CHAPTER 9

1. With $A^{+} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{pmatrix}$

we have

$$(A^{+})^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{12} & 0 & 0 \end{pmatrix}$$

It follows that

$$(A^{+})^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{12} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sqrt{3.2.1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{4.3.2} & 0 & 0 & 0 \end{pmatrix}$$

The next step is obvious: In the 5 x 5 format, there is only one entry in the bottom left-most corner, and it is $\sqrt{4.3.2.1}$.

2. [The reference should be to Eq. (6-36) instead of Eq. (6-4)

$$x = \sqrt{\frac{\hbar}{2m\omega}}(A + A^{+}) = \sqrt{\frac{\hbar}{2m\omega}}\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{pmatrix}$$

from which it follows that

$$x^2 = \left(\frac{\hbar}{2m\omega}\right)\begin{pmatrix} 1 & 0 & \sqrt{2.1} & 0 & 0 \\ 0 & 3 & 0 & \sqrt{3.2} & 0 \\ \sqrt{2.1} & 0 & 5 & 0 & \sqrt{4.3} \\ 0 & \sqrt{3,2} & 0 & 7 & 0 \\ 0 & 0 & \sqrt{4.3} & 0 & 9 \end{pmatrix}$$

3. The procedure here is exactly the same.

We have

$$p = i\sqrt{\frac{m\hbar\omega}{2}}(A^+ - A) = i\sqrt{\frac{m\hbar\omega}{2}}\begin{pmatrix} 0 & -\sqrt{1} & 0 & 0 & 0 \\ \sqrt{1} & 0 & -\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{4} \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{pmatrix}$$

from which it follows that

$$p^2 = \frac{m\hbar\omega}{2}\begin{pmatrix} 1 & 0 & -\sqrt{2.1} & 0 & 0 \\ 0 & 3 & 0 & -\sqrt{3.2} & 0 \\ -\sqrt{2.1} & 0 & 5 & 0 & -\sqrt{4.3} \\ 0 & -\sqrt{3.2} & 0 & 7 & 0 \\ 0 & 0 & -\sqrt{4.3} & 0 & 9 \end{pmatrix}$$

4. We have

$$u_1 = A^+ u_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

5. We write

$$u_2 = \frac{1}{\sqrt{2!}}(A^+)^2 u_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{12} & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Similarly

$$u_3 = \frac{1}{\sqrt{3!}}(A^+)^3 u_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sqrt{3.2.1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{4.3.2} & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and

$$u_4 = \frac{1}{\sqrt{4!}}(A^+)^4 u_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sqrt{4.3.2.1} & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The pattern is clear. u_n is represented by a column vector with all zeros, except a 1 in the (n+1)-*th* place.

6. (a)

$$\langle H \rangle = \frac{1}{\sqrt{6}}(1 \quad 2 \quad 1 \quad 0)\hbar\omega \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 5/2 & 0 \\ 0 & 0 & 0 & 7/2 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \frac{3}{2}\hbar\omega$$

(b) $$\langle x^2 \rangle = \frac{1}{\sqrt{6}}(1 \quad 2 \quad 1 \quad 0)\frac{\hbar}{2m\omega} \begin{pmatrix} 1 & 0 & \sqrt{2} & 0 \\ 0 & 3 & 0 & \sqrt{6} \\ \sqrt{2} & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2m\omega}(3 + \frac{\sqrt{2}}{3})$$

$$\langle x\rangle = \frac{1}{\sqrt{6}}(1 \quad 2 \quad 1 \quad 0)\sqrt{\frac{\hbar}{2m\omega}}\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}\frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\\0\end{pmatrix} = \sqrt{\frac{\hbar}{2m\omega}}\frac{2}{3}(1+\sqrt{2})$$

$$\langle p^2\rangle = \frac{1}{\sqrt{6}}(1 \quad 2 \quad 1 \quad 0)\frac{m\hbar\omega}{2}\begin{pmatrix} 1 & 0 & -\sqrt{2} & 0 \\ 0 & 3 & 0 & -\sqrt{6} \\ -\sqrt{2} & 0 & 5 & 0 \\ 0 & -\sqrt{6} & 0 & 7 \end{pmatrix}\frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\\0\end{pmatrix} = \frac{m\hbar\omega}{2}(3-\frac{\sqrt{2}}{3})$$

$$\langle p\rangle = \frac{1}{\sqrt{6}}(1 \quad 2 \quad 1 \quad 0)i\sqrt{\frac{m\hbar\omega}{2}}\begin{pmatrix} 0 & -\sqrt{1} & 0 & 0 \\ \sqrt{1} & 0 & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}\frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\\0\end{pmatrix} = 0$$

(c) We get

$$(\Delta x)^2 = \frac{\hbar}{2m\omega}\frac{5}{3}(1-\frac{\sqrt{2}}{3});(\Delta p)^2 = \frac{\hbar}{2m\omega}(3-\frac{\sqrt{2}}{3})$$

$$(\Delta x)(\Delta p) = 2.23\hbar$$

7. Consider

$$\begin{pmatrix} -3 & \sqrt{19/4}e^{i\pi/3} \\ \sqrt{19/4}e^{-i\pi/3} & 6 \end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \lambda\begin{pmatrix}u_1\\u_2\end{pmatrix}$$

Suppose we choose u_1=1. The equations then lead to

$$(\lambda+3)+\sqrt{19/4}e^{i\pi/3}u_2 = 0$$
$$\sqrt{19/4}e^{-i\pi/3} + (6-\lambda)u_2 = 0$$

(a) Dividing one equation by the other leads to

$$(\lambda + 3)(\lambda - 6) = - 19/4$$

The roots of this equation are λ =- 7/2 and λ = 13/2. The values of u_2 corresponding to the two eigenvalues are

$$u_2(-7/2) = \frac{1}{\sqrt{19}}\, e^{-i\pi/3} \quad ; u_2(13/2) = -\sqrt{19}e^{-i\pi/3}$$

(b) The normalized eigenvectors are

$$\frac{1}{\sqrt{20}}\begin{pmatrix} \sqrt{19} \\ -e^{-i\pi/3} \end{pmatrix} \; ; \; \frac{1}{\sqrt{20}}\begin{pmatrix} e^{i\pi/3} \\ \sqrt{19} \end{pmatrix}$$

It is easy to see that these are orthogonal.

(c) The matrix that diagonalizes the original matrix is, according to Eq. (9-55)

$$U = \frac{1}{\sqrt{20}}\begin{pmatrix} 1 & -\sqrt{19}e^{i\pi/3} \\ \sqrt{19}e^{-i\pi/3} & 1 \end{pmatrix}$$

It is easy to check that

$$U^{+}AU = \begin{pmatrix} 13/2 & 0 \\ 0 & -7/2 \end{pmatrix}$$

8. We have, as a result of problem **7**,

$$A = UA_{diag}U^{+}$$

From this we get

$$e^{A} = Ue^{A_{diag}}U^{+} = U\begin{pmatrix} e^{13/2} & 0 \\ 0 & e^{-7/2} \end{pmatrix}U^{+}$$

The rest is rather trivial matrix multiplication.

9, The solution of

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

is equivalent to solving

$$a + b + c + d = \lambda a = \lambda b = \lambda c = \lambda d$$

One solution is clearly $a = b = c = d$ with $\lambda = 4$. The eigenvector is $\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

We next observe that if any two (or more) of a , b , c , d are *not* equal, then $\lambda = 0$. These are the only possibilities, so that we have *three* eigenvalues all equal to zero. The Eigenvectors must satisfy $a + b + c + d = 0$, and they all must be mutually orthogonal. The following choices will work

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}; \ \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}; \ \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

10. An hermitian matrix *A* can always be diagonalized by a particular unitary matrix *U*, such that

$$UAU^{+} = A_{diag}$$

Let us now take traces on both sides: $TrUAU^{+} = TrU^{+}UA = TrA$ while $TrA_{diag} = \sum_{n} a_{n}$

Where the a_n are the eigenvalues of *A*.

11. The product of two *N* x *N* matrices of the form $M = \begin{pmatrix} 1 & 1 & 1 & 1 & \ldots \\ 1 & 1 & 1 & 1 & \ldots \\ 1 & 1 & 1 & 1 & \ldots \\ 1 & 1 & 1 & 1 & \ldots \\ . & . & . & . & \ldots \end{pmatrix}$ is

$\begin{pmatrix} N & N & N & N & \ldots \\ N & N & N & N & \ldots \\ N & N & N & N & \ldots \\ N & N & N & N & \ldots \\ . & . & . & . & \ldots \end{pmatrix}$. Thus $M^2 = N M$. This means that the eigenvalues can only be *N* or zero. Now the sum of the eigenvalues is the trqice of *M* which is *N* (see problem 10). Thus there is one eigenvalue *N* and $(N-1)$ eigenvalues 0.

12. We found that the matrix $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix}$ has the property that

$M_3 = U(L_z/\hbar)U^+$. We may now calculate

$$M_1 \equiv U(L_x/\hbar)U^+ =$$
$$= \frac{1}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$M_2 \equiv U(L_y/\hbar)U^+ =$$
$$= \frac{1}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

We can easily check that

$$[M_1, M_2] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = iM_3$$

This was to be expected. The set M_1, M_2 and M_3 give us another representation of angular momentum matrices.

13. We have $AB = BA$. Now let U be a unitary matrix that diagonalizes A. In our case we have the additional condition that in

$$UAU^+ = A_{diag} = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

all the diagonal elements are different. (We wrote this out for a 4 x 4 matrix)
Consider now

$$U[A,B]U^+ = UAU^+UBU^+ - UBU^+UAU^+ = 0$$

This reads as follows (for a 4 x 4 matrix)

$$\begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}\begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

If we look at the (12) matrix elements of the two products, we get, for example

$$a_1 b_{12} = a_2 b_{12}$$

and since we require that the eigenvalues are all different, we find that $b_{12} = 0$. This argument extends to all off-diagonal elements in the products, so that the only matrix elements in $UBU^=$ are the diagonal elements b_{ii}.

14. If M and M^+ commute, so do the *hermitian* matrices $(M + M^+)$ and $i(M - M^+)$. Suppose we find the matrix U that diagonalizes $(M + M^+)$. Then that same matrix will diagonalize $i(M - M^+)$, provided that the eigenvalues of $M + M^+$ are all

different. This then shows that the same matrix U diagonalizes both M and M^+ separately.
(The problem is not really solved, till we learn how to deal with the situation when the eigenvalues of A in problem **13** are not all different).