CHAPTER 3.

1. The linear operators are (a), (b), (f)

2.We have

$$\int_{-\infty}^{x} dx' x' \psi(x') = \lambda \psi(x)$$

To solve this, we differentiate both sides with respect to *x*, and thus get

$$\lambda \frac{d\psi(x)}{dx} = x\psi(x)$$

A solution of this is obtained by writing $d\psi / \psi = (1/\lambda) x dx$ from which we can immediately state that

$$\psi(x) = Ce^{\lambda x^2/2}$$

The existence of the integral that defines $O_6\psi(x)$ requires that $\lambda < 0$.

3, (a)

$$\begin{aligned}
&O_2 O_6 \psi(x) - O_6 O_2 \psi(x) \\
&= x\frac{d}{dx}\int_{-\infty}^{x} dx' x' \psi(x') - \int_{-\infty}^{x} dx' x'^2 \frac{d\psi(x')}{dx'} \\
&= x^2\psi(x) - \int_{-\infty}^{x} dx' \frac{d}{dx'}\left(x'^2 \psi(x')\right) + 2\int_{-\infty}^{x} dx' x' \psi(x') \\
&= 2O_6\psi(x)
\end{aligned}$$

Since this is true for every $\psi(x)$ that vanishes rapidly enough at infinity, we conclude that

$$[O_2 , O_6] = 2O_6$$

(b)

$$\begin{aligned}
&O_1 O_2 \psi(x) - O_2 O_1 \psi(x) \\
&= O_1\left(x\frac{d\psi}{dx}\right) - O_2\left(x^3\psi\right) = x^4\frac{d\psi}{dx} - x\frac{d}{dx}\left(x^3\psi\right) \\
&= -3x^3\psi(x) = -3O_1\psi(x)
\end{aligned}$$

so that

$$[O_1, O_2] = -3O_1$$

4. We need to calculate

$$\langle x^2 \rangle = \frac{2}{a}\int_0^a dx x^2 \sin^2 \frac{n\pi x}{a}$$

With $\pi x/a = u$ we have

$$\langle x^2 \rangle = \frac{2}{a}\frac{a^3}{\pi^3}\int_0^\pi du u^2 \sin^2 nu = \frac{a^2}{\pi^3}\int_0^\pi du u^2 (1-\cos 2nu)$$

The first integral is simple. For the second integral we use the fact that

$$\int_0^\pi du u^2 \cos\alpha u = -\left(\frac{d}{d\alpha}\right)^2 \int_0^\pi du \cos\alpha u = -\left(\frac{d}{d\alpha}\right)^2 \frac{\sin\alpha\pi}{\alpha}$$

At the end we set $\alpha = n\pi$. A little algebra leads to

$$\langle x^2 \rangle = \frac{a^2}{3} - \frac{a^2}{2\pi^2 n^2}$$

For large n we therefore get $\Delta x = \frac{a}{\sqrt{3}}$. Since $\langle p^2 \rangle = \frac{\hbar^2 n^2 \pi^2}{a^2}$, it follows that $\Delta p = \frac{\hbar \pi n}{a}$, so that

$$\Delta p \Delta x \approx \frac{n\pi\hbar}{\sqrt{3}}$$

The product of the uncertainties thus grows as n increases.

5. With $E_n = \frac{\hbar^2\pi^2}{2ma^2} n^2$ we can calculate

$$E_2 - E_1 = 3\frac{(1.05\times 10^{-34} J.s)^2}{2(0.9\times 10^{-30}kg)(10^{-9}m)^2}\frac{1}{(1.6\times 10^{-19} J/eV)} = 0.115 eV$$

We have $\Delta E = \frac{hc}{\lambda}$ so that $\lambda = \frac{2\pi\hbar c}{\Delta E} = \frac{2\pi(2.6\times 10^{-7} ev.m)}{0.115 eV} = 1.42\times 10^{-5} m$

where we have converted $\hbar c$ from J.m units to eV.m units.

6. (a) Here we write

$$n^2 = \frac{2ma^2E}{\hbar^2\pi^2} = \frac{2(0.9\times10^{-30}kg)(2\times10^{-2}m)^2(1.5eV)(1.6\times10^{-19}J/eV)}{(1.05\times10^{-34}J.s)^2\pi^2} = 1.59\times10^{15}$$

so that $n = 4 \times 10^7$.

(b) We have

$$\Delta E = \frac{\hbar^2\pi^2}{2ma^2}2n\Delta n = \frac{(1.05\times10^{-34}J.s)^2\pi^2}{2(0.9\times10^{-30}kg)(2\times10^{-2}m)^2}2(4\times10^7) = 1.2\times10^{-26}J$$
$$= 7.6\times10^{-8}eV$$

7. The longest wavelength corresponds to the lowest frequency. Since ΔE is proportional to $(n + 1)^2 - n^2 = 2n + 1$, the lowest value corresponds to $n = 1$ (a state with $n = 0$ does not exist). We therefore have

$$h\frac{c}{\lambda} = 3\frac{\hbar^2\pi^2}{2ma^2}$$

If we assume that we are dealing with electrons of mass $m = 0.9 \times 10^{-30}$ kg, then

$$a^2 = \frac{3\hbar\pi\lambda}{4mc} = \frac{3\pi(1.05\times10^{-34}J.s)(4.5\times10^{-7}m)}{4(0.9\times10^{-30}kg)(3\times10^8 m/s)} = 4.1\times10^{-19}m^2$$

so that $a = 6.4 \times 10^{-10}$ m.

8. The solutions for a box of width a have energy eigenvalues $E_n = \frac{\hbar^2\pi^2n^2}{2ma^2}$ with $n = 1,2,3,\ldots$The odd integer solutions correspond to solutions even under $x \to -x$, while the even integer solutions correspond to solutions that are odd under reflection. These solutions vanish at $x = 0$, and it is these solutions that will satisfy the boundary conditions for the "half-well" under consideration. Thus the energy eigenvalues are given by E_n above *with n even*.

9. The general solution is

$$\psi(x,t) = \sum_{n=1}^{\infty} C_n u_n(x) e^{-iE_n t/\hbar}$$

with the C_n defined by

$$C_n = \int_{-a/2}^{a/2} dx u_n^*(x)\psi(x,0)$$

(a) It is clear that the wave function does not remain localized on the l.h.s. of the box at later times, since the special phase relationship that allows for a total interference for $x > 0$ no longer persists for $t \neq 0$.

(b) With our wave function we have $C_n = \sqrt{\frac{2}{a}}\int_{-q/2}^{0} dx u_n(x)$ .We may work this out by using the solution of the box extending from $x = 0$ to $x = a$, since the shift has no physical consequences. We therefore have

$$C_n = \sqrt{\frac{2}{a}}\int_0^{a/2} dx\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} = \frac{2}{a}\left[-\frac{a}{n\pi}\cos\frac{n\pi x}{a}\right]_0^{a/2} = \frac{2}{n\pi}\left[1-\cos\frac{n\pi}{2}\right]$$

Therefore $P_1 = |C_1|^2 = \frac{4}{\pi^2}$ and $P_2 = |C_2|^2 = \frac{1}{\pi^2}|(1-(-1))|^2 = \frac{4}{\pi^2}$

10. (a) We use the solution of the above problem to get

$$P_n = |C_n|^2 = \frac{4}{n^2\pi^2} f_n$$

where $f_n = 1$ for n = odd integer; $f_n = 0$ for $n = 4,8,12,\ldots$ and $f_n = 4$ for $n = 2,6,10,\ldots$

(b) We have

$$\sum_{n=1}^{\infty} P_n = \frac{4}{\pi^2}\sum_{odd}\frac{1}{n^2} + \frac{4}{\pi^2}\sum_{n=2,6,10,,,}\frac{4}{n^2} = \frac{8}{\pi^2}\sum_{odd}\frac{1}{n^2} = 1$$

Note. There is a typo in the statement of the problem. The sum should be restricted to *odd* integers.

11. We work this out by making use of an identity. The hint tells us that

$$(\sin x)^5 = \left(\frac{1}{2i}\right)^5 (e^{ix} - e^{-ix})^5 = \frac{1}{16}\frac{1}{2i}(e^{5ix} - 5e^{3ix} + 10e^{ix} - 10e^{-ix} + 5e^{-3ix} - e^{-5ix})$$
$$= \frac{1}{16}(\sin 5x - 5\sin 3x + 10\sin x)$$

Thus

$$\psi(x,0) = A\sqrt{\frac{a}{2}}\frac{1}{16}\left(u_5(x) - 5u_3(x) + 10u_1(x)\right)$$

(a) It follows that

$$\psi(x,t) = A\sqrt{\frac{a}{2}}\frac{1}{16}\left(u_5(x)e^{-iE_5t/\hbar} - 5u_3(x)e^{-iE_3t/\hbar} + 10u_1(x)e^{-iE_1t/\hbar}\right)$$

(b) We can calculate A by noting that $\int_0^a dx\,|\psi(x,0)|^2 = 1$. This however is equivalent to the statement that the sum of the probabilities of finding *any* energy eigenvalue adds up to 1. Now we have

$$P_5 = \frac{a}{2}A^2\frac{1}{256}; P_3 = \frac{a}{2}A^2\frac{25}{256}; P_1 = \frac{a}{2}A^2\frac{100}{256}$$

so that

$$A^2 = \frac{256}{63a}$$

The probability of finding the state with energy E_3 is 25/126.

12. The initial wave function vanishes for $x \le -a$ and for $x \ge a$. In the region in between it is proportional to $\cos\frac{\pi x}{2a}$, since this is the first nodeless trigonometric function that vanishes at $x = \pm a$. The normalization constant is obtained by requiring that

$$1 = N^2\int_{-a}^{a} dx\cos^2\frac{\pi x}{2a} = N^2\left(\frac{2a}{\pi}\right)\int_{-\pi/2}^{\pi/2} du\cos^2 u = N^2 a$$

so that $N = \sqrt{\frac{1}{a}}$. We next expand this in eigenstates of the infinite box potential with boundaries at $x = \pm b$. We write

$$\sqrt{\frac{1}{a}}\cos\frac{\pi x}{2a} = \sum_{n=1}^{\infty} C_n u_n(x;b)$$

so that

$$C_n = \int_{-b}^{b} dx u_n(x;b)\psi(x) = \int_{-a}^{a} dx u_n(x;b)\sqrt{\frac{1}{a}}\cos\frac{\pi x}{2a}$$

In particular, after a little algebra, using cosu cosv=(1/2)[cos(u-v)+cos(u+v)], we get

$$C_1 = \sqrt{\frac{1}{ab}}\int_{-a}^{a} dx \cos\frac{\pi x}{2b}\cos\frac{\pi x}{2a} = \sqrt{\frac{1}{ab}}\int_{-a}^{a} dx \frac{1}{2}\left[\cos\frac{\pi x(b-a)}{2ab} + \cos\frac{\pi x(b+a)}{2ab}\right]$$
$$= \frac{4b\sqrt{ab}}{\pi(b^2-a^2)}\cos\frac{\pi a}{2b}$$

so that

$$P_1 = |C_1|^2 = \frac{16ab^3}{\pi^2(b^2-a^2)^2}\cos^2\frac{\pi a}{2b}$$

The calculation of C_2 is trivial. The reason is that while $\psi(x)$ is an *even* function of x, $u_2(x)$ is an *odd* function of x, and the integral over an interval symmetric about $x = 0$ is zero. Hence P_2 will be zero.

13. We first calculate

$$\phi(p) = \int_0^a dx\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}\frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}} = \frac{1}{i}\sqrt{\frac{1}{4\pi\hbar a}}\left(\int_0^a dx e^{ix(n\pi/a+p/\hbar)} - (n \leftrightarrow -n)\right)$$
$$= \sqrt{\frac{1}{4\pi\hbar a}}\left(\frac{e^{iap/\hbar}(-1)^n - 1}{p/\hbar - n\pi/a} - \frac{e^{iap/\hbar}(-1)^n - 1}{p/\hbar + n\pi/a}\right)$$
$$= \sqrt{\frac{1}{4\pi\hbar a}}\frac{2n\pi/a}{(n\pi/a)^2 - (p/\hbar)^2}\left\{(-1)^n\cos pa/\hbar - 1 + i(-1)^n\sin pa/\hbar\right\}$$

From this we get

$$P(p) = |\phi(p)|^2 = \frac{2n^2\pi}{a^3\hbar}\frac{1-(-1)^n\cos pa/\hbar}{\left[(n\pi/a)^2 - (p/\hbar)^2\right]^2}$$

The function $P(p)$ does not go to infinity at $p = n\pi\hbar/a$, but if definitely peaks there. If we write $p/\hbar = n\pi/a + \varepsilon$, then the numerator becomes $1-\cos a\varepsilon \approx a^2\varepsilon^2/2$ and the denominator becomes $(2n\pi\varepsilon/a)^2$, so that at the peak $P\left(\frac{n\pi\hbar}{a}\right) = a/4\pi\hbar$. The fact that the peaking occurs at

$$\frac{p^2}{2m} = \frac{\hbar^2\pi^2 n^2}{2ma^2}$$

suggests agreement with the correspondence principle, since the kinetic energy of the particle is, as the r.h.s. of this equation shows, just the energy of a particle in the infinite box of width a. To confirm this, we need to show that the distribution is strongly peaked for large n. We do this by looking at the numerator, which vanishes when $a\varepsilon = \pi/2$, that is, when $p/\hbar = n\pi/a + \pi/2a = (n+1/2)\pi/a$. This implies that the width of the distribution is $\Delta p = \pi\hbar / 2a$. Since the x-space wave function is localized to $0 \le x \le a$ we only know that $\Delta x = a$. The result $\Delta p \Delta x \approx (\pi/2)\hbar$ is consistent with the uncertainty principle.

14. We calculate

$$\begin{aligned}\phi(p) &= \int_{-\infty}^{\infty} dx \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2} \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \\ &= \left(\frac{\alpha}{\pi}\right)^{1/4} \left(\frac{1}{2\pi\hbar}\right)^{1/2} \int_{-\infty}^{\infty} dx e^{-\alpha(x-ip/\alpha\hbar)^2} e^{-p^2/2\alpha\hbar^2} \\ &= \left(\frac{1}{\pi\alpha\hbar^2}\right)^{1/4} e^{-p^2/2\alpha\hbar^2}\end{aligned}$$

From this we find that the probability the momentum is in the range $(p, p + dp)$ is

$$|\phi(p)|^2 \, dp = \left(\frac{1}{\pi\alpha\hbar^2}\right)^{1/2} e^{-p^2/\alpha\hbar^2}$$

To get the expectation value of the energy we need to calculate

$$\begin{aligned}\langle\frac{p^2}{2m}\rangle &= \frac{1}{2m}\left(\frac{1}{\pi\alpha\hbar^2}\right)^{1/2} \int_{-\infty}^{\infty} dp p^2 e^{-p^2/\alpha\hbar^2} \\ &= \frac{1}{2m}\left(\frac{1}{\pi\alpha\hbar^2}\right)^{1/2} \frac{\sqrt{\pi}}{2} (\alpha\hbar^2)^{3/2} = \frac{\alpha\hbar^2}{2m}\end{aligned}$$

An estimate on the basis of the uncertainty principle would use the fact that the "width" of the packet is $1/\sqrt{\alpha}$. From this we estimate $\Delta p \approx \hbar / \Delta x = \hbar\sqrt{\alpha}$, so that

$$E \approx \frac{(\Delta p)^2}{2m} = \frac{\alpha\hbar^2}{2m}$$

The *exact* agreement is fortuitous, since both the definition of the width and the numerical statement of the uncertainty relation are somewhat elastic.

15. We have

$$j(x) = \frac{\hbar}{2im}\left(\psi^*(x)\frac{d\psi(x)}{dx} - \frac{d\psi^*(x)}{dx}\psi(x)\right)$$

$$= \frac{\hbar}{2im}\left[(A^* e^{-ikx} + B^* e^{ikx})(ikAe^{ikx} - ikBe^{-ikx}) - c.c)\right]$$

$$= \frac{\hbar}{2im}[ik\,|A|^2 - ik\,|B|^2 + ikAB^* e^{2ikx} - ikA^* Be^{-2ikx}$$

$$-(-ik)\,|A|^2 - (ik)\,|B|^2 - (-ik)A^* Be^{-2ikx} - ikAB^* e^{2ikx}]$$

$$= \frac{\hbar k}{m}[|A|^2 - |B|^2]$$

This is a sum of a flux to the right associated with $A\,e^{ikx}$ and a flux to the left associated with Be^{-ikx}..

16. Here

$$j(x) = \frac{\hbar}{2im}\left[u(x)e^{-ikx}(iku(x)e^{ikx} + \frac{du(x)}{dx}e^{ikx}) - c.c\right]$$

$$= \frac{\hbar}{2im}[(iku^2(x) + u(x)\frac{du(x)}{dx}) - c.c] = \frac{\hbar k}{m}u^2(x)$$

(c) Under the reflection $x \rightarrow -x$ both x and $p = -i\hbar\frac{\partial}{\partial x}$ change sign, and since the function consists of an odd power of x and/or p, it is an odd function of x. Now the eigenfunctions for a box symmetric about the x axis have a definite parity. So that $u_n(-x) = \pm u_n(x)$. This implies that the integrand is *antisymmetric* under $x \rightarrow -x$. Since the integral is over an interval symmetric under this exchange, it is zero.

(d) We need to prove that

$$\int_{-\infty}^{\infty} dx (P\psi(x))^* \psi(x) = \int_{-\infty}^{\infty} dx \psi(x)^* P\psi(x)$$

The left hand side is equal to

$$\int_{-\infty}^{\infty} dx \psi^*(-x)\psi(x) = \int_{-\infty}^{\infty} dy \, \psi^*(y)\psi(-y)$$

with a change of variables $x \rightarrow -y$, and this is equal to the right hand side.

The eigenfunctions of P with eigenvalue +1 are functions for which $u(x) = u(-x)$, while those with eigenvalue –1 satisfy $v(x) = -v(-x)$. Now the scalar product is

$$\int_{-\infty}^{\infty} dx u^*(x)v(x) = \int_{-\infty}^{\infty} dy u^*(-x)v(-x) = -\int_{-\infty}^{\infty} dx u^*(x)v(x)$$

so that

$$\int_{-\infty}^{\infty} dx u^*(x)v(x) = 0$$

(e) A simple sketch of $\psi(x)$ shows that it is a function symmetric about $x = a/2$.
This means that the integral $\int_0^a dx \psi(x) u_n(x)$ will vanish for the $u_n(x)$ which are *odd* under the reflection about this axis. This means that the integral vanishes for $n = 2,4,6,\ldots$