CHAPTER 10

1. We need to solve $\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}=\pm\frac{\hbar}{2}\begin{pmatrix}u\\ v\end{pmatrix}$

For the + eigenvalue we have *u* = -*iv* , s that the normalized eigenstate is $\chi_+=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$

The – eigenstate can be obtained by noting that it must be orthogonal to the + state, and this leads to $\chi_-=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$.

2, We note that the matrix has the form

$$\sigma_z\cos\alpha+\sigma_x\sin\alpha\cos\beta+\sigma_y\sin\alpha\sin\beta\equiv\sigma\bullet\mathbf{n}$$
$$\mathbf{n}=(\sin\alpha\cos\beta,\sin\alpha\sin\beta,\cos\alpha)$$

This implies that the eigenvalues must be ± 1. We can now solve

$$\begin{pmatrix}\cos\alpha & \sin\alpha e^{-i\beta}\\ \sin\alpha e^{i\beta} & -\cos\alpha\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}=\pm\begin{pmatrix}u\\ v\end{pmatrix}$$

For the + eigenvalue we have $u\cos\alpha+v\sin\alpha\, e^{-I\beta} = u$. We may rewrite this in the form

$$2v\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}e^{-i\beta}=2u\sin^2\frac{\alpha}{2}$$

From this we get

$$\chi_+=\begin{pmatrix}\cos\frac{\alpha}{2}\\ e^{i\beta}\sin\frac{\alpha}{2}\end{pmatrix}$$

The – eigenstate can be obtained in a similar way, *or* we may use the requirement of orthogonality, which directly leads to

$$\chi_-=\begin{pmatrix}e^{-i\beta}\sin\frac{\alpha}{2}\\ -\cos\frac{\alpha}{2}\end{pmatrix}$$

The matrix $U = \begin{pmatrix} \cos\frac{\alpha}{2} & e^{-i\beta}\sin\frac{\alpha}{2} \\ e^{i\beta}\sin\frac{\alpha}{2} & -\cos\frac{\alpha}{2} \end{pmatrix}$

has the property that

$$U^{+}\begin{pmatrix} \cos\alpha & \sin\alpha e^{-i\beta} \\ \sin\alpha e^{i\beta} & -\cos\alpha \end{pmatrix}U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as is easily checked.

The construction is quite simple.

$$S_z = \hbar\begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix}$$

To construct S_+ we use $(S_+)_{mn} = \hbar\delta_{m,n+1}\sqrt{(l-m+1)(l+m)}$ and get

$$S_+ = \hbar\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We can easily construct $S_- = (S_+)^+$. We can use these to construct

$$S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

and

$$S_y = \frac{i}{2}(S_- - S_+) = \frac{\hbar}{2}\begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix}$$

The eigenstates in the above representation are very simple:

$$\chi_{3/2} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \chi_{1/2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \chi_{-1/2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \chi_{-3/2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

5. We first need the eigenstates of $(3S_x + 4S_y)/5$. The eigenvalues will be $\pm\ \hbar/2$ since the operator is of the form $\mathbf{S} \bullet \mathbf{n}$, where $\mathbf{n}$ is a unit vector (3/5,4/5,0).The equation to be solved is

$$\frac{\hbar}{2}(\frac{3}{5}\sigma_x + \frac{4}{5}\sigma_y)\chi_\pm = \pm\frac{\hbar}{2}\chi_\pm$$

In paricular we want the eigenstate for the –ve eigenvalue, that is, we want to solve

$$\begin{pmatrix} 0 & \dfrac{3-4i}{5} \\ \dfrac{3+4i}{5} & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} u \\ v \end{pmatrix}$$

This is equivalent to $(3\text{-}4i)\, v = -5u$ A normalized state is $\frac{1}{\sqrt{50}}\begin{pmatrix} 3-4i \\ -5 \end{pmatrix}$.

The required probability is the square of

$$\frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \end{pmatrix}\frac{1}{\sqrt{50}}\begin{pmatrix} 3-4i \\ -5 \end{pmatrix} = \frac{1}{\sqrt{250}}(6-8i-5) = \frac{1}{\sqrt{250}}(1-8i)$$

This number is 65/250 = 13/50.

6. The normalized eigenspinor of S_y corresponding to the negative eigenvalue was found in problem 1. It is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$. The answer is thus the square of

$$\frac{1}{\sqrt{65}}\begin{pmatrix}4 & 7\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}=\frac{1}{\sqrt{130}}(4-7i)$$

which is 65/130 = 1/2.

7. We make use of $\sigma_x\sigma_y = i\sigma_z = -\sigma_y\sigma_x$ and so on, as well as $\sigma_x^2 = 1$ and so on, to work out

$$\begin{aligned}
&(\sigma_x A_x + \sigma_y A_y + \sigma_z A_z)(\sigma_x B_x + \sigma_y B_y + \sigma_z B_z)\\
&= A_x B_x + A_y B_y + A_z B_z + i\sigma_z(A_x B_y - A_y B_x) + i\sigma_y(A_z B_x - A_x B_z) + i\sigma_x(A_y B_z - A_z B_y)\\
&= \mathbf{A}\bullet\mathbf{B} + i\sigma\bullet\mathbf{A}\times\mathbf{B}
\end{aligned}$$

8. We may use the material in Eq. (10-26,27)., so that at time *T*, we start with

$$\psi(T)=\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\omega T}\\ e^{i\omega T}\end{pmatrix}$$

with $\omega = egB/4m_e$. This now serves as an initial state for a spin 1/2 particle placed in a magnetic field pointing in the *y* direction. The equation for ψ is according to Eq. (10-23)

$$i\frac{d\psi(t)}{dt}=\omega\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\psi(t)$$

Thus with $\psi(t)=\begin{pmatrix}a(t)\\ b(t)\end{pmatrix}$ we get $\frac{da}{dt}=-\omega b; \frac{db}{dt}=\omega a$. The solutions are in general

$$\begin{aligned}
a(t) &= a(T)\cos\omega(t-T) - b(T)\sin\omega(t-T)\\
b(t) &= b(T)\cos\omega(t-T) + a(T)\sin\omega(t-T)
\end{aligned}$$

We know that $a(T)=\frac{e^{-i\omega T}}{\sqrt{2}}; b(T)=\frac{e^{i\omega T}}{\sqrt{2}}$

So that

$$\psi(2T)=\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\omega T}\cos\omega T - e^{i\omega T}\sin\omega T\\ e^{i\omega T}\cos\omega T + e^{-i\omega T}\sin\omega T\end{pmatrix}$$

The amplitude that a measurement of S_x yields $\hbar/2$ is

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\omega T}\cos\omega T - e^{i\omega T}\sin\omega T \\ e^{i\omega T}\cos\omega T + e^{-i\omega T}\sin\omega T\end{pmatrix} =$$
$$= \left(\cos^2\omega T - i\sin^2\omega T\right)$$

Thus the probability is $P = \cos^4\omega T + \sin^4\omega T = \frac{1}{2}(1+\cos^2 2\omega T)$

9. If we set an arbitrary matrix $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$ equal to $A + \sigma \bullet \mathbf{B} = \begin{pmatrix}A + B_z & B_x - iB_y \\ B_x + iB_y & A - B_z\end{pmatrix}$

we see that allowing A, B_x … to be complex we can match all of the $\alpha,\beta,\ldots$.

(b) If the matrix $M = A + \sigma \bullet \mathbf{B}$ is to be unitary, then we require that

$$(A + \sigma \bullet \mathbf{B})(A^* + \sigma \bullet \mathbf{B}^*) =$$
$$|A|^2 + A\sigma \bullet \mathbf{B}^* + A^*\sigma \bullet \mathbf{B} + \mathbf{B} \bullet \mathbf{B}^* + i\sigma \bullet \mathbf{B} \times \mathbf{B}^* = \mathbf{1}$$

which can be satisfied if

$$|A|^2 + |B_x|^2 + |B_y|^2 + |B_z|^2 = 1$$
$$AB_x^* + A^*B_x + i(B_yB_z^* - B_y^*B_z) = 0$$
$$\ldots\ldots\ldots$$

If the matrix M is to be hermitian, we must require that A and all the components of **B** be real.

10. Here we make use of the fact that $(\sigma \bullet \mathbf{a})(\sigma \bullet \mathbf{a}) = \mathbf{a} \bullet \mathbf{a} \equiv a^2$ in the expansion

$$e^{i\sigma\bullet\mathbf{a}} = 1 + i(\sigma \bullet \mathbf{a}) + \frac{i^2}{2!}(\sigma \bullet \mathbf{a})^2 + \frac{i^3}{3!}(\sigma \bullet \mathbf{a})^3 + \frac{i^4}{4!}(\sigma \bullet \mathbf{a})^4 + \ldots$$
$$= 1 - \frac{1}{2!}a^2 + \frac{1}{4!}(a^2)^2 + \ldots + i\sigma \bullet \hat{\mathbf{a}}(a - \frac{a^3}{3!} + \ldots)$$
$$= \cos a + i\sigma \bullet \hat{\mathbf{a}}\sin a = \cos a + i\sigma \bullet \mathbf{a}\frac{\sin a}{a}$$

11. We begin with the relation

$$\mathbf{S}^2 = \left(\frac{\hbar}{2}\sigma_1 + \frac{\hbar}{2}\sigma_2\right)^2 = \frac{\hbar^2}{4}\left(\sigma_1^2 + \sigma_2^2 + 2\sigma_1 \bullet \sigma_2\right)$$

from which we obtain $\sigma_1 \bullet \sigma_2 = 2S(S+1) - 3$. This = -3 for a singlet and +1 for a triplet state.
We now choose $\hat{\mathbf{e}}$ to point in the z direction, so that the first term in S_{12} is equal to $3\sigma_{1z}\sigma_{2z}$.

(a) for a singlet state the two spins are always in opposite directions so that the first term is –6 and the second is +3. Thus

$$S_{12}X_{singlet} = 0$$

(b) For a triplet the first term is +1 when S_z = 1 and S_z = -1 and –1 when S_z =0. This means that S_{12} acting on a triplet state in the first case is 3-1= 2, and in the second case it is –3-1= - 4. Thus

$$(S_{12} - 2)(S_{12} + 4)X_{triplet} = 0$$

12. The potential may be written in the form

$$V(r) = V_1(r) + V_2(r)S_{12} + V_3(r)[2S(S+1) - 3]$$

For a singlet state S_{12} has expectation value zero, so that

$$V(r) = V_1(r) - 3V_3(r)$$

For the triplet state S_{12} has a value that depends on the z component of the total spin. What may be relevant
for a potential energy is an average, assuming that the two particles have equal probability of being in any one of the three S_z states. In that case the average value of S_z is (2+2-4)/3= 0

13. (a) It is clear that for the singlet state, $\psi_{singlet} = \frac{1}{\sqrt{2}}(\chi_+^{(1)}\chi_-^{(2)} - \chi_-^{(1)}\chi_+^{(2)})$, if one of the electrons is in the "up" state, the other must be in the "down" state.

(b). Suppose that we denote the eigenstates of S_y by $\xi_\pm$. These are, as worked out in problem **1**,

$$\xi_+^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}; \xi_-^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$$

The spinors for particle (1) may be expanded in terms of the $\xi_\pm$ thus:

$$\chi_+ = \begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\left(\xi_+ + \xi_-\right)$$

$$\chi_- = \begin{pmatrix}0\\1\end{pmatrix} = \frac{i}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}\right) = \frac{i}{\sqrt{2}}(\xi_+ - \xi_-)$$

Similarly, for particle (2), we want to expand the spinors in terms of the $\eta_\pm$, the eigenstates of S_x

$$\eta_+^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \;;\; \eta_-^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

thus

$$\chi_+ = \begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\left(\eta_+ + \eta_-\right)$$

$$\chi_- = \begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\left(\eta_+ - \eta_-\right)$$

We now pick out, in the expansion of the singlet wave function the coefficient of $\xi_+^{(1)}\eta_+^{(2)}$ and take its absolute square.Some simple algebra shows that it is

$$\left|\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\frac{1-i}{\sqrt{2}}\right|^2 = \frac{1}{4}$$

9. The state is $(\cos\alpha_1\chi_+^{(1)} + \sin\alpha_1 e^{i\beta_1}\chi_-^{(1)})(\cos\alpha_2\chi_+^{(2)} + \sin\alpha_2 e^{i\beta_2}\chi_-^{(2)})$. We need to calculate the scalar product of this with the three triplet wave functions of the two-electgron system. It is easier to calculate the probability that the state is found in a singlet state, and then subtract that from unity.

The calculation is simple

$$\langle\frac{1}{\sqrt{2}}(\chi_+^{(1)}\chi_-^{(2)} - \chi_-^{(1)}\chi_+^{(2)})\,|\,(\cos\alpha_1\chi_+^{(1)} + \sin\alpha_1 e^{i\beta_1}\chi_-^{(1)})(\cos\alpha_2\chi_+^{(2)} + \sin\alpha_2 e^{i\beta_2}\chi_-^{(2)})\rangle$$

$$= \frac{1}{\sqrt{2}}(\cos\alpha_1\sin\alpha_2 e^{i\beta_2} - \sin\alpha_1 e^{i\beta_1}\cos\alpha_2)$$

The absolute square of this is the singlet probability. It is

$$P_s = \frac{1}{2}(\cos^2\alpha_1\sin^2\alpha_2 + \cos^2\alpha_2\sin^2\alpha_1 + 2\sin\alpha_1\cos\alpha_1\sin\alpha_1\cos\alpha_2\cos(\beta_1 - \cos\beta_2))$$

and

$$P_t = 1 - P_s$$

14. We use **J = L +S** so that $\mathbf{J}^2 = \mathbf{L}^2 + \mathbf{S}^2 + 2\mathbf{L}.\mathbf{S}$, from which we get

$$\mathbf{L} \bullet \mathbf{S} = \frac{1}{2}\left[J(J+1) - L(L+1) - 2\right]$$

since $S = 1$. Note that we have taken the division by $\hbar^2$ into account. For $J = L + 1$ this takes on the value L; for $J = L$, it takes on the value –1, and for $J = L - 1$ it is $-L - 1$. We therefore find

$$J = L + 1: \quad V = V_1 + LV_2 + L^2 V_3$$

$$J = L \qquad V = V_1 - V_2 + V_3$$

$$J = L - 1 \quad V = V_1 - (L+1)V_2 + (L+1)^2 V_3$$