$$

$$= -\hbar^2\left[\frac{iq}{\hbar}(\vec{\nabla}\Lambda\cdot\vec{\nabla}\Psi)e^{iq\Lambda/\hbar} + e^{iq\Lambda/\hbar}\nabla^2\Psi\right] - \frac{\hbar q}{i}(\vec{\nabla}\cdot\vec{A})e^{iq\Lambda/\hbar}\Psi - q^2(\vec{A}\cdot\vec{\nabla}\Lambda)e^{iq\Lambda/\hbar}\Psi$$

$$-\frac{q\hbar}{i}e^{iq\Lambda/\hbar}\vec{A}\cdot(\vec{\nabla}\Psi) - \frac{q\hbar}{i}e^{iq\Lambda/\hbar}(\vec{A}\cdot\vec{\nabla}\Psi) + q^2A^2e^{iq\Lambda/\hbar}\Psi - \frac{q\hbar}{i}e^{iq\Lambda/\hbar}(\vec{\nabla}\Lambda\cdot\vec{\nabla}\Psi)$$

$$+q^2(\vec{A}\cdot\vec{\nabla}\Lambda)e^{iq\Lambda/\hbar}\Psi$$

$$= e^{iq\Lambda/\hbar}\Big\{\left[-\hbar^2\nabla^2\Psi + i\hbar q(\vec{\nabla}\cdot\vec{A})\Psi + 2iq\hbar(\vec{A}\cdot\vec{\nabla}\Psi) + q^2A^2\Psi\right]$$

$$-iq\hbar(\vec{\nabla}\Lambda)(\vec{\nabla}\Psi) - q^2(\vec{A}\cdot\vec{\nabla}\Lambda)\Psi + iq\hbar(\vec{\nabla}\Lambda)\cdot(\vec{\nabla}\Psi) + q^2(\vec{A}\cdot\vec{\nabla}\Lambda)\Psi\Big\}$$

$$= e^{iq\Lambda/\hbar}\left[\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}\right)^2\Psi\right].$$

So: $\left[\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}'\right)^2 + q\varphi'\right]\Psi' = e^{iq\Lambda/\hbar}\left[\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}\right)^2 + q\varphi - q\frac{\partial\Lambda}{\partial t}\right]\Psi$

[using 4.202] $\quad = e^{iq\Lambda/\hbar}\left(i\hbar\frac{\partial\Psi}{\partial t} - q\frac{\partial\Lambda}{\partial t}\Psi\right) = i\hbar\frac{\partial}{\partial t}\left(e^{iq\Lambda/\hbar}\Psi\right) = i\hbar\frac{\partial\Psi'}{\partial t}$. QED

Chapter 5

Problem 5.1 (a) $(m_1+m_2)\vec R = m_1\vec r_1 + m_2\vec r_2 = m_1\vec r_1 + m_2(\vec r_1 - \vec r) = (m_1+m_2)\vec r_1 - m_2\vec r \Rightarrow$

$$\vec r_1 = \vec R + \frac{m_2}{m_1+m_2}\vec r = \vec R + \frac{\mu}{m_1}\vec r.$$

$(m_1+m_2)\vec R = m_1(\vec r_2+\vec r) + m_2\vec r_2 = (m_1+m_2)\vec r_2 + m_1\vec r \Rightarrow \vec r_2 = \vec R - \frac{m_1}{m_1+m_2}\vec r = \vec R - \frac{\mu}{m_2}\vec r.$

Let $\vec R = (X,Y,Z)$, $\vec r = (x,y,z)$. $(\nabla_1)_x = \frac{\partial}{\partial x_1} = \frac{\partial X}{\partial x_1}\frac{\partial}{\partial X} + \frac{\partial x}{\partial x_1}\frac{\partial}{\partial x} = \left(\frac{m_1}{m_1+m_2}\right)\frac{\partial}{\partial X} + (1)\frac{\partial}{\partial x} = \frac{\mu}{m_2}(\nabla_R)_x + (\nabla_r)_x$

$$\text{So } \vec\nabla_1 = \frac{\mu}{m_2}\vec\nabla_R + \vec\nabla_r$$

$$(\nabla_2)_x = \frac{\partial}{\partial x_2} = \frac{\partial X}{\partial x_2}\frac{\partial}{\partial X} + \frac{\partial x}{\partial x_2}\frac{\partial}{\partial x} = \left(\frac{m_2}{m_1+m_2}\right)\frac{\partial}{\partial X} - (1)\frac{\partial}{\partial x} = \frac{\mu}{m_1}(\nabla_R)_x - (\nabla_r)_x \Rightarrow \vec\nabla_2 = \frac{\mu}{m_1}\vec\nabla_R - \vec\nabla_r.$$

(b) $\nabla_1^2\psi = \vec\nabla_1\cdot(\vec\nabla_1\psi) = \vec\nabla_1\cdot\left[\frac{\mu}{m_2}\vec\nabla_R\psi + \vec\nabla_r\psi\right] = \frac{\mu}{m_2}\vec\nabla_R\cdot\left(\frac{\mu}{m_2}\vec\nabla_R\psi + \vec\nabla_r\psi\right) + \vec\nabla_r\cdot\left(\frac{\mu}{m_2}\vec\nabla_R\psi + \vec\nabla_r\psi\right)$

$= \left(\frac{\mu}{m_2}\right)^2\nabla_R^2\psi + 2\frac{\mu}{m_2}\left(\vec\nabla_r\cdot\vec\nabla_R\right)\psi + \nabla_r^2\psi$. Likewise $\nabla_2^2\psi = \left(\frac{\mu}{m_1}\right)^2\nabla_R^2\psi - 2\frac{\mu}{m_1}\left(\vec\nabla_r\cdot\vec\nabla_R\right)\psi + \nabla_r^2\psi.$

$\therefore H\psi = -\frac{\hbar^2}{2m_1}\nabla_1^2\psi - \frac{\hbar^2}{2m_2}\nabla_2^2\psi + V(\vec r_1,\vec r_2)\psi = -\frac{\hbar^2}{2}\left\{\frac{\mu^2}{m_1m_2^2}\nabla_R^2 + \frac{2\mu}{m_1m_2}\vec\nabla_r\cdot\vec\nabla_R + \frac{1}{m_1}\nabla_r^2 + \frac{\mu^2}{m_2m_1^2}\nabla_R^2 - \frac{2\mu}{m_2m_1}\vec\nabla_r\cdot\vec\nabla_R + \frac{1}{m_2}\nabla_r^2\right\}\psi$

$+V(\vec r)\psi = -\frac{\hbar^2}{2}\left[\frac{\mu^2}{m_1m_2}\left(\frac{1}{m_2}+\frac{1}{m_1}\right)\nabla_R^2 + \left(\frac{1}{m_1}+\frac{1}{m_2}\right)\nabla_r^2\right]\psi + V(\vec r)\psi = E\psi.$ But $\left(\frac{1}{m_1}+\frac{1}{m_2}\right) = \frac{m_1+m_2}{m_1m_2} = \frac{1}{\mu}$, so

$\frac{\mu^2}{m_1m_2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right) = \frac{\mu}{m_1m_2} = \frac{m_1m_2}{m_1m_2(m_1+m_2)} = \frac{1}{m_1+m_2}$. $\therefore$ $-\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\vec r)\psi = E\psi.$

(c) Put in $\psi = \psi_r(\vec r)\psi_R(\vec R)$, and divide by $\psi_r\psi_R$:

$\left[-\frac{\hbar^2}{2(m_1+m_2)}\frac{1}{\psi_R}\nabla_R^2\psi_R\right] + \left[-\frac{\hbar^2}{2\mu}\frac{1}{\psi_r}\nabla_r^2\psi_r + V(\vec r)\right] = E$. First term depends only on $\vec R$, second only on $\vec r$,

so each must be a constant — call them E_R and E_r, respectively. Then:

$$\boxed{-\frac{\hbar^2}{2(m_1+m_2)}\nabla^2\psi_R = E_R\psi_R}\;;\quad \boxed{-\frac{\hbar^2}{2\mu}\nabla^2\psi_r + V(\vec r)\psi_r = E_r\psi_r},\ \text{with } \boxed{E_R + E_r = E}.$$

Problem 5.2 (a) From [4.77], E_1 is proportional to mass, so $\frac{\Delta E_1}{E_1} = \frac{\Delta m}{\mu} = \frac{m-\mu}{\mu} = \frac{m(m+M)}{mM} - \frac{M}{M} = \frac{m}{M}$.

So the fractional error is the ratio of the electron mass to the proton mass: $\frac{9.109\times10^{-31}\,\text{kg}}{1.673\times10^{-27}\,\text{kg}} = 5.44\times10^{-4}$.

The percent error is $\boxed{0.054\%}$ (pretty small).

(b) From [4.94], R is proportional to m, so $\frac{\Delta(1/\lambda)}{(1/\lambda)} = \frac{\Delta R}{R} = \frac{\Delta\mu}{\mu} = \frac{-(1/\lambda^2)\Delta\lambda}{(1/\lambda)} = -\frac{\Delta\lambda}{\lambda}$.

So (in magnitude) $\frac{\Delta\lambda}{\lambda} = \frac{\Delta\mu}{\mu}$. $\mu = \frac{mM}{m+M}$, where m = electron mass, and M = nuclear mass.

$$\Delta\mu = \frac{m\,2m_p}{m+2m_p} - \frac{m\,m_p}{m+m_p} = \frac{m\,m_p}{(m+m_p)(m+2m_p)}\left[2m+2m_p-m-2m_p\right] = \frac{m^2m_p}{(m+m_p)(m+2m_p)} = \frac{m\mu}{m+2m_p}.$$

$\frac{\Delta\lambda}{\lambda} = \frac{\Delta\mu}{\mu} = \frac{m}{m+2m_p} \cong \frac{m}{2m_p}$, so $\boxed{\Delta\lambda = \frac{m}{2m_p}\lambda_h}$, where λ_h is the hydrogen wavelength:

$\frac{1}{\lambda} = R\left(\frac{1}{4}-\frac{1}{9}\right) = \frac{5}{36}R \Rightarrow \lambda = \frac{36}{5R} = \frac{36}{5(1.097\times10^7)}\,\text{m} = 6.563\times10^{-7}\,\text{m}$. $\Delta\lambda = \frac{9.109\times10^{-31}}{2(1.673\times10^{-27})}(6.563\times10^{-7})\,\text{m}$

$$= \boxed{1.79\times10^{-10}\,\text{m}}$$

(b) $\langle x\rangle_{mn} = \frac{2}{a}\int_0^a x\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right)dx = \frac{1}{a}\int_0^a x\left[\cos\left(\frac{(m-n)\pi}{a}x\right)-\cos\left(\frac{(m+n)\pi}{a}x\right)\right]dx$

$= \frac{1}{a}\left\{\left(\frac{a}{(m-n)\pi}\right)^2\cos\left(\frac{(m-n)\pi}{a}x\right)+\left(\frac{ax}{(m-n)\pi}\right)\sin\left(\frac{(m-n)\pi}{a}x\right)-\left(\frac{a}{(m+n)\pi}\right)^2\cos\left(\frac{(m+n)\pi}{a}x\right)-\left(\frac{ax}{(m+n)\pi}\right)\sin\left(\frac{(m+n)\pi}{a}x\right)\right\}\Big|_0^a$

$= \frac{1}{a}\left\{\left(\frac{a}{(m-n)\pi}\right)^2\left(\cos((m-n)\pi)-1\right)-\left(\frac{a}{(m+n)\pi}\right)^2\left(\cos((m+n)\pi)-1\right)\right\}$. But $\cos((m\pm n)\pi)=(-1)^{m+n}$.

$= \frac{1}{a}\frac{a^2}{\pi^2}\left((-1)^{m+n}-1\right)\left(\frac{1}{(m-n)^2}-\frac{1}{(m+n)^2}\right)=\begin{cases}\frac{a}{\pi^2}\frac{(-8mn)}{(m^2-n^2)^2}, & \text{if } m,n \text{ have opposite parity}\\ 0, & \text{if } m,n \text{ have same parity (i.e. both even or both odd).}\end{cases}$

So [5.21] $\Rightarrow$ $\langle(x_1-x_2)^2\rangle = \boxed{a^2\left[\frac{1}{6}-\frac{1}{2\pi^2}\left(\frac{1}{n^2}+\frac{1}{m^2}\right)\right]-\frac{128a^2m^2n^2}{\pi^4(m^2-n^2)^4}}$. (Last term present only when m,n have opposite parity).

(c) [5.21] $\Rightarrow$ $\langle(x_1-x_2)^2\rangle = \boxed{a^2\left[\frac{1}{6}-\frac{1}{2\pi^2}\left(\frac{1}{n^2}+\frac{1}{m^2}\right)\right]+\frac{128a^2m^2n^2}{\pi^4(m^2-n^2)^4}}$. (Last term present only when m,n have opposite parity.)

Problem 5.6 (a) $\boxed{\psi(x_1,x_2,x_3)=\psi_a(x_1)\psi_b(x_2)\psi_c(x_3)}$.

(b) $\psi(x_1,x_2,x_3)=\boxed{\frac{1}{\sqrt{6}}\left[\psi_a(x_1)\psi_b(x_2)\psi_c(x_3)-\psi_a(x_1)\psi_c(x_2)\psi_b(x_3)-\psi_b(x_1)\psi_a(x_2)\psi_c(x_3)+\psi_b(x_1)\psi_c(x_2)\psi_a(x_3)-\psi_c(x_1)\psi_b(x_2)\psi_a(x_3)+\psi_c(x_1)\psi_a(x_2)\psi_b(x_3)\right]}$

(c) $\psi(x_1,x_2,x_3)=\boxed{\frac{1}{\sqrt{6}}\left[\psi_a(x_1)\psi_b(x_2)\psi_c(x_3)-\psi_a(x_1)\psi_c(x_2)\psi_b(x_3)-\psi_b(x_1)\psi_a(x_2)\psi_c(x_3)+\psi_b(x_1)\psi_c(x_2)\psi_a(x_3)-\psi_c(x_1)\psi_b(x_2)\psi_a(x_3)+\psi_c(x_1)\psi_a(x_2)\psi_b(x_3)\right]}$

Problem 5.7 $\psi = A\left[\psi(\vec r_1,\vec r_2,\vec r_3,\dots,\vec r_Z)\pm\psi(\vec r_2,\vec r_1,\vec r_3,\dots,\vec r_Z)+\psi(\vec r_2,\vec r_3,\vec r_1,\dots,\vec r_Z)+\text{etc.}\right]$,

where "etc" runs over all permutations of the arguments $\vec r_1,\vec r_2,\dots,\vec r_Z$, with + sign for all even permutations (even number of transpositions $\vec r_i\leftrightarrow\vec r_j$, starting from $\vec r_1,\vec r_2,\dots,\vec r_Z$), and $\pm$ for all odd permutations (+ for bosons, − for fermions). At the end of the process, normalize the result to determine A. (Typically $A=1/\sqrt{Z!}$, but this may not be right if the starting function is already symmetric under some interchanges.)

Problem 5.8 (a) Energy of each electron: $E=Z^2E_1/n^2=4E_1/4=E_1=-13.6\,\text{eV}$, so the total initial energy is $2\times(-13.6)\,\text{eV}=-27.2\,\text{eV}$. One electron drops to the ground state $Z^2E_1/1=4E_1$, so the other is left with $2E_1-4E_1=-2E_1=\boxed{27.2\,\text{eV}}$.

(b) He^+ has one electron — it's a hydrogenic ion (Problem 4.17) with $Z=2$, so the spectrum is $\boxed{\frac{1}{\lambda}=4R\left(\frac{1}{n_f^2}-\frac{1}{n_i^2}\right)}$, where R is the Hydrogen Rydberg constant [4.94], and n_i, n_f are the initial and final quantum numbers (1,2,3,...).

(c) $\mu = \frac{mm}{m+m} = \frac{m}{2}$, so the energy is <u>half</u> what it would be for hydrogen: $\frac{13.6\text{eV}}{2} = \boxed{6.8\text{eV}}$.

(d) $\mu = \frac{m_p m_\mu}{m_p + m_\mu}$; $R \propto \mu$, so R is changed by a factor $\frac{m_p m_\mu}{m_p+m_\mu} \cdot \frac{m_p + m_e}{m_p m_e} = \frac{m_\mu (m_p+m_e)}{m_e(m_p+m_\mu)}$, as compared with hydrogen. For hydrogen, $\frac{1}{\lambda} = R\left(1-\frac{1}{4}\right) = \frac{3}{4}R \Rightarrow \lambda = \frac{4}{3R} = \frac{4}{3(1.097\times 10^7)}\text{m} = 1.215\times 10^{-7}\text{m}$, and $\lambda \propto \frac{1}{R}$, so for muonic hydrogen the Lyman-alpha line is at

$$\lambda = \frac{m_e(m_p+m_\mu)}{m_\mu(m_p+m_e)}(1.215\times 10^{-7}\text{m}) = \frac{1}{206.77}\frac{(1.673\times 10^{-27} + 206.77\times 9.109\times 10^{-31})}{(1.673\times 10^{-27} + 9.109\times 10^{-31})}(1.215\times 10^{-7}\text{m}) =$$

$$\boxed{6.54\times 10^{-10}\text{m}}$$

<u>PROBLEM 5.3</u> (a) $1 = \int |\psi_\pm|^2 d^3r_1 d^3r_2 = |A|^2 \int [\psi_a(r_1)\psi_b(r_2) \pm \psi_b(r_1)\psi_a(r_2)]^* [\psi_a(r_1)\psi_b(r_2) \pm \psi_b(r_1)\psi_a(r_2)] d^3r_1 d^3r_2$

$= |A|^2 \Big\{ \int |\psi_a(r_1)|^2 d^3r_1 \int |\psi_b(r_2)|^2 d^3r_2 \pm \int \psi_a(r_1)^* \psi_b(r_1) d^3r_1 \int \psi_b(r_2)^* \psi_a(r_2) d^3r_2 \pm \int \psi_b(r_1)^* \psi_a(r_1) d^3r_1 \int \psi_a(r_2)^* \psi_b(r_2) d^3r_2$

$+ \int |\psi_b(r_1)|^2 d^3r_1 \int |\psi_a(r_2)|^2 d^3r_2 \Big\} = |A|^2 (1\cdot 1 \pm 0\cdot 0 \pm 0\cdot 0 + 1\cdot 1) = 2|A|^2 \Rightarrow \boxed{A = \frac{1}{\sqrt{2}}}$.

(b) $1 = |A|^2 \int [2\psi_a(r_1)\psi_a(r_2)]^* [2\psi_a(r_1)\psi_a(r_2)] d^3r_1 d^3r_2 = 4|A|^2 \int |\psi_a(r_1)|^2 d^3r_1 \int |\psi_a(r_2)|^2 d^3r_2 = 4|A|^2$. $\boxed{A = \frac{1}{2}}$.

<u>PROBLEM 5.4</u> (a) $\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx_1^2} - \frac{\hbar^2}{2m}\frac{d^2\psi}{dx_2^2} = E\psi}$ (for $0 \le x_1, x_2 \le a$ — otherwise $\psi = 0$).

$$\psi = \frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) - \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right] \Rightarrow \frac{d^2\psi}{dx_1^2} = \frac{\sqrt{2}}{a}\left[-\left(\frac{\pi}{a}\right)^2 \sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) - \left(-\frac{2\pi}{a}\right)^2 \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right]$$

$$\frac{d^2\psi}{dx_2^2} = \frac{\sqrt{2}}{a}\left[-\left(\frac{2\pi}{a}\right)^2 \sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) - \left(-\frac{\pi}{a}\right)^2 \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right], \text{ so } \left(\frac{d^2\psi}{dx_1^2} + \frac{d^2\psi}{dx_2^2}\right) = -\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{2\pi}{a}\right)^2\right]\psi = -5\frac{\pi^2}{a^2}\psi.$$

$$\therefore -\frac{\hbar^2}{2m}\left(\frac{d^2\psi}{dx_1^2} + \frac{d^2\psi}{dx_2^2}\right) = \frac{5\hbar^2}{2m}\frac{\pi^2}{a^2}\psi = E\psi, \text{ with } E = \frac{5\pi^2\hbar^2}{2ma^2} = 5K. ✓$$

(b) <u>Distinguishable</u>: $\boxed{\psi_{22} = \frac{2}{a}\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right), \text{ with } E_{22} = 8K}$ (nondegenerate)

$\boxed{\left.\begin{aligned}\psi_{13} &= \frac{2}{a}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{3\pi x_2}{a}\right)\\ \psi_{31} &= \frac{2}{a}\sin\left(\frac{3\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\end{aligned}\right\}, \text{ with } E_{13} = E_{31} = 10K}$ (doubly degenerate)

<u>Identical Bosons</u>: $\boxed{\psi_{22} = \frac{2}{a}\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right), E_{22} = 8K}$ (nondegenerate)

$\boxed{\psi_{13} = \frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{3\pi x_2}{a}\right) + \sin\left(\frac{3\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right], E_{13} = 10K}$ (nondegenerate)

<u>Identical Fermions</u>: $\boxed{\psi_{13} = \frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{3\pi x_2}{a}\right) - \sin\left(\frac{3\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right], E_{13} = 10K}$ (nondegenerate)

$\boxed{\psi_{23} = \frac{\sqrt{2}}{a}\left[\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{3\pi x_2}{a}\right) - \sin\left(\frac{3\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right)\right], E_{23} = 13K}$ (nondegenerate)

<u>PROBLEM 5.5</u> (a) [5.19], with $\langle x \rangle_n = \frac{a}{2}$ (Problem 2.5) and $\langle x^2 \rangle_n = a^2\left(\frac{1}{3} - \frac{1}{2(n\pi)^2}\right)$ (Problem 2.5) $\Rightarrow$

$$\langle (x_1 - x_2)^2 \rangle = a^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right) + a^2\left(\frac{1}{3} - \frac{1}{2m^2\pi^2}\right) - 2\cdot\frac{a}{2}\cdot\frac{a}{2} = \boxed{a^2\left\{\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{m^2}\right)\right\}}.$$

PROBLEM 5.9 (a) The ground state [5.30] is spatially symmetric, so it goes with the symmetric (triplet) spin configuration. Thus the ground state is orthohelium, and it is triply degenerate. The excited states [5.32] come in ortho (triplet) and para (singlet) form; since the former go with the symmetric spatial wave function, the orthohelium states are higher in energy than the corresponding (nondegenerate) para states.

(b) Ground state [5.30] and all excited states [5.32] come in both ortho and para form. All are quadruply degenerate (or at any rate we have no way a priori of knowing whether ortho or para are higher in energy, since we don't know which goes with the symmetric spatial configuration).

PROBLEM 5.10 (a) $\left\langle \frac{1}{|\mathbf{r}_1-\mathbf{r}_2|}\right\rangle = \left(\frac{8}{\pi a^3}\right)^2 \int\left\{\int \frac{e^{-4(r_1+r_2)/a}}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta_2}}\, d^3\mathbf{r}_2\right\} d^3\mathbf{r}_1$

$$\{\} = 2\pi \int_0^\infty e^{-4(r_1+r_2)/a}\left\{\underbrace{\int_0^\pi \frac{\sin\theta_2}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta_2}}\,d\theta_2}\right\} r_2^2\,dr_2$$

$$\frac{1}{r_1r_2}\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta_2}\,\Big|_0^\pi = \frac{1}{r_1r_2}\left[\sqrt{r_1^2+r_2^2+2r_1r_2}-\sqrt{r_1^2+r_2^2-2r_1r_2}\right] = \frac{1}{r_1r_2}\left[(r_1+r_2)-|r_1-r_2|\right]$$

$$= \begin{cases} 2/r_1 & (r_2<r_1)\\ 2/r_2 & (r_2>r_1)\end{cases}$$

$$= 4\pi e^{-4r_1/a}\left\{\frac{1}{r_1}\int_0^{r_1} r_2^2 e^{-4r_2/a}\,dr_2 + \int_{r_1}^\infty r_2 e^{-4r_2/a}\,dr_2\right\}.$$

$$\frac{1}{r_1}\int_0^{r_1} r_2^2 e^{-4r_2/a}\,dr_2 = \frac{1}{r_1}\left[-\frac{a}{4}r_2^2 e^{-4r_2/a} + \frac{a}{2}\cdot\left(\frac{a}{4}\right)^2 e^{-4r_2/a}\left(-\frac{4r_2}{a}-1\right)\right]\Big|_0^{r_1}$$

$$= -\frac{a}{4r_1}\left[r_1^2 e^{-4r_1/a} + \frac{ar_1}{2}e^{-4r_1/a} + \frac{a^2}{8}e^{-4r_1/a} - \frac{a^2}{8}\right].$$

$$\int_{r_1}^\infty r_2 e^{-4r_2/a}\,dr_2 = \left(\frac{a}{4}\right)^2 e^{-4r_2/a}\left(-\frac{4r_2}{a}-1\right)\Big|_{r_1}^\infty = \frac{ar_1}{4}e^{-4r_1/a} + \frac{a^2}{16}e^{-4r_1/a}.$$

$$= 4\pi\left\{\frac{a^3}{32r_1}e^{-4r_1/a} + \left[-\frac{ar_1}{4}-\frac{a^2}{8}-\frac{a^3}{32r_1}+\frac{ar_1}{4}+\frac{a^2}{16}\right]e^{-8r_1/a}\right\}$$

$$= \frac{\pi a^2}{8}\left\{\frac{a}{r_1}e^{-4r_1/a} - \left(1+\frac{a}{r_1}\right)e^{-8r_1/a}\right\}.$$

$$\left\langle\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|}\right\rangle = \frac{8}{\pi a^4}\cdot 4\pi\int_0^\infty\left[\frac{a}{r_1}e^{-4r_1/a} - \left(1+\frac{a}{r_1}\right)e^{-8r_1/a}\right]r_1^2\,dr_1$$

$$= \frac{32}{a^4}\left\{a\int_0^\infty r_1 e^{-4r_1/a}\,dr_1 - \int_0^\infty r_1^2 e^{-8r_1/a}\,dr_1 - a\int_0^\infty r_1 e^{-8r_1/a}\,dr_1\right\}$$

$$= \frac{32}{a^4}\left\{a\cdot\left(\frac{a}{4}\right)^2 - 2\left(\frac{a}{8}\right)^3 - a\cdot\left(\frac{a}{8}\right)^2\right\} = \frac{32}{a}\left(\frac{1}{16}-\frac{1}{128}-\frac{1}{64}\right) = \frac{1}{a}\left(2-\frac{1}{4}-\frac{1}{2}\right)$$

$$= \boxed{5/4a}.$$

(b) $V_{ee} \equiv \frac{e^2}{4\pi\epsilon_0}\left\langle \frac{1}{|\mathbf{r}_1-\mathbf{r}_2|}\right\rangle = \boxed{\frac{5}{4}\frac{e^2}{4\pi\epsilon_0}\frac{1}{a}} = \frac{5}{4}\frac{m}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 = \frac{5}{2}(-E_1) = \frac{5}{2}(13.6\text{eV}) = \boxed{34\text{eV}}.$

$E_0 + V_{ee} = -109\text{eV} + 34\text{eV} = \boxed{-75\text{eV}}$, which is pretty close to the experimental value (-79eV).

PROBLEM 5.11 (a) hydrogen: $(1s)$; helium: $(1s)^2$; lithium: $(1s)^2$ $(2s)$; beryllium: $(1s)^2(2s)^2$; boron: $(1s)^2(2s)^2(2p)$; carbon: $(1s)^2(2s)^2(2p)^2$; nitrogen: $(1s)^2(2s)^2(2p)^3$; oxygen: $(1s)^2(2s)^2(2p)^4$; fluorine: $(1s)^2(2s)^2(2p)^5$; neon: $(1s)^2(2s)^2(2p)^6$. They agree with Table 5.1 — no surprises here.

(b) hydrogen: ${}^2S_{1/2}$, helium: ${}^1S_0$, lithium: ${}^2S_{1/2}$, beryllium: ${}^1S_0$. (Unambiguous, because the orbital angular momentum is zero in all four cases.) For boron, the spin (1/2) and orbital angular momenta (1) could add to give 3/2 or 1/2, so possibilities are $\boxed{{}^2P_{3/2} \text{ or } {}^2P_{1/2}}$. For carbon, the two p electrons could combine for orbital angular momentum 2, 1, or 0, and the spins could add to 1 or 0: $\boxed{{}^1S_0, {}^3S_1, {}^1P_1, {}^3P_2, {}^3P_1, {}^3P_0, {}^1D_2, {}^3D_3, {}^3D_2, {}^3D_1}$. For nitrogen, the 3 p electrons can add to orbital angular momentum 3, 2, 1, or 0, and the spins to 3/2 or 1/2:

$$\boxed{\begin{array}{l} {}^2S_{1/2}, {}^4S_{3/2}, {}^2P_{1/2}, {}^2P_{3/2}, {}^4P_{1/2}, {}^4P_{3/2}, {}^4P_{5/2}, {}^2D_{3/2}, {}^2D_{5/2}, {}^4D_{1/2}, {}^4D_{3/2}, {}^4D_{5/2}, {}^4D_{7/2}, {}^2F_{5/2}, {}^2F_{7/2}, \\ {}^4F_{3/2}, {}^4F_{5/2}, {}^4F_{7/2}, {}^4F_{9/2}. \end{array}}$$

(c) Orthohelium should have lower energy than parahelium, for corresponding states. (Which is true.)

(d) For boron, $L=1$ and $S=1/2$, so $J=|L-S|=1/2$, and it's the $\boxed{{}^2P_{1/2}}$ configuration.

(e) For carbon, the 2p subshell is less than half filled, so $J=|L-S|$. This leaves ${}^1S_0, {}^3S_1, {}^1P_1, {}^3P_0, {}^1D_2, {}^3D_1$ as possibilities. Hund's first rule says ${}^3S_1$, ${}^3P_0$, or ${}^3D_1$. But each of these is the (symmetric) triplet spin state, and hence the spatial state must be antisymmetric. Now, the spherical harmonics carry parity $(-1)^\ell$, and parity $(x,y,z \to -x,-y,-z)$ corresponds (for the two-particle system) to symmetry under interchange, so S and D are symmetric, P is antisymmetric. $\therefore$ $\boxed{{}^3P_0}$ is the answer.

For nitrogen, the 2p subshell is half filled, so $J=|L-S|$. This leaves ${}^2S_{1/2}, {}^4S_{3/2}, {}^2P_{1/2}, {}^4P_{1/2}, {}^4D_{1/2}, {}^2D_{3/2}, {}^2F_{5/2}, {}^4F_{3/2}$. The first rule narrows this down to ${}^4S_{3/2}, {}^4P_{1/2}, {}^4D_{1/2}$, or ${}^4F_{3/2}$. But the 3/2 spin state is symmetric (it includes, in particular, $\uparrow\uparrow\uparrow$), so we need an antisymmetric spatial state. When we combine the first two ($\ell=1$) electrons, we get $\ell=2$ (symmetric), $\ell=1$ (antisymmetric) and $\ell=0$ (symmetric); we need the $\ell=1$ combination. Combining the 3rd $\ell=1$ electron with these we get again 2, 1, and 0, but their symmetry is not clear. Referring to the Clebsch-Gordan table (4.7), we have:

$$\left\{\begin{array}{l} |11\rangle_2 = \frac{1}{\sqrt{2}}|11\rangle|10\rangle - \frac{1}{\sqrt{2}}|10\rangle|11\rangle \\ |10\rangle_2 = \frac{1}{\sqrt{2}}|11\rangle|1{-1}\rangle - \frac{1}{\sqrt{2}}|1{-1}\rangle|11\rangle \\ |1{-1}\rangle_2 = \frac{1}{\sqrt{2}}|10\rangle|1{-1}\rangle - \frac{1}{\sqrt{2}}|1{-1}\rangle|10\rangle \end{array}\right\}$$ (the antisymmetric $l=1$ combination of the first two electrons)

Using 1x1 table again:

$|22\rangle = |11\rangle|11\rangle_2 = \frac{1}{\sqrt{2}}|11\rangle|11\rangle|10\rangle - \frac{1}{\sqrt{2}}|11\rangle|10\rangle|11\rangle$, which is not fully antisymmetric.

$$|11\rangle = \frac{1}{\sqrt{2}}|11\rangle|10\rangle_2 - \frac{1}{\sqrt{2}}|10\rangle|11\rangle_2 = \frac{1}{\sqrt{2}}|11\rangle\left[\frac{1}{\sqrt{2}}|11\rangle|1{-1}\rangle - \frac{1}{\sqrt{2}}|1{-1}\rangle|11\rangle\right]$$
$$- \frac{1}{\sqrt{2}}|10\rangle\left[\frac{1}{\sqrt{2}}|11\rangle|10\rangle - \frac{1}{\sqrt{2}}|10\rangle|11\rangle\right]$$
$$= \frac{1}{2}\left\{|11\rangle|11\rangle|1{-1}\rangle - |11\rangle|1{-1}\rangle|11\rangle - |10\rangle|11\rangle|10\rangle + |10\rangle|10\rangle|11\rangle\right\}$$, which is not fully antisymmetric.

$$|00\rangle = \frac{1}{\sqrt{3}}|11\rangle|1{-1}\rangle_2 - \frac{1}{\sqrt{3}}|10\rangle|10\rangle_2 + \frac{1}{\sqrt{3}}|1{-1}\rangle|11\rangle_2$$
$$= \frac{1}{\sqrt{6}}\{|11\rangle|10\rangle|1{-1}\rangle - |11\rangle|1{-1}\rangle|10\rangle - |10\rangle|11\rangle|1{-1}\rangle + |10\rangle|1{-1}\rangle|11\rangle + |1{-1}\rangle|11\rangle|10\rangle$$
$$- |1{-1}\rangle|10\rangle|11\rangle\}$$, which is completely antisymmetric.

[Indeed: to make $|22\rangle$ from 3 $l=1$ states we need permutations of $|11\rangle|11\rangle|10\rangle$, and there is no way such a state can be fully antisymmetric. Likewise, to make $|11\rangle$, we need permutations of $|11\rangle|11\rangle|1{-1}\rangle$ and/or $|11\rangle|10\rangle|10\rangle$ — and such combinations also cannot be fully antisymmetric. $|00\rangle$ is the only possibility, built on the antisymmetric combination of permutations of $|11\rangle|10\rangle|1{-1}\rangle$, as indicated.]

Conclusion: The ground state of nitrogen must be $\boxed{{}^4S_{3/2}}$. (Table 5.1 confirms this.)

Problem 5.12 $\boxed{S=2;\ L=6;\ J=8}$ $\underbrace{(1s)^2(2s)^2(2p)^6(3s)^2(3p)^6(3d)^{10}(4s)^2(4p)^6}_{\text{definite (36 electrons)}}\underbrace{(4d)^{10}(5s)^2(5p)^6(4f)^{12}}_{\text{likely (30 electrons)}}$

Problem 5.13 (a) $E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3}$. $\rho = \frac{Nq}{V} = \frac{N}{V} = \frac{\text{atoms}}{\text{mole}} \times \frac{\text{moles}}{\text{gm}} \times \frac{\text{gm}}{\text{volume}} = \frac{N_A}{M}\cdot d$,

where N_A = Avogadro's number (6.02×10^{23}), M = atomic mass = 63.5 gm/mole, d = density = 8.96 gm/cm^3.

$$\therefore \rho = \frac{(6.02\times10^{23})(8.96\ \text{gm/cm}^3)}{(63.5\ \text{gm})} = 8.49\times10^{22}/\text{cm}^3 = 8.49\times10^{28}/\text{m}^3.$$

$$E_F = \frac{(1.055\times10^{-34}\ \text{J}\cdot\text{s})(6.58\times10^{-16}\ \text{eV}\cdot\text{s})}{(2)\ (9.109\times10^{-31}\ \text{kg})}\left(3\pi^2\, 8.49\times10^{28}/\text{m}^3\right)^{2/3} = \boxed{7.04\ \text{eV}}.$$

(b) $7.04\ \text{eV} = \frac{1}{2}(.511\times10^6\ \text{eV}/c^2)v^2 \Rightarrow \frac{v^2}{c^2} = \frac{14.08}{.511\times10^6} = 2.76\times10^{-5} \Rightarrow \frac{v}{c} = 5.25\times10^{-3}$, so $\boxed{\text{nonrelativistic}}$.

$v = (5.25\times10^{-3})\times(3\times10^8) = \boxed{1.57\times10^6\ \text{m/s}}$

(c) $T = \dfrac{7.04\,\text{eV}}{8.62\times10^{-5}\,\text{eV/K}} = \boxed{8.17\times10^{4}\,\text{K}}$.

(d) $P = \dfrac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3} = \dfrac{(3\pi^2)^{2/3}(1.055\times10^{-34})^2}{5\,(9.109\times10^{-31})}\left(8.49\times10^{28}\right)^{5/3}\ \text{N/m}^2 = \boxed{3.84\times10^{10}\,\text{N/m}^2}$.

Problem 5.14 $P = \dfrac{(3\pi^2)^{2/3}\hbar^2}{5m}\left(\dfrac{Nq}{V}\right)^{5/3} = AV^{-5/3} \Rightarrow B = -V\dfrac{dP}{dV} = -VA\left(-\tfrac{5}{3}\right)V^{-\frac{5}{3}-1} = \tfrac{5}{3}AV^{-5/3} = \tfrac{5}{3}P.$ ✓

$\therefore$ for copper, $B = \left(\tfrac{5}{3}\right)(3.84\times10^{10}\,\text{N/m}^2) = \boxed{6.4\times10^{10}\,\text{N/m}^2}$.

Problem 5.15 (a) $\psi = A\sin(kx) + B\cos(kx)$; $A\sin(ka) = \left[e^{iKa} - \cos(ka)\right]B \Rightarrow$

$$\psi = A\sin(kx) + \frac{A\sin(ka)}{\left[e^{iKa} - \cos(ka)\right]}\cos kx = \frac{A}{\left[e^{iKa} - \cos(ka)\right]}\left\{e^{iKa}\sin(kx) - \sin(kx)\cos(ka) + \cos(kx)\sin(ka)\right\}$$

$$= C\left\{\sin(kx) + e^{-iKa}\sin\left[k(a-x)\right]\right\}, \text{ where } C \equiv \frac{Ae^{iKa}}{e^{iKa} - \cos(ka)}.$$

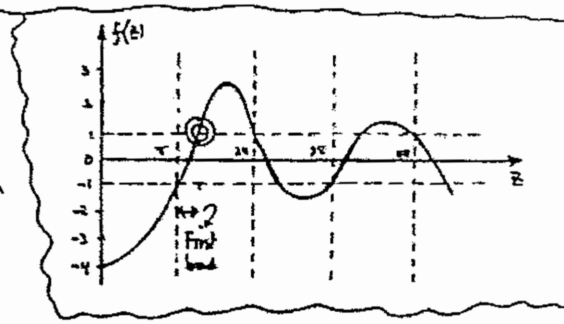
(b) If $ka = Ka = j\pi$, then $[5.63] \Rightarrow A\sin(j\pi) = \left[e^{ij\pi} - \cos(j\pi)\right]B \Rightarrow 0 = \left[(-1)^j - (-1)^j\right]B = 0$ — no condition on A or B. In this case [5.61] holds automatically, and [5.62] gives

$$kA - (-1)^j k\left[A(-1)^j - 0\right] = -\frac{2m\alpha}{\hbar^2}B \Rightarrow \underline{B=0}. \text{ So } \boxed{\psi = A\sin(kx)}.$$

ψ is <u>zero</u> at each delta function, so the wave function never "feels" the potential at all.

Problem 5.16 We're looking for a solution to

$f(z) = \cos z - 5\dfrac{\sin z}{z} = 1$, somewhere in the range $\pi < z < 2\pi$.



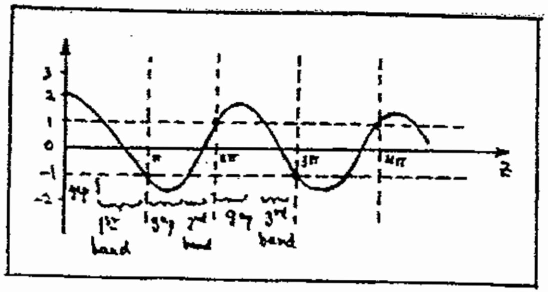
z	$\cos z - 5\sin z/z$
4	0.292
5	1.243
4.6	0.968
4.7	1.051
4.63	0.994
4.64	1.002
4.636	0.9991
4.637	0.9999
4.638	1.0007
4.6375	1.0003
4.6371	0.999988
4.6372	1.000072

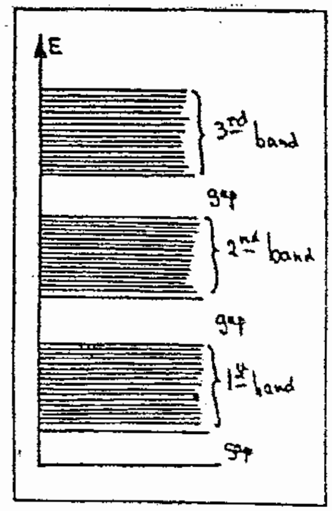
I did it by trial-and-error, on a calculator (Table left): $\boxed{z = 4.6371} = ka.$

$$E = \frac{\hbar^2k^2}{2m} = \frac{\hbar^2 z^2}{2ma^2} = \frac{\alpha z^2}{2\beta a}$$

$$= (4.6371)^2\,\frac{1}{10}\,(1\,\text{eV}) = \boxed{2.15\,\text{eV}}.$$

Problem 5.17





There are N allowed energies in every band — including the first.
The energy at the top of the j^{th} band is given by $z = j\pi$, or

$$E_j = \frac{\hbar^2 k^2 a^2}{2ma^2} = \frac{\hbar^2 z^2}{2ma^2} = \boxed{\frac{\hbar^2 j^2 \pi^2}{2ma^2}}.$$

Problem 5.18 $K = \frac{2\pi n}{Na} \Rightarrow Ka = 2\pi \frac{n}{N}$; $n = 0, 1, 2, \ldots, N-1$ (eq. [5.56] and page 202).

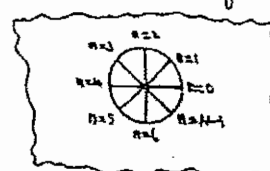
$\underline{N=1} \Rightarrow n = 0 \Rightarrow \cos(Ka) = 1$. Nondegenerate.

$\underline{N=2} \Rightarrow n = 0,1 \Rightarrow \cos(Ka) = 1, -1$. Nondegenerate.

$\underline{N=3} \Rightarrow n = 0,1,2 \Rightarrow \cos(Ka) = 1, -\frac{1}{2}, -\frac{1}{2}$. First $\Rightarrow$ nondegenerate, other two $\Rightarrow$ degenerate.

$\underline{N=4} \Rightarrow n = 0,1,2,3 \Rightarrow \cos(Ka) = 1, 0, -1, 0$. First and third $\Rightarrow$ nondegenerate, other two $\Rightarrow$ degenerate.

Evidently they are doubly degenerate (two different n's give same $\cos(Ka)$) except when $\cos(Ka) = \pm 1$ — i.e. at the $\boxed{\text{top or bottom of a band}}$. The point is that the Bloch factors e^{iKa} lie at equal angles in the complex plane, starting with 1 (see figure); by symmetry, there is always one with negative imaginary part symmetrically opposite each one with positive imaginary part — and the two have the same real part ($\cos(Ka)$). Only points which fall on the real axis have no twins.



(example for $N=8$)

Problem 5.19 (a)

$$\psi(x_A,x_B,x_C) = \frac{1}{\sqrt{6}}\left(\sqrt{\frac{2}{a}}\right)^3 \Big\{ \sin\left(\frac{5\pi x_A}{a}\right)\sin\left(\frac{7\pi x_B}{a}\right)\sin\left(\frac{13\pi x_C}{a}\right) - \sin\left(\frac{5\pi x_A}{a}\right)\sin\left(\frac{13\pi x_B}{a}\right)\sin\left(\frac{7\pi x_C}{a}\right)$$
$$- \sin\left(\frac{7\pi x_A}{a}\right)\sin\left(\frac{5\pi x_B}{a}\right)\sin\left(\frac{13\pi x_C}{a}\right) + \sin\left(\frac{7\pi x_A}{a}\right)\sin\left(\frac{13\pi x_B}{a}\right)\sin\left(\frac{5\pi x_C}{a}\right) - \sin\left(\frac{13\pi x_A}{a}\right)\sin\left(\frac{7\pi x_B}{a}\right)\sin\left(\frac{5\pi x_C}{a}\right)$$
$$+ \sin\left(\frac{13\pi x_A}{a}\right)\sin\left(\frac{5\pi x_B}{a}\right)\sin\left(\frac{7\pi x_C}{a}\right)\Big\}$$

(b)(i)
$$\psi = \left(\sqrt{\frac{2}{a}}\right)^3 \Big\{ \sin\left(\frac{9\pi x_A}{a}\right)\sin\left(\frac{9\pi x_B}{a}\right)\sin\left(\frac{9\pi x_C}{a}\right)\Big\}.$$

(ii)
$$\psi = \frac{1}{\sqrt{3}}\left(\sqrt{\frac{2}{a}}\right)^3 \Big\{ \sin\left(\frac{3\pi x_A}{a}\right)\sin\left(\frac{3\pi x_B}{a}\right)\sin\left(\frac{13\pi x_C}{a}\right) + \sin\left(\frac{3\pi x_A}{a}\right)\sin\left(\frac{13\pi x_B}{a}\right)\sin\left(\frac{3\pi x_C}{a}\right) + \sin\left(\frac{13\pi x_A}{a}\right)\sin\left(\frac{3\pi x_B}{a}\right)\sin\left(\frac{3\pi x_C}{a}\right)\Big\}$$

(iii)
$$\psi = \frac{1}{\sqrt{6}}\left(\sqrt{\frac{2}{a}}\right)^3 \Big\{ \sin\left(\frac{5\pi x_A}{a}\right)\sin\left(\frac{7\pi x_B}{a}\right)\sin\left(\frac{13\pi x_C}{a}\right) + \sin\left(\frac{5\pi x_A}{a}\right)\sin\left(\frac{13\pi x_B}{a}\right)\sin\left(\frac{7\pi x_C}{a}\right) + \sin\left(\frac{7\pi x_A}{a}\right)\sin\left(\frac{5\pi x_B}{a}\right)\sin\left(\frac{13\pi x_C}{a}\right)$$
$$+ \sin\left(\frac{7\pi x_A}{a}\right)\sin\left(\frac{13\pi x_B}{a}\right)\sin\left(\frac{5\pi x_C}{a}\right) + \sin\left(\frac{13\pi x_A}{a}\right)\sin\left(\frac{7\pi x_B}{a}\right)\sin\left(\frac{5\pi x_C}{a}\right) + \sin\left(\frac{13\pi x_A}{a}\right)\sin\left(\frac{5\pi x_B}{a}\right)\sin\left(\frac{7\pi x_C}{a}\right)\Big\}$$

PROBLEM 5.20 (a) $E_{n_1 n_2 n_3} = (n_1+n_2+n_3+3/2)\hbar\omega = \frac{9}{2}\hbar\omega \Rightarrow n_1+n_2+n_3 = 3$. $(n_1, n_2, n_3 = 0, 1, 2, 3, \ldots)$

n_1	n_2	n_3	Configuration $(N_0, N_1, N_2, \ldots)$	# of States
0	0	3	$(2,0,0,1,0,0,\ldots)$	3
0	3	0		
3	0	0		
0	1	2	$(1,1,1,0,0,0,\ldots)$	6
0	2	1		
1	0	2		
1	2	0		
2	0	1		
2	1	0		
1	1	1	$(0,3,0,0,0,\ldots)$	1

Most probable configuration: $\boxed{(1,1,1,0,0,0,\ldots)}$

Possible one-particle energies:

$$\boxed{\begin{aligned} E_0 &= \hbar\omega/2 : P_0 = \tfrac{12}{30} = \tfrac{2}{5}. \\ E_1 &= 3\hbar\omega/2 : P_1 = \tfrac{9}{30} = \tfrac{3}{10}. \\ E_2 &= 5\hbar\omega/2 : P_2 = \tfrac{6}{30} = \tfrac{1}{5}. \\ E_3 &= 7\hbar\omega/2 : P_3 = \tfrac{3}{30} = \tfrac{1}{10}. \end{aligned}}$$

Most probable energy: $\boxed{E_0 = \frac{1}{2}\hbar\omega}$.

(b) For identical fermions the only configuration is $\boxed{(1,1,1,0,0,0,\ldots) \text{ (one state)}}$, so this is also the most probable configuration. The possible one-particle energies are $\boxed{E_0\,(P_0 = 1/3), E_1\,(P_1 = 1/3), E_2\,(P_2 = 1/3)}$, and they are all equally likely, so it's a 3-way tie for most probable energy.

(c) For identical bosons all three configurations are possible, and there is $\boxed{\text{one state for each}}$.

Possible one-particle energies: $\boxed{E_0\,(P_0 = 1/3),\ E_1\,(P_1 = 4/9),\ E_2\,(P_2 = 1/9),\ E_3\,(P_3 = 1/9)}$.

Most probable energy: $\boxed{E_1}$.

PROBLEM 5.21 Here $d_n = 1$ for all states, so:

$$\begin{cases} [5.73] \Rightarrow Q = N! \prod_{n=1}^{\infty} \frac{1}{N_n!} & \text{(distinguishable)} \\ [5.74] \Rightarrow Q = \prod_{n=1}^{\infty} \frac{1}{N_n!(1-N_n)!} & \text{(fermions)} \\ [5.76] \Rightarrow Q = 1 & \text{(bosons)} \end{cases}$$

(In the products most factors are $\frac{1}{0!}$ or $\frac{1}{1!}$, both of which are 1, so I won't write them.) $N = 3$.

Configuration 1 ($N_9 = 3$, all others 0):
$$\begin{cases} Q = 3! \times \frac{1}{3!} = \boxed{1} & \text{(distinguishable)}, \\ Q = \frac{1}{3!} \times \frac{1}{(-2)!} = \boxed{0} & \text{(fermions)}, \\ Q = \boxed{1} & \text{(bosons)}. \end{cases}$$

Configuration 2 ($N_3 = 2$, $N_{15} = 1$, else 0):
$$\begin{cases} Q = 3! \times \frac{1}{2!} \times \frac{1}{1!} = \boxed{3} & \text{(distinguishable)}, \\ Q = \frac{1}{2!(-1)!} \times \frac{1}{1!\,0!} = \boxed{0} & \text{(fermions)}, \\ Q = \boxed{1} & \text{(bosons)}. \end{cases}$$

Configuration 3 ($N_5 = N_7 = N_{13} = 1$):
$$\begin{cases} Q = 3! \times \frac{1}{1!} \times \frac{1}{1!} \times \frac{1}{1!} = \boxed{6} & \text{(distinguishable)}, \\ Q = \frac{1}{1!\,0!} \times \frac{1}{1!\,0!} \times \frac{1}{1!\,0!} = \boxed{1} & \text{(fermions)}, \\ Q = \boxed{1} & \text{(bosons)}. \end{cases}$$

All of these agree with what we got "by hand" in the Example.

Problem 5.22 <u>$N=1$</u>: can put the ball in any of d baskets, so $\boxed{d}$ ways.

<u>$N=2$</u>: $\left\{\begin{array}{l}\text{could put both balls in any of the } d \text{ baskets — } d \text{ ways — or} \\ \text{could put one in one basket } (d \text{ ways}), \text{ the other in another } (d-1 \text{ ways}) \text{ — but it doesn't matter which} \end{array}\right.$
is which, so divide by 2;

<u>total</u>: $d + \frac{1}{2}d(d-1) = \frac{1}{2}d(2+d-1) = \boxed{\frac{1}{2}d(d+1)}$ ways.

<u>$N=3$</u>: $\left\{\begin{array}{l}\text{could put all three in one basket — } d \text{ ways, or} \\ \text{two in one basket, one in another — } d(d-1) \text{ ways, or} \\ \text{one each in 3 baskets: } d(d-1)(d-2)/3! \text{ ways}\end{array}\right\}$ <u>total</u>: $d + d(d-1) + d(d-1)(d-2)/6$

$$= \tfrac{1}{6}d\left[6 + 6d - 6 + d^2 - 3d + 2\right]$$
$$= \tfrac{1}{6}d(d^2 + 3d + 2) = \boxed{\frac{d(d+1)(d+2)}{6}}.$$

<u>$N=4$</u>: $\left\{\begin{array}{l}\text{all in one basket: } d \text{ ways, or} \\ \text{3 in one basket, 1 in another: } d(d-1) \text{ ways, or} \\ \text{2 in one basket, 2 in another: } d(d-1)/2 \text{ ways, or} \\ \text{2 in one basket, one each in others: } d(d-1)(d-2)/2, \text{ or} \\ \text{all in different baskets: } d(d-1)(d-2)(d-3)/4!\end{array}\right\}$ <u>total</u>: $d + d(d-1) + d(d-1)/2 + d(d-1)(d-2)/2 + d(d-1)(d-2)(d-3)/24$
$= \frac{d}{24}\left[24 + 24d - 24 + 12d - 12 + 12d^2 - 36d + 24 + d^3 - 6d^2 + 11d - 6\right]$

$$\text{total} = \tfrac{1}{24}d(d^3 + 6d^2 + 11d + 6) = \boxed{\frac{d(d+1)(d+2)(d+3)}{24}}.$$

The general formula seems to be $f(N,d) = \dfrac{d(d+1)(d+2)\cdots(d+N-1)}{N!} = \dfrac{(d+N-1)!}{N!\,(d-1)!} = \boxed{\dbinom{d+N-1}{N}}$.

<u>Proof</u>: How many ways to put N identical balls in d baskets? Call it $f(N,d)$.

Could put all of them in the first basket: 1 way.
Could put all but one in the first basket; there remains 1 ball for $d-1$ baskets: $f(1,d-1)$ ways.
Could put all but two in the first basket; there remain 2 for $d-1$ baskets: $f(2,d-1)$ ways.
⋮
Could put zero in the first basket, leaving N for $d-1$ baskets: $f(N,d-1)$ ways.

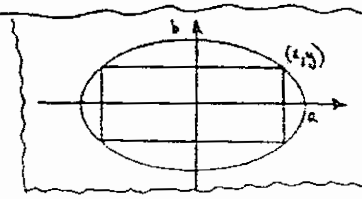
Thus: $f(N,d) = f(0,d-1) + f(1,d-1) + f(2,d-1) + \cdots + f(N,d-1) = \sum_{j=0}^{N} f(j,d-1)$ (where $f(0,d) \equiv 1$).

It follows that $f(N,d) = \sum_{j=0}^{N-1} f(j,d-1) + f(N,d-1) = f(N-1,d) + f(N,d-1)$. Use this recursion formula to confirm the conjectured formula by induction:

$$\binom{d+N-1}{N} \overset{?}{=} \binom{d+N-2}{N-1} + \binom{d+N-2}{N} = \frac{(d+N-2)!}{(N-1)!\,(d-1)!} + \frac{(d+N-2)!}{N!\,(d-2)!} = \frac{(d+N-2)!}{(N)!\,(d-1)!}\left[N+d-1\right] = \frac{(d+N-1)!}{N!\,(d-1)!}$$

$= \dbinom{d+N-1}{d-1}$ ✓. [And it works for $N=0$: $\binom{d-1}{0} = 1$ ✓, and for $d=1$: $\binom{N}{N} = 1$ (which is obviously correct for just one basket.] QED

Problem 5.23 $A(x,y) = (2x)(2y) = 4xy$; maximize, subject to the constraint
$(x/a)^2 + (y/b)^2 = 1$.



$G(x,y,\lambda) \equiv 4xy + \lambda\left[(x/a)^2 + (y/b)^2 - 1\right]$. $\frac{\partial G}{\partial x} = 4y + \frac{2\lambda x}{a^2} = 0 \Rightarrow y = -\frac{\lambda x}{2a^2}$.

$\frac{\partial G}{\partial y} = 4x + \frac{2\lambda y}{b^2} = 0 \Rightarrow 4x = -\frac{2\lambda}{b^2}\left(-\frac{\lambda x}{2a^2}\right) \Rightarrow 4x = \frac{\lambda^2}{a^2b^2}x \Rightarrow x = 0$ (obviously a minimum), or else

$\lambda = \pm 2ab$. So $y = \mp \frac{2abx}{2a^2} = \mp \frac{b}{a}x$. We may as well pick x and y positive (as in the figure); then $y = bx/a$ (and $\lambda = -2ab$). $\frac{\partial G}{\partial \lambda} = 0 \Rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ (of course), so

$\frac{x^2}{a^2} + \frac{b^2x^2}{a^2b^2} = 1$, or $\frac{2}{a^2}x^2 = 1$, or $\underline{\underline{x = a/\sqrt{2}}}$, and hence $y = \frac{b}{a}\frac{a}{\sqrt{2}} \Rightarrow \underline{\underline{y = b/\sqrt{2}}}$.

$\therefore A = 4\frac{a}{\sqrt{2}}\frac{b}{\sqrt{2}} = \boxed{2ab}$.

PROBLEM 5.24 (a) $\ln(10!) = \ln(3628800) = \underline{\underline{15.1044}}$; $10\ln(10) - 10 = 23.026 - 10 = \underline{\underline{13.0259}}$

$15.1044 - 13.0259 = 2.0785$. $\frac{2.0785}{15.1044} = 0.1376$, or $\boxed{14\%}$.

(b) The percent error is: $\left[\frac{\ln(z!) - z\ln z + z}{\ln(z)!}\right] \times 100 \equiv \%$

Since my calculator can't compute factorials greater than $69!$, I used Mathematica to construct the table.

Evidently the smallest integer for which the error is $< 1\%$ is $\boxed{90}$.

z	%
20	5.7
100	0.89
50	1.9
90	0.996
85	1.06
89	1.009

PROBLEM 5.25 $[5.107] \Rightarrow N = \frac{V}{2\pi^2}\int_0^\infty k^2 n(\epsilon)\,dk$, where $n(\epsilon)$ is given (as $T \to 0$) by $[5.103]$. So

$$N = \frac{V}{2\pi^2}\int_0^{k_{max}} k^2\,dk = \frac{V}{2\pi^2}\frac{k_{max}^3}{3},$$ where k_{max} is given by $\frac{\hbar^2 k_{max}^2}{2m} = \mu(0) = E_F \Rightarrow k_{max} = \frac{\sqrt{2mE_F}}{\hbar}$.

$\therefore \boxed{N = \frac{V}{6\pi^2\hbar^3}(2mE_F)^{3/2}}$. Compare $[5.43]$, which says $E_F = \frac{\hbar^2}{2m}\left(3\pi^2\frac{Nq}{V}\right)^{2/3}$, or

$\frac{(2mE_F)^{3/2}}{\hbar^3} = 3\pi^2 Nq/V$, or $N = \frac{V}{3\pi^2 q\hbar^3}(2mE_F)^{3/2}$. Here $q = 1$, and $[5.107]$ needs an extra factor of 2 on the right, to account for spin, so the two formulas agree.

$$[5.108] \Rightarrow E_{tot} = \frac{V\hbar^2}{4\pi^2 m}\int_0^{k_{max}} k^4\,dk = \frac{V\hbar^2}{4\pi^2 m}\frac{k_{max}^5}{5} \Rightarrow E_{tot} = \boxed{\frac{V}{20\pi^2 m\hbar^3}(2mE_F)^{5/2}}.$$

Compare $[5.45]$, which says $E_{tot} = \frac{V\hbar^2}{10\pi^2 m}k_{max}^5$. Again, $[5.108]$ for electrons has an extra factor of 2, so the two agree.

Problem 5.26 (a) [5.102], $n(\epsilon)>0 \Rightarrow \dfrac{1}{e^{(\epsilon-\mu)/k_BT}-1} > 0 \Rightarrow e^{(\epsilon-\mu)/k_BT} > 1 \Rightarrow \dfrac{\epsilon-\mu}{k_BT} > 0 \Rightarrow \boxed{\epsilon > \mu(T)}$, for all allowed energies ϵ.

(b) For free particle gas, $E = \frac{\hbar^2}{2m}k^2 \to 0$ (as $k\to 0$, in the continuum limit), so $\underline{\mu(T)}$ is always negative. [Technically, the lowest energy is $\frac{\hbar^2\pi^2}{2m}\left(\frac{1}{\ell_x^2}+\frac{1}{\ell_y^2}+\frac{1}{\ell_z^2}\right)$, but we take the dimensions $\ell_x\, \ell_y\, \ell_z$ to be very large, in the continuum approximation.]

$[5.107] \Rightarrow \quad \dfrac{N}{V} = \dfrac{1}{2\pi^2}\displaystyle\int_0^\infty \dfrac{k^2}{e^{[\hbar^2k^2/2m-\mu]/k_BT}-1}\,dk.$ The integrand is always positive, and the only T dependence is in $\mu(T)$ and k_BT. So, as T decreases, $\frac{\hbar^2k^2}{2m}-\mu(T)$ must also decrease, and hence $-\mu(T)$ decreases, or $\underline{\mu(T)\text{ increases}}$ (always negative).

(c) $\dfrac{N}{V} = \dfrac{1}{2\pi^2}\displaystyle\int_0^\infty \dfrac{k^2}{e^{\hbar^2k^2/2mk_BT}-1}\,dk.$ Let $x \equiv \dfrac{\hbar^2k^2}{2mk_BT}$, so $k = \dfrac{\sqrt{2mk_BT}}{\hbar}x^{1/2}$; $dk = \dfrac{\sqrt{2mk_BT}}{\hbar}\frac{1}{2}x^{-1/2}dx$.

$$\frac{N}{V} = \frac{1}{2\pi^2}\left(\frac{2mk_BT}{\hbar^2}\right)^{3/2}\frac{1}{2}\underbrace{\int_0^\infty \frac{x^{1/2}}{e^x-1}\,dx}$$

$$\int_0^\infty \frac{x^{\frac{3}{2}-1}}{e^x-1}\,dx = \Gamma\left(\tfrac{3}{2}\right)\zeta\left(\tfrac{3}{2}\right). \quad \Gamma\left(\tfrac{3}{2}\right)=\tfrac{\sqrt{\pi}}{2};\ \zeta\left(\tfrac{3}{2}\right)=2.61238$$

$$\frac{N}{V} = \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}(2.61); \qquad \boxed{T_c = \frac{2\pi\hbar^2}{mk_B}\left(\frac{N}{2.61V}\right)^{2/3}}$$

(d) $$\frac{N}{V} = \frac{\text{mass/volume}}{\text{mass/atom}} = \frac{0.15\times10^3\ \text{kg/m}^3}{4(1.67\times10^{-27}\ \text{kg})} = 2.2\times10^{28}/\text{m}^3.$$

$$T_c = \frac{2\pi(1.05\times10^{-34}\ \text{J}\cdot\text{s})^2}{4(1.67\times10^{-27}\ \text{kg})(1.38\times10^{-23}\ \text{J/K})}\left(\frac{2.2\times10^{28}}{2.61\ \text{m}^3}\right)^{2/3} = \boxed{3.1\ \text{K}}.$$

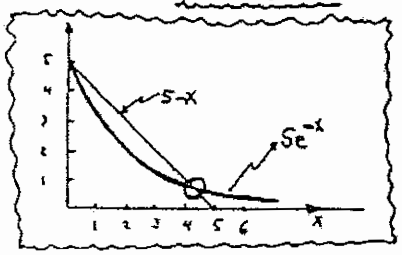
Problem 5.27 $\omega = 2\pi\nu = \frac{2\pi c}{\lambda}$, so $d\omega = -\frac{2\pi c}{\lambda^2}d\lambda$, and $\rho(\omega) = \frac{\hbar}{\pi^2 c^3}\frac{(2\pi c)^3}{\lambda^3\left(e^{2\pi\hbar c/k_BT\lambda}-1\right)}$.

$$\rho(\omega)|d\omega| = 8\pi\hbar\frac{1}{\lambda^3\left(e^{2\pi\hbar c/k_BT\lambda}-1\right)}\left|\left(-\frac{2\pi c}{\lambda^2}d\lambda\right)\right| = \bar{\rho}(\lambda)d\lambda \Rightarrow \boxed{\bar{\rho}(\lambda) = \frac{16\pi^2\hbar c}{\lambda^5\left[e^{2\pi\hbar c/k_BT\lambda}-1\right]}}.$$

(For density we want only the size of the interval, not its sign.) To maximise, set $d\bar{\rho}/d\lambda = 0$:

$$0 = 16\pi^2\hbar c\left\{\frac{-5}{\lambda^6\left(e^{2\pi\hbar c/k_BT\lambda}-1\right)} - \frac{e^{2\pi\hbar c/k_BT\lambda}\left(\frac{2\pi\hbar c}{k_BT}\right)}{\lambda^5\left(e^{2\pi\hbar c/k_BT\lambda}-1\right)^2}\left(-\frac{1}{\lambda^2}\right)\right\} \Rightarrow 5\left(e^{2\pi\hbar c/k_BT\lambda}-1\right) = e^{2\pi\hbar c/k_BT\lambda}\left[\frac{2\pi\hbar c}{k_BT\lambda}\right].$$

Let $x \equiv \frac{2\pi\hbar c}{k_BT\lambda}$; then $5(e^x-1) = xe^x$, or $5(1-e^{-x}) = x$, or $5e^{-x} = 5-x$.



From the graph, the solution occurs slightly below $x=5$. By calculator:

x	$5e^{-x}$	$5-x$
4.5	.0555	.5
4.9	.0372	.1
4.98	.0344	.02
4.96	.0351	.04
4.97	.0347	.03
4.965	.03489	.035
4.966	.03485	.034
4.9655	.03487	.0345

Evidently $x = 4.966$.

So $\boxed{\lambda_{max} = \frac{2\pi\hbar c}{(4.966)k_B}\frac{1}{T}}$

$$= \frac{(6.626\times10^{-34}\,\mathrm{J\,s})(2.998\times10^8\,\mathrm{m/s})}{(4.966)(1.3807\times10^{-23}\,\mathrm{J/K})}\frac{1}{T}$$

$$= \boxed{2.897\times10^{-3}\,\mathrm{m\,K}/T}.$$

Problem 5.28 From [5.112]: $\frac{E}{V} = \int_0^\infty \rho(\omega)d\omega = \frac{\hbar}{\pi^2c^3}\int_0^\infty\frac{\omega^3}{\left(e^{\hbar\omega/k_BT}-1\right)}d\omega$. Let $x \equiv \frac{\hbar\omega}{k_BT}$. Then

$$\frac{E}{V} = \frac{\hbar}{\pi^2c^3}\left(\frac{k_BT}{\hbar}\right)^4\int_0^\infty\frac{x^3}{e^x-1}dx = \frac{(k_BT)^4}{\pi^2c^3\hbar^3}\Gamma(4)\zeta(4) = \frac{(k_BT)^4}{\pi^2c^3\hbar^3}\cdot 6\cdot\frac{\pi^4}{90} = \left(\frac{\pi^2k_B^4}{15c^3\hbar^3}\right)T^4.$$

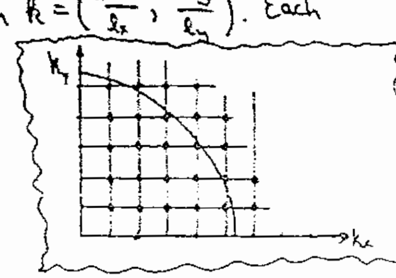
$$= \left\{\frac{\pi^2(1.3807\times10^{-23}\,\mathrm{J/K})^4}{15(2.998\times10^8\,\mathrm{m/s})^3(1.0546\times10^{-34}\,\mathrm{J\cdot s})^3}\right\}T^4 = 7.566\times10^{-16}\frac{\mathrm{J}}{\mathrm{m^3K^4}}T^4. \quad \text{QED.}$$

Problem 5.29 (a) Each particle has 3 possible states: $3\times3\times3 = \boxed{27}$.

(b) all in same state: aaa, bbb, ccc $\Rightarrow$ 3
  2 in one state: aab, aac, bba, bbc, cca, ccb $\Rightarrow$ 6 (each symmetrized)
  3 different states: abc (symmetrized) $\Rightarrow$ 1
  total: $\boxed{10}$.

(c) only abc (antisymmetrized) $\Rightarrow \boxed{1}$.

Problem 5.30 Eqn. [5.39] $\Rightarrow$ $E_{n_xn_y} = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_x^2}{l_x^2}+\frac{n_y^2}{l_y^2}\right) = \frac{\hbar^2k^2}{2m}$, with $\vec{k} = \left(\frac{\pi n_x}{l_x}, \frac{\pi n_y}{l_y}\right)$. Each state is represented by an intersection on a grid in "k-space" – this time a plane – and each state occupies an area $\pi^2/l_xl_y = \pi^2/A$ of k-space (where $A \equiv l_xl_y$ is the area of the well). Two electrons per state $\Rightarrow$



$\frac{1}{4}\pi k_F^2 = \frac{Nq}{2}\left(\frac{\pi^2}{A}\right)$, or $k_F = \left(2\pi \frac{Nq}{A}\right)^{1/2} = (2\pi\sigma)^{1/2}$, where $\sigma \equiv \frac{Nq}{A}$ is the number of free electrons per unit area. $\therefore\ E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}2\pi\sigma = \boxed{\frac{\pi\hbar^2\sigma}{m}}$.

PROBLEM 5.31 (a) $V = \frac{4}{3}\pi R^3$, so $E = \frac{\hbar^2(3\pi^2 Nq)^{5/3}}{10\pi^2 m}\left(\frac{4}{3}\pi R^3\right)^{-2/3} = \boxed{\frac{2\hbar^2}{15\pi m R^2}\left(\frac{9}{4}\pi Nq\right)^{5/3}}$.

(b) Imagine building up a sphere by layers. When it has reached mass m, and radius r, the work necessary to bring in the next increment (dm) is: $dW = -G\frac{m\,dm}{r}$. In terms of the mass density ρ, $m = \frac{4}{3}\pi r^3\rho$, and $dm = 4\pi r^2 dr\,\rho$, where dr is the resulting increase in radius.

Thus:

$$dW = -G\,\tfrac{4}{3}\pi r^3\rho\,4\pi r^2\rho\,dr/r = -\frac{16\pi^2}{3}\rho^2 G r^4 dr,$$

and the total energy of a sphere of radius R is therefore $E_{grav} = -\frac{16\pi^2}{3}\rho^2 G\int_0^R r^4 dr = -\frac{16\pi^2\rho^2 R^5}{15}G$. But $\rho = \frac{NM}{\frac{4}{3}\pi R^3}$, so

$$E_{grav} = -\frac{16\pi^2 R^5}{15}G\,\frac{9N^2M^2}{16\pi^2 R^6} = \boxed{-\frac{3}{5}G\frac{N^2M^2}{R}}.$$

(c) $E_{tot} = \frac{A}{R^2} - \frac{B}{R}$, where $A \equiv \frac{2\hbar^2}{15\pi m}\left(\frac{9}{4}\pi Nq\right)^{5/3}$ and $B \equiv \frac{3}{5}GN^2M^2$.

$$\frac{dE_{tot}}{dR} = -\frac{2A}{R^3} + \frac{B}{R^2} = 0 \Rightarrow 2A = BR, \text{ or } R = \frac{2A}{B} = \frac{4\hbar^2}{15\pi m}\left(\frac{9}{4}\pi Nq\right)^{5/3}\frac{5}{3GN^2M^2}.$$

$$R = \left[\left(\frac{4}{9\pi}\right)\left(\frac{9\pi}{4}\right)^{5/3}\right]\left[\frac{N^{5/3}}{N^2}\right]\frac{\hbar^2}{GmM^2}q^{5/3} = \boxed{\left(\frac{9\pi}{4}\right)^{2/3}\frac{\hbar^2}{GmM^2}\frac{q^{5/3}}{N^{1/3}}}.$$

$$R = \left(\frac{9\pi}{4}\right)^{2/3}\frac{(1.055\times10^{-34}\,J.s)^2\left(\frac{1}{2}\right)^{5/3}}{(6.673\times10^{-11}\,N\,m^2/kg^2)(9.109\times10^{-31}\,kg)(1.674\times10^{-27}\,kg)^2}N^{-1/3} = \boxed{(7.58\times10^{25}\,m)N^{-1/3}}.$$

(d) Mass of sun: $1.989\times10^{30}\,kg$, so $N = \frac{1.989\times10^{30}}{1.674\times10^{-27}} = 1.188\times10^{57}$; $N^{-1/3} = 9.44\times10^{-20}$.

$R = (7.58\times10^{25})(9.44\times10^{-20})\,m = \boxed{7.16\times10^6\,m}$ (slightly larger than the earth).

(e) From eq. [5.43]: $E_F = \frac{\hbar^2}{2m}\left(3\pi^2\frac{Nq}{\frac{4}{3}\pi R^3}\right)^{2/3} = \frac{\hbar^2}{2mR^2}\left(\frac{9\pi}{4}Nq\right)^{2/3}$; numerically:

$$E_F = \frac{(1.055\times10^{-34}\,J.s)^2}{2\,(9.109\times10^{-31}\,kg)(7.16\times10^6\,m)^2}\left(\frac{9\pi}{4}(1.188\times10^{57})\frac{1}{2}\right)^{2/3} = 3.102\times10^{-14}\,J,$$

or, in electron volts:

$E_F = \frac{3.102\times10^{-14}}{1.602\times10^{-19}} = \boxed{1.94\times10^5\,eV}$. $E_{rest} = mc^2 = 5.11\times10^5\,eV$, so the Fermi energy (which is the energy of the most energetic electrons) is comparable to the rest energy — i.e. they are getting relativistic.

PROBLEM 5.32 (a) $dE = (\hbar c k)\frac{V}{\pi^2}k^2 dk \Rightarrow E_{TOT} = \frac{\hbar c V}{\pi^2}\int_0^{k_F} k^3 dk = \frac{\hbar c V}{4\pi^2}k_F^4$. $k_F = \left(3\pi^2 \frac{Nq}{V}\right)^{1/3}$.

So $E_{TOT} = \boxed{\frac{\hbar c}{4\pi^2}\left(3\pi^2 Nq\right)^{4/3} V^{-1/3}}$.

(b) $V = \frac{4}{3}\pi R^3 \Rightarrow E_{deg} = \frac{\hbar c}{4\pi^2 R}\left(3\pi^2 Nq\right)^{4/3}\left(\frac{4\pi}{3}\right)^{-1/3} = \frac{\hbar c}{3\pi R}\left(\frac{9}{4}\pi Nq\right)^{4/3}$. Adding in the gravitational energy, from Problem 5.31(b),

$$E_{TOT} = \frac{A}{R} - \frac{B}{R}, \text{ where } A \equiv \frac{\hbar c}{3\pi}\left(\frac{9}{4}\pi Nq\right)^{4/3} \text{ and } B \equiv \frac{3}{5}GN^2M^2.$$

$\frac{dE_{TOT}}{dR} = -\frac{(A-B)}{R^2} = 0 \Rightarrow A = B$, but there is no special value of R for which E_{TOT} is minimal.

Critical value: $A = B\ (E_{TOT} = 0) \Rightarrow \frac{\hbar c}{3\pi}\left(\frac{9}{4}\pi Nq\right)^{4/3} = \frac{3}{5}GN^2M^2$, or

$$\boxed{N_c = \frac{15}{16}\sqrt{5\pi}\left(\frac{\hbar c}{G}\right)^{3/2}\frac{q^2}{M^3}} = \frac{15}{16}\sqrt{5\pi}\left(\frac{1.055\times10^{-34}\,J\cdot s\times 2.998\times10^{8}\,m/s}{6.673\times10^{-11}\,N\cdot m^2/kg^2}\right)^{3/2}\frac{(1/2)^2}{(1.674\times10^{-27}\,kg)^3}$$

$= \boxed{2.04\times10^{57}}$. (About twice the value for the sun – Problem 5.31(d))

(c) Same as Problem 2.31(c), with $m \to M$ and $q \to 1$, so multiply old answer by $(2)^{5/3}\frac{m}{M}$:

$R = 2^{5/3}\frac{(9.109\times10^{-31})}{(1.674\times10^{-27})}(7.58\times10^{25}\,m)N^{-1/3} = (1.31\times10^{23}\,m)N^{-1/3}$. Using $N = 1.188\times10^{57}$,

$R = (1.31\times10^{23}\,m)(9.44\times10^{-20}) = \boxed{12.4\text{ km}}$. To get E_F, use Problem 5.31(e) with $q=1$, the new R, and the neutron mass in place of m:

$$E_F = (2)^{2/3}\left(\frac{7.16\times10^{6}}{1.24\times10^{4}}\right)^2\left(\frac{9.11\times10^{-31}}{1.67\times10^{-27}}\right)(1.94\times10^{5}\,eV) = 5.60\times10^{7}\,eV = \boxed{56.0\text{ MeV}}.$$

The rest energy of a neutron is 940 MeV, so this is reasonably nonrelativistic.

PROBLEM 5.33 (a) From Problem 4.39: $E_n = (n+\frac{3}{2})\hbar\omega$, with $n = 0,1,2,\dots$; $d_n = \frac{1}{2}(n+1)(n+2)$.

From [5.102], $n(\epsilon) = e^{-(\epsilon-\mu)/k_BT}$, so $N_n = \frac{1}{2}(n+1)(n+2)\,e^{(\mu-\frac{3}{2}\hbar\omega)/k_BT}\,e^{-n\hbar\omega/k_BT}$.

$$N = \sum_{n=0}^{\infty} N_n = \frac{1}{2}e^{(\mu-\frac{3}{2}\hbar\omega)/k_BT}\sum_{n=0}^{\infty}(n+1)(n+2)x^n, \quad \text{where } x \equiv e^{-\hbar\omega/k_BT}.$$

Now $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow \frac{x}{1-x} = \sum_{n=0}^{\infty} x^{n+1} \Rightarrow \frac{d}{dx}\left(\frac{x}{1-x}\right) = \sum_{n=0}^{\infty}(n+1)x^n$, or $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$.

$\therefore \frac{x^2}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^{n+2}$, and hence $\frac{d}{dx}\left(\frac{x^2}{(1-x)^2}\right) = \sum_{n=0}^{\infty}(n+1)(n+2)x^{n+1} = \frac{2x}{(1-x)^3}$

$\therefore \sum_{n=0}^{\infty}(n+1)(n+2)x^n = \frac{2}{(1-x)^3}$. So $N = e^{\mu/k_BT}e^{-\frac{3}{2}\hbar\omega/k_BT}\frac{1}{(1-e^{-\hbar\omega/k_BT})^3}$.

$e^{\mu/k_BT} = N\left(1-e^{-\hbar\omega/k_BT}\right)^3 e^{\frac{3}{2}\hbar\omega/k_BT}$; $\boxed{\mu = k_BT\left\{\ln N + 3\ln\left(1-e^{-\hbar\omega/k_BT}\right) + \frac{3}{2}\hbar\omega/k_BT\right\}}$

$E = \sum_{n=0}^{\infty} N_n E_n = \frac{1}{2}\hbar\omega\, e^{(\mu-\frac{3}{2}\hbar\omega)/k_BT}\sum_{n=0}^{\infty}(n+\frac{3}{2})(n+1)(n+2)x^n$. From above,

$\frac{2x^{3/2}}{(1-x)^3} = \sum_{n=0}^{\infty}(n+1)(n+2)x^{n+3/2} \Rightarrow \frac{d}{dx}\left(\frac{2x^{3/2}}{(1-x)^3}\right) = \sum_{n=0}^{\infty}(n+\frac{3}{2})(n+1)(n+2)x^{n+1/2}$, or

$$\sum_{n=0}^{\infty}(n+\tfrac{3}{2})(n+1)(n+2)x^n = \frac{1}{x^{1/2}}\frac{d}{dx}\left(\frac{2x^{3/2}}{(1-x)^3}\right) = \frac{2}{x^{1/2}}\left\{\frac{\frac{3}{2}x^{1/2}}{(1-x)^3} + \frac{3x^{3/2}}{(1-x)^4}\right\} = \frac{3(1+x)}{(1-x)^4}.$$

$\therefore E = \frac{1}{2}\hbar\omega\, e^{(\mu-\frac{3}{2}\hbar\omega)/k_BT}\frac{3(1+e^{-\hbar\omega/k_BT})}{(1-e^{-\hbar\omega/k_BT})^4}$. But $e^{(\mu-\frac{3}{2}\hbar\omega)/k_BT} = N\left(1-e^{-\hbar\omega/k_BT}\right)^3$, so

$$\boxed{E = \frac{3}{2}N\hbar\omega\frac{(1+e^{-\hbar\omega/k_BT})}{(1-e^{-\hbar\omega/k_BT})}}.$$

(b) $k_BT \ll \hbar\omega$ (low temperature) $\Rightarrow e^{-\hbar\omega/k_BT} \approx 0$, so $\boxed{E \cong \frac{3}{2}N\hbar\omega}$ $(\mu \cong \frac{3}{2}\hbar\omega)$.

In this limit all particles are in the ground state, $E_0 = \frac{3}{2}\hbar\omega$.

(c) $k_BT \gg \hbar\omega$ (high temperature) $\Rightarrow e^{-\hbar\omega/k_BT} \approx 1 - \frac{\hbar\omega}{k_BT} \Rightarrow \boxed{E \cong 3Nk_BT}$ $\left(\mu \cong k_BT\left(\ln N + 3\ln\left(\frac{\hbar\omega}{k_BT}\right)\right)\right)$.

Equipartition theorem says $E = N\#\frac{1}{2}k_BT$, where $\#$ = number of degrees of freedom for each particle. So in this case $\frac{\#}{2} = 3$, or $\boxed{\# = 6}$ (3 kinetic, 3 potential, for each particle — one of each for each direction in space.)

CHAPTER 6

PROBLEM 6.1 $\psi_n^0(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$, so $E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle = \frac{2}{a}\alpha\int_0^a \sin^2\left(\frac{n\pi}{a}x\right)\delta(x-a/2)\,dx.$

$E_n^1 = \frac{2\alpha}{a}\sin^2\left(\frac{n\pi}{a}\frac{a}{2}\right) = \frac{2\alpha}{a}\sin^2\left(\frac{n\pi}{2}\right) = \boxed{\begin{Bmatrix} 0, \text{ if } n \text{ is even,} \\ 2\alpha/a, \text{ if } n \text{ is odd.}\end{Bmatrix}}$ For even n the wave function is zero at the location of the perturbation $(x = a/2)$, so it never "feels" H'.

PROBLEM 6.2 (a) $E_n = (n+\frac{1}{2})\hbar\omega'$, where $\omega' \equiv \sqrt{k(1+\epsilon)/m} = \omega\sqrt{1+\epsilon} = \omega(1+\frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{16}\epsilon^3\cdots)$.

$\therefore \boxed{E_n = (n+\frac{1}{2})\hbar\omega\sqrt{1+\epsilon} = (n+\frac{1}{2})\hbar\omega(1+\frac{\epsilon}{2} - \frac{1}{8}\epsilon^2 + \cdots)}$

(b) $H' = \frac{1}{2}k'x^2 - \frac{1}{2}kx^2 = \frac{1}{2}kx^2(1+\epsilon-1) = \epsilon(\frac{1}{2}kx^2) = \epsilon V$, where V is the unperturbed potential energy. So $E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle = \epsilon\langle n|V|n\rangle$, with $\langle n|V|n\rangle$ the expectation value of the (unperturbed) potential energy in the n^{th} unperturbed state. This is most easily obtained from the Virial Theorem (Problem 3.53), but it can also be derived algebraically (Problem 2.37). In this case the Virial Theorem says $\langle T\rangle = \langle V\rangle$. But $\langle T\rangle + \langle V\rangle = E_n$. So $\langle V\rangle = \frac{1}{2}E_n^0 = \frac{1}{2}(n+\frac{1}{2})\hbar\omega$. So $\boxed{E_n^1 = \frac{\epsilon}{2}(n+\frac{1}{2})\hbar\omega}$, which is precisely the ϵ^1 term in the power series from part (a).

PROBLEM 6.3 (a) In terms of the one-particle states [2.24] and energies [2.23]:

Ground state: $\psi_1^0(x_1,x_2) = \psi_1(x_1)\psi_1(x_2) = \boxed{\frac{2}{a}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)}$; $E_1^0 = 2E_1 = \boxed{\frac{\pi^2\hbar^2}{ma^2}}$.

First excited state: $\psi_2^0(x_1,x_2) = \frac{1}{\sqrt{2}}\left(\psi_1(x_1)\psi_2(x_2) + \psi_2(x_1)\psi_1(x_2)\right) = \boxed{\frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) + \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right]}$;

$E_2^0 = E_1 + E_2 = \boxed{\frac{5}{2}\frac{\pi^2\hbar^2}{ma^2}}$.

(b) $E_1^1 = \langle\psi_1^0|H'|\psi_1^0\rangle = (-aV_0)\left(\frac{2}{a}\right)^2\iint_0^a \sin^2\left(\frac{\pi x_1}{a}\right)\sin^2\left(\frac{\pi x_2}{a}\right)\delta(x_1-x_2)\,dx_1dx_2 = -\frac{4V_0}{a}\int_0^a \sin^4\left(\frac{\pi x}{a}\right)dx$

$= -\frac{4V_0}{a}\frac{a}{\pi}\int_0^\pi \sin^4 y\,dy = -\frac{4V_0}{\pi}\cdot\frac{3\pi}{8} = \boxed{-\frac{3}{2}V_0}$.

$E_2^1 = \langle\psi_2^0|H'|\psi_2^0\rangle = (-aV_0)\left(\frac{2}{a^2}\right)\iint_0^a\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) + \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right]^2\delta(x_1-x_2)\,dx_1dx_2$

$= -\frac{2V_0}{a}\int_0^a\left[\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right) + \sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\right]^2dx = -\frac{8V_0}{a}\int_0^a \sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{2\pi x}{a}\right)dx$

$= -\frac{8V_0}{a}\cdot\frac{a}{\pi}\int_0^\pi \sin^2 y\,\sin^2(2y)\,dy = -\frac{8V_0}{\pi}\cdot 4\int_0^\pi \sin^2 y\,\sin^2 y\cos^2 y\,dy = -\frac{32V_0}{\pi}\int_0^\pi(\sin^4 y - \sin^6 y)\,dy$

$= -\frac{32V_0}{\pi}\left(\frac{3\pi}{8} - \frac{5\pi}{16}\right) = \boxed{-2V_0}$.

PROBLEM 6.4 (a) $\langle\psi_m^0|H'|\psi_n^0\rangle = \frac{2}{a}\alpha\int_0^a \sin\left(\frac{m\pi}{a}x\right)\delta(x-\frac{a}{2})\sin\left(\frac{n\pi}{a}x\right)dx = \frac{2\alpha}{a}\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right)$,

which is zero unless both m and n are odd — in which case it is $\pm 2\alpha/a$. So [6.14] says

$$E_n^2 = \sum_{\substack{m\neq n\\(odd)}} \left(\frac{2\alpha}{a}\right)^2 \frac{1}{(E_n^0 - E_m^0)}.$$ But [2.23] $E_n^0 = \frac{\pi^2\hbar^2}{2ma^2}n^2$, so

$$\boxed{E_n^2 = \begin{cases} 0, & \text{if } n \text{ is even;}\\ 2m\left(\frac{2\alpha}{\pi\hbar}\right)^2 \sum\limits_{\substack{m\neq n\\ odd}} \frac{1}{(n^2-m^2)}, & \text{if } n \text{ is odd.}\end{cases}}$$

To sum the series, note that $\frac{1}{(n^2-m^2)} = \frac{1}{2n}\left(\frac{1}{(m+n)} - \frac{1}{(m-n)}\right)$. Thus,

for $n=1$: $\sum = \frac{1}{2}\sum\limits_{3,5,7,\ldots}\left(\frac{1}{m+1} - \frac{1}{m-1}\right) = \frac{1}{2}\left(\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots - \frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}\cdots\right) = \frac{1}{2}\left(-\frac{1}{2}\right) = -\frac{1}{4}$;

for $n=3$: $\sum = \frac{1}{6}\sum\limits_{1,5,7,\ldots}\left(\frac{1}{m+3} - \frac{1}{m-3}\right) = \frac{1}{6}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{10}+\cdots + \frac{1}{2} - \frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}-\frac{1}{10}\cdots\right) = \frac{1}{6}\left(-\frac{1}{6}\right) = \frac{-1}{36}$.

In general, there is perfect cancellation except for the "missing" term $\frac{1}{2n}$ in the first sum, so the total is $\frac{1}{2n}\left(-\frac{1}{2n}\right) = -\frac{1}{(2n)^2}$. Therefore:

$$\boxed{E_n^2 = \begin{cases} 0, & \text{if } n \text{ is even;}\\ -2m\left(\alpha/\pi\hbar n\right)^2, & \text{if } n \text{ is odd.}\end{cases}}$$

(b) $H' = \frac{1}{2}\epsilon k x^2$; $\langle\psi_m^0|H'|\psi_n^0\rangle = \frac{1}{2}k\epsilon\langle m|x^2|n\rangle$. Following Problem [2.37]:

$$\langle m|x^2|n\rangle = -\frac{1}{2m\omega^2}\langle m|(a_+^2 - a_+a_- - a_-a_+ + a_-^2)|n\rangle$$

$$= -\frac{1}{2m\omega^2}\left[\sqrt{(n+1)(n+2)}\,\hbar\omega\langle m|n+2\rangle - n\hbar\omega\langle m|n\rangle - (n+1)\hbar\omega\langle m|n\rangle + \sqrt{n(n-1)}\,\hbar\omega\langle m|n-2\rangle\right]$$

So, for $m\neq n$, $\langle\psi_m^0|H'|\psi_n^0\rangle = \left(\frac{1}{2}k\epsilon\right)\left(-\frac{\hbar}{2m\omega}\right)\left[\sqrt{(n+1)(n+2)}\,\delta_{m,n+2} + \sqrt{n(n-1)}\,\delta_{m,n-2}\right]$.

$$\therefore\ E_n^2 = \left(\frac{\epsilon\hbar\omega}{4}\right)^2 \sum_{m\neq n}\frac{\left[\sqrt{(n+1)(n+2)}\,\delta_{m,n+2} + \sqrt{n(n-1)}\,\delta_{m,n-2}\right]^2}{(n+\frac{1}{2})\hbar\omega - (m+\frac{1}{2})\hbar\omega} = \frac{\epsilon^2\hbar\omega}{16}\sum_{m\neq n}\frac{[(n+1)(n+2)\delta_{m,n+2} + n(n-1)\delta_{m,n-2}]}{(n-m)}$$

$$= \frac{\epsilon^2\hbar\omega}{16}\left[\frac{(n+1)(n+2)}{n-(n+2)} + \frac{n(n-1)}{n-(n-2)}\right] = \frac{\epsilon^2\hbar\omega}{16}\left[-\frac{1}{2}(n+1)(n+2) + \frac{1}{2}n(n-1)\right]$$

$$= \frac{\epsilon^2\hbar\omega}{32}\left(-n^2 - 3n - 2 + n^2 - n\right) = \frac{\epsilon^2\hbar\omega}{32}(-4n-2) = \boxed{-\epsilon^2\frac{1}{8}\hbar\omega\left(n+\frac{1}{2}\right)}$$ (which agrees with the ϵ^2 term in the exact solution — Problem 15(a)).

PROBLEM 6.5 (a) $E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle = -qE\langle n|x|n\rangle = \boxed{0}$. (Problem 2.37.)

From [6.14] and Problem 3.50:

$$E_n^2 = (qE)^2\sum_{m\neq n}\frac{|\langle m|x|n\rangle|^2}{(n-m)\hbar\omega} = \frac{(qE)^2}{\hbar\omega}\frac{\hbar}{2m\omega}\sum_{m\neq n}\frac{\left[\sqrt{n+1}\,\delta_{m,n+1} - \sqrt{n}\,\delta_{m,n-1}\right]^2}{(n-m)}$$

$$E_n^2 = \frac{(qE)^2}{2m\omega^2}\sum_{m\neq n}\frac{[(n+1)\delta_{m,n+1}+n\,\delta_{m,n-1}]}{(n-m)} = \frac{(qE)^2}{2m\omega^2}\left[\frac{(n+1)}{n-(n+1)}+\frac{n}{n-(n-1)}\right] = \frac{(qE)^2}{2m\omega^2}[-(n+1)+n]$$

$$= \boxed{\frac{-(qE)^2}{2m\omega^2}}.$$

(b) $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \left(\frac{1}{2}m\omega^2x^2 - qEx\right)\psi = E\psi$. With the suggested change of variables,

$$\left(\tfrac{1}{2}m\omega^2x^2 - qEx\right) = \tfrac{1}{2}m\omega^2\left(x' + \left(\frac{qE}{m\omega^2}\right)\right)^2 - qE\left(x' + \left(\frac{qE}{m\omega^2}\right)\right) = \tfrac{1}{2}m\omega^2x'^2 + m\omega^2x'\frac{qE}{m\omega^2} + \tfrac{1}{2}m\omega^2\frac{(qE)^2}{m^2\omega^4}$$

$$-qEx' - (qE)^2/m\omega^2 = \tfrac{1}{2}m\omega^2x'^2 - \tfrac{1}{2}(qE)^2/m\omega^2.$$ So the Schrödinger equation says

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx'^2} + \tfrac{1}{2}m\omega^2x'^2\psi = \left(E + \tfrac{1}{2}(qE)^2/m\omega^2\right)\psi$$ — which is the Schrodinger equation for a simple harmonic oscillator in the variable x'. The constant on the right must therefore be $(n+\frac{1}{2})\hbar\omega$, and we conclude that

$$\boxed{E_n = (n+\tfrac{1}{2})\hbar\omega - \tfrac{1}{2}(qE)^2/m\omega^2}.$$ The subtracted term is exactly what we got in part (a) using perturbation theory. Evidently all the higher corrections (like the first-order correction) are zero, in this case.

Problem 6.6 (a) $\langle\psi_+^0|\psi_-^0\rangle = \langle(\alpha_+\psi_a^0 + \beta_+\psi_b^0)|(\alpha_-\psi_a^0 + \beta_-\psi_b^0)\rangle$

$$= \alpha_+^*\alpha_-\langle\psi_a^0|\psi_a^0\rangle + \alpha_+^*\beta_-\langle\psi_a^0|\psi_b^0\rangle + \beta_+^*\alpha_-\langle\psi_b^0|\psi_a^0\rangle + \beta_+^*\beta_-\langle\psi_b^0|\psi_b^0\rangle$$

$$= \alpha_+^*\alpha_- + \beta_+^*\beta_-.$$ But [6.21] $\Rightarrow$ $\beta_\pm = \alpha_\pm(E_\pm^1 - W_{aa})/W_{ab}$.

So $$\langle\psi_+^0|\psi_-^0\rangle = \alpha_+^*\alpha_-\left\{1 + \frac{(E_+^1-W_{aa})(E_-^1-W_{aa})}{W_{ab}^*W_{ab}}\right\} = \frac{\alpha_+^*\alpha_-}{|W_{ab}|^2}\left[|W_{ab}|^2 + (E_+^1-W_{aa})(E_-^1-W_{aa})\right].$$

The term in square brackets is:

$[\;] = E_+^1E_-^1 - W_{aa}(E_+^1+E_-^1) + |W_{ab}|^2 + W_{aa}^2$. But [6.26]: $E_\pm^1 = \frac{1}{2}\left[(W_{aa}+W_{bb}) \pm \sqrt{\;}\right]$, where $\sqrt{\;}$ is shorthand for the square root term. $E_+^1 + E_-^1 = W_{aa} + W_{bb}$, and

$$E_+^1E_-^1 = \tfrac{1}{4}\left[(W_{aa}+W_{bb})^2 - (\sqrt{\;})^2\right] = \tfrac{1}{4}\left[(W_{aa}+W_{bb})^2 - (W_{aa}-W_{bb})^2 - 4|W_{ab}|^2\right] = W_{aa}W_{bb} - |W_{ab}|^2.$$

So $[\;] = W_{aa}W_{bb} - |W_{ab}|^2 - W_{aa}(W_{aa}+W_{bb}) + |W_{ab}|^2 + W_{aa}^2 = 0$. $\therefore$ $\langle\psi_+^0|\psi_-^0\rangle = 0$. QED.

(b) $\langle\psi_+^0|H'|\psi_-^0\rangle = \alpha_+^*\alpha_-\langle\psi_a^0|H'|\psi_a^0\rangle + \alpha_+^*\beta_-\langle\psi_a^0|H'|\psi_b^0\rangle + \beta_+^*\alpha_-\langle\psi_b^0|H'|\psi_a^0\rangle + \beta_+^*\beta_-\langle\psi_b^0|H'|\psi_b^0\rangle$

$$= \alpha_+^*\alpha_-W_{aa} + \alpha_+^*\beta_-W_{ab} + \beta_+^*\alpha_-W_{ba} + \beta_+^*\beta_-W_{bb} = \alpha_+^*\alpha_-\left\{W_{aa} + W_{ab}\frac{(E_-^1-W_{aa})}{W_{ab}} + W_{ba}\frac{(E_+^1-W_{aa})}{W_{ab}^*}\right.$$

$$+W_{bb}\frac{(E_+^1-W_{aa})}{W_{ab}^*}\frac{(E_-^1-W_{aa})}{W_{ab}}\Big\} = \alpha_+^*\alpha_-\Big\{W_{aa}+E_-^1-W_{aa}+E_+^1-W_{aa}+W_{bb}\frac{(E_+^1-W_{aa})(E_-^1-W_{aa})}{|W_{ab}|^2}\Big\}.$$

But we know from (a) that $\frac{(E_+^1-W_{aa})(E_-^1-W_{aa})}{|W_{ab}|^2} = -1$, so

$$\langle\psi_+^0|H'|\psi_-^0\rangle = \alpha_+^*\alpha_-\left[E_-^1+E_+^1-W_{aa}-W_{bb}\right] = 0. \quad \text{QED}.$$

(c) $\langle\psi_\pm^0|H'|\psi_\pm^0\rangle = \alpha_\pm^*\alpha_\pm\langle\psi_a^0|H'|\psi_a^0\rangle + \alpha_\pm^*\beta_\pm\langle\psi_a^0|H'|\psi_b^0\rangle + \beta_\pm^*\alpha_\pm\langle\psi_b^0|H'|\psi_a^0\rangle + \beta_\pm^*\beta_\pm\langle\psi_b^0|H'|\psi_b^0\rangle$

$$= |\alpha_\pm|^2\left\{W_{aa}+W_{ab}\frac{(E_\pm^1-W_{aa})}{W_{ab}}\right\} + |\beta_\pm|^2\left\{W_{ba}\frac{(E_\pm^1-W_{bb})}{W_{ba}}+W_{bb}\right\}$$ (this time I used [6.23] to express α in terms of β, in the third term).

$$\therefore \langle\psi_\pm^0|H'|\psi_\pm^0\rangle = |\alpha_\pm|^2(E_\pm^1)+|\beta_\pm|^2(E_\pm^1) = \left(|\alpha_\pm|^2+|\beta_\pm|^2\right)E_\pm^1 = E_\pm^1. \quad \text{QED}$$

PROBLEM 6.7 (a) (with slight change in notation this is precisely the solution obtained in Problem 2.43.)

(b) With $a \to n$, $b \to -n$, we have:

$$W_{aa} = W_{bb} = -\frac{V_0}{L}\int_{-L/2}^{L/2} e^{-x^2/a^2}dx \cong -\frac{V_0}{L}\int_{-\infty}^{\infty} e^{-x^2/a^2}dx = -\frac{V_0}{L}a\sqrt{\pi}.$$

$$W_{ab} = -\frac{V_0}{L}\int_{-L/2}^{L/2} e^{-x^2/a^2}e^{-4\pi n i x/L}dx \cong -\frac{V_0}{L}\int_{-\infty}^{\infty} e^{-(x^2/a^2+4\pi n i x/L)}dx$$

$$= -\frac{V_0}{L}a\sqrt{\pi}\, e^{-(2\pi n/L)^2 a}.$$ (We did this integral in Problem 2.22.)

In this case $W_{aa} = W_{bb}$, and W_{ab} is real, so $[6.26] \Rightarrow E_\pm^1 = W_{aa} \pm |W_{ab}|$, or

$$E_\pm^1 = \boxed{-\sqrt{\pi}\,\frac{V_0 a}{L}\left(1 \mp e^{-(2\pi n/L)^2 a}\right)}.$$

(c) $[6.21] \Rightarrow \beta = \alpha\frac{(E^1-W_{aa})}{W_{ab}} = \alpha\left\{\frac{\pm\sqrt{\pi}\,\frac{V_0 a}{L}e^{-(2\pi n/L)^2 a}}{-\sqrt{\pi}\,\frac{V_0 a}{L}e^{-(2\pi n/L)^2 a}}\right\} = \mp\alpha$. Evidently the "good" linear combinations are: $\psi_+ = \alpha\psi_n - \alpha\psi_{-n} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{L}}\left[e^{i2\pi n x/L} - e^{-i2\pi n x/L}\right] = \boxed{\sqrt{\frac{2}{L}}\,i\sin\left(\frac{2\pi n x}{L}\right)}$

and $\psi_- = \alpha\psi_n + \alpha\psi_{-n} = \boxed{\sqrt{\frac{2}{L}}\cos\left(\frac{2\pi n x}{L}\right)}$. Using [6.9], we have:

$$\left.\begin{aligned} E_+^1 &= \langle\psi_+|H'|\psi_+\rangle = \frac{2}{L}(-V_0)\int_{-L/2}^{L/2} e^{-x^2/a^2}\sin^2\left(\frac{2\pi n x}{L}\right)dx \\ E_-^1 &= \langle\psi_-|H'|\psi_-\rangle = \frac{2}{L}(-V_0)\int_{-L/2}^{L/2} e^{-x^2/a^2}\cos^2\left(\frac{2\pi n x}{L}\right)dx \end{aligned}\right\}$$ Use $\left\{\begin{aligned}\sin^2\theta &= \tfrac{1}{2}(1-\cos 2\theta)\\ \cos^2\theta &= \tfrac{1}{2}(1+\cos 2\theta)\end{aligned}\right\}$ to obtain:

$$E_\pm^1 \cong -\frac{V_0}{L}\int_{-\infty}^{\infty} e^{-x^2/a^2}\left(1 \mp \cos\left(\frac{4\pi n x}{L}\right)\right)dx.$$

$$E^1_\pm = -\frac{V_0}{L}\left\{\int_{-\infty}^{\infty} e^{-x^2/a^2}dx \mp \int_{-\infty}^{\infty} e^{-x^2/a^2}\cos\left(\frac{4\pi n x}{L}\right)dx\right\}$$

$$= -\frac{V_0}{L}\left\{\sqrt{\pi}\,a \mp a\sqrt{\pi}\,e^{-(2\pi n/L)^2 a}\right\} = -\sqrt{\pi}\,\frac{V_0 a}{L}\left(1 \mp e^{-(2\pi n/L)^2 a^2}\right) \text{ — same as (b).}$$

(d) $Af(x) = f(-x)$ (the parity operator). The eigenstates are <u>even</u> functions (with eigenvalue $+1$) and <u>odd</u> functions (with eigenvalue -1). The linear combinations we found in (c) are precisely the odd and even linear combinations of ψ_n and ψ_{-n}.

Problem 6.8 <u>Ground state</u> is nondegenerate; [6.9] and [6.30] $\Rightarrow$

$$E^1 = \left(\frac{2}{a}\right)^3 a^3 V_0 \iiint_0^a \sin^2\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{\pi}{a}y\right)\sin^2\left(\frac{\pi}{a}z\right)\delta\left(x-\frac{a}{4}\right)\delta\left(y-\frac{a}{2}\right)\delta\left(z-\frac{3a}{4}\right)dx\,dy\,dz$$

$$= 8V_0\sin^2\left(\frac{\pi}{4}\right)\sin^2\left(\frac{\pi}{2}\right)\sin^2\left(\frac{3\pi}{4}\right) = 8V_0\left(\frac{1}{2}\right)(1)\left(\frac{1}{2}\right) = \boxed{2V_0}.$$

<u>First excited states</u> [6.33]:

$$W_{aa} = 8V_0\iiint \sin^2\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{\pi}{a}y\right)\sin^2\left(\frac{2\pi}{a}z\right)\delta\left(x-\frac{a}{4}\right)\delta\left(y-\frac{a}{2}\right)\delta\left(z-\frac{3a}{4}\right)dx\,dy\,dz = 8V_0\left(\frac{1}{2}\right)(1)(1) = 4V_0.$$

$$W_{bb} = 8V_0\iiint \sin^2\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{2\pi}{a}y\right)\sin^2\left(\frac{\pi}{a}z\right)\delta\left(x-\frac{a}{4}\right)\delta\left(y-\frac{a}{2}\right)\delta\left(z-\frac{3a}{4}\right)dx\,dy\,dz = 8V_0\left(\frac{1}{2}\right)(0)\left(\frac{1}{2}\right) = 0.$$

$$W_{cc} = 8V_0\iiint \sin^2\left(\frac{2\pi}{a}x\right)\sin^2\left(\frac{\pi}{a}y\right)\sin^2\left(\frac{\pi}{a}z\right)\delta\left(x-\frac{a}{4}\right)\delta\left(y-\frac{a}{2}\right)\delta\left(z-\frac{3a}{4}\right)dx\,dy\,dz = 8V_0(1)(1)\left(\frac{1}{2}\right) = 4V_0.$$

$$W_{ab} = 8V_0\sin^2\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{2}\right)\sin(\pi)\sin\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{4}\right) = 0.$$

$$W_{ac} = 8V_0\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{2}\right)\sin^2\left(\frac{\pi}{2}\right)\sin\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{4}\right) = 8V_0\left(\frac{1}{\sqrt{2}}\right)(1)(1)(-1)\left(\frac{1}{\sqrt{2}}\right) = -4V_0$$

$$W_{bc} = 8V_0\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{2}\right)\sin(\pi)\sin\left(\frac{\pi}{2}\right)\sin^2\left(\frac{3\pi}{4}\right) = 0.$$

$$W = 4V_0\begin{pmatrix}1 & 0 & -1\\ 0 & 0 & 0\\ -1 & 0 & 1\end{pmatrix} = 4V_0 D.$$

$$\det(D-\lambda) = \begin{vmatrix}(1-\lambda) & 0 & -1\\ 0 & -\lambda & 0\\ -1 & 0 & (1-\lambda)\end{vmatrix} = -\lambda(1-\lambda)^2 + \lambda = 0 \Rightarrow \lambda = 0 \text{ or } (1-\lambda)^2 = 1 \Rightarrow 1-\lambda = \pm 1 \Rightarrow \lambda = 0, \lambda = 2.$$

So the first-order corrections to the energies are $\boxed{0, 0, 8V_0}$.

Problem 6.9 (a) $\boxed{\chi_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}}$, with eigenvalue $\boxed{V_0}$; $\boxed{\chi_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}}$, with eigenvalue $\boxed{V_0}$;

$\boxed{\chi_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}}$, with eigenvalue $\boxed{2V_0}$.

(b) Characteristic equation:
$$\det(H-\lambda) = \begin{vmatrix}[V_0(1-\epsilon)-\lambda] & 0 & 0\\ 0 & [V_0-\lambda] & \epsilon V_0\\ 0 & \epsilon V_0 & [2V_0-\lambda]\end{vmatrix} = 0$$

$$[V_0(1-\epsilon)-\lambda]\left\{(V_0-\lambda)(2V_0-\lambda) - (\epsilon V_0)^2\right\} = 0 \Rightarrow \boxed{\lambda_1 = V_0(1-\epsilon)}.$$

$$(V_0-\lambda)(2V_0-\lambda)-(\epsilon V_0)^2=0 \Rightarrow \lambda^2-3V_0\lambda+(2V_0^2-\epsilon^2V_0^2)=0 \Rightarrow$$

$$\lambda = \frac{3V_0 \pm \sqrt{9V_0^2-4(2V_0^2-\epsilon^2V_0^2)}}{2} = \frac{V_0}{2}\left[3\pm\sqrt{1+4\epsilon^2}\right] \cong \frac{V_0}{2}\left[3\pm(1+2\epsilon^2)\right].$$

$$\boxed{\lambda_2 = \frac{V_0}{2}\left(3-\sqrt{1+4\epsilon^2}\right) \cong V_0(1-\epsilon^2)} ; \quad \boxed{\lambda_3 = \frac{V_0}{2}\left(3+\sqrt{1+4\epsilon^2}\right) \cong V_0(2+\epsilon^2)}.$$

(c) $E_3^1 = \langle\chi_3|H'|\chi_3\rangle$; $H' = \epsilon V_0\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}$; $E_3^1 = \epsilon V_0(0\;0\;1)\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \epsilon V_0(0\,0\,1)\begin{pmatrix}0\\1\\0\end{pmatrix}=0.$

$\boxed{E_3^1=0}$ (no first-order correction).

$$E_3^2 = \sum_{m=1,2}\frac{|\langle\chi_m|H'|\chi_3\rangle|^2}{E_3^0-E_m^0}.\quad \begin{cases}\langle\chi_1|H'|\chi_3\rangle = \epsilon V_0(1\,0\,0)\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \epsilon V_0(1\,0\,0)\begin{pmatrix}0\\1\\0\end{pmatrix}=0.\\ \langle\chi_2|H'|\chi_3\rangle = \epsilon V_0(0\,1\,0)\begin{pmatrix}0\\1\\0\end{pmatrix} = \epsilon V_0.\end{cases}$$

$E_3^0-E_2^0 = 2V_0-V_0=V_0$. So $E_3^2 = \dfrac{(\epsilon V_0)^2}{V_0} = \boxed{\epsilon^2V_0}$. Through second-order, then,

$E_3 = E_3^0+E_3^1+E_3^2 = 2V_0+0+\epsilon^2V_0 = V_0(2+\epsilon^2)$ (same as we got for λ_3 in (b)).

(d) $W_{aa} = \langle\chi_1|H'|\chi_1\rangle = \epsilon V_0(1\,0\,0)\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \epsilon V_0(1\,0\,0)\begin{pmatrix}-1\\0\\0\end{pmatrix} = -\epsilon V_0.$

$W_{bb} = \langle\chi_2|H'|\chi_2\rangle = \epsilon V_0(0\,1\,0)\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \epsilon V_0(0\,1\,0)\begin{pmatrix}0\\0\\1\end{pmatrix}=0.$ Plug into [6.26].

$W_{ab} = \langle\chi_1|H'|\chi_2\rangle = \epsilon V_0(1\,0\,0)\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \epsilon V_0(1\,0\,0)\begin{pmatrix}0\\0\\1\end{pmatrix}=0.$

$$E_\pm^1 = \frac{1}{2}\left[-\epsilon V_0+0\pm\sqrt{\epsilon^2V_0^2+0}\right] = \frac{1}{2}(-\epsilon V_0\pm\epsilon V_0) = \begin{Bmatrix}0\\-\epsilon V_0\end{Bmatrix}.$$ To first order, then,

$\boxed{E_1=V_0-\epsilon V_0}$, $\boxed{E_2=V_0}$ — and these are consistent (to first order in ϵ) with what we got in (b).

Problem 6.10 (a) From [4.70]: $E_n = -\left[\dfrac{m}{2\hbar^2}\left(\dfrac{e^2}{4\pi\epsilon_0}\right)^2\right]\dfrac{1}{n^2} = -\dfrac{1}{2}mc^2\left(\dfrac{1}{\hbar c}\dfrac{e^2}{4\pi\epsilon_0}\right)^2\dfrac{1}{n^2} = \boxed{-\dfrac{\alpha^2mc^2}{2n^2}}.$

(b) I have a wonderful solution — unfortunately, there isn't enough room on this page for the proof.

Problem 6.11 [4.191] $\Rightarrow \langle V\rangle = 2E_n$, for hydrogen. $V = -\dfrac{e^2}{4\pi\epsilon_0}\dfrac{1}{r}$; $E_n = -\left[\dfrac{m}{2\hbar^2}\left(\dfrac{e^2}{4\pi\epsilon_0}\right)^2\right]\dfrac{1}{n^2}.$

$$\therefore -\frac{e^2}{4\pi\epsilon_0}\left\langle\frac{1}{r}\right\rangle = -2\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2} \Rightarrow \left\langle\frac{1}{r}\right\rangle = \left(\frac{me^2}{4\pi\epsilon_0\hbar^2}\right)\frac{1}{n^2} = \frac{1}{an^2}$$ (eq. [4.72]). QED.

Problem 6.12 In Problem 4.43, we found (for $n=3, l=2, m=1$) that $\langle r^s\rangle = \dfrac{(s+6)!}{6!}\left(\dfrac{3a}{2}\right)^s$.

$\underline{s=0}$: $\langle 1\rangle = \dfrac{6!}{6!}(1) = \boxed{1}$ ✓; $\underline{s=-1}$: $\left\langle\dfrac{1}{r}\right\rangle = \dfrac{5!}{6!}\left(\dfrac{3a}{2}\right)^{-1} = \dfrac{1}{6}\cdot\dfrac{2}{3a} = \boxed{\dfrac{1}{9a}}$ ([6.54] says $\dfrac{1}{3^2a} = \dfrac{1}{9a}$ ✓);

$\underline{s=-2}$: $\left\langle\dfrac{1}{r^2}\right\rangle = \dfrac{4!}{6!}\left(\dfrac{3a}{2}\right)^{-2} = \dfrac{1}{6\cdot5}\cdot\dfrac{4}{9a^2} = \boxed{\dfrac{2}{135a^2}}$ ([6.55] says $\dfrac{1}{(\frac{5}{2})\cdot27\cdot a^2} = \dfrac{2}{135a^2}$ ✓);

$\underline{s=-3}$: $\left\langle \frac{1}{r^3} \right\rangle = \frac{3!}{6!}\left(\frac{3a}{2}\right)^{-3} = \frac{1}{6\cdot5\cdot4}\cdot\frac{8}{27a^3} = \boxed{\frac{1}{405a^3}}$ ([6.63] says $\frac{1}{2(3/2)3\cdot27a^3} = \frac{1}{405a^3}$ ✓).

For $s=-7$ (or smaller) the integral does not converge: $\left\langle \frac{1}{r^7}\right\rangle = \infty$, in this state; this is reflected in the fact that $(-1)! = \infty$.

PROBLEM 6.13 [6.52] $\Rightarrow$ $E_r^1 = -\frac{1}{2mc^2}\left[E^2 - 2E\langle V\rangle + \langle V^2\rangle\right]$. Here $E = (n+\frac{1}{2})\hbar\omega$, $V = \frac{1}{2}m\omega^2x^2$.

$\therefore E_r^1 = -\frac{1}{2mc^2}\left[(n+\frac{1}{2})^2\hbar^2\omega^2 - 2(n+\frac{1}{2})\hbar\omega\,\frac{1}{2}m\omega^2\langle x^2\rangle + \frac{1}{4}m^2\omega^4\langle x^4\rangle\right]$. But Problem 2.37 $\Rightarrow$ $\langle x^2\rangle = (n+\frac{1}{2})\frac{\hbar}{m\omega}$,

so $E_r^1 = -\frac{1}{2mc^2}\left[(n+\frac{1}{2})^2\hbar^2\omega^2 - (n+\frac{1}{2})^2\hbar^2\omega^2 + \frac{1}{4}m^2\omega^4\langle x^4\rangle\right] = -\frac{m\omega^4}{8c^2}\langle x^4\rangle$. Again from Problem 2.37:

$$x^4 = \frac{1}{4m^2\omega^4}\left(a_+^2 - a_+a_- - a_-a_+ + a_-^2\right)\left(a_+^2 - a_+a_- - a_-a_+ + a_-^2\right), \text{ so}$$

$$\langle x^4\rangle = \frac{1}{4m^2\omega^4}\langle n|\left(a_+^2a_-^2 + a_+a_-a_+a_- + a_+a_-a_-a_+ + a_-a_+a_+a_- + a_-a_+a_-a_+ + a_-^2a_+^2\right)|n\rangle$$ (note that only terms with equal numbers of raising and lowering operators will survive).

$$\langle x^4\rangle = \frac{1}{4m^2\omega^4}\langle n|\Big\{a_+^2\left(\sqrt{n(n-1)}\,\hbar\omega|n-2\rangle\right) + a_+a_-\left(n\hbar\omega|n\rangle\right) + a_+a_-\left((n+1)\hbar\omega|n\rangle\right) + a_-a_+\left(n\hbar\omega|n\rangle\right)$$
$$+ a_-a_+\left((n+1)\hbar\omega|n\rangle\right) + a_-^2\left(\sqrt{(n+1)(n+2)}\,\hbar\omega|n+2\rangle\right)\Big\}$$

$$= \frac{\hbar}{4m^2\omega^3}\langle n|\Big\{\sqrt{n(n-1)}\left(\sqrt{n(n-1)}\,|n\rangle\right)\hbar\omega + n\left(n\hbar\omega|n\rangle\right) + (n+1)\left(n\hbar\omega|n\rangle\right) + n\left((n+1)\hbar\omega|n\rangle\right)$$
$$+ (n+1)\left((n+1)\hbar\omega|n\rangle\right) + \sqrt{(n+1)(n+2)}\left(\sqrt{(n+1)(n+2)}\,\hbar\omega|n\rangle\right)\Big\}$$

$$= \frac{\hbar^2}{4m^2\omega^2}\left\{n(n-1) + n^2 + (n+1)n + n(n+1) + (n+1)^2 + (n+1)(n+2)\right\}$$

$$= \left(\frac{\hbar}{2m\omega}\right)^2\left(n^2 - n + n^2 + n^2 + n + n^2 + n + n^2 + 2n + 1 + n^2 + 3n + 2\right) = \left(\frac{\hbar}{2m\omega}\right)^2\left(6n^2 + 6n + 3\right)$$

$$\therefore E_r^1 = -\frac{m\omega^4}{8c^2}\cdot\frac{\hbar^2}{4m^2\omega^2}\cdot 3\left(2n^2 + 2n + 1\right) = \boxed{-\frac{3}{32}\left(\frac{\hbar^2\omega^2}{mc^2}\right)\left(2n^2 + 2n + 1\right)}.$$

PROBLEM 6.14 (a) $[\vec{L}\cdot\vec{S}, L_x] = [L_xS_x + L_yS_y + L_zS_z, L_x] = S_x[L_x, L_x] + S_y[L_y, L_x] + S_z[L_z, L_x]$
$= S_x(0) + S_y(-i\hbar L_z) + S_z(i\hbar L_y) = i\hbar(L_yS_z - L_zS_y) = i\hbar(\vec{L}\times\vec{S})_x$.

Same goes for the other two components, so $\boxed{[\vec{L}\cdot\vec{S}, \vec{L}] = i\hbar(\vec{L}\times\vec{S})}$. (b) $[\vec{L}\cdot\vec{S}, \vec{S}]$ is identical, only with $\vec{L}\leftrightarrow\vec{S}$: $\boxed{[\vec{L}\cdot\vec{S}, \vec{S}] = i\hbar(\vec{S}\times\vec{L})}$. (c) $[\vec{L}\cdot\vec{S}, \vec{J}] = [\vec{L}\cdot\vec{S}, \vec{L}] + [\vec{L}\cdot\vec{S}, \vec{S}] = i\hbar(\vec{L}\times\vec{S} + \vec{S}\times\vec{L}) = \boxed{0}$.

(d) L^2 commutes with all components of $\vec{L}$ (and $\vec{S}$), so $\boxed{[\vec{L}\cdot\vec{S}, L^2] = 0}$. (e) Likewise $\boxed{[\vec{L}\cdot\vec{S}, S^2] = 0}$.

(f) $[\vec{L}\cdot\vec{S}, J^2] = [\vec{L}\cdot\vec{S}, L^2] + [\vec{L}\cdot\vec{S}, S^2] + 2[\vec{L}\cdot\vec{S}, \vec{L}\cdot\vec{S}] = 0 + 0 + 0 \Rightarrow \boxed{[\vec{L}\cdot\vec{S}, J^2] = 0}$.

Problem 6.15 With the plus sign, $j = l + 1/2$ $(l = j - 1/2)$: [6.56] $\Rightarrow E_r^1 = -\frac{E_n^2}{2mc^2}\left(\frac{4n}{j} - 3\right)$.

$$[6.64] \Rightarrow E_{so}^1 = \frac{E_n^2}{mc^2}\,\frac{n\left[j(j+1) - (j-\frac{1}{2})(j+\frac{1}{2}) - \frac{3}{4}\right]}{(j-\frac{1}{2})\, j\, (j+\frac{1}{2})} = \frac{E_n^2}{mc^2}\,\frac{n\left(j^2 + j - j^2 + \frac{1}{4} - \frac{3}{4}\right)}{(j-\frac{1}{2})j(j+\frac{1}{2})} = \frac{E_n^2}{mc^2}\,\frac{n}{j(j+\frac{1}{2})}.$$

$$E_{fs}^1 = E_r^1 + E_{so}^1 = \frac{E_n^2}{2mc^2}\left(-\frac{4n}{j} + 3 + \frac{2n}{j(j+\frac{1}{2})}\right) = \frac{E_n^2}{2mc^2}\left(3 + \frac{2n}{j(j+1/2)}\left(1 - 2(j+1/2)\right)\right) = \frac{E_n^2}{2mc^2}\left(3 - \frac{4n}{j+1/2}\right).$$

With the minus sign, $j = l - 1/2$ $(l = j + 1/2)$: [6.56] $\Rightarrow E_r^1 = -\frac{E_n^2}{2mc^2}\left(\frac{4n}{j+1} - 3\right)$.

$$[6.64] \Rightarrow E_{so}^1 = \frac{E_n^2}{mc^2}\,\frac{n\left[j(j+1) - (j+\frac{1}{2})(j+\frac{3}{2}) - \frac{3}{4}\right]}{(j+\frac{1}{2})(j+1)(j+\frac{3}{2})} = \frac{E_n^2}{mc^2}\,\frac{n\left(j^2 + j - j^2 - 2j - \frac{3}{4} - \frac{3}{4}\right)}{(j+\frac{1}{2})(j+1)(j+3/2)} = \frac{E_n^2}{mc^2}\,\frac{-n}{(j+\frac{1}{2})(j+1)}.$$

$$E_{fs}^1 = \frac{E_n^2}{2mc^2}\left(\frac{-4n}{j+1} - 3 - \frac{2n}{(j+\frac{1}{2})(j+1)}\right) = \frac{E_n^2}{2mc^2}\left(3 - \frac{2n}{(j+1)(j+\frac{1}{2})}\left(1 + 2(j+\tfrac{1}{2})\right)\right) = \frac{E_n^2}{2mc^2}\left(3 - \frac{4n}{j+1/2}\right).$$

For both signs, then, $E_{fs}^1 = \frac{E_n^2}{2mc^2}\left(3 - \frac{4n}{(j+1/2)}\right)$. QED.

Problem 6.16 $E_3^0 - E_2^0 = h\nu = \frac{2\pi\hbar c}{\lambda} = E_1\left(\frac{1}{9} - \frac{1}{4}\right) = -\frac{5}{36}E_1 \Rightarrow \lambda = -\frac{36}{5}\,\frac{2\pi\hbar c}{E_1}$. $E_1 = -13.6\text{ eV}$,

$\hbar c = 1.97\times10^{-11}$ MeV·cm, so $\lambda = \frac{36}{5}\,\frac{(2\pi)(1.97\times10^{-11}\times10^6\text{ eV}\cdot\text{cm})}{(13.6\text{ eV})} = 6.55\times10^{-5}\text{cm} = \boxed{655\text{ nm}}$.

$\nu = \frac{c}{\lambda} = \frac{3.00\times10^8\text{ m/s}}{6.55\times10^{-7}\text{ m}} = \boxed{4.58\times10^{14}\text{Hz}}$. [6.65] $\Rightarrow E_{fs}^1 = \frac{E_n^2}{2mc^2}\left(3 - \frac{4n}{j+1/2}\right)$:

For $n=2$: $l=0$ or $l=1$, so $j = \frac{1}{2}$ or $\frac{3}{2}$. Thus $n=2$ splits into two levels:

$j = 1/2$: $E_2^1 = \frac{E_2^2}{2mc^2}\left(3 - \frac{8}{1}\right) = -\frac{5}{2}\frac{E_2^2}{mc^2} = -\frac{5}{2}\left(\frac{1}{4}\right)^2\frac{E_1^2}{mc^2} = -\frac{5}{32}\frac{(13.6\text{eV})^2}{(.511\times10^6\text{eV})} = -5.66\times10^{-5}\text{eV}$.

$j = 3/2$: $E_2^1 = \frac{E_2^2}{2mc^2}\left(3 - \frac{8}{2}\right) = -\frac{1}{2}\frac{E_2^2}{mc^2} = -\frac{1}{32}(3.62\times10^{-4}\text{eV}) = -1.13\times10^{-5}\text{eV}$.

For $n=3$: $l = 0, 1,$ or 2, so $j = \frac{1}{2}, \frac{3}{2},$ or $\frac{5}{2}$. Thus $n=3$ splits into three levels:

$j = 1/2$: $E_3^1 = \frac{E_3^2}{2mc^2}\left(3 - \frac{12}{1}\right) = -9\frac{E_3^2}{2mc^2} = -\frac{9}{2}\cdot\frac{1}{9^2}\frac{E_1^2}{mc^2} = -\frac{1}{18}(3.62\times10^{-4}\text{eV}) = -2.01\times10^{-5}\text{eV}$.

$j = 3/2$: $E_3^1 = \frac{E_3^2}{2mc^2}\left(3 - \frac{12}{2}\right) = -\frac{3}{2}\frac{E_3^2}{mc^2} = -\frac{1}{54}(3.62\times10^{-4}\text{eV}) = -0.67\times10^{-5}\text{eV}$.

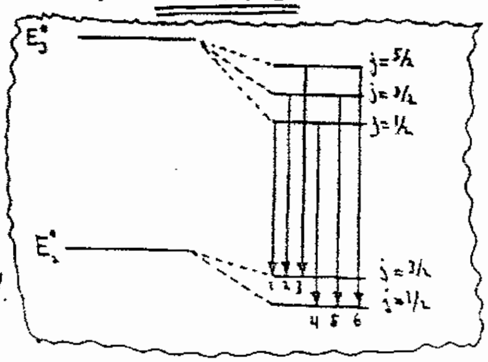
$j = 5/2$: $E_3^1 = \frac{E_3^2}{2mc^2}\left(3 - \frac{12}{3}\right) = -\frac{1}{2}\frac{E_3^2}{mc^2} = -\frac{1}{162}(3.62\times10^{-4}\text{eV}) = -0.22\times10^{-5}\text{eV}$.

There are six transitions here; their energies are $(E_3^0 + E_3^1) - (E_2^0 + E_2^1) = (E_3^0 - E_2^0) + \Delta E$, where

$\Delta E \equiv E_3^1 - E_2^1$. Let $\beta \equiv E_1^2/mc^2 = 3.62\times10^{-4}\text{eV}$.

Then: ① $\frac{1}{2}\to\frac{3}{2}$: $\Delta E = \left[\left(-\frac{1}{18}\right) - \left(-\frac{1}{32}\right)\right]\beta = -\frac{7}{288}\beta = -8.80\times10^{-6}\text{eV}$.

② $\frac{3}{2}\to\frac{3}{2}$: $\Delta E = \left[\left(-\frac{1}{54}\right) - \left(-\frac{1}{32}\right)\right]\beta = \frac{11}{864}\beta = 4.61\times10^{-6}\text{eV}$.



③ $\frac{5}{2}\to\frac{3}{2}$: $\Delta E = \left[-\frac{1}{162}+\frac{1}{32}\right]\beta = \frac{65}{2592}\beta = \underline{\underline{9.08\times10^{-6}\,eV}}.$

④ $\frac{1}{2}\to\frac{1}{2}$: $\Delta E = \left[\frac{5}{32}-\frac{1}{18}\right]\beta = \frac{29}{288}\beta = \underline{\underline{36.45\times10^{-6}\,eV}}.$

⑤ $\frac{3}{2}\to\frac{1}{2}$: $\Delta E = \left[-\frac{1}{54}+\frac{5}{32}\right]\beta = \frac{119}{864}\beta = \underline{\underline{49.86\times10^{-6}\,eV}}.$

⑥ $\frac{5}{2}\to\frac{1}{2}$: $\Delta E = \left[-\frac{1}{162}+\frac{5}{32}\right]\beta = \frac{389}{2592}\beta = \underline{\underline{54.33\times10^{-6}\,eV}}.$

Conclusion: There are six lines; one of them ($\frac{1}{2}\to\frac{3}{2}$) has a frequency less than the unperturbed line, the other five have (slightly) higher frequencies. In order, they are: #2: $\frac{3}{2}\to\frac{3}{2}$; #3: $\frac{5}{2}\to\frac{3}{2}$; #4: $\frac{1}{2}\to\frac{1}{2}$; #5: $\frac{3}{2}\to\frac{1}{2}$; #6: $\frac{5}{2}\to\frac{1}{2}$. The frequency spacings are:

$\nu_2-\nu_1 = \frac{\Delta E_2-\Delta E_1}{2\pi\hbar} =$	3.23×10^{9} Hz	$\nu_5-\nu_4 = 3.23\times10^{9}$ Hz
$\nu_3-\nu_2 = \frac{\Delta E_3-\Delta E_2}{2\pi\hbar} =$	1.08×10^{9} Hz	$\nu_6-\nu_5 = 1.08\times10^{9}$ Hz
$\nu_4-\nu_3 = \frac{\Delta E_4-\Delta E_3}{2\pi\hbar} =$	6.60×10^{9} Hz	

Problem 6.17 $\sqrt{(j+1/2)^2-\alpha^2} = (j+1/2)\sqrt{1-(\alpha/(j+1/2))^2} \cong (j+1/2)\left[1-\frac{1}{2}\left(\frac{\alpha}{j+1/2}\right)^2\right] = (j+\frac{1}{2}) - \frac{\alpha^2}{2(j+1/2)}.$

$$\therefore \frac{\alpha}{n-(j+1/2)+\sqrt{(j+1/2)^2-\alpha^2}} \cong \frac{\alpha}{n-(j+1/2)+(j+\frac{1}{2})-\frac{\alpha^2}{2(j+1/2)}} = \frac{\alpha}{n-\frac{\alpha^2}{2(j+1/2)}} = \frac{\alpha}{n\left[1-\frac{\alpha^2}{2n(j+1/2)}\right]}$$

$$\cong \frac{\alpha}{n}\left[1+\frac{\alpha^2}{2n(j+1/2)}\right].$$

$$\left[1+\left(\frac{\alpha}{n-(j+1/2)+\sqrt{(j+1/2)^2-\alpha^2}}\right)^2\right]^{-1/2} \cong \left[1+\frac{\alpha^2}{n^2}\left(1+\frac{\alpha^2}{n(j+1/2)}\right)\right]^{-1/2} \cong 1-\frac{1}{2}\frac{\alpha^2}{n^2}\left(1+\frac{\alpha^2}{n(j+1/2)}\right)+\frac{3}{8}\frac{\alpha^4}{n^4}$$

$$= 1-\frac{\alpha^2}{2n^2}+\frac{\alpha^4}{2n^4}\left(\frac{-n}{j+1/2}+\frac{3}{4}\right).$$

$$\therefore E_{nj} \cong mc^2\left\{1-\frac{\alpha^2}{2n^2}+\frac{\alpha^4}{2n^4}\left(\frac{-n}{j+1/2}+\frac{3}{4}\right)-1\right\} = \underbrace{-\frac{\alpha^2mc^2}{2n^2}}_{E_n\ \text{(Problem 6.10)}}\left[1+\frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2}-\frac{3}{4}\right)\right]$$

$$= -\frac{13.6\,eV}{n^2}\left[1+\frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2}-\frac{3}{4}\right)\right],$$ confirming [6.66].

Problem 6.18 [6.58] $\Rightarrow \vec{B} = \frac{1}{4\pi\epsilon_0}\frac{e}{mc^2r^3}\vec{L}$. Say $L=\hbar$, $r=a$; then

$$B = \frac{1}{4\pi\epsilon_0}\frac{e\hbar}{mc^2a^3} = \frac{(1.60\times10^{-19}C)(1.05\times10^{-34}J\cdot s)}{(4\pi)(8.9\times10^{-12}\frac{C^2}{Nm^2})(9.1\times10^{-31}kg)(3\times10^8 m/s)^2(0.53\times10^{-10}m)^3} = \boxed{12\,T}.$$

So a "strong" Zeeman field is $B_{ext} \gg 10\,T$, and a "weak" one is $B_{ext} \ll 10\,T$. (Incidentally, the earth's field ($10^{-4}T$) is definitely weak.)

PROBLEM 6.19 For $n=2$, $\ell=0$ ($j=1/2$) or $\ell=1$ ($j=1/2$ or $3/2$). The eight states are:

$$\left.\begin{array}{l}|1\rangle = |2\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle \\ |2\rangle = |2\,0\,\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle\end{array}\right\} g_J = \left[1+\frac{(1/2)(3/2)+(3/4)}{2(1/2)(3/2)}\right] = 1+\frac{3/2}{3/2} = 2$$

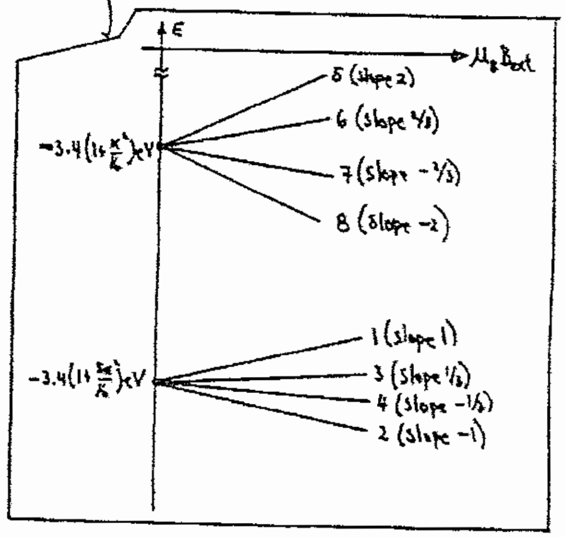
$$\left.\begin{array}{l}|3\rangle = |2\,1\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle \\ |4\rangle = |2\,1\,\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle\end{array}\right\} g_J = \left[1+\frac{(1/2)(3/2)-(1)(2)+(3/4)}{2(1/2)(3/2)}\right] = 1+\frac{-1/2}{3/2} = \frac{2}{3}$$

$$\left.\right\} E_{nj} = -\frac{13.6\text{eV}}{4}\left[1+\frac{\alpha^2}{4}\left(\frac{2}{1}-\frac{3}{4}\right)\right] = -3.4\text{eV}\left(1+\frac{5}{16}\alpha^2\right).$$

$$\left.\begin{array}{l}|5\rangle = |2\,1\,\tfrac{3}{2}\,\tfrac{3}{2}\rangle \\ |6\rangle = |2\,1\,\tfrac{3}{2}\,\tfrac{1}{2}\rangle \\ |7\rangle = |2\,1\,\tfrac{3}{2}\,{-\tfrac{1}{2}}\rangle \\ |8\rangle = |2\,1\,\tfrac{3}{2}\,{-\tfrac{3}{2}}\rangle\end{array}\right\} g_J = \left[1+\frac{(3/2)(5/2)-(1)(2)+(3/4)}{2(3/2)(5/2)}\right] = 1+\frac{5/2}{15/2} = \frac{4}{3}\left\} E_{nj} = -3.4\text{eV}\left[1+\frac{\alpha^2}{4}\left(\frac{2}{2}-\frac{3}{4}\right)\right] = -3.4\text{eV}\left(1+\frac{1}{16}\alpha^2\right).\right.$$

The energies are:

$$\boxed{\begin{aligned}
E_1 &= -3.4\text{eV}\left(1+\tfrac{5}{16}\alpha^2\right) + \mu_B B_{ext}\\
E_2 &= -3.4\text{eV}\left(1+\tfrac{5}{16}\alpha^2\right) - \mu_B B_{ext}\\
E_3 &= -3.4\text{eV}\left(1+\tfrac{5}{16}\alpha^2\right) + \tfrac{1}{3}\mu_B B_{ext}\\
E_4 &= -3.4\text{eV}\left(1+\tfrac{5}{16}\alpha^2\right) - \tfrac{1}{3}\mu_B B_{ext}\\
E_5 &= -3.4\text{eV}\left(1+\tfrac{1}{16}\alpha^2\right) + 2\mu_B B_{ext}\\
E_6 &= -3.4\text{eV}\left(1+\tfrac{1}{16}\alpha^2\right) + \tfrac{2}{3}\mu_B B_{ext}\\
E_7 &= -3.4\text{eV}\left(1+\tfrac{1}{16}\alpha^2\right) - \tfrac{2}{3}\mu_B B_{ext}\\
E_8 &= -3.4\text{eV}\left(1+\tfrac{1}{16}\alpha^2\right) - 2\mu_B B_{ext}
\end{aligned}}$$



PROBLEM 6.20 $E^1_{fs} = \langle n\,\ell\,m_\ell\,m_s|(H^1_r + H^1_{so})|n\,\ell\,m_\ell\,m_s\rangle = -\dfrac{E_n^2}{2mc^2}\left[\dfrac{4n}{\ell+1/2}-3\right] + \dfrac{e^2}{8\pi\epsilon_0 m^2c^2}\,\dfrac{\hbar^2 m_\ell m_s}{\ell(\ell+1/2)(\ell+1)n^3a^3}$.

Now:
$$\left\{\begin{aligned}
&\frac{2E_n^2}{mc^2} = \left(-\frac{2E_1}{mc^2}\right)\left(-\frac{E_1}{n^4}\right) = \frac{\alpha^2}{n^4}(13.6\text{eV}). \quad (\text{Problem 6.10.})\\
&\frac{e^2\hbar^2}{8\pi\epsilon_0 m^2c^2a^3} = \frac{e^2\hbar^2\,(me^2)^3}{2\cdot 4\pi\epsilon_0\, m^2c^2(4\pi\epsilon_0\hbar^2)^3} = \left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\left(\frac{e^2}{4\pi\epsilon_0\hbar c}\right)^2 = \alpha^2(13.6\text{eV}).
\end{aligned}\right.$$

So:
$$E^1_{fs} = \frac{13.6\text{eV}}{n^3}\alpha^2\left\{-\frac{1}{(\ell+1/2)}+\frac{3}{4n}+\frac{m_\ell m_s}{\ell(\ell+\frac{1}{2})(\ell+1)}\right\} = \frac{13.6\text{eV}}{n^3}\alpha^2\left\{\frac{3}{4n}-\frac{\ell(\ell+1)-m_\ell m_s}{\ell(\ell+1/2)(\ell+1)}\right\} \quad \text{QED}$$

PROBLEM 6.21 The Bohr energy is the same for all of them: $E_2 = -\dfrac{13.6\text{eV}}{2^2} = -3.4\text{eV}$.

The Zeeman contribution is the second term in [6.78]: $\mu_B B_{ext}(m_\ell + 2m_s)$.

The fine structure is given by [6.81]: $\quad E^1_{fs} = \dfrac{13.6\text{eV}}{8}\alpha^2\{\} = (1.7\text{eV})\alpha^2\{\}$.

In the table below I record the 8 states, the value of $(m_\ell+2m_s)$, the value of

$\{\} \equiv \dfrac{3}{8} - \left[\dfrac{\ell(\ell+1)-m_\ell m_s}{\ell(\ell+1/2)(\ell+1)}\right]$, and (in the last column) the total energy, $-3.4\text{eV}\left(1-\frac{\alpha^2}{2}\{\}\right)+(m_\ell+2m_s)\mu_B B_{ext}$.

STATE $\lvert n\ell m_\ell m_s\rangle$	$(m_\ell+2m_s)$	$\{\}$	TOTAL ENERGY
$\lvert 1\rangle=\lvert 2\,0\,0\,\frac{1}{2}\rangle$	1	$-5/8$	$-3.4\,\text{eV}\left(1+\frac{5}{16}\alpha^2\right)+\mu_B B_{ext}$
$\lvert 2\rangle=\lvert 2\,0\,0\,-\frac{1}{2}\rangle$	-1	$-5/8$	$-3.4\,\text{eV}\left(1+\frac{5}{16}\alpha^2\right)-\mu_B B_{ext}$
$\lvert 3\rangle=\lvert 2\,1\,1\,\frac{1}{2}\rangle$	2	$-1/8$	$-3.4\,\text{eV}\left(1+\frac{1}{16}\alpha^2\right)+2\mu_B B_{ext}$
$\lvert 4\rangle=\lvert 2\,1\,-1\,-\frac{1}{2}\rangle$	-2	$-1/8$	$-3.4\,\text{eV}\left(1+\frac{1}{16}\alpha^2\right)-2\mu_B B_{ext}$
$\lvert 5\rangle=\lvert 2\,1\,0\,\frac{1}{2}\rangle$	1	$-7/24$	$-3.4\,\text{eV}\left(1+\frac{7}{48}\alpha^2\right)+\mu_B B_{ext}$
$\lvert 6\rangle=\lvert 2\,1\,0\,-\frac{1}{2}\rangle$	-1	$-7/24$	$-3.4\,\text{eV}\left(1+\frac{7}{48}\alpha^2\right)-\mu_B B_{ext}$
$\lvert 7\rangle=\lvert 2\,1\,1\,-\frac{1}{2}\rangle$	0	$-11/24$	$-3.4\,\text{eV}\left(1+\frac{11}{48}\alpha^2\right)$
$\lvert 8\rangle=\lvert 2\,1\,-1\,\frac{1}{2}\rangle$	0	$-11/24$	$-3.4\,\text{eV}\left(1+\frac{11}{48}\alpha^2\right)$

Ignoring fine structure there are $\boxed{\text{five}}$ distinct levels — corresponding to the possible values of $(m_\ell+2m_s)$:

2	$(d=1)$
1	$(d=2)$
0	$(d=2)$
-1	$(d=2)$
-2	$(d=1)$.

Problem 6.22 [6.71] $\Rightarrow E_Z^1 = \frac{e}{2m}\vec{B}_{ext}\cdot\langle\vec{L}+2\vec{S}\rangle = \frac{e}{2m}B_{ext}\,2m_s\hbar = 2m_s\mu_B B_{ext}$ (same as the Zeeman term in [6.78], with $m_\ell=0$). [6.66] $\Rightarrow E_{nj} = -\frac{13.6\text{eV}}{n^2}\left[1+\frac{\alpha^2}{n^2}\left(n-\frac{3}{4}\right)\right]$ (since $j=1/2$).

Total energy:
$$\boxed{E = -\frac{13.6\text{eV}}{n^2}\left[1+\frac{\alpha^2}{n^2}\left(n-\frac{3}{4}\right)\right]+2m_s\mu_B B_{ext}}.$$

Fine structure is the α^2 term: $E_{fs}^1 = -\frac{13.6\text{eV}}{n^4}\alpha^2\left(n-\frac{3}{4}\right) = \frac{13.6\text{eV}}{n^3}\alpha^2\left\{\frac{3}{4n}-1\right\}$, which is the same as [6.81], with the term in square brackets set equal to 1. QED.

Problem 6.23 [6.65] $\Rightarrow E_{fs}^1 = \frac{E_2^2}{2mc^2}\left(3-\frac{8}{j+1/2}\right) = \frac{E_1^2}{32mc^2}\left(3-\frac{8}{j+1/2}\right)$. $\frac{E_1}{mc^2} = -\alpha^2/2$ (Problem 6.10),

so $E_{fs}^1 = -\frac{E_1}{32}\left(\frac{\alpha^2}{2}\right)\left(3-\frac{8}{j+1/2}\right) = \frac{13.6\text{eV}}{64}\alpha^2\left(3-\frac{8}{j+1/2}\right) = \gamma\left(3-\frac{8}{j+1/2}\right)$.

For $j=\frac{1}{2}$ $(\psi_1,\psi_2,\psi_6,\psi_8)$, $H_{fs}^1 = \gamma(3-8) = -5\gamma$. For $j=\frac{3}{2}$ $(\psi_3,\psi_4,\psi_5,\psi_7)$, $H_{fs}^1 = \gamma\left(3-\frac{8}{2}\right) = -\gamma$.

This confirms all the γ terms in $-W$ (p. 248).

Meanwhile $H_Z^1 = \frac{e}{2m}B_{ext}(L_z+2S_z)$ [6.70]; $\psi_1,\psi_2,\psi_3,\psi_4$ are eigenstates of L_z and S_z — for these there are only diagonal elements: $\langle H_Z^1\rangle = \frac{e\hbar}{2m}B_{ext}(m_\ell+2m_s) = (m_\ell+2m_s)\beta$.

$\therefore \langle H_Z^1\rangle_{11} = \beta$; $\langle H_Z^1\rangle_{22} = -\beta$; $\langle H_Z^1\rangle_{33} = 2\beta$; $\langle H_Z^1\rangle_{44} = -2\beta$. This confirms the upper left corner of $-W$. Finally:

$$
\left.
\begin{aligned}
(L_z+2S_z)\lvert\psi_5\rangle &= \hbar\sqrt{\tfrac{2}{3}}\,\lvert 10\rangle\lvert\tfrac{1}{2}\,\tfrac{1}{2}\rangle\\
(L_z+2S_z)\lvert\psi_6\rangle &= -\hbar\sqrt{\tfrac{1}{3}}\,\lvert 10\rangle\lvert\tfrac{1}{2}\,\tfrac{1}{2}\rangle\\
(L_z+2S_z)\lvert\psi_7\rangle &= -\hbar\sqrt{\tfrac{2}{3}}\,\lvert 10\rangle\lvert\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\\
(L_z+2S_z)\lvert\psi_8\rangle &= -\hbar\sqrt{\tfrac{1}{3}}\,\lvert 10\rangle\lvert\tfrac{1}{2}\,-\tfrac{1}{2}\rangle
\end{aligned}
\right\}
\quad\text{So}\quad
\begin{aligned}
\langle H_Z^1\rangle_{55} &= \tfrac{2}{3}\beta,\\
\langle H_Z^1\rangle_{66} &= \tfrac{1}{3}\beta,\\
\langle H_Z^1\rangle_{77} &= -\tfrac{2}{3}\beta,\\
\langle H_Z^1\rangle_{88} &= -\tfrac{1}{3}\beta,
\end{aligned}
$$

$\langle H_Z^1\rangle_{56} = \langle H_Z^1\rangle_{65} = -\frac{\sqrt{2}}{3}\beta$, $\langle H_Z^1\rangle_{78} = \langle H_Z^1\rangle_{87} = -\frac{\sqrt{2}}{3}\beta$; which confirms the remaining elements.

PROBLEM 6.24 There are eighteen $n=3$ states (in general, $2n^2$).

WEAK FIELD [6.66] $\Rightarrow$ $E_{3j} = -\frac{13.6\,\text{eV}}{9}\left[1+\frac{\alpha^2}{9}\left(\frac{3}{j+1/2}-\frac{3}{4}\right)\right] = -1.51\,\text{eV}\left[1+\frac{\alpha^2}{3}\left(\frac{1}{j+1/2}-\frac{1}{4}\right)\right]$.

[6.75] $\Rightarrow$ $E_Z^1 = g_J m_j \mu_B B_{ext}$.

	State $\|3 \ell j m_j\rangle$	g_J [6.74]	$\frac{1}{3}\left(\frac{1}{j+1/2}-\frac{1}{4}\right)$	Total Energy
$\ell=0$: $j=\frac{1}{2}$	$\|3\,0\,\frac{1}{2}\,\frac{1}{2}\rangle$	2	1/4	$-1.51\,\text{eV}(1+\frac{\alpha^2}{4})+\mu_B B_{ext}$
	$\|3\,0\,\frac{1}{2}\,-\frac{1}{2}\rangle$	2	1/4	$-1.51\,\text{eV}(1+\frac{\alpha^2}{4})-\mu_B B_{ext}$
$\ell=1$: $j=\frac{1}{2}$	$\|3\,1\,\frac{1}{2}\,\frac{1}{2}\rangle$	2/3	1/4	$-1.51\,\text{eV}(1+\frac{\alpha^2}{4})+\frac{1}{3}\mu_B B_{ext}$
	$\|3\,1\,\frac{1}{2}\,-\frac{1}{2}\rangle$	2/3	1/4	$-1.51\,\text{eV}(1+\frac{\alpha^2}{4})-\frac{1}{3}\mu_B B_{ext}$
$j=\frac{3}{2}$	$\|3\,1\,\frac{3}{2}\,\frac{3}{2}\rangle$	4/3	1/12	$-1.51\,\text{eV}(1+\frac{\alpha^2}{12})+2\mu_B B_{ext}$
	$\|3\,1\,\frac{3}{2}\,\frac{1}{2}\rangle$	4/3	1/12	$-1.51\,\text{eV}(1+\frac{\alpha^2}{12})+\frac{2}{3}\mu_B B_{ext}$
	$\|3\,1\,\frac{3}{2}\,-\frac{1}{2}\rangle$	4/3	1/12	$-1.51\,\text{eV}(1+\frac{\alpha^2}{12})-\frac{2}{3}\mu_B B_{ext}$
	$\|3\,1\,\frac{3}{2}\,-\frac{3}{2}\rangle$	4/3	1/12	$-1.51\,\text{eV}(1+\frac{\alpha^2}{12})-2\mu_B B_{ext}$
$\ell=2$: $j=\frac{3}{2}$	$\|3\,2\,\frac{3}{2}\,\frac{3}{2}\rangle$	4/5	1/12	$-1.51\,\text{eV}(1+\frac{\alpha^2}{12})+\frac{6}{5}\mu_B B_{ext}$
	$\|3\,2\,\frac{3}{2}\,\frac{1}{2}\rangle$	4/5	1/12	$-1.51\,\text{eV}(1+\frac{\alpha^2}{12})+\frac{2}{5}\mu_B B_{ext}$
	$\|3\,2\,\frac{3}{2}\,-\frac{1}{2}\rangle$	4/5	1/12	$-1.51\,\text{eV}(1+\frac{\alpha^2}{12})-\frac{2}{5}\mu_B B_{ext}$
	$\|3\,2\,\frac{3}{2}\,-\frac{3}{2}\rangle$	4/5	1/12	$-1.51\,\text{eV}(1+\frac{\alpha^2}{12})-\frac{6}{5}\mu_B B_{ext}$
$j=\frac{5}{2}$	$\|3\,2\,\frac{5}{2}\,\frac{5}{2}\rangle$	6/5	1/36	$-1.51\,\text{eV}(1+\frac{\alpha^2}{36})+3\mu_B B_{ext}$
	$\|3\,2\,\frac{5}{2}\,\frac{3}{2}\rangle$	6/5	1/36	$-1.51\,\text{eV}(1+\frac{\alpha^2}{36})+\frac{9}{5}\mu_B B_{ext}$
	$\|3\,2\,\frac{5}{2}\,\frac{1}{2}\rangle$	6/5	1/36	$-1.51\,\text{eV}(1+\frac{\alpha^2}{36})+\frac{3}{5}\mu_B B_{ext}$
	$\|3\,2\,\frac{5}{2}\,-\frac{1}{2}\rangle$	6/5	1/36	$-1.51\,\text{eV}(1+\frac{\alpha^2}{36})-\frac{3}{5}\mu_B B_{ext}$
	$\|3\,2\,\frac{5}{2}\,-\frac{3}{2}\rangle$	6/5	1/36	$-1.51\,\text{eV}(1+\frac{\alpha^2}{36})-\frac{9}{5}\mu_B B_{ext}$
	$\|3\,2\,\frac{5}{2}\,-\frac{5}{2}\rangle$	6/5	1/36	$-1.51\,\text{eV}(1+\frac{\alpha^2}{36})-3\mu_B B_{ext}$

STRONG FIELD [6.78] $\Rightarrow$ $-1.51\,\text{eV}+(m_\ell+2m_s)\mu_B B_{ext}$;

$$[6.81] \Rightarrow \frac{13.6\,\text{eV}}{27}\alpha^2\left\{\frac{1}{4}-\left[\frac{\ell(\ell+1)-m_\ell m_s}{\ell(\ell+1/2)(\ell+1)}\right]\right\} = -1.51\,\text{eV}\,\alpha^2\,\frac{1}{3}\left\{\left[\frac{\ell(\ell+1)-m_\ell m_s}{\ell(\ell+1/2)(\ell+1)}\right]-\frac{1}{4}\right\}.$$

So $E_{TOT} = -1.51\,\text{eV}(1+\alpha^2 A)+(m_\ell+2m_s)\mu_B B_{ext}$, where $A \equiv \frac{1}{3}\left[\frac{\ell(\ell+1)-m_\ell m_s}{\ell(\ell+1/2)(\ell+1)}-\frac{1}{4}\right]$.

These terms are given in the table below:

	STATE $\lvert n\ell m_\ell m_s\rangle$	$(m_\ell+2m_s)$	A	TOTAL ENERGY
$\ell=0$:	$\lvert 3 0 0 \tfrac{1}{2}\rangle$	1	1/4	$-1.51\,\text{eV}\left(1+\frac{\alpha^2}{4}\right)+\mu_B B_{ext}$
	$\lvert 3 0 0 -\tfrac{1}{2}\rangle$	−1	1/4	$-1.51\,\text{eV}\left(1+\frac{\alpha^2}{4}\right)-\mu_B B_{ext}$
$\ell=1$:	$\lvert 3 1 1 \tfrac{1}{2}\rangle$	2	1/12	$-1.51\,\text{eV}\left(1+\frac{\alpha^2}{12}\right)+2\mu_B B_{ext}$
	$\lvert 3 1 -1 -\tfrac{1}{2}\rangle$	−2	1/12	$-1.51\,\text{eV}\left(1+\frac{\alpha^2}{12}\right)-2\mu_B B_{ext}$
	$\lvert 3 1 0 \tfrac{1}{2}\rangle$	1	5/36	$-1.51\,\text{eV}\left(1+\frac{5\alpha^2}{36}\right)+\mu_B B_{ext}$
	$\lvert 3 1 0 -\tfrac{1}{2}\rangle$	−1	5/36	$-1.51\,\text{eV}\left(1+\frac{5\alpha^2}{36}\right)-\mu_B B_{ext}$
	$\lvert 3 1 -1 \tfrac{1}{2}\rangle$	0	7/36	$-1.51\,\text{eV}\left(1+\frac{7\alpha^2}{36}\right)$
	$\lvert 3 1 1 -\tfrac{1}{2}\rangle$	0	7/36	$-1.51\,\text{eV}\left(1+\frac{7\alpha^2}{36}\right)$
$\ell=2$:	$\lvert 3 2 2 \tfrac{1}{2}\rangle$	3	1/36	$-1.51\,\text{eV}\left(1+\frac{\alpha^2}{36}\right)+3\mu_B B_{ext}$
	$\lvert 3 2 -2 -\tfrac{1}{2}\rangle$	−3	1/36	$-1.51\,\text{eV}\left(1+\frac{\alpha^2}{36}\right)-3\mu_B B_{ext}$
	$\lvert 3 2 1 \tfrac{1}{2}\rangle$	2	7/180	$-1.51\,\text{eV}\left(1+\frac{7\alpha^2}{180}\right)+2\mu_B B_{ext}$
	$\lvert 3 2 -1 -\tfrac{1}{2}\rangle$	−2	7/180	$-1.51\,\text{eV}\left(1+\frac{7\alpha^2}{180}\right)-3\mu_B B_{ext}$
	$\lvert 3 2 0 \tfrac{1}{2}\rangle$	1	1/20	$-1.51\,\text{eV}\left(1+\frac{\alpha^2}{20}\right)+\mu_B B_{ext}$
	$\lvert 3 2 0 -\tfrac{1}{2}\rangle$	−1	1/20	$-1.51\,\text{eV}\left(1+\frac{\alpha^2}{20}\right)-\mu_B B_{ext}$
	$\lvert 3 2 -1 \tfrac{1}{2}\rangle$	0	11/180	$-1.51\,\text{eV}\left(1+\frac{11\alpha^2}{180}\right)$
	$\lvert 3 2 1 -\tfrac{1}{2}\rangle$	0	11/180	$-1.51\,\text{eV}\left(1+\frac{11\alpha^2}{180}\right)$
	$\lvert 3 2 -2 \tfrac{1}{2}\rangle$	−1	13/180	$-1.51\,\text{eV}\left(1+\frac{13\alpha^2}{180}\right)-\mu_B B_{ext}$
	$\lvert 3 2 2 -\tfrac{1}{2}\rangle$	1	13/180	$-1.51\,\text{eV}\left(1+\frac{13\alpha^2}{180}\right)+\mu_B B_{ext}$

INTERMEDIATE FIELD As in the book, I'll use the basis $\lvert n\ell j m_j\rangle$ (same as for weak field); then the fine structure matrix elements are diagonal: [6.65] $\Rightarrow$

$$E^1_{fs} = \frac{E_3^2}{2mc^2}\left(3-\frac{12}{j+1/2}\right) = \frac{E_1^2}{54mc^2}\left(1-\frac{4}{j+1/2}\right) = -\frac{E_1\alpha^2}{108}\left(1-\frac{4}{j+1/2}\right) = 3\gamma\left(1-\frac{4}{j+1/2}\right), \quad \gamma \equiv \frac{13.6\,\text{eV}}{324}\alpha^2.$$

For $j=\frac{1}{2}$, $E^1_{fs}=-9\gamma$; for $j=\frac{3}{2}$, $E^1_{fs}=-3\gamma$; for $j=\frac{5}{2}$, $E^1_{fs}=-\gamma$.

The Zeeman Hamiltonian is [6.70]: $H'_Z = \frac{1}{\hbar}(L_z+2S_z)\mu_B B_{ext}$. The first eight states ($\ell=0$ and $\ell=1$) are the same as before (p. 248), so the β terms in W are unchanged; recording just the nonzero blocks of $-W$:

$$(9\gamma-\beta),\ (9\gamma+\beta),\ (3\gamma-2\beta),\ (3\gamma+2\beta),\ \begin{pmatrix}(3\gamma-\frac{2}{3}\beta) & \frac{\sqrt{2}}{3}\beta \\ \frac{\sqrt{2}}{3}\beta & (9\gamma-\frac{1}{3}\beta)\end{pmatrix},\ \begin{pmatrix}(3\gamma+\frac{2}{3}\beta) & \frac{\sqrt{2}}{3}\beta \\ \frac{\sqrt{2}}{3}\beta & (9\gamma+\frac{1}{3}\beta)\end{pmatrix}.$$

The other 10 states ($\ell=2$) must first be decomposed into eigenstates of L_z and S_z:

$$|\tfrac{5}{2}\,\tfrac{5}{2}\rangle = |2\,2\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle \quad\Longrightarrow\quad (\gamma - 3\beta)$$

$$|\tfrac{5}{2}\,{-\tfrac{5}{2}}\rangle = |2\,{-2}\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle \quad\Longrightarrow\quad (\gamma + 3\beta)$$

$$\left.\begin{aligned} |\tfrac{5}{2}\,\tfrac{3}{2}\rangle &= \sqrt{\tfrac{1}{5}}\,|2\,2\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle + \sqrt{\tfrac{4}{5}}\,|2\,1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle \\ |\tfrac{3}{2}\,\tfrac{3}{2}\rangle &= \sqrt{\tfrac{4}{5}}\,|2\,2\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle - \sqrt{\tfrac{1}{5}}\,|2\,1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle \end{aligned}\right\} \Rightarrow \begin{pmatrix} (\gamma - \frac{9}{5}\beta) & \frac{2}{5}\beta \\ \frac{2}{5}\beta & (3\gamma - \frac{6}{5}\beta) \end{pmatrix}$$

$$\left.\begin{aligned} |\tfrac{5}{2}\,\tfrac{1}{2}\rangle &= \sqrt{\tfrac{2}{5}}\,|2\,1\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle + \sqrt{\tfrac{3}{5}}\,|2\,0\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle \\ |\tfrac{3}{2}\,\tfrac{1}{2}\rangle &= \sqrt{\tfrac{3}{5}}\,|2\,1\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle - \sqrt{\tfrac{2}{5}}\,|2\,0\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle \end{aligned}\right\} \Rightarrow \begin{pmatrix} (\gamma - \frac{3}{5}\beta) & \frac{\sqrt{6}}{5}\beta \\ \frac{\sqrt{6}}{5}\beta & (3\gamma - \frac{2}{5}\beta) \end{pmatrix}$$

$$\left.\begin{aligned} |\tfrac{5}{2}\,{-\tfrac{1}{2}}\rangle &= \sqrt{\tfrac{3}{5}}\,|2\,0\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle + \sqrt{\tfrac{2}{5}}\,|2\,{-1}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle \\ |\tfrac{3}{2}\,{-\tfrac{1}{2}}\rangle &= \sqrt{\tfrac{2}{5}}\,|2\,0\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle - \sqrt{\tfrac{3}{5}}\,|2\,{-1}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle \end{aligned}\right\} \Rightarrow \begin{pmatrix} (\gamma + \frac{3}{5}\beta) & \frac{\sqrt{6}}{5}\beta \\ \frac{\sqrt{6}}{5}\beta & (3\gamma + \frac{2}{5}\beta) \end{pmatrix}$$

$$\left.\begin{aligned} |\tfrac{5}{2}\,{-\tfrac{3}{2}}\rangle &= \sqrt{\tfrac{4}{5}}\,|2\,{-1}\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle + \sqrt{\tfrac{1}{5}}\,|2\,{-2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle \\ |\tfrac{3}{2}\,{-\tfrac{3}{2}}\rangle &= \sqrt{\tfrac{1}{5}}\,|2\,{-1}\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle - \sqrt{\tfrac{4}{5}}\,|2\,{-2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle \end{aligned}\right\} \Rightarrow \begin{pmatrix} (\gamma + \frac{9}{5}\beta) & \frac{2}{5}\beta \\ \frac{2}{5}\beta & (3\gamma + \frac{6}{5}\beta) \end{pmatrix}$$

[<u>Sample calculation</u>: for the last two, letting $Q \equiv \frac{1}{\hbar}(L_z + 2S_z)$, we have

$$Q|\tfrac{5}{2}\,{-\tfrac{3}{2}}\rangle = -2\sqrt{\tfrac{4}{5}}\,|2\,{-1}\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle - \sqrt{\tfrac{1}{5}}\,|2\,{-2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle;\quad Q|\tfrac{3}{2}\,{-\tfrac{3}{2}}\rangle = -2\sqrt{\tfrac{1}{5}}\,|2\,{-1}\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle + \sqrt{\tfrac{4}{5}}\,|2\,{-2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle.$$

So $\langle\tfrac{5}{2}\,{-\tfrac{3}{2}}|Q|\tfrac{5}{2}\,{-\tfrac{3}{2}}\rangle = (-2)(4/5) - (1/5) = -9/5$; $\langle\tfrac{3}{2}\,{-\tfrac{3}{2}}|Q|\tfrac{3}{2}\,{-\tfrac{3}{2}}\rangle = -2(\tfrac{1}{5}) - \tfrac{4}{5} = -\tfrac{6}{5}$;

$$\langle\tfrac{5}{2}\,{-\tfrac{3}{2}}|Q|\tfrac{3}{2}\,{-\tfrac{3}{2}}\rangle = -2\sqrt{\tfrac{4}{5}}\sqrt{\tfrac{1}{5}} + \sqrt{\tfrac{1}{5}}\sqrt{\tfrac{4}{5}} = -\tfrac{4}{5} + \tfrac{2}{5} = -\tfrac{2}{5} = \langle\tfrac{3}{2}\,{-\tfrac{3}{2}}|Q|\tfrac{5}{2}\,{-\tfrac{3}{2}}\rangle.\ ]$$

So the 18×18 matrix $-W$ splits into six 1×1 blocks and six 2×2 blocks. We need the eigenvalues of the 2×2 blocks. This means solving 3 characteristic equations (the other 3 are obtained trivially by changing the sign of β):

$$(3\gamma - \tfrac{2}{3}\beta - \lambda)(9\gamma - \tfrac{1}{3}\beta - \lambda) - \tfrac{2}{9}\beta^2 = 0 \Rightarrow \lambda^2 + \lambda(\beta - 12\gamma) + \gamma(27\gamma - 7\beta) = 0.$$

$$(\gamma - \tfrac{9}{5}\beta - \lambda)(3\gamma - \tfrac{6}{5}\beta - \lambda) - \tfrac{4}{25}\beta^2 = 0 \Rightarrow \lambda^2 + \lambda(3\beta - 4\gamma) + (3\gamma^2 - \tfrac{33}{5}\gamma\beta + 2\beta^2) = 0.$$

$$(\gamma - \tfrac{3}{5}\beta - \lambda)(3\gamma - \tfrac{2}{5}\beta - \lambda) - \tfrac{6}{25}\beta^2 = 0 \Rightarrow \lambda^2 + \lambda(\beta - 4\gamma) + \gamma(3\gamma - \tfrac{11}{5}\beta) = 0.$$

The solutions are: $\lambda = -\beta/2 + 6\gamma \pm \sqrt{(\beta/2)^2 + \beta\gamma + 9\gamma^2}$

$$\lambda = -3\beta/2 + 2\gamma \pm \sqrt{(\beta/2)^2 + \tfrac{3}{5}\beta\gamma + \gamma^2}$$

$$\lambda = -\beta/2 + 2\gamma \pm \sqrt{(\beta/2)^2 + \tfrac{1}{5}\gamma\beta + \gamma^2}$$

$\epsilon_1 = E_3 - 9\gamma + \beta$	$\epsilon_9 = E_3 - 2\gamma + \beta/2 - \sqrt{\gamma^2 + \frac{1}{5}\gamma\beta + \beta^2/4}$
$\epsilon_2 = E_3 - 3\gamma + 2\beta$	
$\epsilon_3 = E_3 - \gamma + 3\beta$	(and the other 9 are the same, but with $\beta \to -\beta$).
$\epsilon_4 = E_3 - 6\gamma + \beta/2 + \sqrt{9\gamma^2 + \beta\gamma + \beta^2/4}$	
$\epsilon_5 = E_3 - 6\gamma + \beta/2 - \sqrt{9\gamma^2 + \beta\gamma + \beta^2/4}$	Here $\begin{cases} \gamma = \frac{13.6\text{ eV}}{324}\alpha^2; \\ \beta = \mu_B B_{\text{ext}}. \end{cases}$
$\epsilon_6 = E_3 - 2\gamma + 3\beta/2 + \sqrt{\gamma^2 + \frac{3}{5}\gamma\beta + \beta^2/4}$	
$\epsilon_7 = E_3 - 2\gamma + 3\beta/2 - \sqrt{\gamma^2 + \frac{3}{5}\gamma\beta + \beta^2/4}$	
$\epsilon_8 = E_3 - 2\gamma + \beta/2 + \sqrt{\gamma^2 + \frac{1}{5}\gamma\beta + \beta^2/4}$	

In the weak-field limit ($\beta \ll \gamma$):

$$\epsilon_4 \cong E_3 - 6\gamma + \beta/2 + 3\gamma\sqrt{1+\beta/9\gamma} \cong E_3 - 6\gamma + \beta/2 + 3\gamma\left(1+\frac{\beta}{18\gamma}\right) = E_3 - 3\gamma + \frac{2}{3}\beta.$$

$$\epsilon_5 \cong E_3 - 6\gamma + \beta/2 - 3\gamma(1+\beta/18\gamma) = E_3 - 9\gamma + \frac{1}{3}\beta.$$

$$\epsilon_6 \cong E_3 - 2\gamma + 3\beta/2 + \gamma(1+3\beta/10\gamma) = E_3 - \gamma + \frac{9}{5}\beta.$$

$$\epsilon_7 \cong E_3 - 2\gamma + 3\beta/2 - \gamma(1+3\beta/10\gamma) = E_3 - 3\gamma + \frac{6}{5}\beta.$$

$$\epsilon_8 \cong E_3 - 2\gamma + \beta/2 + \gamma(1+\beta/10\gamma) = E_3 - \gamma + \frac{3}{5}\beta.$$

$$\epsilon_9 \cong E_3 - 2\gamma + \beta/2 - \gamma(1+\beta/10\gamma) = E_3 - 3\gamma + \frac{2}{5}\beta.$$

Noting that $\gamma = -(E_2/36)\alpha^2 = \frac{1.51\,eV}{36}\alpha^2$, we see that the weak field energies are recovered as in the first table.

In the strong-field limit ($\beta \gg \gamma$):

$$\epsilon_4 \cong E_3 - 6\gamma + \beta/2 + \beta/2\sqrt{1+4\gamma/\beta} \cong E_3 - 6\gamma + \beta/2 + \beta/2\left(1+\frac{2\gamma}{\beta}\right) = E_3 - 5\gamma + \beta.$$

$$\epsilon_5 \cong E_3 - 6\gamma + \beta/2 - \beta/2(1+2\gamma/\beta) = E_3 - 7\gamma.$$

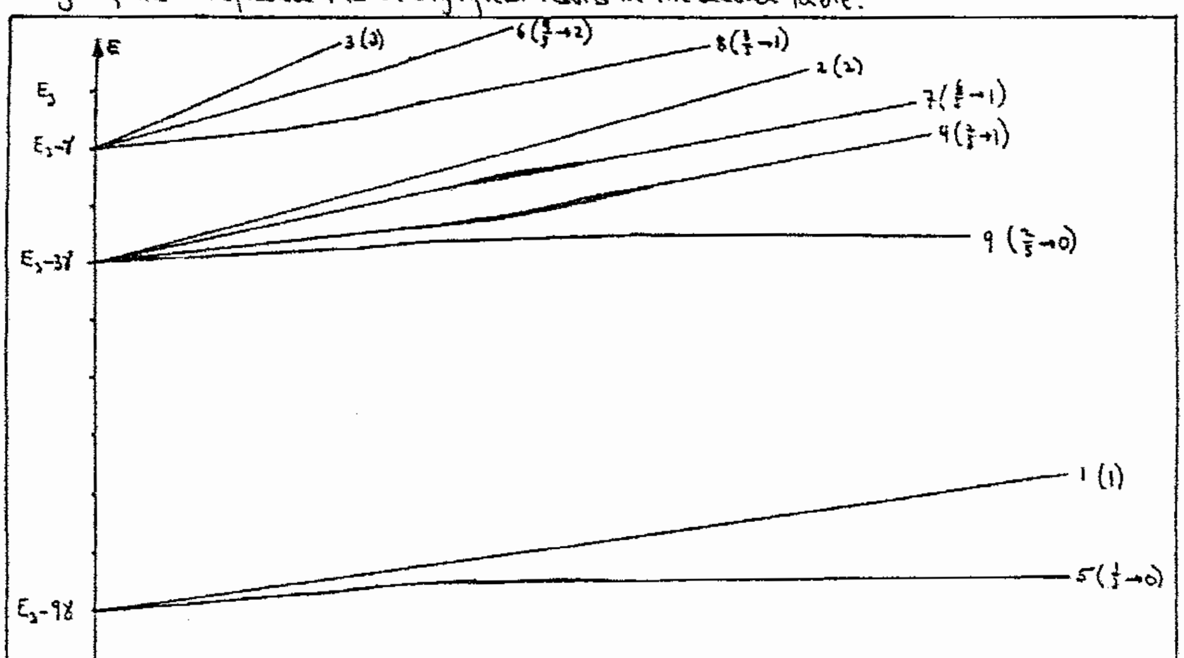
$$\epsilon_6 \cong E_3 - 2\gamma + 3\beta/2 + \beta/2(1+6\gamma/5\beta) = E_3 - \frac{7}{5}\gamma + 2\beta.$$

$$\epsilon_7 \cong E_3 - 2\gamma + 3\beta/2 - \beta/2(1+6\gamma/5\beta) = E_3 - \frac{13}{5}\gamma + \beta.$$

$$\epsilon_8 \cong E_3 - 2\gamma + \beta/2 + \beta/2(1+2\gamma/5\beta) = E_3 - \frac{9}{5}\gamma + \beta.$$

$$\epsilon_9 \cong E_3 - 2\gamma + \beta/2 - \beta/2(1+2\gamma/5\beta) = E_3 - \frac{11}{5}\gamma.$$

Again, these reproduce the strong-field results in the second table.



(For each of these lines there is a corresponding one starting from the same point, but sloping down. Lines are labeled by level number and (in parentheses), starting and ending slopes.)

Problem 6.25 $I \equiv \int (\vec{a}\cdot\hat{r})(\vec{b}\cdot\hat{r}) \sin\theta \, d\theta \, d\phi = \int (a_x \sin\theta\cos\phi + a_y \sin\theta\sin\phi + a_z\cos\theta)(b_x\sin\theta\cos\phi + b_y \sin\theta\sin\phi + b_z\cos\theta)\sin\theta\, d\theta\, d\phi$

But $\int_0^{2\pi}\sin\phi\, d\phi = \int_0^{2\pi}\cos\phi\, d\phi = \int_0^{2\pi}\sin\phi\cos\phi\, d\phi = 0$, so only three terms survive:

$$I = \int (a_x b_x \sin^2\theta\cos^2\phi + a_y b_y \sin^2\theta\sin^2\phi + a_z b_z\cos^2\theta)\sin\theta\, d\theta\, d\phi.$$ But $\int_0^{2\pi}\sin^2\phi\, d\phi = \int_0^{2\pi}\cos^2\phi\, d\phi = \pi$, $\int_0^{2\pi} d\phi = 2\pi$, so

$$I = \int_0^\pi \left[\pi(a_x b_x + a_y b_y)\sin^2\theta + 2\pi a_z b_z \cos^2\theta\right]\sin\theta\, d\theta.$$ But $\int_0^\pi \sin^3\theta\, d\theta = \frac{4}{3}$, $\int_0^\pi \cos^2\theta\sin\theta\, d\theta = \frac{2}{3}$, so

$$I = \pi(a_x b_x + a_y b_y)\frac{4}{3} + 2\pi a_z b_z \frac{2}{3} = \frac{4\pi}{3}(a_x b_x + a_y b_y + a_z b_z) = \frac{4\pi}{3}(\vec{a}\cdot\vec{b}).$$ QED

[Alternatively, noting that I has to be a <u>scalar</u> bilinear in $\vec{a}$ and $\vec{b}$, we know immediately that $I = A(\vec{a}\cdot\vec{b})$, where A is some <u>constant</u> (same for all $\vec{a}$ and $\vec{b}$). To determine A, pick $\vec{a} = \vec{b} = \hat{k}$; then $I = A = \int \cos^2\theta \sin\theta\, d\theta\, d\phi = 4\pi/3$.]

For states with $l=0$, the wave function is independent of θ and ϕ ($Y_0^0 = \frac{1}{\sqrt{4\pi}}$), so

$$\left\langle \frac{3(\vec{S}_p\cdot\hat{r})(\vec{S}_e\cdot\hat{r}) - \vec{S}_p\cdot\vec{S}_e}{r^3}\right\rangle = \left\{\int_0^\infty \frac{1}{r^3}|\psi(r)|^2 r^2\, dr\right\}\int\left[3(\vec{S}_p\cdot\hat{r})(\vec{S}_e\cdot\hat{r}) - \vec{S}_p\cdot\vec{S}_e\right]\sin\theta\, d\theta\, d\phi.$$

The first angular integral is $3\cdot\frac{4\pi}{3}(\vec{S}_p\cdot\vec{S}_e) = 4\pi(\vec{S}_p\cdot\vec{S}_e)$, while the second is $-(\vec{S}_p\cdot\vec{S}_e)\int\sin\theta\, d\theta\, d\phi = -4\pi(\vec{S}_p\cdot\vec{S}_e)$, so the two cancel, and the result is zero. QED. [Actually, there is a little slight-of-hand here, since for $l=0$, $\psi \to$ constant as $r\to 0$, and hence the radial integral diverges logarithmically at the origin. Technically, the first term in [6.85] is the field outside an infinitesimal <u>sphere</u> — the delta-function gives the field <u>inside</u>. For this reason it is correct to do the angular integral first (getting zero) and not worry about the radial integral.]

Problem 6.26 From [6.88] we see that $\Delta E \propto \left(\frac{g}{m_p m_e a^3}\right)$; we want reduced mass in a, but <u>not</u> in $m_p m_e$ (which come from [6.84]) — the notation in [6.92] obscures this point.

(a) g and m_p are unchanged; $m_e \to m_\mu = 207\, m_e$, and $a \to a_\mu$: from [4.72], $a \propto \frac{1}{m}$, so

$$\frac{a}{a_\mu} = \frac{m_\mu(\text{reduced})}{m_e} = \frac{m_\mu m_p}{m_\mu + m_p}\cdot\frac{1}{m_e} = \frac{207}{1 + 207\frac{m_e}{m_p}} = \frac{207}{1 + 207\frac{(9.11\times10^{-31})}{(1.67\times10^{-27})}} = \frac{207}{1.11} = 186.$$

$$\Delta E = (5.88\times10^{-6}\, eV)\left(\frac{1}{207}\right)(186)^3 = \boxed{0.183\, eV}.$$

(b) $g: 5.59 \to 2$; $m_p \to m_e$; $\frac{a}{a_p} = \frac{m_p(\text{red})}{m_e} = \frac{m_e^2}{m_e + m_e}\cdot\frac{1}{m_e} = \frac{1}{2}$. $\Delta E = (5.88\times10^{-6}\, eV)\left(\frac{2}{5.59}\right)\left(\frac{1.67\times10^{-27}}{9.11\times10^{-31}}\right)\left(\frac{1}{2}\right)^3$

$= \boxed{4.82\times10^{-4}\, eV}$.

(c) $g: 5.59 \to 2$; $m_p \to m_\mu$; $\frac{a}{a_m} = \frac{m_m(\text{red})}{m_e} = \frac{m_e m_\mu}{m_e + m_\mu}\cdot\frac{1}{m_e} = \frac{207}{208}$. $\Delta E = (5.88\times10^{-6})\left(\frac{2}{5.59}\right)\left(\frac{1.67\times10^{-27}}{(207)(9.11\times10^{-31})}\right)\left(\frac{207}{208}\right)^3$;

$\boxed{\Delta E = 1.84\times10^{-5}\, eV}$

PROBLEM 6.27 (a) Let the unperturbed Hamiltonian be $H(\lambda_0)$, for some fixed value λ_0. Now tweak λ to $\lambda_0+d\lambda$. The perturbing Hamiltonian is

$$H' = H(\lambda_0+d\lambda) - H(\lambda_0) = \frac{\partial H}{\partial \lambda}\,d\lambda \quad \text{(derivative evaluated at } \lambda_0\text{)}.$$

The change in energy is given by [6.9]:

$$dE_n = E_n^1 = \langle \psi_n^0 | H' | \psi_n^0 \rangle = \langle \psi_n | \frac{\partial H}{\partial \lambda} | \psi_n \rangle d\lambda \quad \text{(all evaluated at } \lambda_0\text{)}.$$

$$\therefore \quad \frac{\partial E_n}{\partial \lambda} = \langle \psi_n | \frac{\partial H}{\partial \lambda} | \psi_n \rangle.$$ [Note: even though we used perturbation theory, the result is exact, since all we needed is the infinitesimal change in E_n.]

(b) $E_n = (n+\frac{1}{2})\hbar\omega$; $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2$.

(i) $\frac{\partial E_n}{\partial \omega} = (n+\frac{1}{2})\hbar$; $\frac{\partial H}{\partial \omega} = m\omega x^2$; so theorem $\Rightarrow$ $(n+\frac{1}{2})\hbar = \langle n | m\omega x^2 | n\rangle$. $V = \frac{1}{2}m\omega^2x^2$,

So $\langle V \rangle = \langle n | \frac{1}{2}m\omega^2x^2 | n\rangle = \frac{1}{2}\omega(n+\frac{1}{2})\hbar = \boxed{\frac{1}{2}(n+\frac{1}{2})\hbar\omega = \langle V\rangle}$.

(ii) $\frac{\partial E_n}{\partial \hbar} = (n+\frac{1}{2})\omega$; $\frac{\partial H}{\partial \hbar} = -\frac{\hbar}{m}\frac{d^2}{dx^2} = \frac{2}{\hbar}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right) = \frac{2}{\hbar}T$; so theorem $\Rightarrow$ $(n+\frac{1}{2})\omega = \frac{2}{\hbar}\langle n|T|n\rangle$,

or $\boxed{\langle T\rangle = \frac{1}{2}(n+\frac{1}{2})\hbar\omega}$.

(iii) $\frac{\partial E_n}{\partial m} = 0$; $\frac{\partial H}{\partial m} = \frac{\hbar^2}{2m^2}\frac{d^2}{dx^2} + \frac{1}{2}\omega^2x^2 = -\frac{1}{m}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right) + \frac{1}{m}\left(\frac{1}{2}m\omega^2x^2\right) = -\frac{1}{m}T + \frac{1}{m}V$.

So theorem $\Rightarrow$ $0 = -\frac{1}{m}\langle T\rangle + \frac{1}{m}\langle V\rangle$, or $\boxed{\langle T\rangle = \langle V\rangle}$.

These results are consistent with what we found in Problems 2.37 and 3.53.

PROBLEM 6.28 (a) $\frac{\partial E_n}{\partial e} = \frac{-4me^3}{32\pi^2\epsilon_0^2\hbar^2(j_{max}+l+1)^2} = \frac{4}{e}E_n$; $\frac{\partial H}{\partial e} = -\frac{2e}{4\pi\epsilon_0}\frac{1}{r}$.

So the theorem says: $\frac{4}{e}E_n = -\frac{e}{2\pi\epsilon_0}\langle\frac{1}{r}\rangle$, or $\langle\frac{1}{r}\rangle = -\frac{8\pi\epsilon_0}{e^2}E_n = -\frac{8\pi\epsilon_0 E_1}{e^2\,n^2}$

$\therefore \langle\frac{1}{r}\rangle = -\frac{8\pi\epsilon_0}{e^2}\left[-\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} = \frac{e^2m}{4\pi\epsilon_0\hbar^2}\frac{1}{n^2}$. But $\frac{4\pi\epsilon_0\hbar^2}{me^2} = a$ [4.72], so $\boxed{\langle\frac{1}{r}\rangle = \frac{1}{n^2a}}$.

(Agrees with [6.54].)

(b) $\frac{\partial E_n}{\partial l} = \frac{2me^4}{32\pi^2\epsilon_0^2\hbar^2(j_{max}+l+1)^3} = -\frac{2E_n}{n}$; $\frac{\partial H}{\partial l} = \frac{\hbar^2}{2mr^2}(2l+1)$; theorem says

$-\frac{2E_n}{n} = \frac{\hbar^2(2l+1)}{2m}\langle\frac{1}{r^2}\rangle$, or $\langle\frac{1}{r^2}\rangle = -\frac{4mE_n}{n(2l+1)\hbar^2} = -\frac{4mE_1}{n^3(2l+1)\hbar^2}$. But $-\frac{4mE_1}{\hbar^2} = \frac{2}{a^2}$, so

$\boxed{\langle\frac{1}{r^2}\rangle = \frac{1}{n^3(l+\frac{1}{2})a^2}}$. (Agrees with [6.55].)

Problem 6.29 [4.53] $\Rightarrow u'' = \left[\frac{l(l+1)}{r^2} - \frac{2mE_n}{\hbar^2} - \frac{2m}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{r}\right]u$.

But $\frac{me^2}{4\pi\epsilon_0\hbar^2} = \frac{1}{a}$ (eq. [4.72]), and $-\frac{2mE_n}{\hbar^2} = \frac{2m}{\hbar^2}\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2} = \frac{1}{a^2n^2}$. So

$$\maltese \quad u'' = \left[\frac{l(l+1)}{r^2} - \frac{2}{ar} + \frac{1}{n^2a^2}\right]u.$$

$$\therefore \int (ur^s u'')dr = \int ur^s\left[\frac{l(l+1)}{r^2} - \frac{2}{ar} + \frac{1}{n^2a^2}\right]u\,dr = l(l+1)\langle r^{s-2}\rangle - \frac{2}{a}\langle r^{s-1}\rangle + \frac{1}{n^2a^2}\langle r^s\rangle$$

$$\# \quad = -\int \frac{d}{dr}(ur^s)\,u'\,dr = -\int (u'r^su')dr - s\int (ur^{s-1}u')dr.$$

Lemma 1: $\int (ur^su')dr = -\int \frac{d}{dr}(ur^s)u\,dr = -\int (u'r^su)dr - s\int ur^{s-1}u\,dr$

$\therefore\ 2\int (ur^su')dr = -s\langle r^{s-1}\rangle$, or $\int (ur^su')dr = -\frac{s}{2}\langle r^{s-1}\rangle$.

Lemma 2: $\int (u''r^{s+1}u')dr = -\int u'\frac{d}{dr}(r^{s+1}u')dr = -(s+1)\int u'r^su'dr - \int u'r^{s+1}u''dr$.

$\therefore\ 2\int (u''r^{s+1}u')dr = -(s+1)\int (u'r^su')dr$, or:

$$\int (u'r^su')dr = -\frac{2}{(s+1)}\int (u''r^{s+1}u')\,dr.$$

Lemma 3: Use $\maltese$ in Lemma 2, and exploit Lemma 1:

$$\int (u'r^su')dr = -\frac{2}{(s+1)}\int\left[\frac{l(l+1)}{r^2} - \frac{2}{ar} + \frac{1}{n^2a^2}\right]ur^{s+1}u'\,dr$$

$$= -\frac{2}{(s+1)}\left\{l(l+1)\int ur^{s-1}u'dr - \frac{2}{a}\int ur^su'dr + \frac{1}{n^2a^2}\int ur^{s+1}u'dr\right\}$$

$$= -\frac{2}{(s+1)}\left\{l(l+1)\left(-\frac{(s-1)}{2}\langle r^{s-2}\rangle\right) - \frac{2}{a}\left(-\frac{s}{2}\langle r^{s-1}\rangle\right) + \frac{1}{n^2a^2}\left(-\frac{(s+1)}{2}\langle r^s\rangle\right)\right\}$$

$$= l(l+1)\left(\frac{s-1}{s+1}\right)\langle r^{s-2}\rangle - \frac{2}{a}\left(\frac{s}{s+1}\right)\langle r^{s-1}\rangle + \frac{1}{n^2a^2}\langle r^s\rangle.$$

Plug Lemmas 1 and 3 into #:

$$l(l+1)\langle r^{s-2}\rangle - \frac{2}{a}\langle r^{s-1}\rangle + \frac{1}{n^2a^2}\langle r^s\rangle = -l(l+1)\left(\frac{s-1}{s+1}\right)\langle r^{s-2}\rangle + \frac{2}{a}\left(\frac{s}{s+1}\right)\langle r^{s-1}\rangle - \frac{1}{n^2a^2}\langle r^s\rangle + \frac{s(s-1)}{2}\langle r^{s-2}\rangle.$$

$$\frac{2}{n^2a^2}\langle r^s\rangle - \frac{2}{a}\underbrace{\left[1 + \frac{s}{s+1}\right]}_{\frac{2s+1}{s+1}}\langle r^{s-1}\rangle + \left\{l(l+1)\underbrace{\left[1 + \frac{s-1}{s+1}\right]}_{\frac{2s}{s+1}} - \frac{s(s-1)}{2}\right\}\langle r^{s-2}\rangle = 0.$$

$$\frac{2(s+1)}{n^2a^2}\langle r^s\rangle - \frac{2}{a}(2s+1)\langle r^{s-1}\rangle + 2s\left[l^2 + l - \frac{(s^2-1)}{4}\right]\langle r^{s-2}\rangle = 0, \text{ or:}$$

$$\frac{(s+1)}{n^2}\langle r^s\rangle - a(2s+1)\langle r^{s-1}\rangle + \frac{sa^2}{4}\big(\underbrace{4l^2+4l+1}_{(2l+1)^2} - s^2\big)\langle r^{s-2}\rangle = 0. \quad \text{QED.}$$

PROBLEM 6.30 (a) $\frac{1}{n^2}\langle 1\rangle - a\langle \frac{1}{r}\rangle + 0 = 0 \Rightarrow \boxed{\langle \frac{1}{r}\rangle = \frac{1}{n^2 a}}$.

$$\frac{2}{n^2}\langle r\rangle - 3a\langle 1\rangle + \frac{1}{4}\left[(2l+1)^2-1\right]a^2\langle \tfrac{1}{r}\rangle = 0 \Rightarrow \frac{2}{n^2}\langle r\rangle = 3a - l(l+1)a^2\frac{1}{n^2 a} = \frac{a}{n^2}\left(3n^2 - l(l+1)\right).$$

$$\therefore \boxed{\langle r\rangle = \frac{a}{2}\left[3n^2 - l(l+1)\right]}.$$

$$\frac{3}{n^2}\langle r^2\rangle - 5a\langle r\rangle + \frac{1}{2}\left[(2l+1)^2-4\right]a^2 = 0 \Rightarrow \frac{3}{n^2}\langle r^2\rangle = 5a\,\frac{a}{2}\left[3n^2 - l(l+1)\right] - \frac{a^2}{2}\left[(2l+1)^2-4\right]$$

$$\frac{3}{n^2}\langle r^2\rangle = \frac{a^2}{2}\left\{15n^2 - 5l(l+1) - 4l(l+1) - 1 + 4\right\} = \frac{a^2}{2}\left[15n^2 - 9l(l+1) + 3\right] = \frac{3a^2}{2}\left[5n^2 - 3l(l+1) + 1\right]$$

$$\boxed{\langle r^2\rangle = \frac{n^2 a^2}{2}\left[5n^2 - 3l(l+1) + 1\right]}.$$

$$\frac{4}{n^2}\langle r^3\rangle - 7a\langle r^2\rangle + \frac{3}{4}\left[(2l+1)^2-9\right]a^2\langle r\rangle = 0 \Rightarrow$$

$$\frac{4}{n^2}\langle r^3\rangle = 7a\,\frac{n^2a^2}{2}\left[5n^2 - 3l(l+1) + 1\right] - \frac{3}{4}\left[4l(l+1) - 8\right]a^2\,\frac{a}{2}\left[3n^2 - l(l+1)\right]$$

$$= \frac{a^3}{2}\left\{35n^4 - 21l(l+1)n^2 + 7n^2 - \left(3l(l+1) - 6\right)\left(3n^2 - l(l+1)\right)\right\}$$

$$= \frac{a^3}{2}\left\{35n^4 - 21l(l+1)n^2 + 7n^2 - 9l(l+1)n^2 + 3l^2(l+1)^2 + 18n^2 - 6l(l+1)\right\}$$

$$= \frac{a^3}{2}\left\{35n^4 + 25n^2 - 30l(l+1)n^2 + 3l^2(l+1)^2 - 6l(l+1)\right\}.$$

$$\boxed{\langle r^3\rangle = \frac{n^2a^3}{8}\left[35n^4 + 25n^2 - 30l(l+1)n^2 + 3l^2(l+1)^2 - 6l(l+1)\right]}.$$

(b) $0 + a\langle \frac{1}{r^2}\rangle - \frac{1}{4}\left[(2l+1)^2-1\right]a^2\langle \frac{1}{r^3}\rangle = 0 \Rightarrow \boxed{\langle \frac{1}{r^2}\rangle = a\,l(l+1)\langle \frac{1}{r^3}\rangle}$.

(c) $a\,l(l+1)\langle \frac{1}{r^3}\rangle = \frac{1}{(l+\frac{1}{2})n^3a^2} \Rightarrow \boxed{\langle \frac{1}{r^3}\rangle = \frac{1}{l(l+\frac{1}{2})(l+1)n^3a^3}}$. (Agrees with [6.63].)

PROBLEM 6.31 (a) $|100\rangle = \frac{1}{\sqrt{\pi a^3}}\,e^{-r/a}$ [4.80], so

$E_s^1 = \langle 100|H_s'|100\rangle = -eE_{ext}\frac{1}{\pi a^3}\int e^{-2r/a}(r\cos\theta)r^2\sin\theta\,dr\,d\theta\,d\phi$. But the θ integral is zero:

$$\int_0^\pi \cos\theta\sin\theta\,d\theta = \left.\frac{\sin^2\theta}{2}\right|_0^\pi = 0. \quad \text{So } E_s^1 = 0. \text{ QED}$$

(b) From Problem 4.11:

$$\begin{cases} |1\rangle = \psi_{200} = \frac{1}{\sqrt{2\pi a}}\frac{1}{2a}\left(1-\frac{r}{2a}\right)e^{-r/2a} \\ |2\rangle = \psi_{211} = -\frac{1}{\sqrt{\pi a}}\frac{1}{8a^2}\,r e^{-r/2a}\sin\theta e^{i\phi} \\ |3\rangle = \psi_{210} = \frac{1}{\sqrt{2\pi a}}\frac{1}{4a^2}\,r e^{-r/2a}\cos\theta \\ |4\rangle = \psi_{21-1} = \frac{1}{\sqrt{\pi a}}\frac{1}{8a^2}\,r e^{-r/2a}\sin\theta e^{-i\phi} \end{cases}$$

$$\left.\begin{aligned}
\langle 1|H_s'|1\rangle &= \{\cdots\}\int_0^\pi \cos\theta\sin\theta\,d\theta = 0\\
\langle 2|H_s'|2\rangle &= \{\cdots\}\int_0^\pi \sin^2\theta\cos\theta\sin\theta\,d\theta = 0\\
\langle 3|H_s'|3\rangle &= \{\cdots\}\int_0^\pi \cos^2\theta\cos\theta\sin\theta\,d\theta = 0\\
\langle 4|H_s'|4\rangle &= \{\cdots\}\int_0^\pi \sin^2\theta\cos\theta\sin\theta\,d\theta = 0\\
\langle 1|H_s'|2\rangle &= \{\cdots\}\int_0^{2\pi} e^{i\phi}\,d\phi = 0\\
\langle 1|H_s'|4\rangle &= \{\cdots\}\int_0^{2\pi} e^{-i\phi}\,d\phi = 0\\
\langle 2|H_s'|3\rangle &= \{\cdots\}\int_0^{2\pi} e^{-i\phi}\,d\phi = 0\\
\langle 2|H_s'|4\rangle &= \{\cdots\}\int_0^{2\pi} e^{-2i\phi}\,d\phi = 0\\
\langle 3|H_s'|4\rangle &= \{\cdots\}\int_0^{2\pi} e^{-i\phi}\,d\phi = 0
\end{aligned}\right\}$$

All matrix elements of H_s' are zero except $\langle 1|H_s'|3\rangle$ and $\langle 3|H_s'|1\rangle$ (which are complex conjugates, so only one needs to be evaluated).

$$\langle 1|H_s'|3\rangle = -eE_{ext}\frac{1}{\sqrt{2\pi a}}\frac{1}{2a}\frac{1}{\sqrt{2\pi a}}\frac{1}{4a^2}\int\left(1-\frac{r}{2a}\right)e^{-r/2a}\,re^{-r/2a}\cos\theta\,r\cos\theta\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{-eE_{ext}}{2\pi a\;8a^3}(2\pi)\Big\{\underbrace{\int_0^\pi\cos^2\theta\sin\theta\,d\theta}_{-\frac{\cos^3\theta}{3}\Big|_0^\pi=\frac{2}{3}}\Big\}\int_0^\infty\left(1-\frac{r}{2a}\right)e^{-r/a}r^4\,dr$$

$$= \frac{-eE_{ext}}{8a^4}\cdot\frac{2}{3}\cdot\left\{\int_0^\infty r^4e^{-r/a}dr - \frac{1}{2a}\int_0^\infty r^5e^{-r/a}dr\right\} = \frac{-eE_{ext}}{4\cdot 3\,a^4}\left(4!a^5 - \frac{1}{2a}5!a^6\right)$$

$$= -\frac{eE_{ext}}{4\cdot 3\,a^4}\,4\cdot 3\cdot 2\,a^5\left(1-\frac{5}{2}\right) = -eaE_{ext}(-3) = 3aeE_{ext}.$$

$$\therefore\; W = 3aeE_{ext}\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}.$$

We need the eigenvalues of this matrix. The characteristic equation is

$$\det\begin{pmatrix}-\lambda&0&1&0\\0&-\lambda&0&0\\1&0&-\lambda&0\\0&0&0&-\lambda\end{pmatrix} = -\lambda\begin{vmatrix}-\lambda&0&0\\0&-\lambda&0\\0&0&-\lambda\end{vmatrix} + \begin{vmatrix}0&-\lambda&0\\1&0&0\\0&0&-\lambda\end{vmatrix} = -\lambda(-\lambda)^3 + (-\lambda^2) = \lambda^2(\lambda^2-1) = 0.$$

The eigenvalues are $0, 0, 1$, and -1, so the perturbed energies are

$$\boxed{E_2,\;E_2,\;E_2+3aeE_{ext},\;E_2-3aeE_{ext}}\;;\;\boxed{\text{Three levels}}.$$

(c) $|2\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ and $|4\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ are the eigenvectors with eigenvalue 0; $|\pm\rangle \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\\pm1\\0\end{pmatrix}$ are eigenvectors with eigenvalues ±1. So the "good" states are $\boxed{\psi_{211},\;\psi_{21-1},\;\frac{1}{\sqrt{2}}(\psi_{200}+\psi_{210}),\;\frac{1}{\sqrt{2}}(\psi_{200}-\psi_{210})}$.

$$\langle \vec{p}_e\rangle_4 = -e\,\frac{1}{\pi a}\,\frac{1}{64a^4}\int r^2 e^{-r/a}\sin^2\theta\left(r\sin\theta\cos\phi\,\hat{i}+r\sin\theta\sin\phi\,\hat{j}+r\cos\theta\,\hat{k}\right)r^2\sin\theta\,dr\,d\theta\,d\phi,$$

but $\int_0^{2\pi}\cos\phi\,d\phi = \int_0^{2\pi}\sin\phi\,d\phi = 0$, $\int_0^{\pi}\sin^3\theta\cos\theta\,d\theta = \frac{\sin^4\theta}{4}\Big|_0^{\pi} = 0$,

So $\boxed{\langle \vec{p}_e\rangle_4 = 0}$. Likewise $\boxed{\langle \vec{p}_e\rangle_2 = 0}$.

$$\langle \vec{p}_e\rangle_{\pm} = -\tfrac{1}{2}e\int(\psi_1\pm\psi_3)^2\,\vec{r}\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= -\tfrac{1}{2}e\,\frac{1}{2\pi a}\,\frac{1}{4a^2}\int\left[\left(1-\frac{r}{2a}\right)\pm\frac{r}{2a}\cos\theta\right]^2 e^{-r/a}\,r\,(\underbrace{\sin\theta\cos\phi\,\hat{i}+\sin\theta\sin\phi\,\hat{j}}_{\phi\text{ integral}\to 0}+\cos\theta\,\hat{k})\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= -\frac{e}{2}\,\frac{\hat{k}}{2\pi a}\,\frac{1}{4a^2}\,2\pi\int\left[\left(1-\frac{r}{2a}\right)\pm\frac{r}{2a}\cos\theta\right]^2 r^3 e^{-r/a}\cos\theta\sin\theta\,dr\,d\theta.$$

But $\int_0^{\pi}\cos\theta\sin\theta\,d\theta = \int_0^{\pi}\cos^3\theta\sin\theta\,d\theta = 0$, so only the cross-term survives:

$$\langle \vec{p}_e\rangle_{\pm} = -\frac{e}{8a^3}\,\hat{k}\left(\pm\frac{1}{a}\right)\int\left(1-\frac{r}{2a}\right)r\cos\theta\,r^3 e^{-r/a}\cos\theta\sin\theta\,dr\,d\theta$$

$$= \pm\frac{-e}{8a^4}\,\hat{k}\left\{\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta\right\}\int_0^{\infty}\left(1-\frac{r}{2a}\right)r^4 e^{-r/a}\,dr = \pm\frac{(-e)}{8a^4}\hat{k}\,\frac{2}{3}\left[4!\,a^5-\frac{1}{2a}5!\,a^6\right]$$

$$= \pm e\hat{k}\,\frac{-1}{4\cdot 3a^4}\,4\cdot 3\cdot 2a^5\left(1-\frac{5}{2}\right) = \boxed{\pm 3ae\hat{k}}.$$

Problem 6.32 (a) The nine states are:

$l=0$: $|300\rangle = R_{30}Y_0^0$

$l=1$: $|311\rangle = R_{31}Y_1^1$, $|310\rangle = R_{31}Y_1^0$, $|31\,{-1}\rangle = R_{31}Y_1^{-1}$

$l=2$: $|322\rangle = R_{32}Y_2^2$, $|321\rangle = R_{32}Y_2^1$, $|320\rangle = R_{32}Y_2^0$, $|32\,{-1}\rangle = R_{32}Y_2^{-1}$, $|32\,{-2}\rangle = R_{32}Y_2^{-2}$

H_s' contains no ϕ dependence, so the ϕ integral will be: $\langle nlm|H_s'|n'l'm'\rangle = \{\cdots\}\int_0^{2\pi}e^{-im\phi}e^{im'\phi}d\phi$,

which is zero unless $m'=m$.

For diagonal elements: $\langle nlm|H_s'|nlm\rangle = \{\cdots\}\int_0^{\pi}\left(P_l^m(\cos\theta)\right)^2\cos\theta\sin\theta\,d\theta$. But (p.126) P_l^m is a polynomial (even or odd) in $\cos\theta$ — multiplied (if m is odd) by $\sin\theta$. Since $\sin^2\theta = 1-\cos^2\theta$, $(P_l^m(\cos\theta))^2$ is a polynomial in even powers of $\cos\theta$. So the θ integral is of the form

$$\int_0^{\pi}(\cos(\theta))^{2j+1}\sin\theta\,d\theta = -\frac{(\cos\theta)^{2j+2}}{(2j+2)}\Big|_0^{\pi} = 0. \quad \therefore \text{All diagonal elements are zero.}$$

There remain just 4 elements to calculate:

$m=m'=0$: $\langle 300|H_s'|310\rangle$, $\langle 300|H_s'|320\rangle$, and $\langle 310|H_s'|320\rangle$;

$m=m'=\pm1$: $\langle 31\pm1|H_s'|32\pm1\rangle$.

$$\langle 300|H_s'|310\rangle = -eE_{ext}\int R_{30}R_{31}r^3dr\int Y_0^0 Y_1^0\cos\theta\sin\theta\,d\theta\,d\phi \quad . \quad \text{From Table 4.6:}$$

$$\int R_{30}R_{31}r^3dr = \frac{2}{\sqrt{27}}\frac{1}{a^{3/2}}\,\frac{8}{27\sqrt{6}}\frac{1}{a^{3/2}}\frac{1}{a}\int\left(1-\frac{2r}{3a}+\frac{2r^2}{27a^2}\right)e^{-r/3a}\left(1-\frac{r}{6a}\right)re^{-r/3a}r^3dr. \quad \text{Let } x\equiv\frac{2r}{3a}.$$

$$= \frac{2^4}{3^5\sqrt{2}\,a^4}\left(\frac{3a}{2}\right)^5\int_0^\infty\left(1-x+\frac{x^2}{6}\right)\left(1-\frac{x}{4}\right)x^4e^{-x}dx = \frac{a}{2\sqrt{2}}\int_0^\infty\left(1-\frac{5}{4}x+\frac{5}{12}x^2-\frac{1}{24}x^3\right)x^4e^{-x}dx$$

$$= \frac{a}{2\sqrt{2}}\left(4!-\frac{5}{4}5!+\frac{5}{12}6!-\frac{1}{24}7!\right) = -9\sqrt{2}\,a.$$

$$\int Y_0^0Y_1^0\cos\theta\sin\theta\,d\theta\,d\phi = \frac{1}{\sqrt{4\pi}}\sqrt{\frac{3}{4\pi}}\int\cos\theta\cos\theta\sin\theta\,d\theta\,d\phi = \frac{\sqrt{3}}{4\pi}2\pi\int_0^\pi\cos^2\theta\sin\theta\,d\theta = \frac{\sqrt{3}}{2}\cdot\frac{2}{3} = \frac{\sqrt{3}}{3}.$$

$$\therefore\ \langle 300|H_s'|310\rangle = (-eE_{ext})(-9\sqrt{2}a)\left(\frac{\sqrt{3}}{3}\right) = \boxed{3\sqrt{6}\,aeE_{ext}}.$$

$$\langle 300|H_s'|320\rangle = -eE_{ext}\int R_{30}R_{32}r^3dr\int Y_0^0Y_2^0\cos\theta\sin\theta\,d\theta\,d\phi.$$

$$\int Y_0^0Y_2^0\cos\theta\sin\theta\,d\theta\,d\phi = \frac{1}{\sqrt{4\pi}}\sqrt{\frac{5}{16\pi}}\int(3\cos^2\theta-1)\cos\theta\sin\theta\,d\theta\,d\phi = 0. \quad \boxed{\langle 300|H_s'|320\rangle = 0.}$$

$$\langle 310|H_s'|320\rangle = -eE_{ext}\int R_{31}R_{32}r^3dr\int Y_1^0Y_2^0\cos\theta\sin\theta\,d\theta\,d\phi.$$

$$\int R_{31}R_{32}r^3dr = \frac{8}{27\sqrt{6}}\frac{1}{a^{3/2}}\frac{1}{a}\,\frac{4}{81\sqrt{30}}\frac{1}{a^{3/2}}\frac{1}{a^2}\int\left(1-\frac{r}{6a}\right)re^{-r/3a}r^2e^{-r/3a}r^3dr$$

$$= \frac{2^4}{3^8\sqrt{5}\,a^6}\left(\frac{3a}{2}\right)^7\int_0^\infty\left(1-\frac{x}{4}\right)x^6e^{-x}dx = \frac{a}{24\sqrt{5}}\left(6!-\frac{1}{4}7!\right) = -\frac{9\sqrt{5}}{2}a.$$

$$\int Y_1^0Y_2^0\sin\theta\cos\theta\,d\theta\,d\phi = \sqrt{\frac{3}{4\pi}}\sqrt{\frac{5}{16\pi}}\int\cos\theta(3\cos^2\theta-1)\cos\theta\sin\theta\,d\theta\,d\phi$$

$$= \frac{\sqrt{15}}{8\pi}2\pi\int_0^\pi(3\cos^4\theta-\cos^2\theta)\sin\theta\,d\theta = \frac{\sqrt{15}}{4}\left[-\frac{3}{5}\cos^5\theta+\frac{1}{3}\cos^3\theta\right]\Big|_0^\pi = \frac{2}{\sqrt{15}}.$$

$$\therefore\ \langle 310|H_s'|320\rangle = (-eE_{ext})\left(-\frac{9\sqrt{5}}{2}a\right)\left(\frac{2}{\sqrt{15}}\right) = \boxed{3\sqrt{3}\,aeE_{ext}}.$$

$$\langle 31\pm1|H_s'|32\pm1\rangle = -eE_{ext}\int R_{31}R_{32}r^3dr\int Y_1^{\pm1*}Y_2^{\pm1}\cos\theta\sin\theta\,d\theta\,d\phi.$$

$$\int Y_1^{\pm1*}Y_2^{\pm1}\cos\theta\sin\theta\,d\theta\,d\phi = \left(\mp\sqrt{\frac{3}{8\pi}}\right)\left(\mp\sqrt{\frac{15}{8\pi}}\right)\int\sin\theta\,e^{\mp i\phi}\sin\theta\cos\theta\,e^{\pm i\phi}\cos\theta\sin\theta\,d\theta\,d\phi$$

$$= \frac{3\sqrt{5}}{8\pi}2\pi\int_0^\pi\cos^2\theta(1-\cos^2\theta)\sin\theta\,d\theta = \frac{3}{4}\sqrt{5}\left(-\frac{\cos^3\theta}{3}+\frac{\cos^5\theta}{5}\right)\Big|_0^\pi = \frac{1}{\sqrt{5}}.$$

$$\therefore\ \langle 31\pm1|H_s'|32\pm1\rangle = (-eE_{ext})\left(-9\frac{\sqrt{5}}{2}a\right)\left(\frac{1}{\sqrt{5}}\right) = \boxed{\frac{9}{2}\,aeE_{ext}}$$

So the matrix representing H_s' is: (I have ordered the states in a convenient way, as indicated)

	300	310	320	311	321	31-1	32-1	322	32-2
300		$3\sqrt{6}$							
310	$3\sqrt{6}$		$3\sqrt{3}$						
320		$3\sqrt{3}$							
311					$9/2$				
321				$9/2$					
31-1							$9/2$		
32-1						$9/2$			
322									
32-2									

(aeE_{ext})

(All empty boxes are zero.)

(b) The perturbing matrix breaks into a 3x3 block, two 2x2 blocks, and two 1x1 blocks, so we can work out the eigenvalues in each block separately.

3x3: $3\sqrt{3}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$; $\det\begin{pmatrix} -\lambda & \sqrt{2} & 0 \\ \sqrt{2} & -\lambda & 1 \\ 0 & 1 & -\lambda \end{pmatrix} = -\lambda^3 + \lambda + 2\lambda = -\lambda(\lambda^2-3) = 0$. $\lambda = 0, \sqrt{3}, -\sqrt{3}$.

$E_1^1 = 0$, $E_2^1 = 9aeE_{ext}$, $E_3^1 = -9aeE_{ext}$.

2x2: $\frac{9}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0 \Rightarrow \lambda = 1, -1$. $E_4^1 = \frac{9}{2}aeE_{ext}$, $E_5^1 = -\frac{9}{2}aeE_{ext}$.

From the other 2x2 we get $E_6^1 = E_4^1$, $E_7^1 = E_5^1$, and from the 1x1's we get $E_8^1 = E_9^1 = 0$.

Thus the perturbations to the energy (E_3) are:

0 (degeneracy 3)
$\frac{9}{2}aeE_{ext}$ (degeneracy 2)
$-\frac{9}{2}aeE_{ext}$ (degeneracy 2)
$9aeE_{ext}$ (degeneracy 1)
$-9aeE_{ext}$ (degeneracy 1)

Problem 6.33 $[6.88] \Rightarrow E^1_{hf} = \dfrac{\mu_0 g_d e^2}{3\pi m_d m_e a^3}\langle \vec{S}_d \cdot \vec{S}_e \rangle$; $[6.90] \Rightarrow \vec{S}_d\cdot\vec{S}_e = \frac{1}{2}(S^2 - S_e^2 - S_d^2)$.

Electron has spin $\frac{1}{2}$, so $S_e^2 = \frac{1}{2}(\frac{3}{2})\hbar^2 = \frac{3}{4}\hbar^2$; deuteron has spin 1, so $S_d^2 = 1(2)\hbar^2 = 2\hbar^2$.

Total spin could be $\frac{3}{2}$ — in which case $S^2 = \frac{3}{2}(\frac{5}{2})\hbar^2 = \frac{15}{4}\hbar^2$ — or $\frac{1}{2}$ — in which case $S^2 = \frac{3}{4}\hbar^2$.

Thus $\langle \vec{S}_d \cdot \vec{S}_e \rangle = \begin{cases} \frac{1}{2}\left(\frac{15}{4}\hbar^2 - \frac{3}{4}\hbar^2 - 2\hbar^2\right) = \frac{1}{2}\hbar^2 \\ \frac{1}{2}\left(\frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2 - 2\hbar^2\right) = -\hbar^2 \end{cases}$; the difference is $\frac{3}{2}\hbar^2$, so $\Delta E = \dfrac{\mu_0 g_d e^2 \hbar^2}{2\pi m_d m_e a^3}$.

But $\mu_0\epsilon_0 = 1/c^2 \Rightarrow \mu_0 = \frac{1}{\epsilon_0 c^2}$, so $\Delta E = \frac{2 g_d e^2 \hbar^2}{4\pi\epsilon_0 m_d m_e c^2 a^3} = \frac{2 g_d \hbar^4}{m_d m_e^2 c^2 a^4} = \frac{3}{2}\frac{g_d}{g_p}\frac{m_p}{m_d}\Delta E_{\text{hydrogen}}$

(eq. [6.92]). Now $\lambda = \frac{c}{\nu} = \frac{ch}{\Delta E}$, so $\lambda_d = \frac{2}{3}\frac{g_p}{g_d}\frac{m_d}{m_p}\lambda_h$, and since $m_d = 2m_p$,

$\lambda_d = \frac{4}{3}\left(\frac{5.59}{1.71}\right)(21\text{ cm}) = \boxed{92\text{ cm}}$.

PROBLEM 6.34 (a) The potential energy of the electron (charge $-e$) at (x,y,z) due to q's at $x=\pm d$ alone is:

$$V = \frac{-eq}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{(x+d)^2+y^2+z^2}} + \frac{1}{\sqrt{(x-d)^2+y^2+z^2}}\right\}.$$ Expanding (with $d \gg x,y,z$):

$$\frac{1}{\sqrt{(x\pm d)^2+y^2+z^2}} = (x^2 \pm 2dx + d^2+y^2+z^2)^{-1/2} = (d^2 \pm 2dx + r^2)^{-1/2} = \frac{1}{d}\left(1 \pm \frac{2x}{d} + \frac{r^2}{d^2}\right)^{-1/2} \cong \frac{1}{d}\left(1 \mp \frac{x}{d} - \frac{r^2}{2d^2} + \frac{3}{8}\frac{4x^2}{d^2}\right)$$

$$= \frac{1}{d}\left[1 \mp \frac{x}{d} + \frac{1}{2d^2}(3x^2-r^2)\right].$$

$$\therefore V = \frac{-eq}{4\pi\epsilon_0 d}\left[1 - \frac{x}{d} + \frac{1}{2d^2}(3x^2-r^2) + 1 + \frac{x}{d} + \frac{1}{2d^2}(3x^2-r^2)\right] = -\frac{2eq}{4\pi\epsilon_0 d} - \frac{eq}{4\pi\epsilon_0 d^3}(3x^2-r^2)$$

$$= 2\beta d^2 + 3\beta x^2 - \beta r^2, \text{ where } \beta \equiv -\frac{e}{4\pi\epsilon_0}\frac{q}{d^3}.$$

Thus with all six charges in place, $H' = 2(\beta_1 d_1^2 + \beta_2 d_2^2 + \beta_3 d_3^2) + 3(\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) - r^2(\beta_1+\beta_2+\beta_3)$. QED.

(b) $$\langle 100|H'|100\rangle = \frac{1}{\pi a^3}\int e^{-2r/a} H' r^2 \sin\theta\, dr\, d\theta\, d\phi = V_0 + \frac{3}{\pi a^3}\int e^{-2r/a}(\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$- \frac{(\beta_1+\beta_2+\beta_3)}{\pi a^3}\underbrace{\int r^2 e^{-2r/a} r^2 \sin\theta\, dr\, d\theta\, d\phi}_{4\pi\int_0^\infty r^4 e^{-2r/a}dr = 4\pi\, 4!\left(\frac{a}{2}\right)^5 = 3\pi a^5}.$$

$$I \equiv \int e^{-2r/a}(\beta_1 x^2 + \beta_2 y^2 + \beta_3 z^2) r^2 \sin\theta\, dr\, d\theta\, d\phi = \int r^4 e^{-2r/a}(\beta_1 \sin^2\theta\cos^2\phi + \beta_2 \sin^2\theta \sin^2\phi + \beta_3\cos^2\theta)\, dr \sin\theta\, d\theta\, d\phi.$$

But $\int_0^{2\pi}\cos^2\phi\, d\phi = \int_0^{2\pi}\sin^2\phi\, d\phi = \pi$, $\int_0^{2\pi} d\phi = 2\pi$. So

$$I = \int_0^\infty r^4 e^{-2r/a} dr \int_0^\pi \left[\pi(\beta_1+\beta_2)\sin^2\theta + 2\pi\beta_3\cos^2\theta\right]\sin\theta\, d\theta.$$ But $\int_0^\pi \sin^3\theta\, d\theta = \frac{4}{3}$, $\int_0^\pi \cos^2\theta\sin\theta\, d\theta = \frac{2}{3}$.

$$= 4!\left(\frac{a}{2}\right)^5\left[\frac{4\pi}{3}(\beta_1+\beta_2) + \frac{4\pi}{3}\beta_3\right] = \pi a^5(\beta_1+\beta_2+\beta_3).$$

$$\therefore \langle 100|H'|100\rangle = V_0 + \frac{3}{\pi a^3}\pi a^5(\beta_1+\beta_2+\beta_3) - \frac{(\beta_1+\beta_2+\beta_3)}{\pi a^3}3\pi a^5 = \boxed{V_0}.$$

(c) The four states are
$$\left\{\begin{array}{l} |200\rangle = R_{20}Y_0^0; \\ |211\rangle = R_{21}Y_1^1; \quad |21\,-1\rangle = R_{21}Y_1^{-1} \\ |210\rangle = R_{21}Y_1^0; \end{array}\right\}$$ (Functional forms in Problem 4.11.)

CHAPTER 7

PROBLEM 7.1 (a) $\langle V\rangle = 2\alpha A^2\int_0^\infty x e^{-2bx^2}dx = 2\alpha A^2\left(-\frac{1}{4b}e^{-2bx^2}\right)\Big|_0^\infty = \frac{\alpha A^2}{2b} = \frac{\alpha}{2b}\sqrt{\frac{2b}{\pi}} = \frac{\alpha}{\sqrt{2b\pi}}$.

$\therefore \langle H\rangle = \frac{\hbar^2 b}{2m} + \frac{\alpha}{\sqrt{2\pi b}}$. $\quad \frac{\partial\langle H\rangle}{\partial b} = \frac{\hbar^2}{2m} - \frac{1}{2}\frac{\alpha}{\sqrt{2\pi}}b^{-3/2} = 0 \Rightarrow b^{3/2} = \frac{\alpha}{\sqrt{2\pi}}\frac{m}{\hbar^2}$; $\; b = \left(\frac{m\alpha}{\sqrt{2\pi}\hbar^2}\right)^{2/3}$.

$\langle H\rangle_{min} = \frac{\hbar^2}{2m}\left(\frac{m\alpha}{\sqrt{2\pi}\hbar^2}\right)^{2/3} + \frac{\alpha}{\sqrt{2\pi}}\left(\frac{\sqrt{2\pi}\,\hbar^2}{m\alpha}\right)^{1/3} = \frac{\alpha^{2/3}\hbar^{2/3}}{m^{1/3}(2\pi)^{1/3}}\left(\frac{1}{2}+1\right) = \boxed{\frac{3}{2}\left(\frac{\alpha^2\hbar^2}{2\pi m}\right)^{1/3}}$.

(b) $\langle V\rangle = 2\alpha A^2\int_0^\infty x^4 e^{-2bx^2}dx = 2\alpha A^2\frac{3}{8(2b)^2}\sqrt{\frac{\pi}{2b}} = \frac{3\alpha}{16b^2}\sqrt{\frac{\pi}{2b}}\sqrt{\frac{2b}{\pi}} = \frac{3\alpha}{16b^2}$.

$\therefore \langle H\rangle = \frac{\hbar^2 b}{2m} + \frac{3\alpha}{16b^2}$. $\quad \frac{\partial\langle H\rangle}{\partial b} = \frac{\hbar^2}{2m} - \frac{3\alpha}{8b^3} = 0 \Rightarrow b^3 = \frac{3\alpha m}{4\hbar^2}$; $\; b = \left(\frac{3\alpha m}{4\hbar^2}\right)^{1/3}$.

$\langle H\rangle_{min} = \frac{\hbar^2}{2m}\left(\frac{3\alpha m}{4\hbar^2}\right)^{1/3} + \frac{3\alpha}{16}\left(\frac{4\hbar^2}{3\alpha m}\right)^{2/3} = \frac{\alpha^{1/3}\hbar^{4/3}}{m^{2/3}}3^{1/3}4^{-1/3}\left(\frac{1}{2}+\frac{1}{4}\right) = \boxed{\frac{3}{4}\left(\frac{3\alpha\hbar^4}{4m^2}\right)^{1/3}}$.

PROBLEM 7.2 Normalize: $1 = 2|A|^2\int_0^\infty \frac{1}{(x^2+b^2)^2}dx = 2|A|^2\frac{\pi}{4b^3} = \frac{\pi}{2b^3}|A|^2$. $\quad A = \sqrt{\frac{2b^3}{\pi}}$.

Kinetic energy: $\langle T\rangle = -\frac{\hbar^2}{2m}|A|^2\int_{-\infty}^\infty \frac{1}{(x^2+b^2)}\underbrace{\frac{d^2}{dx^2}\left(\frac{1}{(x^2+b^2)}\right)}dx$

$$\frac{d}{dx}\left(\frac{-2x}{(x^2+b^2)^2}\right) = \frac{-2}{(x^2+b^2)^2} + 2x\frac{4x}{(x^2+b^2)^3} = \frac{2(3x^2-b^2)}{(x^2+b^2)^3}$$

$$\langle T\rangle = -\frac{\hbar^2}{2m}\cdot\frac{2b^3}{\pi}\cdot 4\int_0^\infty \frac{(3x^2-b^2)}{(x^2+b^2)^4}dx = -\frac{4\hbar^2 b^3}{\pi m}\left\{3\int_0^\infty\frac{1}{(x^2+b^2)^3}dx - 4b^2\int_0^\infty\frac{1}{(x^2+b^2)^4}dx\right\}$$

$$= -\frac{4\hbar^2 b^3}{\pi m}\left\{3\cdot\frac{3\pi}{16b^5} - 4b^2\cdot\frac{5\pi}{32b^7}\right\} = \frac{\hbar^2}{4mb^2}.$$

Potential energy: $\langle V\rangle = \frac{1}{2}m\omega^2|A|^2\,2\int_0^\infty\frac{x^2}{(x^2+b^2)^2}dx = m\omega^2\frac{2b^3}{\pi}\cdot\frac{\pi}{4b} = \frac{1}{2}m\omega^2b^2$.

$\therefore \langle H\rangle = \frac{\hbar^2}{4mb^2} + \frac{1}{2}m\omega^2b^2$. $\quad \frac{\partial\langle H\rangle}{\partial b} = -\frac{\hbar^2}{2mb^3} + m\omega^2 b = 0 \Rightarrow b^4 = \frac{\hbar^2}{2m^2\omega^2} \Rightarrow b^2 = \frac{1}{\sqrt{2}}\frac{\hbar}{m\omega}$.

$\therefore \langle H\rangle_{min} = \frac{\hbar^2}{4m}\frac{\sqrt{2}\,m\omega}{\hbar} + \frac{1}{2}m\omega^2\frac{1}{\sqrt{2}}\frac{\hbar}{m\omega} = \hbar\omega\left(\frac{\sqrt{2}}{4}+\frac{1}{2\sqrt{2}}\right) = \boxed{\frac{\sqrt{2}}{2}\hbar\omega} = 0.707\hbar\omega > \frac{1}{2}\hbar\omega$. ✓

PROBLEM 7.3 $V(x) = -\alpha\delta(x-\frac{a}{2})$. $\quad \langle T\rangle = \frac{6\hbar^2}{ma^2}$ [7.13]; $\langle V\rangle = -\alpha\int\delta(x-\frac{a}{2})|\psi(x)|^2dx = -\alpha|\psi(\frac{a}{2})|^2$,

or [7.10]: $\langle V\rangle = -\alpha|A|^2(\frac{a}{2})^2 = -\alpha\frac{a^2}{4}\cdot\frac{4}{a^2}\cdot\frac{3}{a}$ [7.11]. $\quad \langle V\rangle = -\frac{3\alpha}{a}$. $\quad \langle H\rangle = \frac{6\hbar^2}{ma^2} - \frac{3\alpha}{a}$.

$\frac{\partial\langle H\rangle}{\partial a} = -\frac{12\hbar^2}{ma^3} + \frac{3\alpha}{a^2} = 0 \Rightarrow a = \frac{4\hbar^2}{m\alpha}$. $\quad \therefore \langle H\rangle_{min} = \frac{6\hbar^2}{m}\frac{m^2\alpha^2}{16\hbar^4} - 3\alpha\frac{m\alpha}{4\hbar^2} = \boxed{-\frac{3m\alpha^2}{8\hbar^2}} > -\frac{m\alpha^2}{2\hbar^2}$. ✓

$\langle 21\text{-}1|x^2|211\rangle = -\frac{1}{64\pi a^5}\cdot 6!\,a^7\cdot\frac{16}{15}\cdot\frac{\pi}{2} = -6a^2$. For y^2 the ϕ integral is

$$\int_0^{2\pi} e^{2i\phi}\sin^2\phi\,d\phi = -\frac{1}{4}\int_0^{2\pi} e^{2i\phi}\left(e^{i\phi}-e^{-i\phi}\right)^2 d\phi = -\frac{1}{4}\int_0^{2\pi} e^{2i\phi}\left(e^{2i\phi}-2+e^{-2i\phi}\right)d\phi = -\frac{\pi}{2},\ \text{so}$$

$\langle 21\text{-}1|y^2|211\rangle = 6a^2$, and hence $\langle 21\text{-}1|z^2|211\rangle = 0$.

$\therefore\ \langle 21\text{-}1|H'|211\rangle = 3\left(\beta_1(-6a^2)+\beta_2(6a^2)\right) = \boxed{-18a^2(\beta_1-\beta_2)}$.

The perturbation matrix is:

	200	210	211	21-1
200	V_0	0	0	0
210	0	$V_0-12a^2(\beta_1+\beta_2)+24a^2\beta_3$	0	0
211	0	0	$V_0+6a^2(\beta_1+\beta_2)-12a^2\beta_3$	$-18a^2(\beta_1-\beta_2)$
21-1	0	0	$-18a^2(\beta_1-\beta_2)$	$V_0+6a^2(\beta_1+\beta_2)-12a^2\beta_3$

The 2x2 block has the form $\begin{pmatrix} A & B \\ B & A\end{pmatrix}$; its characteristic equation is $(A-\lambda)^2-B^2=0$, so $A-\lambda=\pm B$,

or $\lambda = A\mp B = V_0+6a^2(\beta_1+\beta_2)-12a^2\beta_3 \pm 18a^2(\beta_1-\beta_2) = \begin{cases} V_0+24a^2\beta_1-12a^2\beta_2-12a^2\beta_3 \\ V_0-12a^2\beta_1+24a^2\beta_2-12a^2\beta_3.\end{cases}$

The first-order corrections to the energy (E_2) are therefore:

$$\boxed{\begin{aligned} \epsilon_1 &= V_0 \\ \epsilon_2 &= V_0-12a^2(\beta_1+\beta_2-2\beta_3) \\ \epsilon_3 &= V_0-12a^2(-2\beta_1+\beta_2+\beta_3) \\ \epsilon_4 &= V_0-12a^2(\beta_1-2\beta_2+\beta_3)\end{aligned}}$$

(i) If $\beta_1=\beta_2=\beta_3$, $\epsilon_1=\epsilon_2=\epsilon_3=\epsilon_4=V_0$. $\boxed{\text{ONE LEVEL}}$ (still 4-fold degenerate).

(ii) If $\beta_1=\beta_2\neq\beta_3$, $\epsilon_1=V_0$, $\epsilon_2=V_0-24a^2(\beta_1-\beta_3)$, $\epsilon_3=\epsilon_4=V_0+12a^2(\beta_1-\beta_3)$. $\boxed{\text{THREE LEVELS}}$ (one remains doubly degenerate).

(iii) $\boxed{\text{FOUR LEVELS}}$.

Diagonal elements: $\langle n l m|H'|n l m\rangle = V_0 + 3\left(\beta_1\langle x^2\rangle + \beta_2\langle y^2\rangle + \beta_3\langle z^2\rangle\right) - (\beta_1+\beta_2+\beta_3)\langle r^2\rangle$.

For $|200\rangle$, $\langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle = \frac{1}{3}\langle r^2\rangle$ (Y_0^0 does not depend on ϕ, θ – the state has spherical symmetry)

So $\boxed{\langle 200|H'|200\rangle = V_0}$. (I could have used the same argument in (b).)

From Problem 6.30(a), $\langle r^2\rangle = \frac{n^2a^2}{2}\left[5n^2 - 3l(l+1)+1\right]$, so for $n=2, l=1$: $\langle r^2\rangle = 30a^2$.

Moreover, since $\langle x^2\rangle = \{\cdots\}\int_0^{2\pi}\cos^2\phi\, d\phi = \{\cdots\}\int_0^{2\pi}\sin^2\phi\, d\phi = \langle y^2\rangle$, and $\langle x^2\rangle + \langle y^2\rangle + \langle z^2\rangle = \langle r^2\rangle$,

it follows that $\langle x^2\rangle = \langle y^2\rangle = \frac{1}{2}\left(\langle r^2\rangle - \langle z^2\rangle\right) = 15a^2 - \frac{1}{2}\langle z^2\rangle$. So all we need to calculate is $\langle z^2\rangle$.

$$\langle 210|z^2|210\rangle = \frac{1}{2\pi a}\frac{1}{16a^4}\int r^2 e^{-r/a}\cos^2\theta\,(r^2\cos^2\theta)\, r^2\sin\theta\, dr\, d\theta\, d\phi = \frac{1}{16a^5}\int_0^\infty r^6 e^{-r/a}dr\int_0^\pi \cos^4\theta\sin\theta\, d\theta$$

$$= \frac{1}{16a^5}\, 6!\, a^7\, \frac{2}{5} = 18a^2;\quad \langle x^2\rangle = \langle y^2\rangle = 15a^2 - 9a^2 = 6a^2.$$

$$\therefore \langle 210|H'|210\rangle = V_0 + 3\left(6a^2\beta_1 + 6a^2\beta_2 + 18a^2\beta_3\right) - 30a^2(\beta_1+\beta_2+\beta_3) = \boxed{V_0 - 12a^2(\beta_1+\beta_2+\beta_3) + 36a^2\beta_3}$$

$$\langle 21\pm1|z^2|21\pm1\rangle = \frac{1}{\pi a}\frac{1}{64a^4}\int r^2 e^{-r/a}\sin^2\theta\,(r^2\cos^2\theta)\, r^2\sin\theta\, dr\, d\theta\, d\phi = \frac{1}{32a^5}\int_0^\infty r^6 e^{-r/a}dr\int_0^\pi (1-\cos^2\theta)\cos^2\theta\sin\theta\, d\theta$$

$$= \frac{1}{32a^5}\, 6!\, a^7\left(\frac{2}{3}-\frac{2}{5}\right) = 6a^2;\quad \langle x^2\rangle = \langle y^2\rangle = 15a^2 - 3a^2 = 12a^2.$$

$$\langle 21\pm1|H'|21\pm1\rangle = V_0 + 3\left(12a^2\beta_1 + 12a^2\beta_2 + 6a^2\beta_3\right) - 30a^2(\beta_1+\beta_2+\beta_3) = \boxed{V_0 + 6a^2(\beta_1+\beta_2+\beta_3) - 18a^2\beta_3}$$

Off-diagonal elements: We need $\langle 200|H'|210\rangle$, $\langle 200|H'|21\pm1\rangle$, $\langle 210|H'|21\pm1\rangle$, $\langle 21\,{-1}|H'|211\rangle$.

Now $\langle n l m|V_0|n'l'm'\rangle = 0$, by orthogonality, and $\langle n l m|r^2|n'l'm'\rangle = 0$ by orthogonality of Y_l^m,

so all we need are the matrix elements of x^2 and y^2 $\left(\langle|z^2|\rangle = -\langle|x^2|\rangle - \langle|y^2|\rangle\right)$.

For $\langle 200|x^2|21\pm1\rangle$ and $\langle 210|x^2|21\pm1\rangle$ the ϕ integral is $\int_0^{2\pi}\cos^2\phi\, e^{\pm i\phi}d\phi = \int_0^{2\pi}\cos^3\phi\, d\phi \pm i\int_0^{2\pi}\cos^2\phi\sin\phi\, d\phi$

$= 0$, and the same goes for y^2. So $\boxed{\langle 200|H'|21\pm1\rangle = \langle 210|H'|21\pm1\rangle = 0}$.

For $\langle 200|x^2|210\rangle$ and $\langle 200|y^2|210\rangle$ the θ integral is $\int_0^\pi \cos\theta\,(\sin^2\theta)\sin\theta\, d\theta = \left.\frac{\sin^4\theta}{4}\right|_0^\pi = 0$, so

$\boxed{\langle 200|H'|210\rangle = 0}$. Finally:

$$\langle 21\,{-1}|x^2|211\rangle = -\frac{1}{\pi a}\frac{1}{64a^4}\int r^2 e^{-r/a}\sin^2\theta\, e^{2i\phi}\left(r^2\sin^2\theta\cos^2\phi\right) r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= -\frac{1}{64\pi a^5}\underbrace{\int_0^\infty r^6 e^{-r/a}dr}_{6!\,a^7}\ \underbrace{\int_0^\pi \sin^5\theta\, d\theta}_{\frac{16}{15}}\ \underbrace{\int_0^{2\pi} e^{2i\phi}\cos^2\phi\, d\phi}_{\frac{1}{4}\int_0^{2\pi} e^{2i\phi}\left(e^{i\phi}+e^{-i\phi}\right)^2 d\phi = \frac{1}{4}\int_0^{2\pi} e^{2i\phi}\left(e^{2i\phi}+2+e^{-2i\phi}\right)d\phi = \frac{\pi}{2}}.$$

Problem 7.4 (a) Follow the proof in §7.1: $\psi=\sum_{n=1}^{\infty} c_n\psi_n$, where ψ_1 is the ground state. Since $\langle\psi_1|\psi\rangle=0$, we have: $\sum_{n=1}^{\infty} c_n\langle\psi_1|\psi_n\rangle = c_1 = 0$ — the coefficient of the ground state is zero. So

$$\langle H\rangle = \sum_{n=2}^{\infty} E_n|c_n|^2 \geq E_f \sum_{n=2}^{\infty}|c_n|^2 = E_f, \text{ since } E_n \geq E_f \text{ for all } n \text{ except } 1.$$

(b) $1=|A|^2\int_{-\infty}^{\infty} x^2 e^{-2bx^2}dx = |A|^2\, 2\cdot\frac{1}{8b}\sqrt{\frac{\pi}{2b}} \Rightarrow |A|^2 = 4b\sqrt{\frac{2b}{\pi}}$.

$$\langle T\rangle = -\frac{\hbar^2}{2m}|A|^2\int_{-\infty}^{\infty} xe^{-bx^2}\underbrace{\frac{d^2}{dx^2}\left(xe^{-bx^2}\right)}dx$$

$$\frac{d}{dx}\left(e^{-bx^2}-2bx^2e^{-bx^2}\right) = -2bxe^{-bx^2}-4bxe^{-bx^2}+4b^2x^3e^{-bx^2}$$

$$\langle T\rangle = -\frac{\hbar^2}{2m}4b\sqrt{\frac{2b}{\pi}}\,2\int_0^{\infty}\left(-6bx^2+4b^2x^4\right)e^{-2bx^2}dx = -\frac{2\hbar^2 b}{m}\sqrt{\frac{2b}{\pi}}\,2\left\{-6b\frac{1}{8b}\sqrt{\frac{\pi}{2b}}+4b^2\frac{3}{32b^2}\sqrt{\frac{\pi}{2b}}\right\}$$

$$= -\frac{4\hbar^2 b}{m}\left(-\frac{3}{4}+\frac{3}{8}\right) = \frac{3\hbar^2 b}{2m}.$$

$$\langle V\rangle = \frac{1}{2}m\omega^2|A|^2\int_{-\infty}^{\infty} x^2e^{-2bx^2}x^2dx = \frac{1}{2}m\omega^2 4b\sqrt{\frac{2b}{\pi}}\,2\cdot\frac{3}{32b^2}\sqrt{\frac{\pi}{2b}} = \frac{3m\omega^2}{8b}.$$

$$\langle H\rangle = \frac{3\hbar^2 b}{2m}+\frac{3m\omega^2}{8b};\quad \frac{\partial\langle H\rangle}{\partial b} = \frac{3\hbar^2}{2m}-\frac{3m\omega^2}{8b^2}=0 \Rightarrow b^2=\frac{m^2\omega^2}{4\hbar^2} \Rightarrow b=\frac{m\omega}{2\hbar}.$$

$\therefore \langle H\rangle_{min} = \frac{3\hbar^2}{2m}\frac{m\omega}{2\hbar}+\frac{3m\omega^2}{8}\frac{2\hbar}{m\omega} = \hbar\omega\left(\frac{3}{4}+\frac{3}{4}\right) = \boxed{\frac{3}{2}\hbar\omega}$. (which is exact, since the trial wave function is in the form of the true first excited state.)

Problem 7.5 (a) Use the unperturbed ground state (ψ_g^0) as the trial wave function. The variational principle says $\langle\psi_g^0|H|\psi_g^0\rangle \geq E_g$. But $H=H^0+H'$, so $\langle\psi_g^0|H|\psi_g^0\rangle = \langle\psi_g^0|H^0|\psi_g^0\rangle + \langle\psi_g^0|H'|\psi_g^0\rangle$. But $\langle\psi_g^0|H^0|\psi_g^0\rangle = E_g^0$ (the unperturbed ground state energy), and $\langle\psi_g^0|H'|\psi_g^0\rangle$ is precisely the first order correction to the ground state energy [6.9], so

$$E_g^0+E_g^1 \geq E_g. \quad \text{QED}.$$

(b) The second-order correction (E_g^2) is $E_g^2 = \sum_{m\neq g}\frac{|\langle\psi_m^0|H'|\psi_g^0\rangle|^2}{E_g^0-E_m^0}$. But the numerator is clearly positive, and the denominator is always negative (since $E_g^0 < E_m^0$ for all m), so E_g^2 is negative.

Problem 7.6 He^+ is a hydrogenic ion (see Problem 4.17); its ground state energy is $(2)^2(-13.6\text{eV})$, or -54.4eV. Takes $79.0-54.4 = \boxed{24.6\text{eV}}$ to remove one electron.

PROBLEM 7.7 I'll do the general case of a nucleus with Z_0 protons. Ignoring electron–electron repulsion altogether gives

$$\psi_0 = \frac{Z_0^3}{\pi a^3} e^{-Z_0(r_1+r_2)/a}, \qquad \text{(generalizing [7.17])}$$

and the energy is $2Z_0^2 E_1$.

$\langle V_{ee}\rangle$ goes like $\frac{1}{a}$ (eq. [7.20] and [7.25]), so the generalization of [7.25] is

$$\langle V_{ee}\rangle = -\frac{5}{4} Z_0 E_1,$$

and the generalization of [7.26] is $\langle H\rangle = \left(2Z_0^2 - \frac{5}{4}Z_0\right)E_1$.

If we include shielding, the only change is that $(Z-2)$ in equations [7.28], [7.29], and [7.32] is replaced by $(Z-Z_0)$. Thus [7.32] generalizes to

$$\langle H\rangle = \left[2Z^2 - 4Z(Z-Z_0) - \frac{5}{4}Z\right]E_1 = \left[-2Z^2 + 4ZZ_0 - \frac{5}{4}Z\right]E_1.$$

$$\frac{\partial\langle H\rangle}{\partial Z} = \left[-4Z + 4Z_0 - \frac{5}{4}\right]E_1 = 0 \Rightarrow \boxed{Z = Z_0 - \frac{5}{16}}.$$

$$\langle H\rangle_{min} = \left[-2\left(Z_0-\frac{5}{16}\right)^2 + 4\left(Z_0 - \frac{5}{16}\right)Z_0 - \frac{5}{4}\left(Z_0 - \frac{5}{16}\right)\right]E_1 = \left(-2Z_0^2 + \frac{5}{4}Z_0 - \frac{25}{128} + 4Z_0^2 - \frac{5}{4}Z_0 - \frac{5}{4}Z_0 + \frac{25}{64}\right)E_1.$$

$$\boxed{\langle H\rangle_{min} = \left(2Z_0^2 - \frac{5}{4}Z_0 + \frac{25}{128}\right)E_1} \quad \text{(generalizing [7.34])}.$$

The first term is the naive estimate ignoring electron-electron repulsion altogether; the second term is $\langle V_{ee}\rangle$ in the unscreened state, and the third term is the effect of screening.

For $Z_0 = 1$ (H^-): $Z = 1 - \frac{5}{16} = \boxed{\frac{11}{16} = 0.688}$. The effective nuclear charge is less than 1, as expected.

$$\langle H\rangle_{min} = \left(2 - \frac{5}{4} + \frac{25}{128}\right)E_1 = \boxed{\frac{121}{128}E_1 = -12.9\,\text{eV}}.$$

For $Z_0 = 2$ (He): $Z = 2 - \frac{5}{16} = \frac{27}{16} = 1.69$ (as before); $\langle H\rangle_{min} = \left(8 - \frac{5}{2} + \frac{25}{128}\right)E_1 = \frac{729}{128}E_1 = -77.5\,\text{eV}$.

For $Z_0 = 3$ (Li^+): $Z = 3 - \frac{5}{16} = \boxed{\frac{43}{16} = 2.69}$ (somewhat less than 3).

$$\langle H\rangle_{min} = \left(18 - \frac{15}{4} + \frac{25}{128}\right)E_1 = \frac{1849}{128}E_1 = \boxed{-196\,\text{eV}}.$$

PROBLEM 7.8 $D = a\langle\psi_g(r_1)|\frac{1}{r_2}|\psi_g(r_1)\rangle = a\langle\psi_g(r_2)|\frac{1}{r_1}|\psi_g(r_2)\rangle = a\frac{1}{\pi a^3}\int e^{-2r_2/a}\frac{1}{r_1}d^3r$

$$= \frac{1}{\pi a^3}\int e^{-\frac{2}{a}\sqrt{r^2+R^2-2rR\cos\theta}}\frac{1}{r}r^2\sin\theta\, dr\, d\theta\, d\phi = \frac{2\pi}{\pi a^3}\int_0^\infty r\left\{\int_0^\pi e^{-\frac{2}{a}\sqrt{r^2+R^2-2rR\cos\theta}}\sin\theta\, d\theta\right\}dr.$$

$$\{\} = \frac{1}{rR}\int_{|r-R|}^{(r+R)} e^{-2y/a}y\,dy = -\frac{a}{2rR}\left[e^{-2(r+R)/a}\left(r+R+\frac{a}{2}\right) - e^{-2|r-R|/a}\left(|r-R| + \frac{a}{2}\right)\right]$$

$$D = \frac{2}{a^2}\left(-\frac{a}{2R}\right)\left\{e^{-2R/a}\int_0^\infty e^{-2r/a}\left(r+R+\frac{a}{2}\right)dr - e^{-2R/a}\int_0^R e^{2r/a}\left(R-r+\frac{a}{2}\right)dr - e^{2R/a}\int_R^\infty e^{-2r/a}\left(r-R+\frac{a}{2}\right)dr\right\}$$

$$= -\frac{1}{aR}\left\{e^{-2R/a}\left[\left(\frac{a}{2}\right)^2+\left(R+\frac{a}{2}\right)\left(\frac{a}{2}\right)\right] - e^{-2R/a}\left(R+\frac{a}{2}\right)\left(\frac{a}{2}e^{2r/a}\right)\Big|_0^R + e^{-2R/a}\left(\frac{a}{2}\right)^2 e^{2r/a}\left(\frac{2r}{a}-1\right)\Big|_0^R\right.$$

$$\left. - e^{2R/a}\left(-R+\frac{a}{2}\right)\left(-\frac{a}{2}e^{-2r/a}\right)\Big|_R^\infty - e^{2R/a}\left(\frac{a}{2}\right)^2 e^{-2r/a}\left(-\frac{2r}{a}-1\right)\Big|_R^\infty\right\}$$

$$= -\frac{1}{aR}\left\{e^{-2R/a}\left[\frac{a^2}{4}+\frac{aR}{2}+\frac{a^2}{4}+\frac{aR}{2}+\frac{a^2}{4}+\frac{a^2}{4}\right]+\left[-\frac{aR}{2}-\frac{a^2}{4}+\frac{a^2}{4}\frac{2R}{a}-\frac{a^2}{4}+\frac{aR}{2}-\frac{a^2}{4}-\frac{a^2}{4}\frac{2R}{a}-\frac{a^2}{4}\right]\right\}$$

$$= -\frac{1}{aR}\left\{e^{-2R/a}\left[a^2+aR\right]+\left[-a^2\right]\right\} \Rightarrow \boxed{D = \frac{a}{R} - \left(1+\frac{a}{R}\right)e^{-2R/a}} \text{ (confirms [7.47]).}$$

$$X = a\langle\psi_0(r_1)|\frac{1}{r_1}|\psi_0(r_2)\rangle = a\frac{1}{\pi a^3}\int e^{-r_1/a}e^{-r_2/a}\frac{1}{r_1}d^3r = \frac{1}{\pi a^2}\int e^{-r/a}e^{-\sqrt{r^2+R^2-2rR\cos\theta}/a}\frac{1}{r}r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{2\pi}{\pi a^2}\int_0^\infty r e^{-r/a}\left\{\int_0^\pi e^{-\sqrt{r^2+R^2-2rR\cos\theta}/a}\sin\theta\,d\theta\right\}dr.$$

$$\{\,\} = -\frac{a}{rR}\left[e^{-(r+R)/a}(r+R+a) - e^{-|r-R|/a}(|r-R|+a)\right]$$

$$X = \frac{2}{a^2}\left(-\frac{a}{R}\right)\left\{e^{-R/a}\int_0^\infty e^{-2r/a}(r+R+a)dr - e^{-R/a}\int_0^R (R-r+a)dr - e^{R/a}\int_R^\infty e^{-2r/a}(r-R+a)dr\right\}$$

$$= -\frac{2}{aR}\left\{e^{-R/a}\left[\left(\frac{a}{2}\right)^2+(R+a)\left(\frac{a}{2}\right)\right] - e^{-R/a}\left[(R+a)R-\frac{R^2}{2}\right] - e^{R/a}(-R+a)\left(-\frac{a}{2}e^{-2r/a}\right)\Big|_R^\infty\right.$$

$$\left. - e^{R/a}\left(\frac{a}{2}\right)^2 e^{-2r/a}\left(-\frac{2r}{a}-1\right)\Big|_R^\infty\right\}$$

$$= -\frac{2}{aR}\left\{e^{-R/a}\left[\frac{a^2}{4}+\frac{aR}{2}+\frac{a^2}{2}-R^2-aR+\frac{R^2}{2}+\frac{aR}{2}-\frac{a^2}{2}-\frac{a^2}{4}\frac{2R}{a}-\frac{a^2}{4}\right]\right\}$$

$$= -\frac{2}{aR}e^{-R/a}\left(-\frac{aR}{2}-\frac{R^2}{2}\right) \Rightarrow \boxed{X = e^{-R/a}(1+R/a)} \text{ (confirms [7.48]).}$$

<u>PROBLEM 7.9</u> There are two changes: (1) the 2 in [7.38] changes sign ... which amounts to changing the sign of I in [7.43]; (2) the last term in [7.44] changes sign ... which amounts to reversing the sign of X. Thus [7.49] becomes

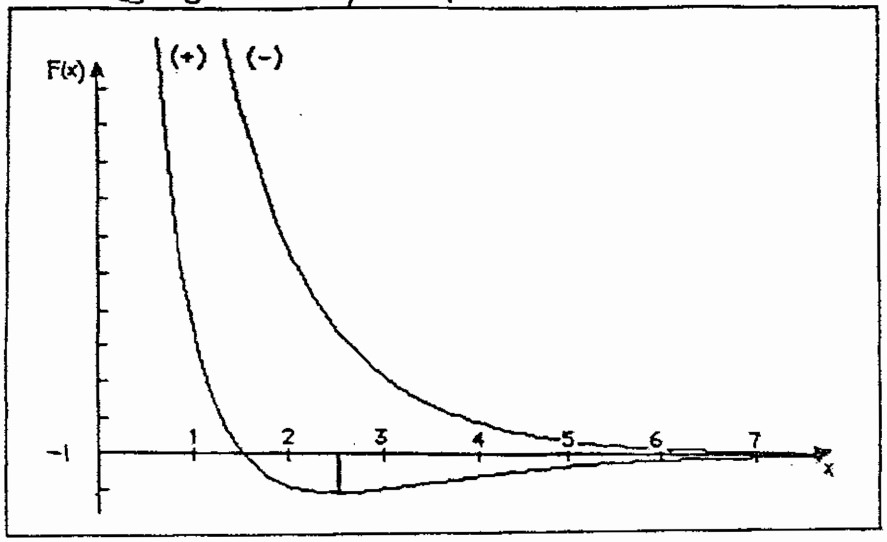
$$\langle H\rangle = \left[1+2\frac{(D-X)}{(1-I)}\right]E_1,$$

and hence [7.51] becomes

$$F(x) = \frac{E_{tot}}{-E_1} = \frac{2a}{R} - 1 - 2\frac{(D-X)}{(1-I)} = -1+\frac{2}{x}-2\frac{\frac{1}{x}-\left(1+\frac{1}{x}\right)e^{-2x}-(1+x)e^{-x}}{1-(1+x+x^2/3)e^{-x}}$$

$$= -1+\frac{2}{x}\left\{\frac{1-(1+x+x^2/3)e^{-x}-1+(x+1)e^{-2x}+(x+x^2)e^{-x}}{1-(1+x+x^2/3)e^{-x}}\right\} = \boxed{-1+\frac{2}{x}\left\{\frac{(1+x)e^{-2x}+\left(\frac{2}{3}x^2-1\right)e^{-x}}{1-(1+x+x^2/3)e^{-x}}\right\}}.$$

The graph (next page, with plus sign too for comparison) has no minimum, and remains above −1, indicating that the energy is greater than for the proton and atom dissociated. ∴ No evidence of bonding here.



PROBLEM 7.10 According to Mathematica the minimum occurs at $x = 2.493$, and at this point $F'' = 0.1257$.

$m\omega^2 = V'' = -\frac{E_1}{a^2}F''$, so $\omega = \frac{1}{a}\sqrt{\frac{-(0.1257)E_1}{m}}$. Here m is the reduced mass of the proton: $m = \frac{m_p m_p}{m_p + m_p}$,

so $m = \frac{1}{2}m_p$. ∴ $\omega = \frac{3\times10^8\,\text{m/s}}{(0.529\times10^{-10}\,\text{m})}\sqrt{\frac{(0.1257)(13.6\,\text{eV})}{(938\times10^6\,\text{eV})/2}} = \underline{\underline{3.42\times10^{14}/\text{s}}}$.

$\frac{1}{2}\hbar\omega = \frac{1}{2}(6.58\times10^{-16}\,\text{eV}\cdot\text{s})(3.42\times10^{14}/\text{s}) = \boxed{0.113\,\text{eV}}$ (ground state vibrational energy).

Mathematica says that at the minimum $F = -1.1297$, so the binding energy is $(0.1297)(13.6\,\text{eV}) =$ $\underline{1.76\,\text{eV}}$. Since this is substantially greater than the vibrational energy, it stays bound. The highest vibrational level is given by

$(n+\frac{1}{2})\hbar\omega = 1.76\,\text{eV}$, so $n = \frac{1.76}{0.226} - \frac{1}{2} = 7.29$. I'd estimate $\boxed{\text{EIGHT}}$ bound vibrational states (including $n=0$).

PROBLEM 7.11 $1 = |A|^2 \int e^{-2br^2} r^2 \sin\theta\, dr\, d\theta\, d\phi = 4\pi|A|^2 \int_0^\infty r^2 e^{-2br^2} dr = 4\pi|A|^2 \frac{1}{8b}\sqrt{\frac{\pi}{2b}} = |A|^2\left(\frac{\pi}{2b}\right)^{3/2}$.

$\therefore A = \left(\frac{2b}{\pi}\right)^{3/4}$.

$$\langle V\rangle = -\frac{e^2}{4\pi\epsilon_0}|A|^2 4\pi \int_0^\infty e^{-2br^2}\frac{1}{r}r^2 dr = -\frac{e^2}{4\pi\epsilon_0}\left(\frac{2b}{\pi}\right)^{3/2} 4\pi\frac{1}{4b} = -\frac{e^2}{4\pi\epsilon_0}2\sqrt{\frac{2b}{\pi}}.$$

$$\langle T\rangle = -\frac{\hbar^2}{2m}|A|^2 \int e^{-br^2}\underbrace{(\nabla^2 e^{-br^2})}\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}e^{-br^2}\right) = \frac{1}{r^2}\frac{d}{dr}\left(-2br^3 e^{-br^2}\right) = \frac{-2b}{r^2}\left(3r^2 - 2br^4\right)e^{-br^2}$$

$$\langle T\rangle = -\frac{\hbar^2}{2m}\left(\frac{2b}{\pi}\right)^{3/2}(4\pi)(-2b)\int_0^\infty (3r^2 - 2br^4)e^{-2br^2} dr = \frac{\hbar^2}{m}\pi b 4\left(\frac{2b}{\pi}\right)^{3/2}\left[3\frac{1}{8b}\sqrt{\frac{\pi}{2b}} - 2b\frac{3}{32b^2}\sqrt{\frac{\pi}{2b}}\right]$$

$$\langle T\rangle = \frac{\hbar^2}{m}4\pi b\left(\frac{2b}{\pi}\right)\left(\frac{3}{8b}-\frac{3}{16b}\right) = \frac{3\hbar^2 b}{2m}.$$

$$\langle H\rangle = \frac{3\hbar^2 b}{2m} - \frac{e^2}{4\pi\epsilon_0}2\sqrt{\frac{2b}{\pi}}. \qquad \frac{\partial\langle H\rangle}{\partial b} = \frac{3\hbar^2}{2m} - \frac{e^2}{4\pi\epsilon_0}\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{b}} = 0 \Rightarrow \sqrt{b} = \frac{e^2}{4\pi\epsilon_0}\sqrt{\frac{2}{\pi}}\frac{2m}{3\hbar^2}.$$

$$\therefore \langle H\rangle_{min} = \frac{3\hbar^2}{2m}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{2}{\pi}\frac{4m^2}{9\hbar^4} - \frac{e^2}{4\pi\epsilon_0}2\sqrt{\frac{2}{\pi}}\left(\frac{e^2}{4\pi\epsilon_0}\right)\sqrt{\frac{2}{\pi}}\frac{2m}{3\hbar^2} = \left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{m}{\hbar^2}\left(\frac{4}{3\pi}-\frac{8}{3\pi}\right)$$

$$= -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{8}{3\pi} = \boxed{\frac{8}{3\pi}E_1 = -11.5\,\text{eV}}.$$

<u>Problem 7.12</u> Let $\psi = \frac{1}{\sqrt{\pi b^3}}e^{-r/b}$ (same as hydrogen, but with $a \to b$ adjustable).

From [4.91] we have $\langle T\rangle = -E_1 = \frac{\hbar^2}{2ma^2}$ for hydrogen, so in this case $\langle T\rangle = \frac{\hbar^2}{2mb^2}$.

$$\langle V\rangle = -\frac{e^2}{4\pi\epsilon_0}\frac{4\pi}{\pi b^3}\int_0^\infty e^{-2r/b}\frac{e^{-\mu r}}{r}r^2\,dr = -\frac{e^2}{4\pi\epsilon_0}\frac{4}{b^3}\int_0^\infty e^{-(\mu+2/b)r}r\,dr = \frac{-e^2}{4\pi\epsilon_0}\frac{4}{b^3}\frac{1}{(\mu+2/b)^2}$$

$$\therefore \langle V\rangle = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{b(1+\mu b/2)^2}. \qquad \therefore \langle H\rangle = \frac{\hbar^2}{2mb^2} - \frac{e^2}{4\pi\epsilon_0}\frac{1}{b(1+\mu b/2)^2}.$$

$$\frac{\partial\langle H\rangle}{\partial b} = -\frac{\hbar^2}{mb^3} + \frac{e^2}{4\pi\epsilon_0}\left[\frac{1}{b^2(1+\mu b/2)^2} + \frac{\mu}{b(1+\mu b/2)^3}\right] = -\frac{\hbar^2}{mb^3} + \frac{e^2}{4\pi\epsilon_0}\frac{(1+3\mu b/2)}{b^2(1+\mu b/2)^3} = 0 \Rightarrow$$

$$\frac{\hbar^2}{m}\left(\frac{4\pi\epsilon_0}{e^2}\right) = b\frac{(1+3\mu b/2)}{(1+\mu b/2)^3}, \text{ or } \underline{b\frac{(1+3\mu b/2)}{(1+\mu b/2)^3} = a}.$$ This determines b ... but unfortunately

it gives a cubic equation. So we use the fact that μ is small to obtain a suitable approximate solution. If $\mu = 0$, then $b = a$ (of course), so $\mu a \ll 1 \Rightarrow \mu b \ll 1$ too. So let's expand in powers of μb:

$$a \cong b\left(1+3\mu b/2\right)\left(1-3\mu b/2+6(\mu b/2)^2\right) = b\left(1-\tfrac{9}{4}(\mu b)^2+\tfrac{6}{4}(\mu b)^2\right) = b\left(1-\tfrac{3}{4}(\mu b)^2\right).$$ Since the

$\frac{3}{4}(\mu b)^2$ term is already a second-order correction, we can replace b by a:

$$b \cong \frac{a}{\left(1-\frac{3}{4}(\mu b)^2\right)} \cong a\left(1+\tfrac{3}{4}(\mu a)^2\right).$$

$$\therefore \langle H\rangle_{min} = \frac{\hbar^2}{2ma^2\left(1+\frac{3}{4}(\mu a)^2\right)^2} - \frac{e^2}{4\pi\epsilon_0}\frac{1}{a\left(1+\frac{3}{4}(\mu a)^2\right)\left[1+\mu a/2\right]^2}$$

$$\cong \frac{\hbar^2}{2ma^2}\left(1-2\tfrac{3}{4}(\mu a)^2\right) - \frac{e^2}{4\pi\epsilon_0}\frac{1}{a}\left(1-\tfrac{3}{4}(\mu a)^2\right)\left(1-2\frac{\mu a}{2}+3\left(\frac{\mu a}{2}\right)^2\right)$$

$$= -E_1\left(1-\tfrac{3}{2}(\mu a)^2\right) + 2E_1\left(1-\mu a+\tfrac{3}{4}(\mu a)^2-\tfrac{3}{4}(\mu a)^2\right) = \boxed{E_1\left\{1-2(\mu a)+\tfrac{3}{2}(\mu a)^2\right\}}.$$

PROBLEM 7.13 (a) $H = \begin{pmatrix} E_a & h \\ h & E_b \end{pmatrix}$; $\det(H-\lambda) = (E_a-\lambda)(E_b-\lambda) - h^2 = 0 \Rightarrow \lambda^2 - \lambda(E_a+E_b) + E_aE_b - h^2 = 0.$

$$\lambda = \frac{1}{2}\left\{E_a + E_b \pm \sqrt{E_a^2 + 2E_aE_b + E_b^2 - 4E_aE_b + 4h^2}\right\} \Rightarrow \boxed{E_\pm = \frac{1}{2}\left\{E_a + E_b \pm \sqrt{(E_a-E_b)^2 + 4h^2}\right\}}.$$

(b) Zeroth order: $E_a^0 = E_a$, $E_b^0 = E_b$. First order: $E_a^1 = \langle\psi_a|H'|\psi_a\rangle = 0$, $E_b^1 = \langle\psi_b|H'|\psi_b\rangle = 0$.

Second order: $E_a^2 = \dfrac{|\langle\psi_b|H'|\psi_a\rangle|^2}{E_a - E_b} = -\dfrac{h^2}{E_b - E_a}$; $E_b^2 = \dfrac{|\langle\psi_a|H'|\psi_b\rangle|^2}{E_b - E_a} = \dfrac{h^2}{E_b - E_a}$.

So $\boxed{E_- \cong E_a - \dfrac{h^2}{(E_b - E_a)}}$; $\boxed{E_+ \cong E_b + \dfrac{h^2}{(E_b - E_a)}}$.

(c) $\langle H\rangle = \langle \cos\phi\,\psi_a + \sin\phi\,\psi_b | (H^0 + H') | \cos\phi\,\psi_a + \sin\phi\,\psi_b\rangle = \cos^2\phi\,\langle\psi_a|H^0|\psi_a\rangle + \sin^2\phi\,\langle\psi_b|H^0|\psi_b\rangle$
$+ \sin\phi\cos\phi\,\langle\psi_b|H'|\psi_a\rangle + \sin\phi\cos\phi\,\langle\psi_a|H'|\psi_b\rangle = E_a\cos^2\phi + E_b\sin^2\phi + 2h\sin\phi\cos\phi.$

$$\frac{\partial\langle H\rangle}{\partial\phi} = -E_a\,2\cos\phi\sin\phi + E_b\,2\sin\phi\cos\phi + 2h(\cos^2\phi - \sin^2\phi) = (E_b - E_a)\sin 2\phi + 2h\cos 2\phi = 0 \Rightarrow$$

$$\tan 2\phi = -\frac{2h}{E_b - E_a} = -\epsilon \quad \left(\text{let } \epsilon \equiv \frac{2h}{E_b - E_a}\right).$$

$\therefore \dfrac{\sin 2\phi}{\sqrt{1-\sin^2 2\phi}} = -\epsilon \Rightarrow \sin^2 2\phi = \epsilon^2(1 - \sin^2 2\phi)$, or $\sin^2 2\phi(1+\epsilon^2) = \epsilon^2$; $\sin 2\phi = \dfrac{\pm\epsilon}{\sqrt{1+\epsilon^2}}$.

$\cos^2 2\phi = 1 - \sin^2 2\phi = 1 - \dfrac{\epsilon^2}{1+\epsilon^2} = \dfrac{1}{1+\epsilon^2}$; $\cos 2\phi = \dfrac{\mp 1}{\sqrt{1+\epsilon^2}}$ (sign dictated by $\tan 2\phi = \dfrac{\sin 2\phi}{\cos 2\phi} = -\epsilon$).

$\cos^2\phi = \frac{1}{2}(1 + \cos 2\phi) = \frac{1}{2}\left(1 \mp \dfrac{1}{\sqrt{1+\epsilon^2}}\right)$; $\sin^2\phi = \frac{1}{2}(1 - \cos 2\phi) = \frac{1}{2}\left(1 \pm \dfrac{1}{\sqrt{1+\epsilon^2}}\right)$. So

$$\langle H\rangle_{min} = \frac{1}{2}E_a\left(1 \mp \frac{1}{\sqrt{1+\epsilon^2}}\right) + \frac{1}{2}E_b\left(1 \pm \frac{1}{\sqrt{1+\epsilon^2}}\right) \pm h\frac{\epsilon}{\sqrt{1+\epsilon^2}} = \frac{1}{2}\left\{E_a + E_b \pm \frac{(E_b - E_a + 2h\epsilon)}{\sqrt{1+\epsilon^2}}\right\}$$

But $$\frac{(E_b - E_a + 2h\epsilon)}{\sqrt{1+\epsilon^2}} = \frac{(E_b - E_a) + 2h\dfrac{2h}{(E_b-E_a)}}{\sqrt{1 + \dfrac{4h^2}{(E_b-E_a)^2}}} = \frac{(E_b-E_a)^2 + 4h^2}{\sqrt{(E_b-E_a)^2 + 4h^2}} = \sqrt{(E_b-E_a)^2 + 4h^2}.$$

So $\langle H\rangle_{min} = \frac{1}{2}\left\{E_a + E_b \pm \sqrt{(E_b-E_a)^2 + 4h^2}\right\}$. For minimum, we want the minus sign (+ is maximum).

$$\boxed{\langle H\rangle_{min} = \frac{1}{2}\left\{E_a + E_b - \sqrt{(E_b-E_a)^2 + 4h^2}\right\}}.$$

(d) If h is small, the exact result (a) can be expanded: $E_\pm = \frac{1}{2}\left\{(E_a+E_b) \pm (E_b - E_a)\sqrt{1 + \dfrac{4h^2}{(E_b-E_a)^2}}\right\} \Rightarrow$

$E_\pm \cong \frac{1}{2}\left\{E_a + E_b \pm (E_b - E_a)\left[1 + \dfrac{2h^2}{(E_b-E_a)^2}\right]\right\} = \frac{1}{2}\left\{E_a + E_b \pm (E_b - E_a) \pm \dfrac{2h^2}{(E_b-E_a)}\right\}$, so $E_+ \cong E_b + \dfrac{h^2}{(E_b - E_a)}$,

$E_- \cong E_a - \dfrac{h^2}{(E_b - E_a)}$, confirming the perturbation theory results (b). The variational principle (c) gets the ground state (E_-) exactly — not too surprising since the trial wave function [7.56] is almost the most general state (could be a relative phase factor $e^{i\theta}$).

Problem 7.14 For the electron, $\gamma = -e/m$, so $E_\pm = \pm e B_z \hbar/2m$ (eq. [4.161]). For consistency with Problem 7.13, $E_b > E_a$, so $\chi_b = \chi_+ = \begin{pmatrix}1\\0\end{pmatrix}$, $\chi_a = \chi_- = \begin{pmatrix}0\\1\end{pmatrix}$, $E_b = E_+ = \frac{eB_z\hbar}{2m}$, $E_a = E_- = -\frac{eB_z\hbar}{2m}$.

(a) $\langle\chi_a|H'|\chi_a\rangle = \frac{eB_x}{m}\frac{\hbar}{2}(0\;1)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{eB_x\hbar}{2m}(0\;1)\begin{pmatrix}1\\0\end{pmatrix} = 0$; $\langle\chi_b|H'|\chi_b\rangle = \frac{eB_x\hbar}{2m}(1\;0)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 0$;

$\langle\chi_a|H'|\chi_b\rangle = \frac{eB_x\hbar}{2m}(0\;1)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{eB_x\hbar}{2m}(0\;1)\begin{pmatrix}0\\1\end{pmatrix} = \frac{eB_x\hbar}{2m}$; $\langle\chi_b|H'|\chi_a\rangle = \frac{eB_x\hbar}{2m}(1\;0)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{eB_x\hbar}{2m}$.

$\therefore \boxed{h = \frac{eB_x\hbar}{2m}}$, and the conditions of Problem 7.13 are met.

(b) From Problem 7.13 (b), $E_g \cong E_a - \frac{h^2}{(E_b-E_a)} = -\frac{eB_z\hbar}{2m} - \frac{(eB_x\hbar/2m)^2}{(eB_z\hbar/m)} = \boxed{-\frac{e\hbar}{2m}\left(B_z + \frac{B_x^2}{2B_z}\right)}$.

(c) From Problem 7.13(c), $E_g = \frac{1}{2}\left\{E_a + E_b - \sqrt{(E_b-E_a)^2 + 4h^2}\right\}$ (it's actually the exact ground state).

$E_g = -\frac{1}{2}\sqrt{\left(\frac{eB_z\hbar}{m}\right)^2 + 4\left(\frac{eB_x\hbar}{2m}\right)^2} = \boxed{-\frac{e\hbar}{2m}\sqrt{B_z^2 + B_x^2}}$ (which was obvious from the start, since the square root is simply the magnitude of the total field).

Problem 7.15 (a) $\vec r_1 = \frac{1}{\sqrt2}(\vec u + \vec v)$; $\vec r_2 = \frac{1}{\sqrt2}(\vec u - \vec v)$. $\therefore r_1^2 + r_2^2 = \frac12(u^2 + 2\vec u\cdot\vec v + v^2 + u^2 - 2\vec u\cdot\vec v + v^2) = u^2 + v^2$.

$(\nabla_1^2 + \nabla_2^2)f(\vec r_1, \vec r_2) = \left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial y_1^2} + \frac{\partial^2 f}{\partial z_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial y_2^2} + \frac{\partial^2 f}{\partial z_2^2}\right)$. $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial u_x}\frac{\partial u_x}{\partial x_1} + \frac{\partial f}{\partial v_x}\frac{\partial v_x}{\partial x_1} = \frac{1}{\sqrt2}\left(\frac{\partial f}{\partial u_x} + \frac{\partial f}{\partial v_x}\right)$

$\frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial u_x}\frac{\partial u_x}{\partial x_2} + \frac{\partial f}{\partial v_x}\frac{\partial v_x}{\partial x_2} = \frac{1}{\sqrt2}\left(\frac{\partial f}{\partial u_x} - \frac{\partial f}{\partial v_x}\right)$. $\frac{\partial^2 f}{\partial x_1^2} = \frac{1}{\sqrt2}\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial u_x} + \frac{\partial f}{\partial v_x}\right) = \frac{1}{\sqrt2}\left[\frac{\partial^2 f}{\partial u_x^2}\frac{\partial u_x}{\partial x_1} + \frac{\partial^2 f}{\partial u_x\partial v_x}\frac{\partial v_x}{\partial x_1} + \frac{\partial^2 f}{\partial v_x\partial u_x}\frac{\partial u_x}{\partial x_1} + \frac{\partial^2 f}{\partial v_x^2}\frac{\partial v_x}{\partial x_1}\right]$

$= \frac12\left(\frac{\partial^2 f}{\partial u_x^2} + 2\frac{\partial^2 f}{\partial u_x\partial v_x} + \frac{\partial^2 f}{\partial v_x^2}\right)$; $\frac{\partial^2 f}{\partial x_2^2} = \frac{1}{\sqrt2}\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial u_x} - \frac{\partial f}{\partial v_x}\right) = \frac{1}{\sqrt2}\left[\frac{\partial^2 f}{\partial u_x^2}\frac{\partial u_x}{\partial x_2} + \frac{\partial^2 f}{\partial u_x\partial v_x}\frac{\partial v_x}{\partial x_2} - \frac{\partial^2 f}{\partial v_x\partial u_x}\frac{\partial u_x}{\partial x_2} - \frac{\partial^2 f}{\partial v_x^2}\frac{\partial v_x}{\partial x_2}\right]$

$= \frac12\left(\frac{\partial^2 f}{\partial u_x^2} - 2\frac{\partial^2 f}{\partial u_x\partial v_x} + \frac{\partial^2 f}{\partial v_x^2}\right)$. So $\left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}\right) = \left(\frac{\partial^2 f}{\partial u_x^2} + \frac{\partial^2 f}{\partial v_x^2}\right)$ — and likewise for y and z.

$\therefore \nabla_1^2 + \nabla_2^2 = \nabla_u^2 + \nabla_v^2$, and hence $H = -\frac{\hbar^2}{2m}(\nabla_u^2 + \nabla_v^2) + \frac12 m\omega^2(u^2+v^2) - \frac{\lambda}{4}m\omega^2 2v^2$, or

$H = \left[-\frac{\hbar^2}{2m}\nabla_u^2 + \frac12 m\omega^2 u^2\right] + \left[-\frac{\hbar^2}{2m}\nabla_v^2 + \frac12 m\omega^2 v^2 - \frac12\lambda m\omega^2 v^2\right]$. QED

(b) $\frac32\hbar\omega$ (for u part) and $\frac32\hbar\omega\sqrt{1-\lambda}$ (for v part): $\boxed{E_g = \frac32\hbar\omega\left(1 + \sqrt{1-\lambda}\right)}$.

(c) The ground state for a one-dimensional oscillator is $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$ [2.48 + 2.54].

So for 3-D oscillator it is: $\psi_0(\vec r) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar}$, and for two particles

$\psi(\vec r_1, \vec r_2) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/2} e^{-\frac{m\omega}{2\hbar}(r_1^2 + r_2^2)}$. (This is the analog to [7.17].)

$\therefore \langle H\rangle = \frac32\hbar\omega + \frac32\hbar\omega + \langle V_{ee}\rangle = 3\hbar\omega + \langle V_{ee}\rangle$ (the analog to [7.19]).

$\langle V_{ee}\rangle = -\frac{\lambda}{4}m\omega^2\left(\frac{m\omega}{\pi\hbar}\right)^3 \int e^{-\frac{m\omega}{\hbar}(r_1^2 + r_2^2)} \underbrace{(\vec r_1 - \vec r_2)^2}_{r_1^2 - 2\vec r_1\cdot\vec r_2 + r_2^2} d^3 r_1\, d^3 r_2$ (the analog to [7.20]).

The $\vec{r}_1\cdot\vec{r}_2$ term integrates to zero, by symmetry, and the r_2^2 term is the same as the r_1^2 term, so

$$\langle V_{ee}\rangle = -\frac{\lambda}{4}m\omega^2\left(\frac{m\omega}{\pi\hbar}\right)^3 2\int e^{-\frac{m\omega}{\hbar}(r_1^2+r_2^2)}r_1^2\,d^3r_1\,d^3r_2 = -\frac{\lambda}{2}m\omega^2\left(\frac{m\omega}{\pi\hbar}\right)^3(4\pi)^2\int_0^\infty e^{-m\omega r_2^2/\hbar}r_2^2dr_2\int_0^\infty e^{-m\omega r_1^2/\hbar}r_1^4dr_1$$

$$= -\lambda\frac{8m^4\omega^5}{\pi\hbar^3}\left[\frac{1}{4}\frac{\hbar}{m\omega}\sqrt{\frac{\pi\hbar}{m\omega}}\right]\left[\frac{3}{8}\left(\frac{\hbar}{m\omega}\right)^2\sqrt{\frac{\pi\hbar}{m\omega}}\right] = -\frac{3}{4}\lambda\hbar\omega. \quad \therefore \langle H\rangle = 3\hbar\omega - \frac{3}{4}\lambda\hbar\omega, \text{ or}$$

$\boxed{\langle H\rangle = 3\hbar\omega\left(1-\frac{\lambda}{4}\right)}$. The variational principle says this must exceed the exact ground-state energy (b).

$$3\hbar\omega\left(1-\frac{\lambda}{4}\right) \overset{?}{>} \frac{3}{2}\hbar\omega\left(1+\sqrt{1-\lambda}\right) \Leftrightarrow 2-\frac{\lambda}{2} \overset{?}{>} 1+\sqrt{1-\lambda} \Leftrightarrow 1-\frac{\lambda}{2} \overset{?}{>} \sqrt{1-\lambda} \Leftrightarrow 1-\lambda+\frac{\lambda^2}{4} \overset{?}{>} 1-\lambda \text{ (yes, it checks).}$$

In fact, if we expand the exact answer in powers of λ:

$$E_g \cong \frac{3}{2}\hbar\omega\left(1+1-\frac{1}{2}\lambda\right) = 3\hbar\omega\left(1-\frac{\lambda}{4}\right), \text{ which is the variational result.}$$

Problem 7.16 Normalization: $1 = \int|\psi|^2d^3r_1\,d^3r_2 = |A|^2\left\{\int \psi_1^2d^3r_1\int\psi_2^2d^3r_2 + 2\int\psi_1\psi_2d^3r_1\int\psi_1\psi_2d^3r_2 + \int\psi_2^2d^3r_1\int\psi_1^2d^3r_2\right\}$

$$= |A|^2\left(1+2S^2+1\right), \text{ where}$$

$$S \equiv \int\psi_1(r)\psi_2(r)d^3r = \frac{\sqrt{(Z_1Z_2)^3}}{\pi a^3}\int e^{-(Z_1+Z_2)r/a}4\pi r^2dr = \frac{4}{a^3}\left(\frac{y}{2}\right)^3 \quad \cdot \quad \left[\frac{2a^3}{(Z_1+Z_2)^3}\right]$$

$$= \left(y/x\right)^3. \qquad \therefore \boxed{A^2 = \frac{1}{2\left(1+(y/x)^6\right)}}.$$

$$H = -\frac{\hbar^2}{2m}\left(\nabla_1^2+\nabla_2^2\right) - \frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{r_1}+\frac{1}{r_2}\right) + \frac{e^2}{4\pi\epsilon_0}\frac{1}{|\vec{r}_1-\vec{r}_2|}, \text{ so}$$

$$H\psi = A\left\{\left[-\frac{\hbar^2}{2m}\left(\nabla_1^2+\nabla_2^2\right)-\frac{e^2}{4\pi\epsilon_0}\left(\frac{Z_1}{r_1}+\frac{Z_2}{r_2}\right)\right]\psi_1(r_1)\psi_2(r_2) + \left[-\frac{\hbar^2}{2m}\left(\nabla_1^2+\nabla_2^2\right)-\frac{e^2}{4\pi\epsilon_0}\left(\frac{Z_2}{r_1}+\frac{Z_1}{r_2}\right)\right]\psi_2(r_1)\psi_1(r_2)\right\}$$

$$+A\frac{e^2}{4\pi\epsilon_0}\left\{\left[\frac{Z_1-1}{r_1}+\frac{Z_2-1}{r_2}\right]\psi_1(r_1)\psi_2(r_2) + \left[\frac{Z_2-1}{r_1}+\frac{Z_1-1}{r_2}\right]\psi_2(r_1)\psi_1(r_2)\right\} + V_{ee}\psi, \text{ where } V_{ee} \equiv \frac{e^2}{4\pi\epsilon_0}\frac{1}{|\vec{r}_1-\vec{r}_2|}.$$

The term in first curly brackets is $(Z_1^2+Z_2^2)E_1\psi_1(r_1)\psi_2(r_2) + (Z_2^2+Z_1^2)\psi_2(r_1)\psi_1(r_2)$, so

$$H\psi = (Z_1^2+Z_2^2)E_1\psi + A\frac{e^2}{4\pi\epsilon_0}\left\{\left[\frac{Z_1-1}{r_1}+\frac{Z_2-1}{r_2}\right]\psi_1(r_1)\psi_2(r_2)+\left[\frac{Z_2-1}{r_1}+\frac{Z_1-1}{r_2}\right]\psi_2(r_1)\psi_1(r_2)\right\} + V_{ee}\psi.$$

$$\therefore \langle H\rangle = (Z_1^2+Z_2^2)E_1 + A^2\left(\frac{e^2}{4\pi\epsilon_0}\right)\left\{\left\langle\left(\psi_1(r_1)\psi_2(r_2)+\psi_2(r_1)\psi_1(r_2)\right)\right|\left(\left[\frac{Z_1-1}{r_1}+\frac{Z_2-1}{r_2}\right]\left|\psi_1(r_1)\psi_2(r_2)\right\rangle\right.\right.$$

$$\left.\left. + \left[\frac{Z_2-1}{r_1}+\frac{Z_1-1}{r_2}\right]\right)\left|\psi_2(r_1)\psi_1(r_2)\right\rangle\right\} + \langle V_{ee}\rangle.$$

$$\{\,\} = (Z_1-1)\langle\psi_1(r_1)|\tfrac{1}{r_1}|\psi_1(r_1)\rangle + (Z_2-1)\langle\psi_2(r_2)|\tfrac{1}{r_2}|\psi_2(r_2)\rangle + (Z_1-1)\langle\psi_2(r_1)|\tfrac{1}{r_1}|\psi_1(r_1)\rangle\langle\psi_1(r_2)|\psi_2(r_2)\rangle$$

$$+ (Z_2-1)\langle\psi_2(r_1)|\psi_1(r_1)\rangle\langle\psi_1(r_2)|\tfrac{1}{r_2}|\psi_2(r_2)\rangle + (Z_2-1)\langle\psi_1(r_1)|\tfrac{1}{r_1}|\psi_2(r_1)\rangle \cdot \langle\psi_2(r_2)|\psi_1(r_2)\rangle$$

$$+ (Z_1-1)\langle\psi_1(r_1)|\psi_2(r_1)\rangle\langle\psi_2(r_2)|\tfrac{1}{r_2}|\psi_1(r_2)\rangle + (Z_2-1)\langle\psi_2(r_1)|\tfrac{1}{r_1}|\psi_2(r_1)\rangle + (Z_1-1)\langle\psi_1(r_2)|\tfrac{1}{r_2}|\psi_1(r_2)\rangle$$

$$= 2(Z_1-1)\left\langle\tfrac{1}{r}\right\rangle_1 + 2(Z_2-1)\left\langle\tfrac{1}{r}\right\rangle_2 + 2(Z_1-1)\langle\psi_1|\psi_2\rangle\langle\psi_1|\tfrac{1}{r}|\psi_2\rangle + 2(Z_2-1)\langle\psi_1|\psi_2\rangle\langle\psi_1|\tfrac{1}{r}|\psi_2\rangle.$$

But $\langle \frac{1}{r} \rangle_1 = \langle \psi_1(r) | \frac{1}{r} | \psi_1(r) \rangle = \frac{Z_1}{a}$; $\langle \frac{1}{r} \rangle_2 = \frac{Z_2}{a}$, so

$$\langle H \rangle = (Z_1^2 + Z_2^2) E_1 + A^2 \left(\frac{e^2}{4\pi\epsilon_0} \right) 2 \left[\frac{1}{a}(Z_1 - 1) Z_1 + \frac{1}{a}(Z_2 - 1) Z_2 + (Z_1 + Z_2 - 2) \langle \psi_1 | \psi_2 \rangle \langle \psi_1 | \frac{1}{r} | \psi_2 \rangle \right] + \langle V_{ee} \rangle.$$

But $\langle \psi_1 | \psi_2 \rangle = S = (y/x)^3$, and $\langle \psi_1 | \frac{1}{r} | \psi_2 \rangle = \frac{\sqrt{(Z_1 Z_2)^3}}{\pi a^3} 4\pi \int e^{-(Z_1+Z_2)r/a} r\,dr = \frac{y^3}{2a^3} \left[\frac{a}{Z_1 + Z_2} \right]^2 = \frac{y^3}{2ax^2}$.

$$\langle H \rangle = (x^2 - \tfrac{1}{2}y^2) E_1 + A^2 \left(\frac{e^2}{4\pi\epsilon_0} \right) \frac{2}{a} \left\{ \left[Z_1^2 + Z_2^2 - (Z_1 + Z_2) \right] + (x-2) \left(\frac{y}{x} \right)^3 \frac{y^3}{2x^2} \right\} + \langle V_{ee} \rangle$$

$$= (x^2 - \tfrac{1}{2}y^2) E_1 - 4E_1 A^2 \left[x^2 - \tfrac{1}{2}y^2 - x + \tfrac{1}{2}(x-2) \frac{y^6}{x^5} \right] + \langle V_{ee} \rangle$$

$$\langle V_{ee} \rangle = \frac{e^2}{4\pi\epsilon_0} \langle \psi | \frac{1}{|\vec{r}_1 - \vec{r}_2|} | \psi \rangle = \left(\frac{e}{4\pi\epsilon_0} \right) A^2 \langle (\psi_1(r_1)\psi_2(r_2) + \psi_2(r_1)\psi_1(r_2)) | \frac{1}{|\vec{r}_1 - \vec{r}_2|} | \psi_1(r_1)\psi_2(r_2) + \psi_2(r_1)\psi_1(r_2) \rangle$$

$$= \left(\frac{e}{4\pi\epsilon_0} \right) A^2 \left\{ 2 \langle \psi_1(r_1)\psi_2(r_2) | \frac{1}{|\vec{r}_1 - \vec{r}_2|} | \psi_1(r_1)\psi_2(r_2) \rangle + 2 \langle \psi_1(r_1)\psi_2(r_2) | \frac{1}{|\vec{r}_1 - \vec{r}_2|} | \psi_2(r_1)\psi_1(r_2) \rangle \right\}$$

$$= 2 \left(\frac{e}{4\pi\epsilon_0} \right) A^2 (B + C), \text{ where}$$

$$B \equiv \langle \psi_1(r_1)\psi_2(r_2) | \frac{1}{|\vec{r}_1 - \vec{r}_2|} | \psi_1(r_1)\psi_2(r_2) \rangle \; ; \; C \equiv \langle \psi_1(r_1)\psi_2(r_2) | \frac{1}{|\vec{r}_1 - \vec{r}_2|} | \psi_2(r_1)\psi_1(r_2) \rangle.$$

$B = \frac{Z_1^3 Z_2^3}{(\pi a^3)^2} \int e^{-2Z_1 r_1/a} e^{-2Z_2 r_2/a} \frac{1}{|\vec{r}_1 - \vec{r}_2|} d^3r_1 d^3r_2$. As on p. 263, the r_2 integral is

$$\int e^{-2Z_2 r_2/a} \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta_2}} d^3r_2 = \left([7.24], \text{but with } a \to \tfrac{2}{Z_2} a \right) = \frac{\pi a^3}{Z_2^3 r_1} \left[1 - \left(1 + \frac{Z_2 r_1}{a} \right) e^{-2Z_2 r_1/a} \right]$$

$$\therefore B = \frac{Z_1^3 Z_2^3}{(\pi a^3)^2} \frac{(\pi a^3)}{Z_2^3} 4\pi \int_0^\infty e^{-2Z_1 r_1/a} \frac{1}{r_1} \left[1 - \left(1 + \frac{Z_2 r_1}{a} \right) e^{-2Z_2 r_1/a} \right] r_1^2 dr_1$$

$$= \frac{4Z_1^3}{a^3} \int_0^\infty \left\{ r_1 e^{-2Z_1 r_1/a} - r_1 e^{-2(Z_1+Z_2)r_1/a} - \frac{Z_2}{a} r_1^2 e^{-2(Z_1+Z_2)r_1/a} \right\} dr_1$$

$$= \frac{4Z_1^3}{a^3} \left\{ \left(\frac{a}{2Z_1} \right)^2 - \left(\frac{a}{2(Z_1+Z_2)} \right)^2 - \frac{Z_2}{a} 2 \left(\frac{a}{2(Z_1+Z_2)} \right)^3 \right\} = \frac{Z_1^3}{a} \left(\frac{1}{Z_1^2} - \frac{1}{(Z_1+Z_2)^2} - \frac{Z_2}{(Z_1+Z_2)^3} \right)$$

$$= \frac{Z_1 Z_2}{a(Z_1+Z_2)} \left[1 + \frac{Z_1 Z_2}{(Z_1+Z_2)^2} \right] = \frac{y^2}{4ax} \left(1 + \frac{y^2}{4x^2} \right).$$

$$C = \frac{Z_1^3 Z_2^3}{(\pi a^3)^2} \int e^{-Z_1 r_1/a} e^{-Z_1 r_2/a} e^{-Z_2 r_1/a} e^{-Z_1 r_2/a} \frac{1}{|\vec{r}_1 - \vec{r}_2|} d^3r_1 d^3r_2 = \frac{(Z_1 Z_2)^3}{(\pi a^3)^2} \int e^{-(Z_1+Z_2)(r_1+r_2)/a} \frac{1}{|\vec{r}_1 - \vec{r}_2|} d^3r_1 d^3r_2$$

The integral is same as in [7.20], only with $a \to \frac{4}{Z_1+Z_2} a$. Comparing [7.20] and [7.25], we see that the integral itself was $\frac{5}{4a} \left(\frac{\pi a^3}{8} \right)^2 = \frac{5}{256} \pi^2 a^5$. So $C = \frac{(Z_1 Z_2)^3}{(\pi a^3)^2} \frac{5\pi^2}{256} \frac{4^5 a^5}{(Z_1+Z_2)^5}$.

$$C = \frac{20}{a}\frac{(Z_1Z_2)^3}{(Z_1+Z_2)^5} = \frac{5}{16a}\frac{y^6}{x^5}.$$

$$\langle V_{ee}\rangle = 2\left(\frac{e}{4\pi\epsilon_0}\right)A^2\left[\frac{y^2}{4ax}\left(1+\frac{y^2}{4x^2}\right)+\frac{5}{16a}\frac{y^6}{x^5}\right] = 2A^2(-2E_1)\frac{y^2}{4x}\left(1+\frac{y^2}{4x^2}+\frac{5}{4}\frac{y^4}{x^4}\right)$$

$$\langle H\rangle = E_1\left\{x^2-\tfrac{1}{2}y^2 - \frac{2}{(1+(y/x)^6)}\left[x^2-\tfrac{1}{2}y^2-x+\tfrac{1}{2}(x-2)\frac{y^6}{x^5}\right] - \frac{2}{(1+(y/x)^6)}\frac{y^2}{4x}\left(1+\frac{y^2}{4x^2}+\frac{5}{4}\frac{y^4}{x^4}\right)\right\}$$

$$= \frac{E_1}{(x^6+y^6)}\left\{(x^2-\tfrac{1}{2}y^2)(x^6+y^6) - 2x^6\left[x^2-\tfrac{1}{2}y^2-x+\tfrac{1}{2}\frac{y^6}{x^4}-\frac{y^6}{x^5}+\frac{y^2}{4x}+\frac{y^4}{16x^3}+\frac{5y^6}{16x^5}\right]\right\}$$

$$= \frac{E_1}{(x^6+y^6)}\left(x^8+x^2y^6-\tfrac{1}{2}x^6y^2-\tfrac{1}{2}y^8-2x^8+x^6y^2+2x^7-x^2y^6+2xy^6-\tfrac{1}{2}x^5y^2-\tfrac{1}{8}x^3y^4-\tfrac{5}{8}xy^6\right)$$

$$= \boxed{\frac{E_1}{(x^6+y^6)}\left(-x^8+2x^7+\tfrac{1}{2}x^6y^2-\tfrac{1}{2}x^5y^2-\tfrac{1}{8}x^3y^4+\tfrac{11}{8}xy^6-\tfrac{1}{2}y^8\right)}.$$

Mathematica finds the minimum of $\langle H\rangle$ at $x = 1.32245$, $y = 1.08505$, corresponding to $Z_1 = 1.0392$, $Z_2 = 0.2832$.
At this point $\boxed{\langle H\rangle_{min} = 1.0266\,E_1 = -13.962\,\text{eV}}$, which is less than -13.6 eV (but not by much!).

Problem 7.17 Calculation is same as before, but with $m_e \to m_\mu(\text{reduced})$, where

$$m_\mu(\text{red}) = \frac{m_\mu m_d}{m_\mu + m_d} = \frac{m_\mu 2m_p}{m_\mu + 2m_p} = \frac{m_\mu}{1+\frac{m_\mu}{2m_p}}.$$ From Problem 6.26, $m_\mu = 207\,m_e$, so

$$1+\frac{m_\mu}{2m_p} = 1+\left(\frac{207}{2}\right)\frac{(9.11\times10^{-31})}{(1.67\times10^{-27})} = 1.056;\quad m_\mu(\text{red}) = \frac{207\,m_e}{1.056} = 196\,m_e.$$ This shrinks the whole molecule down by a factor of almost 200 — bringing the deuterons much closer together, as desired. The equilibrium separation for the electron case was $2.493a$ (Problem 7.10), so for muons

$$R = \frac{2.493}{196}(.529\times10^{-10}\,\text{m}) = \boxed{6.73\times10^{-13}\,\text{m}}$$

Chapter 8

Problem 8.1 $\int_0^a p(x)dx = n\pi\hbar$, with $n = 1,2,3,\ldots$ and $p(x) = \sqrt{2m(E-V(x))}$ [8.16].

Here $\int_0^a p(x)dx = \sqrt{2mE}\left(\frac{a}{2}\right) + \sqrt{2m(E-V_0)}\left(\frac{a}{2}\right) = \sqrt{2m}\left(\frac{a}{2}\right)\left(\sqrt{E}+\sqrt{E-V_0}\right) = n\pi\hbar \Rightarrow$

$$E + E - V_0 + 2\sqrt{E(E-V_0)} = \frac{4}{2m}\left(\frac{n\pi\hbar}{a}\right)^2 = 4E_n^0;\quad 2\sqrt{E(E-V_0)} = (4E_n^0 - 2E + V_0).$$ Square again:

$$4E(E-V_0) = \cancel{4E^2} - \cancel{4EV_0} = 16E_n^{0\,2} + \cancel{4E^2} + V_0^2 - 16EE_n^0 + 8E_n^0V_0 - \cancel{4EV_0} \Rightarrow 16EE_n^0 = 16E_n^{0\,2} + 8E_n^0V_0 + V_0^2$$

$\boxed{E_n = E_n^0 + \frac{V_0}{2} + \frac{V_0^2}{16E_n^0}}$. Perturbation theory gave $E_n = E_n^0 + \frac{V_0}{2}$; the extra term goes to zero for very small V_0 (or, since $E_n^0 \sim n^2$) for large n.

Problem 8.2 (a) $\frac{d\psi}{dx} = \frac{i}{\hbar}f'e^{if/\hbar}$; $\frac{d^2\psi}{dx^2} = \frac{i}{\hbar}\left(f''e^{if/\hbar} + \frac{i}{\hbar}(f')^2e^{if/\hbar}\right) = \left[\frac{i}{\hbar}f'' - \frac{1}{\hbar^2}(f')^2\right]e^{if/\hbar}$.

$$\frac{d^2\psi}{dx^2} = -\frac{p^2}{\hbar^2}\psi \Rightarrow \left[\frac{i}{\hbar}f'' - \frac{1}{\hbar^2}(f')^2\right]e^{if/\hbar} = -\frac{p^2}{\hbar^2}e^{if/\hbar} \Rightarrow i\hbar f'' - (f')^2 + p^2 = 0.\ \text{QED}$$

(b) $f' = f_0' + \hbar f_1' + \hbar^2 f_2' + \cdots \Rightarrow (f')^2 = (f_0' + \hbar f_1' + \hbar^2 f_2' + \cdots)^2 = (f_0')^2 + 2\hbar f_0'f_1' + \hbar^2(2f_0'f_2' + (f_1')^2) + \cdots$

$f'' = f_0'' + \hbar f_1'' + \hbar^2 f_2'' + \cdots$. $\therefore i\hbar(f_0'' + \hbar f_1'' + \hbar^2 f_2'') - (f_0')^2 - 2\hbar f_0'f_1' - \hbar^2(2f_0'f_2' + (f_1')^2) + p^2 + \cdots = 0$

$\underline{\hbar^0}$: $(f_0')^2 = p^2$; $\underline{\hbar^1}$: $if_0'' = 2f_0'f_1'$; $\underline{\hbar^2}$: $if_1'' = 2f_0'f_2' + (f_1')^2$; $\cdots$

(c) $\frac{df_0}{dx} = \pm p \Rightarrow f_0 = \pm\int p(x)dx + \text{constant}$; $\frac{df_1}{dx} = \frac{i}{2}\frac{f_0''}{f_0'} = \frac{i}{2}\frac{\pm p'}{\pm p} = \frac{i}{2}\frac{p'}{p} = \frac{i}{2}\frac{d}{dx}\ln p \Rightarrow f_1 = \frac{i}{2}\ln p + \text{constant}$.

$$\therefore \psi = e^{if/\hbar} = e^{\frac{i}{\hbar}\left(\pm\int p(x)dx + \hbar\frac{i}{2}\ln p + K\right)} = e^{\pm\frac{i}{\hbar}\int p\,dx}\,p^{-1/2}\,e^{iK/\hbar} = \frac{C}{\sqrt{p}}e^{\pm\frac{i}{\hbar}\int p\,dx}.\ \text{QED}.$$

Problem 8.3 $\gamma = \frac{1}{\hbar}\int|p(x)|dx = \frac{1}{\hbar}\int_0^{2a}\sqrt{2m(V_0-E)}\,dx = \frac{2a}{\hbar}\sqrt{2m(V_0-E)}$. $\boxed{T \cong e^{-4a\sqrt{2m(V_0-E)}/\hbar}}$.

From Problem 2.32, the exact answer is:
$$T = \frac{1}{1 + \frac{V_0^2}{4E(V_0-E)}\sinh^2\gamma}.$$
Now, the WKB approximation assumes the tunneling probability is small (p. 281) — which is to say, that γ is large. In this case $\sinh\gamma = \frac{1}{2}(e^\gamma - e^{-\gamma}) \cong \frac{1}{2}e^\gamma$, and $\sinh^2\gamma \cong \frac{1}{4}e^{2\gamma}$, and the exact result becomes

$$T \cong \frac{1}{1 + \frac{V_0^2}{16E(V_0-E)}e^{2\gamma}} \cong \left\{\frac{16E(V_0-E)}{V_0^2}\right\}e^{-2\gamma}.$$ The coefficient in {} is of order 1 — the dominant dependence on E is in the exponential factor. In this sense $T \approx e^{-2\gamma}$ (the WKB result).

PROBLEM 8.4 I take the masses from Thornton and Rex, *Modern Physics*, Appendix 8. They are all atomic masses, but the electron masses subtract out in the calculation of E. All masses are in atomic units (u) – $1\,u = 931\ \text{MeV}/c^2$. The mass of He^4 is $4.002602\ u$, and that of the α-particle is $3727\ \text{MeV}/c^2$.

U^{238}: $Z=92,\ A=238,\ m=238.050784\ u \longrightarrow Th^{234}$: $m=234.043593\ u$.

$$r_1 = (1.07\times10^{-15}\text{m})(238)^{1/3} = 6.63\times10^{-15}\text{m}.$$

$$E = (238.050784 - 234.043593 - 4.002602)(931)\ \text{MeV} = 4.27\ \text{MeV}.$$

$$v = \sqrt{\frac{2E}{m}} = \sqrt{\frac{(2)(4.27)}{3727}}\times 3\times10^8\ \text{m/s} = 1.44\times10^7\ \text{m/s}.$$

$$\gamma = 1.980\frac{90}{\sqrt{4.27}} - 1.485\sqrt{90(6.63)} = \qquad 86.19 - 36.28 = 49.9.$$

$$\tau = \frac{(2)(6.63\times10^{-15})}{1.44\times10^7}e^{188}\ \text{s} = 7.46\times10^{21}\ \text{s} = \frac{7.46\times10^{21}}{3.15\times10^7}\ \text{yr} = \boxed{2.4\times10^{14}\ \text{yrs.}}$$

Po^{212}: $Z=84,\ A=212,\ m=211.988842\ u \longrightarrow Pb^{208}$: $m=207.976627\ u$.

$$r_1 = (1.07\times10^{-15}\text{m})(212)^{1/3} = 6.38\times10^{-15}\text{m}.$$

$$E = (211.988842 - 207.976627 - 4.002602)(931)\ \text{MeV} = 8.95\ \text{MeV}.$$

$$v = \sqrt{\frac{2E}{m}} = \sqrt{\frac{(2)(8.95)}{3727}}\times3\times10^8\ \text{m/s} = 2.08\times10^7\ \text{m/s}.$$

$$\gamma = 1.980\frac{82}{\sqrt{8.95}} - 1.485\sqrt{82(6.38)} = 54.37 - 33.97 = 20.4.$$

$$\tau = \frac{(2)(6.38\times10^{-15})}{2.08\times10^7}e^{40.8}\ \text{s} = \boxed{3.2\times10^{-4}\ \text{s}}.$$

These results are way off – but note the extraordinary sensitivity to nuclear masses – a tiny change in E produces enormous changes in τ.

Problem 8.5 (a) $\boxed{V(x) = mgx}$. (b) $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + mgx\psi = E\psi \Rightarrow \frac{d^2\psi}{dx^2} = \frac{2m^2g}{\hbar^2}\left(x - \frac{E}{mg}\right)$. Let $y \equiv x - \frac{E}{mg}$, and $\alpha \equiv \left(\frac{2m^2g}{\hbar^2}\right)^{1/3}$. Then $\frac{d^2\psi}{dy^2} = \alpha^3 y\psi$. Let $z \equiv \alpha y = \alpha\left(x - \frac{E}{mg}\right)$, so $\frac{d^2\psi}{dz^2} = z\psi$. This is the Airy equation [8.36], and the general solution is $\psi = a\,Ai(z) + b\,Bi(z)$. However, $Bi(z)$ blows up for large z, so $b=0$ (to make ψ normalizable). Hence $\boxed{\psi(x) = a\,Ai\left(\alpha\left(x - \frac{E}{mg}\right)\right)}$.

(c) Since $V(x) = \infty$ for $x<0$, we require $\psi(0) = 0$; hence $Ai(\alpha(-E/mg)) = 0$. Now, the zeros of Ai are a_n ($n = 1,2,3,\dots$). Abromowitz and Stegun list $a_1 = -2.338$, $a_2 = -4.088$, $a_3 = -5.521$, $a_4 = -6.787$, etc. Here $-\frac{\alpha E_n}{mg} = a_n$, or $E_n = -\frac{mg}{\alpha}a_n = -mg\left(\frac{\hbar^2}{2m^2g}\right)^{1/3}a_n$, or $\boxed{E_n = -\left(\tfrac{1}{2}mg^2\hbar^2\right)^{1/3}a_n}$. In this case

$$\tfrac{1}{2}mg^2\hbar^2 = \tfrac{1}{2}(0.1\text{kg})(9.8\text{m/s}^2)^2(1.055\times10^{-34}\text{J}\cdot\text{s})^2 = 5.34\times10^{-68}\text{J}^3;\quad \left(\tfrac{1}{2}mg^2\hbar^2\right)^{1/3} = 3.77\times10^{-23}\text{J}.$$

$$\therefore \boxed{E_1 = 8.81\times10^{-23}\text{J},\ E_2 = 1.54\times10^{-22}\text{J},\ E_3 = 2.08\times10^{-22}\text{J};\ E_4 = 2.56\times10^{-22}\text{J}}$$

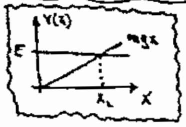
(d) $2\langle T\rangle = \langle x\frac{dV}{dx}\rangle$ [3.159]; here $\frac{dV}{dx} = mg$, so $\langle x\frac{dV}{dx}\rangle = \langle mgx\rangle = \langle V\rangle$. $\therefore \langle T\rangle = \frac{1}{2}\langle V\rangle$.

But $\langle T\rangle + \langle V\rangle = \langle H\rangle = E_n$, so $\frac{3}{2}\langle V\rangle = E_n$, or $\langle V\rangle = \frac{2}{3}E_n$. But $\langle V\rangle = mg\langle x\rangle$, so

$\langle x\rangle = \frac{2E_n}{3mg}$. For the electron, $\left(\frac{1}{2}mg^2\hbar^2\right)^{1/3} = \left(\frac{1}{2}(9.11\times10^{-31})(9.8)^2(1.055\times10^{-34})^2\right)^{1/3} = 7.87\times10^{-33}$ J.

So $E_1 = 1.84\times10^{-32}$ J $= \boxed{1.15\times10^{-13}\text{ eV}}$. $\therefore \langle x\rangle = \frac{2(1.84\times10^{-32})}{3(9.11\times10^{-31})(9.8)} = 1.37\times10^{-3}$ m, or $\boxed{1.37\text{ mm}}$.

PROBLEM 8.6 (a)



[8.47] $\Rightarrow \int_0^{x_2} p(x)\,dx = (n-\frac{1}{4})\pi\hbar$, where $p(x) = \sqrt{2m(E-mgx)}$, and

$E = mgx_2 \Rightarrow x_2 = E/mg$.

$$\int_0^{x_2} p(x)\,dx = \sqrt{2m}\int_0^{x_2}\sqrt{E-mgx}\,dx = \sqrt{2m}\left[-\frac{2}{3mg}(E-mgx)^{3/2}\right]\Big|_0^{x_2} = -\frac{2}{3}\sqrt{\frac{2}{m}}\frac{1}{g}\left\{(E-mgx_2)^{3/2}-E^{3/2}\right\} = \frac{2}{3}\sqrt{\frac{2}{m}}\frac{1}{g}E^{3/2}.$$

$$\therefore \frac{1}{3\sqrt{m}\,g}(2E)^{3/2} = (n-\tfrac{1}{4})\pi\hbar, \text{ or } \boxed{E_n = \left[\frac{9}{8}\pi^2 mg^2\hbar^2(n-\tfrac{1}{4})^2\right]^{1/3}}.$$

(b) $\left(\frac{9}{8}\pi^2 mg^2\hbar^2\right)^{1/3} = \left[\frac{9}{8}\pi^2(0.1)(9.8)^2(1.055\times10^{-34})^2\right]^{1/3} = 1.0588\times10^{-22}$ J.

$$\begin{aligned} E_1 &= (1.0588\times10^{-22})(3/4)^{2/3} = 8.74\times10^{-23}\text{ J} \\ E_2 &= (1.0588\times10^{-22})(7/4)^{2/3} = 1.54\times10^{-22}\text{ J} \\ E_3 &= (1.0588\times10^{-22})(11/4)^{2/3} = 2.08\times10^{-22}\text{ J} \\ E_4 &= (1.0588\times10^{-22})(15/4)^{2/3} = 2.56\times10^{-22}\text{ J} \end{aligned}$$

{These are in very close agreement with the exact results (Problem 8.5(c)). In fact, they agree precisely (to 3 significant digits), except for E_1 (for which the exact result was 8.81×10^{-23} J).

(c) From Problem 8.5(d), $\langle x\rangle = \frac{2E_n}{3mg}$, so $1 = \frac{2}{3}\frac{(1.0588\times10^{-22})}{(0.1)(9.8)}(n-\frac{1}{4})^{2/3}$, or $(n-\frac{1}{4})^{2/3} = 1.388\times10^{22}$

$n = \frac{1}{4} + (1.388\times10^{22})^{3/2} = \boxed{1.64\times10^{33}}$.

PROBLEM 8.7 $\int_{x_1}^{x_2} p(x)\,dx = (n-\frac{1}{2})\pi\hbar$. $p(x) = \sqrt{2m(E-\frac{1}{2}m\omega^2x^2)}$; $x_2 = -x_1 = \frac{1}{\omega}\sqrt{\frac{2E}{m}}$.

$$(n-\tfrac{1}{2})\pi\hbar = m\omega\int_{-x_2}^{x_2}\sqrt{\frac{2E}{m\omega^2}-x^2}\,dx = 2m\omega\int_0^{x_2}\sqrt{x_2^2-x^2}\,dx = m\omega\left[x\sqrt{x_2^2-x^2}+x_2^2\sin^{-1}(x/x_2)\right]_0^{x_2}$$

$$= m\omega x_2^2\sin^{-1}(1) = \frac{\pi}{2}m\omega x_2^2 = \frac{\pi}{2}m\omega\frac{2E}{m\omega^2} = \frac{\pi E}{\omega}. \quad \therefore \boxed{E_n = (n-\tfrac{1}{2})\hbar\omega} \quad (n=1,2,3,\ldots)$$

(Since the WKB numbering starts with $n=1$, whereas for oscillator states we usually start with $n=0$, letting $n\to n+1$ converts this to the usual formula $E_n = (n+\frac{1}{2})\hbar\omega$. In this case the WKB approximation yields the *exact* energies.)

PROBLEM 8.8 (a) $\frac{1}{2}m\omega^2x_2^2 = E_n = (n+\frac{1}{2})\hbar\omega$ (counting $n = 0,1,2,\ldots$); $\boxed{x_2 = \sqrt{\frac{(2n+1)\hbar}{m\omega}}}$.

(b) $V_{lin}(x) = \frac{1}{2}m\omega^2x_2^2 + (m\omega^2x_2)(x-x_2) \Rightarrow V_{lin}(x_2+d) = \frac{1}{2}m\omega^2x_2^2 + m\omega^2x_2 d$.

$$\therefore\ \frac{V(x_2+d)-V_{lin}(x_2+d)}{V(x_2)} = \frac{\frac{1}{2}m\omega^2(x_2+d)^2-\frac{1}{2}m\omega^2x_2^2-m\omega^2x_2d}{\frac{1}{2}m\omega^2x_2^2} = \frac{x_2^2+2x_2d+d^2-x_2^2-2x_2d}{x_2^2} = \left(\frac{d}{x_2}\right)^2 = 0.01.$$

$$\therefore\ \boxed{d = 0.1\,x_2}$$

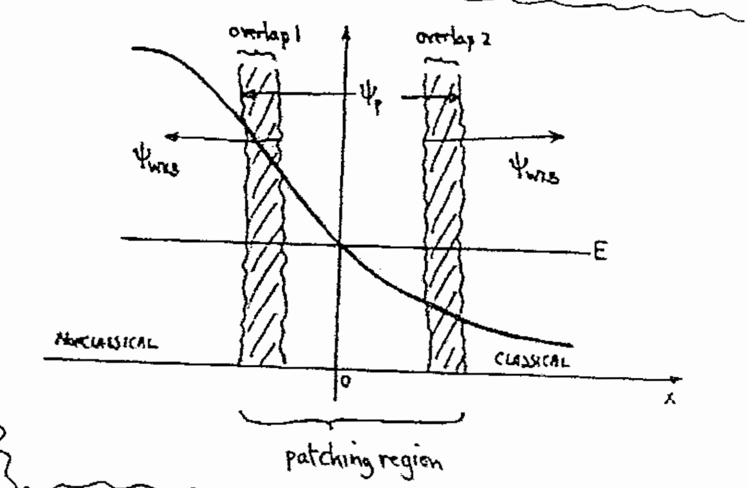
(c) $\alpha = \left[\frac{2m}{\hbar^2}m\omega^2 x_2\right]^{1/3}$ (eq. [8.34]), so $0.1\,x_2\left[\frac{2m^2\omega^2}{\hbar^2}x_2\right]^{1/3} \geq 5 \Rightarrow \left[\frac{2m^2\omega^2}{\hbar^2}x_2^4\right]^{1/3} \geq 50$

$$\frac{2m^2\omega^2}{\hbar^2}\frac{(2n+1)^2\hbar^2}{m^2\omega^2} \geq (50)^3;\ \text{or}\ (2n+1)^2 \geq \frac{(50)^3}{2} = 62500;\ 2n+1 \geq 250;\ n \geq \frac{249}{2} = 124.5$$

$\boxed{n_{min} = 125}$. (However, as we saw in Problems 8.6 and 8.7, WKB may be valid at much smaller n.)

PROBLEM 8.9 Shift origin to the turning point.

$$\psi_{WKB}(x) = \begin{cases} \frac{1}{\sqrt{|p(x)|}}\,D\,e^{-\frac{1}{\hbar}\int_x^0 |p(x')|\,dx'} & (x<0) \\ \frac{1}{\sqrt{p(x)}}\left\{B\,e^{\frac{i}{\hbar}\int_0^x p(x')dx'} + C\,e^{-\frac{i}{\hbar}\int_0^x p(x')dx'}\right\} & (x>0) \end{cases}$$



Linearized potential in the patching region:

$V(x) \cong E + V'(0)\,x$. Note: $V'(0)$ negative.

$$\frac{d^2\psi_p}{dx^2} = \frac{2mV'(0)}{\hbar^2}x\,\psi_p = -\alpha^3 x\,\psi_p,\ \text{where}$$

$$\alpha \equiv \left(\frac{2m|V'(0)|}{\hbar^2}\right)^{1/3}. \quad \psi_p(x) = a\,Ai(-\alpha x) + b\,Bi(-\alpha x).$$ (Note change of sign, as compared to [8.37].)

$$p(x) = \sqrt{2m(E-E-V'(0)x)} = \sqrt{-2mV'(0)x} = \sqrt{2m|V'(0)|x} = \sqrt{\alpha^3\hbar^2 x} = \hbar\alpha^{3/2}\sqrt{x}.$$

Overlap region 1: $x<0$. $\int_x^0 |p(x')|\,dx' = \hbar\alpha^{3/2}\int_x^0\sqrt{-x'}\,dx' = \hbar\alpha^{3/2}\left(-\frac{2}{3}(-x')^{3/2}\right)\Big|_x^0 = \frac{2}{3}\hbar\alpha^{3/2}(-x)^{3/2} = \frac{2}{3}\hbar(-\alpha x)^{3/2}.$

$$\therefore\ \psi_{WKB} \cong \frac{1}{\hbar^{1/2}\alpha^{3/4}(-x)^{1/4}}\,D\,e^{-\frac{2}{3}(-\alpha x)^{3/2}}.$$ For large positive argument $(-\alpha x \gg 1)$:

$$\psi_p \cong a\,\frac{1}{2\sqrt{\pi}(-\alpha x)^{1/4}}e^{-\frac{2}{3}(-\alpha x)^{3/2}} + b\,\frac{1}{\sqrt{\pi}(-\alpha x)^{1/4}}e^{\frac{2}{3}(-\alpha x)^{3/2}}.$$ Comparing the two expressions $\Rightarrow$

$a = 2D\sqrt{\frac{\pi}{\alpha\hbar}};\ b = 0.$

Overlap region 2: $x>0$. $\int_0^x p(x')dx' = \hbar\alpha^{3/2}\int_0^x\sqrt{x'}\,dx' = \hbar\alpha^{3/2}\left(\frac{2}{3}x'^{3/2}\right)\Big|_0^x = \frac{2}{3}\hbar(\alpha x)^{3/2}.$

$$\therefore\ \psi_{WKB} \cong \frac{1}{\hbar^{1/2}\alpha^{3/4}x^{1/4}}\left\{B\,e^{i\frac{2}{3}(\alpha x)^{3/2}} + C\,e^{-i\frac{2}{3}(\alpha x)^{3/2}}\right\}.$$ For large negative argument $(-\alpha x \ll -1)$:

$$\psi_t(x) \cong a\frac{1}{\sqrt{\pi}(\alpha x)^{1/4}}\sin\left[\frac{2}{3}(\alpha x)^{3/2}+\frac{\pi}{4}\right] = \frac{a}{\sqrt{\pi}(\alpha x)^{1/4}}\frac{1}{2i}\left\{e^{i\pi/4}e^{i\frac{2}{3}(\alpha x)^{3/2}} - e^{-i\pi/4}e^{-i\frac{2}{3}(\alpha x)^{3/2}}\right\} \quad \text{(remember: } b=0\text{)}$$

Comparing the two: $\underline{\underline{B=\frac{a}{2i}\sqrt{\frac{\alpha\hbar}{\pi}}e^{i\pi/4}}}$, $\underline{\underline{C=-\frac{a}{2i}\sqrt{\frac{\alpha\hbar}{\pi}}e^{-i\pi/4}}}$. Or, inserting the expression for a from overlap region 1: $\underline{\underline{B=-ie^{i\pi/4}D;\ C=ie^{-i\pi/4}D}}$. Therefore, for $x>0$,

$$\psi_{WKB} = \frac{-iD}{\sqrt{p(x)}}\left\{e^{\frac{i}{\hbar}\int_0^x p(x')dx'+i\frac{\pi}{4}} - e^{-\frac{i}{\hbar}\int_0^x p(x')dx'-i\pi/4}\right\} = \frac{2D}{\sqrt{p(x)}}\sin\left[\frac{1}{\hbar}\int_0^x p(x')dx'+\frac{\pi}{4}\right].$$

Finally, switching the origin back to x_1:

$$\psi_{WKB}(x) = \left\{\begin{array}{ll} \frac{D}{\sqrt{|p(x)|}}e^{-\frac{1}{\hbar}\int_x^{x_1}|p(x')|dx'}, & (x<x_1); \\ \frac{2D}{\sqrt{p(x)}}\sin\left[\frac{1}{\hbar}\int_{x_1}^x p(x')dx'+\frac{\pi}{4}\right], & (x>x_1). \end{array}\right\} \qquad \text{QED}$$

PROBLEM 8.10 At x_1 we have an upward-sloping turning point. Follow method in book. Shifting origin to x_1:

$$\psi_{WKB}(x) = \left\{\begin{array}{ll} \frac{1}{\sqrt{p(x)}}\left[Ae^{\frac{i}{\hbar}\int_x^0 p(x')dx'} + Be^{-\frac{i}{\hbar}\int_x^0 p(x')dx'}\right] & (x<0) \\ \frac{1}{\sqrt{|p(x)|}}\left[Ce^{\frac{1}{\hbar}\int_0^x |p(x')|dx'} + De^{-\frac{1}{\hbar}\int_0^x |p(x')|dx'}\right] & (x>0) \end{array}\right.$$

In overlap region 2, [8.39] becomes $\psi_{WKB} \cong \frac{1}{\hbar^{1/2}\alpha^{3/4}x^{1/4}}\left[Ce^{\frac{2}{3}(\alpha x)^{3/2}} + De^{-\frac{2}{3}(\alpha x)^{3/2}}\right]$, whereas [8.40] is unchanged. Comparing them $\Rightarrow$ $a = 2D\sqrt{\frac{\pi}{\alpha\hbar}}$, $b = C\sqrt{\frac{\pi}{\alpha\hbar}}$.

In overlap region 1, [8.43] becomes $\psi_{WKB} \cong \frac{1}{\hbar^{1/2}\alpha^{3/4}(-x)^{1/4}}\left[Ae^{i\frac{2}{3}(-\alpha x)^{3/2}} + Be^{-i\frac{2}{3}(-\alpha x)^{3/2}}\right]$, and

[8.44] (with $b\neq 0$) generalizes to

$$\psi_t(x) \cong \frac{a}{\sqrt{\pi}(-\alpha x)^{1/4}}\sin\left[\frac{2}{3}(-\alpha x)^{3/2}+\frac{\pi}{4}\right] + \frac{b}{\sqrt{\pi}(-\alpha x)^{1/4}}\cos\left[\frac{2}{3}(-\alpha x)^{3/2}+\frac{\pi}{4}\right]$$

$$= \frac{1}{2\sqrt{\pi}(-\alpha x)^{1/4}}\left\{(-ia+b)e^{i\frac{2}{3}(-\alpha x)^{3/2}}e^{i\pi/4} + (ia+b)e^{-i\frac{2}{3}(-\alpha x)^{3/2}}e^{-i\pi/4}\right\}.$$ Comparing them $\Rightarrow$

$A = \sqrt{\frac{\hbar\alpha}{\pi}}\left(\frac{-ia+b}{2}\right)e^{i\pi/4}$; $B = \sqrt{\frac{\hbar\alpha}{\pi}}\left(\frac{ia+b}{2}\right)e^{-i\pi/4}$. Putting in the expressions above for a and b:

$\underline{\underline{A=\left(\frac{C}{2}-iD\right)e^{i\pi/4};\ B=\left(\frac{C}{2}+iD\right)e^{-i\pi/4}}}$. These are the connection formulas relating A, B, C, D, at x_1.

At x_2, we have a downward-sloping turning point, and follow the method of Problem 8.9. First rewrite the middle expression in [8.52]:

$$\psi_{WKB} = \frac{1}{\sqrt{|p(x)|}}\left\{Ce^{\frac{1}{\hbar}\int_{x_1}^{x_2}|p(x')|dx' + \frac{1}{\hbar}\int_{x_2}^{x}|p(x')|dx'} + De^{-\frac{1}{\hbar}\int_{x_1}^{x_2}|p(x')|dx' - \frac{1}{\hbar}\int_{x_2}^{x}|p(x')|dx'}\right\}.$$

Let $\gamma \equiv \int_{x_1}^{x_2} |p(x)|\,dx$, as before [8.22], and let $C' \equiv De^{-\gamma}$, $D' \equiv Ce^{\gamma}$. Then (shifting the origin to x_2):

$$\psi_{WKB} = \begin{cases} \frac{1}{\sqrt{|p(x)|}}\left[C' e^{\frac{1}{\hbar}\int_x^0 |p(x')|dx'} + D' e^{-\frac{1}{\hbar}\int_x^0 |p(x')|dx'}\right], & (x<0); \\ \frac{1}{\sqrt{p(x)}} F e^{\frac{i}{\hbar}\int_0^x p(x')dx'}, & (x>0). \end{cases}$$

In the patching region $\psi_p(x) = a\,Ai(-\alpha x) + b\,Bi(-\alpha x)$, where $\alpha \equiv \left(\frac{2m|V'(0)|}{\hbar^2}\right)^{1/3}$; $p(x) = \hbar\alpha^{3/2}\sqrt{x}$.

In overlap region 1: $(x<0)$ $\int_x^0 |p(x')|dx' = \frac{2}{3}\hbar(-\alpha x)^{3/2}$, so

$$\left.\begin{aligned} \psi_{WKB} &\cong \frac{1}{\hbar^{1/2}\alpha^{3/4}(-x)^{1/4}}\left[C' e^{\frac{2}{3}(-\alpha x)^{3/2}} + D' e^{-\frac{2}{3}(-\alpha x)^{3/2}}\right] \\ \psi_p &\cong \frac{a}{2\sqrt{\pi}(-\alpha x)^{1/4}} e^{-\frac{2}{3}(-\alpha x)^{3/2}} + \frac{b}{\sqrt{\pi}(-\alpha x)^{1/4}} e^{\frac{2}{3}(-\alpha x)^{3/2}} \end{aligned}\right\} \text{Comparing} \Rightarrow \begin{cases} a = 2\sqrt{\frac{\pi}{\hbar\alpha}}\,D' \\ b = \sqrt{\frac{\pi}{\hbar\alpha}}\,C' \end{cases}$$

In overlap region 2: $(x>0)$ $\int_0^x p(x')dx' = \frac{2}{3}\hbar(\alpha x)^{3/2} \Rightarrow$ $\psi_{WKB} \cong \frac{1}{\hbar^{1/2}\alpha^{3/4}x^{1/4}} F e^{i\frac{2}{3}(\alpha x)^{3/2}}$

$$\psi_p \cong \frac{a}{\sqrt{\pi}(\alpha x)^{1/4}}\sin\left[\frac{2}{3}(\alpha x)^{3/2} + \frac{\pi}{4}\right] + \frac{b}{\sqrt{\pi}(\alpha x)^{1/4}}\cos\left[\frac{2}{3}(\alpha x)^{3/2} + \frac{\pi}{4}\right]$$

$$= \frac{1}{2\sqrt{\pi}(\alpha x)^{1/4}}\left\{(-ia+b)e^{i\frac{\pi}{4}} e^{i\frac{2}{3}(\alpha x)^{3/2}} + (ia+b)e^{-i\pi/4} e^{-i\frac{2}{3}(\alpha x)^{3/2}}\right\}. \text{ Comparing} \Rightarrow (ia+b)=0;$$

$F = \sqrt{\frac{\hbar\alpha}{\pi}}\left(\frac{-ia+b}{2}\right)e^{i\pi/4} = b\sqrt{\frac{\hbar\alpha}{\pi}}e^{i\pi/4}$. $\therefore$ $b = \sqrt{\frac{\pi}{\hbar\alpha}}e^{-i\pi/4}F$; $a = i\sqrt{\frac{\pi}{\hbar\alpha}}e^{-i\pi/4}F$.

$\therefore$ $C' = \sqrt{\frac{\hbar\alpha}{\pi}}\,b = e^{-i\pi/4}F$, $D' = \frac{1}{2}\sqrt{\frac{\hbar\alpha}{\pi}}\,a = \frac{i}{2}e^{-i\pi/4}F$. $\therefore$ $D = e^{\gamma}e^{-i\pi/4}F$; $C = \frac{i}{2}e^{-\gamma}e^{-i\pi/4}F$.

These are the connection formulas at x_2. Putting them into the equation for A:

$$A = \left(\frac{C}{2} - iD\right)e^{i\pi/4} = \left(\frac{i}{4}e^{-\gamma}e^{-i\pi/4}F - ie^{\gamma}e^{-i\pi/4}F\right)e^{i\pi/4} = i\left(\frac{e^{-\gamma}}{4} - e^{\gamma}\right)F$$

$$T = \left|\frac{F}{A}\right|^2 = \frac{1}{\left(e^{\gamma} - \frac{e^{-\gamma}}{4}\right)^2} = \boxed{\frac{e^{-2\gamma}}{\left[1 - \left(e^{-\gamma}/2\right)^2\right]^2}}.$$ If $\gamma \gg 1$, the denominator is essentially 1, and we recover $T = e^{-2\gamma}$ [8.22].

PROBLEM 8.11 [8.51] $\Rightarrow$ $(n-\frac{1}{2})\pi\hbar = 2\int_0^{x_2}\sqrt{2m(E-\alpha x^{\nu})}\,dx = 2\sqrt{2mE}\int_0^{x_2}\sqrt{1-\frac{\alpha}{E}x^{\nu}}\,dx$. $E = \alpha x_2^{\nu}$. Let $z \equiv \frac{\alpha}{E}x^{\nu}$, so $x = \left(\frac{zE}{\alpha}\right)^{1/\nu}$; $dx = \left(\frac{E}{\alpha}\right)^{1/\nu}\frac{1}{\nu}z^{\frac{1}{\nu}-1}dz$. $(n-\frac{1}{2})\pi\hbar = 2\sqrt{2mE}\left(\frac{E}{\alpha}\right)^{1/\nu}\frac{1}{\nu}\int_0^1 z^{\frac{1}{\nu}-1}\sqrt{1-z}\,dz$. From integral table:

$$(n-\tfrac{1}{2})\pi\hbar = 2\sqrt{2mE}\left(\frac{E}{\alpha}\right)^{1/\nu}\frac{1}{\nu}\frac{\Gamma(1/\nu)\Gamma(3/2)}{\Gamma(\frac{1}{\nu}+\frac{3}{2})} = 2\sqrt{2mE}\left(\frac{E}{\alpha}\right)^{1/\nu}\frac{\Gamma(\frac{1}{\nu}+1)\frac{1}{2}\sqrt{\pi}}{\Gamma(\frac{1}{\nu}+\frac{3}{2})} = \sqrt{2\pi mE}\left(\frac{E}{\alpha}\right)^{1/\nu}\frac{\Gamma(\frac{1}{\nu}+1)}{\Gamma(\frac{1}{\nu}+\frac{3}{2})}.$$

$$E^{\frac{1}{\nu}+\frac{1}{2}} = \frac{(n-\frac{1}{2})\pi\hbar}{\sqrt{2\pi m}}\alpha^{\frac{1}{\nu}}\frac{\Gamma(\frac{1}{\nu}+\frac{3}{2})}{\Gamma(\frac{1}{\nu}+1)}; \quad \boxed{E_n = \left\{(n-\tfrac{1}{2})\hbar\sqrt{\frac{\pi}{2m\alpha}}\frac{\Gamma(\frac{1}{\nu}+\frac{3}{2})}{\Gamma(\frac{1}{\nu}+1)}\right\}^{\left(\frac{2\nu}{\nu+2}\right)}\alpha}.$$ For $\nu=2$:

$E_n = \left\{(n-\frac{1}{2})\hbar\sqrt{\frac{\pi}{2m\alpha}}\frac{\Gamma(2)}{\Gamma(3/2)}\right\}\alpha = (n-\frac{1}{2})\hbar\sqrt{\frac{2\alpha}{m}}$. Oscillator, with $\alpha = \frac{1}{2}m\omega^2$, so $E_n = (n-\frac{1}{2})\hbar\omega$ $(n=1,2,3,\ldots)$ ✓

Problem 8.12 $V(x) = -\frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax)$. [8.51] $\Rightarrow (n-\frac{1}{2})\pi\hbar = 2\int_0^{x_2}\sqrt{2m\left(E+\frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax)\right)}\,dx$

$$= 2\sqrt{2}\,\hbar a\int_0^{x_2}\sqrt{\operatorname{sech}^2(ax)+\frac{mE}{\hbar^2 a^2}}\,dx$$

$E = -\frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax_2)$ defines x_2. Let $b \equiv -\frac{mE}{\hbar^2 a^2}$, $z \equiv \operatorname{sech}^2(ax)$, so that $x = \frac{1}{a}\operatorname{sech}^{-1}\sqrt{z}$, and hence

$dx = \frac{1}{a}\left(\frac{-1}{\sqrt{z}\sqrt{1-z}}\right)\frac{1}{2\sqrt{z}}dz = -\frac{1}{2a}\frac{1}{z\sqrt{1-z}}dz$. Then $(n-\frac{1}{2})\pi = 2\sqrt{2}a\left(-\frac{1}{2a}\right)\int_{z_1}^{z_2}\frac{\sqrt{z-b}}{z\sqrt{1-z}}dz$.

Limits: $\begin{cases} x=0 \Rightarrow z = \operatorname{sech}^2(0) = 1 \\ x = x_2 \Rightarrow z = \operatorname{sech}^2(ax_2) = -\frac{mE}{\hbar^2 a^2} = b\end{cases}$. $\therefore (n-\frac{1}{2})\pi = \sqrt{2}\int_b^1 \frac{1}{z}\sqrt{\frac{z-b}{1-z}}\,dz$.

$$\frac{1}{z}\sqrt{\frac{z-b}{1-z}} = \frac{1}{z}\frac{(z-b)}{\sqrt{(1-z)(z-b)}} = \frac{1}{\sqrt{(1-z)(z-b)}} - \frac{b}{z\sqrt{(1-z)(z-b)}}.$$

$$(n-\tfrac{1}{2})\pi = \sqrt{2}\left\{\int_b^1\frac{1}{\sqrt{(1-z)(z-b)}}dz - b\int_b^1\frac{1}{z\sqrt{-b+(1+b)z-z^2}}dz\right\} = \sqrt{2}\left\{-2\tan^{-1}\sqrt{\frac{1-z}{z-b}} - \sqrt{b}\sin^{-1}\left[\frac{(1+b)z-2b}{z(1-b)}\right]\right\}\Bigg|_b^1$$

$$= \sqrt{2}\left\{-2\tan^{-1}(0)+2\tan^{-1}(\infty)-\sqrt{b}\sin^{-1}(1)+\sqrt{b}\sin^{-1}(-1)\right\} = \sqrt{2}\left(0+2\frac{\pi}{2}-\sqrt{b}\frac{\pi}{2}-\sqrt{b}\frac{\pi}{2}\right)$$

$= \sqrt{2}\pi(1-\sqrt{b})$; $\frac{(n-\frac{1}{2})}{\sqrt{2}} = 1-\sqrt{b}$; $\sqrt{b} = 1-\frac{1}{\sqrt{2}}(n-\frac{1}{2})$. Since the left side is positive, the right side must also be. $\therefore (n-\frac{1}{2}) < \sqrt{2}$, $n < \frac{1}{2}+\sqrt{2} = .5+1.414 = 1.914$. So the only possible n is 1 — looks as though there is only one bound state — which is correct — see Problem 2.48.

For $n=1$, $\sqrt{b} = 1-\frac{1}{2\sqrt{2}}$; $b = 1-\frac{1}{\sqrt{2}}+\frac{1}{8} = \frac{9}{8}-\frac{1}{\sqrt{2}}$. $\therefore \boxed{E_1 = -\frac{\hbar^2 a^2}{m}\left(\frac{9}{8}-\frac{1}{\sqrt{2}}\right)}$.

Numerically, $E = -0.418\frac{\hbar^2 a^2}{m}$. The exact answer (Problem 2.48(c)) is $-0.5\frac{\hbar^2 a^2}{m}$. Not bad.

Problem 8.13 $(n-\frac{1}{4})\pi\hbar = \int_0^{r_0}\sqrt{2m(E-V_0\ln(r/a))}\,dr$. $E = V_0\ln(r_0/a)$ defines r_0.

$$= \sqrt{2m}\int_0^{r_0}\sqrt{V_0\ln(r_0/a)-V_0\ln(r/a)}\,dr = \sqrt{2mV_0}\int_0^{r_0}\sqrt{\ln(r_0/r)}\,dr.$$

Let $x \equiv \ln(r_0/r)$, so $e^x = r_0/r$, or $r = r_0 e^{-x} \Rightarrow dr = -r_0 e^{-x}dx$. $(n-\frac{1}{4})\pi\hbar = \sqrt{2mV_0}(-r_0)\int_{x_1}^{x_2}\sqrt{x}\,e^{-x}dx$.

Limits: $\begin{cases} r=0 \Rightarrow x_1=\infty \\ r=r_0 \Rightarrow x_2=0\end{cases}$. $\therefore (n-\frac{1}{4})\pi\hbar = \sqrt{2mV_0}\,r_0\int_0^\infty\sqrt{x}\,e^{-x}dx = \sqrt{2mV_0}\,r_0\,\Gamma(3/2) = \sqrt{2mV_0}\,r_0\frac{\sqrt{\pi}}{2}$.

$r_0 = \sqrt{\frac{2\pi}{mV_0}}\hbar(n-\frac{1}{4})$. $\therefore \boxed{E_n = V_0\ln\left[\frac{\hbar}{a}\sqrt{\frac{2\pi}{mV_0}}(n-\frac{1}{4})\right]} = V_0\ln(n-\frac{1}{4})+V_0\ln\left[\frac{\hbar}{a}\sqrt{\frac{2\pi}{mV_0}}\right]$.

$\therefore E_{n+1}-E_n = V_0\ln(n+\frac{3}{4})-V_0\ln(n-\frac{1}{4}) = V_0\ln\left(\frac{n+3/4}{n-1/4}\right)$, which is indeed independent of m (or a).

Problem 8.14 $(n-\frac{1}{2})\pi\hbar = \int_{r_1}^{r_2}\sqrt{2m\left(E+\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}-\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right)}\,dr = \sqrt{-2mE}\int_{r_1}^{r_2}\sqrt{-1+\frac{A}{r}-\frac{B}{r^2}}\,dr$, where

$A \equiv -\frac{e^2}{4\pi\epsilon_0}\frac{1}{E}$ and $B \equiv -\frac{\hbar^2}{2m}\frac{l(l+1)}{E}$ are positive constants, since E is negative.

$$\left(n-\tfrac{1}{2}\right)\pi\hbar = \sqrt{-2mE}\int_{r_1}^{r_2}\frac{\sqrt{-r^2+Ar-B}}{r}\,dr.$$ r_1 and r_2 are the roots of the polynomial in the numerator:

$$-r^2+Ar-B=(r-r_1)(r_2-r).\quad \therefore\ \left(n-\tfrac{1}{2}\right)\pi\hbar = \sqrt{-2mE}\int_{r_1}^{r_2}\frac{\sqrt{(r-r_1)(r_2-r)}}{r}\,dr = \sqrt{-2mE}\,\frac{\pi}{2}\left(\sqrt{r_2}-\sqrt{r_1}\right)^2.$$

$$2\left(n-\tfrac{1}{2}\right)\hbar = \sqrt{-2mE}\left(r_2+r_1-2\sqrt{r_1r_2}\right).$$ Now $-r^2+Ar-B = -r^2+(r_1+r_2)r-r_1r_2 \Rightarrow \left\{\begin{matrix} r_1+r_2=A \\ r_1r_2=B\end{matrix}\right\}$

$$\therefore\ 2\left(n-\tfrac{1}{2}\right)\hbar = \sqrt{-2mE}\left(A-2\sqrt{B}\right) = \sqrt{-2mE}\left(-\frac{e^2}{4\pi\epsilon_0}\frac{1}{E}-2\sqrt{\frac{-\hbar^2}{2m}\frac{\ell(\ell+1)}{E}}\right) = \frac{e^2}{4\pi\epsilon_0}\sqrt{-\frac{2m}{E}}-2\hbar\sqrt{\ell(\ell+1)}.$$

$$\therefore\ \frac{e^2}{4\pi\epsilon_0}\sqrt{-\frac{2m}{E}} = 2\hbar\left[n-\tfrac{1}{2}+\sqrt{\ell(\ell+1)}\right];\quad -\frac{E}{2m} = \frac{\left(e^2/4\pi\epsilon_0\right)^2}{4\hbar^2\left[n-\frac{1}{2}+\sqrt{\ell(\ell+1)}\right]^2}.$$

$$E = \frac{-\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2}{\left[n-\frac{1}{2}+\sqrt{\ell(\ell+1)}\right]^2} = \boxed{\frac{-13.6\text{ eV}}{\left(n-\frac{1}{2}+\sqrt{\ell(\ell+1)}\right)^2}}.$$

PROBLEM 8.15 (a) $\boxed{\text{(i) } \psi_{WKB}(x) = \frac{D}{\sqrt{|p(x)|}}\,e^{-\frac{1}{\hbar}\int_{x_2}^{x}|p(x')|dx'}}$ $\quad (x>x_2)$;

(ii) $\psi_{WKB}(x) = \frac{1}{\sqrt{p(x)}}\left\{B e^{\frac{i}{\hbar}\int_x^{x_2}p(x')dx'} + C e^{-\frac{i}{\hbar}\int_x^{x_2}p(x')dx'}\right\}$ $\quad (x_1<x<x_2)$;

(iii) $\psi_{WKB}(x) = \frac{1}{\sqrt{|p(x)|}}\left\{F e^{\frac{1}{\hbar}\int_x^{x_1}|p(x')|dx'} + G e^{-\frac{1}{\hbar}\int_x^{x_1}|p(x')|dx'}\right\}$ $\quad (0<x<x_1)$.

$[8.46] \Rightarrow \boxed{\text{(ii) } \psi_{WKB} = \frac{2D}{\sqrt{p(x)}}\sin\left[\frac{1}{\hbar}\int_x^{x_2}p(x')dx'+\frac{\pi}{4}\right]}$ $\quad (x_1<x<x_2)$.

To effect the joint at x_1, first rewrite (ii):

$$\text{(ii) } \psi_{WKB} = \frac{2D}{\sqrt{p(x)}}\sin\left[\frac{1}{\hbar}\int_{x_1}^{x_2}p(x')dx'-\frac{1}{\hbar}\int_{x_1}^{x}p(x')dx'+\frac{\pi}{4}\right] = -\frac{2D}{\sqrt{p(x)}}\sin\left[\frac{1}{\hbar}\int_{x_1}^{x}p(x')dx'-\theta-\frac{\pi}{4}\right],$$ where

θ is defined in [8.58]. Now shift the origin to x_1:

$$\psi_{WKB} = \left\{\begin{matrix} \frac{1}{\sqrt{|p(x)|}}\left\{F e^{\frac{1}{\hbar}\int_x^0|p(x')|dx'} + G e^{-\frac{1}{\hbar}\int_x^0|p(x')|dx'}\right\} & (x<0) \\ -\frac{2D}{\sqrt{p(x)}}\sin\left[\frac{1}{\hbar}\int_0^x p(x')dx'-\theta-\frac{\pi}{4}\right] & (x>0)\end{matrix}\right\}$$

Following Problem 8.9: $\psi_p(x) = a\,Ai(-\alpha x)+b\,Bi(-\alpha x)$, with $\alpha \equiv \left(\frac{2m\,|V'(0)|}{\hbar^2}\right)^{1/3}$; $p(x) = \hbar\alpha^{3/2}\sqrt{x}$.

Overlap region 1 ($x<0$): $\int_x^0|p(x')|dx' = \frac{2}{3}\hbar(-\alpha x)^{3/2}$.

$$\left.\begin{matrix} \psi_{WKB} \cong \frac{1}{\hbar^{1/2}\alpha^{3/4}(-x)^{1/4}}\left\{F e^{\frac{2}{3}(-\alpha x)^{3/2}} + G e^{-\frac{2}{3}(-\alpha x)^{3/2}}\right\} \\ \psi_p \cong \frac{a}{2\sqrt{\pi}(-\alpha x)^{1/4}}e^{-\frac{2}{3}(-\alpha x)^{3/2}} + \frac{b}{\sqrt{\pi}(-\alpha x)^{1/4}}e^{\frac{2}{3}(-\alpha x)^{3/2}}\end{matrix}\right\} \Rightarrow a = 2G\sqrt{\frac{\pi}{\hbar\alpha}};\ b = F\sqrt{\frac{\pi}{\hbar\alpha}}.$$

Overlap region 2 ($x>0$): $\int_0^x p(x')dx' = \frac{2}{3}\hbar(\alpha x)^{3/2} \Rightarrow \psi_{WKB} \cong \frac{-2D}{\hbar^{1/2}\alpha^{3/4}x^{1/4}}\sin\left[\frac{2}{3}(\alpha x)^{3/2}-\theta-\frac{\pi}{4}\right]$,

$$\psi_p \cong \frac{a}{\sqrt{\pi}(\alpha x)^{1/4}}\sin\left[\frac{2}{3}(\alpha x)^{3/2}+\frac{\pi}{4}\right] + \frac{b}{\sqrt{\pi}(\alpha x)^{1/4}}\cos\left[\frac{2}{3}(\alpha x)^{3/2}+\frac{\pi}{4}\right].$$

Equating the two expressions: $\frac{-2D}{\hbar^{1/2}\alpha^{3/4}}\frac{1}{2i}\left\{e^{i\frac{2}{3}(\alpha x)^{3/2}}e^{-i\theta}e^{-i\pi/4} - e^{-i\frac{2}{3}(\alpha x)^{3/2}}e^{i\theta}e^{i\pi/4}\right\} =$

$$= \frac{1}{\sqrt{\pi}\,\alpha^{1/4}}\left\{\frac{a}{2i}\left[e^{i\frac{2}{3}(\alpha x)^{3/2}}e^{i\pi/4} - e^{-i(\frac{2}{3})(\alpha x)^{3/2}}e^{-i\pi/4}\right] + \frac{b}{2}\left[e^{i\frac{2}{3}(\alpha x)^{3/2}}e^{i\pi/4} + e^{-i\frac{2}{3}(\alpha x)^{3/2}}e^{-i\pi/4}\right]\right\} \Rightarrow$$

$$\left\{\begin{array}{l} -2D\sqrt{\frac{\pi}{\alpha\hbar}}\,e^{-i\theta}e^{-i\pi/4} = (a+ib)e^{i\pi/4}, \text{ or } (a+ib) = 2D\sqrt{\frac{\pi}{\alpha\hbar}}\,i\,e^{-i\theta} \\ 2D\sqrt{\frac{\pi}{\alpha\hbar}}\,e^{i\theta}e^{i\pi/4} = (-a+ib)e^{-i\pi/4}, \text{ or } (a-ib) = -2D\sqrt{\frac{\pi}{\alpha\hbar}}\,i\,e^{i\theta} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 2a = 2D\sqrt{\frac{\pi}{\alpha\hbar}}\,i\left(e^{-i\theta}-e^{i\theta}\right) \\ 2ib = 2D\sqrt{\frac{\pi}{\alpha\hbar}}\,i\left(e^{-i\theta}+e^{i\theta}\right) \end{array}\right.$$

or $a = 2D\sqrt{\frac{\pi}{\alpha\hbar}}\sin\theta$, $b = 2D\sqrt{\frac{\pi}{\alpha\hbar}}\cos\theta$. Combining these with the results from overlap region 1 $\rightarrow$

$2G\sqrt{\frac{\pi}{\alpha\hbar}} = 2D\sqrt{\frac{\pi}{\alpha\hbar}}\sin\theta$, or $G = D\sin\theta$; $\quad F\sqrt{\frac{\pi}{\alpha\hbar}} = 2D\sqrt{\frac{\pi}{\alpha\hbar}}\cos\theta$, or $F = 2D\cos\theta$. Putting these into

(iii): $\boxed{\psi_{WKB}(x) = \frac{D}{\sqrt{|p(x)|}}\left\{2\cos\theta\, e^{\frac{1}{\hbar}\int_x^{x_1}|p(x')|dx'} + \sin\theta\, e^{-\frac{1}{\hbar}\int_x^{x_1}|p(x')|dx'}\right\}}$ $\quad (0<x<x_1)$

(b) Odd (−) case: (iii) $\rightarrow \psi(0) = 0 \Rightarrow \quad 2\cos\theta\, e^{\frac{1}{\hbar}\int_0^{x_1}|p(x')|dx'} + \sin\theta\, e^{-\frac{1}{\hbar}\int_0^{x_1}|p(x')|dx'} = 0.$ $\quad \frac{1}{\hbar}\int_0^{x_1}|p(x')|dx' = \frac{1}{2}\phi$,

with ϕ defined by [8.60]. So $\quad \sin\theta\, e^{-\phi/2} = -2\cos\theta\, e^{\phi/2}$, or $\tan\theta = -2e^{\phi}$.

Even (+) case: (iii) $\Rightarrow \psi'(0) = 0 \rightarrow$

$$-\frac{1}{2}\frac{D}{(|p(0)|)^{3/2}}\left.\frac{d|p(x)|}{dx}\right|_0\left\{2\cos\theta\, e^{\phi/2} + \sin\theta\, e^{-\phi/2}\right\} + \frac{D}{\sqrt{|p(0)|}}\left\{2\cos\theta\, e^{\frac{1}{\hbar}\int_0^{x_1}|p(x')|dx'}\left(-\frac{1}{\hbar}|p(0)|\right)\right.$$

$$\left. + \sin\theta\, e^{-\frac{1}{\hbar}\int_0^{x_1}|p(x')|dx'}\left(\frac{1}{\hbar}|p(0)|\right)\right\} = 0$$

Now $\frac{d|p(x)|}{dx} = \frac{d}{dx}\sqrt{2m(V(x)-E)} = \sqrt{2m}\,\frac{1}{2}\frac{1}{\sqrt{V-E}}\frac{dV}{dx}$, and $\left.\frac{dV}{dx}\right|_0 = 0$, so $\left.\frac{d|p(x)|}{dx}\right|_0 = 0$.

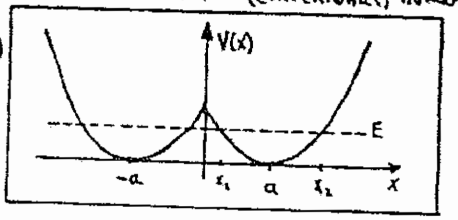
$\therefore \quad 2\cos\theta\, e^{\phi/2} = \sin\theta\, e^{-\phi/2}$, or $\tan\theta = 2e^{\phi}$. Combining the two results: $\tan\theta = \pm 2e^{\phi}$. QED

(c) $\tan\theta = \tan\left[(n+\frac{1}{2})\pi + \epsilon\right] = \frac{\sin[(n+\frac{1}{2})\pi+\epsilon]}{\cos[(n+\frac{1}{2})\pi+\epsilon]} = \frac{(-1)^n\cos\epsilon}{(-1)^{n+1}\sin\epsilon} = -\frac{\cos\epsilon}{\sin\epsilon} \cong -\frac{1}{\epsilon}$. So $-\frac{1}{\epsilon} \cong \pm 2e^{\phi}$,

or $\epsilon \cong \mp\frac{1}{2}e^{-\phi}$, or $\theta - (n+\frac{1}{2})\pi \cong \mp\frac{1}{2}e^{-\phi}$, so $\theta \cong (n+\frac{1}{2})\pi \mp \frac{1}{2}e^{-\phi}$. QED.

[Note: Since θ [8.58] is positive, n must be a nonnegative integer: $n = 0, 1, 2, \ldots$ This is like harmonic oscillator (conventional) numbering, since it starts with $n=0$.]

(d)



$\theta = \frac{1}{\hbar}\int_{x_1}^{x_2}\sqrt{2m\left(E - \frac{1}{2}m\omega^2(x-a)^2\right)}\,dx$. Let $z \equiv x - a$ (shifts the origin to a):

$\theta = \frac{2}{\hbar}\int_0^{z_2}\sqrt{2m\left(E - \frac{1}{2}m\omega^2 z^2\right)}\,dz$, where $E = \frac{1}{2}m\omega^2 z_2^2$.

$$\therefore\ \theta = \frac{2}{\hbar} m\omega \int_0^{z_2} \sqrt{z_2^2 - z^2}\, dz = \frac{m\omega}{\hbar}\left[z\sqrt{z_2^2 - z^2} + z_2^2 \sin^{-1}(z/z_2)\right]\Big|_0^{z_2} = \frac{m\omega}{\hbar} z_2^2 \sin^{-1}(1) = \frac{\pi}{2}\frac{m\omega}{\hbar} z_2^2,\ \text{or}$$

$$\theta = \frac{\pi}{2}\frac{m\omega}{\hbar}\frac{2E}{m\omega^2} = \boxed{\frac{\pi E}{\hbar\omega}}.$$ Putting this into [8.61] yields $\frac{\pi E}{\hbar\omega} \cong (n+\frac{1}{2})\pi \mp \frac{1}{2}e^{-\phi}$, or

$$E_n^{\pm} \cong (n+\tfrac{1}{2})\hbar\omega \mp \frac{\hbar\omega}{2\pi} e^{-\phi}.\quad \text{QED}$$

(e) $\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_n^+ e^{-iE_n^+ t/\hbar} + \psi_n^- e^{-iE_n^- t/\hbar}\right) \Rightarrow$

$$|\Psi(x,t)|^2 = \frac{1}{2}\left[|\psi_n^+|^2 + |\psi_n^-|^2 + \psi_n^+\psi_n^-\left(e^{i(E_n^- - E_n^+)t/\hbar} + e^{-i(E_n^- - E_n^+)t/\hbar}\right)\right]$$

(note that the wave functions (i), (ii), (iii) are real). But $\frac{E_n^- - E_n^+}{\hbar} \cong \frac{1}{\hbar} 2 \frac{\hbar\omega}{2\pi} e^{-\phi} = \frac{\omega}{\pi} e^{-\phi}$.

$\therefore\ |\Psi(x,t)|^2 = \frac{1}{2}\left[\psi_n^+(x)^2 + \psi_n^-(x)^2\right] + \psi_n^+(x)\psi_n^-(x)\cos\left(\frac{\omega}{\pi}e^{-\phi}t\right)$. It oscillates back and forth,

with period $\tau = \frac{2\pi}{\left(\frac{\omega}{\pi}e^{-\phi}\right)} = \frac{2\pi^2}{\omega}e^{\phi}$. QED

(f) $\phi = 2\frac{1}{\hbar}\int_0^{x_1}\sqrt{2m\left(\frac{1}{2}m\omega^2(x-a)^2 - E\right)}\,dx = \frac{2}{\hbar}\sqrt{2mE}\int_0^{x_1}\sqrt{\frac{m\omega^2}{2E}(x-a)^2 - 1}\,dx$. Let $z \equiv \sqrt{\frac{m}{2E}}\,\omega(a-x)$.

$dx = -\sqrt{\frac{2E}{m}}\frac{1}{\omega}dz$. Limits: $\left\{\begin{array}{l} x=0 \Rightarrow z = \sqrt{\frac{m}{2E}}\,\omega a \equiv z_0 \\ x = x_1 \Rightarrow \text{radicand} = 0 \Rightarrow z = 1\end{array}\right\}$. $\phi = \frac{2}{\hbar}\sqrt{2mE}\sqrt{\frac{2E}{m}}\frac{1}{\omega}\int_1^{z_0}\sqrt{z^2-1}\,dz$.

$$\phi = \frac{4E}{\hbar\omega}\int_1^{z_0}\sqrt{z^2-1}\,dz = \frac{4E}{\hbar\omega}\frac{1}{2}\left[z\sqrt{z^2-1} - \ln\left(z+\sqrt{z^2-1}\right)\right]\Big|_1^{z_0} = \boxed{\frac{2E}{\hbar\omega}\left[z_0\sqrt{z_0^2-1} - \ln\left(z_0+\sqrt{z_0^2-1}\right)\right]},$$

where $\boxed{z_0 = a\omega\sqrt{\frac{m}{2E}}}$. $V(0) = \frac{1}{2}m\omega^2 a^2$, so $V(0) \gg E \Rightarrow \frac{m}{2}\omega^2 a^2 \gg E \Rightarrow a\omega\sqrt{\frac{m}{2E}} \gg 1$, or

$z_0 \gg 1$. In that case $\phi \cong \frac{2E}{\hbar\omega}\left(z_0^2 - \ln(2z_0)\right) \cong \frac{2E}{\hbar\omega}z_0^2 = \frac{2E}{\hbar\omega}a^2\omega^2\frac{m}{2E} = \frac{m\omega a^2}{\hbar}$. ✓

This, together with [8.64] gives us the period of oscillation in a double well.

CHAPTER 9

PROBLEM 9.1 $\psi_{n\ell m} = R_{n\ell} Y_\ell^m$. From Tables 4.2 and 4.5:

$$\psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \; ; \; \psi_{200} = \frac{1}{\sqrt{8\pi a^3}}\left(1-\frac{r}{2a}\right)e^{-r/2a} \; ; \; \psi_{210} = \frac{1}{\sqrt{32\pi a^3}}\frac{r}{a}e^{-r/a}\cos\theta ; \; \psi_{21\pm1} = \mp\frac{1}{\sqrt{64\pi a^3}}\frac{r}{a}e^{-r/2a}\sin\theta e^{\pm i\phi}$$

But $r\cos\theta = z$ and $r\sin\theta e^{\pm i\phi} = r\sin\theta(\cos\phi \pm i\sin\phi) = r\sin\theta\cos\phi \pm i r\sin\theta\sin\phi = x \pm iy$. So $|\psi|^2$ is an even function of z in all cases, and hence $\int z|\psi|^2 dx\,dy\,dz = 0$. $\therefore$ $\boxed{H'_{ii} = 0}$. Moreover, ψ_{100} is even in z, and so are ψ_{200}, ψ_{211}, and ψ_{21-1}, so $\boxed{H'_{ij} = 0}$ for all except

$$H'_{100,210} = -eE\frac{1}{\sqrt{\pi a^3}}\frac{1}{\sqrt{32\pi a^3}}\frac{1}{a}\int e^{-r/a}e^{-r/2a}z^2 d^3r = -\frac{eE}{4\sqrt{2}\pi a^4}\int e^{-3r/2a} r^2\cos^2\theta\, r^2\sin\theta\, dr\,d\theta\,d\phi$$

$$= -\frac{eE}{4\sqrt{2}\pi a^4}\int_0^\infty r^4 e^{-3r/2a}dr\int_0^\pi \cos^2\theta\sin\theta\,d\theta\int_0^{2\pi}d\phi = -\frac{eE}{4\sqrt{2}\pi a^4}\cdot 4!\left(\frac{2a}{3}\right)^5\cdot\frac{2}{3}\cdot 2\pi = \boxed{-\left(\frac{2^8}{\sqrt{2}\,3^5}\right)eEa},$$

or $\boxed{-0.7449\,eEa}$.

PROBLEM 9.2 $\dot{c}_a = -\frac{i}{\hbar}H'_{ab}e^{-i\omega_0 t}c_b$; $\dot{c}_b = -\frac{i}{\hbar}H'_{ba}e^{i\omega_0 t}c_a$.

$$\therefore \ddot{c}_b = -\frac{i}{\hbar}H'_{ba}\left[i\omega_0 e^{i\omega_0 t}c_a + e^{i\omega_0 t}\dot{c}_a\right] = i\omega_0\left[-\frac{i}{\hbar}H'_{ba}e^{i\omega_0 t}c_a\right] - \frac{i}{\hbar}H'_{ba}e^{i\omega_0 t}\left[-\frac{i}{\hbar}H'_{ab}e^{-i\omega_0 t}c_b\right], \text{ or}$$

$\ddot{c}_b = i\omega_0\dot{c}_b - \frac{1}{\hbar^2}|H'_{ab}|^2 c_b$. Let $\underline{\alpha^2 \equiv \frac{1}{\hbar^2}|H'_{ab}|^2}$. Then $\ddot{c}_b - i\omega_0\dot{c}_b + \alpha^2 c_b = 0$. This is a linear differential equation with constant coefficients, so it can be solved by a function of the form $c_b = e^{\lambda t}$:

$\lambda^2 - i\omega_0\lambda + \alpha^2 = 0 \Rightarrow \lambda = \frac{1}{2}\left[i\omega_0 \pm \sqrt{-\omega_0^2 - 4\alpha^2}\right] = \frac{i}{2}\left[\omega_0 \pm \omega\right]$, where $\underline{\omega \equiv \sqrt{\omega_0^2 + 4\alpha^2}}$. The general solution is therefore $c_b(t) = Ae^{\frac{i}{2}(\omega_0+\omega)} + Be^{\frac{i}{2}(\omega_0-\omega)} = e^{i\omega_0 t/2}\left(Ae^{i\omega t/2} + Be^{-i\omega t/2}\right)$, or

$c_b(t) = e^{i\omega_0 t/2}\left[C\cos(\omega t/2) + D\sin(\omega t/2)\right]$. But $c_b(0) = 0$, so $C = 0$, and hence $\underline{c_b(t) = De^{i\omega_0 t/2}\sin(\omega t/2)}$.

$$\therefore \dot{c}_b = D\left\{\frac{i\omega_0}{2}e^{i\omega_0 t/2}\sin(\omega t/2) + \frac{\omega}{2}e^{i\omega_0 t/2}\cos(\omega t/2)\right\} = \frac{\omega}{2}De^{i\omega_0 t/2}\left[\cos(\omega t/2) + i\frac{\omega_0}{\omega}\sin(\omega t/2)\right] = -\frac{i}{\hbar}H'_{ba}e^{i\omega_0 t}c_a.$$

$\therefore c_a = \frac{i\hbar}{H'_{ba}}\frac{\omega}{2}e^{-i\omega_0 t/2}D\left[\cos(\omega t/2) + i\frac{\omega_0}{\omega}\sin(\omega t/2)\right]$. But $c_a(0) = 1$, so $\frac{i\hbar}{H'_{ba}}\frac{\omega}{2}D = 1$.

$$\therefore \boxed{\begin{aligned} c_a(t) &= e^{-i\omega_0 t/2}\left[\cos(\omega t/2) + i\frac{\omega_0}{\omega}\sin(\omega t/2)\right] \\ c_b(t) &= \frac{2H'_{ba}}{i\hbar\omega}e^{i\omega_0 t/2}\sin(\omega t/2)\end{aligned}}, \text{ where } \boxed{\omega \equiv \sqrt{\omega_0^2 + 4|H'_{ab}|^2/\hbar^2}}.$$

$$|c_a|^2 + |c_b|^2 = \cos^2(\omega t/2) + \frac{\omega_0^2}{\omega^2}\sin^2(\omega t/2) + \frac{4|H'_{ab}|^2}{\hbar^2\omega^2}\sin^2(\omega t/2) = \cos^2(\omega t/2) + \frac{1}{\omega^2}\left[\omega_0^2 + 4|H'_{ab}|^2/\hbar^2\right]\sin^2(\omega t/2)$$

$$= \cos^2(\omega t/2) + \sin^2(\omega t/2) = 1 \checkmark$$

PROBLEM 9.3 [9.13] $\Rightarrow \frac{dc_a}{dt} = -\frac{i\alpha}{\hbar}\delta(t-t_0)e^{-i\omega_0 t}c_b \Rightarrow c_a(t) = c_a(-\infty) - \frac{i\alpha}{\hbar}\int_{-\infty}^{t}\delta(t'-t_0)e^{-i\omega_0 t'}c_b(t')dt' \Rightarrow$

$c_a(t) = 1 - \frac{i\alpha}{\hbar}e^{-i\omega_0 t_0}c_b(t_0)\theta(t-t_0)$. Likewise $\frac{dc_b}{dt} = -\frac{i\alpha^*}{\hbar}\delta(t-t_0)e^{i\omega_0 t}c_a \Rightarrow$

$c_b(t) = c_b(-\infty) - \frac{i\alpha^*}{\hbar}\int_{-\infty}^{t}\delta(t'-t_0)e^{i\omega_0 t'}c_a(t')dt' = -\frac{i\alpha^*}{\hbar}e^{i\omega_0 t_0}c_a(t_0)\theta(t-t_0)$. This is slippery, because c_a and c_b are discontinuous at t_0, but we obtain consistent results by interpreting $\theta(0)$ as $\frac{1}{2}$; then

$c_a(t_0) = 1 - \frac{i\alpha}{2\hbar}e^{-i\omega_0 t_0}c_b(t_0)$; $c_b(t_0) = -\frac{i\alpha^*}{2\hbar}e^{i\omega_0 t_0}c_a(t_0)$, so $c_a(t_0) = 1 - \frac{|\alpha|^2}{4\hbar^2}c_a(t_0)$, or

$c_a(t_0) = \frac{1}{1+|\alpha|^2/4\hbar^2}$; $c_b(t_0) = -\frac{i\alpha^*}{2\hbar}\frac{e^{i\omega_0 t_0}}{1+|\alpha|^2/4\hbar^2}$. Putting these back into the formulas for c_a and c_b:

$$\boxed{c_a(t) = 1 - \left(\frac{|\alpha|^2/2\hbar^2}{1+|\alpha|^2/4\hbar^2}\right)\theta(t-t_0)}\,;\ \boxed{c_b(t) = -\frac{i\alpha^*}{\hbar}e^{i\omega_0 t_0}\frac{1}{1+|\alpha|^2/4\hbar^2}\theta(t-t_0)}.$$

For $t<t_0$, $c_a(t)=1$, $c_b(t)=0$, so obviously $|c_a|^2+|c_b|^2=1$.

For $t>t_0$, $c_a(t) = 1 - \frac{|\alpha|^2/2\hbar^2}{1+|\alpha|^2/4\hbar^2} = \frac{1-|\alpha|^2/4\hbar^2}{1+|\alpha|^2/4\hbar^2}$, $c_b(t) = -\frac{i\alpha^*}{\hbar}\frac{e^{i\omega_0 t_0}}{(1+|\alpha|^2/4\hbar^2)}$, so

$$|c_a|^2+|c_b|^2 = \frac{(1-|\alpha|^2/2\hbar^2+|\alpha|^4/16\hbar^4)+|\alpha|^2/\hbar^2}{(1+|\alpha|^2/4\hbar^2)^2} = \frac{(1+|\alpha|^2/4\hbar^2)^2}{(1+|\alpha|^2/4\hbar^2)^2} = 1 \checkmark$$

The probability of a transition is $|c_b|^2 = \boxed{\frac{|\alpha|^2/\hbar^2}{(1+|\alpha|^2/4\hbar^2)^2}}$.

PROBLEM 9.4 (a) [9.10] $\Rightarrow \dot{c}_a = -\frac{i}{\hbar}\left[c_a H'_{aa} + c_b H'_{ab}e^{-i\omega_0 t}\right]$

[9.11] $\Rightarrow \dot{c}_b = -\frac{i}{\hbar}\left[c_b H'_{bb} + c_a H'_{ba}e^{i\omega_0 t}\right]$ (these are exact, and replace [9.13]).

Initial conditions: $c_a(0)=1$, $c_b(0)=0$.

Zeroth order: $c_a(t)=1$, $c_b(t)=0$.

First order: $\dot{c}_a = -\frac{i}{\hbar}H'_{aa} \Rightarrow \boxed{c_a(t) = 1 - \frac{i}{\hbar}\int_0^t H'_{aa}(t')dt'}$

$\dot{c}_b = -\frac{i}{\hbar}H'_{ba}e^{i\omega_0 t} \Rightarrow \boxed{c_b(t) = -\frac{i}{\hbar}\int_0^t H'_{ba}(t')e^{i\omega_0 t'}dt'}$.

$|c_a|^2 = \left(1 - \frac{i}{\hbar}\int_0^t H'_{aa}(t')dt'\right)\left(1 + \frac{i}{\hbar}\int_0^t H'_{aa}(t')dt'\right) = 1 + \left(\frac{1}{\hbar}\int_0^t H'_{aa}(t')dt'\right)^2 = 1$ (to first order in H').

$|c_b|^2 = \left(\frac{-i}{\hbar}\int_0^t H'_{ba}(t')e^{i\omega_0 t'}dt'\right)\left(\frac{i}{\hbar}\int_0^t H'_{ab}(t')e^{-i\omega_0 t'}dt'\right) = 0$ (to first order in H').

So $|c_a|^2+|c_b|^2=1$ (to first order).

(b) $\dot{d}_a = \cancel{e^{\frac{i}{\hbar}\int_0^t H'_{aa}(t')dt'}\left(\frac{i}{\hbar}H'_{aa}\right)c_a} + e^{\frac{i}{\hbar}\int_0^t H'_{aa}(t')dt'}\dot{c}_a$

$\hookrightarrow -\frac{i}{\hbar}\left[\cancel{c_a H'_{aa}} + c_b H'_{ab}e^{-i\omega_0 t}\right]$

The indicated terms cancel, leaving $\dot{d}_a = -\frac{i}{\hbar} e^{\frac{i}{\hbar}\int_0^t H'_{aa}(t')dt'} c_b H'_{ab} e^{-i\omega_0 t}$, so
$\quad c_b \to e^{-\frac{i}{\hbar}\int_0^t H'_{bb}(t')dt'} d_b$

$$\dot{d}_a = -\frac{i}{\hbar} e^{\frac{i}{\hbar}\int_0^t [H'_{aa}(t')-H'_{bb}(t')]dt'} H'_{ab} e^{-i\omega_0 t} d_b, \text{ or}$$

$$\dot{d}_a = -\frac{i}{\hbar} e^{i\phi} H'_{ab} e^{-i\omega_0 t} d_b.$$

Similarly:

$$\dot{d}_b = \cancel{e^{\frac{i}{\hbar}\int_0^t H'_{bb}(t')dt'}\left(\frac{i}{\hbar}H'_{bb}\right)c_b} + e^{\frac{i}{\hbar}\int_0^t H'_{bb}(t')dt'} \dot{c}_b$$
$$\dot{c}_b \to -\frac{i}{\hbar}\left[\cancel{c_b H'_{bb}} + c_a H'_{ba} e^{i\omega_0 t}\right]$$

$$= -\frac{i}{\hbar} e^{\frac{i}{\hbar}\int_0^t H'_{bb}(t')dt'} c_a H'_{ba} e^{i\omega_0 t}$$
$$c_a \to e^{-\frac{i}{\hbar}\int_0^t H'_{aa}(t')dt'} d_a$$

$$\dot{d}_b = -\frac{i}{\hbar} e^{\frac{i}{\hbar}\int_0^{t} [H'_{bb}(t')-H'_{aa}(t')]dt'} H'_{ba} e^{i\omega_0 t} d_a = -\frac{i}{\hbar} e^{-i\phi} H'_{ba} e^{i\omega_0 t} d_a. \quad \text{QED}$$

(c) Initial conditions: $c_a(0)=1 \Rightarrow d_a(0)=1$; $c_b(0)=0 \Rightarrow d_b(0)=0$.

Zeroth order: $d_a(t)=1$, $d_b(t)=0$.

First order: $\dot{d}_a = 0 \Rightarrow d_a(t)=1 \Rightarrow \boxed{c_a(t) = e^{-\frac{i}{\hbar}\int_0^t H'_{aa}(t')dt'}}$.

$$\dot{d}_b = -\frac{i}{\hbar} e^{-i\phi} H'_{ba} e^{i\omega_0 t} \Rightarrow d_b = -\frac{i}{\hbar}\int_0^t e^{-i\phi(t')} H'_{ba}(t') e^{i\omega_0 t'} dt' \Rightarrow$$

$$\boxed{c_b(t) = -\frac{i}{\hbar} e^{-\frac{i}{\hbar}\int_0^t H'_{bb}(t')dt'} \int_0^t e^{-i\phi(t')} H'_{ba}(t') e^{i\omega_0 t'} dt'}.$$

These don't look much like the results in (a), but remember: we're only working to first order in H', so $c_a(t) \cong 1 - \frac{i}{\hbar}\int_0^t H'_{aa}(t')dt'$ (to this order), while for c_b, the factor H_{ba} in the integral means it is already first order and hence both the exponential factor in front and $e^{-i\phi}$ should be replaced by 1. Then $c_b(t) \cong -\frac{i}{\hbar}\int_0^t H'_{ba}(t') e^{i\omega_0 t'} dt'$, and we recover the results in (a).

Problem 9.5 Zeroth order: $c_a^{(0)}(t) = a$, $c_b^{(0)}(t) = b$.

First order: $\dot{c}_a = -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} b \Rightarrow c_a^{(1)}(t) = a - \frac{ib}{\hbar}\int_0^t H'_{ab}(t') e^{-i\omega_0 t'} dt'$.

$$\dot{c}_b = -\frac{i}{\hbar} H'_{ba} e^{i\omega_0 t} a \Rightarrow c_b^{(1)}(t) = b - \frac{ia}{\hbar}\int_0^t H'_{ba}(t') e^{i\omega_0 t'} dt'.$$

Second order: $\dot{c}_a = -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t}\left\{ b - \frac{ia}{\hbar}\int_0^t H'_{ba}(t') e^{i\omega_0 t'} dt' \right\} \Rightarrow$

$$\boxed{c_a^{(2)}(t) = a - \frac{ib}{\hbar}\int_0^t H'_{ab}(t') e^{-i\omega_0 t'} dt' - \frac{a}{\hbar^2}\int_0^t H'_{ab}(t') e^{-i\omega_0 t'} \left\{ \int_0^{t'} H'_{ba}(t'') e^{i\omega_0 t''} dt'' \right\} dt'}.$$

To get c_b, just switch $a \leftrightarrow b$ (which entails also changing the sign of ω_0):

$$\boxed{c_b^{(2)}(t) = b - \frac{ia}{\hbar}\int_0^t H'_{ba}(t')e^{i\omega_0 t'}dt' - \frac{b}{\hbar^2}\int_0^t H'_{ba}(t')e^{i\omega_0 t'}\left\{\int_0^{t'} H'_{ab}(t'')e^{-i\omega_0 t''}dt''\right\}dt'}$$

PROBLEM 9.6 For H' independent of t, [9.17] $\Rightarrow$ $c_b^{(2)}(t) = c_b^{(1)}(t) = -\frac{i}{\hbar}H'_{ba}\int_0^t e^{i\omega_0 t'}dt' \Rightarrow$

$$c_b^{(2)}(t) = -\frac{i}{\hbar}H'_{ba}\left.\frac{e^{i\omega_0 t'}}{i\omega_0}\right|_0^t = \boxed{-\frac{H'_{ba}}{\hbar\omega_0}\left(e^{i\omega_0 t}-1\right)}.$$ Meanwhile [9.18] $\Rightarrow$

$$c_a^{(2)}(t) = 1 - \frac{1}{\hbar^2}|H'_{ab}|^2\int_0^t e^{-i\omega_0 t'}\Big\{\underbrace{\int_0^{t'} e^{i\omega_0 t''}dt''}_{\left.\frac{e^{i\omega_0 t''}}{i\omega_0}\right|_0^{t'} = \frac{1}{i\omega_0}\left(e^{i\omega_0 t'}-1\right)}\Big\}dt' = 1 - \frac{1}{\hbar^2}|H'_{ab}|^2\frac{1}{i\omega_0}\int_0^t\left(1-e^{-i\omega_0 t'}\right)dt'$$

$$c_a^{(2)}(t) = 1 + \frac{i}{\omega_0\hbar^2}|H'_{ab}|^2\left.\left(t' + \frac{e^{-i\omega_0 t'}}{i\omega_0}\right)\right|_0^t = \boxed{1 + \frac{i}{\omega_0\hbar^2}|H'_{ab}|^2\left[t + \frac{1}{i\omega_0}\left(e^{-i\omega_0 t}-1\right)\right]}.$$

For comparison with the exact answers (Problem 9.2), note first that $c_b(t)$ is already first order (because of the H'_{ba} in front), whereas ω differs from ω_0 only in second order, so it suffices to replace $\omega \to \omega_0$ in the exact formula to get the second-order result:

$$c_b(t) \cong \frac{2H'_{ba}}{i\hbar\omega_0}e^{i\omega_0 t/2}\sin(\omega_0 t/2) = \frac{2H'_{ba}}{i\hbar\omega_0}e^{i\omega_0 t/2}\frac{1}{2i}\left(e^{i\omega_0 t/2}-e^{-i\omega_0 t/2}\right) = -\frac{H'_{ba}}{\hbar\omega_0}\left(e^{i\omega_0 t}-1\right),$$ in

agreement with the result above. Checking c_a is more difficult. Note that

$$\omega = \omega_0\sqrt{1+\frac{4|H'_{ab}|^2}{\omega_0^2\hbar^2}} \cong \omega_0\left(1+2\frac{|H'_{ab}|^2}{\omega_0^2\hbar^2}\right) = \omega_0 + \frac{2|H'_{ab}|^2}{\omega_0\hbar^2};\quad \frac{\omega_0}{\omega} \cong 1 - 2\frac{|H'_{ab}|^2}{\omega_0^2\hbar^2};$$

Taylor expansion:
$$\left\{\begin{array}{l}\cos(x+\epsilon) = \cos x - \epsilon\sin x \Rightarrow \cos(\omega t/2) = \cos\left(\frac{\omega_0 t}{2} + \frac{|H'_{ab}|^2 t}{\omega_0\hbar^2}\right) \cong \cos(\omega_0 t/2) - \frac{|H'_{ab}|^2 t}{\omega_0\hbar^2}\sin(\omega_0 t/2)\\ \sin(x+\epsilon) = \sin x + \epsilon\cos x \Rightarrow \sin(\omega t/2) = \sin\left(\frac{\omega_0 t}{2} + \frac{|H'_{ab}|^2 t}{\omega_0\hbar^2}\right) \cong \sin(\omega_0 t/2) + \frac{|H'_{ab}|^2 t}{\omega_0\hbar^2}\cos(\omega_0 t/2)\end{array}\right\}$$

Putting this into the exact expression (Problem 9.2), we expand to second order in H:

$$c_a(t) \cong e^{-i\omega_0 t/2}\left\{\cos\left(\frac{\omega_0 t}{2}\right) - \frac{|H'_{ab}|^2 t}{\omega_0\hbar^2}\sin(\omega_0 t/2) + i\left(1-2\frac{|H'_{ab}|^2}{\omega_0^2\hbar^2}\right)\left[\sin\left(\frac{\omega_0 t}{2}\right) + \frac{|H'_{ab}|^2 t}{\omega_0\hbar^2}\cos(\omega_0 t/2)\right]\right\}$$

$$= e^{-i\omega_0 t/2}\left\{\left[\cos\left(\frac{\omega_0 t}{2}\right) + i\sin\left(\frac{\omega_0 t}{2}\right)\right] - \frac{|H'_{ab}|^2}{\omega_0\hbar^2}\left[t\left(\sin\left(\frac{\omega_0 t}{2}\right) - i\cos\left(\frac{\omega_0 t}{2}\right)\right) + \frac{2i}{\omega_0}\sin(\omega_0 t/2)\right]\right\}$$

$$= e^{-i\omega_0 t/2}\left\{e^{i\omega_0 t/2} - \frac{|H'_{ab}|^2}{\omega_0\hbar^2}\left[-it\,e^{i\omega_0 t/2} + \frac{2i}{\omega_0}\frac{1}{2i}\left(e^{i\omega_0 t/2}-e^{-i\omega_0 t/2}\right)\right]\right\}$$

$$= 1 - \frac{|H'_{ab}|^2}{\omega_0\hbar^2}\left[-it + \frac{1}{\omega_0}\left(1-e^{-i\omega_0 t}\right)\right] = 1 + \frac{i}{\omega_0\hbar^2}|H'_{ab}|^2\left[t + \frac{1}{i\omega_0}\left(e^{-i\omega_0 t}-1\right)\right],$$ as above. ✓

PROBLEM 9.7 (a) $\dot{c}_a = -\frac{i}{2\hbar}V_{ab}e^{i\omega t}e^{-i\omega_0 t}c_b$; $\dot{c}_b = -\frac{i}{2\hbar}V_{ba}e^{-i\omega t}e^{i\omega_0 t}c_a$. Differentiate the latter, and substitute in the former:

$$\ddot{c}_b = -i\frac{V_{ba}}{2\hbar}\left[i(\omega_0-\omega)e^{i(\omega_0-\omega)t}c_a + e^{i(\omega_0-\omega)t}\dot{c}_a\right]$$

$$= i(\omega_0-\omega)\left[-i\frac{V_{ba}}{2\hbar}e^{i(\omega_0-\omega)t}c_a\right] - i\frac{V_{ba}}{2\hbar}e^{i(\omega_0-\omega)t}\left[-\frac{i}{2\hbar}V_{ab}e^{-i(\omega_0-\omega)t}c_b\right] = i(\omega_0-\omega)\dot{c}_b - \frac{|V_{ab}|^2}{(2\hbar)^2}c_b.$$

$\frac{d^2c_b}{dt^2} + i(\omega-\omega_0)\frac{dc_b}{dt} + \frac{|V_{ab}|^2}{4\hbar^2}c_b = 0$. Solution is of the form $c_b = e^{\lambda t}$: $\lambda^2 + i(\omega-\omega_0)\lambda + \frac{|V_{ab}|^2}{4\hbar^2} = 0 \Rightarrow$

$\lambda = \frac{1}{2}\left[-i(\omega-\omega_0) \pm \sqrt{-(\omega-\omega_0)^2 - |V_{ab}|^2/\hbar^2}\right] = i\left[-\frac{(\omega-\omega_0)}{2} \pm \omega_r\right]$, with ω_r defined in [9.30].

General solution: $c_b(t) = Ae^{i\left[-\frac{(\omega-\omega_0)}{2}+\omega_r\right]t} + Be^{i\left[\frac{(\omega-\omega_0)}{2}-\omega_r\right]t} = e^{-i(\omega-\omega_0)t/2}\left[Ae^{i\omega_r t} + Be^{-i\omega_r t}\right]$,

or, more conveniently: $c_b(t) = e^{-i(\omega-\omega_0)t/2}\left[C\cos(\omega_r t) + D\sin(\omega_r t)\right]$. But $c_b(0)=0$, so $C=0$:

$c_b(t) = De^{i(\omega_0-\omega)t/2}\sin(\omega_r t)$. $\therefore \dot{c}_b = D\left\{i\left(\frac{\omega_0-\omega}{2}\right)e^{i(\omega_0-\omega)t/2}\sin(\omega_r t) + \omega_r e^{i(\omega_0-\omega)t/2}\cos(\omega_r t)\right\}$; so

$c_a(t) = i\frac{2\hbar}{V_{ba}}e^{i(\omega-\omega_0)t}\dot{c}_b = i\frac{2\hbar}{V_{ba}}e^{i(\omega-\omega_0)t/2}D\left[i\left(\frac{\omega_0-\omega}{2}\right)\sin(\omega_r t) + \omega_r\cos(\omega_r t)\right]$. But $c_a(0) = 1$, so

$1 = i\frac{2\hbar}{V_{ba}}D\omega_r$, or $D = \frac{-iV_{ba}}{2\hbar\omega_r}$. $\therefore$ $\boxed{c_b(t) = -\frac{i}{2\hbar\omega_r}V_{ba}e^{i(\omega_0-\omega)t/2}\sin(\omega_r t)}$, and

$\boxed{c_a(t) = e^{i(\omega-\omega_0)t/2}\left[\cos(\omega_r t) + i\left(\frac{\omega_0-\omega}{2\omega_r}\right)\sin(\omega_r t)\right]}$.

(b) $P_{a\to b}(t) = |c_b(t)|^2 = \boxed{\left(\frac{|V_{ab}|}{2\hbar\omega_r}\right)^2\sin^2(\omega_r t)}$. The largest this gets (when $\sin^2 = 1$) is $\frac{|V_{ab}|^2/\hbar^2}{4\omega_r^2}$,

and the denominator, $4\omega_r^2 = (\omega-\omega_0)^2 + |V_{ab}|^2/\hbar^2$, exceeds the numerator, so $P \leq 1$ (and 1 only if $\omega = \omega_0$).

$|c_a|^2 + |c_b|^2 = \cos^2(\omega_r t) + \left(\frac{\omega_0-\omega}{2\omega_r}\right)^2\sin^2(\omega_r t) + \left(\frac{|V_{ab}|}{2\hbar\omega_r}\right)^2\sin^2(\omega_r t) = \cos^2(\omega_r t) + \frac{(\omega-\omega_0)^2 + (|V_{ab}|/\hbar)^2}{4\omega_r^2}\sin^2(\omega_r t)$

$= \cos^2(\omega_r t) + \sin^2(\omega_r t) = 1$ ✓.

(c) If $\boxed{|V_{ab}|^2 \ll \hbar^2(\omega-\omega_0)^2}$, then $\omega_r \cong \frac{1}{2}|\omega-\omega_0|$, and $P_{a\to b} \cong \frac{|V_{ab}|^2}{\hbar^2}\frac{\sin^2\left(\frac{\omega-\omega_0}{2}t\right)}{(\omega-\omega_0)^2}$, confirming [9.28].

(d) $\omega_r t = \pi \Rightarrow \boxed{t = \pi/\omega_r}$.

Problem 9.8 Spontaneous emission rate [9.56]: $A = \frac{\omega^3|\vec{p}|^2}{3\pi\epsilon_0\hbar c^3}$. Thermally stimulated emission rate [9.47]: $R = \frac{\pi}{3\epsilon_0\hbar^2}|\vec{p}|^2\rho(\omega)$, with $\rho(\omega) = \frac{\hbar}{\pi^2c^3}\frac{\omega^3}{(e^{\hbar\omega/k_BT}-1)}$ [9.52]. So the ratio is

$\frac{A}{R} = \frac{\omega^3|\vec{p}|^2}{3\pi\epsilon_0\hbar c^3}\cdot\frac{3\epsilon_0\hbar^2}{\pi|\vec{p}|^2}\cdot\frac{\pi^2c^3(e^{\hbar\omega/k_BT}-1)}{\hbar\omega^3} = e^{\hbar\omega/k_BT} - 1$. The ratio is a monotonically increasing function of ω, and is 1 when

$e^{\hbar\omega/k_BT} = 2$; i.e. $\frac{\hbar\omega}{k_BT} = \ln 2$, or $\omega = \frac{k_BT}{\hbar}\ln 2$, or $\nu = \frac{\omega}{2\pi} = \frac{k_BT}{h}\ln 2$. For $T = 300K$,

$\nu = \frac{(1.38\times10^{-23}\,J/K)(300\,K)}{(6.63\times10^{-34}\,J\cdot s)}\ln 2 = 4.35\times10^{12}\,Hz$. {For higher frequencies (including light, at 10^{14} Hz) spontaneous emission dominates.

Problem 9.9 $N(t) = e^{-t/\tau}N(0)$ (equations [9.58] and [9.59]). After $t_{1/2}$, $N(t) = \frac{1}{2}N(0)$, so $\frac{1}{2} = e^{-t/\tau}$, or $2 = e^{t/\tau}$, so $\frac{t}{\tau} = \ln 2$, or $\boxed{t_{1/2} = \tau \ln 2}$.

Problem 9.10 In Problem 9.1 we calculated the matrix elements of z; all of them are zero except $\langle 100|z|210\rangle = \frac{2^8}{\sqrt{2}\,3^5}a$. As for x and y, we noted that $|100\rangle$, $|200\rangle$, and $|210\rangle$ are <u>even</u> (in x, y), whereas $|21\pm1\rangle$ is odd. So the only nonzero matrix elements are $\langle 100|x|21\pm1\rangle$ and $\langle 100|y|21\pm1\rangle$. Using the wave functions in Problem 9.1:

$$\langle 100|x|21\pm1\rangle = \frac{1}{\sqrt{\pi a^3}}\left(\frac{\mp 1}{8\sqrt{\pi a^3}}\right)\frac{1}{a}\int e^{-r/a}\, r e^{-r/2a}\sin\theta\, e^{\pm i\phi}(r\sin\theta\cos\phi)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \mp\frac{1}{8\pi a^4}\int_0^\infty r^4 e^{-3r/2a}dr\int_0^\pi \sin^3\theta\, d\theta\int_0^{2\pi}(\cos\phi \pm i\sin\phi)\cos\phi\, d\phi = \frac{\mp 1}{8\pi a^4}\left(4!\left(\frac{2a}{3}\right)^5\right)\left(\frac{4}{3}\right)(\pi) = \mp\frac{2^7}{3^5}a.$$

$$\langle 100|y|21\pm1\rangle = \mp\frac{1}{8\pi a^4}\left(4!\left(\frac{2a}{3}\right)^5\right)\left(\frac{4}{3}\right)\int_0^{2\pi}(\cos\phi \pm i\sin\phi)\sin\phi\, d\phi = \frac{\mp 1}{8\pi a^4}\left(4!\left(\frac{2a}{3}\right)^5\right)\left(\frac{4}{3}\right)(\pm i\pi) = -i\frac{2^7}{3^5}a.$$

So: $\langle 100|\vec{r}|200\rangle = 0$; $\langle 100|\vec{r}|210\rangle = \frac{2^7\sqrt{2}}{3^5}a\,\hat{k}$; $\langle 100|\vec{r}|21\pm1\rangle = \frac{2^7}{3^5}a(\mp\hat{\imath} - i\hat{\jmath})$, and hence

$\wp^2 = 0$ (for $200 \to 100$), and $|\wp|^2 = (qa)^2\frac{2^{15}}{3^{10}}$ (for $210 \to 100$ and $21\pm1 \to 100$). Meanwhile,

$$\omega = \frac{E_2 - E_1}{\hbar} = \frac{1}{\hbar}\left(\frac{E_1}{4} - E_1\right) = -\frac{3E_1}{4\hbar},$$ so for the three $l=1$ states: $A = -\frac{3^3E_1^3}{2^6\hbar^3}\cdot\frac{(ea)^2 2^{15}}{3^{10}}\cdot\frac{1}{3\pi\epsilon_0\hbar c^3}$

$$A = -\frac{2^9}{3^8\pi}\cdot\frac{E_1^3e^2a^2}{\epsilon_0\hbar^4c^3} = \frac{2^{10}}{3^8}\left(\frac{E_1}{mc^2}\right)^2\frac{c}{a} = \frac{2^{10}}{3^8}\left(\frac{13.6}{0.511\times10^6}\right)^2\frac{(3.00\times10^8\,\text{m/s})}{(0.529\times10^{-10}\,\text{m})} = 6.27\times10^8/\text{s}.$$

$\tau = \frac{1}{A} = \boxed{1.60\times10^{-9}\,\text{s}}$ for the three $l=1$ states (all have the <u>same</u> lifetime); $\boxed{\tau = \infty}$ for the $l=0$ state.

Problem 9.11 $[L^2, z] = [L_x^2, z] + [L_y^2, z] + [L_z^2, z] = L_x[L_x, z] + [L_x, z]L_x + L_y[L_y, z] + [L_y, z]L_y + L_z[L_z, z] + [L_z, z]L_z$

But
$$\begin{cases}[L_x, z] = [yp_z - zp_y, z] = [yp_z, z] - [zp_y, z] = y[p_z, z] = -i\hbar y,\\ [L_y, z] = [zp_x - xp_z, z] = [zp_x, z] - [xp_z, z] = -x[p_z, z] = i\hbar x,\\ [L_z, z] = [xp_y - yp_x, z] = [xp_y, z] - [yp_x, z] = 0.\end{cases}$$

So: $[L^2, z] = L_x(-i\hbar y) + (-i\hbar y)L_x + L_y(i\hbar x) + (i\hbar x)L_y = i\hbar(-L_xy - yL_x + L_yx + xL_y)$.

But
$$\begin{cases}L_xy = L_xy - yL_x + yL_x = [L_x, y] + yL_x = i\hbar z + yL_x,\\ L_yx = L_yx - xL_y + xL_y = [L_y, x] + xL_y = -i\hbar z + xL_y.\end{cases}$$

So: $[L^2, z] = i\hbar(2xL_y - i\hbar z - 2yL_x - i\hbar z) \Rightarrow \boxed{[L^2, z] = 2i\hbar(xL_y - yL_x - i\hbar z)}$.

$[L^2, [L^2, z]] = 2i\hbar\{[L^2, xL_y] - [L^2, yL_x] - i\hbar[L^2, z]\} = 2i\hbar\{[L^2, x]L_y + x[L^2, L_y] - [L^2, y]L_x - y[L^2, L_x] - i\hbar(L^2z - zL^2)\}$

But $[L^2, L_y] = [L^2, L_x] = 0$ [4.103], so

$[L^2, [L^2, z]] = 2i\hbar\{2i\hbar(yL_z - zL_y - i\hbar x)L_y - 2i\hbar(zL_x - xL_z - i\hbar y)L_x - i\hbar(L^2z - zL^2)\}$, or

$$[L^2,[L^2,z]] = -2\hbar^2\big(2yL_zL_y \underbrace{-2zL_y^2 - 2i\hbar xL_y - 2zL_x^2}_{-2z(L_x^2+L_y^2+L_z^2)+2zL_z^2} + 2xL_zL_x + 2i\hbar yL_x - L^2z + zL^2\big)$$

$$= -2\hbar^2\big(2yL_zL_y - 2i\hbar xL_y + 2xL_zL_x + 2i\hbar yL_x + 2zL_z^2 - 2zL^2 - L^2z + zL^2\big)$$

$$= 2\hbar^2(zL^2 + L^2z) - 4\hbar^2\Big\{\underbrace{(yL_z - i\hbar x)}_{L_zy}L_y + \underbrace{(xL_z + i\hbar y)}_{L_zx}L_x + zL_zL_z\Big\}$$

$$= 2\hbar^2(zL^2 + L^2z) - 4\hbar^2\underbrace{(L_zyL_y + L_zxL_x + L_zzL_z)}_{L_z(\vec{r}\cdot\vec{L})=0} = 2\hbar^2(zL^2+L^2z). \text{ QED.}$$

PROBLEM 9.12 $|n00\rangle = R_{n0}(r)Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}R_{n0}(r)$, so $\langle n'00|\vec{r}|n00\rangle = \frac{1}{4\pi}\int R_{n'0}(r)R_{n0}(r)(x\hat{i}+y\hat{j}+z\hat{k})\,dx\,dy\,dz$.

But the integrand is odd in x, y, or z, so the integral is zero.

PROBLEM 9.13 (a) $|300\rangle \rightarrow \left\{\begin{matrix}|210\rangle\\ |211\rangle\\ |21\,{-1}\rangle\end{matrix}\right\} \rightarrow |100\rangle$. ($|300\rangle \rightarrow |200\rangle$ or $|300\rangle \rightarrow |100\rangle$ violate $\Delta l = \pm 1$ rule).

(b) From [9.72]: $\langle 210|\vec{r}|300\rangle = \langle 210|z|300\rangle\hat{k}$; from [9.69]: $\langle 21\pm1|\vec{r}|300\rangle = \langle 21\pm1|x|300\rangle\hat{i} + \langle 21\pm1|y|300\rangle\hat{j}$;

from [9.70]: $\pm\langle 21\pm1|x|300\rangle = i\langle 21\pm1|y|300\rangle$. So $|\langle 210|\vec{r}|300\rangle|^2 = |\langle 210|z|300\rangle|^2$ and

$|\langle 21\pm1|\vec{r}|300\rangle|^2 = 2\,|\langle 21\pm1|x|300\rangle|^2$, so there are really just two matrix elements to calculate.

$$\psi_{300} = \frac{1}{\sqrt{4\pi}}\frac{2}{\sqrt{27}}\frac{1}{a^{3/2}}\left(1 - \frac{2}{3}\frac{r}{a} + \frac{2}{27}\frac{r^2}{a^2}\right)e^{-r/3a}; \quad \psi_{210} = \sqrt{\frac{3}{4\pi}}\cos\theta\frac{1}{\sqrt{24}}\frac{1}{a^{3/2}}\frac{r}{a}e^{-r/2a};$$

$$\psi_{21\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}\frac{1}{\sqrt{24}}\frac{1}{a^{3/2}}\frac{r}{a}e^{-r/2a}.$$

$$\langle 21\pm1|x|300\rangle = \mp\sqrt{\frac{3}{8\pi}}\frac{1}{\sqrt{24}}\frac{1}{a^{5/2}}\frac{1}{\sqrt{4\pi}}\frac{2}{\sqrt{27}}\frac{1}{a^{3/2}}\int \sin\theta\, e^{\pm i\phi}re^{-r/2a}\left(1-\frac{2}{3}\frac{r}{a}+\frac{2}{27}\frac{r^2}{a^2}\right)e^{-r/3a}r\sin\theta\cos\phi\,\sin\theta\, r^2dr\,d\theta\,d\phi$$

$$= \mp\frac{1}{24\sqrt{3}\pi a^4}\int_0^\infty r^4\left(1-\frac{2}{3a}r+\frac{2}{27a^2}r^2\right)e^{-5r/6a}dr\int_0^\pi \sin^3\theta\,d\theta\int_0^{2\pi}(\cos\phi \pm i\sin\phi)\cos\phi\,d\phi$$

$$= \mp\frac{1}{24\sqrt{3}\pi a^4}\left\{4!\left(\frac{6a}{5}\right)^5 - \frac{2}{3a}5!\left(\frac{6a}{5}\right)^6 + \frac{2}{27a^2}6!\left(\frac{6a}{5}\right)^7\right\}\left(\frac{4}{3}\right)(\pi)$$

$$= \mp\frac{1}{18\sqrt{3}a^4}4!\frac{6^5a^5}{5^7}\left(25 - \frac{2}{3a}\cdot5\cdot5\cdot6a + \frac{2}{27a^2}\cdot6\cdot5\cdot6^2a^2\right) = \mp\frac{4a}{3\sqrt{3}}\frac{6^5}{5^7}(25-100+80)$$

$$= \mp\frac{2^7 3^4}{5^6\sqrt{3}}a.$$

$$\langle 210|z|300\rangle = \sqrt{\frac{3}{4\pi}}\frac{1}{\sqrt{24}}\frac{1}{a^{5/2}}\frac{1}{\sqrt{4\pi}}\frac{2}{\sqrt{27}}\frac{1}{a^{3/2}}\int\cos\theta\, re^{-r/2a}\left(1-\frac{2}{3a}r+\frac{2}{27a^2}r^2\right)e^{-r/3a}r\cos\theta\, r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{(1/a^4)}{12\sqrt{6}\pi}\int_0^\infty r^4\left(1-\frac{2}{3a}r+\frac{2}{27a^2}r^2\right)e^{-5r/6a}dr\int_0^\pi\cos^2\theta\sin\theta\,d\theta\int_0^{2\pi}d\phi = \frac{(1/a^4)}{12\sqrt{6}\pi}\left(\frac{4!\,6^5a^5}{5^6}\right)\left(\frac{2}{3}\right)(2\pi)$$

$$= \frac{2^8 3^4}{5^6\sqrt{6}}a.$$

$\therefore\ |\langle 210|\vec{r}|300\rangle|^2 = |\langle 21\pm1|\vec{r}|300\rangle|^2 = \underline{\underline{\dfrac{2^{15}3^7}{5^{12}}a^2}} = 0.2935a^2$. Evidently the three transition rates are equal, and hence $\boxed{1/3}$ go by each route.

(c) For each mode, $A = \dfrac{\omega^3 e^2 |\langle\vec{r}\rangle|^2}{3\pi\epsilon_0 \hbar c^3}$; here $\omega = \dfrac{E_3 - E_2}{\hbar} = \dfrac{1}{\hbar}\left(\dfrac{E_1}{9} - \dfrac{E_1}{4}\right) = -\dfrac{5}{36}\dfrac{E_1}{\hbar}$, so the total decay

rate is
$$R = 3\left(-\frac{5}{36}\frac{E_1}{\hbar}\right)^3 \frac{e^2}{3\pi\epsilon_0\hbar c^3}\left(\frac{2^{15}3^7}{5^{12}}a^2\right) = 6\left(\frac{2}{5}\right)^9\left(\frac{E_1}{mc^2}\right)^2\left(\frac{c}{a}\right)$$
$$= 6\left(\frac{2}{5}\right)^9\left(\frac{13.6}{.511\times10^6}\right)^2\left(\frac{3\times10^8}{.529\times10^{-10}}\right)/s = 6.32\times10^6/s. \quad \tau = \frac{1}{R} = \boxed{1.58\times10^{-7}\,s}.$$

Problem 9.14 (a) $\Psi(t) = \sum c_n(t)e^{-iE_nt/\hbar}\psi_n$. $H\Psi = i\hbar\dfrac{\partial\Psi}{\partial t}$; $H = H_0 + H'(t)$; $H_0\psi_n = E_n\psi_n$. So
$$\cancel{\sum c_n e^{-iE_nt/\hbar}E_n\psi_n} + \sum c_n e^{-iE_nt/\hbar}H'\psi_n = i\hbar\sum \dot{c}_n e^{-iE_nt/\hbar}\psi_n + \cancel{i\hbar\left(-\frac{i}{\hbar}\right)\sum c_n E_n e^{-iE_nt/\hbar}\psi_n}.$$
The first and last terms cancel, so $\sum c_n e^{-iE_nt/\hbar}H'\psi_n = i\hbar\sum\dot{c}_n e^{-iE_nt/\hbar}\psi_n$. Take the inner product with ψ_m:
$$\sum c_n e^{-iE_nt/\hbar}\langle\psi_m|H'|\psi_n\rangle = i\hbar\sum\dot{c}_n e^{-iE_nt/\hbar}\langle\psi_m|\psi_n\rangle.$$
Assume orthonormality of the unperturbed states: $\langle\psi_m|\psi_n\rangle = \delta_{mn}$, and define $H'_{mn} \equiv \langle\psi_m|H'|\psi_n\rangle$: $\sum c_n e^{-iE_nt/\hbar}H'_{mn} = i\hbar\dot{c}_m e^{-iE_mt/\hbar}$, or
$$\boxed{\dot{c}_m = -\frac{i}{\hbar}\sum_n c_n H'_{mn}e^{i(E_m-E_n)t/\hbar}}.$$

(b) Zeroth order: $c_N(t) = 1$, $c_m(t) = 0$ for $m \neq N$. Then in first order: $\dot{c}_N = -\dfrac{i}{\hbar}H'_{NN}$, or
$$\boxed{c_N(t) = 1 - \frac{i}{\hbar}\int_0^t H'_{NN}(t')dt'},$$
whereas for $m\neq N$: $\dot{c}_m = -\dfrac{i}{\hbar}H'_{mN}e^{i(E_m-E_N)t/\hbar}$, or
$$\boxed{c_m(t) = -\frac{i}{\hbar}\int_0^t H'_{mN}(t')e^{i(E_m-E_N)t'/\hbar}dt'}.$$

(c)
$$c_m(t) = -\frac{i}{\hbar}H'_{mN}\int_0^t e^{i(E_m-E_N)t'/\hbar}dt' = -\frac{i}{\hbar}H'_{mN}\left\{\frac{e^{i(E_m-E_N)t'/\hbar}}{i(E_m-E_N)/\hbar}\right\}\Bigg|_0^t = -H'_{mN}\left[\frac{e^{i(E_m-E_N)t/\hbar}-1}{E_m-E_N}\right]$$
$$= -\frac{H'_{mN}}{(E_m-E_N)}e^{i(E_m-E_N)t/2\hbar}\,2i\sin\left(\frac{E_m-E_N}{2\hbar}t\right). \quad P_{N\to m} = |c_m|^2 = \boxed{\frac{4|H'_{mN}|^2}{(E_m-E_N)^2}\sin^2\left(\frac{E_m-E_N}{2\hbar}t\right)}.$$

(d)
$$c_m(t) = -\frac{i}{\hbar}V_{mN}\frac{1}{2}\int_0^t\left(e^{i\omega t'}+e^{-i\omega t'}\right)e^{i(E_m-E_N)t'/\hbar}dt'$$
$$= -\frac{iV_{mN}}{2\hbar}\left\{\frac{e^{i(\omega\hbar+E_m-E_N)t'/\hbar}}{i(\hbar\omega+E_m-E_N)/\hbar} + \frac{e^{i(-\hbar\omega+E_m-E_N)t'/\hbar}}{i(-\hbar\omega+E_m-E_N)/\hbar}\right\}\Bigg|_0^t.$$

If $E_m > E_N$, the second term dominates, and transitions occur only for $\omega \approx (E_m-E_N)/\hbar$:
$$c_m(t) \cong -\frac{iV_{mN}}{2\hbar}\frac{1}{(i/\hbar)(E_m-E_N-\hbar\omega)}e^{i(E_m-E_N-\hbar\omega)t/2\hbar}\,2i\sin\left(\frac{E_m-E_N-\hbar\omega}{2\hbar}t\right),$$
so

$$P_{N\to m} = |c_m|^2 = |V_{mN}|^2 \frac{\sin^2\left(\frac{E_m-E_N-\hbar\omega}{2\hbar}t\right)}{(E_m-E_N-\hbar\omega)^2}.$$

If $E_m < E_N$ the first term dominates, and transitions occur only for $\omega \approx (E_N-E_m)/\hbar$:

$$c_m(t) \cong -\frac{i V_{mN}}{2\hbar}\frac{1}{(i/\hbar)(E_m-E_N+\hbar\omega)} e^{i(E_m-E_N+\hbar\omega)t/2\hbar} 2i\sin\left(\frac{E_m-E_N+\hbar\omega}{2\hbar}t\right), \text{ and hence}$$

$$P_{N\to m} = |V_{mN}|^2 \frac{\sin^2\left(\frac{E_m-E_N+\hbar\omega}{2\hbar}t\right)}{(E_m-E_N+\hbar\omega)^2}.$$ Combining the two results, we conclude that transitions occur to states with energy $E_m \cong E_N \pm \hbar\omega$, and $\boxed{P_{N\to m} = |V_{mN}|^2 \frac{\sin^2\left(\frac{E_N-E_m\pm\hbar\omega}{2\hbar}t\right)}{(E_N-E_m\pm\hbar\omega)^2}}$.

(e) For light, $V_{ba} = -\wp E_0$ (eq. [9.34]). The rest is as before (§ 9.2.3), leading to [9.47]:

$$\boxed{R_{N\to m} = \frac{\pi}{3\epsilon_0\hbar^2}|\vec{\wp}|^2\rho(\omega) \quad \text{, with } \omega = \pm(E_m-E_N)/\hbar} \left\{\begin{array}{l}+\text{sign} \Rightarrow \text{absorption}\\ -\text{sign} \Rightarrow \text{stimulated emission}\end{array}\right\}.$$

PROBLEM 9.15 For example (c): $\boxed{c_N(t) = 1 - \frac{i}{\hbar}H'_{NN}t \;;\; c_m(t) = -2i\frac{H'_{mN}}{(E_m-E_N)} e^{i(E_m-E_N)t/2\hbar}\sin\left(\frac{E_m-E_N}{2\hbar}t\right) \quad (m\neq N).}$

$|c_N|^2 = 1 + \frac{1}{\hbar^2}|H'_{NN}|^2t^2$, $|c_m|^2 = 4|H'_{mN}|^2 \frac{\sin^2\left(\frac{E_m-E_N}{2\hbar}t\right)}{(E_m-E_N)^2}$, so

$$\sum_m |c_m|^2 = 1 + \frac{t^2}{\hbar^2}|H'_{NN}|^2 + 4\sum_{m\neq N}|H'_{mN}|^2\frac{\sin^2\left(\frac{E_m-E_N}{2\hbar}t\right)}{(E_m-E_N)^2}.$$ This is plainly greater than 1 — but remember: the c's are accurate only to first order in H' — to this order the $|H'|^2$ terms do not belong. Only if terms of first order appeared in the sum would there be a real problem with normalization.

For example (d): $c_N = 1 - \frac{i}{\hbar}V_{NN}\int_0^t \cos(\omega t')dt' = 1 - \frac{i}{\hbar}V_{NN}\left.\frac{\sin(\omega t')}{\omega}\right|_0^t \Rightarrow \boxed{c_N(t) = 1 - \frac{i}{\hbar\omega}V_{NN}\sin(\omega t)}$.

$$\boxed{c_m(t) = -\frac{V_{mN}}{2}\left\{\frac{e^{i(E_m-E_N+\hbar\omega)t/\hbar}-1}{(E_m-E_N+\hbar\omega)} + \frac{e^{i(E_m-E_N-\hbar\omega)t/\hbar}-1}{(E_m-E_N-\hbar\omega)}\right\} \quad (m\neq N)}.$$

$|c_N|^2 = 1 + \frac{|V_{NN}|^2}{(\hbar\omega)^2}\sin^2(\omega t)$; in the rotating wave approximation $|c_m|^2 = |V_{mN}|^2 \frac{\sin^2\left(\frac{E_N-E_m\pm\hbar\omega}{2\hbar}t\right)}{(E_N-E_m\pm\hbar\omega)^2} \quad (m\neq N)$.

Again, ostensibly $\sum|c_m|^2 > 1$ — but the "extra" terms are of second order in H', and hence do not belong (to first order).

You would do $\boxed{\text{better to use } 1-\sum_{m\neq N}|c_m|^2}$. Schematically: $c_m = a_1H + a_2H^2 + \cdots$, so $|c_m|^2 = a_1^2H^2 + 2a_1a_2H^3 + \cdots$, whereas $c_N = 1 + b_1H + b_2H^2 + \cdots$, so $|c_N|^2 = 1 + 2b_1H + (2b_2+b_1^2)H^2 + \cdots$. Thus

... (knowing c_N to first order gets you $|c_m|$ to second order, but knowing c_N to first order (i.e. b_1) does NOT get you $|c_N|^2$ to second order (you'd also need b_2). It is precisely this b_2 term that would cancel the "extra" (second-order) terms in the calculation of $\sum |c_m|^2$ above.

PROBLEM 9.16 (a) [9.82] $\Rightarrow$ $\dot{c}_m = -\frac{i}{\hbar}\sum_n c_n H'_{mn} e^{i(E_m-E_n)t/\hbar}$. Here $H'_{mn} = \langle\psi_m|V_0(t)|\psi_n\rangle = \delta_{mn}V_0(t)$.

$\therefore \dot{c}_m = -\frac{i}{\hbar}c_m V_0(t)$; $\frac{dc_m}{c_m} = -\frac{i}{\hbar}V_0(t)dt \Rightarrow \ln c_m = -\frac{i}{\hbar}\int V_0(t')dt' + \text{constant}.$

$c_m(t) = c_m(0)e^{-\frac{i}{\hbar}\int_0^t V_0(t')dt'}$. Let $\Phi(t) \equiv -\frac{1}{\hbar}\int_0^t V_0(t')dt'$; $c_m(t) = e^{i\Phi}c_m(0)$ — hence

$|c_m(t)|^2 = |c_m(0)|^2$, and there are no transitions. $\boxed{\Phi(T) = -\frac{1}{\hbar}\int_0^T V_0(t)dt}$.

(b) [9.84] $\Rightarrow$ $c_N(t) \cong 1 - \frac{i}{\hbar}\int_0^t V_0(t')dt' = 1 + i\Phi.$

[9.85] $\Rightarrow$ $c_m(t) = -\frac{i}{\hbar}\int_0^t \delta_{mN}V_0(t')e^{i(E_m-E_N)t'/\hbar}dt' = 0 \quad (m\neq N).$ $\boxed{\begin{array}{l} c_N(t) = 1 + i\Phi(t) \\ c_m(t) = 0 \quad (m\neq N)\end{array}}$

The exact answer is $c_N(t) = e^{i\Phi(t)}$; $c_m(t) = 0$, and they are consistent, since $e^{i\Phi} \cong 1 + i\Phi$, to first order.

PROBLEM 9.17 Use result of Problem 9.14(c). Here $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$, so $E_2 - E_1 = \frac{3\pi^2\hbar^2}{2ma^2}$.

$$H'_{12} = \frac{2}{a}\int_0^{a/2}\sin\left(\frac{\pi}{a}x\right)V_0\sin\left(\frac{2\pi}{a}x\right)dx = \frac{2V_0}{a}\left[\frac{\sin\left(\frac{\pi}{a}x\right)}{2(\pi/a)} - \frac{\sin\left(\frac{3\pi}{a}x\right)}{2(3\pi/a)}\right]\Bigg|_0^{a/2} = \frac{V_0}{\pi}\left[\sin\left(\frac{\pi}{2}\right) - \frac{1}{3}\sin\left(\frac{3\pi}{2}\right)\right] = \frac{4V_0}{3\pi}.$$

$$\therefore [9.86] \Rightarrow P_{1\to2} = 4\left(\frac{4V_0}{3\pi}\right)^2\left(\frac{2ma^2}{3\pi^2\hbar^2}\right)^2\sin^2\left(\frac{3\pi^2\hbar}{4ma^2}t\right) = \boxed{\left[\frac{16ma^2V_0}{9\pi^3\hbar^2}\sin\left(\frac{3\pi^2\hbar T}{4ma^2}\right)\right]^2}.$$

PROBLEM 9.18 [9.86] $\Rightarrow$ $P_{N\to m} = 4|H'_{mN}|^2\frac{\sin^2[(E_N-E_m)t/2\hbar]}{(E_N-E_m)^2}$. This first hits its maximum value at time t given by

$$\frac{(E_N-E_m)t}{2\hbar} = \frac{\pi}{2}. \quad \text{Let } \Delta E \equiv E_N - E_m,\ \Delta t \equiv t. \text{ Then } \Delta E\Delta t = \pi\hbar.$$

Of course, you could wait longer than this — but it wouldn't improve the chances of a transition occurring. And the probability of a transition even in this amount of time is generally very much less than 1 — and depends on the nature of the perturbation, not just the energy gap. Still, it is clear that the smaller the energy gap, the longer the time for a transition. Anyhow, it is certainly true (in first-order perturbation theory) that $\Delta E\Delta t \geq \hbar/2$, if ΔE = energy gap and $\Delta t = \frac{1}{2\pi}$ times the time you must leave a constant perturbation on to obtain maximum probability of a transition.

PROBLEM 9.19 (a) $H = \frac{e}{m}\vec{B}\cdot\vec{S} = \frac{e\hbar}{2m}(B_x\sigma_x + B_y\sigma_y + B_z\sigma_z) = \frac{e\hbar}{2m}\left(\begin{pmatrix}0 & B_x\\ B_x & 0\end{pmatrix} + \begin{pmatrix}0 & -iB_y\\ iB_y & 0\end{pmatrix} + \begin{pmatrix}B_z & 0\\ 0 & -B_z\end{pmatrix}\right)$

$$H = \frac{e\hbar}{2m}\begin{pmatrix} B_z & (B_x - iB_y) \\ (B_x + iB_y) & -B_z \end{pmatrix}. \quad B_z = B_0\cos\alpha;\; B_x \pm iB_y = B_0\sin\alpha(\cos\omega t \pm i\sin\omega t) = B_0 e^{\pm i\omega t}\sin\alpha.$$

$$\therefore \boxed{H = \frac{e\hbar B_0}{2m}\begin{pmatrix} \cos\alpha & e^{-i\omega t}\sin\alpha \\ e^{i\omega t}\sin\alpha & -\cos\alpha \end{pmatrix}}.$$

(b) $i\hbar\frac{\partial\chi}{\partial t} = H\chi$; $\boxed{\chi = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}}$. $i\hbar\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = -\frac{\hbar\omega_1}{2}\begin{pmatrix} \cos\alpha & e^{-i\omega t}\sin\alpha \\ e^{i\omega t}\sin\alpha & -\cos\alpha \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow$

$\dot{a} = i\frac{\omega_1}{2}\left[\cos\alpha\, a + e^{-i\omega t}\sin\alpha\, b\right]$; $\dot{b} = i\frac{\omega_1}{2}\left[e^{i\omega t}\sin\alpha\, a - \cos\alpha\, b\right]$. Differentiate the second:

$\ddot{b} = i\frac{\omega_1}{2}\left[i\omega e^{i\omega t}\sin\alpha\, a + e^{i\omega t}\sin\alpha\,\dot{a} - \cos\alpha\,\dot{b}\right]$. Substitute the first for $\dot{a}$:

$$= i\frac{\omega_1}{2}\left[i\omega e^{i\omega t}\sin\alpha\, a + i\frac{\omega_1}{2}\sin\alpha\left(e^{i\omega t}\cos\alpha\, a + \sin\alpha\, b\right) - \cos\alpha\,\dot{b}\right]$$

$$= i\frac{\omega_1}{2}\, i e^{i\omega t}\sin\alpha\left[\omega + \frac{\omega_1}{2}\cos\alpha\right]a - \frac{\omega_1^2}{4}\sin^2\alpha\, b - \frac{i\omega_1}{2}\cos\alpha\,\dot{b}.$$ But $e^{i\omega t}\sin\alpha\, a = \cos\alpha\, b - \frac{2i}{\omega_1}\dot{b}$, so

$$= -\frac{\omega_1}{2}\left(\omega + \frac{\omega_1}{2}\cos\alpha\right)\left(\cos\alpha\, b - \frac{2i}{\omega_1}\dot{b}\right) - \frac{\omega_1^2}{4}\sin^2\alpha\, b - i\frac{\omega_1}{2}\cos\alpha\,\dot{b}$$

$$= \dot{b}\left\{-\frac{i\omega_1}{2}\cos\alpha + i\left(\omega + \frac{\omega_1}{2}\cos\alpha\right)\right\} + b\left\{-\frac{\omega_1^2}{4}\sin^2\alpha - \frac{\omega_1}{2}\cos\alpha\left(\omega + \frac{\omega_1}{2}\cos\alpha\right)\right\}$$

$$= i\omega\dot{b} + b\left(-\frac{\omega_1^2}{4} - \frac{\omega\omega_1}{2}\cos\alpha\right) = i\omega\dot{b} - \frac{b}{4}\left(\omega_1^2 + 2\omega\omega_1\cos\alpha\right) = i\omega\dot{b} - \frac{1}{4}(\lambda^2 - \omega^2)b.$$

$\ddot{b} - i\omega\dot{b} + \frac{1}{4}(\lambda^2 - \omega^2)b = 0$. Solutions are of the form $e^{\mu t}$: $\mu^2 - i\omega\mu + \frac{1}{4}(\lambda^2 - \omega^2) = 0 \Rightarrow$

$\mu = \frac{1}{2}\left[i\omega \pm\sqrt{-\omega^2 - \lambda^2 + \omega^2}\right] = \frac{i}{2}(\omega \pm \lambda)$. General solution: $b(t) = Ae^{\frac{i}{2}(\omega+\lambda)t} + Be^{\frac{i}{2}(\omega-\lambda)t}$, or

$b(t) = e^{i\omega t/2}\left[Ae^{i\lambda t/2} + Be^{-i\lambda t/2}\right] = e^{i\omega t/2}\left(C\cos(\lambda t/2) + D\sin(\lambda t/2)\right)$. But $\chi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so $b(0) = 0$.

$\therefore C = 0$, and $\underline{\underline{b(t) = De^{i\omega t/2}\sin(\lambda t/2)}}$. $\quad a(t) = \frac{1}{\sin\alpha}e^{-i\omega t}\left(\cos\alpha\, b - \frac{2i}{\omega_1}\dot{b}\right) \Rightarrow$

$$a = \frac{1}{\sin\alpha}e^{-i\omega t}\left\{D\cos\alpha\, e^{i\omega t/2}\sin(\lambda t/2) - \frac{2i}{\omega_1}D\left[\frac{i\omega}{2}e^{i\omega t/2}\sin(\lambda t/2) + \frac{\lambda}{2}e^{i\omega t/2}\cos(\lambda t/2)\right]\right\}$$

$$= \frac{D}{\sin\alpha}e^{-i\omega t/2}\left\{\left(\cos\alpha + \frac{\omega}{\omega_1}\right)\sin(\lambda t/2) - \frac{i\lambda}{\omega_1}\cos(\lambda t/2)\right\}.$$ But $a(0) = 1$, so

$1 = \frac{D}{\sin\alpha}\left(-\frac{i\lambda}{\omega_1}\right)$, or $D = \frac{i\omega_1\sin\alpha}{\lambda}$. $\quad\therefore \boxed{b(t) = \frac{i\omega_1\sin\alpha}{\lambda}e^{i\omega t/2}\sin(\lambda t/2)}$.

$$\boxed{a(t) = e^{-i\omega t/2}\left[\cos(\lambda t/2) + i\frac{(\omega_1\cos\alpha + \omega)}{\lambda}\sin(\lambda t/2)\right]}.$$

(c) $[9.17] \Rightarrow c_b^{(1)} = -\frac{i}{\hbar}\int_0^t H'_{ba}(t')e^{i\omega_0 t'}dt'$. Here $H^0 = -\frac{\hbar\omega_1}{2}\cos\alpha\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; $E_a = -\frac{\hbar\omega_1}{2}\cos\alpha$, $E_b = \frac{\hbar\omega_1}{2}\cos\alpha$.

So $\omega_0 = \frac{E_b - E_a}{\hbar} = \omega_1\cos\alpha$. $H' = -\frac{\hbar\omega_1}{2}\sin\alpha\begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{pmatrix}$, so $H'_{ba} = -\frac{\hbar\omega_1}{2}\sin\alpha\, e^{i\omega t}$.

$$\therefore c_b^{(1)} = -\frac{i}{\hbar}\left(-\frac{\hbar\omega_1}{2}\sin\alpha\right)\int_0^t e^{i\omega t'}e^{i\omega_0 t'}dt' = \frac{i\omega_1}{2}\sin\alpha\left.\frac{e^{i(\omega+\omega_0)t'}}{i(\omega+\omega_0)}\right|_0^t = \frac{\omega_1}{2}\sin\alpha\,\frac{e^{i(\omega+\omega_0)t} - 1}{(\omega+\omega_0)}.$$

$$c_b^{(1)} = \frac{\omega_1}{2}\sin\alpha\,\frac{e^{i(\omega+\omega_0)t/2}}{(\omega+\omega_0)}\,2i\sin((\omega+\omega_0)t/2);\quad \boxed{c_b^{(1)}(t) = \frac{i\omega_1\sin\alpha}{(\omega+\omega_0)}\sin((\omega+\omega_0)t/2)\,e^{i(\omega+\omega_0)t/2}};$$

$$\boxed{P_{a\to b} = \left[\frac{\omega_1\sin\alpha}{\omega+\omega_1\cos\alpha}\sin((\omega+\omega_1\cos\alpha)t/2)\right]^2}.$$

To compare the exact and perturbation theory results, note first that $b(t)$ is not the same as $c_b(t)$ — see parenthetical remark at the bottom of p 299. Rather,

$$e^{-iE_1t/\hbar}c_1(t) = b(t)\ ;\ \text{or}\ e^{-i\frac{\omega_1\cos\alpha}{2}t}c_b(t) = b(t),\ \text{or}\ e^{-i\omega_0t/2}c_b(t) = b(t).$$

Now $e^{-i\omega_0t/2}c_b^{(1)}(t) = \dfrac{i\omega_1\sin\alpha}{(\omega+\omega_0)}e^{i\omega t/2}\sin((\omega+\omega_0)t/2)$, which differs from the exact $b(t)$ in the replacement $\lambda \to \omega+\omega_0 = \omega+\omega_1\cos\alpha$. But $\lambda = \sqrt{\omega^2+\omega_1^2+2\omega\omega_1\cos\alpha} = \sqrt{\omega^2+\omega_1^2\cos^2\alpha+2\omega\omega_1\cos\alpha+\omega_1^2(1-\cos^2\alpha)}$, or $\lambda = \sqrt{(\omega+\omega_1\cos\alpha)^2+(\omega_1\sin\alpha)^2}$. So the requirement is $\boxed{\omega_1\sin\alpha \ll (\omega+\omega_1\cos\alpha)}$, or

$$\boxed{|B_0| \ll \frac{\omega m}{|e(\sin\alpha-\cos\alpha)|}}.$$

PROBLEM 9.20 (a) $H' = -q\vec{E}\cdot\vec{r} = -q(\vec{E}_0\cdot\vec{r})(\vec{k}\cdot\vec{r})\sin(\omega t)$. Write $\vec{E}_0 = E_0\hat{n}$, $\vec{k} = \frac{\omega}{c}\hat{k}$. Then

$$H' = -q\frac{E_0\omega}{c}(\hat{n}\cdot\vec{r})(\hat{k}\cdot\vec{r})\sin(\omega t).\quad\therefore\quad H'_{ba} = -q\frac{E_0\omega}{c}\langle b|(\hat{n}\cdot\vec{r})(\hat{k}\cdot\vec{r})|a\rangle\sin(\omega t).$$

This is the analog to [9.33]: $H'_{ba} = -qE_0\langle b|\hat{n}\cdot\vec{r}|a\rangle_A^{\cos t}$. The rest of the analysis is identical to the dipole case (except that it is $\sin(\omega t)$ instead of $\cos(\omega t)$ — but this amounts to resetting the clock, and clearly has no effect on the transition rate). We can skip therefore to [9.56] — except for the factor of 1/3, which came from the averaging in [9.45] and [9.46]:

$$A = \frac{\omega^3}{\pi\epsilon_0\hbar c^3}q^2\frac{\omega^2}{c^2}\left|\langle b|(\hat{n}\cdot\vec{r})(\hat{k}\cdot\vec{r})|a\rangle\right|^2 = \boxed{\frac{q^2\omega^5}{\pi\epsilon_0\hbar c^5}\left|\langle b|(\hat{n}\cdot\vec{r})(\hat{k}\cdot\vec{r})|a\rangle\right|^2}.$$

(b) Let the oscillator lie along the x direction, so $(\hat{n}\cdot\vec{r}) = \hat{n}_x x$ and $\hat{k}\cdot\vec{r} = \hat{k}_x x$. For a transition from n to n', we have

$$A = \frac{q^2\omega^5}{\pi\epsilon_0\hbar c^5}(\hat{k}_x\hat{n}_x)^2\left|\langle n'|x^2|n\rangle\right|^2.$$

From Problem 2.37,

$$\langle n'|x^2|n\rangle = -\frac{\hbar}{2m\bar{\omega}}\langle n'|(a_+^2-a_+a_--a_-a_++a_-^2)|n\rangle,$$

where $\bar{\omega}$ is the frequency of the oscillator — not to be confused with ω, the frequency of the electromagnetic wave. Now for spontaneous emission the final state must be lower in energy, so $n'<n$, and hence the only surviving term is a_-^2:

$$\langle n'|x^2|n\rangle = -\frac{1}{2m\bar{\omega}^2}\langle n'|\sqrt{n(n-1)}\,\hbar\bar{\omega}|n-2\rangle = -\frac{\hbar}{2m\bar{\omega}}\sqrt{n(n-1)}\,\delta_{n',n-2}$$

Evidently transitions only go from $|n\rangle$ to $|n-2\rangle$, and hence $\omega = \dfrac{E_n-E_{n-2}}{\hbar} = \dfrac{1}{\hbar}\left[(n+\tfrac{1}{2})\hbar\bar{\omega}-(n-2+\tfrac{1}{2})\hbar\bar{\omega}\right] = 2\bar{\omega}$.

$\langle n'|x^2|n\rangle = -\frac{\hbar}{2m\omega}\sqrt{n(n-1)}\,\delta_{n',n-2}$; $R_{n\to n-2} = \frac{q^2\omega^5}{\pi\epsilon_0\hbar c^5}(\hat{k}_x\hat{n}_x)^2\frac{\hbar^2}{m^2\omega^2}n(n-1)$. It remains to calculate the average of $(\hat{k}_x\hat{n}_x)^2$. It's easiest to reorient the oscillator along a direction $\hat{r}$, making angle θ with the z axis, and let the radiation be incident from the z direction (so $\hat{k}_x \to \hat{k}_r = \cos\theta$). Averaging over the two polarizations ($\hat{i}$ and $\hat{j}$): $\langle \hat{n}_r^2\rangle = \frac{1}{2}(\hat{i}_r^2 + \hat{j}_r^2) = \frac{1}{2}(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) = \frac{1}{2}\sin^2\theta$. Now average over all directions:

$$\langle \hat{k}_r^2\hat{n}_r^2\rangle = \frac{1}{4\pi}\int \frac{1}{2}\sin^2\theta\cos^2\theta(\sin\theta\, d\theta\, d\phi) = \frac{1}{8\pi}2\pi\int_0^\pi (1-\cos^2\theta)\cos^2\theta\sin\theta\, d\theta = \frac{1}{4}\left[-\frac{\cos^3\theta}{3}+\frac{\cos^5\theta}{5}\right]_0^\pi = \frac{1}{4}\left(\frac{2}{3}-\frac{2}{5}\right) = \frac{1}{15}.$$

$\therefore \boxed{R = \frac{1}{15}\frac{q^2\hbar\omega^3}{\pi\epsilon_0 m^2c^5}n(n-1)}$. Comparing [9.63] $\frac{R(\text{forbidden})}{R(\text{allowed})} = \boxed{\frac{2}{5}(n-1)\frac{\hbar\omega}{mc^2}}$. For a nonrelativistic system, $\hbar\omega \ll mc^2$ – hence the term "forbidden".

(c) If both the initial state and the final state have $l=0$, the wave function is independent of angle ($Y_0^0 = \frac{1}{\sqrt{4\pi}}$), and the angular part of the integral is:

$$\langle a|(\hat{n}\cdot\vec{r})(\hat{k}\cdot\vec{r})|b\rangle = \cdots\int(\hat{n}\cdot\hat{r})(\hat{k}\cdot\hat{r})\sin\theta\, d\theta\, d\phi = \cdots\frac{4\pi}{3}(\hat{n}\cdot\hat{k}).\qquad [6.94].$$

But $\hat{n}\cdot\hat{k}=0$, since electromagnetic waves are transverse. So $R=0$ in this case (both for allowed and) for forbidden transitions.

Chapter 10

Problem 10.1 (a) Let $(mvx^2 - 2E_n^i a t)/2\hbar w = \phi(x,t)$. $\Phi_n = \sqrt{\frac{2}{w}}\sin\left(\frac{n\pi}{w}x\right)e^{i\phi}$, so

$$\frac{\partial\Phi_n}{\partial t} = \sqrt{2}\left(-\frac{1}{2}\frac{1}{w^{3/2}}v\right)\sin\left(\frac{n\pi}{w}x\right)e^{i\phi} + \sqrt{\frac{2}{w}}\left(-\frac{n\pi x}{w^2}v\cos\left(\frac{n\pi}{w}x\right)\right)e^{i\phi} + \sqrt{\frac{2}{w}}\sin\left(\frac{n\pi}{w}x\right)\left(i\frac{\partial\phi}{\partial t}\right)e^{i\phi}$$

$$= \left[-\frac{v}{2w} - \frac{n\pi x v}{w^2}\cot\left(\frac{n\pi}{w}x\right) + i\frac{\partial\phi}{\partial t}\right]\Phi_n. \quad \frac{\partial\phi}{\partial t} = \frac{1}{2\hbar}\left[-\frac{2E_n^i a}{w} - \frac{v}{w^2}\left(mvx^2 - 2E_n^i a t\right)\right] = -\frac{E_n^i a}{\hbar w} - \frac{v}{w}\phi.$$

$$\therefore i\hbar\frac{\partial\Phi_n}{\partial t} = -i\hbar\left[\frac{v}{2w} + \frac{n\pi x v}{w^2}\cot\left(\frac{n\pi}{w}x\right) + i\frac{E_n^i a}{\hbar w} + i\frac{v}{w}\phi\right]\Phi_n.$$

$H\Phi_n = -\frac{\hbar^2}{2m}\frac{\partial^2\Phi_n}{\partial x^2}$. $\frac{\partial\Phi_n}{\partial x} = \sqrt{\frac{2}{w}}\left(\frac{n\pi}{w}\cos\left(\frac{n\pi}{w}x\right)\right)e^{i\phi} + \sqrt{\frac{2}{w}}\sin\left(\frac{n\pi}{w}x\right)e^{i\phi}\left(i\frac{\partial\phi}{\partial x}\right)$. $\frac{\partial\phi}{\partial x} = \frac{mvx}{\hbar w}$.

$$\frac{\partial\Phi_n}{\partial x} = \left[\frac{n\pi}{w}\cot\left(\frac{n\pi}{w}x\right) + i\frac{mvx}{\hbar w}\right]\Phi_n. \quad \frac{\partial^2\Phi_n}{\partial x^2} = \left[-\left(\frac{n\pi}{w}\right)^2\csc^2\left(\frac{n\pi}{w}x\right) + \frac{imv}{\hbar w}\right]\Phi_n + \left[\frac{n\pi}{w}\cot\left(\frac{n\pi}{w}x\right) + i\frac{mvx}{\hbar w}\right]^2\Phi_n$$

So Schrodinger equation ($i\hbar\,\partial\Phi_n/\partial t = H\Phi_n$) is satisfied $\iff$

$$-i\hbar\left[\frac{v}{2w} + \frac{n\pi x v}{w^2}\cot\left(\frac{n\pi}{w}x\right) + i\frac{E_n^i a}{\hbar w} + i\frac{v}{w}\phi\right] = -\frac{\hbar^2}{2m}\left\{-\left(\frac{n\pi}{w}\right)^2\csc^2\left(\frac{n\pi}{w}x\right) + \frac{imv}{\hbar w} + \left[\frac{n\pi}{w}\cot\left(\frac{n\pi}{w}x\right) + i\frac{mvx}{\hbar w}\right]^2\right\}$$

Cot terms: $-i\hbar\left(\frac{n\pi x v}{w^2}\right) \stackrel{?}{=} -\frac{\hbar^2}{2m}\left(2\frac{n\pi}{w}i\frac{mvx}{\hbar w}\right) = -i\hbar\frac{n\pi v x}{w^2}$ ✓

Remaining trig terms on right: $-\left(\frac{n\pi}{w}\right)^2\csc^2\left(\frac{n\pi}{w}x\right) + \left(\frac{n\pi}{w}\right)^2\cot^2\left(\frac{n\pi}{w}x\right) = -\left(\frac{n\pi}{w}\right)^2\left[\frac{1-\cos^2\left(\frac{n\pi}{w}x\right)}{\sin^2\left(\frac{n\pi}{w}x\right)}\right] = -\left(\frac{n\pi}{w}\right)^2$.

This leaves:

$$i\left[\frac{v}{2w} + i\frac{E_n^i a}{\hbar w} + i\frac{v}{w}\left(\frac{mvx^2 - 2E_n^i a t}{2\hbar w}\right)\right] \stackrel{?}{=} \frac{\hbar}{2m}\left[-\left(\frac{n\pi}{w}\right)^2 + \frac{imv}{\hbar w} - \frac{m^2v^2x^2}{\hbar^2 w^2}\right]$$

$$\cancel{\frac{iv}{2}} - \frac{E_n^i a}{\hbar} - \cancel{\frac{mv^2x^2}{2\hbar w}} + \frac{vE_n^i a t}{\hbar w} \stackrel{?}{=} -\frac{\hbar n^2\pi^2}{2mw} + \cancel{\frac{iv}{2}} - \cancel{\frac{mv^2x^2}{2\hbar w}}$$

$$-\frac{E_n^i a}{\hbar w}(w - vt) = -\frac{E_n^i a^2}{\hbar w} \stackrel{?}{=} -\frac{\hbar n^2\pi^2}{2mw} \iff -\frac{n^2\pi^2\hbar^2}{2ma^2}\frac{a^2}{\hbar w} = -\frac{\hbar n^2\pi^2}{2mw} = \text{rhs.} ✓$$

So Φ_n does satisfy the Schrodinger equation, and since $\Phi_n(x,t) = (\dots)\sin\left(\frac{n\pi}{w}x\right)$, it fits the boundary conditions: $\Phi_n(0,t) = \Phi_n(w,t) = 0$.

(b) [10.4] $\Rightarrow$ $\Psi(x,0) = \sum c_n\Phi_n(x,0) = \sum c_n\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)e^{imvx^2/2\hbar a}$. Multiply by $\sqrt{\frac{2}{a}}\sin\left(\frac{n'\pi}{a}x\right)e^{-imvx^2/2\hbar a}$ and integrate:

$$\sqrt{\frac{2}{a}}\int_0^a \Psi(x,0)\sin\left(\frac{n'\pi}{a}x\right)e^{-imvx^2/2\hbar a}dx = \sum c_n\underbrace{\left[\frac{2}{a}\int_0^\pi \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n'\pi}{a}x\right)dx\right]}_{\delta_{nn'}} = c_{n'}$$

So, in general: $c_n = \sqrt{\frac{2}{a}}\int_0^a e^{-imvx^2/2\hbar a}\sin\left(\frac{n\pi}{a}x\right)\Psi(x,0)\,dx$. In this particular case,

$$c_n = \frac{2}{a}\int_0^a e^{-imvx^2/2\hbar a}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{\pi}{a}x\right)dx. \quad \text{Let } \frac{\pi}{a}x \equiv z;\ dx = \frac{a}{\pi}dz;\ \frac{mvx^2}{2\hbar a} = \frac{mvz^2a^2}{2\hbar a\pi^2} = \frac{mva}{2\pi^2\hbar}z^2.$$

$$\therefore c_n = \frac{2}{\pi}\int_0^\pi e^{-i\alpha z^2}\sin(nz)\sin(z)dz. \text{ QED}$$

(c) $w(T_e) = 2a \Rightarrow a + vT_e = 2a \Rightarrow vT_e = a \Rightarrow \boxed{T_e = a/v}$; $e^{-iE_1t/\hbar} \Rightarrow \omega = \frac{E_1}{\hbar} \Rightarrow T_i = \frac{2\pi}{\omega} = 2\pi\frac{\hbar}{E_1}$, or

$T_i = \frac{2\pi\hbar}{\pi^2\hbar^2}2ma^2 = \frac{4}{\pi}\frac{ma^2}{\hbar}$. $\boxed{T_i = 4ma^2/\pi\hbar}$. Adiabatic $\Rightarrow T_e \gg T_i \Rightarrow \frac{a}{v} \gg \frac{4ma^2}{\pi\hbar} \Rightarrow \frac{4}{\pi}\frac{mav}{\hbar} \ll 1$, or

$8\pi\left(\frac{mav}{2\pi^2\hbar}\right) = 8\pi\alpha \ll 1$, so $\alpha \ll 1$. Then $c_n = \frac{2}{\pi}\int_0^\pi \sin(nz)\sin(z)\,dz = \boxed{\delta_{n1}}$.

$\therefore \boxed{\Psi(x,t) = \sqrt{\frac{2}{w}}\sin\left(\frac{\pi x}{w}\right)e^{i(mvx^2 - 2E_1^i at)/2\hbar w}}$, which (apart from a phase factor) is the ground state of the instantaneous well, of width w, as required by the adiabatic theorem. (Actually, the first term in the exponent, which is at most $\frac{mva^2}{2\hbar a} = \frac{mva}{2\hbar} \ll 1$ and could be dropped, in the adiabatic regime.)

(d) $\theta(t) = -\frac{1}{\hbar}\left(\frac{\pi^2\hbar^2}{2m}\right)\int_0^t \frac{1}{(a+vt')^2}dt' = -\frac{\pi^2\hbar}{2m}\left(-\frac{1}{v}\left(\frac{1}{a+vt'}\right)\right)\Big|_0^t = -\frac{\pi^2\hbar}{2mv}\left(\frac{1}{a}-\frac{1}{w}\right) = -\frac{\pi^2\hbar}{2mv}\left(\frac{vt}{aw}\right) = -\frac{\pi^2\hbar t}{2maw}$.

So (dropping the $\frac{mvx^2}{2\hbar w}$ term, as explained in (c)) $\Psi(x,t) = \sqrt{\frac{2}{w}}\sin\left(\frac{\pi x}{w}\right)e^{-iE_1^i at/\hbar w}$

can be written (since $-E_1^i at/\hbar w = -\frac{\pi^2\hbar^2}{2ma^2}\frac{at}{\hbar w} = -\frac{\pi^2\hbar t}{2maw} = \theta$): $\boxed{\Psi(x,t) = \sqrt{\frac{2}{w}}\sin\left(\frac{\pi x}{w}\right)e^{i\theta}}$.

This is exactly what one might naively expect: for a fixed well (of width a) we'd have $\Psi(x,t) = \psi_1(x)e^{-iE_1t/\hbar}$; for the (adiabatically) expanding well, simply replace a by the (time-dependent) width w, and integrate to get the accumulated phase factor, noting that E_1 is now a function of t.

PROBLEM 10.2 $\boxed{T_{ext} \text{ is the minimum value of } f/\left(\frac{df}{dt}\right)}$; $\boxed{T_{int} \text{ is the maximum of } \frac{\hbar}{|E_\ell - E_n|}}$.

(In the first case the minimum is over the period of change — see Figure 10.4; in the second the maximum is over other states ℓ to which a transition could be made — i.e. for which $V_{\ell n} \neq 0$.) Then $T_{ext} \gg T_{int}$ (the adiabatic condition) ensures that [10.15] always holds.

PROBLEM 10.3 To show: $i\hbar\frac{\partial\chi}{\partial t} = H\chi$, where χ is given by [10.31] and H is given by [10.25].

$$\frac{\partial\chi}{\partial t} = \begin{pmatrix} \frac{\lambda}{2}\left[-\sin\left(\frac{\lambda t}{2}\right) + i\frac{(\omega_1+\omega)}{\lambda}\cos\left(\frac{\lambda t}{2}\right)\right]\cos(\alpha/2)e^{-i\omega t/2} - \frac{i\omega}{2}\left[\cos(\lambda t/2) + \frac{i(\omega_1+\omega)}{\lambda}\sin(\lambda t/2)\right]\cos(\alpha/2)e^{-i\omega t/2} \\ \frac{\lambda}{2}\left[-\sin(\lambda t/2) + i\frac{(\omega_1-\omega)}{\lambda}\cos(\lambda t/2)\right]\sin(\alpha/2)e^{i\omega t/2} + \frac{i\omega}{2}\left[\cos(\lambda t/2) + i\frac{(\omega_1-\omega)}{\lambda}\sin(\lambda t/2)\right]\sin(\alpha/2)e^{i\omega t/2} \end{pmatrix}$$

$$H\chi = -\frac{\hbar\omega_1}{2}\begin{pmatrix} \cos\alpha\left[\cos(\lambda t/2) + i\frac{(\omega_1+\omega)}{\lambda}\sin(\lambda t/2)\right]\cos\frac{\alpha}{2}e^{-i\omega t/2} + e^{-i\omega t}\sin\alpha\left[\cos\left(\frac{\lambda t}{2}\right) + \frac{i(\omega_1-\omega)}{\lambda}\sin(\lambda t/2)\right]\sin\frac{\alpha}{2}e^{i\omega t/2} \\ e^{i\omega t}\sin\alpha\left[\cos(\lambda t/2) + \frac{i(\omega_1+\omega)}{\lambda}\sin\left(\frac{\lambda t}{2}\right)\right]\cos\frac{\alpha}{2}e^{-i\omega t/2} - \cos\alpha\left[\cos\left(\frac{\lambda t}{2}\right) + \frac{i(\omega_1-\omega)}{\lambda}\sin(\lambda t/2)\right]\sin\frac{\alpha}{2}e^{i\omega t/2} \end{pmatrix}$$

So the conditions are:

(1) $i\cancel{\hbar}\left\{\frac{\lambda}{\cancel{2}}\left[-\sin(\lambda t/2) + i\frac{(\omega_1+\omega)}{\lambda}\cos(\lambda t/2)\right]\cancel{\cos\frac{\alpha}{2}} - \frac{i\omega}{\cancel{2}}\left[\cos\left(\frac{\lambda t}{2}\right) + \frac{i(\omega_1+\omega)}{\lambda}\sin(\lambda t/2)\right]\cancel{\cos\frac{\alpha}{2}}\right\}$

$\stackrel{?}{=} -\frac{\cancel{\hbar}\omega_1}{\cancel{2}}\left\{\left[\cos\left(\frac{\lambda t}{2}\right) + i\frac{(\omega_1+\omega)}{\lambda}\sin\left(\frac{\lambda t}{2}\right)\right]\cos\alpha\cancel{\cos\frac{\alpha}{2}} + \left[\cos\left(\frac{\lambda t}{2}\right) + \frac{i(\omega_1-\omega)}{\lambda}\sin\left(\frac{\lambda t}{2}\right)\right]\underbrace{\sin\alpha}_{2\sin\frac{\alpha}{2}\cancel{\cos\frac{\alpha}{2}}}\sin\frac{\alpha}{2}\right\}$

(2) $\sin\left(\frac{\lambda t}{2}\right)\left\{-i\lambda + \frac{i\omega(\omega_1+\omega)}{\lambda} + \frac{\omega_1(\omega_1+\omega)}{\lambda}\cos\alpha + \frac{i\omega_1(\omega_1-\omega)}{\lambda}2\sin^2\frac{\alpha}{2}\right\} \stackrel{?}{=} 0.$

$$= \frac{i}{\lambda}\sin(\lambda t/2)\left[-\cancel{\omega_1^2} - 2\omega\omega_1\cos\alpha + \cancel{\omega\omega_1} + \omega_1^2\cancel{\cos\alpha} + \omega\omega_1\cos\alpha + \cancel{\omega_1^2} - \cancel{\omega\omega_1} - \omega_1^2\cancel{\cos\alpha} + \omega\omega_1\cos\alpha\right] = 0 \checkmark$$

(b) $\cos(\lambda t/2)\left\{-(\omega_1 + \cancel{\omega}) + \cancel{\omega} + \omega_1\cos\alpha + \omega_1 2\sin^2\frac{\alpha}{2}\right\} = \omega_1\cos(\lambda t/2)\left[-1 + \cos\alpha + (1-\cos\alpha)\right] = 0 \checkmark$

(2) $i\cancel{\hbar}\left\{\frac{\lambda}{\cancel{2}}\left[-\sin(\lambda t/2) + i\frac{(\omega_1-\omega)}{\lambda}\cos(\lambda t/2)\right]\cancel{\sin(\alpha/2)} + \frac{i\omega}{\cancel{2}}\left[\cos(\lambda t/2) + \frac{i(\omega_1-\omega)}{\lambda}\sin(\lambda t/2)\right]\cancel{\sin(\alpha/2)}\right\}$

$$\stackrel{?}{=} -\frac{\cancel{\hbar}\omega_1}{\cancel{2}}\left\{\left[\cos\left(\frac{\lambda t}{2}\right) + i\frac{(\omega_1+\omega)}{\lambda}\sin\left(\frac{\lambda t}{2}\right)\right]2\cancel{\sin\frac{\alpha}{2}}\cos^2\frac{\alpha}{2} - \left[\cos\left(\frac{\lambda t}{2}\right) + \frac{i(\omega_1-\omega)}{\lambda}\sin\left(\frac{\lambda t}{2}\right)\right]\cos\alpha\,\cancel{\sin\frac{\alpha}{2}}\right\}$$

(a) $\sin(\lambda t/2)\left\{-i\lambda - \frac{i\omega(\omega_1-\omega)}{\lambda} + \frac{i\omega_1(\omega_1+\omega)}{\lambda}2\cos^2\left(\frac{\alpha}{2}\right) - \frac{i\omega_1(\omega_1-\omega)}{\lambda}\cos\alpha\right\} \stackrel{?}{=} 0$

$$\frac{i}{\lambda}\sin(\lambda t/2)\left\{-\cancel{\omega^2} - \omega_1^2 - 2\omega\omega_1\cos\alpha - \omega\omega_1 + \cancel{\omega^2} + (\omega_1^2 + \omega\omega_1)(1+\cos\alpha) - (\omega_1^2 - \omega\omega_1)\cos\alpha\right\}$$

$$= \frac{i}{\lambda}\sin(\lambda t/2)\left[-\cancel{\omega_1^2} - 2\omega\omega_1\cos\alpha - \cancel{\omega\omega_1} + \cancel{\omega_1^2} + \cancel{\omega\omega_1} + \omega_1^2\cancel{\cos\alpha} + \omega\omega_1\cos\alpha - \omega_1^2\cancel{\cos\alpha} + \omega\omega_1\cos\alpha\right] = 0 \checkmark$$

(b) $\cos(\lambda t/2)\left\{-(\omega_1 - \cancel{\omega}) - \cancel{\omega} + \omega_1 2\cos^2\frac{\alpha}{2} - \omega_1\cos\alpha\right\} = \cos(\lambda t/2)\left[-\omega_1 + \omega_1(1+\cos\alpha) \quad -\omega_1\cos\alpha\right] = 0 \checkmark$

As for [10.33]:

$$\left[\cos(\lambda t/2) + i\frac{(\omega_1 + \omega\cos\alpha)}{\lambda}\sin(\lambda t/2)\right]e^{-i\omega t/2}\begin{pmatrix}\cos\alpha/2 \\ e^{i\omega t}\sin\alpha/2\end{pmatrix} + i\left[\frac{\omega}{\lambda}\sin\alpha\sin\left(\frac{\lambda t}{2}\right)\right]e^{-i\omega t/2}\begin{pmatrix}\sin\alpha/2 \\ -e^{i\omega t}\cos\alpha/2\end{pmatrix} = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}, \text{ with}$$

$$\alpha = \left\{\left[\cos\left(\frac{\lambda t}{2}\right) + \frac{i\omega_1}{\lambda}\sin\left(\frac{\lambda t}{2}\right)\right]\cos\frac{\alpha}{2} + \frac{i\omega}{\lambda}\Big[\underbrace{\cos\alpha\cos\frac{\alpha}{2} + \sin\alpha\sin\frac{\alpha}{2}}_{\cos(\alpha - \frac{\alpha}{2}) = \cos\frac{\alpha}{2}}\Big]\sin\left(\frac{\lambda t}{2}\right)\right\}e^{-i\omega t/2}$$

$$= \left[\cos\left(\frac{\lambda t}{2}\right) + \frac{i(\omega_1+\omega)}{\lambda}\sin\left(\frac{\lambda t}{2}\right)\right]\cos\frac{\alpha}{2}e^{-i\omega t/2} \quad \text{(confirming the top entry).}$$

$$\beta = \left\{\left[\cos(\lambda t/2) + \frac{i\omega_1}{\lambda}\sin(\lambda t/2)\right]\sin\frac{\alpha}{2} + \frac{i\omega}{\lambda}\Big[\underbrace{\cos\alpha\sin\frac{\alpha}{2} - \sin\alpha\cos\frac{\alpha}{2}}_{\sin(\frac{\alpha}{2} - \alpha) = -\sin\frac{\alpha}{2}}\Big]\sin\frac{\lambda t}{2}\right\}e^{i\omega t/2}$$

$$= \left[\cos(\lambda t/2) + \frac{i(\omega_1-\omega)}{\lambda}\sin(\lambda t/2)\right]\sin\frac{\alpha}{2}e^{i\omega t/2} \quad \text{(confirming the bottom term).}$$

$$|c_+|^2 + |c_-|^2 = \cos^2(\lambda t/2) + \frac{(\omega_1 + \omega\cos\alpha)^2}{\lambda^2}\sin^2\left(\frac{\lambda t}{2}\right) + \frac{\omega^2}{\lambda^2}\sin^2\alpha\sin^2\left(\frac{\lambda t}{2}\right)$$

$$= \cos^2(\lambda t/2) + \frac{1}{\lambda^2}\Big(\underbrace{\omega_1^2 + 2\omega\omega_1\cos\alpha + \omega^2\cos^2\alpha + \omega^2\sin^2\alpha}_{\omega^2 + \omega_1^2 + 2\omega\omega_1\cos\alpha = \lambda^2}\Big)\sin^2(\lambda t/2) = \cos^2(\lambda t/2) + \sin^2(\lambda t/2) = 1 \checkmark$$

Problem 10.4 (a) $\psi_n(x) = \sqrt{\frac{2}{W}}\sin\left(\frac{n\pi}{W}x\right)$. In this case $R = W$.

$$\frac{\partial\psi_n}{\partial R} = \sqrt{2}\left(-\frac{1}{2}\frac{1}{W^{3/2}}\right)\sin\left(\frac{n\pi}{W}x\right) + \sqrt{\frac{2}{W}}\left(-\frac{n\pi x}{W^2}\right)\cos\left(\frac{n\pi}{W}x\right); \quad \left\langle\psi_n\middle|\frac{\partial\psi_n}{\partial R}\right\rangle = \int_0^W \psi_n\frac{\partial\psi_n}{\partial R}dx$$

$$= -\frac{1}{W^2}\int_0^W \sin^2\left(\frac{n\pi}{W}x\right)dx - \frac{2n\pi}{W^3}\int_0^W x\underbrace{\sin\left(\frac{n\pi}{W}x\right)\cos\left(\frac{n\pi}{W}x\right)}_{\frac{1}{2}\sin\left(\frac{2n\pi}{W}x\right)}dx = -\frac{1}{W^2}\left(\frac{W}{2}\right) - \frac{n\pi}{W^3}\int_0^W x\sin\left(\frac{2n\pi}{W}x\right)dx$$

$$= -\frac{1}{2W} - \frac{n\pi}{W^3}\left[\left(\frac{W}{2n\pi}\right)^2\sin\left(\frac{2n\pi}{W}x\right) - \frac{Wx}{2n\pi}\cos\left(2n\pi x/W\right)\right]\Big|_0^W = -\frac{1}{2W} - \frac{n\pi}{W^3}\left(-\frac{W^2}{2n\pi}\cos(2n\pi)\right) = -\frac{1}{2W} + \frac{1}{2W} = 0.$$

$\therefore$ [10.46] $\Rightarrow$ $\boxed{\gamma_n(t)=0}$. (An example of the comment following [10.51]: if the eigenfunctions are real, the geometric phase vanishes.)

(b) $[10.41] \Rightarrow \theta_n(t) = -\frac{1}{\hbar}\int_0^t \frac{n^2\pi^2\hbar^2}{2mw^2}\,dt' = -\frac{n^2\pi^2\hbar}{2m}\int \frac{1}{w^2}\frac{dt'}{dw}\,dw$; $\theta_n = -\frac{n^2\pi^2\hbar}{2mv}\int_{w_1}^{w_2}\frac{1}{w^2}dw = \frac{n^2\pi^2\hbar}{2mv}\left(\frac{1}{w}\right)\Big|_{w_1}^{w_2}$

$$\boxed{\theta_n = \frac{n^2\pi^2\hbar}{2mv}\left(\frac{1}{w_2}-\frac{1}{w_1}\right)}$$

(c) $\boxed{\text{Zero}}$.

Problem 10.5 $\quad \psi = \frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2}$. Here $R=\alpha$, so

$$\frac{\partial\psi}{\partial R} = \frac{\sqrt{m}}{\hbar}\left(\frac{1}{2}\frac{1}{\sqrt{\alpha}}\right)e^{-m\alpha|x|/\hbar^2} + \frac{\sqrt{m\alpha}}{\hbar}\left(-\frac{m|x|}{\hbar^2}\right)e^{-m\alpha|x|/\hbar^2}.$$

$$\psi\frac{\partial\psi}{\partial R} = \frac{\sqrt{m\alpha}}{\hbar}\left[\frac{1}{2\hbar}\sqrt{\frac{m}{\alpha}} - \frac{m\sqrt{m\alpha}}{\hbar^3}|x|\right]e^{-2m\alpha|x|/\hbar^2} = \left(\frac{m}{2\hbar^2} - \frac{m^2\alpha}{\hbar^4}|x|\right)e^{-2m\alpha|x|/\hbar^2}.$$

$$\langle\psi|\frac{\partial\psi}{\partial R}\rangle = 2\left\{\frac{m}{2\hbar^2}\int_0^\infty e^{-2m\alpha x/\hbar^2}dx - \frac{m^2\alpha}{\hbar^4}\int_0^\infty x\,e^{-2m\alpha x/\hbar^2}dx\right\} = \frac{m}{\hbar^2}\left(\frac{\hbar^2}{2m\alpha}\right) - \frac{2m^2\alpha}{\hbar^4}\left(\frac{\hbar^2}{2m\alpha}\right)^2$$

$= \frac{1}{2\alpha} - \frac{1}{2\alpha} = 0$. So (eq. [10.46]) $\boxed{\gamma(t)=0}$.

$E = -\frac{m\alpha^2}{2\hbar^2}$, so $\quad \theta(t) = -\frac{1}{\hbar}\int_0^T\left(-\frac{m\alpha^2}{2\hbar^2}\right)dt' = \frac{m}{2\hbar^3}\int_{\alpha_1}^{\alpha_2}\alpha^2\frac{dt'}{d\alpha}d\alpha = \frac{m}{2\hbar^3 c}\int_{\alpha_1}^{\alpha_2}\alpha^2 d\alpha = \boxed{\frac{m}{6\hbar^3 c}(\alpha_2^3-\alpha_1^3)}$.

Problem 10.6 $\quad \psi_n' = e^{i\phi_n(\vec{R})}\psi_n$, where ψ_n is real. Then $\vec{\nabla}_R\psi_n' = e^{i\phi_n}\vec{\nabla}_R\psi_n + i(\vec{\nabla}_R\phi_n)e^{i\phi_n}\psi_n$.

$\therefore \langle\psi_n'|\vec{\nabla}_R\psi_n'\rangle = e^{-i\phi_n}e^{i\phi_n}\langle\psi_n|\vec{\nabla}_R\psi_n\rangle + ie^{-i\phi_n}(\vec{\nabla}_R\phi_n)e^{i\phi_n}\langle\psi_n|\psi_n\rangle$. But $\langle\psi_n|\psi_n\rangle=1$, and

$\langle\psi_n|\vec{\nabla}_R\psi_n\rangle = 0$ (p. 338), so $\quad \langle\psi_n'|\vec{\nabla}_R\psi_n'\rangle = i\,\vec{\nabla}_R\phi_n$. $\quad\therefore$ [10.48] $\Rightarrow$

$\gamma_n'(t) = i\int_{R_i}^{R_f} i\,\vec{\nabla}_R(\phi_n)\cdot d\vec{R} = -[\phi_n(\vec{R}_f)-\phi_n(\vec{R}_i)]$, and [10.40]:

$$\Psi_n'(x,t) = \psi_n'(x,t)e^{-\frac{i}{\hbar}\int_0^t E_n(t')dt'}e^{-i[\phi_n(\vec{R}_f)-\phi_n(\vec{R}_i)]}$$

— the wave function picks up a (trivial, time-independent) phase factor — whose only function is precisely to kill the phase factor we put in "by hand": $\Psi_n'(x,t) = \left[\psi_n(x,t)e^{-\frac{i}{\hbar}\int_0^t E_n(t')dt'}\right]e^{i\phi_n(\vec{R}_i)}$

$$= \Psi_n(x,t)\,e^{i\phi_n(\vec{R}_i)}.$$

In particular, for a closed loop $\phi_n(\vec{R}_f)=\phi_n(\vec{R}_i)$, so $\gamma_n'(T)=0$.

Problem 10.7 (a) $[10.27] \Rightarrow \frac{\partial\chi_+}{\partial t} = \begin{pmatrix}0\\ i\omega e^{i\omega t}\sin\frac{\alpha}{2}\end{pmatrix}$; $\langle\chi_+|\frac{\partial\chi_+}{\partial t}\rangle = \left(\cos\frac{\alpha}{2}\quad e^{-i\omega t}\sin\frac{\alpha}{2}\right)\begin{pmatrix}0\\ i\omega e^{i\omega t}\sin\frac{\alpha}{2}\end{pmatrix} = i\omega\sin^2\frac{\alpha}{2}$.

$[10.44] \Rightarrow \frac{d\gamma_+}{dt} = -\omega\sin^2\frac{\alpha}{2} \Rightarrow \gamma_+(t) = -\omega t\sin^2\frac{\alpha}{2} + \text{constant}$. $\gamma_+(0)=0 \Rightarrow \text{constant}=0 \Rightarrow$

$\gamma_+(t) = -\omega t\sin^2\frac{\alpha}{2} = -\frac{1}{2}\omega t(1-\cos\alpha)$, in agreement with [10.64].

(b) $\frac{\partial\chi_+}{\partial t} + i\chi_+\frac{d\gamma_+}{dt} = \begin{pmatrix} 0 \\ i\omega e^{i\omega t}\sin(\alpha/2)\end{pmatrix} + i\begin{pmatrix}\cos(\alpha/2) \\ e^{i\omega t}\sin(\alpha/2)\end{pmatrix}\left(-\omega\sin^2(\alpha/2)\right) = \begin{pmatrix} -i\omega\sin^2(\alpha/2)\cos(\alpha/2) \\ i\omega e^{i\omega t}\sin(\alpha/2)\underbrace{[1-\sin^2(\alpha/2)]}_{\cos^2(\alpha/2)}\end{pmatrix}$

$= -i\omega\sin(\alpha/2)\cos(\alpha/2)\begin{pmatrix}\sin(\alpha/2) \\ -e^{i\omega t}\cos(\alpha/2)\end{pmatrix} = -\frac{i\omega}{2}\sin\alpha\,\chi_- \neq 0$, so [10.43] does not hold.

Comparing [10.62] and [10.54], and noting that $\epsilon = \omega/\omega_1$: $\boxed{c_- = i\sin\alpha\sin\left(\frac{\omega_1 t}{2}\right)e^{-i\omega t/2}}$.

The last term in [10.56] is

$$-e^{-i\theta_+}e^{-i\gamma_+}\frac{\omega}{\omega_1}c_-\frac{\partial\chi_-}{\partial t} = -e^{-i\theta_+}e^{-i\gamma_+}\frac{\omega}{\omega_1}\left(i\sin\alpha\sin\left(\frac{\omega_1 t}{2}\right)e^{-i\omega t/2}\right)\begin{pmatrix}0 \\ -i\omega e^{i\omega t}\cos(\alpha/2)\end{pmatrix}$$

$$= -\frac{\omega^2}{\omega_1}e^{-i\theta_+}e^{-i\gamma_+}e^{i\omega t/2}\sin(\omega_1 t/2)\sin\alpha\cos(\alpha/2)\begin{pmatrix}0\\1\end{pmatrix},$$ which is indeed second-order in ω.

The right side of [10.56] reads (inserting [10.63] for θ_+ and [10.29] for E_-):

$$-e^{-i\theta_+}\frac{\omega}{\omega_1}\left[\frac{i}{\hbar}c_-E_- + \frac{dc_-}{dt}\right]\chi_- = -e^{-i\omega_1 t/2}\frac{\omega}{\omega_1}\left[\frac{i}{\hbar}\left(i\sin\alpha\sin\left(\frac{\omega_1 t}{2}\right)e^{-i\omega t/2}\right)\frac{\hbar\omega_1}{2} + i\sin\alpha\left(\frac{\omega_1}{2}\right)\cos\left(\frac{\omega_1 t}{2}\right)e^{-i\omega t/2}\right.$$

$$\left. + i\sin\alpha\sin(\omega_1 t/2)(-i\omega/2)e^{-i\omega t/2}\right]\chi_- = -\frac{\omega}{2}e^{-i\omega_1 t/2}\sin\alpha\, e^{-i\omega t/2}\left[\left(\frac{\omega}{\omega_1}-1\right)\sin(\omega_1 t/2) + i\cos(\omega_1 t/2)\right]\chi_-.$$

But we want only terms first order in ω, so $\frac{\omega}{\omega_1} \ll 1$ drops out, and $e^{-i\omega t/2} \approx 1$ (see comment preceding [10.56]), so right side of [10.56] is

$$\cong -\frac{\omega}{2}e^{-i\omega_1 t/2}\sin\alpha\; i e^{i\omega_1 t/2}\chi_- = -\frac{i\omega}{2}\sin\alpha\,\chi_-,$$ in agreement with the left side (calculated in the top two lines of this page).

PROBLEM 10.8 $H = \frac{e}{m}\vec{B}\cdot\vec{S}$. $\vec{B} = B_0[\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}]$; Spin matrices from Problem 4.32.

$$H = \frac{eB_0}{m}\frac{\hbar}{\sqrt{2}}\left\{\sin\theta\cos\phi\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} + \sin\theta\sin\phi\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix} + \cos\theta\begin{pmatrix}\sqrt{2}&0&0\\0&0&0\\0&0&-\sqrt{2}\end{pmatrix}\right\}.$$

$$= \frac{eB_0\hbar}{\sqrt{2}\,m}\begin{pmatrix}\sqrt{2}\cos\theta & e^{-i\phi}\sin\theta & 0 \\ e^{i\phi}\sin\theta & 0 & e^{-i\phi}\sin\theta \\ 0 & e^{i\phi}\sin\theta & -\sqrt{2}\cos\theta\end{pmatrix}.$$ We need the "spin up" eigenvector: $H\chi_+ = \frac{eB_0}{m}\hbar\chi_+$.

$$\begin{pmatrix}\sqrt{2}\cos\theta & e^{-i\phi}\sin\theta & 0 \\ e^{i\phi}\sin\theta & 0 & e^{-i\phi}\sin\theta \\ 0 & e^{i\phi}\sin\theta & -\sqrt{2}\cos\theta\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \sqrt{2}\begin{pmatrix}a\\b\\c\end{pmatrix} \Rightarrow \begin{cases}\text{(i) } \sqrt{2}\cos\theta\, a + e^{-i\phi}\sin\theta\, b = \sqrt{2}\,a. \\ \text{(ii) } e^{i\phi}\sin\theta\, a + e^{-i\phi}\sin\theta\; c = \sqrt{2}\,b. \\ \text{(iii) } e^{i\phi}\sin\theta\, b - \sqrt{2}\cos\theta\, c = \sqrt{2}\,c.\end{cases}$$

(i) $\Rightarrow b = \sqrt{2}e^{i\phi}\left(\frac{1-\cos\theta}{\sin\theta}\right)a = \sqrt{2}e^{i\phi}\tan(\theta/2)a$; (iii) $\Rightarrow b = \sqrt{2}e^{-i\phi}\left(\frac{1+\cos\theta}{\sin\theta}\right)c = \sqrt{2}e^{-i\phi}\cot(\theta/2)\,c$

$\therefore c = e^{2i\phi}\tan^2(\theta/2)a$; ((ii) is redundant). <u>Normalize</u>: $|a|^2 + 2\tan^2\frac{\theta}{2}|a|^2 + \tan^4\frac{\theta}{2}|a|^2 = 1 \Rightarrow$

$|a|^2(1+\tan^2\theta/2)^2 = |a|^2\left(\frac{1}{\cos\theta/2}\right)^4 = 1 \Rightarrow |a|^2 = \cos^4(\theta/2)$. Pick $a = e^{-i\phi}\cos^2\theta/2$. Then $b = \sqrt{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

and $c = e^{i\phi}\sin^2\theta/2$. So
$$\boxed{\chi_+ = \begin{pmatrix} e^{-i\phi}\cos^2(\theta/2) \\ \sqrt{2}\ \sin(\theta/2)\cos(\theta/2) \\ e^{i\phi}\ \sin^2(\theta/2)\end{pmatrix}}.$$
This is the spin-1 analog to [10.66].

$$\vec{\nabla}\chi_+ = \frac{\partial\chi_+}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\chi_+}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\chi_+}{\partial\phi}\hat{\phi}$$
$$= \frac{1}{r}\begin{pmatrix} -e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ \sqrt{2}\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)/2 \\ e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\end{pmatrix}\hat{\theta} + \frac{1}{r\sin\theta}\begin{pmatrix} -ie^{-i\phi}\cos^2\frac{\theta}{2} \\ 0 \\ ie^{i\phi}\sin^2\frac{\theta}{2}\end{pmatrix}\hat{\phi}.$$

$$\langle\chi_+|\vec{\nabla}\chi_+\rangle = \frac{1}{r}\left[-\cos^2\frac{\theta}{2}\left(\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right) + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) + \sin^2\frac{\theta}{2}\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\right]\hat{\theta}$$
$$+ \frac{1}{r\sin\theta}\left[\cos^2\frac{\theta}{2}\left(-i\cos^2\frac{\theta}{2}\right) + \sin^2\frac{\theta}{2}\left(i\sin^2\frac{\theta}{2}\right)\right]\hat{\phi} = \frac{i}{r\sin\theta}\left[\sin^4\frac{\theta}{2} - \cos^4\frac{\theta}{2}\right]\hat{\phi}$$
$$= \frac{i}{r\sin\theta}\left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right)\left(\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\right)\hat{\phi} = \frac{i}{r\sin\theta}(1)(-\cos\theta)\hat{\phi} = -\frac{i}{r}\cot\theta\,\hat{\phi}.$$

$$\vec{\nabla}\times\langle\chi_+|\vec{\nabla}\chi_+\rangle = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left(-\frac{i}{r}\cot\theta\right)\right)\hat{r} = \frac{-i}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\cos\theta)\hat{r} = \frac{i\sin\theta}{r^2\sin\theta}\hat{r} = \frac{i}{r^2}\hat{r}.$$

$$\therefore [10.60] \Rightarrow \gamma_+(T) = i\int\frac{i}{r^2}r^2\,d\Omega = \boxed{-\Omega}.$$

PROBLEM 10.9 (a) Giving H a test function f to act upon:

$$Hf = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\vec{A}\right)\cdot\left(\frac{\hbar}{i}\nabla f - q\vec{A}f\right) + q\varphi f = \frac{1}{2m}\Big[-\hbar^2\,\vec{\nabla}\cdot(\nabla f) - \frac{q\hbar}{i}\underbrace{\nabla\cdot(\vec{A}f)}_{(\nabla\cdot\vec{A})f + \vec{A}\cdot(\nabla f)} - \frac{q\hbar}{i}\vec{A}\cdot(\nabla f) + q^2\vec{A}\cdot\vec{A}f\Big] + q\varphi f$$

But $\nabla\cdot\vec{A} = 0$ and $\varphi = 0$ (see comments on eq. [10.75]), so
$$Hf = \frac{1}{2m}\left[-\hbar^2\nabla^2 f + 2iq\hbar\vec{A}\cdot\nabla f + q^2A^2 f\right], \text{ or } H = \frac{1}{2m}\left[-\hbar^2\nabla^2 + q^2A^2 + 2iq\hbar\vec{A}\cdot\vec{\nabla}\right]. \quad \text{QED.}$$

(b) Apply $\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}\right)\cdot$ to both sides of [10.87]:
$$\left(\frac{\hbar}{i}\nabla - q\vec{A}\right)^2\Psi = \left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}\right)\cdot\left(\frac{\hbar}{i}e^{ig}\vec{\nabla}\Psi'\right) = -\hbar^2\underbrace{\vec{\nabla}\cdot\left(e^{ig}\vec{\nabla}\Psi'\right)}_{ie^{ig}(\nabla g)\cdot(\nabla\Psi') + e^{ig}\nabla\cdot(\nabla\Psi')} - \frac{q\hbar}{i}e^{ig}\vec{A}\cdot\nabla\Psi'.$$

But $\vec{\nabla}g = \frac{q}{\hbar}\vec{A}$, so right side is
$$-i\hbar^2\frac{q}{\hbar}e^{ig}\vec{A}\cdot\nabla\Psi' - \hbar^2e^{ig}\nabla^2\Psi' + iq\hbar e^{ig}\vec{A}\cdot\nabla\Psi' = -\hbar^2e^{ig}\nabla^2\Psi'. \quad \text{QED}$$

PROBLEM 10.10 (a) Check the answer given: $x_c = \omega\int_0^t f(t')\sin[\omega(t-t')]\,dt' \Rightarrow x_c(0) = 0$ ✓

$$\dot{x}_c = \omega f(t)\sin[\omega(t-t)] + \omega^2\int_0^t f(t')\cos[\omega(t-t')]\,dt' = \omega^2\int_0^t f(t')\cos[\omega(t-t')]\,dt' \Rightarrow \dot{x}_c(0) = 0 \checkmark$$

$$\ddot{x}_c = \omega^2 f(t)\cos[\omega(t-t)] - \omega^3\int_0^t f(t')\sin[\omega(t-t')]\,dt' = \omega^2 f(t) - \omega^2 x_c.$$

Now, the classical equation of motion is: $m\frac{d^2x}{dt^2} = -m\omega^2 x + m\omega^2 f$. For the proposed solution $m\frac{d^2x_c}{dt^2} = m\omega^2 f - m\omega^2 x_c$, so it does satisfy the equation of motion, with appropriate boundary conditions.

(b) Let $z \equiv x - x_c$ (so $\psi_n(x-x_c) = \psi_n(z)$, and z depends on t as well as x).

$$\frac{\partial\Psi}{\partial t} = \frac{d\psi_n}{dz}(-\dot{x}_c)e^{i\{\}} + \psi_n e^{i\{\}}\frac{i}{\hbar}\Big[-(n+\tfrac{1}{2})\hbar\omega + m\underbrace{\ddot{x}_c}_{\omega^2 f-\omega^2 x_c}\big(x-\tfrac{x_c}{2}\big) - \tfrac{m}{2}\dot{x}_c^2 + \tfrac{m\omega^2}{2}f x_c\Big]$$

$$[\;] = -(n+\tfrac{1}{2})\hbar\omega + \frac{m\omega^2}{2}\left(2x(f-x_c) + x_c^2 - \dot{x}_c^2/\omega^2\right).$$

$$\therefore \frac{\partial\Psi}{\partial t} = -\dot{x}_c\frac{d\psi_n}{dz}e^{i\{\}} + i\Psi\left[-(n+\tfrac{1}{2})\omega + \frac{m\omega^2}{2\hbar}\left(2x(f-x_c)+x_c^2-\frac{1}{\omega^2}\dot{x}_c^2\right)\right].$$

$$\frac{\partial\Psi}{\partial x} = \frac{d\psi_n}{dz}e^{i\{\}} + \psi_n e^{i\{\}}\frac{i}{\hbar}[m\dot{x}_c];\quad \frac{\partial^2\Psi}{\partial x^2} = \frac{d^2\psi_n}{dz^2}e^{i\{\}} + 2\frac{d\psi_n}{dz}e^{i\{\}}\frac{i}{\hbar}(m\dot{x}_c) - \left(\frac{m\dot{x}_c}{\hbar}\right)^2\psi_n e^{i\{\}}$$

$$\therefore H\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \tfrac{1}{2}m\omega^2x^2\Psi - m\omega^2 f x\Psi = -\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dz^2}e^{i\{\}} - \frac{\hbar^2}{2m}2\frac{d\psi_n}{dz}e^{i\{\}}\frac{im\dot{x}_c}{\hbar} + \frac{\hbar^2}{2m}\left(\frac{m\dot{x}_c}{\hbar}\right)^2\Psi + \tfrac{1}{2}m\omega^2x^2\Psi$$

$-m\omega^2 f x\Psi$. But $-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dz^2} + \frac{1}{2}m\omega^2 z^2\psi_n = (n+\frac{1}{2})\hbar\omega\psi_n$, so

$$H\Psi = \cancel{(n+\tfrac{1}{2})\hbar\omega}\,\Psi - \tfrac{1}{2}m\omega^2z^2\Psi - \cancel{i\hbar\dot{x}_c\frac{d\psi_n}{dz}}e^{i\{\}} + \tfrac{m}{2}\dot{x}_c^2\Psi + \tfrac{1}{2}m\omega^2x^2\Psi - m\omega^2 f x\Psi$$

$$\stackrel{?}{=} i\hbar\frac{\partial\Psi}{\partial t} = -i\hbar\cancel{\dot{x}_c\frac{d\psi_n}{dz}}e^{i\{\}} - \hbar\Psi\left[-\cancel{(n+\tfrac{1}{2})}\omega + \frac{m\omega^2}{2\hbar}\left(2xf - 2xx_c + x_c^2 - \frac{1}{\omega^2}\dot{x}_c^2\right)\right]$$

$$-\tfrac{1}{2}m\omega^2z^2 + \cancel{\tfrac{m}{2}\dot{x}_c^2} + \tfrac{1}{2}m\omega^2x^2 - \cancel{m\omega^2 f x} \stackrel{?}{=} -\frac{m\omega^2}{2}\left(\cancel{2xf} - 2xx_c + x_c^2 - \cancel{\tfrac{1}{\omega^2}\dot{x}_c^2}\right)$$

$$z^2 - x^2 \stackrel{?}{=} -2xx_c + x_c^2\;;\quad z^2 \stackrel{?}{=} (x^2 - 2xx_c + x_c^2) = (x-x_c)^2 \checkmark$$

(c) $[10.99] \Rightarrow H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2(x^2 - 2xf + f^2) - \frac{1}{2}m\omega^2 f^2$. Shift origin: $u \equiv x - f$.

$$H = \underbrace{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial u^2} + \tfrac{1}{2}m\omega^2u^2}_{\text{simple harmonic oscillator in variable } u} - \underbrace{\tfrac{1}{2}m\omega^2 f^2}_{\text{constant (with respect to position)}}.$$ So eigenfunctions are $\psi_n(u) = \psi_n(x-f)$, and

eigenvalues are harmonic oscillator ones $(n+\frac{1}{2})\hbar\omega$, less the added constant: $E_n = (n+\frac{1}{2})\hbar\omega - \frac{1}{2}m\omega^2 f^2$.

(d) Note that $\sin[\omega(t-t')] = \frac{1}{\omega}\frac{d}{dt'}\cos[\omega(t-t')]$, so $x_c(t) = \int_0^t f(t')\frac{d}{dt'}\cos[\omega(t-t')]\,dt'$, or

$$x_c(t) = f(t')\cos[\omega(t-t')]\Big|_0^t - \int_0^t\left(\frac{df}{dt'}\right)\cos[\omega(t-t')]\,dt' = f(t) - \int_0^t\left(\frac{df}{dt'}\right)\cos[\omega(t-t')]\,dt' \quad (\text{since } f(0)=0).$$

Now, for adiabatic process we want df/dt very small – specifically, $\boxed{\frac{df}{dt'} \ll \omega f(t)}$ $(0 < t' \le t)$. Then the integral is negligible compared to $f(t)$, and we have $\boxed{x_c(t) \cong f(t)}$. (Physically, this says that if you pull on the spring very gently, no fancy oscillations will occur – the mass just moves along as though attached to a string of fixed length.)

(e) Put $x_c \cong f$ into [10.101], using [10.102]:

$$\Psi(x,t) = \psi_n(x,t)\, e^{\frac{i}{\hbar}\left[-(n+\frac{1}{2})\hbar\omega t + m\dot{f}(x - f/2) + \frac{m\omega^2}{2}\int_0^t f^2(t')dt'\right]}$$

The dynamic phase [10.41] is $\theta_n(t) = -\frac{1}{\hbar}\int_0^t E_n(t')dt' = -(n+\frac{1}{2})\hbar\omega t + \frac{m\omega^2}{2\hbar}\int_0^t f^2(t')dt'$, so

$$\Psi(x,t) = \psi_n(x,t)\, e^{i\theta_n(t)}\, e^{i\gamma_n(t)}$$, confirming [10.103], with the geometric phase given (ostensibly) by

$$\gamma_n(t) = \frac{m}{\hbar}\dot{f}(x - f/2).$$

But the eigen-functions here are real, and hence (see p. 338) the geometric phase should be zero. The point is that (in the adiabatic approximation) $\dot{f}$ is extremely small (see above), and hence in this limit $\frac{m}{\hbar}\dot{f}(x - f/2) \cong 0$ (at least, in the only region of x where $\psi_n(x,t)$ is nonzero).

PROBLEM 10.11 (a) $i\hbar \frac{\partial\Psi}{\partial t} = H\Psi \Rightarrow$

$$i\hbar \sum_n \left[\dot{c}_n \psi_n e^{i\theta_n} + c_n \frac{\partial\psi_n}{\partial t} e^{i\theta_n} + \cancel{c_n\psi_n i \frac{d\theta_n}{dt}} e^{i\theta_n}\right] = \sum_n c_n [H\psi_n] e^{i\theta_n} = \cancel{\sum_n c_n E_n(t)} e^{i\theta_n}$$

$$\frac{d\theta_n}{dt} \to -\frac{1}{\hbar}E_n(t)$$

The indicated terms cancel, leaving: $\sum_n \left[\dot{c}_n \psi_n e^{i\theta_n} + c_n \frac{\partial\psi_n}{\partial t} e^{i\theta_n}\right] = 0$. Take the inner product with ψ_m:

$$\sum_n \left\{ \dot{c}_n \langle\psi_m|\psi_n\rangle e^{i\theta_n} + c_n \langle\psi_m|\frac{\partial\psi_n}{\partial t}\rangle e^{i\theta_n}\right\} = 0.$$ Assuming the (instantaneous) eigenfunctions have been orthonormalized, $\langle\psi_m|\psi_n\rangle = \delta_{mn}$, so

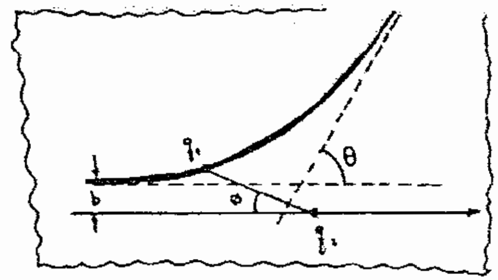
$$\dot{c}_m e^{i\theta_m} + \sum_n c_n \langle\psi_m|\frac{\partial\psi_n}{\partial t}\rangle e^{i\theta_n} = 0, \text{ or } \quad \dot{c}_m = -\sum_n c_n \langle\psi_m|\frac{\partial\psi_n}{\partial t}\rangle e^{i(\theta_n - \theta_m)}.$$

(b) $\dot{c}_m = -\sum_n \langle\psi_m|\frac{\partial\psi_n}{\partial t}\rangle \delta_{nN} e^{i\gamma_N} e^{i(\theta_n - \theta_m)} = -\langle\psi_m|\frac{\partial\psi_N}{\partial t}\rangle e^{i\gamma_N} e^{i(\theta_N - \theta_m)}$

$$\therefore \boxed{c_m(t) = c_m(0) - \int_0^t \langle\psi_m|\frac{\partial\psi_N}{\partial t'}\rangle e^{i\gamma_N} e^{i(\theta_N - \theta_m)} dt'}$$

(c) $\psi_n(x,t) \Rightarrow \psi_n(x-f) = \psi_n(z)$, where $z \equiv x - f$, and $\psi_n(z)$ is the n^{th} state of the ordinary harmonic oscillator; $\frac{\partial\psi_N}{\partial t} \Rightarrow \frac{\partial\psi_N}{\partial z}\frac{\partial z}{\partial t} = -\dot{f}\frac{\partial\psi_N}{\partial z}$. But $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial z}$, so $\langle\psi_m|\frac{\partial\psi_N}{\partial t}\rangle \Rightarrow -\frac{i}{\hbar}\dot{f}\langle m|p|N\rangle$, where

CHAPTER 11



PROBLEM 11.1 (a) Conservation of energy: $E=\frac{1}{2}m(\dot{r}^2+r^2\dot{\phi}^2)+V(r)$,

where $V(r)=\frac{q_1q_2}{4\pi\epsilon_0}\frac{1}{r}$; conservation of angular momentum:

$J=mr^2\dot{\phi}$. So $\dot{\phi}=\frac{J}{mr^2}$.

$\therefore \dot{r}^2+\frac{J^2}{m^2r^2}=\frac{2}{m}(E-V)$. We want r as a function of ϕ (not t). Also, let $u\equiv 1/r$. Then

$\dot{r}=\frac{dr}{dt}=\frac{dr}{du}\frac{du}{d\phi}\frac{d\phi}{dt}=\left(-\frac{1}{u^2}\right)\frac{du}{d\phi}\frac{J}{m}u^2=-\frac{J}{m}\frac{du}{d\phi}$. Then: $\left(-\frac{J}{m}\frac{du}{d\phi}\right)^2+\frac{J^2}{m^2}u^2=\frac{2}{m}(E-V)$, or

$\left(\frac{du}{d\phi}\right)^2=\frac{2m}{J^2}(E-V)-u^2$; $\frac{du}{d\phi}=\sqrt{\frac{2m}{J^2}(E-V)-u^2}$; $d\phi=\frac{du}{\sqrt{\frac{2m}{J^2}(E-V)-u^2}}=\frac{du}{\sqrt{I(u)}}$, where

$I(u)\equiv\frac{2m}{J^2}(E-V)-u^2$. Now, the particle q_1 starts out at $r=\infty$ $(u=0)$, $\phi=0$, and the point of closest approach is $r_{min}(u_{max})$, Φ: $\Phi=\int_0^{u_{max}}\frac{du}{\sqrt{I}}$. It now swings through an equal angle Φ on the way out, so $\Phi+\Phi+\theta=\pi$, or $\theta=\pi-2\Phi$. $\theta=\pi-2\int_0^{u_{max}}\frac{du}{\sqrt{I(u)}}$.

So far this is general — now we put in the specific potential:

$I(u)=\frac{2mE}{J^2}-\frac{2m}{J^2}\frac{q_1q_2}{4\pi\epsilon_0}u-u^2=(u_2-u)(u-u_1)$, where u_1 and u_2 are the two roots.

(Since $\frac{du}{d\phi}=\sqrt{I(u)}$, u_{max} = one of the roots — setting $u_2>u_1$, $u_{max}=u_2$.) $\therefore$

$$\theta=\pi-2\int_0^{u_2}\frac{du}{\sqrt{(u_2-u)(u-u_1)}}=\pi+2\sin^{-1}\left(\frac{-2u+u_1+u_2}{u_2-u_1}\right)\Bigg|_0^{u_2}=\pi+2\left(\sin^{-1}(-1)-\sin^{-1}\left(\frac{u_1+u_2}{u_2-u_1}\right)\right)$$

$\theta=\pi+2\left(-\frac{\pi}{2}-\sin^{-1}\left(\frac{u_1+u_2}{u_2-u_1}\right)\right)=-2\sin^{-1}\left(\frac{u_1+u_2}{u_2-u_1}\right)$. Now $J=mvb$, $E=\frac{1}{2}mv^2$, where v is the incoming velocity, so $J^2=m^2b^2\frac{2E}{m}=2mb^2E$, and hence $\frac{2m}{J^2}=\frac{1}{b^2E}$. So

$I(u)=\frac{1}{b^2}-\frac{1}{b^2}\left(\frac{1}{E}\frac{q_1q_2}{4\pi\epsilon_0}\right)u-u^2$. Let $A\equiv\frac{q_1q_2}{4\pi\epsilon_0E}$, so $-I(u)=u^2+\frac{A}{b^2}u-\frac{1}{b^2}$. To get the roots: $u^2+\frac{A}{b^2}u-\frac{1}{b^2}=0\Rightarrow u=\frac{1}{2}\left\{-\frac{A}{b^2}\pm\sqrt{\frac{A^2}{b^4}+\frac{4}{b^2}}\right\}=\frac{A}{2b^2}\left[-1\pm\sqrt{1+(2b/A)^2}\right]$.

Thus $u_2=\frac{A}{2b^2}\left[-1+\sqrt{1+(2b/A)^2}\right]$, $u_1=\frac{A}{2b^2}\left[-1-\sqrt{1+(2b/A)^2}\right]$; $\frac{u_1+u_2}{u_2-u_1}=\frac{-1}{\sqrt{1+(2b/A)^2}}$.

$\therefore\theta=2\sin^{-1}\left(\frac{1}{\sqrt{1+(2b/A)^2}}\right)$, or $\frac{1}{\sqrt{1+(2b/A)^2}}=\sin(\theta/2)$; $1+\left(\frac{2b}{A}\right)^2=\frac{1}{\sin^2\theta/2}$; $\left(\frac{2b}{A}\right)^2=\frac{1-\sin^2\theta/2}{\sin^2\theta/2}=\frac{\cos^2\theta/2}{\sin^2\theta/2}$

$\frac{2b}{A}=\cot\theta/2$, or $\boxed{b=\frac{q_1q_2}{8\pi\epsilon_0E}\cot(\theta/2)}$.

(b) $D(\theta)=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$. Here $\frac{db}{d\theta}=\frac{q_1q_2}{8\pi\epsilon_0E}\left(-\frac{1}{2\sin^2\theta/2}\right)$, so $D=\frac{1}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\frac{q_1q_2}{8\pi\epsilon_0E}\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\frac{q_1q_2}{8\pi\epsilon_0E}\frac{1}{2\sin^2\theta/2}$,

or $\boxed{D(\theta)=\left[\frac{q_1q_2}{16\pi\epsilon_0E\sin^2\theta/2}\right]^2}$.

(from Problem 3.50): $\langle m|p|N\rangle = \sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{N}\,\delta_{m,N-1} + \sqrt{m}\,\delta_{N,m-1}\right)$. Thus:

$\langle \psi_m | \frac{\partial \psi_N}{\partial t}\rangle \Rightarrow -i\dot{f}\sqrt{\frac{m\omega}{2\hbar}}\left(\sqrt{N}\,\delta_{m,N-1} + \sqrt{m}\,\delta_{N,m-1}\right)$. Evidently transitions occur only to the immediately adjacent states $N \pm 1$, and:

$\underline{m = N+1}$: $C_{N+1} = -\int_0^t \left(-i\dot{f}\sqrt{\frac{m\omega}{2\hbar}}\sqrt{N+1}\right)e^{i\gamma_N}e^{i(\theta_N - \theta_{N+1})}\,dt'$. But $\gamma_N = 0$, because the eigenfunctions are real, and $\theta_n = -\frac{1}{\hbar}(n+\frac{1}{2})\hbar\omega t \Rightarrow \theta_N - \theta_{N+1} = \left[-(N+\frac{1}{2}) + (N+1+\frac{1}{2})\right]\omega t = \omega t$.

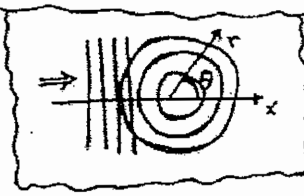
So $$\boxed{C_{N+1} = i\sqrt{\frac{m\omega}{2\hbar}}\sqrt{N+1}\int_0^t \dot{f}\,e^{i\omega t'}\,dt'}.$$

$\underline{m = N-1}$: $C_{N-1} = -\int_0^t \left(-i\dot{f}\sqrt{\frac{m\omega}{2\hbar}}\sqrt{N}\right)e^{i\gamma_N}e^{i(\theta_N - \theta_{N-1})}\,dt'$. $\theta_N - \theta_{N-1} = \left[-(N+\frac{1}{2}) + (N-1+\frac{1}{2})\right]\omega t = -\omega t$.

$$\boxed{C_{N-1} = i\sqrt{\frac{m\omega}{2\hbar}}\sqrt{N}\int_0^t \dot{f}\,e^{-i\omega t'}\,dt'}.$$

(c) $\sigma = \int D(\theta)\sin\theta\, d\theta\, d\phi = 2\pi \left(\frac{q_1 q_2}{8\pi\epsilon_0 E}\right)^2 \int_0^{\pi} \frac{\sin\theta}{\sin^4(\theta/2)}\, d\theta$. This integral does not converge, for near $\theta = 0$ (and again near π) we have $\sin\theta \approx \theta$, $\sin(\theta/2) \approx \theta/2$, so the integral goes like $16\int_0^{\epsilon} \frac{1}{\theta^3} d\theta = -\frac{8}{\theta^2}\Big|_0^{\epsilon} \to \infty$.

PROBLEM 11.2 Two dimensions: $\boxed{\psi(r,\theta) \approx A\left\{e^{ikx} + f(\theta)\frac{e^{ikr}}{\sqrt{r}}\right\}}$.



One dimension: $\boxed{\psi(x) \approx A\left\{e^{ikx} + f\, e^{-ikx}\right\}}$. [See [2.155] and [2.156].]

PROBLEM 11.3 Use the orthogonality of the Legendre polynomials [4.34]. Formally, multiply [11.34] by $P_{\ell'}(\cos\theta)\sin\theta\, d\theta$ and integrate: $\int_0^{\pi} P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta\, d\theta = \int_{-1}^{1} P_\ell(x)P_{\ell'}(x)\, dx = \frac{2}{2\ell+1}\delta_{\ell\ell'}$. This collapses the sum to a single term:

$$\left[i^{\ell'}(2\ell'+1)\, j_{\ell'}(ka) + \sqrt{\frac{2\ell'+1}{4\pi}}\, C_{\ell'}\, h^{(1)}_{\ell'}(ka)\right] = 0,$$ or, dropping the prime and solving for C_ℓ:

$$C_\ell = -\, i^\ell (2\ell+1)\, j_\ell(ka)\sqrt{\frac{4\pi}{2\ell+1}}\,\frac{1}{h^{(1)}_\ell(ka)} = -i^\ell\sqrt{4\pi(2\ell+1)}\,\frac{j_\ell(ka)}{h^{(1)}_\ell(ka)}.\quad \text{QED}$$

PROBLEM 11.4 $[11.31] \Rightarrow \psi \equiv A\left\{j_0(kr)P_0(\cos\theta) + \frac{C_0}{\sqrt{4\pi}}h^{(1)}_0(kr)P_0(\cos\theta)\right\} = A\left\{\frac{\sin(kr)}{kr} - i\frac{C_0}{\sqrt{4\pi}}\frac{e^{ikr}}{kr}\right\}$, $r > a$.

[11.18] (with n_ℓ eliminated because it blows up at the origin) $\Rightarrow \psi(r) \equiv b\, j_0(kr) = b\,\frac{\sin(kr)}{kr}$, $r < a$.

(The exact solution inside is $\sum b_\ell\, j_\ell(kr)P_\ell(\cos\theta)$.) The boundary conditions (which, incidentally, hold independently for each ℓ, as you can check by keeping the summation over ℓ and exploiting the orthogonality of the Legendre polynomials) are:

(1) ψ continuous at $r = a$.

(2) ψ' discontinuous at $r = a$ – in fact, integrating the radial equation across the delta function gives:

$$-\frac{\hbar^2}{2m}\int \frac{d^2u}{dr^2}dr + \int\left[\alpha\delta(r-a) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u\, dr = \int E\, u\, dr \Rightarrow$$

$$-\frac{\hbar^2}{2m}\Delta u' + \alpha u(a) = 0,\ \text{or}\ \ \Delta u' = \frac{2m\alpha}{\hbar^2}u(a).$$ Now $u = rR$, so $u' = R + rR'$, so

$$\Delta u' = \Delta R + a\Delta R' = a\Delta R' = \frac{2m\alpha}{\hbar^2}aR(a),\ \text{so}\ \ \underline{\underline{\Delta\psi' = \frac{2m\alpha}{\hbar^2}\psi(a)}} = \frac{\phi}{a}\psi(a).$$

So $(1) \Rightarrow A\left[\frac{\sin(ka)}{ka} - i\frac{C_0}{\sqrt{4\pi}}\frac{e^{ika}}{ka}\right] = b\,\frac{\sin(ka)}{ka}$, while $(2) \Rightarrow$

$$\frac{A}{ka}\left[k\cos(ka) - \frac{iC_0}{\sqrt{4\pi}}ik e^{ika}\right] - \frac{A}{ka^2}\cancel{\left[\sin(ka) - i\frac{C_0}{\sqrt{4\pi}}e^{ika}\right]} - \frac{b}{ka}k\cos(ka) + \cancel{\frac{b}{ka^2}\sin ka} = \frac{\phi}{a}b\,\frac{\sin(ka)}{ka}.$$

The indicated terms cancel (by (1)), leaving $A\left[\cos(ka) + \frac{C_0}{\sqrt{4\pi}}e^{ika}\right] = b\left[\cos(ka) + \frac{\phi}{ka}\sin(ka)\right]$

Using (1) to eliminate b: $A\left[\cos(ka) + \frac{C_0}{\sqrt{4\pi}}e^{ika}\right] = \left[\cot(ka) + \frac{\phi}{ka}\right]\left[\sin(ka) - \frac{iC_0}{\sqrt{4\pi}}e^{ika}\right]A$.

$$\cancel{\cos(ka)} + \frac{C_0}{\sqrt{4\pi}}e^{ika} = \cancel{\cos(ka)} + \frac{\phi}{ka}\sin(ka) - \frac{iC_0}{\sqrt{4\pi}}\cot(ka)e^{ika} - i\frac{C_0}{\sqrt{4\pi}}\frac{\phi}{ka}e^{ika}.$$

$$\frac{C_0}{\sqrt{4\pi}} e^{ika}\left[1 + i\cot(ka) + i\frac{\phi}{ka}\right] = \frac{\phi}{ka}\sin(ka).$$ But $ka \ll 1$, so $\sin(ka) \cong ka$, and $\cot(ka) = \frac{\cos(ka)}{\sin(ka)} \cong \frac{1}{ka}$.

$$\frac{C_0}{\sqrt{4\pi}}(1+ika)\left(1 + \frac{i}{ka}(1+\phi)\right) = \phi\ ; \quad \frac{C_0}{\sqrt{4\pi}}\left(1 + \frac{i}{ka}(1+\phi) + ika - 1 - \phi\right) \cong \frac{C_0}{\sqrt{4\pi}}\left(\frac{i}{ka}(1+\phi)\right) = \phi.$$

$\therefore \boxed{C_0 = -i\sqrt{4\pi}\, ka\, \frac{\phi}{1+\phi}}$. $f(\theta) = \frac{1}{k}(-i)\frac{1}{\sqrt{4\pi}} C_0$ [eq. 11.28], so $\boxed{f(\theta) = -\frac{a\phi}{1+\phi}}$.

$$D = |f|^2 = \boxed{\left(\frac{a\phi}{1+\phi}\right)^2}; \quad \boxed{\sigma = 4\pi a^2\left(\frac{\phi}{1+\phi}\right)^2}.$$

PROBLEM 11.5 $G = -\frac{e^{ikr}}{4\pi r} \Rightarrow \vec\nabla G = -\frac{1}{4\pi}\left(\frac{1}{r}\vec\nabla e^{ikr} + e^{ikr}\vec\nabla\frac{1}{r}\right) \Rightarrow \nabla^2 G = \vec\nabla\cdot(\vec\nabla G) = -\frac{1}{4\pi}\left[2(\vec\nabla\frac{1}{r})\cdot(\vec\nabla e^{ikr}) + \frac{1}{r}\nabla^2(e^{ikr}) + e^{ikr}\nabla^2(\frac{1}{r})\right]$.

But $\vec\nabla\frac{1}{r} = -\frac{1}{r^2}\hat r$; $\vec\nabla(e^{ikr}) = ike^{ikr}\hat r$; $\nabla^2 e^{ikr} = ik\vec\nabla\cdot(e^{ikr}\hat r) = ik\frac{1}{r^2}\frac{d}{dr}(r^2 e^{ikr})$ [See reference in footnote 7] $\Rightarrow \nabla^2 e^{ikr} = \frac{ik}{r^2}(2re^{ikr} + ikr^2e^{ikr}) = ike^{ikr}\left(\frac{2}{r} + ik\right)$; $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec r)$. So

$$\nabla^2 G = -\frac{1}{4\pi}\left\{2\left(-\frac{1}{r^2}\hat r\right)\cdot\left(ike^{ikr}\hat r\right) + \frac{1}{r}ike^{ikr}\left(\frac{2}{r}+ik\right) - 4\pi e^{ikr}\delta^3(\vec r)\right\}.$$ But $e^{ikr}\delta^3(\vec r) = \delta^3(\vec r)$, so

$$\nabla^2 G = \delta^3(\vec r) - \frac{1}{4\pi}e^{ikr}\left[-\frac{2ik}{r^2} + \frac{2ik}{r^2} - \frac{k^2}{r}\right] = \delta^3(\vec r) + k^2\frac{e^{ikr}}{4\pi r} = \delta^3(\vec r) - k^2 G.$$

$\therefore (\nabla^2 + k^2)G = \delta^3(\vec r)$. QED

PROBLEM 11.6 $\psi = \frac{1}{\sqrt{\pi a^3}}e^{-r/a}$; $V = -\frac{e^2}{4\pi\epsilon_0 r} = -\frac{\hbar^2}{ma}\frac{1}{r}$ [4.72]; $k = i\frac{\sqrt{-2mE}}{\hbar} = \frac{i}{a}$.

In this case there is no "incoming" wave, and $\psi_0(\vec r) = 0$. Our problem is to show that

$$-\frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\vec r - \vec r_0|}}{|\vec r - \vec r_0|}V(\vec r_0)\psi(\vec r_0)d^3\vec r_0 = \psi(\vec r).$$ So we proceed to evaluate the left side (call it I):

$$I = \left(-\frac{m}{2\pi\hbar^2}\right)\left(-\frac{\hbar^2}{ma}\right)\frac{1}{\sqrt{\pi a^3}}\int\frac{e^{-|\vec r - \vec r_0|/a}}{|\vec r - \vec r_0|}\frac{1}{r_0}e^{-r_0/a}d^3 r_0 = \frac{1}{2\pi a}\frac{1}{\sqrt{\pi a^3}}\int\frac{e^{-\sqrt{r^2+r_0^2-2rr_0\cos\theta}/a}}{\sqrt{r^2+r_0^2-2rr_0\cos\theta}}\frac{e^{-r_0/a}}{r_0}r_0^2\sin\theta\, dr_0\, d\theta\, d\phi.$$

(I have set the z_0 axis along the (fixed) direction $\vec r$, for convenience.) Doing the ϕ integral (2π):

$$I = \frac{1}{a\sqrt{\pi a^3}}\int_0^\infty r_0 e^{-r_0/a}\left\{\int_0^\pi \frac{e^{-\sqrt{\ }/a}}{\sqrt{\ }}\sin\theta\, d\theta\right\}dr_0.$$ The θ integral is

$$\int_0^\pi \frac{e^{-\sqrt{r_0^2+r^2-2rr_0\cos\theta}/a}}{\sqrt{r_0^2+r^2-2rr_0\cos\theta}}\sin\theta\, d\theta = -\frac{a}{rr_0}e^{-\sqrt{r_0^2+r^2-2rr_0\cos\theta}/a}\Big|_0^\pi = -\frac{a}{rr_0}\left[e^{-(r+r_0)/a} - e^{-|r-r_0|/a}\right]$$

$$I = -\frac{1}{r\sqrt{\pi a^3}}\int_0^\infty e^{-r_0/a}\left[e^{-(r_0+r)/a} - e^{-|r_0-r|/a}\right]dr_0 = -\frac{1}{r\sqrt{\pi a^3}}\left\{e^{-r/a}\int_0^\infty e^{-2r_0/a}dr_0 - e^{-r/a}\int_0^r dr - e^{r/a}\int_r^\infty e^{-2r_0/a}dr_0\right\}$$

$$\therefore I = -\frac{1}{r\sqrt{\pi a^3}}\left\{e^{-r/a}\left(\frac{a}{2}\right) - e^{-r/a}(r) - e^{r/a}\left(-\frac{a}{2}e^{-2r/a}\right)\Big|_r^\infty\right\} = -\frac{1}{r\sqrt{\pi a^3}}\left\{\frac{a}{2}e^{-r/a} - re^{-r/a} - \frac{a}{2}e^{r/a}e^{-2r/a}\right\}$$

$$= \frac{1}{\sqrt{\pi a^3}}e^{-r/a} = \psi(r). \text{ QED}$$

PROBLEM 11.7 For the potential [11.71], [11.78] $\Rightarrow f(\theta) = -\frac{2m}{\hbar^2\kappa}V_0\int_0^a r\sin(\kappa r)dr = -\frac{2mV_0}{\hbar^2\kappa}\left[\frac{1}{\kappa^2}\sin\kappa r - \frac{r}{\kappa}\cos(\kappa r)\right]\Big|_0^a$

$\boxed{f(\theta) = -\frac{2mV_0}{\hbar^2\kappa^3}\left(\sin(\kappa a) - (\kappa a)\cos(\kappa a)\right)}$, where [11.79] $\kappa = 2k\sin(\theta/2)$. For low-energy scattering ($ka \ll 1$)

$\sin(\kappa a) \cong \kappa a - \frac{1}{3!}(\kappa a)^3$; $\cos(\kappa a) = 1 - \frac{1}{2}(\kappa a)^2$; so $f(\theta) \cong -\frac{2mV_0}{\hbar^2\kappa^3}\left[\kappa a - \frac{1}{6}(\kappa a)^3 - \kappa a + \frac{1}{2}(\kappa a)^3\right] = \boxed{-\frac{2}{3}\frac{mV_0a^3}{\hbar^2}}$,

in agreement with [11.72].

PROBLEM 11.8 $\sin(\kappa r) = \frac{1}{2i}\left(e^{i\kappa r} - e^{-i\kappa r}\right)$, so $\int_0^\infty e^{-\mu r}\sin(\kappa r)dr = \frac{1}{2i}\int_0^\infty\left[e^{-(\mu-i\kappa)r} - e^{-(\mu+i\kappa)r}\right]dr$

$$= \frac{1}{2i}\left[\frac{e^{-(\mu-i\kappa)r}}{-(\mu-i\kappa)} - \frac{e^{-(\mu+i\kappa)r}}{-(\mu+i\kappa)}\right]\Big|_0^\infty = \frac{1}{2i}\left[\frac{1}{\mu-i\kappa} - \frac{1}{\mu+i\kappa}\right] = \frac{1}{2i}\left(\frac{\mu+i\kappa-\mu+i\kappa}{\mu^2+\kappa^2}\right) = \frac{\kappa}{\mu^2+\kappa^2}.$$

So $f(\theta) = -\frac{2m\beta}{\hbar^2\kappa}\frac{\kappa}{\mu^2+\kappa^2} = -\frac{2m\beta}{\hbar^2(\mu^2+\kappa^2)}$. QED

PROBLEM 11.9 [11.81] $\Rightarrow D(\theta) = |f(\theta)|^2 = \left(\frac{2m\beta}{\hbar^2}\right)^2\frac{1}{(\mu^2+\kappa^2)^2}$, where [11.79] $\Rightarrow \kappa = 2k\sin(\theta/2)$.

$\therefore \sigma = \int D(\theta)\sin\theta\,d\theta\,d\phi = 2\pi\left(\frac{2m\beta}{\hbar^2}\right)^2\frac{1}{\mu^4}\int_0^\pi\frac{1}{\left[1+\left(\frac{2k}{\mu}\sin\frac{\theta}{2}\right)^2\right]^2}2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)d\theta$. Let $\frac{2k}{\mu}\sin\frac{\theta}{2} = x$, so

$2\sin\frac{\theta}{2} = \frac{\mu}{k}x$, and $\cos\frac{\theta}{2}d\theta = \frac{\mu}{k}dx$. $\sigma = 2\pi\left(\frac{2m\beta}{\hbar^2}\right)^2\frac{1}{\mu^4}\left(\frac{\mu}{k}\right)^2\int_{x_0}^{x_1}\frac{x}{[1+x^2]^2}dx$. The limits are:

$\left\{\begin{matrix}\theta = 0 \Rightarrow x = x_0 = 0\\ \theta = \pi \Rightarrow x = x_1 = \frac{2k}{\mu}\end{matrix}\right\}$. So: $\sigma = 2\pi\left(\frac{2m\beta}{\hbar^2}\right)^2\frac{1}{(\mu k)^2}\left[-\frac{1}{2}\frac{1}{(1+x^2)}\right]\Big|_0^{2k/\mu} = \pi\left(\frac{2m\beta}{\hbar^2}\right)^2\frac{1}{(\mu k)^2}\left[1 - \frac{1}{1+(2k/\mu)^2}\right]$.

$\sigma = \pi\left(\frac{2m\beta}{\hbar^2}\right)^2\frac{1}{(\mu k)^2}\left[\frac{4(k/\mu)^2}{1+4k^2/\mu^2}\right] = \pi\left(\frac{4m\beta}{\hbar^2}\right)^2\frac{1}{\mu^2}\frac{1}{\mu^2+4k^2}$. But $k^2 = 2mE/\hbar^2$, so

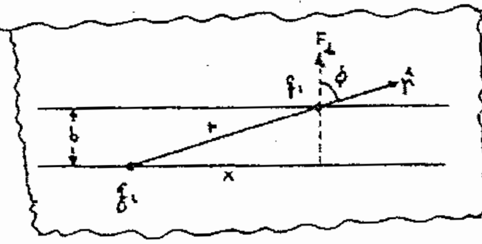
$$\boxed{\sigma = \pi\left(\frac{4m\beta}{\mu\hbar}\right)^2\frac{1}{(\mu\hbar)^2+8mE}}.$$

PROBLEM 11.10 (a) $V(\vec{r}) = \alpha\delta(r-a)$. [11.70] $\Rightarrow f = -\frac{m}{2\pi\hbar^2}\int V(\vec{r})d^3\vec{r} = -\frac{m}{2\pi\hbar^2}\alpha 4\pi\int_0^\infty\delta(r-a)r^2dr$

$\boxed{f = -\frac{2m\alpha}{\hbar^2}a^2}$; $D = |f|^2 = \boxed{\left(\frac{2m\alpha}{\hbar^2}a^2\right)^2}$; $\sigma = 4\pi D = \boxed{\pi\left(\frac{4m\alpha}{\hbar^2}a^2\right)^2}$.

(b) [11.78] $\Rightarrow f = -\frac{2m}{\hbar^2\kappa}\alpha\int_0^\infty r\delta(r-a)\sin(\kappa r)dr = \boxed{-\frac{2m\alpha}{\hbar^2\kappa}a\sin(\kappa a)}$. $(\kappa = 2k\sin(\theta/2))$.

(c) Note first that (b) reduces to (a) in the low-energy regime ($ka \ll 1 \Rightarrow \kappa a \ll 1$). Since Problem 11.4 was also for low energy, what we must confirm is that 11.4 reproduces (a) in the regime for which the Born approximation holds. Comparison shows that 11.4 does reduce to $f = -2m\alpha a^3/\hbar^2$ when $\phi \ll 1$ — i.e. when $\frac{f}{a} \ll 1$. This is the appropriate condition, since [11.12] $\frac{f}{a}$ is a measure of the relative size of the scattered wave, in the interaction region.



PROBLEM 11.11 $\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$; $F_\perp = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\cos\phi$.

$\cos\phi = \frac{b}{r}$, so $F_\perp = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2 b}{r^3}$. $dt = dx/v$, so

$$I_\perp = \int F_\perp dt = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2 b}{v}\underbrace{\int_{-\infty}^{\infty}\frac{dx}{(x^2+b^2)^{3/2}}}$$

$$2\int_0^\infty \frac{dx}{(x^2+b^2)^{3/2}} = \frac{2x}{b^2\sqrt{x^2+b^2}}\Big|_0^\infty = \frac{2}{b^2}$$

$$\Big\} \; I_\perp = \frac{1}{4\pi\epsilon_0}\frac{2q_1 q_2}{bv}.$$

$$\tan\theta = \frac{I_\perp}{mv} = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{b(\frac{1}{2}mv^2)} = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{bE}. \quad \boxed{\theta = \tan^{-1}\left[\frac{q_1 q_2}{4\pi\epsilon_0 bE}\right]}.$$

$b = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{E\tan\theta} = \left(\frac{q_1 q_2}{8\pi\epsilon_0 E}\right)(2\cot\theta)$. The exact answer is the same, only with $\cot(\theta/2)$ in place of $2\cot\theta$. So I must show that $\cot(\theta/2) \cong 2\cot\theta$, for small θ (that's the régime in which the impulse approximation should work). Well: $\cot(\theta/2) = \frac{\cos(\theta/2)}{\sin(\theta/2)} \cong \frac{1}{\theta/2} = \frac{2}{\theta}$, for small θ, while $2\cot\theta = 2\frac{\cos\theta}{\sin\theta} \cong 2\frac{1}{\theta}$. So it works.

PROBLEM 11.12 First let's set up the general formalism. From [11.91]:

$$\psi(\vec{r}) = \psi_0(\vec{r}) + \int g(\vec{r}-\vec{r}_0)V(\vec{r}_0)\psi_0(\vec{r}_0)d^3r_0 + \int g(\vec{r}-\vec{r}_0)V(\vec{r}_0)\left\{\int g(\vec{r}_0-\vec{r}_{00})V(\vec{r}_{00})\psi_0(\vec{r}_{00})d^3r_{00}\right\}d^3r_0 + \cdots$$

Put in $\psi_0(\vec{r}) = Ae^{ikz}$, $g(\vec{r}) = -\frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}$:

$$\psi(\vec{r}) = Ae^{ikz} - \frac{mA}{2\pi\hbar^2}\int\frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|}V(\vec{r}_0)e^{ikz_0}d^3r_0 + \left(\frac{m}{2\pi\hbar^2}\right)^2 A\int\frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|}V(\vec{r}_0)\left\{\int\frac{e^{ik|\vec{r}_0-\vec{r}_{00}|}}{|\vec{r}_0-\vec{r}_{00}|}V(\vec{r}_{00})e^{ikz_{00}}d^3r_{00}\right\}d^3r_0$$

In the scattering region $r \gg r_0$, [11.63] $\Rightarrow \frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} \cong \frac{e^{ikr}}{r}e^{-i\vec{k}\cdot\vec{r}_0}$, with $\vec{k} \equiv k\hat{r}$, so

$$\psi(\vec{r}) = A\left\{e^{ikz} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\vec{k}\cdot\vec{r}_0}V(\vec{r}_0)e^{ikz_0}d^3r_0 + \left(\frac{m}{2\pi\hbar^2}\right)^2\frac{e^{ikr}}{r}\int e^{-i\vec{k}\cdot\vec{r}_0}V(\vec{r}_0)\left\{\int\frac{e^{ik|\vec{r}_0-\vec{r}_{00}|}}{|\vec{r}_0-\vec{r}_{00}|}V(\vec{r}_{00})e^{ikz_{00}}d^3r_{00}\right\}d^3r_0\right\}$$

$$\therefore \boxed{f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{i(\vec{k}'-\vec{k})\cdot\vec{r}}V(\vec{r})d^3r + \left(\frac{m}{2\pi\hbar^2}\right)^2\int e^{-i\vec{k}\cdot\vec{r}}V(\vec{r})\left\{\int\frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|}V(\vec{r}_0)e^{ikz_0}d^3r_0\right\}d^3r}$$

(I dropped one subscript "o", since the original $\vec{r}$ occurs only in the direction of $\vec{k}$; $\vec{k}' = k\hat{z}$.)

For low energy scattering we drop the exponentials (see p. 368):

$$\boxed{f(\theta,\phi) \cong -\frac{m}{2\pi\hbar^2}\int V(\vec{r})d^3r + \left(\frac{m}{2\pi\hbar^2}\right)^2\int V(\vec{r})\left\{\int\frac{1}{|\vec{r}-\vec{r}_0|}V(\vec{r}_0)d^3r_0\right\}d^3r}.$$

Now apply this to the potential [11.71]:

$$\int \frac{1}{|\vec r-\vec r_0|} V(\vec r_0)\,d^3r_0 = V_0 \int_0^a \frac{1}{|\vec r-\vec r_0|} r_0^2 \sin\theta_0\, dr_0\, d\theta_0\, d\phi_0 .$$ Orient z_0 axis along $\vec r$, so $|\vec r-\vec r_0| = r^2+r_0^2-2rr_0\cos\theta_0$.

$$= V_0\, 2\pi \int_0^a r_0^2 \Bigg\{ \underbrace{\int_0^\pi \frac{1}{\sqrt{r_0^2+r^2-2rr_0\cos\theta_0}} \sin\theta_0\, d\theta_0}_{\frac{1}{rr_0}\sqrt{r_0^2+r^2-2rr_0\cos\theta_0}\Big|_0^\pi = \frac{1}{rr_0}\{(r_0+r)-|r_0-r|\} = \begin{cases} 2/r, & r_0<r \\ 2/r_0, & r_0>r \end{cases}} \Bigg\} dr_0$$

Here $r<a$ (from the "outer" integral), so

$$\int \frac{1}{|\vec r-\vec r_0|} V(\vec r_0)\,d^3r_0 = 4\pi V_0 \left\{ \frac{1}{r}\int_0^r r_0^2\,dr_0 + \int_r^a r_0\,dr_0 \right\} = 4\pi V_0 \left(\frac{1}{r}\cdot\frac{r^3}{3} + \frac{1}{2}(a^2-r^2) \right) = 2\pi V_0 \left(a^2 - \frac{1}{3}r^2 \right).$$

$$\therefore\ \int V(\vec r)\left\{ \int \frac{1}{|\vec r-\vec r_0|} V(\vec r_0)\,d^3r_0 \right\} d^3r = V_0(2\pi V_0)\,4\pi \int_0^a \left(a^2-\frac{1}{3}r^2\right) r^2\,dr = 8\pi^2 V_0^2 \left[a^2\cdot\frac{a^3}{3} - \frac{1}{3}\cdot\frac{a^5}{5} \right] = \frac{32}{15}\pi^2 V_0^2 a^5.$$

$$f(\theta) = -\frac{m}{2\pi\hbar^2} V_0 \frac{4}{3}\pi a^3 + \left(\frac{m}{2\pi\hbar^2}\right)^2 \frac{32}{15}\pi^2 V_0^2 a^5 = \boxed{-\left(\frac{2mV_0a^3}{3\hbar^2}\right)\left[1-\frac{4}{5}\left(\frac{mV_0a^2}{\hbar^2}\right)\right]}.$$

PROBLEM 11.13 $\left(\frac{d^2}{dx^2}+k^2\right)G(x) = \delta(x)$ (analog to [11.42]). $G(x) = \frac{1}{\sqrt{2\pi}}\int e^{isx} g(s)\,ds$ (analog to [11.44]).

$$\left(\frac{d^2}{dx^2}+k^2\right)G = \frac{1}{\sqrt{2\pi}}\int (-s^2+k^2) g(s) e^{isx}\,ds = \delta(x) = \frac{1}{2\pi}\int e^{isx}\,ds \Rightarrow g(s) = \frac{1}{\sqrt{2\pi}(k^2-s^2)}.$$

$\therefore\ G(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{isx}}{k^2-s^2}\,ds.$ Skirt the poles as in Figure 11.8. $x>0 \Rightarrow$ close above:

$$G(x) = -\frac{1}{2\pi}\oint \left(\frac{e^{isx}}{s+k}\right)\frac{1}{s-k}\,ds = -\frac{1}{2\pi}2\pi i\left(\frac{e^{isx}}{s+k}\right)\Big|_{s=k} = -i\frac{e^{ikx}}{2k}.$$ $x<0 \Rightarrow$ close below:

$$G(x) = +\frac{1}{2\pi}\oint \left(\frac{e^{isx}}{s-k}\right)\frac{1}{s+k}\,ds = \frac{1}{2\pi}2\pi i\left(\frac{e^{isx}}{s-k}\right)\Big|_{s=-k} = -i\frac{e^{-ikx}}{2k}.$$

In general, then, $\boxed{G(x) = -\frac{i}{2k}e^{ik|x|}}$. [Analog to [11.55].]

$$\psi(x) = \int G(x-x_0)\frac{2m}{\hbar^2}V(x_0)\psi(x_0)\,dx_0 = -\frac{i}{2k}\frac{2m}{\hbar^2}\int e^{ik|x-x_0|}V(x_0)\psi(x_0)\,dx_0$$ — plus any solution $\psi_0(x)$

to the homogeneous Schrödinger equation: $\left(\frac{d^2}{dx^2}+k^2\right)\psi_0(x)=0$. So:

$$\boxed{\psi(x) = \psi_0(x) - \frac{im}{\hbar^2k}\int_{-\infty}^{\infty} e^{ik|x-x_0|}V(x_0)\psi(x_0)\,dx_0}.$$

PROBLEM 11.14 For the Born approximation let $\psi_0(x) = Ae^{ikx}$, and $\psi(x) \cong Ae^{ikx}$

$$\psi(x) \cong A\left\{ e^{ikx} - \frac{im}{\hbar^2k}\int_{-\infty}^{\infty} e^{ik|x-x_0|}V(x_0)e^{ikx_0}\,dx_0 \right\} = A\left\{ e^{ikx} - \frac{im}{\hbar^2k}\int_{-\infty}^{x} e^{ik(x-x_0)}V(x_0)e^{ikx_0}\,dx_0 \right.$$
$$\left. - \frac{im}{\hbar^2k}\int_x^{\infty} e^{ik(x_0-x)}V(x_0)e^{ikx_0}\,dx_0 \right\}$$

$$\boxed{\psi(x) = A\left\{ e^{ikx} - \frac{im}{\hbar^2k}e^{ikx}\int_{-\infty}^{x} V(x_0)\,dx_0 - \frac{im}{\hbar^2k}e^{-ikx}\int_x^{\infty} e^{2ikx_0}V(x_0)\,dx_0 \right\}}.$$

Now assume $V(x)$ is localized; for large positive x, the third term is zero, and

$$\psi(x) = A e^{ikx}\left[1 - \frac{im}{\hbar^2 k}\int_{-\infty}^{\infty} V(x_0)\,dx_0\right].$$ This is the transmitted wave.

For large negative x the middle term is zero:

$$\psi(x) = A\left\{e^{ikx} - \frac{im}{\hbar^2 k} e^{-ikx}\int_{-\infty}^{\infty} e^{2ikx_0} V(x_0)\,dx_0\right\}.$$

Evidently the first term is the incident wave and the second the reflected wave:

$$\boxed{R = \left(\frac{m}{\hbar^2 k}\right)^2 \left|\int_{-\infty}^{\infty} e^{2ikx} V(x)\,dx\right|^2}.$$ [If you try in the same spirit to calculate T, you get

$$T = \left|1 - \frac{im}{\hbar^2 k}\int_{-\infty}^{\infty} V(x)\,dx\right|^2 = 1 + \left(\frac{m}{\hbar^2 k}\right)^2\left[\int_{-\infty}^{\infty} V(x)\,dx\right]^2,$$ which is nonsense (greater than 1).

The first Born approximation gets R right, but all you can say is $T \approx 1$, to this order (or get it by using $T = 1 - R$). See Problem 9.15 for a more complete explanation.]

PROBLEM 11.15 Delta function: $V(x) = -\alpha\delta(x)$. $\int_{-\infty}^{\infty} e^{2ikx} V(x)\,dx = -\alpha$, so $R = \left(\frac{m\alpha}{\hbar^2 k}\right)^2$ — or, in terms of energy ($k^2 = 2mE/\hbar^2$): $R = \frac{m^2\alpha^2}{2mE\hbar^2} = \frac{m\alpha^2}{2\hbar^2 E}$; $T = 1 - R = \boxed{1 - \frac{m\alpha^2}{2\hbar^2 E}}$.

The exact answer [2.123] is $\frac{1}{1 + \frac{m\alpha^2}{2\hbar^2 E}} \cong 1 - \frac{m\alpha^2}{2\hbar^2 E}$, so they agree provided $\boxed{E \gg \frac{m\alpha^2}{2\hbar^2}}$.

Finite square well: $V(x) = \left\{\begin{matrix} -V_0 & (-a<x<a) \\ 0 & (\text{otherwise}) \end{matrix}\right\}$. $\int_{-\infty}^{\infty} e^{2ikx} V(x)\,dx = -V_0\int_{-a}^{a} e^{2ikx}\,dx$

$$= -V_0 \left.\frac{e^{2ikx}}{2ik}\right|_{-a}^{a} = -\frac{V_0}{k}\left(\frac{e^{2ika} - e^{-2ika}}{2i}\right) = -\frac{V_0}{k}\sin(2ka).$$ So $R = \left(\frac{m}{\hbar^2 k}\right)^2\left(\frac{V_0}{k}\sin(2ka)\right)^2$.

$$\boxed{T = 1 - \left(\frac{V_0}{2E}\sin\left(\frac{2a}{\hbar}\sqrt{2mE}\right)\right)^2}.$$ If $\boxed{E \gg V_0}$, the exact answer [2.151] becomes

$$T^{-1} = 1 + \left(\frac{V_0}{2E}\sin\left(\frac{2a}{\hbar}\sqrt{2mE}\right)\right)^2 \Rightarrow T \cong 1 - \left(\frac{V_0}{2E}\sin\left(\frac{2a}{\hbar}\sqrt{2mE}\right)\right)^2,$$ so they agree provided $E \gg V_0$.
