CHAPTER 16.

1. The perturbation caused by the magnetic field changes the simple harmonic oscillator Hamiltonian H_0 to the new Hamiltonian H

$$H = H_0 + \frac{q}{2m}\mathbf{B}\bullet\mathbf{L}$$

If we choose **B** to define the direction of the *z* axis, then the additional term involves $B\,L_z$. When H acts on the eigenstates of the harmonic oscillator, labeled by $|n_r, l, m_l>$, we get

$$H\,|\,n_r,l,m_l\rangle = \left(\hbar\omega(2n_r + l + \frac{3}{2} + \frac{qB\hbar}{2m}m_l\right)|\,n_r,l,m_l\rangle$$

Let us denote $qB/2m$ by ω_B . Consider the three lowest energy states:

$n_r = 0$, $l = 0$, the energy is $3\hbar\omega\,/\,2$.

$n_r = 0$, $l = 1$ This three-fold degenerate level with unperturbed energy $5\hbar\omega\,/\,2$, splits into three nondegenerate energy levels with energies

$$E = 5\hbar\omega\,/\,2 + \hbar\omega_B\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

The next energy level has quantum numbers $n_r = 2$, $l = 0$ *or* $n_r = 0$, $l = 2$. We thus have a four-fold degeneracy with energy $7\hbar\omega\,/\,2$. The magnetic field splits these into the levels according to the m_l value. The energies are

$$E = 7\hbar\omega\,/\,2 + \hbar\omega_B\begin{pmatrix}2\\1\\0,0\\-1\\-2\end{pmatrix} \qquad n_r = 1,0$$

2, The system has only one degree of freedom, the angle of rotation θ. In the absence of torque, the angular velocity $\omega = d\theta/\mathrm{dt}$ is constant. The kinetic energy is

$$E = \frac{1}{2}Mv^2 = \frac{1}{2}\frac{(M^2v^2R^2)}{MR^2} = \frac{1}{2}\frac{L^2}{I}$$

where $L = MvR$ is the angular momentum, and I the moment of inertia. Extending this to a quantum system implies the replacement of L^2 by the corresponding operator. This suggests that

$$H = \frac{\mathbf{L}^2}{2I}$$

(b) The operator **L** can also be written as **p x R**.
When the system is placed in a constant magnetic field, we make the replacement

$$\mathbf{p} \to \mathbf{p} - q\mathbf{A} = \mathbf{p} - q(-\frac{1}{2}\mathbf{r} \times \mathbf{B}) = \mathbf{p} + \frac{q}{2}\mathbf{r} \times \mathbf{B}$$

The operator **r** represents the position of the particle relative to the axis of rotation, and this is equal to **R**. We may therefore write

$$\mathbf{L} = \mathbf{R} \times \mathbf{p} \to \mathbf{R} \times (\mathbf{p} + \frac{q}{2}\mathbf{R} \times \mathbf{B}) = \mathbf{L} + \frac{q}{2}\left(\mathbf{R}(\mathbf{R} \bullet \mathbf{B}) - R^2\mathbf{B}\right)$$

If we square this, and only keep terms linear in **B** , then it follows from (**R.B) =** 0, that

$$H = \frac{1}{2I}\left(\mathbf{L}^2 - qR^2\mathbf{L} \bullet \mathbf{B}\right) = \frac{\mathbf{L}^2}{2I} - \frac{q}{2M}\mathbf{L} \bullet \mathbf{B} = \frac{\mathbf{L}^2}{2I} - \frac{qB}{2M}L_z$$

The last step is taken because we choose the direction of **B** to define the z axis.

The energy eigenvalues are therefore

$$E = \frac{\hbar^2 l(l+1)}{2I} - \frac{qB\hbar}{2M}m_l$$

where $m_l = l, l-1, l-2, \ldots -l$. Note that the lowest of the levels corresponds to $m_l = l$.

3. In the absence of a magnetic field, the frequency for the transition $n = 3$ to $n = 2$ is determined by

$$2\pi\hbar\nu = \frac{1}{2}mc^2\alpha^2\left(\frac{1}{4} - \frac{1}{9}\right)$$

so that

$$\nu = \frac{mc^2\alpha^2}{4\pi\hbar}\frac{5}{36}$$

The lines with $\Delta m_l = \pm 1$ are shifted upward (and downward) relative to the $\Delta m_l = 0$ (unperturbed) line. The amount of the shift is given by

$$h\Delta\nu = \frac{e\hbar B}{2mc}$$

so that

$$\Delta\nu = \frac{eB}{4\pi mc}$$

Numerically $\nu = 0.4572 \times 10^{15}$ Hz and with $B = 1$ T, $\Delta\nu = 1.40 \times 10^{10}$ Hz. Thus the frequencies are ν and $\nu(1 \pm \Delta\nu/\nu)$. Thus the wavelengths are c/ν and $(c/\nu)(1 \mp \Delta\nu/\nu)$. This leads to the three values $\lambda = 655.713$ nm, with the other lines shifted down/up by 0.02 nm.

4. The Hamiltonian is

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 - q\mathbf{E} \bullet \mathbf{r}$$

Let us choose $\mathbf{E} = (E, 0, 0)$ and $\mathbf{B} = (0, 0, B)$, but now we choose the gauge such that $\mathbf{A} = (0, Bx, 0)$. This leads to

$$H = \frac{1}{2m}\left(p_x^2 + (p_y - qBx)^2 + p_z^2\right) - qEx =$$
$$= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2 - 2qBp_y x + q^2B^2x^2 - 2mqEx)$$

Let us now choose the eigenstate to be a simultaneous eigenstate of H, p_z (with eigenvalue zero) and p_y (with eigenvalue $\hbar k$). Then the Hamiltonian takes the form

$$H = \frac{\hbar^2k^2}{2m} + \frac{1}{2m}p_x^2 + \frac{1}{2m}(qBx - \hbar k - mE/B)^2 - \frac{1}{2m}\left((\hbar k + mE/B)^2\right)$$

This is the Hamiltonian for a shifted harmonic oscillator with a constant energy added on. We may write this in the form

$$H = -\frac{\hbar kE}{B} - \frac{mE^2}{2B^2} + \frac{1}{2}m\left(\frac{q^2B^2}{m^2}\right)\left(x - \frac{\hbar k - mE/B}{qB}\right)^2$$

Thus the energy is

$$E = -\frac{\hbar k E}{B} - \frac{mE^2}{2B^2} + \hbar\left(\frac{qB}{m}\right)(n + \frac{1}{2})$$

with $n = 0,1,2,3,\ldots$

5. We first need to express everything in cylindrical coordinates. Since we are dealing with an infinite cylinder which we choose to be aligned with the z axis,, nothing depends on z, and we only deal with the ρ and ϕ coordinates. We only need to consider the Schrodinger equation in the region $a \le \rho \le b$.

We start with $H = \frac{1}{2m_e}\left(\Pi_x^2 + \Pi_y^2\right)$

where

$$\Pi_x = -i\hbar\frac{\partial}{\partial x} + eA_x;\ \ \Pi_y = -i\hbar\frac{\partial}{\partial y} + eA_y$$

To write this in cylindrical coordinates we use Eq. (16-33) and the fact that for the situation at hand

$$A_x = -sin\varphi\ \ A_\varphi;\ \ A_y = cos\varphi\ \ A_\varphi;\ \ A_\varphi = \frac{\Phi}{2\pi\rho}$$

where Φ is the magnetic flux in the interior region. When all of this is put together, the equation

$$H\psi(\rho,\varphi) = E\,\psi(\rho,\varphi)$$

takes the form

$$E\psi = -\frac{\hbar^2}{2m_e}\left(\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2}\right) - 2i\hbar e\frac{\Phi}{2\pi}\frac{1}{\rho^2}\frac{\partial\psi}{\partial\varphi} + \frac{e^2}{\rho^2}\left(\frac{\Phi}{2\pi}\right)^2\psi$$

To solve this, we use the separation of variables technique. Based on previous experience, we write

$$\psi(\rho,\varphi) = f(\rho)e^{im\varphi}$$

The single-valuedness of the solution implies that $m = 0,\pm1,\pm2,\pm3,\ldots$

With the notation $k^2 = 2m_eE/\hbar^2$ the equation for $f(\rho)$ becomes

$$-k^2 f(\rho) = \frac{d^2 f}{d\rho^2} + \frac{1}{\rho}\frac{df}{d\rho} - \left(m + \frac{e\Phi}{2\pi\hbar}\right)^2 f$$

If we now introduce $z = k\rho$ and $\nu = m + \frac{e\Phi}{2\pi\hbar}$ the equation takes the form

$$\frac{d^2 f(z)}{dz^2} + \frac{1}{z}\frac{df(z)}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) f(z) = 0$$

This is Bessel's equation. The most general solution has the form

$$f(\rho) = AJ_\nu(k\rho) + BN_\nu(k\rho)$$

If we now impose the boundary conditions $f(ka) = f(kb) = 0$ we end up with

$$AJ_\nu(ka) + BN_\nu(ka) = 0$$

and

$$AJ_\nu(kb) + BN_\nu(kb) = 0$$

The two equations can only be satisfied if

$$J_\nu(ka)N_\nu(kb) - J_\nu(kb)N_\nu(ka) = 0$$

This is the *eigenvalue equation,* and the solution *k* clearly depends on the order ν of the Bessel functions, that is, on the flux enclosed in the interior cylinder.