CHAPTER 7

1. (a) The system under consideration has rotational degrees of freedom, allowing it to rotate about two orthogonal axes perpendicular to the rigid rod connecting the two masses. If we define the *z* axis as represented by the rod, then the Hamiltonian has the form

$$H = \frac{L_x^2 + L_y^2}{2I} = \frac{\mathbf{L}^2 - L_z^2}{2I}$$

where *I* is the moment of inertia of the dumbbell.

(b) Since there are no rotations about the *z* axis, the eigenvalue of L_z is zero, so that the eigenvalues of the Hamiltonian are

$$E = \frac{\hbar^2 \ell(\ell+1)}{2I}$$

with $\ell = 0,1,2,3,\ldots$

(c) To get the energy spectrum we need an expression for the moment of inertia. We use the fact that

$$I = M_{red}\, a^2$$

where the reduced mass is given by

$$M_{red} = \frac{M_C M_N}{M_C + M_N} = \frac{12 \times 14}{26} M_{nucleon} = 6.46 M_{nucleon}$$

If we express the separation *a* in Angstroms, we get

$$I = 6.46 \times (1.67 \times 10^{-27} kg)(10^{-10} m / A)^2 a_A^2 = 1.08 \times 10^{-46} a_A^2$$

The energy difference between the ground state and the first excited state is $2\hbar^2 / 2I$ which leads to the numerical result

$$\Delta E = \frac{(1.05 \times 10^{-34} J.s)^2}{1.08 \times 10^{-46} a_A^2 kg.m^2} \times \frac{1}{(1.6 \times 10^{-19} J / eV)} = \frac{6.4 \times 10^{-4}}{a_A^2} eV$$

2. We use the connection $\frac{x}{r} = \sin\theta\cos\phi;\ \frac{y}{r} = \sin\theta\sin\phi;\ \frac{z}{r} = \cos\theta$ to write

$$Y_{1,1} = -\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta = -\sqrt{\frac{3}{8\pi}}(\frac{x+iy}{r})$$
$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}(\frac{z}{r})$$
$$Y_{1,-1} = (-1)Y_{1,1}^{*} = \sqrt{\frac{3}{8\pi}}e^{-i\phi}\sin\theta = \sqrt{\frac{3}{8\pi}}(\frac{x-iy}{r})$$

Next we have

$$Y_{2,2} = \sqrt{\frac{15}{32\pi}}e^{2i\phi}\sin^2\theta = \sqrt{\frac{15}{32\pi}}(\cos 2\phi + i\sin 2\phi)\sin^2\theta$$
$$= \sqrt{\frac{15}{32\pi}}(\cos^2\phi - \sin^2\phi + 2i\sin\phi\cos\phi)\sin^2\theta$$
$$= \sqrt{\frac{15}{32\pi}}\left(\frac{x^2 - y^2 + 2ixy}{r^2}\right)$$

$$Y_{2,1} = -\sqrt{\frac{15}{8\pi}}e^{i\phi}\sin\theta\cos\theta = -\sqrt{\frac{15}{8\pi}}\frac{(x+iy)z}{r^2}$$

and

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1) = \sqrt{\frac{5}{16\pi}}\left(\frac{2z^2 - x^2 - y^2}{r^2}\right)$$

We may use Eq. (7-46) to obtain the form for $Y_{2,-1}$ *and* $Y_{2,-2}$.

3. We use $L_{\pm} = L_x \pm iL_y$ to calculate $L_x = \frac{1}{2}(L_+ + L_-);\ L_y = \frac{i}{2}(L_- - L_+)$. We may now proceed

$$\langle l,m_1|L_x|l,m_2\rangle = \frac{1}{2}\langle l,m_1|L_+|l,m_2\rangle + \frac{1}{2}\langle l,m_1|L_-|l,m_2\rangle$$
$$\langle l,m_1|L_y|l,m_2\rangle = \frac{i}{2}\langle l,m_1|L_-|l,m_2\rangle - \frac{i}{2}\langle l,m_1|L_+|l,m_2\rangle$$

and on the r.h.s. we insert

$$\langle l,m_1|L_+|l,m_2\rangle = \hbar\sqrt{(l-m_2)(l+m_2+1)}\,\delta_{m_1,m_2+1}$$
$$\langle l,m_1|L_-|l,m_2\rangle = \hbar\sqrt{(l+m_2)(l-m_2+1)}\,\delta_{m_1,m_2-1}$$

4. Again we use $L_x = \frac{1}{2}(L_+ + L_-);\ L_y = \frac{i}{2}(L_- - L_+)$ to work out

$$L_x^2 = \frac{1}{4}(L_+ + L_-)(L_+ + L_-) =$$
$$= \frac{1}{4}(L_+^2 + L_-^2 + \mathbf{L}^2 - L_z^2 + \hbar L_z + \mathbf{L}^2 - L_z^2 - \hbar L_z)$$
$$= \frac{1}{4}L_+^2 + \frac{1}{4}L_-^2 + \frac{1}{2}\mathbf{L}^2 - \frac{1}{2}L_z^2$$

We calculate

$$\langle l,m_1 | L_+^2 | l,m_2 \rangle = \hbar\sqrt{(l-m_2)(l+m_2+1)}\langle l,m_1 | L_+ | l,m_2+1 \rangle$$
$$= \hbar^2\big((l-m_2)(l+m_2+1)(l-m_2-1)(l+m_2+2)\big)^{1/2}\delta_{m_1,m_2+2}$$

and

$$\langle l,m_1 | L_-^2 | l,m_2 \rangle = \langle l,m_2 | L_+^2 | l,m_1 \rangle *$$

which is easily obtained from the preceding result by interchanging m_1 and m_2.

The remaining two terms yield

$$\frac{1}{2}\langle l,m_1 | (\mathbf{L}^2 - L_z^2) | l,m_2 \rangle = \frac{\hbar^2}{2}(l(l+1) - m_2^2)\delta_{m_1,m_2}$$

The remaining calculation is simple, since

$$\langle l,m_1 | L_y^2 | l,m_2 \rangle = \langle l,m_1 | \mathbf{L}^2 - L_z^2 - L_x^2 | l,m_2 \rangle$$

5. The Hamiltonian may be written as

$$H = \frac{\mathbf{L}^2 - L_z^2}{2I_1} + \frac{L_z^2}{2I_3}$$

whose eigenvalues are

$$\hbar^2\left[\frac{l(l+1)}{2I_1} + m^2\left(\frac{1}{2I_3} - \frac{1}{2I_1}\right)\right]$$

where $-l \le m \le l$.

(b) The plot is given on the right.

(c) The spectrum in the limit that $I_1 >> I_3$ is just $E = \frac{\hbar^2}{2I_3} m^2$,

with $m = 0,1,2,\ldots l$. The $m = 0$ eigenvalue is nondegenerate, while the other ones are doubly degenerate (corresponding to the negative values of m).

6. We will use the lowering operator $L_- = \hbar e^{-i\phi}(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi})$ acting on Y_{44}. Since we are not interested in the normalization, we will not carry the $\hbar$ factor.

$$Y_{43} \propto e^{-i\phi}(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi})\left[e^{4i\phi}\sin^4\theta\right]$$
$$= e^{3i\phi}\left\{-4\sin^3\theta\cos\theta - 4\cot\theta\sin^4\theta\right\} = -8e^{3i\phi}\sin^3\theta\cos\theta$$

$$Y_{42} \propto e^{-i\phi}(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi})\left[e^{3i\phi}\sin^3\theta\cos\theta\right]$$
$$= e^{2i\phi}\left\{-3\sin^2\theta\cos^2\theta + \sin^4\theta - 3\sin^2\theta\cos^2\theta\right\} =$$
$$= e^{2i\phi}\left\{-6\sin^2\theta + 7\sin^4\theta\right\}$$

$$Y_{41} \propto e^{-i\phi}(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi})\left[e^{2i\phi}(-6\sin^2\theta + 7\sin^4\theta\right]$$
$$= e^{i\phi}\left\{12\sin\theta\cos\theta - 28\sin^3\theta\cos\theta - 2\left(-6\sin\theta\cos\theta + 7\sin^3\theta\cos\theta\right)\right\}$$
$$= e^{i\phi}\left\{24\sin\theta\cos\theta - 42\sin^3\cos\theta\right\}$$

$$Y_{40} \propto e^{-i\phi}(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi})\left[e^{i\phi}(4\sin\theta - 7\sin^3\theta)\cos\theta\right]$$
$$= \{(-4\cos\theta + 21\sin^2\theta\cos\theta)\cos\theta + (4\sin^2\theta - 7\sin^4\theta) - (4\cos^2\theta - 7\sin^2\theta\cos^2\theta\}$$
$$= \{-8 + 40\sin^2\theta - 35\sin^4\theta\}$$

7. Consider the H given. The angular momentum eigenstates $|\ell,m\rangle$ are eigenstates of the Hamiltonian, and the eigenvalues are

$$E = \frac{\hbar^2\ell(\ell+1)}{2I} + \alpha\hbar m$$

with $-\ell \le m \le \ell$. Thus for every value of ℓ there will be $(2\ell+1)$ states, no longer degenerate.

8. We calculate

$$[x,L_x] = [x, yp_z - zp_y] = 0$$
$$[y,L_x] = [y, yp_z - zp_y] = z[p_y, y] = -i\hbar z$$
$$[z,L_x] = [z, yp_z - zp_y] = -y[p_z, z] = i\hbar y$$
$$[x,L_y] = [x, zp_x - xp_z] = -z[p_x, x] = i\hbar z$$
$$[y,L_y] = [y, zp_x - xp_z] = 0$$
$$[z,L_y] = [z, zp_x - xp_z] = x[p_z, z] = -i\hbar x$$

The pattern is cyclical $(x\,,\mathrm{y})\rightarrow i\hbar z$ and so on, so that we expect (and can check) that

$$[x,L_z] = -i\hbar y$$
$$[y,L_z] = i\hbar x$$
$$[z,L_z] = 0$$

9. We again expect a cyclical pattern. Let us start with

$$[p_x, L_y] = [p_x, zp_x - xp_z] = -[p_x, x]p_z = i\hbar p_z$$

and the rest follows automatically.

10. (a) The eigenvalues of L_z are known to be 2,1,0,-1,-2 in units of $\hbar$.

(b) We may write

$$(3/5)L_x - (4/5)L_y = \mathbf{n}\bullet\mathbf{L}$$

where **n** is a *unit vector*, since $n_x^2 + n_y^2 = (3/5)^2 + (-4/5)^2 = 1$. However, we may well have chosen the **n** direction as our selected *z* direction, and the eigenvalues for this are again 2,1,0,-1,-2.

(c) We may write

$$2L_x - 6L_y + 3L_z = \sqrt{2^2 + 6^2 + 3^2}\left(\frac{2}{7}L_x - \frac{6}{7}L_y + 3L_z\right)$$
$$= 7\mathbf{n}\bullet\mathbf{L}$$

Where **n** is yet another unit vector. By the same argument we can immediately state that the eigenvalues are 7*m* i.e. 14,7,0,-7,-14.

11. For our purposes, the only part that is relevant is

$$\frac{xy + yz + zx}{r^2} = \sin^2\theta\sin\phi\cos\phi + (\sin\phi + \cos\phi)\sin\theta\cos\theta$$
$$= \frac{1}{2}\sin^2\theta\frac{e^{2i\phi} - e^{-2i\phi}}{2i} + \sin\theta\cos\theta\left(\frac{e^{i\phi} - e^{-i\phi}}{2i} + \frac{e^{i\phi} + e^{-i\phi}}{2}\right)$$

Comparison with the table of Spherical Harmonics shows that all of these involve combinations of $\ell = 2$ functions. We can therefore immediately conclude that the probability of finding $\ell = 0$ is *zero*, and the probability of finding $6\hbar^2$ iz one, since this value corresponds to $\ell = 2$.A look at the table shows that

$$e^{2i\phi}\sin^2\theta = \sqrt{\frac{32\pi}{15}}Y_{2,2};\ \ e^{-2i\phi}\sin^2\theta = \sqrt{\frac{32\pi}{15}}Y_{2,-2}$$
$$e^{i\phi}\sin\theta\cos\theta = -\sqrt{\frac{8\pi}{15}}Y_{2,1};\ \ e^{-i\phi}\sin\theta\cos\theta = \sqrt{\frac{8\pi}{15}}Y_{2,-1}$$

Thus

$$\frac{xy + yz + zx}{r^2} = \frac{1}{2}\sin^2\theta\frac{e^{2i\phi} - e^{-2i\phi}}{2i} + \sin\theta\cos\theta\left(\frac{e^{i\phi} - e^{-i\phi}}{2i} + \frac{e^{i\phi} + e^{-i\phi}}{2}\right)$$
$$= \frac{1}{4i}\sqrt{\frac{32\pi}{15}}Y_{2,2} - \frac{1}{4i}\sqrt{\frac{32\pi}{15}}Y_{2,-2} - \frac{-i+1}{2}\sqrt{\frac{8\pi}{15}}Y_{2,1} + \frac{i+1}{2}\sqrt{\frac{8\pi}{15}}Y_{2,-1}$$

This is not normalized. The sum of the squares of the coefficients is

$\frac{2\pi}{15}+\frac{2\pi}{15}+\frac{4\pi}{15}+\frac{4\pi}{15}=\frac{12\pi}{15}=\frac{4\pi}{5}$, so that for normalization purposes we must multiply by $\sqrt{\frac{5}{4\pi}}$. Thus the probability of finding m = 2 is the same as getting m = -2, and it is

$$P_{\pm 2}=\frac{5}{4\pi}\frac{2\pi}{15}=\frac{1}{6}$$

Similarly $P_1 = P_{-1}$, and since all the probabilities have add up to 1,

$$P_{\pm 1}=\frac{1}{3}$$

12. Since the particles are identical, the wave function $e^{im\phi}$ must be unchanged under the rotation $\phi \rightarrow \phi + 2\pi/N$. This means that $m(2\pi/N) = 2n\pi$, so that $m = nN$, with n = 0,±1,±2,±3,…
The energy is

$$E=\frac{\hbar^2 m^2}{2MR^2}=\frac{\hbar^2 N^2}{2MR^2}n^2$$

The gap between the ground state (n = 0) and the first excited state (n =1) is

$$\Delta E=\frac{\hbar^2 N^2}{2MR^2}\rightarrow\infty \quad as \quad N\rightarrow\infty$$

If the cylinder is nicked, then there is no such symmetry and m = 0,±1,±23,…and

$$\Delta E=\frac{\hbar^2}{2MR^2}$$