CHAPTER 2

1. We have

$$\psi(x) = \int_{-\infty}^{\infty} dk A(k) e^{ikx} = \int_{-\infty}^{\infty} dk \frac{N}{k^2 + \alpha^2} e^{ikx} = \int_{-\infty}^{\infty} dk \frac{N}{k^2 + \alpha^2} \cos kx$$

because only the even part of $e^{ikx} = \cos kx + i \sin kx$ contributes to the integral. The integral can be looked up. It yields

$$\psi(x) = N \frac{\pi}{\alpha} e^{-\alpha|x|}$$

so that

$$|\psi(x)|^2 = \frac{N^2 \pi^2}{\alpha^2} e^{-2\alpha|x|}$$

If we look at $|A(k)|^2$ we see that this function drops to 1/4 of its peak value at $k = \pm \alpha$.. We may therefore estimate the width to be $\Delta k = 2\alpha$. The square of the wave function drops to about 1/3 of its value when
$x = \pm 1/2\alpha$. This choice then gives us $\Delta k \, \Delta x = 1$. Somewhat different choices will give slightly different numbers, but in all cases the product of the widths is independent of α.

2. the definition of the group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{2\pi d\nu}{2\pi d(1/\lambda)} = \frac{d\nu}{d(1/\lambda)} = -\lambda^2 \frac{d\nu}{d\lambda}$$

The relation between wavelength and frequency may be rewritten in the form

$$\nu^2 - \nu_0^2 = \frac{c^2}{\lambda^2}$$

so that

$$-\lambda^2 \frac{d\nu}{d\lambda} = \frac{c^2}{\nu\lambda} = c\sqrt{1 - (\nu_0/\nu)^2}$$

3. We may use the formula for $\boldsymbol{v_g}$ derived above for

$$\nu = \sqrt{\frac{2\pi T}{\rho}} \lambda^{-3/2}$$

to calculate

$$v_g = -\lambda^2 \frac{d\nu}{d\lambda} = \frac{3}{2}\sqrt{\frac{2\pi T}{\rho\lambda}}$$

4. For deep gravity waves,

$$\nu = \sqrt{g/2\pi}\,\lambda^{-1/2}$$

from which we get, in exactly the same way $v_g = \frac{1}{2}\sqrt{\frac{\lambda g}{2\pi}}$.

5. With $\omega = \hbar k^2/2m$, $\beta = \hbar/m$ and with the original width of the packet $w(0) = \sqrt{2}\alpha$, we have

$$\frac{w(t)}{w(0)} = \sqrt{1+\frac{\beta^2 t^2}{2\alpha^2}} = \sqrt{1+\frac{\hbar^2 t^2}{2m^2\alpha^2}} = \sqrt{1+\frac{2\hbar^2 t^2}{m^2 w^4(0)}}$$

(a) With t = 1 s, $m = 0.9 \times 10^{-30}$ kg and $w(0) = 10^{-6}$ m, the calculation yields $w(1) = 1.7 \times 10^2$ m

With $w(0) = 10^{-10}$ m, the calculation yields $w(1) = 1.7 \times 10^6$ m.

These are very large numbers. We can understand them by noting that the characteristic velocity associated with a particle spread over a range Δx is $v = \hbar/m\Delta x$ and here m is very small.

(b) For an object with mass 10^{-3} kg and $w(0)= 10^{-2}$ m, we get

$$\frac{2\hbar^2 t^2}{m^2 w^4(0)} = \frac{2(1.05\times 10^{-34}\,J.s)^2 t^2}{(10^{-3}\,kg)^2 \times (10^{-2} m)^4} = 2.2\times 10^{-54}$$

for t = 1. This is a totally negligible quantity so that $w(t) = w(0)$.

6. For the 13.6 eV electron v/c = 1/137, so we may use the nonrelativistic expression for the kinetic energy. We may therefore use the same formula as in problem **5**, that is

$$\frac{w(t)}{w(0)} = \sqrt{1+\frac{\beta^2 t^2}{2\alpha^2}} = \sqrt{1+\frac{\hbar^2 t^2}{2m^2\alpha^2}} = \sqrt{1+\frac{2\hbar^2 t^2}{m^2 w^4(0)}}$$

We caclulate t for a distance of 10^4 km = 10^7 m, with speed (3×10^8m/137) to be 4.6 s. We are given that $w(0) = 10^{-3}$ m. In that case

$$w(t) = (10^{-3}\,m)\sqrt{1+\frac{2(1.05\times 10^{-34}\,J.s)^2 (4.6s)^2}{(0.9\times 10^{-30} kg)^2 (10^{-3} m)^4}} = 7.5\times 10^{-2} m$$

For a 100 MeV electron $E = pc$ to a very good approximation. This means that $\beta = 0$ and therefore the packet does not spread.

7. For any massless particle $E = pc$ so that β= 0 and there is no spreading.

8. We have

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx A e^{-\mu|x|} e^{-ipx/\hbar} = \frac{A}{\sqrt{2\pi\hbar}}\left\{\int_{-\infty}^{0} dx e^{(\mu - ik)x} + \int_{0}^{\infty} dx e^{-(\mu+ik)x}\right\}$$

$$= \frac{A}{\sqrt{2\pi\hbar}}\left\{\frac{1}{\mu - ik} + \frac{1}{\mu + ik}\right\} = \frac{A}{\sqrt{2\pi\hbar}}\frac{2\mu}{\mu^2 + k^2}$$

where $k = p/\hbar$.

9. We want

$$\int_{-\infty}^{\infty} dx A^2 e^{-2\mu|x|} = A^2\left\{\int_{-\infty}^{0} dx e^{2\mu x} + \int_{0}^{\infty} dx e^{-2\mu x}\right\} = A^2 \frac{1}{\mu} = 1$$

so that

$$A = \sqrt{\mu}$$

10. Done in text.

11. Consider the Schrodinger equation with $V(x)$ complex. We now have

$$\frac{\partial\psi(x,t)}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} - \frac{i}{\hbar}V(x)\psi(x,t)$$

and

$$\frac{\partial\psi^*(x,t)}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2\psi^*(x,t)}{\partial x^2} + \frac{i}{\hbar}V^*(x)\psi(x,t)$$

Now

$$\frac{\partial}{\partial t}(\psi^*\psi) = \frac{\partial\psi^*}{\partial t}\psi + \psi^*\frac{\partial\psi}{\partial t}$$

$$= (-\frac{i\hbar}{2m}\frac{\partial^2\psi^*}{\partial x^2} + \frac{i}{\hbar}V^*(x)\psi^*)\psi + \psi^*(\frac{i\hbar}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} - \frac{i}{\hbar}V(x)\psi(x,t))$$

$$= -\frac{i\hbar}{2m}(\frac{\partial^2\psi^*}{\partial x^2}\psi - \psi^*\frac{\partial^2\psi(x,t)}{\partial x^2}) + \frac{i}{\hbar}(V^* - V)\psi^*\psi$$

$$= -\frac{i\hbar}{2m}\frac{\partial}{\partial x}\left\{\frac{\partial\psi^*}{\partial x}\psi - \psi^*\frac{\partial\psi}{\partial x}\right\} + \frac{2\,\mathrm{Im}V(x)}{\hbar}\psi^*\psi$$

Consequently

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} dx\,|\psi(x,t)|^2 = \frac{2}{\hbar}\int_{-\infty}^{\infty} dx(\mathrm{Im}V(x))\,|\psi(x,t)|^2$$

We require that the left hand side of this equation is negative. This does not tell us much about Im$V(x)$
except that it cannot be positive everywhere. If it has a fixed sign, it must be *negative.*

12. The problem just involves simple arithmetic. The class average

$$\langle g\rangle = \sum_g g n_g = 38.5$$

$$(\Delta g)^2 = \langle g^2\rangle - \langle g\rangle^2 = \sum_g g^2 n_g - (38.5)^2 = \text{ 1570.8-1482.3= 88.6}$$

The table below is a result of the numerical calculations for this system

g	n_g	$(g - \langle g\rangle)^2/(\Delta g)^2 = \lambda$	$e^{-\lambda}$	$Ce^{-\lambda}$
60	1	5.22	0.0054	0.097
55	2	3.07	0.0463	0.833
50	7	1.49	0.2247	4.04
45	9	0.48	0.621	11.16
40	16	0.025	0.975	17.53
35	13	0.138	0.871	15.66
30	3	0.816	0.442	7.96
25	6	2.058	0.128	2.30
20	2	3.864	0.021	0.38
15	0	6.235	0.002	0.036
10	1	9.70	0.0001	0.002
5	0	12.97	“0”	“0”

(c) We want

$$1 = 4N^2\int_{-\infty}^{\infty} dx\frac{\sin^2 kx}{x^2} = 4N^2 k\int_{-\infty}^{\infty} dt\frac{\sin^2 t}{t^2} = 4\pi N^2 k$$

so that $N = \sqrt{\dfrac{1}{4\pi k}}$

(d) We have

$$\langle x^n \rangle = \left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} dx x^n e^{-\alpha x^2}$$

Note that this integral vanishes for n an odd integer, because the rest of the integrand is even.

For $n = 2m$, an even integer, we have

$$\langle x^{2m} \rangle = \left(\frac{\alpha}{\pi}\right)^{1/2} = \left(\frac{\alpha}{\pi}\right)^{1/2} \left(-\frac{d}{d\alpha}\right)^m \int_{-\infty}^{\infty} dx e^{-\alpha x^2} = \left(\frac{\alpha}{\pi}\right)^{1/2} \left(-\frac{d}{d\alpha}\right)^m \left(\frac{\pi}{\alpha}\right)^{1/2}$$

For $n = 1$ as well as $n = 17$ this is zero, while for $n = 2$, that is, $m = 1$, this is $\frac{1}{2\alpha}$.

(e) $\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx e^{-ipx/\hbar} \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}$

The integral is easily evaluated by rewriting the exponent in the form

$$-\frac{\alpha}{2} x^2 - ix\frac{p}{\hbar} = -\frac{\alpha}{2}\left(x + \frac{ip}{\hbar\alpha}\right)^2 - \frac{p^2}{2\hbar^2\alpha}$$

A shift in the variable x allows us to state the value of the integral as and we end up with

$$\phi(p) = \frac{1}{\sqrt{\pi\hbar}} \left(\frac{\pi}{\alpha}\right)^{1/4} e^{-p^2/2\alpha\hbar^2}$$

We have, for n even, i.e. $n = 2m$,

$$\langle p^{2m} \rangle = \frac{1}{\pi\hbar}\left(\frac{\pi}{\alpha}\right)^{1/2} \int_{-\infty}^{\infty} dp p^{2m} e^{-p^2/\alpha\hbar^2} =$$

$$= \frac{1}{\pi\hbar}\left(\frac{\pi}{\alpha}\right)^{1/2} \left(-\frac{d}{d\beta}\right)^m \left(\frac{\pi}{\beta}\right)^{1/2}$$

where at the end we set $\beta = \frac{1}{\alpha\hbar^2}$. For odd powers the integral vanishes.
Specifically for $m = 1$ we have We have

$$(\Delta x)^2 = \langle x^2 \rangle = \frac{1}{2\alpha}$$

$$(\Delta p)^2 = \langle p^2 \rangle = \frac{\alpha\hbar^2}{2}$$

so that $\Delta p \Delta x = \frac{\hbar}{2}$. This is, in fact, the smallest value possible for the product of the dispersions.

24. We have

$$\int_{-\infty}^{\infty} dx \psi^*(x) x \psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx \, \psi^*(x) x \int_{-\infty}^{\infty} dp \phi(p) e^{ipx/\hbar}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx \psi^*(x) \int_{-\infty}^{\infty} dp \phi(p) \frac{\hbar}{i} \frac{\partial}{\partial p} e^{ipx/\hbar} = \int_{-\infty}^{\infty} dp \phi^*(p) i\hbar \frac{\partial \phi(p)}{\partial p}$$

In working this out we have shamelessly interchanged orders of integration. The justification of this is that the wave functions are expected to go to zero at infinity faster than any power of x , and this is also true of the momentum space wave functions, in their dependence on p.