CHAPTER 8

1. The solutions are of the form $\psi_{n_1 n_2 n_3}(x,y,z) = u_{n_1}(x)u_{n_2}(y)u_{n_3}(z)$ where $u_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$,and so on. The eigenvalues are

$$E = E_{n_1} + E_{n_2} + E_{n_3} = \frac{\hbar^2\pi^2}{2ma^2}(n_1^2 + n_2^2 + n_3^2)$$

2. (a) The lowest energy state corresponds to the lowest values of the integers $\{n_1,n_2,n_3\}$, that is, {1,1,1)Thus

$$E_{ground} = \frac{\hbar^2\pi^2}{2ma^2} \times 3$$

In units of $\frac{\hbar^2\pi^2}{2ma^2}$ the energies are

{1,1,1} → 3 nondegenerate)
{1,1,2},(1,2,1},(2,1,1} → 6 (triple degeneracy)
{1,2,2},{2,1,2}.{2,2,1}→ 9 (triple degeneracy)
{3,1,1},{1,3,1},{1,1,3}→ 11 (triple degeneracy)
{2,2,2}→ 12 (nondegenrate)
{1,2,3},{1,3,2},{2,1,3},{2,3,1},{3,1,2},{3,2,1}→ 14 (6-fold degenerate)
{2,2,3},{2,3,2},{3,2,2}→17 (triple degenerate)
{1,1,4},{1,4,1},{4,1,1}→18 (triple degenerate)
{1,3,3},{3,1,3},{3,3,1}→ 19 (triple degenerate)
{1,2,4},{1,4,2},{2,1,4},{2,4,1},{4,1,2},{4,2,1}→21 (6-fold degenerate)

3. The problem breaks up into three separate, here identical systems. We know that the energy for a one-dimensional oscillator takes the values $\hbar\omega(n+1/2)$, so that here the energy eigenvalues are

$$E = \hbar\omega(n_1 + n_2 + n_3 + 3/2)$$

The ground state energy correspons to the *n* values all zero. It is $\frac{3}{2}\hbar\omega$.

4. The energy eigenvalues in terms of $\hbar\omega$ with the corresponding integers are

(0,0,0)	3/2	degeneracy 1
(0,0,1) etc	5/2	3
(0,1,1) (0,0,2) etc	7/2	6
(1,1,1),(0,0,3),(0,1,2) etc	9/2	10
(1,1,2),(0,0,4),(0,2,2),(0,1,3)	11/2	15
(0,0,5),(0,1,4),(0,2,3)(1,2,2)		
(1,1,3)	13/2	21

(0,0,6),(0,1,5),(0,2,4),(0,3,3)		
(1,1,4),(1,2,3),(2,2,2),	15/2	28
(0,0,7),(0,1,6),(0,2,5),(0,3,4)		
(1,1,5),(1,2,4),(1,3,3),(2,2,3)	17/2	36
(0,0,8),(0,1,7),(0,2,6),(0,3,5)		
(0,4,4),(1,1,6),(1,2,5),(1,3,4)		
(2,2,4),(2,3,3)	19/2	45
(0,0,9),(0,1,8),(0,2,7),(0,3,6)		
(0,4,5)(1,1,7),(1,2,6),(1,3,5)		
(1,4,4),(2,2,5) (2,3,4),(3,3,3)	21/2	55

5. It follows from the relations $x = \rho\cos\phi, y = \rho\sin\phi$ that

$$dx = d\rho\cos\phi - \rho\sin\phi d\phi;\ \mathrm{dy} = \mathrm{d}\rho\sin\phi + \rho\cos\phi\mathrm{d}\phi$$

Solving this we get

$$d\rho = \cos\phi dx + \sin\phi dy; \rho d\phi = -\sin\phi dx + \cos\phi dy$$

so that

$$\frac{\partial}{\partial x} = \frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi} = \cos\phi\frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}$$

and

$$\frac{\partial}{\partial y} = \frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi} = \sin\phi\frac{\partial}{\partial\rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi}$$

We now need to work out

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} =$$

$$(\cos\phi\frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi})(\cos\phi\frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}) + (\sin\phi\frac{\partial}{\partial\rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi})(\sin\phi\frac{\partial}{\partial\rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi})$$

A little algebra leads to the r.h.s. equal to

$$\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}$$

The time-independent Schrodinger equation now reads

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi(\rho,\phi)}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2\Psi(\rho,\phi)}{\partial\phi^2}\right) + V(\rho)\Psi(\rho,\phi) = E\Psi(\rho,\phi)$$

The substitution of $\Psi(\rho,\phi) = R(\rho)\Phi(\phi)$ leads to two separate ordinary differential equations. The equation for $\Phi(\phi)$, when supplemented by the condition that the solution is unchanged when $\phi \rightarrow \phi + 2\pi$ leads to

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \qquad m = 0, \pm 1, \pm 2, \ldots$$

and the radial equation is then

$$\frac{d^2 R(\rho)}{d\rho^2} - \frac{m^2}{\rho^2} R(\rho) + \frac{2mE}{\hbar^2} R(\rho) = \frac{2mV(\rho)}{\hbar^2} R(\rho)$$

6. The relation between energy difference and wavelength is

$$2\pi\hbar \frac{c}{\lambda} = \frac{1}{2} m_{red} c^2 \alpha^2 \left(1 - \frac{1}{4}\right)$$

so that

$$\lambda = \frac{16\pi}{3} \frac{\hbar}{m_e c \alpha^2} \left(1 + \frac{m_e}{M}\right)$$

where M is the mass of the second particle, bound to the electron. We need to evaluate this for the three cases: $M = m_P$; $M = 2m_p$ and $M = m_e$. The numbers are

$$\lambda(in \quad m) = 1215.0226 \times 10^{-10} (1 + \frac{m_e}{M})$$

= 1215.68 for hydrogen

=1215.35 for deuterium

= 2430.45 for positronium

7. The ground state wave function of the electron in tritium ($Z = 1$) is

$$\psi_{100}(\mathbf{r}) = \frac{2}{\sqrt{4\pi}} \left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$$

This is to be expanded in a complete set of eigenstates of the $Z = 2$ hydrogenlike atom, and the probability that an energy measurement will yield the ground state energy of the $Z = 2$ atom is the square of the scalar product

$$\int d^3r \frac{2}{\sqrt{4\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0} \frac{2}{\sqrt{4\pi}} 2\left(\frac{2}{a_0}\right)^{3/2} e^{-2r/a_0}$$
$$= \frac{8\sqrt{2}}{a_0^3}\int_0^\infty r^2 dr e^{-3r/a_0} = \frac{8\sqrt{2}}{a_0^3}\left(\frac{a_0}{3}\right)^3 2! = \frac{16\sqrt{2}}{27}$$

Thus the probability is $P = \dfrac{512}{729}$

8. The equation reads

$$-\nabla^2\psi + (-\frac{E^2 - m^2c^4}{\hbar^2 c^2} - \frac{2Z\alpha E}{\hbar c}\frac{1}{r} - \frac{(Z\alpha)^2}{r^2})\psi(\mathbf{r}) = 0$$

Compare this with the hydrogenlike atom case

$$-\nabla^2\psi(\mathbf{r}) + \left(\frac{2mE_B}{\hbar^2} - \frac{2mZe^2}{4\pi\varepsilon_0\hbar^2}\frac{1}{r}\right)\psi(\mathbf{r}) = 0$$

and recall that

$$-\nabla^2 = -\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{\ell(\ell+1)}{r^2}$$

We may thus make a translation

$$E^2 - m^2c^4 \to -2mc^2E_B$$
$$-\frac{2Z\alpha E}{\hbar c} \to -\frac{2mZe^2}{4\pi\varepsilon_0\hbar^2}$$
$$\ell(\ell+1) - Z^2\alpha^2 \to \ell(\ell+1)$$

Thus in the expression for the hydrogenlike atom energy eigenvalue

$$2mE_B = -\frac{m^2Z^2e^2}{4\pi\varepsilon_0\hbar^2}\frac{1}{(n_r+\ell+1)^2}$$

we replace ℓ by ℓ^*, where $\ell^*(\ell^*+1) = \ell(\ell+1) - (Z\alpha)^2$, that is,

$$\ell^* = -\frac{1}{2} + \left[\left(\ell + \frac{1}{2}\right)^2 - (Z\alpha)^2\right]^{1/2}$$

We also replace $\dfrac{mZe^2}{4\pi\varepsilon_0\hbar}$ by $\dfrac{Z\alpha E}{c}$ and $2mE_B$ by $-\dfrac{E^2 - m^2c^4}{c^2}$

We thus get

$$E^2 = m^2c^4\left[1+\frac{Z^2\alpha^2}{(n_r+\ell^*+1)^2}\right]^{-1}$$

For $(Z\alpha) << 1$ this leads to

$$E - mc^2 = -\frac{1}{2}mc^2(Z\alpha)^2\frac{1}{(n_r+\ell^*+1)^2}$$

This differs from the nonrelativisric result only through the replacement of ℓ by ℓ^*.

9. We use the fact that

$$\langle T\rangle_{nl} - \frac{Ze^2}{4\pi\varepsilon_0}\langle\frac{1}{r}\rangle_{nl} = E_{nl} = -\frac{mc^2(Z\alpha)^2}{2n^2}$$

Since

$$\frac{Ze^2}{4\pi\varepsilon_0}\langle\frac{1}{r}\rangle_{nl} = \frac{Ze^2}{4\pi\varepsilon_0}\frac{Z}{a_0n^2} = \frac{Ze^2}{4\pi\varepsilon_0}\frac{2mc\alpha}{\hbar n^2} = \frac{mc^2Z^2\alpha^2}{n^2}$$

we get

$$\langle T\rangle_{nl} = \frac{mc^2Z^2\alpha^2}{2n^2} = \frac{1}{2}\langle V(r)\rangle_{nl}$$

10. The expectation value of the energy is

$$\langle E\rangle = \left(\frac{4}{6}\right)^2E_1 + \left(\frac{3}{6}\right)^2E_2 + \left(-\frac{1}{6}\right)^2E_2 + \left(\frac{\sqrt{10}}{6}\right)^2E_2$$

$$= -\frac{mc^2\alpha^2}{2}\left[\frac{16}{36}+\frac{20}{36}\frac{1}{2^2}\right] = -\frac{mc^2\alpha^2}{2}\frac{21}{36}$$

Similarly

$$\langle \mathbf{L}^2\rangle = \hbar^2\left[\frac{16}{36}\times 0+\frac{20}{36}\times 2\right] = \frac{40}{36}\hbar^2$$

Finally

$$\langle L_z\rangle = \hbar\left[\frac{16}{36}\times 0+\frac{9}{36}\times 1+\frac{1}{36}\times 0+\frac{10}{36}\times(-1)\right]$$

$$= -\frac{1}{36}\hbar$$

11. We change notation from α to β to avoid confusion with the fine-structure constant that appears in the hydrogen atom wave function. The probability is the square of the integral

$$\int d^3r\left(\frac{\beta}{\sqrt{\pi}}\right)^{3/2} e^{-\beta^2 r^2/2}\frac{2}{\sqrt{4\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$$

$$=\frac{4}{\pi^{1/4}}\left(\frac{Z\beta}{a_0}\right)^{3/2}\int_0^\infty r^2 dr e^{-\beta^2 r^2/2} e^{-Zr/a_0}$$

$$=\frac{4}{\pi^{1/4}}\left(\frac{Z\beta}{a_0}\right)^{3/2}\left(-2\frac{d}{d\beta^2}\right)\int_0^\infty dr e^{-\beta^2 r^2/2} e^{-Zr/a_0}$$

The integral cannot be done in closed form, but it can be discussed for large and small $a_0\beta$.

12. It follows from $\langle\frac{d}{dt}(\mathbf{p}\bullet\mathbf{r})\rangle = 0$ that $\langle[H,\mathbf{p}\bullet\mathbf{r}]\rangle = 0$

Now

$$[\frac{1}{2m}p_i p_i + V(r), x_j p_j] = \frac{1}{m}(-i\hbar)p^2 + i\hbar x_j\frac{\partial V}{\partial x_j} = -i\hbar\left(\frac{p^2}{m} - \mathbf{r}\bullet\nabla\mathbf{V(r)}\right)$$

As a consequence

$$\langle\frac{\mathbf{p}^2}{m}\rangle = \langle\mathbf{r}\bullet\nabla\mathbf{V}(r)\rangle$$

If

$$V(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r}$$

then

$$\langle\mathbf{r}\bullet\nabla V(r)\rangle = \langle\frac{Ze^2}{4\pi\varepsilon_0 r}\rangle$$

so that

$$\langle T\rangle = \frac{1}{2}\langle\frac{Ze^2}{4\pi\varepsilon_0 r}\rangle = -\frac{1}{2}V(r)\rangle$$

13. The radial equation is

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)R(r)+\frac{2m}{\hbar^2}\left(E-\frac{1}{2}m\omega^2r^2-\frac{l(l+1)\hbar^2}{2mr^2}\right)R(r)=0$$

With a change of variables to $\rho=\sqrt{\frac{m\omega}{\hbar}}r$ and with $E=\lambda\hbar\omega/2$ this becomes

$$\left(\frac{d^2}{d\rho^2}+\frac{2}{\rho}\frac{d}{d\rho}\right)R(\rho)+\left(\lambda-\rho^2-\frac{l(l+1)}{\rho^2}\right)R(\rho)=0$$

We can easily check that the large ρ behavior is $e^{-\rho^2/2}$ and the small ρ behavior is ρ^l. The function $H(\rho)$ defined by

$$R(\rho)=\rho^l e^{-\rho^2/2}H(\rho)$$

obeys the equation

$$\frac{d^2H(\rho)}{d\rho^2}+2\left(\frac{l+1}{\rho}-\rho\right)\frac{dH(\rho)}{d\rho}+(\lambda-3-2l)H(\rho)=0$$

Another change of variables to $y=\rho^2$ yields

$$\frac{d^2H(y)}{dy^2}+\left(\frac{l+3/2}{y}-1\right)\frac{dH(y)}{dy}+\frac{\lambda-2l-3}{4y}H(y)=0$$

This is the same as Eq. (8-27), if we make the replacement

$$2l\rightarrow 2l+3/2$$
$$\lambda-1\rightarrow\frac{\lambda-2l-3}{4}$$

This leads to the result that

$$\lambda=4n_r+2l+3$$

or, equivalently

$$E=\hbar\omega(2n_r+l+3/2)$$

While the solution is $L_a^{(b)}(y)$ with $a=n_r$ and $b=(2l+3)/4$