CHAPTER 11

1. The first order contribution is

$$E_n^{(1)} = \lambda\langle n \mid x^2 \mid n\rangle = \lambda\left(\sqrt{\frac{\hbar}{2m\omega}}\right)^2 \langle n \mid (A + A^+)(A + A^+) \mid n\rangle$$

To calculate the matrix element $\langle n \mid A^2 + AA^+ + A^+A + (A^+)^2 \mid n\rangle$ we note that

$A^+ \mid n\rangle = \sqrt{n+1} \mid n+1\rangle$; $\langle n \mid A = \sqrt{n+1}\langle n+1 \mid$ so that (1) the first and last terms give

zero, and the second and third terms yield $(n + 1) + (n – 1)=2n$. Thus the first order shift is

$$E_n^{(1)} = \lambda\left(\frac{\hbar}{m\omega} n\right)$$

The second order calculation is quite complicated. What is involved is the calculation of

$$E_n^{(2)} = \lambda^2\left(\frac{\hbar}{2m\omega}\right)^2 \sum_{m\neq n} \frac{\langle n \mid (A + A^+)^2 \mid m\rangle\langle m \mid (A + A^+)^2 \mid n\rangle}{\hbar\omega(n - m)}$$

This is manageable but quite messy. The suggestion is to write

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 + \lambda x^2$$

This is just a simple harmonic oscillator with frequency

$$\omega^* = \sqrt{\omega^2 + 2\lambda / m} = \omega + \frac{\lambda}{\omega m} - \frac{1}{2}\frac{\lambda^2}{\omega^3 m^2} + \ldots$$

Whose spectrum is

$$E_n = \hbar\omega^*(n + \frac{1}{2}) = \hbar\omega(n + \frac{1}{2}) + \frac{\lambda\hbar}{\omega m}(n + \frac{1}{2}) - \frac{\lambda^2\hbar}{2\omega^3 m^2}(n + \frac{1}{2}) + \ldots$$

The extra factor of 1/2 that goes with each n is the zero-point energy. We are only interested in the change in energy of a given state $|n\rangle$ and thus subtract the zero-point energy to each order of λ. Note that the first order λ calculation is correct.

2. The eigenfunction of the rotator are the spherical harmonics. The first order energy shift for $l = 1$ states is given by

$$\Delta E = \langle 1,m \mid E\cos\theta \mid 1,m\rangle = E\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \cos\theta \mid Y_{1.m} \mid^2$$

For $m = \pm 1$, this becomes

$$2\pi E \int_0^{\pi} \sin\theta d\theta \cos\theta \left(\frac{3}{8\pi}\right) \sin^2\theta = \frac{3E}{4}\int_{-1}^{1} duu(1-u^2) = 0$$

The integral for $m = 0$ is also zero. This result should have been anticipated. The eigenstates of $\mathbf{L}^2$ are also eigenstates of parity. The perturbation $\cos\theta$ is *odd* under the reflection **r** → **- r** and therefore the expectation value of an odd operator will always be zero. Since the perturbation represents the interaction with an electric field, our result states that a symmetric rotator does not have a permanent electric dipole moment.

The second order shift is more complicated. What needs to be evaluated is

$$\Delta E^{(2)} = E^2 \sum_{L,M(L\neq 1)} \frac{|\langle 1,m \mid \cos\theta \mid L,M\rangle|^2}{E_1 - E_L}$$

with $E_L = \frac{\hbar^2}{2I} L(L+1)$. The calculation is simplified by the fact that only $L = 0$ and $L = 2$ terms contribute. This can easily be seen from the table of spherical harmonics. For $L = 1$ we saw that the matrix element vanishes. For the higher values we see that $\cos\theta Y_{1,\pm 1} \propto Y_{2,\pm 1}$ and $\cos\theta Y_{1,0} \propto aY_{2,0} + bY_{0,0}$. The orthogonality of the spherical harmonics for different values of L takes care of the matter. Note that because of the ϕ integration, for $m = \pm 1$ only the L = 2 ,M = ± 1 term contributes, while for the $m = 0$ term, there will be contributions from $L = 0$ and $L = 2$, $M = 0$. Some simple integrations lead to

$$\Delta E^{(2)}_{m=\pm 1} = -\frac{2IE^2}{\hbar^2}\frac{1}{15};\ \Delta E^{(2)}_{m=0} = -\frac{2IE^2}{\hbar^2}\frac{1}{60}$$

3. To lowest order in V_0 the shift is given by

$$\Delta E = \left(\sqrt{\frac{2}{L}}\right)^2 \frac{V_0}{L}\int_0^L dxx\sin^2\frac{n\pi x}{L}$$

$$= \frac{2V_0}{L^2}\left(\frac{L}{\pi}\right)^2 \int_0^{\pi} duu\sin^2 nu = \frac{V_0}{\pi^2}\int_0^{\pi} duu(1-\cos 2nu) = \frac{1}{2}V_0$$

The result that the energy shift is just the value of the perturbation at the mid-point is perhaps not surprising, given that the square of the eigenfunctions do not, on the average, favor one side of the potential over the other.

4. The matrix $\begin{pmatrix} E & \lambda & 0 & 0 \\ \lambda & E & 0 & 0 \\ 0 & 0 & 2E & \sigma \\ 0 & 0 & \sigma & 0 \end{pmatrix}$ consists of two boxes which can be diagonalized separately. The upper left hand box involves solving $\begin{pmatrix} E & \lambda \\ \lambda & E \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \eta\begin{pmatrix} u \\ v \end{pmatrix}$

The result is that the eigenvalues are $\eta = E \pm \lambda$. The corresponding eigenstates are easily worked out and are $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$ for the two cases.

For the lower left hand box we have to solve $\begin{pmatrix} 2E & \sigma \\ \sigma & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \xi\begin{pmatrix} a \\ b \end{pmatrix}$. Here we find that the eigenvalues are $\xi = E \pm \sqrt{E^2 + \sigma^2}$. The corresponding eigenstates are

$$N\begin{pmatrix} \sigma \\ -E \pm \sqrt{E^2 + \sigma^2} \end{pmatrix} \text{ respectively, with } \frac{1}{N^2} = \sigma^2 + (-E \pm \sqrt{E^2 + \sigma^2})^2.$$

5. The *change* in potential energy is given by

$$V_1 = -\frac{3e^2}{8\pi\varepsilon_0 R^3}\left(R^2 - \frac{1}{3}r^2\right) + \frac{e^2}{4\pi\varepsilon_0 r} \qquad r \le R$$
$$= 0 \qquad \textit{elsewhere}$$

Thus

$$\Delta E = \int d^3 r\, \psi_{nl}^*(\mathbf{r}) V_1 \psi_{nl}(\mathbf{r}) \; = \int_0^R r^2 dr V_1 R_{nl}^2(r)$$

We may now calculate this for various states.

$$n = 1 \quad \Delta E_{10} = 4\left(\frac{Z}{a_0}\right)^3 \int_0^R r^2 dr e^{-2Zr/a_0}\left(-\frac{3e^2}{8\pi\varepsilon_0 R^3}\left(R^2 - \frac{1}{3}r^2\right) + \frac{e^2}{4\pi\varepsilon_0 r}\right)$$

With a change of variables to $x = r/Za_0$ and with $\rho = ZR/a_0$ this becomes

$$\Delta E_{10} = 4\left(\frac{Ze^2}{4\pi\varepsilon_0 a_0}\right)\int_0^\rho x^2 dx\left(-\frac{3}{2\rho} + \frac{x^2}{2\rho^3} + \frac{1}{x}\right)e^{-2x}$$

Since $x << 1$ we may approximate $e^{-2x} \approx 1 - 2x$, which simplifies the integrals. What results is

$$\Delta E_{10} = \left(\frac{Ze^2}{4\pi\varepsilon_0 a_0}\right)\left(\frac{4}{10}\rho^2 + ...\right)$$

A similar calculation yields

$$\Delta E_{20} = \frac{1}{2}\left(\frac{Ze^2}{4\pi\varepsilon_0 a_0}\right)\int_0^\rho x^2 dx(1-x)^2\left(-\frac{3}{2\rho} + \frac{x^2}{2\rho^3} + \frac{1}{x}\right)e^{-x} \approx \left(\frac{Ze^2}{4\pi\varepsilon_0 a_0}\right)\left(\frac{1}{20}\rho^2 + ...\right)$$

and

$$\Delta E_{21} = \frac{1}{24}\left(\frac{Ze^2}{4\pi\varepsilon_0 a_0}\right)\int_0^\rho x^2 dx x^2\left(-\frac{3}{2\rho} + \frac{x^2}{2\rho^3} + \frac{1}{x}\right)e^{-x} \approx \left(\frac{Ze^2}{4\pi\varepsilon_0 a_0}\right)\left(\frac{1}{1120}\rho^4 + ...\right)$$

6. We need to calculate $\lambda\langle 0 | x^4 | 0\rangle$. One way of proceeding is to use the expression

$$x = \sqrt{\frac{\hbar}{2m\omega}}(A + A^+)$$

Then

$$\lambda\langle 0 | x^4 | 0\rangle = \lambda\left(\frac{\hbar}{2m\omega}\right)^2 \langle 0 | (A + A^+)(A + A^+)(A + A^+)(A + A^+) | 0\rangle$$

The matrix element is

$$\begin{aligned}
&\langle 0 | (A + A^+)(A + A^+)(A + A^+)(A + A^+) | 0\rangle = \\
&\langle 0 | A^+(A + A^+)(A + A^+)A^+ | 0\rangle = \\
&\langle 1 | (A + A^+)(A + A^+) | 1\rangle = \\
&\left[\langle 0 | + \sqrt{2}\langle 2 |\right]\left[| 0\rangle + \sqrt{2} | 2\rangle\right] = 3
\end{aligned}$$

Thus the energy shift is $\Delta E = 3\lambda\left(\frac{\hbar}{2m\omega}\right)^2$

It is easy to see that the same result is obtained from

$$\int_{-\infty}^{\infty} dx(\lambda x^4)\left[\left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{-m\omega x^2/2\hbar}\right]^2$$

7. The first order perturbation shift is

$$\Delta E_n = \frac{2\varepsilon}{b}\int_0^b dx \sin\frac{\pi x}{b}\left(\sin\frac{n\pi x}{b}\right)^2$$
$$= \frac{2\varepsilon}{\pi}\int_0^\pi du \sin u(\sin nu)^2$$
$$= \frac{2\varepsilon}{\pi}\left(1+\frac{1}{4n^2-1}\right)$$

8. It follows from $[p,x] = -i\hbar$ that

$$-i\hbar = \langle a \mid px - xp \mid a\rangle =$$
$$= \sum_n \{\langle a \mid p \mid n\rangle\langle n \mid x \mid a\rangle - \langle a \mid x \mid n\rangle\langle n \mid p \mid a\rangle\}$$

Now

$$\langle a \mid p \mid n\rangle = m\langle a \mid \frac{dx}{dt} \mid n\rangle = \frac{im}{\hbar}\langle a \mid Hx - xH \mid n\rangle = \frac{im}{\hbar}(E_a - E_n)\langle a \mid x \mid n\rangle$$

Consequently

$$\langle n \mid p \mid a\rangle = \langle a \mid p \mid n\rangle^* = -\frac{im}{\hbar}(E_a - E_n)\langle n \mid x \mid a\rangle$$

Thus

$$-i\hbar = \sum_n \frac{2im}{\hbar}(E_a - E_n)\langle a \mid x \mid n\rangle\langle n \mid x \mid a\rangle$$

from which it follows that

$$\sum_n (E_n - E_a) \mid \langle a \mid x \mid n\rangle \mid^2 = \frac{\hbar^2}{2m}$$

9. For the harmonic oscillator, with $|a\rangle = |0\rangle$, we have

$$\langle n \mid x \mid 0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n \mid A^+ \mid 0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\delta_{n,1}$$

This means that in the sum rule, the left hand side is

$$\hbar\omega\left(\frac{\hbar}{2m\omega}\right)=\frac{\hbar^2}{2m}$$

as expected.

10. For the $n = 3$ Stark effect, we need to consider the following states:

$l = 2:\ m_l = 2,1,0,-1,-2$
$l = 1:\ m_l = 1,0,-1$
$l = 0:\ m_l = 0$

In calculating matrix element of z we have selection rules $\Delta l = 1$ (parity forbids $\Delta l = 0$) and, since we are dealing with z, also $\Delta m_l = 0$. Thus the possible matrix elements that enter are

$$\langle 2,1\,|\,z\,|\,1,1\rangle = \langle 2,-1\,|\,z\,|\,1,-1\rangle \equiv A$$
$$\langle 2,0\,|\,z\,|\,1,0\rangle \equiv B$$
$$\langle 1,0\,|\,z\,|\,0,0\rangle \equiv C$$

The matrix to be diagonalized is

$$\left(\begin{array}{cc|ccc|cc}
0 & A & 0 & 0 & 0 & 0 & 0 \\
A & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & B & 0 & 0 & 0 \\
0 & 0 & B & 0 & C & 0 & 0 \\
0 & 0 & 0 & C & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & A \\
0 & 0 & 0 & 0 & 0 & A & 0
\end{array}\right)$$

The columns and rows are labeled by (2,1),(1,1) (2,0) (1,0),(0,0),(2,-1), (1,-1).

The problem therefore separates into three different matrices. The eigenvalues of the submatrices that couple the (2,1) and (1,1) states, as well as those that couple the (2,-1) and (1,-1) states are

$$\lambda = \pm A$$

where

$$A = \int d\Omega Y_{21}^{*}\cos\theta Y_{11}\int_0^\infty r^2 dr R_{32}(r) r R_{31}(r)$$

The mixing among the $m_l = 0$ states involves the matrix $\begin{pmatrix} 0 & B & 0 \\ B & 0 & C \\ 0 & C & 0 \end{pmatrix}$

Whose eigenvalues are $\lambda = 0,\ \pm\sqrt{B^2 + C^2}$ .. Here

$$B = \int d\Omega Y_{20}^* \cos\theta Y_{10} \int_0^\infty r^2 dr R_{32}(r) r R_{31}(r)$$
$$C = \int d\Omega Y_{10}^* \cos\theta Y_{00} \int_0^\infty r^2 dr R_{31}(r) r R_{30}(r)$$

The eigenstates of the *A* submatrices are those of σ_x, that is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$. The eigenstates of the central 3 x 3 matrix are

$$\frac{1}{\sqrt{B^2 + C^2}}\begin{pmatrix} C \\ 0 \\ -B \end{pmatrix} ; \frac{1}{\sqrt{2(B^2 + C^2)}}\begin{pmatrix} B \\ \pm\sqrt{B^2 + C^2} \\ C \end{pmatrix}$$

with the first one corresponding to the λ = 0 eigenvalue.

11. For a one-dimensional operator (labeled by the *x* variable) we introduced the raising and lowering operators A^+ and *A*. We were able to write the Hamiltonian in the form

$$H_x = \hbar\omega(A^+A + \frac{1}{2})$$

We now do the same thing for the harmonic oscillator labeled by the *y* variable. The raising and lowering operators will be denoted by B^+ and *B*, with

$$H_y = \hbar\omega(B^+B + \frac{1}{2})$$

The eigenstates of $H_x + H_y$ are

$$|m,n\rangle = \frac{(A^+)^n}{\sqrt{n!}} \frac{(B^+)^m}{\sqrt{m!}} |0,0\rangle$$

where the ground state has the property that $A\,|0,0\rangle = B\,|0.0\rangle = 0$

The perturbation may be written in the form

$$H_1 = 2\lambda xy = \frac{\hbar\lambda}{m\omega}(A + A^+)(B + B^+)$$

(a) The first order shift of the ground state is

$$\langle 0,0 \mid H_1 \mid 0,0\rangle = 0$$

since every single of the operators $A,\ldots B^+$ has zero expectation value in the ground state.

(b) Consider the two degenerate states |1,0> and |0,1>. The matrix elements of interest to us are

$$<1,0|(A+A^+)(B\ +B^+)|1,0> = <0,1|(A+A^+)(B\ +B^+)|0,1> = 0$$

$$<1,0|(A+A^+)(B\ +B^+)|0,1> = <0,1|(A+A^+)(B\ +B^+)|1,0> = <1,0|(A+A^+)(B\ +B^+)|1,0> = 1$$

Thus in degenerate perturbation theory we must diagonalize the matrix

$$\begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix}$$

where $h = \frac{\lambda\hbar}{m\omega}$. The eigenvalues are $\pm h$, and the degenerate levels are split to

$$E\ = \hbar\omega(1 \pm \frac{\lambda}{m\omega^2})$$

(c) The second order expression is

$$\left(\frac{\lambda\hbar}{m\omega}\right)^2 \sum_{k,n} \frac{|\langle 0,0 \mid (A + A^+)(B + B^+) \mid k,n\rangle|^2}{-\hbar\omega(k+n)} =$$

$$-\frac{\lambda^2\hbar}{m\omega^3} \sum_{k,n} \frac{|\langle 1,1 \mid k,n\rangle|^2}{(k+n)} = -\frac{\lambda^2\hbar}{2m\omega^3}$$

The exact solution to this problem may be found by working with the potential at a classical level. The potential energy is

$$\frac{1}{2}m\omega^2(x^2 + y^2) + \lambda xy$$

Let us carry out a rotation in the $x - y$ plane. The kinetic energy does not change since $\mathbf{p}^2$ is unchanged under rotations. If we let

$$x = x'\cos\theta + y'\sin\theta$$
$$y = -x'\sin\theta + y'\cos\theta$$

then the potential energy, after a little rearrangement, takes the form

$$(\frac{1}{2}m\omega^2 - \lambda\sin 2\theta)x'^2 + (\frac{1}{2}m\omega^2 + \lambda\sin 2\theta)y'^2 + 2\lambda\cos 2\theta x'y'$$

If we choose $\cos 2\theta = 0$, so that $\sin 2\theta = 1$, this reduces to two decoupled harmonic oscillators. The energy is the sum of the two energies. Since

$$\frac{1}{2}m\omega_x^2 = \frac{1}{2}m\omega^2 - \lambda$$
$$\frac{1}{2}m\omega_y^2 = \frac{1}{2}m\omega^2 + \lambda$$

the total energy for an arbitrary excited state is

$$E_{k,n} = \hbar\omega_x(k + \frac{1}{2}) + \hbar\omega_y(n + \frac{1}{2})$$

where

$$\hbar\omega_x = \hbar\omega(1 - 2\lambda / m\omega^2)^{1/2} = \hbar\omega - \frac{\hbar\lambda}{m\omega} - \frac{\hbar\lambda^2}{2m^2\omega^3} + \ldots$$
$$\hbar\omega_y = \hbar\omega(1 + 2\lambda / m\omega^2)^{1/2} = \hbar\omega + \frac{\hbar\lambda}{m\omega} - \frac{\hbar\lambda^2}{2m^2\omega^3} + \ldots$$

12. Thespectral line corresponds to the transition (n = 4,l = 3) → (n = 3,l = 2). We must therefore examine what happens to these energy levels under the perturbation

$$H_1 = \frac{e}{2m}\mathbf{L}\bullet\mathbf{B}$$

We define the z axis by the direction of **B** , so that the perturbation is $\frac{eB}{2m}L_z$.
In the absence of the perturbation the initial state is ($2l$ + 1) = 7-fold degenerate, with the L_z level unchanged, and the others moved up and down in intervals of $eB/2m$.
The final state is 5-fold degenerate, and the same splitting occurs,
with the same intervals. If transitions with zero or ±1 change in $L_z/\hbar$,
the lines are as shown in the figure on the right.

What will be the effect of a constant electric field parallel to **B**?
The additional perturbation is therefore

$$H_2 = -e\mathbf{E}_0 \bullet \mathbf{r} = -eE_0 z$$

and we are only interested in what this does to the energy level structure. The perturbation acts as in the Stark effect. The effect of H_1 is to mix up levels that are degenerate, corresponding to a given m_l value with *different values of l.* For example, the $l = 3$, $m_l = 2$ and the $l = 2$, $m_l = 2$ degeneracy (for $n = 4$)will be split. There will be a further breakdown of degeneracy.

13. The eigenstates of the unperturbed Hamiltonian are eigenstates of σ_z. They are

$\begin{pmatrix}1\\0\end{pmatrix}$ corresponding to $E = E_0$ and $\begin{pmatrix}0\\1\end{pmatrix}$ corresponding to $E = -E_0$.

The first order shifts are given by

$$\begin{pmatrix}1 & 0\end{pmatrix}\lambda\begin{pmatrix}\alpha & u\\ u^* & \beta\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \lambda\alpha$$

$$\begin{pmatrix}0 & 1\end{pmatrix}\lambda\begin{pmatrix}\alpha & u\\ u^* & \beta\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \lambda\beta$$

for the two energy levels.
The second order shift for the upper state involves summing over intermediate states that differ from the initial state. Thus, for the upper state, the intermediate state is just the lower one, and the energy denominator is $E_0 - (-E_0) = 2E_0$. Thus the second order shift is

$$\frac{\lambda^2}{2E_0}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\alpha & u\\ u^* & \beta\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}\alpha & u\\ u^* & \beta\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\lambda^2 |u|^2}{2E_0}$$

For the lower state we get

$$\frac{\lambda^2}{-2E_0}\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}\alpha & u\\ u^* & \beta\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\alpha & u\\ u^* & \beta\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -\frac{\lambda^2 |u|^2}{2E_0}$$

The exact eigenvalues can be obtained from

$$\det\begin{vmatrix}E_0 + \alpha - \varepsilon & u\\ u^* & -E_0 + \beta - \varepsilon\end{vmatrix} = 0$$

This leads to

$$\varepsilon = \lambda\frac{\alpha+\beta}{2} \pm \sqrt{(E_0 - \lambda\frac{\alpha-\beta}{2})^2 + \lambda^2 \mid u \mid^2}$$
$$= \lambda\frac{\alpha+\beta}{2} \pm (E_0 - \lambda\frac{\alpha-\beta}{2})(1 + \frac{1}{2}\frac{\lambda^2 \mid u \mid^2}{E_0^2} + ...$$

(b) Consider now

$$H = \begin{pmatrix} E_0 & u \\ v & -E_0 \end{pmatrix}$$

where we have dropped the α and β terms. The eigenvalues are easy to determine, and they are

$$\varepsilon = \pm\sqrt{E_0^2 + \lambda^2 uv}$$

The eigenstates are written as $\begin{pmatrix} a \\ b \end{pmatrix}$ and they satisfy

$$\begin{pmatrix} E_0 & u \\ v & -E_0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\sqrt{E_0^2 + \lambda^2 uv}\begin{pmatrix} a \\ b \end{pmatrix}$$

For the upper state we find that the un-normalized eigenstate is

$$\begin{pmatrix} \lambda u \\ \sqrt{E_0^2 + \lambda^2 uv} - E_0 \end{pmatrix}$$

For the lower state it is

$$\begin{pmatrix} -\lambda u \\ \sqrt{E_0^2 + \lambda^2 uv} + E_0 \end{pmatrix}$$

The scalar product

$$-\lambda^2 \mid u \mid^2 + \left[(E_0^2 + \lambda^2 uv) - E_0^2\right] = \lambda^2 u(u^* - v) \neq 0$$

which shows that the eigenstates are not orthogonal unless $v = u^*$.