CHAPTER 5.

1. We are given

$$\int_{-\infty}^{\infty} dx (A\Psi(x))^* \Psi(x) = \int_{-\infty}^{\infty} dx \Psi(x)^* A\Psi(x)$$

Now let $\Psi(x) = \phi(x) + \lambda\psi(x)$, where λ is an arbitrary complex number. Substitution into the above equation yields, on the l.h.s.

$$\int_{-\infty}^{\infty} dx (A\phi(x) + \lambda A\psi(x))^* (\phi(x) + \lambda \psi(x))$$
$$= \int_{-\infty}^{\infty} dx \left[(A\phi)^* \phi + \lambda (A\phi)^* \psi + \lambda^* (A\psi)^* \phi + |\lambda|^2 (A\psi)^* \psi \right]$$

On the r.h.s. we get

$$\int_{-\infty}^{\infty} dx (\phi(x) + \lambda\psi(x))^* (A\phi(x) + \lambda A \psi(x))$$
$$= \int_{-\infty}^{\infty} dx \left[\phi^* A\phi + \lambda^* \psi^* A\phi + \lambda\phi^* A\psi + |\lambda|^2 \psi^* A\psi \right]$$

Because of the hermiticity of *A*, the first and fourth terms on each side are equal. For the rest, sine λ is an arbitrary complex number, the coefficients of λ and λ^* are independent , and we may therefore identify these on the two sides of the equation. If we consider λ, for example, we get

$$\int_{-\infty}^{\infty} dx (A\phi(x))^* \psi(x) = \int_{-\infty}^{\infty} dx \phi(x)^* A \psi(x)$$

the desired result.

2. We have $A^+ = A$ and $B^+ = B$, therefore $(A + B)^+ = (A + B)$. Let us call $(A + B) = X$. We have shown that X is hermitian. Consider now

$$(X^+)^n = X^+ X^+ X^+ \ldots X^+ = X X X \ldots X = (X)^n$$

which was to be proved.

3. We have

$$\langle A^2 \rangle = \int_{-\infty}^{\infty} dx \psi^*(x) A^2 \psi(x)$$

Now define $A\psi(x) = \phi(x)$. Then the above relation can be rewritten as

$$\langle A^2\rangle = \int_{-\infty}^{\infty} dx\, \psi(x) A\phi(x) = \int_{-\infty}^{\infty} dx (A\psi(x))^* \phi(x)$$
$$= \int_{-\infty}^{\infty} dx (A\psi(x))^* A\psi(x) \geq 0$$

4. Let $U = e^{iH} = \sum_{n=0}^{\infty} \frac{i^n H^n}{n!}$. Then $U^+ = \sum_{n=0}^{\infty} \frac{(-i)^n (H^n)^+}{n!} = \sum_{n=0}^{\infty} \frac{(-i)^n (H^n)}{n!} = e^{-iH}$, and thus

the hermitian conjugate of e^{iH} is e^{-iH} provided $H = H^+$.

5. We need to show that

$$e^{iH} e^{-iH} = \sum_{n=0}^{\infty} \frac{i^n}{n!} H^n \sum_{m=o}^{\infty} \frac{(-i)^m}{m!} H^m = 1$$

Let us pick a particular coefficient in the series, say $k = m + n$ and calculate its coefficient. We get, with $m = k - n$, the coefficient of H^k is

$$\sum_{n=0}^{k} \frac{i^n}{n!} \frac{(-i)^{k-n}}{(k-n)!} = \frac{1}{k!} \sum_{n=0}^{k} \frac{k!}{n!(k-n)!} i^n (-i)^{k-n}$$
$$= \frac{1}{k!} (i - i)^k = 0$$

Thus in the product only the $m = n = 0$ term remains, and this is equal to unity.

6. We write $I(\lambda, \lambda^*) = \int_{-\infty}^{\infty} dx \big(\phi(x) + \lambda \psi(x)\big)^* (\phi(x) + \lambda\psi(x)) \geq 0$. The left hand side, in abbreviated notation can be written as

$$I(\lambda, \lambda^*) = \int |\phi|^2 + \lambda^* \int \psi^* \phi + \lambda \int \phi^* \psi + \lambda\lambda^* \int |\psi|^2$$

Since λ and λ^* are independent, he minimum value of this occurs when

$$\frac{\partial I}{\partial \lambda^*} = \int \psi^* \phi + \lambda \int |\psi|^2 = 0$$
$$\frac{\partial I}{\partial \lambda} = \int \phi^* \psi + \lambda^* \int |\psi|^2 = 0$$

When these values of λ and λ^* are inserted in the expression for $I(\lambda,\lambda^*)$ we get

$$I(\lambda_{min}, \lambda^*_{min}) = \int |\phi|^2 - \frac{\int \phi^* \psi \int \psi^* \phi}{\int |\psi|^2} \geq 0$$

from which we get the Schwartz inequality.

7. We have $UU^+ = 1$ and $VV^+ = 1$. Now $(UV)^+ = V^+U^+$ so that

$$(UV)(UV)^+ = UVV^+U^+ = UU^+ = 1$$

8. Let $U\psi(x) = \lambda\psi(x)$, so that λ is an eigenvalue of U. Since U is unitary, $U^+U = 1$. Now

$$\int_{-\infty}^{\infty} dx(U\psi(x))^* U\psi(x) = \int_{-\infty}^{\infty} dx\psi^*(x)U^+U\psi(x) =$$
$$= \int_{-\infty}^{\infty} dx\psi^*(x)\psi(x) = 1$$

On the other hand, using the eigenvalue equation, the integral may be written in the form

$$\int_{-\infty}^{\infty} dx(U\psi(x))^* U\psi(x) = \lambda^*\lambda\int_{-\infty}^{\infty} dx\psi^*(x)\psi(x) = |\lambda|^2$$

It follows that $|\lambda|^2 = 1$, or equivalently $\lambda = e^{ia}$, with a real.

9. We write

$$\int_{-\infty}^{\infty} dx\phi(x)^*\phi(x) = \int_{-\infty}^{\infty} dx(U\psi(x))^* U\psi(x) = \int_{-\infty}^{\infty} dx\psi^*(x)U^+U\psi(x) =$$
$$= \int_{-\infty}^{\infty} dx\psi^*(x)\psi(x) = 1$$

10. We write, in abbreviated notation

$$\int v_a^* v_b = \int (Uu_a)^* Uu_b = \int u_a^* U^+ Uu_b = \int u_a^* u_b = \delta_{ab}$$

11. (a) We are given $A^+ = A$ and $B^+ = B$. We now calculate

$(i\,[A,B])^+ = (iAB - iBA)^+ = -i\,(AB)^+ - (-i)(BA)^+ = -i\,(B^+A^+) + i(A^+B^+)$

$= -iBA + iAB = i[A,B]$

(b) $[AB,C] = ABC\text{-}CAB = ABC - ACB + ACB - CAB = A(BC - CB) - (AC - CA)B$

$= A\,[B,C] - [A,C]B$

(c) The Jacobi identity written out in detail is

$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] =$

$A(BC - CB) - (BC - CB)A + B(CA - AC) - (CA - AC)B + C(AB - BA) - (AB - BA)C$

$= ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB + CAB - CBA - ABC + BAC$

It is easy to see that the sum is zero.

12. We have

$e^A B e^{-A} = (1 + A + A^2/2! + A^3/3! + A^4/4! + \ldots)B\,(1 - A + A^2/2! - A^3/3! + A^4/4! - \ldots)$

Let us now take the term independent of A: it is B.
The terms of first order in A are $AB - BA = [A,B]$.
The terms of second order in A are

$$A^2B/2! - ABA + BA^2/2! = (1/2!)(A^2B - 2ABA + BA^2)$$

$$= (1/2!)(A(AB - BA) - (AB - BA)A) = (1/2!)\{A[A,B]-[A,B]A\}$$

$$= (1/2!)[A,[A,B]]$$

The terms of third order in A are $A^3B/3! - A^2BA/2! + ABA^2/2! - BA^3$. One can again rearrange these and show that this term is $(1/3!)[A,[A,[A,B]]]$.

There is actually a neater way to do this. Consider

$$F(\lambda) = e^{\lambda A} B e^{-\lambda A}$$

Then

$$\frac{dF(\lambda)}{d\lambda} = e^{\lambda A} ABe^{-\lambda A} - e^{\lambda A} BAe^{-\lambda A} = e^{\lambda A}[A,B]e^{-\lambda A}$$

Differentiating again we get

$$\frac{d^2F(\lambda)}{d\lambda^2} = e^{\lambda A}[A,[A,B]]e^{-\lambda A}$$

and so on. We now use the Taylor expansion to calculate $F(1) = e^A B e^{-A}$.

$$F(1) = F(0) + F'(0) + \frac{1}{2!}F''(0) + \frac{1}{3!}F'''(0) + ..,$$
$$= B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + ...$$

13. Consider the eigenvalue equation $Hu = \lambda u$. Applying H to this equation we get

$H^2 u = \lambda^2 u$; $H^3 u = \lambda^3 u$ and $H^4 u = \lambda^4 u$. We are given that $H^4 = 1$, which means that H^4 applied to any function yields 1. In particular this means that $\lambda^4 = 1$. The solutions of this are $\lambda = 1, -1, i$, and $-i$. However, H is hermitian, so that the eigenvalues are real. Thus only $\lambda = \pm 1$ are possible eigenvalues. If H is not hermitian, then all four eigenvalues are acceptable.

14. We have the equations

$$Bu_a^{(1)} = b_{11}u_a^{(1)} + b_{12}u_a^{(2)}$$
$$Bu_a^{(2)} = b_{21}u_a^{(1)} + b_{22}u_a^{(2)}$$

Let us now introduce functions $(v_a^{(1)}, v_a^{(2)})$ that satisfy the equations $Bv_a^{(1)} = b_1 v_a^{(1)}; Bv_a^{(2)} = b_2 v_a^{(2)}$. We write, with simplified notation,

$$v_1 = \alpha\, u_1 + \beta\, u_2$$
$$v_2 = \gamma\, u_1 + \delta\, u_2$$

The b_1 - eigenvalue equation reads

$$b_1 v_1 = B\,(\,\alpha\, u_1 + \beta\, u_2) = \alpha\,(b_{11}\, u_1 + b_{12} u_2) + \beta\,(b_{21} u_1 + b_{22} u_2)$$

We write the l.h.s. as $b_1(\alpha\, u_1 + \beta\, u_2)$. We can now take the coefficients of u_1 and u_2 separately, and get the following equations

$$\alpha\,(b_1 - b_{11}) = \beta b_{21}$$
$$\beta\,(b_1 - b_{22}) = \alpha b_{12}$$

The product of the two equations yields a quadratic equation for b_1, whose solution is

$$b_1 = \frac{b_{11} + b_{22}}{2} \pm \sqrt{\frac{(b_{11} - b_{22})^2}{4} + b_{12} b_{21}}$$

We may choose the + sign for the b_1 eigenvalue. An examination of the equation involving v_2 leads to an identical equation, and we associate the – sign with the b_2 eigenvalue. Once we know the eigenvalues, we can find the ratios α/β and γ/δ. These suffice, since the normalization condition implies that

$$\alpha^2 + \beta^2 = 1 \text{ and } \gamma^2 + \delta^2 = 1$$

15. The equations of motion for the expectation values are

$$\frac{d}{dt}\langle x\rangle = \frac{i}{\hbar}\langle[H,x]\rangle = \frac{i}{\hbar}\langle[\frac{p^2}{2m},x]\rangle = \frac{i}{m\hbar}\langle p[p,x]\rangle = \langle\frac{p}{m}\rangle$$
$$\frac{d}{dt}\langle p\rangle = \frac{i}{\hbar}\langle[H,p]\rangle = -\frac{i}{\hbar}\langle[p,\frac{1}{2}m\omega_1^2x^2+\omega_2 x]\rangle = -m\omega_1^2\langle x\rangle - \omega_2$$

16. We may combine the above equations to get

$$\frac{d^2}{dt^2}\langle x\rangle = -\omega_1^2\langle x\rangle - \frac{\omega_2}{m}$$

The solution of this equation is obtained by introducing the variable

$$X = \langle x\rangle + \frac{\omega_2}{m\omega_1^2}$$

The equation for X reads $d^2X/dt^2 = -\omega_1^2 X$, whose solution is

$$X = A\cos\omega_1 t + B\sin\omega_1 t$$

This gives us

$$\langle x\rangle_t = -\frac{\omega_2}{m\omega_1^2} + A\cos\omega_1 t + B\sin\omega_1 t$$

At $t = 0$

$$\langle x\rangle_0 = -\frac{\omega_2}{m\omega_1^2} + A$$
$$\langle p\rangle_0 = m\frac{d}{dt}\langle x\rangle_{t=0} = mB\omega_1$$

We can therefore write A and B in terms of the initial values of $< x >$ and $< p >$,

$$\langle x\rangle_t = -\frac{\omega_2}{m\omega_1^2} + \left(\langle x\rangle_0 + \frac{\omega_2}{m\omega_1^2}\right)\cos\omega_1 t + \frac{\langle p\rangle_0}{m\omega_1}\sin\omega_1 t$$

17. We calculate as above, but we can equally well use Eq. (5-53) and (5-57), to get

$$\frac{d}{dt}\langle x\rangle = \frac{1}{m}\langle p\rangle$$
$$\frac{d}{dt}\langle p\rangle = -\langle\frac{\partial V(x,t)}{\partial x}\rangle = eE_0 cos\omega t$$

Finally

$$\frac{d}{dt}\langle H\rangle = \langle\frac{\partial H}{\partial t}\rangle = eE_0\omega\sin\omega t\langle x\rangle$$

18. We can solve the second of the above equations to get

$$\langle p\rangle_t = \frac{eE_0}{\omega}\sin\omega t + \langle p\rangle_{t=0}$$

This may be inserted into the first equation, and the result is

$$\langle x\rangle_t = -\frac{eE_0}{m\omega^2}(\cos\omega t - 1) + \frac{\langle p\rangle_{t=0}\,t}{m} + \langle x\rangle_{t=0}$$