CHAPTER 17

1. We start with Eq. (17-19) . We define $\boldsymbol{k}$ as the z axis. This means that the polarization vector, which is perpendicular to $\boldsymbol{k}$ has the general form

$$\varepsilon^{(\lambda)} = \hat{\mathbf{i}}cos\varphi + \hat{\mathbf{j}}sin\varphi$$

This leads to

$$\mathbf{B} = \nabla \times \mathbf{A} = -i\sqrt{\frac{\hbar}{2\varepsilon_0 \omega V}}k\hat{\mathbf{k}} \times (\hat{\mathbf{i}}cos\varphi + \hat{\mathbf{j}}sin\varphi) = B_0(\hat{\mathbf{j}}cos\varphi - \hat{\mathbf{i}}sin\varphi)$$

We are now interested in

$$M = B_0 \frac{g_p - g_n}{2}\frac{\hbar}{2}\overline{X}_0\{(\sigma_y^{(p)} - \sigma_y^{(n)})cos\varphi - (\sigma_x^{(p)} - \sigma_x^{(n)})sin\varphi\}X_1^m$$

The operators are of the form

$$\sigma_y cos\varphi - \sigma_x sin\varphi = \begin{pmatrix} 0 & -icos\varphi \\ icos\varphi & 0 \end{pmatrix} - \begin{pmatrix} 0 & sin\varphi \\ sin\varphi & 0 \end{pmatrix} = \begin{pmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{pmatrix}$$

It is simple to work out the "bra" part of the scalar product

$$\frac{1}{\sqrt{2}}\left(\overline{\chi}_+^{(p)}\overline{\chi}_-^{(n)} - \overline{\chi}_-^{(p)}\overline{\chi}_+^{(n)}\right)\left[\begin{pmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{pmatrix}_p - \begin{pmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{pmatrix}_n\right]$$

with the help of

$$\overline{\chi}_+\begin{pmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -ie^{-i\varphi} \end{pmatrix} = -ie^{-i\varphi}\overline{\chi}_-$$

and

$$\overline{\chi}_-\begin{pmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{pmatrix} = \begin{pmatrix} ie^{i\varphi} & 0 \end{pmatrix} = ie^{i\varphi}\overline{\chi}_+$$

This implies that the "bra" part is

$$\frac{1}{\sqrt{2}}\left(\bar{\chi}_+^{(p)}\bar{\chi}_-^{(n)} - \bar{\chi}_-^{(p)}\bar{\chi}_+^{(n)}\right)\left[\begin{pmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{pmatrix}_p - \begin{pmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{pmatrix}_n\right] =$$
$$= -\sqrt{2}i(e^{-i\varphi}\bar{\chi}_-^{(p)}\bar{\chi}_-^{(n)} + e^{i\varphi}\bar{\chi}_+^{(p)}\bar{\chi}_+^{(n)})$$
$$= -\sqrt{2}i\left(e^{-i\varphi}\bar{X}_1^{-1} + e^{i\varphi}\bar{X}_1^{1}\right)$$

For the "ket" state we may choose $X_{triplet} = \alpha X_1^1 + \beta X_1^0 + \gamma X_1^{-1}$, and then the matrix element is

$$M = -i\sqrt{2}B_0\frac{g_p - g_n}{2}\frac{\hbar}{2}(e^{i\varphi}\alpha + e^{-i\varphi}\gamma)$$

2. We are interested in finding out for what values of *l*, *m*, the matrix element

$$\frac{1}{2}\langle \ell, m \,|\, (\varepsilon.\mathbf{p})(\mathbf{k}.\mathbf{r}) + (\varepsilon.\mathbf{r})(\mathbf{p}.\mathbf{k}) \,|\, 0,0\rangle$$

does not vanish. We use the technique used in Eq. (17-22) to rewrite this in the form

$$\frac{1}{2}\frac{im_e}{\hbar}\langle \ell, m \,|\, [H_0, \varepsilon.\mathbf{r}](\mathbf{k}.\mathbf{r}) + (\varepsilon.\mathbf{r})[H_0, \mathbf{k}.\mathbf{r}] \,|\, 0.0\rangle =$$
$$\frac{im_e}{2\hbar}\langle \ell, m \,|\, H_0(\varepsilon.\mathbf{r})(\mathbf{k}.\mathbf{r}) - (\varepsilon.\mathbf{r})H_0(\mathbf{k}.\mathbf{r}) + (\varepsilon.\mathbf{r})H_0(\mathbf{k}.\mathbf{r}) - (\varepsilon.\mathbf{r})(\mathbf{k}.\mathbf{r})H_0 \,|\, 0,0\rangle =$$
$$\frac{im_e}{2\hbar}(E_{\ell,m} - E_{0,0})\langle \ell, m \,|\, (\varepsilon.\mathbf{r})(\mathbf{k}.\mathbf{r}) \,|\, 0,0\rangle$$

Let us now choose $\boldsymbol{k}$ to define the *z* axis, so that $\boldsymbol{k}$ = (0,0,*k*). Since $\boldsymbol{\varepsilon}$ is perpendicular to $\boldsymbol{k}$, we may choose it to be represented by $\boldsymbol{\varepsilon}$ = (cosα, sinα, 0). Then, with the usual polar coordinates, we have

$$(\varepsilon.\mathbf{r})(\mathbf{k}.\mathbf{r}) = k(cos\alpha sin\theta cos\phi + sin\alpha sin\theta sin\phi)\cos\theta =$$
$$= ksin\theta cos\theta \cos(\phi - \alpha)$$

This is a linear combination of $Y_{21}(\theta,\phi)$ *and* $Y_{2,-1}(\theta,\phi)$. Thus the angular integral is of the form $\int d\Omega Y_{l,m}^* Y_{2,\pm1} Y_{0,0}$, and since $Y_{0,0}$ is just a number, the integral is proportional to $\delta_{\ell,2}$.

There is also a selection rule that requires *m* = ± 1. This comes about because of our choice of axes.

3. In the transition under consideration, the radial part of the transition rate is unchanged. The only change has to do with the part of the matrix element that deals with the dependence on the polarization of the photon emitted in the transition. Eq. (17-44), for example shows that $\delta_{m,1}$ is multiplied by $\varepsilon_x^2 + \varepsilon_y^2 = 1 - \varepsilon_z^2$ and this factor carries some information about the direction of the photon momentum, even though that does not appear explicitly in the matrix element. We proceed as follows: The direction of the polarization of the initial atomic state defines the *z* axis. Let the photon momentum direction be given by

$$\hat{\mathbf{d}} = \hat{\mathbf{i}}\sin\Theta\cos\Phi + \hat{\mathbf{j}}\sin\Theta\sin\Phi + \hat{\mathbf{k}}\cos\Theta$$

We may define two unit vectors perpendicular to this. For the first one we take $\hat{\mathbf{d}} \times \hat{\mathbf{k}}$, which, after being divided by the sine of the angle between these two vectors, i.e. by $\sin\Theta$, yields

$$\hat{\varepsilon}_1 = -\hat{\mathbf{i}}\sin\Phi + \hat{\mathbf{j}}\cos\Phi$$

The other one is $\hat{\varepsilon}_2 = \hat{\mathbf{d}} \times \hat{\varepsilon}_1$ (two vectors perpendicular to each other), which leads to

$$\hat{\varepsilon}_2 = \hat{\mathbf{i}}\cos\Theta\cos\Phi + \hat{\mathbf{j}}\cos\Theta\sin\Phi - \hat{\mathbf{k}}\sin\Theta$$

In the coordinate system in which $\hat{\mathbf{d}}$ represents the *z* axis, the ε_i vectors represent the *x* and *y* axes, and since the photon polarization must lie in that new $x - y$ plane, we see that the polarization vector has the form

$$\varepsilon = \cos\chi\hat{\varepsilon}_1 + \sin\chi\hat{\varepsilon}_2$$

Thus

$$\varepsilon_z = \hat{\mathbf{k}} \bullet \varepsilon = -\sin\chi\sin\Theta,$$
$$\varepsilon_x = \hat{\mathbf{i}} \bullet \varepsilon = \cos\chi\sin\Phi + \sin\chi\cos\Theta\cos\Phi,$$
$$\varepsilon_y = \hat{\mathbf{j}} \bullet \varepsilon = -\cos\chi\cos\Phi + \sin\chi\cos\Theta\sin\Phi$$

and

$$\varepsilon_x^2 + \varepsilon_y^2 = 1 - \varepsilon_z^2 = 1 - \sin^2\chi\sin^2\Theta$$

Thus the final answer (using Eq. (17-44) is

$$d\Gamma = \frac{\alpha}{2\pi}\frac{\omega^3}{c^2}\frac{2^{15}}{3^{10}}\left(\frac{1}{2}\delta_{m,1}\right)\left(1 - \sin^2\chi\sin^2\Theta\right)d(\cos\Theta)d\Phi$$

The dependence on the polarization appears in the $\sin^2\chi$ term.

4. First of all, we need to recognize what $2p \rightarrow 1s$ means for the harmonic oscillator in three dimensions. The numbers "2" and "1" usually refer to the principal quantum number, e.g $n = n_r + \ell + 1$ for the hydrogen atom. Here the energy spectrum is characterized by $2n_r + \ell + 1$, and it is this combination that we call the principal quantum number. Thus we take the $2p \rightarrow 1s$ transition to mean $(n_r = 0, \ell = 1) \rightarrow (n_r = 0, \ell = 0)$.

To solve this problem we recognized that nothing changes in the angular integration that was done for the $2p \rightarrow 1s$ transition in hydrogen. The only change in the matrix element involves the radial functions. In hydrogen we calculated

$$\int_0^\infty r^3 R_{21}(r) R_{10}(r)\, dr$$

using the radial functions for hydrogen. Here the same integral appears, except that the radial functions are those of the three-dimensional harmonic oscillator. Here, the properly normalized eigenfunctions are

$$R_{10}(r) = \frac{2}{\pi^{1/4}} \left(\frac{m\omega}{\hbar}\right)^{3/2} e^{-m\omega r^2/2\hbar}$$

and

$$R_{21}(r) = \left(\frac{8}{3}\right)^{1/2} \frac{1}{\pi^{1/4}} \left(\frac{m\omega}{\hbar}\right)^{5/4} r e^{-m\omega r^2/2\hbar}$$

Note that these functions appear in the solution to problem 8-13. Given these, the integral that yields the matrix element is straightforward. We have

$$\begin{aligned} M &= \left(\frac{8}{3}\right)^{1/2} \frac{2}{\pi^{1/2}} \left(\frac{m\omega}{\hbar}\right)^2 \int_0^\infty dr r^4 e^{-m\omega r^2/\hbar} = \\ &= \frac{4}{\pi^{1/2}} \left(\frac{2}{3}\right)^{1/2} \left(\frac{m\omega}{\hbar}\right)^2 \left(\frac{\hbar}{m\omega}\right)^{5/2} \frac{1}{2} \int_0^\infty dx x^{3/2} e^{-x} \\ &= \frac{4}{\pi^{1/2}} \left(\frac{2}{3}\right)^{1/2} \left(\frac{m\omega}{\hbar}\right)^2 \left(\frac{\hbar}{m\omega}\right)^{5/2} \frac{1}{2} \frac{3\pi^{1/2}}{4} \end{aligned}$$

The square of this is $\frac{3\hbar}{2m\omega}$. We check that this has the dimensions of a (length)^2 as required. To get the decay rate, we just take the hydrogen result and make the substitution

$$|M_{hydrogen}|^2 = \frac{2^{15}}{3^9} a_0^2 \rightarrow |M|^2 = \frac{3\hbar}{2m\omega}$$

This then leads to the rate

$$R = \frac{4}{9}\alpha\frac{\omega^3}{c^2}\,|\,M\,|^2 = \frac{2\alpha}{3}\left(\frac{\hbar\omega}{mc^2}\right)\omega$$