CHAPTER 19

1. We have

$$M_{fi} = \frac{1}{V}\int d^3r e^{-i\Delta.\mathbf{r}} V(\mathbf{r})$$

If $V(\mathbf{r}) = V(r)$, that is, if the p9otential is central, we may work out the angular integration as follows:

$$M_{fi} = \frac{1}{V}\int_0^\infty r^2 V(r)dr \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta e^{-i\Delta r\cos\theta}$$

with the choice of the vector Δ as defining the z axis. The angular integration yields

$$\int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta e^{-i\Delta r\cos\theta} = 2\pi\int_{-1}^1 d(\cos\theta) e^{-i\Delta r\cos\theta} = \frac{4\pi}{\Delta r}\sin\Delta r$$

so that

$$M_{fi} = \frac{1}{V}\frac{4\pi}{\Delta}\int_0^\infty rdrV(r)\sin\Delta r$$

Note that this is an even function of Δ that is, it is a function of $\Delta^2 = (\mathbf{p}_f - \mathbf{p}_i)^2 / \hbar^2$

2. For the gaussian potential

$$M_{fi} = \frac{1}{V}\frac{4\pi V_0}{\Delta}\int_0^\infty rdr\sin\Delta r e^{-r^2/a^2}$$

Note that the integrand is an even function of r. We may therefore rewrite it as

$$\int_0^\infty rdr\sin\Delta r e^{-r^2/a^2} = \frac{1}{2}\int_{-\infty}^\infty rdr\sin\Delta r e^{-r^2/a^2}$$

The integral on the right may be rewritten as

$$\frac{1}{2}\int_{-\infty}^\infty rdr\sin\Delta r e^{-r^2/a^2} = \frac{1}{4i}\int_{-\infty}^\infty rdr\left(e^{-r^2/a^2+i\Delta r} - c.c)\right)$$

Now

$$\frac{1}{4i}\int_{-\infty}^\infty rdre^{-r^2/a^2+i\Delta r} = \frac{1}{i}\frac{\partial}{\partial\Delta}\int_{-\infty}^\infty dre^{-r^2/a^2+i\Delta r} = -i\frac{\partial}{\partial\Delta}a\sqrt{\pi}e^{-a^2\Delta^2/4} = i\frac{\Delta a^3\sqrt{\pi}}{2}e^{-a^2\Delta^2/4}$$

Subtracting the complex conjugate and dividing by $4i$ gives

$$M_{fi} = \frac{1}{V}\left(a\sqrt{\pi}\right)^3 V_0 e^{-a^2\Delta^2/4}$$

The comparable matrix element for the Yukawa potential is

$$\overline{M}_{fi} = \frac{1}{V}\frac{4\pi}{\Delta}V_Y b\int_0^\infty dr e^{-r/b}\sin\Delta r = \frac{1}{V}4\pi V_Y\frac{b^3}{1+b^2\Delta^2}$$

We can easily check that the matrix elements and their derivatives with respect to Δ^2 at $\Delta = 0$ will be equal if $a = 2b$ and $V_Y = 2\sqrt{\pi}V_0$.

The differential cross section takes its simplest form if the scattering involves the same particles in the final state as in the initial state. The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2}{4\pi^2\hbar^4}\,|U(\Delta)|^2$$

where μ is the reduced mass and $U(\Delta) = VM_{fi}$.

We are interested in the comparison

$$\frac{(d\sigma/d\Omega)_{gauss}}{(d\sigma/d\Omega)_{Yukawa}} = \frac{e^{-2b^2\Delta^2}}{(1+b^2\Delta^2)^{-2}} = (1+X)^2 e^{-2X}$$

where we have introduced the notation $X = b^2x^2$. This ratio, as a function of X, starts out at $X = 0$ with the value of 1, and zero slope, but then it drops rapidly, reaching less than 1% of its initial value when $X = 4$, that is, at $\Delta = 2/b$.

3. We use the hint to write

$$\frac{d\sigma}{d\Omega} = \frac{p^2}{\pi\hbar^2}\frac{d\sigma}{d\Delta^2} = \frac{\mu^2}{4\pi^2\hbar^4}\left|4\pi V_0\frac{b^3}{1+b^2\Delta^2}\right|^2$$

The total cross section may be obtained by integrating this over Δ^2 with the range given by $0 \le \Delta^2 \le 4p^2/\hbar^2$, corresponding to the values of $\cos\theta$ between –1 and + 1.. The integral can actually be done analytically. With the notation $k^2 = p^2/\hbar^2$ the integral is

$$\int_0^{4k^2} d\Delta^2 \frac{1}{(1+b^2\Delta^2)^2} = \frac{1}{b^2}\int_0^{4k^2b^2}\frac{dx}{(1+x)^2} = \frac{4k^2}{1+4k^2b^2}$$

This would immediately lead to the cross section if the particles were *not* identical. For identical particles, there are symmetry problems caused by the Pauli Exclusion Principle and the fact that the protons have spin 1/2. The matrix elements are not affected by the

spin because there is no spin-orbit coupling or any other spin dependence in the potential. However:
In the spin *triplet* state, the spatial wave function of the proton is antisymmetric, while for the spin *singlet* state, the spatial wave function is symmetric. This means that in the original Born approximation we have

$$\int d^3r \frac{e^{-i\mathbf{k}'.\mathbf{r}} \mp e^{i\mathbf{k}'.\mathbf{r}}}{\sqrt{2}} V(r) \frac{e^{i\mathbf{k}.\mathbf{r}} \mp e^{-i\mathbf{k}.\mathbf{r}}}{\sqrt{2}} =$$
$$\int d^3r V(r) e^{-i(\mathbf{k}'-\mathbf{k}).\mathbf{r}} \mp \int d^3r V(r) e^{-i(\mathbf{k}+\mathbf{k}').\mathbf{r}}$$

The first term has the familiar form

$$4\pi V_0 \frac{b^3}{1+b^2\Delta^2} = 4\pi V_0 \frac{b^3}{1+2b^2k^2(1-\cos\theta)}$$

and the second term is obtained by changing $\cos\theta$ to $-\cos\theta$.. Thus the cross section involves

$$\int_{-1}^{1} d(\cos\theta)\left(\frac{1}{1+2b^2k^2-2b^2k^2\cos\theta} \mp \frac{1}{1+2b^2k^2+2b^2k^2\cos\theta}\right)^2$$
$$\rightarrow \int_{-1}^{1} dz\left(\frac{1}{1+a-az} \mp \frac{1}{1+a+az}\right)^2$$
$$= \frac{4}{1+2a} \mp \frac{2}{a(1+a)}\ln(1+2a)$$

where $a = 2b^2k^2$.

Thus the total cross section is

$$\sigma = \frac{8\pi\mu^2 b^6}{\hbar^4} V_0^2 \left[\left(\frac{4}{1+4k^2b^2}\right) \mp \frac{1}{k^2b^2(1+2k^2b^2)} \ln(1+4k^2b^2)\right]$$

The relation to the center of mass energy follows from $E = p^2/2\mu = \hbar^2k^2/2\mu$, so that

$$k^2 = \frac{2\mu E}{\hbar^2} = \frac{(1.67\times10^{-27}\,kg)(100\times1.6\times10^{-13}\,J)}{(1.054\times10^{-34}\,J.s)^2}$$

With $b = 1.2 \times 10^{-1`5}$ m, we get $(kb)^2 = 3.5$, so that $\sigma = 4.3 \times 10^{-28}\ \mathrm{m}^2 = 4.3 \times 10^{-24}\ \mathrm{cm}^2 =$ 3.4 barns.

4. To make the table, we first of all make a change of notation: we will represent the proton spinors by $\chi_\pm$ and the neutron spinors by $\eta_\pm$. To work out the action of

$$\sigma_p \bullet \sigma_n = \sigma_{pz}\sigma_{nz} + 2(\sigma_{p+}\sigma_{n-} + \sigma_{p-}\sigma_{n+})$$

on the four initial combinations, we will use $\sigma_+\chi_+ = \sigma_-\chi_- = 0;\ \ \sigma_+\chi_- = \chi_+;\ \ \sigma_-\chi_+ = \chi_-$

and similarly for the neutron spinors. Thus

$$[\sigma_{pz}\sigma_{nz} + 2(\sigma_{p+}\sigma_{n-} + \sigma_{p-}\sigma_{n+})]\chi_+\eta_+ = \chi_+\eta_+$$
$$[\sigma_{pz}\sigma_{nz} + 2(\sigma_{p+}\sigma_{n-} + \sigma_{p-}\sigma_{n+})]\chi_+\eta_- = -\chi_+\eta_- + 2\chi_-\eta_+$$
$$[\sigma_{pz}\sigma_{nz} + 2(\sigma_{p+}\sigma_{n-} + \sigma_{p-}\sigma_{n+})]\chi_-\eta_+ = -\chi_-\eta_+ + 2\chi_+\eta_-$$
$$[\sigma_{pz}\sigma_{nz} + 2(\sigma_{p+}\sigma_{n-} + \sigma_{p-}\sigma_{n+})]\chi_-\eta_- = \chi_-\eta_-$$

From this we get for the matrix $A + B\sigma_p \bullet \sigma_n$, with rows and columns labeled by (++), (+-),(-+). (--)the following

$$A + B\sigma_p \bullet \sigma_n = \begin{pmatrix} A+B & 0 & 0 & 0 \\ 0 & A-B & 2B & 0 \\ 0 & 2B & A-B & 0 \\ 0 & 0 & 0 & A+B \end{pmatrix}$$

The cross sections will form a similar matrix, with the amplitudes replaced by the absolute squares, i.e. $|A+B|^2$, $|2B|^2$, and $|A-B|^2$.

5. Consider $n - p$ scattering again. If the initial proton spin is not specified, then we must add the cross sections for all the possible initial proton states and divide byt 2, since *a priori* there is no reason why in the initial state there should be more or less of up-spin protons. We also need to sum over the final states. Note that we do not sum amplitudes because the spin states of the proton are distinguishable.

Thus, for initial neutron spin up and final neutron spin up we have

$$\sigma(+|+) = \frac{1}{2}\left(\sigma(++,++) + \sigma(++,-+) + \sigma(-+,++) + \sigma(-+,-+)\right)$$

where on the r.h.s. the first label on each side refers to the proton and the second to the neutron. We thus get

$$\sigma(+|+) = \frac{1}{2}\left(|A+B|^2 + |A-B|^2\right) = |A|^2 + |B|^2$$

Similarly

$$\sigma(-|+)=\frac{1}{2}\left(\sigma(+-,++)+\sigma(+-,-+)+\sigma(--,++)+\sigma(--,-+)\right)$$
$$=\frac{1}{2}\left(|2B|^2\right)=2|B|^2$$

Thus

$$P=\frac{|A|^2+|B|^2-2|B|^2}{|A|^2+|B|^2+2|B|^2}=\frac{|A|^2-|B|^2}{|A|^2+3|B|^2}$$

6. For triplet → triplet scattering we have (with the notation $(\boldsymbol{S},S_z)$)

(1,1)→(1,1) $\langle\chi_+\eta_+|\chi_+\eta_+\rangle=A+B$

(1,-1)→(1,-1) $\langle\chi_-\eta_-|\chi_-\eta_-\rangle=A+B$

(1,0)→ (1,0) $\langle\frac{\chi_+\eta_-+\chi_-\eta_+}{\sqrt{2}}|\frac{\chi_+\eta_-+\chi_-\eta_+}{\sqrt{2}}\rangle=\frac{1}{2}\left(A-B+2B+2B+A-B\right)=A+B$

(0,0)→(0.0) $\langle\frac{\chi_+\eta_--\chi_-\eta_+}{\sqrt{2}}|\frac{\chi_+\eta_--\chi_-\eta_+}{\sqrt{2}}\rangle=\frac{1}{2}\left(A-B-2B-2B+A-B\right)=A-3B$

(0,0)→(1,0) $\langle\frac{\chi_+\eta_--\chi_-\eta_+}{\sqrt{2}}|\frac{\chi_+\eta_-+\chi_-\eta_+}{\sqrt{2}}\rangle=\frac{1}{2}\left(A-B+2B-2B-A+B\right)=0$

We can check this by noting that (in units of $\hbar$,

$$A+B\sigma_p\bullet\sigma_n=A+4B\mathbf{s}_p\bullet\mathbf{s}_n=A+2B(\mathbf{S}^2-\mathbf{s}_p^2-\mathbf{s}_n^2)$$
$$=A+2B\left[S(S+1)-\frac{3}{2}\right]$$

For S = 1 this is A + B, For S = 0, it is $A-B$, and since $\langle S=1|\mathbf{S}^2-3/2|S=0\rangle=0$ by orthogonality of the triplet to singlet states, we get the same result as above.

7. We have, with $x = kr$ and $\cos\theta = u$,

$$I(x)=\int_{-1}^{1}du\,g(u)e^{-iux}=\int_{-1}^{1}du\,g(u)\frac{i}{x}\frac{d}{dx}e^{-iux}$$
$$=\frac{i}{x}\int_{-1}^{1}du\frac{d}{du}\left(g(u)e^{-iux}\right)-\frac{i}{x}\int_{-1}^{1}du\left(\frac{dg}{du}\right)e^{-iux}$$

The first term vanishes since $g(\pm1)=0$. We can proceed once more, and using the fact that the derivatives of $g(u)$ also vanish at $u = \pm 1$, we find

$$I(x) = \left(\frac{-i}{x}\right)^2 \int_{-1}^{1} du \left(\frac{d^2 g}{du^2}\right) e^{-ixu}$$

and so on. We can always go beyond any pre-determined power of $1/x$ so that $I(x)$ goes to zero faster than any power of $(1/x)$.

7. We proceed as in the photoelectric effect. There the rate, as given in Eq.(19-111) is

$$R = \frac{2\pi V}{\hbar} \int d\Omega \frac{mp_e}{(2\pi\hbar)^3} |M_{fi}|^2$$

Here m is the electron mass, and p_e is the momentum of the outgoing.electron.The factor arose out of the phase space integral

$$\int dpp^2 \delta\left(\frac{p^2}{2m} - E\gamma\right) = \int d\left(\frac{p^2}{2m}\right) mp\, \delta\left(\frac{p^2}{2m} - E_\gamma\right) = mp_e$$

with p_e determined by the photon energy, as shown in the delta function. In the deuteron photodisintegration process, the energy conservation is manifest in $\delta\left(\frac{p^2}{M} - E_\gamma + E_B\right)$.

The delta function differs in two respects: first, some of the photon energy goes into dissociating the deuteron, which takes an energy E_B; second, in the final state two particles of equal mass move in in equal and opposite directions, both with momentum of magnitude p, so that the reduced mass $M_{red} = M/2$ appears. Thus the factor mp_e will be replaced by $Mp/2$, where the momentum of the particle is determined by the delta function.

Next, we consider the matrix element. The final state is the same as given in Eq. (19-114) with p_e replaced by p , and with the hydrogen-like wave function replaced by the deuteron ground state wave function. We thus have

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{\hbar} \frac{(VMp/2)}{(2\pi\hbar)^3} \frac{V}{c} \left(\frac{e}{M}\right)^2 \frac{\hbar}{2\varepsilon_0 \omega V} \frac{1}{V} (\varepsilon \bullet \mathbf{p})^2 \left| \int d^3 r e^{i(\mathbf{k} - \mathbf{p}/\hbar)\bullet \mathbf{r}} \psi_d(\mathbf{r}) \right|^2$$

We need to determine the magnitude of the factor $e^{i\mathbf{k}\bullet\mathbf{r}}$. The integral is over the wave function of the deuteron. If the ground state wave function behaves as $e^{-\alpha r}$, then the probability distribution goes as $e^{-2\alpha r}$, and we may roughly take $1/2\alpha$ as the "size" of the deuteron. Note that $\alpha^2 = ME_B/\hbar^2$. As far as k is concerned, it is given by

$$k = \frac{p_\gamma}{\hbar} = \frac{E_\gamma}{\hbar c}$$

Numerically we get, with E_B =2.2 MeV, and E_γ = 10 MeV, $k/2\alpha$ = 0.11, which means that we can neglect the oscillating factor. Thus in the matrix element we just need $\int d^3 r e^{i\mathbf{k}\bullet\mathbf{r}}\psi_d(\mathbf{r})$. The wave function to be used is

$$\psi_d(r) = \frac{N}{\sqrt{4\pi}}\frac{e^{-\alpha(r-r_0)}}{r} \qquad r > r_0$$

N is determined by the normalization condition

$$\frac{N^2}{4\pi}\int_{r_0}^{\infty} 4\pi r^2 dr \frac{e^{-2\alpha(r-r_0)}}{r^2} = 1$$

So that

$$N^2 = 2\alpha$$

The matrix element involves

$$\begin{aligned}
&\frac{N}{\sqrt{4\pi}}\frac{4\pi}{k}\int_{r_0}^{\infty} r dr \sin kr \frac{e^{-\alpha(r-r_0)}}{r} = \\
&= \frac{N\sqrt{4\pi}}{k}\int_0^{\infty} dx \sin k(x+r_0) e^{-\alpha x} \\
&= \frac{N\sqrt{4\pi}}{k}\int_0^{\infty} dx \Big(\sin kr_0 \,\mathrm{Re}(e^{-x(\alpha-ik)}\Big) + \cos kr_0 \,\mathrm{Im}(e^{-x(\alpha-ik)})) \\
&= \frac{N\sqrt{4\pi}}{k}\left(\frac{\alpha}{\alpha^2+k^2}\sin kr_0 + \frac{k}{\alpha^2+k^2}\cos kr_0\right)
\end{aligned}$$

The square of this is

$$\frac{4\pi N^2}{k^2} r_0^2 \left(\frac{\alpha r_0}{\alpha^2 r_0^2 + k^2 r_0^2}\sin kr_0 + \frac{kr_0}{\alpha^2 r_0^2 + k^2 r_0^2}\cos kr_0\right)^2$$

It follows that

$$\frac{d\sigma}{d\Omega} = 2\left(\frac{e^2}{4\pi\varepsilon_0 \hbar c}\right)\frac{pr_0}{M\omega}(\alpha r_0)\left[\frac{\alpha r_0}{\alpha^2 r_0^2 + k^2 r_0^2}\sin kr_0 + \frac{kr_0}{\alpha^2 r_0^2 + k^2 r_0^2}\cos kr_0\right]^2$$

We can easily check that this has the correct dimensions of an area.

For numerical work we note that $\alpha r_0 = 0.52$; $kr_0 = 0.26\sqrt{E_{MeV}}$ and $\hbar\omega = E_B + \frac{p^2}{M}$.

9. The change in the calculation consists of replacing the hydrogen wave function

$$\frac{1}{\sqrt{4\pi}} 2\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$$

by

$$\psi(r) = \frac{N}{\sqrt{4\pi}} \frac{\sin qr}{r} \qquad r < r_0$$
$$= \frac{N}{\sqrt{4\pi}} \frac{e^{-\kappa r}}{r} \qquad r > r_0$$

where the binding energy characteristic of the ground state of the electron determines κ as follows

$$\kappa^2 = 2m_e \,|\, E_B \,| / \hbar^2 = (m_e c\alpha / \hbar)^2$$

with α = 1/137. The eigenvalue condition relates q to κ as follows:

$$qr_0 \cot qr_0 = -\kappa r_0$$

where

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$$q^2 = \left(\frac{2m_e V_0}{\hbar^2} - \kappa^2\right)$$

and V_0 is the depth of the square well potential. The expression for the differential cross section is obtained from Eq. (19-116) by dividing by $4(Z/a_0)^2$ and replacing the wave function in the matrix element by the one written out above,

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{\hbar} \frac{m_e p_e}{(2\pi\hbar)^3} \frac{1}{c} \left(\frac{e}{m_e}\right)^2 \frac{\hbar}{2\varepsilon_0 \omega} \frac{p_e^2}{4\pi} (\hat{\varepsilon}.\hat{\mathbf{p}})^2 \left| \int d^3r e^{i(\mathbf{k}-\mathbf{p}_e/\hbar).\mathbf{r}} \psi(r) \right|^2$$

We are interested in the energy-dependence of the cross section, under the assumptions that the photon energy is much larger than the electron binding energy and that the potential has a very short range. The energy conservation law states that under these assumptions $\hbar\omega = p_e^2 / 2m_e$. The factor in front varies as $p_e^3 / \omega \propto p_e \propto \sqrt{E_\gamma}$, and thus we need to analyze the energy dependence of $\left| \int d^3r e^{i(\mathbf{k}-\mathbf{p}_e/\hbar).\mathbf{r}} \psi(r) \right|^2$. The integral has the form

$$\int d^3r e^{i\mathbf{Q}.\mathbf{r}} \psi(r) = \frac{4\pi}{Q} \int_0^\infty r dr \sin Qr \psi(r)$$

where $\mathbf{Q} = \mathbf{k} - \mathbf{p}_e / \hbar$ so that $Q^2 = k^2 + \frac{p_e^2}{\hbar^2} - 2\frac{kp_e}{\hbar}(\hat{\mathbf{k}}.\hat{\mathbf{p}})$.

Now $\hbar^2 k^2 / p_e^2 = \hbar^2\omega^2 / p_e^2 c^2 = \hbar\omega \frac{p_e^2 / 2m}{p_e^2 c^2} = \frac{\hbar\omega}{2m_e c^2}$. We are dealing with the nonrelativistic regime, so that this ratio is much smaller than 1. We will therefore neglect the k –dependence, and replace Q by $p_e/\hbar$. The integral thus becomes

$$\frac{4\pi}{Q}\frac{N}{\sqrt{4\pi}}\left[\int_0^{r_0} dr \sin Qr \sin qr + \int_{r_0}^{\infty} dr \sin Qr e^{-\kappa r}\right]$$

The first integral is

$$\frac{1}{2}\int_0^{r_0} dr\big(\cos(Q-q)r - \cos(Q+q)r\big) =$$
$$\frac{1}{2}\left(\frac{\sin(Q-q)r_0}{Q-q} - \frac{\sin(Q+q)r_0}{Q+q}\right) \approx -\frac{1}{Q}\cos Qr_0 \sin qr_0$$

where, in the last step we used $Q >> q$. The second integral is

$$\mathrm{Im}\int_{r_0}^{\infty} dr e^{-r(\kappa - iQ)} = \mathrm{Im}\frac{e^{-r_0(\kappa - iQ)}}{\kappa - iQ} \approx \frac{\cos Qr_0}{Q} e^{-\kappa r_0}$$

The square of the matrix element is therefore

$$\frac{4\pi N^2}{Q^2}\frac{1}{Q^2}\big(\cos Qr_0 (e^{-\kappa r_0} - \sin qr_0)\big)^2$$

The square of the cosine may be replaced by 1/2, since it is a rapidly oscillating factor, and thus the dominant dependence is $1/Q^4$, i.e. $1/E_\gamma^2$. Thus the total dependence on the photon energy is $1/E_\gamma^{3/2}$ or $1/p_e^3$, in contrast with the atomic $1/p_e^7$ dependence.

10. The differential *rate* for process I, a + A → b + B in the center of momentum frame is

$$\frac{dR_I}{d\Omega} = \frac{1}{(2j_a+1)(2J_A+1)}\frac{1}{(2\pi\hbar)^3} p_b^2 \frac{dp_b}{dE_b}\sum_{spins}|M_I|^2$$

The sum is over all initial and final spin states. Since we have to *average* (rather than sum) over the initial states, the first two factors are there to take that into account. The phase factor is the usual one, written without specification of how E_b depends on p_b. The rate for the inverse process II, b + B → a + A is, similarly

$$\frac{dR_{II}}{d\Omega} = \frac{1}{(2j_b+1)(2J_B+1)} \frac{1}{(2\pi\hbar)^3} p_a^2 \frac{dp_a}{dE_a} \sum_{spins} |M_{II}|^2$$

By the principle of detailed balance the sum over all spin states of the square of the matrix elements for the two reactions are the same provided that these are at the same center of momentum energies. Thus

$$\sum_{spins} |M_I|^2 = \sum_{spins} |M_{II}|^2$$

Use of this leads to the result that

$$\frac{(2j_a+1)(2J_A+1)}{p_b^2(dp_b/dE_b)} \frac{dR_I}{d\Omega} = \frac{(2j_b+1)(2J_B+1)}{p_a^2(dp_a/dE_a)} \frac{dR_{II}}{d\Omega}$$

Let us now apply this result to the calculation of the radiative capture cross section for the process $N + P \rightarrow D + \gamma$. We first need to convert from rate to cross section. This is accomplished by multiplying the rate R by the volume factor V, and dividing by the *relative velocity* of the particles in the initial state. For the process I, the photo-disintegration $\gamma + D \rightarrow N + P$, the relative velocity is c, the speed of light. For process II, the value is $p_b/m_{\mathrm{red}} = 2p_b/M$. Thus

$$\frac{d\sigma_I}{d\Omega} = \frac{V}{c}\frac{dR_I}{d\Omega}; \quad \frac{d\sigma_{II}}{d\Omega} = \frac{MV}{2p_b}\frac{dR_{II}}{d\Omega}$$

Application of the result obtained above leads to

$$\begin{aligned} \frac{d\sigma_{II}}{d\Omega} &= \frac{MV}{2p_b}\frac{dR_{II}}{d\Omega} \\ &= \frac{MV}{2p_b} \frac{p_a^2(dp_a/dE_a)}{(2j_b+1)(2J_B+1)} \times \frac{(2j_a+1)(2J_A+1)}{p_b^2(dp_b/dE_b)} \frac{c}{V} \frac{d\sigma_I}{d\Omega} \end{aligned}$$

We can calculate all the relevant factors. We will neglect the binding energy of the deuteron in our calculation of the kinematics.
First

$$\frac{(2j_\gamma+1)(2J_D+1)}{(2j_P+1)(2J_N+1)} = \frac{2\times 3}{2\times 2} = \frac{3}{2}$$

Next, in the center of momentum frame, the center of mass energy is

$$W = p_a c + \frac{p_a^2}{2M_D} \approx p_a c + \frac{p_a^2}{4M}$$

so that $(dE_a / dp_a) = c + \frac{p_a}{2M}$. In reaction II,

$$W = 2 \times \frac{p_b^2}{2M} = \frac{p_b^2}{M}$$

so that $(dE_b / dp_b) = 2 p_b / M$. There is a relation between p_a and p_b since the values of W are the same in both cases. This can be simplified. For photon energies up to say 50 MeV or so, the deuteron may be viewed as infinitely massive, so that there is no difference between the center of momentum. This means that it is a good approximation to write $W = E_\gamma = p_a c = p_b^2 / M$. We are thus finally led to the result that

$$\frac{d\sigma(NP \to D\gamma)}{d\Omega} = \frac{3}{2}\left(\frac{E_\gamma}{Mc^2}\right)\frac{d\sigma(\gamma D \to NP)}{d\Omega}$$