CHAPTER 6

19. (a) We have

$$A|a\rangle = a|a\rangle$$

It follows that

$$\langle a|A|a\rangle = a\langle a|a\rangle = a$$

if the eigenstate of A corresponding to the eigenvalue a is normalized to unity. The complex conjugate of this equation is

$$\langle a|A|a\rangle^* = \langle a|A^+|a\rangle = a^*$$

If $A^+ = A$, then it follows that $a = a^*$, so that a is real.

13. We have

$$\langle\psi|(AB)^+|\psi\rangle = \langle(AB)\psi|\psi\rangle = \langle B\psi|A^+|\psi\rangle = \langle\psi|B^+A^+|\psi\rangle$$

This is true for every ψ, so that $(AB)^+ = B^+A^+$

2.

$$TrAB = \sum_n \langle n|AB|n\rangle = \sum_n \langle n|A\mathbf{1}B|n\rangle$$
$$= \sum_n\sum_m \langle n|A|m\rangle\langle m|B|n\rangle = \sum_n\sum_m \langle m|B|n\rangle\langle n|A|m\rangle$$
$$= \sum_m \langle m|B\mathbf{1}A|m\rangle = \sum_m \langle m|BA|m\rangle = TrBA$$

3. We start with the definition of $|n\rangle$ as

$$|n\rangle = \frac{1}{\sqrt{n!}}(A^+)^n\,|0\rangle$$

We now take Eq. (6-47) from the text to see that

$$A|n\rangle = \frac{1}{\sqrt{n!}}A(A^+)^n\,|0\rangle = \frac{n}{\sqrt{n!}}(A^+)^{n-1}|0\rangle = \frac{\sqrt{n}}{\sqrt{(n-1)!}}(A^+)^{n-1}|0\rangle = \sqrt{n}\,|n-1\rangle$$

4. Let $f(A^+) = \sum_{n=1}^{N} C_n (A^+)^n$. We then use Eq. (6-47) to obtain

$$Af(A^+)\,|0\rangle = A\sum_{n=1}^{N} C_n (A^+)^n\,|0\rangle = \sum_{n=1}^{N} nC_n (A^+)^{n-1}\,|0\rangle$$
$$= \frac{d}{dA^+}\sum_{n=1}^{N} C_n (A^+)^n\,|0\rangle = \frac{df(A^+)}{dA^+}\,|0\rangle$$

5. We use the fact that Eq. (6-36) leads to

$$x = \sqrt{\frac{\hbar}{2m\omega}}(A + A^+)$$
$$p = i\sqrt{\frac{m\omega\hbar}{2}}(A^+ - A)$$

We can now calculate

$$\langle k\,|\,x\,|\,n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle k\,|\,A + A^+\,|\,n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\langle k\,|\,n-1\rangle + \sqrt{k}\langle k-1\,|\,n\rangle\right)$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\delta_{k,n-1} + \sqrt{n+1}\delta_{k,n+1}\right)$$

which shows that $k = n \pm 1$.

6. In exactly the same way we show that

$$\langle k\,|\,p\,|\,n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\langle k\,|\,A^+ - A\,|\,n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}(\sqrt{n+1}\delta_{k,n+1} - \sqrt{n}\delta_{k,n-1})$$

7. Let us now calculate

$$\langle k\,|\,px\,|\,n\rangle = \langle k\,|\,p\mathbf{1}x\,|\,n\rangle = \sum_q \langle k\,|\,p\,|\,q\rangle\langle q\,|\,x\,|\,n\rangle$$

We may now use the results of problems 5 and 6. We get for the above

$$\frac{i\hbar}{2}\sum_q(\sqrt{k}\delta_{k-1,q} - \sqrt{k+1}\delta_{k+1,q})(\sqrt{n}\delta_{q,n-1} + \sqrt{n+1}\delta_{q,n+1})$$
$$= \frac{i\hbar}{2}(\sqrt{kn}\delta_{kn} - \sqrt{(k+1)n}\delta_{k+1,n-1} + \sqrt{k(n+1)}\delta_{k-1,n+1} - \sqrt{(k+1)(n+1)}\delta_{k+1,n+1})$$
$$= \frac{i\hbar}{2}(-\delta_{kn} - \sqrt{(k+1)(k+2)}\delta_{k+2,n} + \sqrt{n(n+2)}\delta_{k,n+2})$$

To calculate $\langle k\,|\,xp\,|\,n\rangle$ we may proceed in exactly the same way. It is also possible to abbreviate the calculation by noting that since x and p are hermitian operators, it follows that

$$\langle k | xp | n \rangle = \langle n | px | k \rangle^*$$

so that the desired quantity is obtained from what we obtained before by interchanging *k* and *n* and complex-conjugating. The latter only changes the overall sign, so that we get

$$\langle k | xp | n \rangle = -\frac{i\hbar}{2}(-\delta_{kn} - \sqrt{(n+1)(n+2)}\delta_{k,n+2} + \sqrt{(k+1)(k+2)}\delta_{k+2,n})$$

8.The results of problem 7 immediately lead to

$$\langle k | xp - px | n \rangle = i\hbar\delta_{kn}$$

9. This follows immediately from problems 5 and 6.

10. We again use

$$x = \sqrt{\frac{\hbar}{2m\omega}}(A + A^+)$$
$$p = i\sqrt{\frac{m\omega\hbar}{2}}(A^+ - A)$$

to obtain the operator expression for

$$x^2 = \frac{\hbar}{2m\omega}(A + A^+)(A + A^+) = \frac{\hbar}{2m\omega}(A^2 + 2A^+A + (A^+)^2 + 1)$$
$$p^2 = -\frac{m\omega\hbar}{2}(A^+ - A)(A^+ - A) = -\frac{m\omega\hbar}{2}(A^2 - 2A^+A + (A^+)^2 - 1)$$

where we have used $[A, A^+] = 1$.

The quadratic terms change the values of the eigenvalue integer by 2, so that they do not appear in the desired expressions. We get, very simply

$$\langle n | x^2 | n \rangle = \frac{\hbar}{2m\omega}(2n + 1)$$
$$\langle n | p^2 | n \rangle = \frac{m\omega\hbar}{2}(2n + 1)$$

14. Given the results of problem 9, and of 10, we have

$$(\Delta x)^2 = \frac{\hbar}{2m\omega}(2n+1)$$
$$(\Delta p)^2 = \frac{\hbar m\omega}{2}(2n+1)$$

and therefore

$$\Delta x \Delta p = \hbar(n+\frac{1}{2})$$

15. The eigenstate in $A|\alpha> = \alpha|\alpha>$ may be written in the form

$$|\alpha\rangle = f(A^+)\,|0\rangle$$

It follows from the result of problem 4 that the eigenvalue equation reads

$$Af(A^+)\,|0\rangle = \frac{df(A^+)}{dA^+}|0\rangle = \alpha f(A^+)\,|0\rangle$$

The solution of $df(x) = \alpha f(x)$ is $f(x) = C e^{\alpha x}$ so that

$$|\alpha\rangle = Ce^{\alpha A^+}|0\rangle$$

The constant C is determined by the normalization condition $<\alpha|\alpha> = 1$
This means that

$$\frac{1}{C^2} = \langle 0|e^{\alpha^* A}e^{\alpha A^+}|0\rangle = \sum_{n=0}^{\infty}\frac{(\alpha^*)^n}{n!}\langle 0|\left(\frac{d}{dA^+}\right)^n e^{\alpha A^+}|0\rangle$$
$$= \sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}\langle 0|e^{\alpha A^+}|0\rangle = \sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!} = e^{|\alpha|^2}$$

Consequently

$$C = e^{-|\alpha|^2/2}$$

We may now expand the state as follows

$$|\alpha\rangle = \sum_n |n\rangle\langle n|\alpha\rangle = \sum_n |n\rangle\langle 0|\frac{A^n}{\sqrt{n!}}Ce^{\alpha A^+}|0\rangle$$
$$= C\sum_n |n\rangle\frac{1}{\sqrt{n!}}\langle 0|\left(\frac{d}{dA^+}\right)^n e^{\alpha A^+}|0\rangle = C\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

The probability that the state $|\alpha>$ contains n quanta is

$$P_n = |\langle n | \alpha \rangle|^2 = C^2 \frac{|\alpha|^{2n}}{n!} = \frac{(|\alpha|^2)^n}{n!} e^{-|\alpha|^2}$$

This is known as the *Poisson distribution.*

Finally

$$\langle \alpha | N | \alpha \rangle = \langle \alpha | A^+ A | \alpha \rangle = \alpha^* \alpha = |\alpha|^2$$

13. The equations of motion read

$$\frac{dx(t)}{dt} = \frac{i}{\hbar}[H, x(t)] = \frac{i}{\hbar}[\frac{p^2(t)}{2m}, x(t)] = \frac{p(t)}{m}$$

$$\frac{dp(t)}{dt} = \frac{i}{\hbar}[mgx(t), p(t)] = -mg$$

This leads to the equation

$$\frac{d^2x(t)}{dt^2} = -g$$

The general solution is

$$x(t) = \frac{1}{2}gt^2 + \frac{p(0)}{m}t + x(0)$$

14. We have, as always

$$\frac{dx}{dt} = \frac{p}{m}$$

Also

$$\begin{aligned}\frac{dp}{dt} &= \frac{i}{\hbar}[\frac{1}{2}m\omega^2 x^2 + e\xi x, p] \\ &= \frac{i}{\hbar}\left(\frac{1}{2}m\omega^2 x[x,p] + \frac{1}{2}m\omega^2[x,p]x + e\xi[x,p]\right) \\ &= -m\omega^2 x - e\xi\end{aligned}$$

Differentiating the first equation with respect to t and rearranging leads to

$$\frac{d^2x}{dt^2} = -\omega^2 x - \frac{e\xi}{m} = -\omega^2(x + \frac{e\xi}{m\omega^2})$$

The solution of this equation is

$$x + \frac{e\xi}{m\omega^2} = A\cos\omega t + B\sin\omega t$$
$$= \left(x(0) + \frac{e\xi}{m\omega^2} \right)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t$$

We can now calculate the commutator $[x(t_1),x(t_2)]$, which should vanish when $t_1 = t_2$. In this calculation it is only the commutator $[p(0), x(0)]$ that plays a role. We have

$$[x(t_1),x(t_2)] = [x(0)\cos\omega t_1 + \frac{p(0)}{m\omega}\sin\omega t_1, x(0)\cos\omega t_2 + \frac{p(0)}{m\omega}\sin\omega t_2]$$
$$= i\hbar\left(\frac{1}{m\omega}(\cos\omega t_1 \sin\omega t_2 - \sin\omega t_1 \cos\omega t_2) \right) = \frac{i\hbar}{m\omega}\sin\omega(t_2 - t_1)$$

16. We simplify the algebra by writing

$$\sqrt{\frac{m\omega}{2\hbar}} = a; \quad \sqrt{\frac{\hbar}{2m\omega}} = \frac{1}{2a}$$

Then

$$\sqrt{n!}\left(\frac{\hbar\pi}{m\omega}\right)^{1/4} u_n(x) = v_n(x) = \left(ax - \frac{1}{2a}\frac{d}{dx} \right)^n e^{-a^2x^2}$$

Now with the notation $y = ax$ we get

$$v_1(y) = (y - \frac{1}{2}\frac{d}{dy})e^{-y^2} = (y + y)e^{-y^2} = 2ye^{-y^2}$$
$$v_2(y) = (y - \frac{1}{2}\frac{d}{dy})(2ye^{-y^2}) = (2y^2 - 1 + 2y^2)e^{-y^2}$$
$$= (4y^2 - 1)e^{-y^2}$$

Next

$$v_3(y) = (y - \frac{1}{2}\frac{d}{dy})\left[(4y^2 - 1)e^{-y^2}\right]$$
$$= \left(4y^3 - y - 4y + y(4y^2 - 1)\right)e^{-y^2}$$
$$= (8y^3 - 6y)e^{-y^2}$$

The rest is substitution $y = \sqrt{\frac{m\omega}{2\hbar}}x$

17. We learned in problem 4 that

$$Af(A^+)\,|0\rangle = \frac{df(A^+)}{dA^+}\,|0\rangle$$

Iteration of this leads to

$$A^n f(A^+)\,|0\rangle = \frac{d^n f(A^+)}{dA^{+n}}\,|0\rangle$$

We use this to get

$$e^{\lambda A} f(A^+)|0\rangle = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}A^n f(A^+)\,|0\rangle = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\left(\frac{d}{dA^+}\right)^n f(A^+)|0\rangle = f(A^+ + \lambda)|0\rangle$$

18. We use the result of problem **16** to write

$$e^{\lambda A} f(A^+)e^{-\lambda A}g(A^+)|0\rangle = e^{\lambda A} f(A^+)g(A^+ - \lambda)\,|0\rangle = f(A^+ + \lambda)g(A^+)\,|0\rangle$$

Since this is true for *any* state of the form $g(A^+)|0>$ we have

$$e^{\lambda A} f(A^+)e^{-\lambda A} = f(A^+ + \lambda)$$

In the above we used the first formula in the solution to **16**, which depended on the fact that $[A,A^+] = 1$. More generally we have the Baker-Hausdorff form, which we derive as follows:

Define

$$F(\lambda) = e^{\lambda A}A^+e^{-\lambda A}$$

Differentiation w.r.t. λ yields

$$\frac{dF(\lambda)}{d\lambda} = e^{\lambda A}AA^+e^{-\lambda A} - e^{\lambda A}A^+Ae^{-\lambda A} = e^{\lambda A}[A,A^+]e^{-\lambda A} \equiv e^{\lambda A}C_1e^{-\lambda A}$$

Iteration leads to

$$\frac{d^2F(\lambda)}{d\lambda^2} = e^{\lambda A}[A,[A,A^+]]e^{-\lambda A} \equiv e^{\lambda A}C_2e^{-\lambda A}$$

....

$$\frac{d^nF(\lambda)}{d\lambda^n} = e^{\lambda A}[A,[A,[A,[A,....]]..]e^{-\lambda A} \equiv e^{\lambda A}C_ne^{-\lambda A}$$

with A appearing n times in C_n. We may now use a Taylor expansion for

$$F(\lambda+\sigma) = \sum_{n=0}^{\infty}\frac{\sigma^n}{n!}\frac{d^nF(\lambda)}{d\lambda^n} = \sum_{n=0}^{\infty}\frac{\sigma^n}{n!}e^{\lambda A}C_ne^{-\lambda A}$$

If we now set $\lambda = 0$ we get

$$F(\sigma) = \sum_{n=0}^{\infty}\frac{\sigma^n}{n!}C_n$$

which translates into

$$e^{\sigma A}A^+e^{-\sigma A} = A^+ + \sigma[A,A^+] + \frac{\sigma^2}{2!}[A,[A,A^+]] + \frac{\sigma^3}{3!}[A,[A,[A,A^+]]] + ...$$

Note that if $[A,A^+] = 1$ only the first two terms appear, so that

$$e^{\sigma A}f(A^+)e^{-\sigma A} = f(A^+ + \sigma[A,A^+]) = f(A^+ + \sigma)$$

19. We follow the procedure outlined in the hint. We define $F(\lambda)$ by

$$e^{\lambda(aA+bA^+)} = e^{\lambda aA}F(\lambda)$$

Differentiation w.r.t λ yields

$$(aA+bA^+)e^{\lambda aA}F(\lambda) = aAe^{\lambda A}F(\lambda) + e^{\lambda aA}\frac{dF(\lambda)}{d\lambda}$$

The first terms on each side cancel, and multiplication by $e^{-\lambda aA}$ on the left yields

$$\frac{dF(\lambda)}{d\lambda} = e^{-\lambda aA}bA^+e^{\lambda aA}F(\lambda) = bA^+ - \lambda ab[A,A^+]F(\lambda)$$

When $[A,A^+]$ commutes with A. We can now integrate w.r.t. λ and after integration Set $\lambda = 1$. We then get

$$F(1) = e^{bA^+ - ab[A,A^+]/2} = e^{bA^+}e^{-ab/2}$$

so that

$$e^{aA+bA^+} = e^{aA}e^{bA^+}e^{-ab/2}$$

20. We can use the procedure of problem **17**, but a simpler way is to take the hermitian conjugate of the result. For a *real* function f and λ real, this reads

$$e^{-\lambda A^+} f(A)e^{\lambda A^+} = f(A+\lambda)$$

Changing λ to -λ yields

$$e^{\lambda A^+} f(A)e^{-\lambda A^+} = f(A-\lambda)$$

The remaining steps that lead to

$$e^{aA+bA^+} = e^{bA^+}e^{aA}\; e^{ab/2}$$

are identical to the ones used in problem **18**.

20. For the harmonic oscillator problem we have

$$x = \sqrt{\frac{\hbar}{2m\omega}}(A + A^+)$$

This means that e^{ikx} is of the form given in problem **19** with $a = b = ik\sqrt{\hbar / 2m\omega}$

This leads to

$$e^{ikx} = e^{ik\sqrt{\hbar/2m\omega}A^+}e^{ik\sqrt{\hbar/2m\omega}A}e^{-\hbar k^2/4m\omega}$$

Since $A|0\rangle = 0$ and $\langle 0|A^+ = 0$, we get

$$\langle 0|e^{ikx}|0\rangle = e^{-\hbar k^2/4m\omega}$$

21. An alternative calculation, given that $u_0(x) = (m\omega/\pi\hbar)^{1/4}e^{-m\omega x^2/2\hbar}$, is

$$\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int_{-\infty}^{\infty} dx e^{ikx}e^{-m\omega x^2\hbar} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int_{-\infty}^{\infty} dx e^{-\frac{m\omega}{\hbar}(x-\frac{ik\hbar}{2m\omega})^2}e^{-\frac{\hbar k^2}{4m\omega}}$$

The integral is a simple gaussian integral and $\int_{-\infty}^{\infty} dy e^{-m\omega y^2/\hbar} = \sqrt{\frac{\hbar\pi}{m\omega}}$ which just cancels the factor in front. Thus the two results agree.