CHAPTER 13

1. (a) electron-proton system $m_r = \dfrac{m_e}{1 + m_e / M_p} = (1 - 5.45 \times 10^{-4}) m_e$

(b) electron-deuteron system $m_r = \dfrac{m_e}{1 + m_e / M_d} = (1 - 2.722 \times 10^{-4}) m_e$

(c) For two identical particles of mass *m*, we have $m_r = \dfrac{m}{2}$

2. One way to see that P_{12} is hermitian, is to note that the eigenvalues ±1 are both real. Another way is to consider

$$\sum_{i,j} \int dx_1 dx_2 \psi^*_{ij}(x_1, x_2) P_{12} \psi_{ij}(x_1, x_2) =$$
$$\sum_{i,j} \int dx_1 dx_2 \psi^*_{ij}(x_1, x_2) \psi_{ji}(x_2, x_1) =$$
$$\sum_{j,i} \int dy_1 dy_2 \psi^*_{ji}(y_2, y_1) \psi_{ij}(y_1, y_2) = \sum_{j,i} \int dy_1 dy_2 (P_{12} \psi_{ij}(y_1, y_2))^* \psi_{ij}(y_1, y_2)$$

3. If the two electrons are in the same spin state, then the spatial wave function must be antisymmetric. One of the electrons can be in the ground state, corresponding to n = 1, but the other must be in the next lowest energy state, corresponding to n = 2. The wave function will be

$$\psi_{ground}(x_1, x_2) = \frac{1}{\sqrt{2}} \left(u_1(x_1) u_2(x_2) - u_2(x_1) u_1(x_2) \right)$$

4. The energy for the *n*-th level is $E_n = \dfrac{\hbar^2 \pi^2}{2ma^2} n^2 \equiv \varepsilon n^2$

Only two electrons can go into a particular level, so that with *N* electrons, the lowest *N*/2 levels must be filled. The energy thus is

$$E_{tot} = \sum_{n=1}^{N/2} 2\varepsilon n^2 \approx 2\varepsilon \frac{1}{3} \left(\frac{N}{2} \right)^3 = \frac{\varepsilon N^3}{12}$$

If *N* is odd, then the above is uncertain by a factor of εN^2 which differs from the above by (12/*N*)ε, a small number if *N* is very large.

5. The problem is one of two electrons interacting with each other. The form of the interaction is a square well potential. The reduction of the two-body problem to a one-particle system is straightforward. With the notation

$x = x_1 - x_2; X = \frac{x_1 + x_2}{2}; P = p_1 + p_2$, the wave function has the form $\psi(x_1, x_2) = e^{iPX}u(x)$, where $u(x)$ is a solution of

$$-\frac{\hbar^2}{m}\frac{d^2u(x)}{dx^2} + V(x)u(x) = Eu(x)$$

Note that we have taken into account the fact that the reduced mass is $m/2$. The spatial interchange of the two electrons corresponds to the exchange $x \rightarrow -x$ .Let us denote the lowest bound state wave function by $u_0(x)$ and the next lowest one by $u_1(x)$. We know that the lowest state has even parity, that means, it is even under the above interchange, while the next lowest state is odd under the interchange. Hence, for the two electrons in a spin singlet state, the spatial symmetry must be even, and therefor the state is $u_0(x)$, while for the spin triplet states, the spatial wave function is odd, that is, $u_1(x)$.

6. With $P = p_1 + p_2;\ \ p = \frac{1}{2}(p_1 - p_2);\ \ X = \frac{1}{2}(x_1 + x_2);\ \ x = x_1 - x_2$, the Hamiltonian becomes

$$H = \frac{P^2}{2M} + \frac{1}{2}M\omega^2 X^2 + \frac{p^2}{2\mu} + \frac{1}{2}\mu\omega^2 x^2$$

with $M = 2m$ the total mass of the system, and $\mu = m/2$ the reduced mass. The energy spectrum is the sum of the energies of the oscillator describing the motion of the center of mass, and that describing the relative motion. Both are characterized by the same angular frequency ω so that the energy is

$$E = \hbar\omega(N + \frac{1}{2}) + \hbar\omega(n + \frac{1}{2}) = \hbar\omega(N + n + 1) \equiv \hbar\omega(\nu + 1)$$

The degeneracy is given by the number of ways the integer ν can be written as the sum of two non-negative integers. Thus, for a given ν we can have

$$(N,n) = (\nu,0),(\nu-1,1),(\nu-2,2),...(1,\nu-1).(0,\nu)$$

so that the degeneracy is $\nu + 1$.

Note that if we treat the system as two independent harmonic oscillators characterized by the same frequency, then the energy takes the form

$$E = \hbar\omega(n_1 + \frac{1}{2}) + \hbar\omega(n_2 + \frac{1}{2}) = \hbar\omega(n_1 + n_2 + 1) \equiv \hbar\omega(\nu + 1)$$

which is the same result, as expected.

7. When the electrons are in the same spin state, the spatial two-electron wave function must be antisymmetric under the interchange of the electrons. Since the two electrons do not interact, the wave function will be a product of the form

$$\frac{1}{\sqrt{2}}(u_n(x_1)u_k(x_2) - u_k(x_1)u_n(x_2))$$

with energy $E = E_n + E_k = \frac{\hbar^2\pi^2}{2ma^2}(n^2 + k^2)$. The lowest state corresponds to $n = 1$, $k = 2$, with $n^2 + k^2 = 5$. The first excited state would normally be the (2,2) state, but this is not antisymmetric, so that we must choose (1,3) for the quantum numbers.

8. The antisymmetric wave function is of the form

$$N\frac{\pi}{\mu^2}\left(e^{-\mu^2(x_1-a)^2/2}e^{-\mu^2(x_2+a)^2/2} - e^{-\mu^2(x_1+a)^2/2}e^{-\mu^2(x_2-a)^2/2}\right)$$
$$= N\frac{\pi}{\mu^2}e^{-\mu^2a^2}e^{-\mu^2(x_1^2+x_2^2)/2}\left(e^{-\mu^2(x_2-x_1)a} - e^{-\mu^2(x_1-x_2)a}\right)$$

Let us introduce the center of mass variable X and the separation x by

$$x_1 = X + \frac{x}{2};\quad x_2 = X - \frac{x}{2}$$

The wave function then becomes

$$\psi = 2N\frac{\pi}{\mu^2}e^{-\mu^2a^2}e^{-\mu^2X^2}e^{-\mu^2x^2/4}\sinh\mu^2ax$$

To normalize, we require

$$\int_{-\infty}^{\infty}dX\int_{-\infty}^{\infty}dx\,|\psi|^2 = 1$$

Some algebra leads to the result that

$$N\frac{\pi}{\mu^2} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1-e^{-2\mu^2a^2}}}$$

The second factor is present because of the overlap. If we want this to be within 1 part in a 1000 away from 1, then we require that $e^{-2(\mu a)^2} \approx 1/500$, i.e. $\mu a = 1.76$, or $a = 0.353$ nm $R_{fi} = \frac{4}{\pi}(Z\alpha)^3\frac{d^2}{a_0^2}\sqrt{\frac{mc^2}{2\Delta E}}\frac{mc^2}{\hbar}$.

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9. Since

$$\psi = \sqrt{2}\,\frac{e^{-(\mu a)^2}}{\sqrt{1-e^{-2(\mu a)^2}}}\,e^{-(\mu X)^2}e^{-(\mu x)^2/4}\sinh\mu^2 ax$$

the probability density for x is obtained by integrating the square of ψ over all X. This is a simple Gaussian integral, and it leads to

$$P(x)\,dx = \frac{2e^{-2(\mu a)^2}}{1-e^{-2(\mu a)^2}}\sqrt{\frac{\pi}{2}}\,\frac{1}{\mu}\,e^{-(\mu x)^2/2}\sinh^2(\mu^2 ax)dx$$

It is obvious that

$$\langle X\rangle = \int_{-\infty}^{\infty} dX X e^{-2(\mu X)^2} = 0$$

since the integrand os an odd function of X.

10. If we denote μx by y, then the relevant quantities in the plot are $e^{-y^2/2}\sinh^2 2y$ and $e^{-y^2/2}\sinh^2(y/2)$.

11. Suppose that the particles are bosons. Spin is irrelevant, and the wave function for the two particles is symmetric. The changes are minimal. The wave function is

$$\psi = 2N\frac{\pi}{\mu^2}e^{-\mu^2 a^2}e^{-\mu^2 X^2}e^{-\mu^2 x^2/4}\cosh\mu^2 xa$$

with

$$N\frac{\mu^2}{\pi} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+e^{-2\mu^2 a^2}}}$$

and

$$P(x) = \frac{2e^{-2\mu^2 a^2}}{1+e^{-2\mu^2 a^2}}\,\frac{\sqrt{\pi}}{\mu\sqrt{2}}\,e^{-\mu^2 x^2/2}\cosh^2(\mu^2 ax)$$

The relevant form is now $P(y) = e^{-y^2/2}\cosh^2\kappa y$ which peaks at $y = 0$ and has extrema at

$-y\cosh\kappa y + 2\kappa\sinh\kappa y = 0$, that is, when

$$\tanh\kappa y = y/2\kappa$$

which only happens if $2\kappa^2 > 1$. Presumably, when the two centers are close together, then the peak occurs in between; if they are far apart, there is a slight rise in the middle, but most of the time the particles are around their centers at ± *a*.

12. The calculation is almost unchanged. The energy is given by

$$E = pc = \frac{\hbar\pi c}{L}\sqrt{n_1^2 + n_2^2 + n_3^2}$$

so that in Eq. (13-58)

$$R^2 = \sqrt{n_1^2 + n_2^2 + n_3^2} = (E_F/\hbar c\pi)^2 L^2$$

Thus

$$N = \frac{\pi}{3}\left(\frac{E_F L}{\pi\hbar c}\right)^3$$

and

$$E_F = \pi\hbar c\left(\frac{3n}{\pi}\right)^{1/3}$$

13. The number of triplets of positive integers $\{n_1,n_2,n_3\}$ such that

$$n_1^2 + n_2^2 + n_3^2 = R^2 = \frac{2mE}{\hbar^2\pi^2}L^2$$

is equal to the numbers of points that lie on an octant of a sphere of radius *R*, within a thickness of $\Delta n = 1$. We therefore need $\frac{1}{8}4\pi R^2 dR$. To translate this into *E* we use $2RdR = (2mL^2/\hbar^2\pi^2)dE$. Hence the degeneracy of states is

$$N(E)dE = 2\times\frac{1}{8}4\pi R(RdR) = L^3\frac{m\sqrt{2m}}{\hbar^3\pi^2}\sqrt{E}\,dE$$

To get the electron density we had to multiply by 2 to take into account that there are two electrons per state.

14. Since the photons are massless, and there are two photon states per energy state, this problem is identical to problem 12. We thus get

$$n_1^2 + n_2^2 + n_3^2 = R^2 = \left(\frac{E}{\hbar\pi c}\right)^2 L^2$$

or $R = EL/\hbar\pi c$. Hence

$$N(E)dE = \frac{1}{8}4\pi R^2 dE = L^3 \frac{E^2}{\hbar^3 c^3 \pi^2} dE$$

15. The eigenfunctions for a particle in a box of sides L_1,L_2, L_3 are of the form of a product

$$u(x,y,z) = \sqrt{\frac{8}{L_1 L_2 L_3}} \sin\frac{n_1\pi x}{L_1} \sin\frac{n_2\pi y}{L_2} \sin\frac{n_3\pi z}{L_3}$$

and the energy for a massless partticle, for which $E = pc$ is

$$E = \hbar c\pi\sqrt{\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2}} = \hbar c\pi\sqrt{\frac{n_1^2 + n_2^2}{a^2} + \frac{n_3^2}{L^2}}$$

Note that $a << L$. thus the low-lying states will have $n_1 = n_2 = 1$, with n_3 ranging from 1 upwards. At some point the two levels $n_1 = 2$, n_2=1 and n_1=1 and $n_2 = 2$ will provide a new "platform" upon which $n_3 = 1,2,3,\ldots$ are stacked. With $a = 1$ nm and $L = 10^3$ nm, for $n_1 = n_2 = 1$ the n_3 values can go up to 10^3 before the new platform starts.

16. For nonrelativistic particles we have

$$E = \frac{\hbar^2}{2m}\left(\frac{n_1^2 + n_2^2}{a^2} + \frac{n_3^2}{L^2}\right)$$

17. We have

$$E_F = \frac{\hbar^2\pi^2}{2M}\left(\frac{3n}{\pi}\right)^{2/3}$$

where M is the nucleon mass, taken to be the same for protons and for neutrons, and where n is the *number density*. Since there are Z protons in a volume $\frac{4\pi}{3} r_0^3 A$, the number densities for protons and neutrons are

$$n_p = \frac{3}{4\pi r_0^3}\frac{Z}{A}; \quad n_n = \frac{3}{4\pi r_0^3}\frac{A-Z}{A}$$

Putting in numbers, we get

$$E_{Fp} = 65\left(\frac{Z}{A}\right)^{2/3} MeV; \;\; E_{Fn} = 65\left(1 - \frac{Z}{A}\right)^{2/3} MeV$$

For A = 208, Z = 82 these numbers become $E_{Fp} = 35 MeV; \;\; E_{Fn} = 47 MeV$.