ENDORSED BY
CAMBRIDGE
International Examinations

NEW FOR 2013

Cambridge IGCSE®

Core Mathematics

Third Edition

Ric Pimentel
Terry Wall

COMPANION CD

HODDER EDUCATION

ENDORSED BY

CAMBRIDGE
International Examinations

NEW FOR 2013

Cambridge IGCSE®

Core Mathematics

Third Edition

Ric Pimentel
Terry Wall

HODDER EDUCATION
AN HACHETTE UK COMPANY

The Publishers would like to thank the following for permission to reproduce copyright material:
Photo credits p.3 © Inv. Id P. Lund. inv. 35, reproduced with the kind permission of Lund University Library, Sweden; **p.105** © dbimages/Alamy; **p.155** © Alexey Pavluts – Fotolia; **p.225** © BSIP SA/Alamy; **p.251** © Georgios Kollidas – Fotolia; **p.271** © Inter-University Centre for Astronomy and Astrophysics, Pune, India; **p.293** Matěj Bat'ha/http://upload.wikimedia.org/wikipedia/commons/c/c2/Pametni_cedule_Bernard_Bolzano_-_detail.jpg/http://creativecommons.org/licenses/by-sa/2.5/deed.en; **p.315** © Bernard 63 – Fotolia; **p.331** © Jason Butcher/cultura/Corbis.

Orders: please contact Bookpoint Ltd, 130 Milton Park, Abingdon, Oxon OX14 4SB.
Telephone: (44) 01235 827720. Fax: (44) 01235 400454. Lines are open 9.00–5.00, Monday to Saturday, with a 24-hour message answering service. Visit our website at www.hoddereducation.com

First edition published 1997
Second edition published 2006

This third edition published 2013 by
Hodder Education, an Hachette UK Company,
338 Euston Road
London NW1 3BH

Impression number 5 4 3 2 1
Year 2017 2016 2015 2014 2013

Cover photo © Fyle – Fotolia
Illustrations by Pantek Media
Typeset in Times Ten 10/12pt by Pantek Media, Maidstone
Printed in Italy

A catalogue record for this title is available from the British Library
ISBN 978 1 4441 91721

Contents

Introduction

This textbook has been written by two experienced mathematics teachers.

The book is written to cover the Cambridge IGCSE® Mathematics (0580) syllabus (Core only). The syllabus headings (Number, Algebra and graphs, Geometry, Mensuration, Coordinate geometry, Trigonometry, Matrices and transformations, Probability, Statistics) are mirrored in the textbook. Each major topic is divided into a number of chapters, and each chapter has its own discrete exercises and student assessments. Students using this book may follow the Core curriculum.

The syllabus specifically refers to 'Applying mathematical techniques to solve problems', and this is fully integrated into the exercises and assessments. This book also includes a number of such problems so that students develop their skills in this area throughout the course. Ideas for ICT activities are also included, although this is not part of the examination.

The CD included with this book contains Personal Tutor audio-visual worked examples covering all the main concepts.

The study of mathematics crosses all lands and cultures. A mathematician in Africa may be working with another in Japan to extend work done by a Brazilian in the USA; art, music, language and literature belong to the culture of the country of origin. Opera is European. Noh plays are Japanese. It is not likely that people from different cultures could work together on a piece of Indian art for example. But all people in all cultures have tried to understand their world, and mathematics has been a common way of furthering that understanding, even in cultures which have left no written records. Each Topic in this textbook has an introduction which tries to show how, over a period of thousands of years, mathematical ideas have been passed from one culture to another.

The Ishango Bone from Stone-Age Africa has marks suggesting it was a tally stick. It was the start of arithmetic. 4500 years ago in ancient Mesopotamia, clay tablets show multiplication and division problems. An early abacus may have been used at this time. 3600 years ago what is now called The Rhind Papyrus was found in Egypt. It shows simple algebra and fractions. The Moscow Papyrus shows how to find the volume of a pyramid. The Egyptians advanced our knowledge of geometry. The Babylonians worked with arithmetic. 3000 years ago in India the great wise men advanced mathematics and their knowledge travelled to Egypt and later to Greece, then to the rest of Europe when great Arab mathematicians took their knowledge with them to Spain.

Europeans and later Americans made mathematical discoveries from the fifteenth century. It is likely that, with the re-emergence of China and India as major world powers, these countries will again provide great mathematicians and the cycle will be completed. So when you are studying from this textbook remember that you are following in the footsteps of earlier mathematicians who were excited by the discoveries they had made. These discoveries changed our world.

You may find some of the questions in this book difficult. It is easy when this happens to ask the teacher for help. Remember though that mathematics is intended to stretch the mind. If you are trying to get physically fit, you do not stop as soon as things get hard. It is the same with mental fitness. Think logically, try harder. You can solve that difficult problem and get the feeling of satisfaction that comes with learning something new.

Ric Pimentel
Terry Wall

Topic 1

Number

Syllabus

C1.1
Identify and use natural numbers, integers (positive, negative and zero), prime numbers, square numbers, common factors and common multiples, rational and irrational numbers (e.g. π, $\sqrt{2}$), real numbers.

C1.2
Extended curriculum only.

C1.3
Calculate squares, square roots, cubes and cube roots of numbers.

C1.4
Use directed numbers in practical situations. e.g. temperature changes, flood levels.

C1.5
Use the language and notation of simple vulgar and decimal fractions and percentages in appropriate contexts.
Recognise equivalence and convert between these forms.

C1.6
Order quantities by magnitude and demonstrate familiarity with the symbols $=, \neq, >, <, \geq, \leq$

C1.7
Understand the meaning and rules of indices.
Use the standard form $A \times 10^n$ where n is a positive or negative integer, and $1 \leq A < 10$.

C1.8
Use the four rules for calculations with whole numbers, decimals and vulgar (and mixed) fractions, including correct ordering of operations and use of brackets.

C1.9
Make estimates of numbers, quantities and lengths, give approximations to specified numbers of significant figures and decimal places and round off answers to reasonable accuracy in the context of a given problem.

C1.10
Give appropriate upper and lower bounds for data given to a specified accuracy.

C1.11
Demonstrate an understanding of ratio and proportion.
Use common measures of rate.
Calculate average speed.

C1.12
Calculate a given percentage of a quantity.
Express one quantity as a percentage of another.
Calculate percentage increase or decrease.

C1.13
Use a calculator efficiently.
Apply appropriate checks of accuracy.

C1.14
Calculate times in terms of the 24-hour and 12-hour clock.
Read clocks, dials and timetables.

C1.15
Calculate using money and convert from one currency to another.

C1.16
Use given data to solve problems on personal and household finance involving earnings, simple interest and compound interest.
Extract data from tables and charts.

C1.17
Extended curriculum only.

Contents

The development of number

Fragment of a Greek papyrus, showing an early version of the zero sign

In Africa, bones have been discovered with marks cut into them that are probably tally marks. These tally marks may have been used for counting time, such as numbers of days or cycles of the moon, or for keeping records of numbers of animals. A tallying system has no place value, which makes it hard to show large numbers.

The earliest system like ours (known as base 10) dates to 3100BCE in Egypt. Many ancient texts, for example texts from Babylonia (modern Iraq) and Egypt, used zero. Egyptians used the word *nfr* to show a zero balance in accounting. Indian texts used a Sanskrit word, *shunya*, to refer to the idea of the number zero. By the 4th century BCE, the people of south-central Mexico began to use a true zero. It was represented by a shell picture and became a part of Mayan numerals. By AD130, Ptolemy was using a symbol, a small circle, for zero. This Greek zero was the first use of the zero we use today.

The idea of negative numbers was recognised as early as 100BCE in the Chinese text *Jiuzhang Suanshu* (*Nine Chapters on the Mathematical Art*). This is the earliest known mention of negative numbers in the East. In the 3rd century BCE in Greece, Diophantus had an equation whose solution was negative. He said that the equation gave an absurd result. European mathematicians did not use negative numbers until the 17th century, although Fibonacci allowed negative solutions in financial problems where they could be debts or losses.

1 Number and language

Natural numbers

A child learns to count 'one, two, three, four, …'. These are sometimes called the counting numbers or whole numbers.

The child will say 'I am three', or 'I live at number 73'.

If we include the number 0, then we have the set of numbers called the **natural numbers**.

The set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$.

Integers

On a cold day, the temperature may be 4 °C at 10 p.m. If the temperature drops by a further 6 °C, then the temperature is 'below zero'; it is –2 °C.

If you are overdrawn at the bank by \$200, this might be shown as –\$200.

The set of **integers** $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

$\mathbb{Z}$ is therefore an extension of $\mathbb{N}$. Every natural number is an integer.

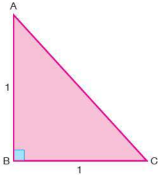
Rational numbers

A child may say 'I am three'; she may also say 'I am three and a half', or even 'three and a quarter'. $3\frac{1}{2}$ and $3\frac{1}{4}$ are **rational numbers**. All rational numbers can be written as a fraction whose denominator is not zero. All terminating and recurring decimals are rational numbers as they can also be written as fractions, e.g.

$$0.2 = \frac{1}{5} \quad 0.3 = \frac{3}{10} \quad 7 = \frac{7}{1} \quad 1.53 = \frac{153}{100} \quad 0.\dot{2} = \frac{2}{9}$$

The set of rational numbers $\mathbb{Q}$ is an extension of the set of integers.

Real numbers



Numbers which cannot be expressed as a fraction are not rational numbers; they are **irrational numbers**.

Using Pythagoras' rule in the diagram to the left, the length of the hypotenuse AC is found as:

$$AC^2 = 1^2 + 1^2$$
$$AC^2 = 2$$
$$AC = \sqrt{2}$$

$\sqrt{2} = 1.41421356\ldots$. The digits in this number do not recur or repeat. This is a property of all irrational numbers. Another example of an irrational number you will come across is π (pi).

It is the ratio of the circumference of a circle to the length of its diameter. Although it is often rounded to 3.142, the digits continue indefinitely never repeating themselves.

The set of rational and irrational numbers together form the set of **real numbers** $\mathbb{R}$.

Prime numbers

A prime number is one whose only factors are 1 and itself. (Note that 1 is not a prime number.)

Exercise 1.1

In a 10 by 10 square, write the numbers 1 to 100.
Cross out number 1.
Cross out all the even numbers after 2 (these have 2 as a factor).
Cross out every third number after 3 (these have 3 as a factor).
Continue with 5, 7, 11 and 13, then list all the prime numbers less than 100.

Square numbers

Exercise 1.2

In a 10 by 10 square write the numbers 1 to 100.
Shade in 1 and then 2×2, 3×3, 4×4, 5×5 etc.
These are the square numbers.

3×3 can be written 3^2 (you say three squared)
7×7 can be written 7^2 (the 2 is called an index; plural indices)

Cube numbers

$3 \times 3 \times 3$ can be written 3^3 (you say three cubed)
$5 \times 5 \times 5$ can be written 5^3
$2 \times 2 \times 2 \times 5 \times 5$ can be written $2^3 \times 5^2$

Exercise 1.3

Write the following using indices:

a) 9×9
b) 12×12
c) 8×8
d) $7 \times 7 \times 7$
e) $4 \times 4 \times 4$
f) $3 \times 3 \times 2 \times 2 \times 2$
g) $5 \times 5 \times 5 \times 2 \times 2$
h) $4 \times 4 \times 3 \times 3 \times 2 \times 2$

Factors

The factors of 12 are all the numbers which will divide exactly into 12,

i.e. 1, 2, 3, 4, 6 and 12.

Exercise 1.4

List all the factors of the following numbers:

a) 6 b) 9 c) 7 d) 15 e) 24
f) 36 g) 35 h) 25 i) 42 j) 100

Prime factors

The factors of 12 are 1, 2, 3, 4, 6 and 12.

Of these, 2 and 3 are prime numbers, so 2 and 3 are the prime factors of 12.

Exercise 1.5

List the prime factors of the following numbers:

a) 15 b) 18 c) 24 d) 16 e) 20
f) 13 g) 33 h) 35 i) 70 j) 56

An easy way to find prime factors is to divide by the prime numbers in order, smallest first.

Worked examples

a) Find the prime factors of 18 and express it as a product of prime numbers:

	18
2	9
3	3
3	1

$18 = 2 \times 3 \times 3$ or 2×3^2

b) Find the prime factors of 24 and express it as a product of prime numbers:

	24
2	12
2	6
2	3
3	1

$24 = 2 \times 2 \times 2 \times 3$ or $2^3 \times 3$

c) Find the prime factors of 75 and express it as a product of prime numbers:

	75
3	25
5	5
5	1

$75 = 3 \times 5 \times 5$ or 3×5^2

Exercise 1.6

Find the prime factors of the following numbers and express them as a product of prime numbers:

a) 12	b) 32	c) 36	d) 40	e) 44
f) 56	g) 45	h) 39	i) 231	j) 63

Highest common factor

The factors of 12 are 1, 2, 3, 4, 6, 12.
The factors of 18 are 1, 2, 3, 6, 9, 18.

So the highest common factor (HCF) can be seen by inspection to be 6.

Exercise 1.7

Find the HCF of the following numbers:

a) 8, 12	b) 10, 25	c) 12, 18, 24
d) 15, 21, 27	e) 36, 63, 108	f) 22, 110
g) 32, 56, 72	h) 39, 52	i) 34, 51, 68
j) 60, 144		

Multiples

Multiples of 5 are 5, 10, 15, 20, etc.

The lowest common multiple (LCM) of 2 and 3 is 6, since 6 is the smallest number divisible by 2 and 3.

The LCM of 3 and 5 is 15.

The LCM of 6 and 10 is 30.

Exercise 1.8

1. Find the LCM of the following:

a) 3, 5	b) 4, 6	c) 2, 7	d) 4, 7
e) 4, 8	f) 2, 3, 5	g) 2, 3, 4	h) 3, 4, 6
i) 3, 4, 5	j) 3, 5, 12		

2. Find the LCM of the following:

a) 6, 14	b) 4, 15	c) 2, 7, 10	d) 3, 9, 10
e) 6, 8, 20	f) 3, 5, 7	g) 4, 5, 10	h) 3, 7, 11
i) 6, 10, 16	j) 25, 40, 100		

Rational and irrational numbers

Earlier in this chapter you learnt about rational and irrational numbers.

A **rational number** is any number which can be expressed as a fraction. Examples of some rational numbers and how they can be expressed as a fraction are shown below:

$$0.2 = \tfrac{1}{5} \quad 0.3 = \tfrac{3}{10} \quad 7 = \tfrac{7}{1} \quad 1.53 = \tfrac{153}{100} \quad 0.\dot{2} = \tfrac{2}{9}$$

An **irrational number** cannot be expressed as a fraction. Examples of irrational numbers include:

$$\sqrt{2}, \quad \sqrt{5}, \quad 6 - \sqrt{3}, \quad \pi$$

In summary:
Rational numbers include:

- whole numbers,
- fractions,
- recurring decimals,
- terminating decimals.

Irrational numbers include:

- the square root of any number other than square numbers,
- a decimal which neither repeats nor terminates (e.g. π).

Exercise 1.9

1. For each of the numbers shown below state whether it is rational or irrational:

a) 1.3 b) $0.\dot{6}$ c) $\sqrt{3}$

d) $-2\frac{3}{5}$ e) $\sqrt{25}$ f) $\sqrt[3]{8}$

g) $\sqrt{7}$ h) 0.625 i) $0.\dot{1}\dot{1}$

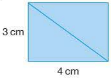
2. For each of the numbers shown below state whether it is rational or irrational:

a) $\sqrt{2} \times \sqrt{3}$ b) $\sqrt{2} + \sqrt{3}$ c) $(\sqrt{2} \times \sqrt{3})^2$

d) $\dfrac{\sqrt{8}}{\sqrt{2}}$ e) $\dfrac{2\sqrt{5}}{\sqrt{20}}$ f) $4 + (\sqrt{9} - 4)$

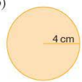
3. In each of the following decide whether the quantity required is rational or irrational. Give reasons for your answer.

a)



The length of the diagonal

b)

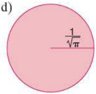


The circumference of the circle

c)

The side length of the square

d)

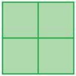


The area of the circle

Calculating squares

This is a square of side 1 cm.

This is a square of side 2 cm.
It has four squares of side 1 cm in it.

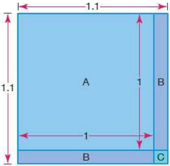


Exercise 1.10 Calculate how many squares of side 1 cm there would be in squares of side:

a) 3 cm b) 5 cm c) 8 cm d) 10 cm
e) 11 cm f) 12 cm g) 7 cm h) 13 cm
i) 15 cm j) 20 cm

In index notation, the square numbers are $1^2, 2^2, 3^2, 4^2$, etc. 4^2 is read as '4 squared'.

Worked example This square is of side 1.1 units.
Its area is 1.1×1.1 units^2.
$A = 1 \times 1 = 1$
$B = 1 \times 0.1 = 0.1$
$B = 1 \times 0.1 = 0.1$
$C = 0.1 \times 0.1 = 0.01$
Total $= 1.21$ units^2



Exercise 1.11

1. Draw diagrams and use them to find the area of squares of side:
   a) 2.1 units b) 3.1 units c) 1.2 units
   d) 2.2 units e) 2.5 units f) 1.4 units
2. Use long multiplication to work out the area of squares of side:
   a) 2.4 b) 3.3 c) 2.8 d) 6.2
   e) 4.6 f) 7.3 g) 0.3 h) 0.8
   i) 0.1 j) 0.9
3. Check your answers to Q.1 and 2 by using the x^2 key on a calculator.

Using a graph

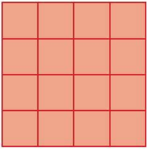
Exercise 1.12

1. Copy and complete the table for the equation $y = x^2$.

x	0	1	2	3	4	5	6	7	8
y				9				49	

Plot the graph of $y = x^2$. Use your graph to find the value of the following:
a) 2.5^2 b) 3.5^2 c) 4.5^2 d) 5.5^2
e) 7.2^2 f) 6.4^2 g) 0.8^2 h) 0.2^2
i) 5.3^2 j) 6.3^2

2. Check your answers to Q1 by using the x^2 key on a calculator.



Square roots

The square on the left contains 16 squares. It has sides of length 4 units.

So the square root of 16 is 4.

This can be written as $\sqrt{16} = 4$.

Note that $4 \times 4 = 16$ so 4 is the square root of 16.

However, -4×-4 is also 16 so -4 is also the square root of 16.

By convention, $\sqrt{16}$ means 'the positive square root of 16' so $\sqrt{16} = 4$ but the square root of 16 is ± 4 i.e. $+4$ or -4.

Note that -16 has no square root since any integer squared is positive.

Exercise 1.13

1. Find the following:

a) $\sqrt{25}$ b) $\sqrt{9}$ c) $\sqrt{49}$ d) $\sqrt{100}$

e) $\sqrt{121}$ f) $\sqrt{169}$ g) $\sqrt{0.01}$ h) $\sqrt{0.04}$

i) $\sqrt{0.09}$ j) $\sqrt{0.25}$

2. Use the $\sqrt{\ }$ key on your calculator to check your answers to Q.1.

3. Calculate the following:

a) $\sqrt{\frac{1}{9}}$ b) $\sqrt{\frac{1}{16}}$ c) $\sqrt{\frac{1}{25}}$ d) $\sqrt{\frac{1}{49}}$

e) $\sqrt{\frac{1}{100}}$ f) $\sqrt{\frac{4}{9}}$ g) $\sqrt{\frac{9}{100}}$ h) $\sqrt{\frac{49}{81}}$

i) $\sqrt{2\frac{7}{9}}$ j) $\sqrt{6\frac{1}{4}}$

Using a graph

Exercise 1.14

1. Copy and complete the table below for the equation $y = \sqrt{x}$.

x	0	1	4	9	16	25	36	49	64	81	100
y											

Plot the graph of $y = \sqrt{x}$. Use your graph to find the approximate values of the following:

a) $\sqrt{70}$ b) $\sqrt{40}$ c) $\sqrt{50}$ d) $\sqrt{90}$

e) $\sqrt{35}$ f) $\sqrt{45}$ g) $\sqrt{55}$ h) $\sqrt{60}$

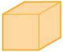
i) $\sqrt{2}$ j) $\sqrt{3}$ k) $\sqrt{20}$ l) $\sqrt{30}$

m) $\sqrt{12}$ n) $\sqrt{75}$ o) $\sqrt{115}$

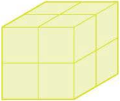
2. Check your answers to Q.1 above by using the $\sqrt{\ }$ key on a calculator.

Cubes of numbers

This cube has sides of 1 unit and occupies 1 cubic unit of space.



The cube below has sides of 2 units and occupies 8 cubic units of space. (That is, $2 \times 2 \times 2$.)



Exercise 1.15 How many cubic units would be occupied by cubes of side:

a) 3 units b) 5 units c) 10 units
d) 4 units e) 9 units f) 100 units?

In index notation, the cube numbers are 1^3, 2^3, 3^3, 4^3, etc.
4^3 is read as '4 cubed'.

Some calculators have an x^3 key. On others, to find a cube you multiply the number by itself three times.

Exercise 1.16 **1.** Copy and complete the table below:

Number	1	2	3	4	5	6	7	8	9	10
Cube			27							

2. Use a calculator to find the following:

a) 11^3 b) 0.5^3 c) 1.5^3 d) 2.5^3
e) 20^3 f) 30^3 g) $3^3 + 2^3$ h) $(3 + 2)^3$
i) $7^3 + 3^3$ j) $(7 + 3)^3$

Cube roots

$\sqrt[3]{\ }$ is read as 'the cube root of …'.

$\sqrt[3]{64}$ is 4, since $4 \times 4 \times 4 = 64$.

Note that $\sqrt[3]{64}$ is not -4

since $-4 \times -4 \times -4 = -64$

but $\sqrt[3]{-64}$ is -4.

Exercise 1.17

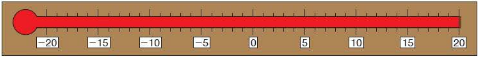
Find the following cube roots:

a) $\sqrt[3]{8}$ b) $\sqrt[3]{125}$ c) $\sqrt[3]{27}$ d) $\sqrt[3]{0.001}$

e) $\sqrt[3]{0.027}$ f) $\sqrt[3]{216}$ g) $\sqrt[3]{1000}$ h) $\sqrt[3]{1\,000\,000}$

i) $\sqrt[3]{-8}$ j) $\sqrt[3]{-27}$ k) $\sqrt[3]{-1000}$ l) $\sqrt[3]{-1}$

Directed numbers



Worked example The diagram above shows the scale of a thermometer. The temperature at 0400 was –3 °C. By 0900 the temperature had risen by 8°C. What was the temperature at 0900?

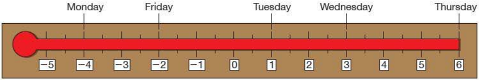
$(-3)^\circ + (8)^\circ = (5)^\circ$

Exercise 1.18

Find the new temperature if:

1. a) The temperature was −5 °C, and rises 9 °C.
b) The temperature was −12 °C, and rises 8 °C.
c) The temperature was +14 °C, and falls 8 °C.
d) The temperature was −3 °C, and falls 4 °C.
e) The temperature was −7 °C, and falls 11 °C.
f) The temperature was 2 °C, it falls 8 °C, then rises 6 °C.
g) The temperature was 5 °C, it falls 8 °C, then falls a further 6 °C.
h) The temperature was −2 °C, it falls 6 °C, then rises 10 °C.
i) The temperature was 20 °C, it falls 18 °C, then falls a further 8 °C.
j) The temperature was 5 °C below zero and falls 8 °C.

2. Mark lives in Canada. Every morning before school he reads a thermometer to find the temperature in the garden. The thermometer below shows the results for 5 days in winter.



Find the change in temperature between:
a) Monday and Friday b) Monday and Thursday
c) Tuesday and Friday d) Thursday and Friday
e) Monday and Tuesday.

3. The highest temperature ever recorded was in Libya. It was 58 °C. The lowest temperature ever recorded was −88 °C in Antarctica. What is the temperature difference?

4. Julius Caesar was born in 100BCE and was 56 years old when he died. In what year did he die?

5. Marcus Flavius was born in 20BCE and died in AD42. How old was he when he died?

6. Rome was founded in 753BCE. Constantinople fell to Mehmet Sultan Ahmet in AD1453, ending the Roman Empire in the East. For how many years did the Roman Empire last?

7. My bank account shows a credit balance of $105. Describe my balance as a positive or negative number after each of these transactions is made in sequence:
   a) rent $140
   b) car insurance $283
   c) 1 week's salary $230
   d) food bill $72
   e) credit transfer $250

8. A lift in the Empire State Building in New York has stopped somewhere close to the halfway point. Call this 'floor zero'. Show on a number line the floors it stops at as it makes the following sequence of journeys:
   a) up 75 floors
   b) down 155 floors
   c) up 110 floors
   d) down 60 floors
   e) down 35 floors
   f) up 100 floors

9. A hang-glider is launched from a mountainside in the Swiss Alps. It climbs 650 m and then starts its descent. It falls 1220 m before landing.
   a) How far below its launch point was the hang-glider when it landed?
   b) If the launch point was at 1650 m above sea level, at what height above sea level did it land?

10. The average noon temperature in Sydney in January is +32 °C. The average midnight temperature in Boston in January is −12 °C. What is the temperature difference between the two cities?

11. The temperature in Madrid on New Year's Day is −2 °C. The temperature in Moscow on the same day is −14 °C. What is the temperature difference between the two cities?

12. The temperature inside a freezer is −8 °C. To defrost it, the temperature is allowed to rise by 12 °C. What will the temperature be after this rise?

13. A plane flying at 8500 m drops a sonar device onto the ocean floor. If the sonar falls a total of 10 200 m, how deep is the ocean at this point?

14. The roof of an apartment block is 130 m above ground level. The car park beneath the apartment is 35 m below ground level. How high is the roof above the floor of the car park?

15. A submarine is at a depth of 165 m. If the ocean floor is 860 m from the surface, how far is the submarine from the ocean floor?

Student assessment 1

1. List all the factors of the following numbers:
a) 45 b) 64

2. Find the highest common factor of the following numbers:
a) 20, 24 b) 42, 72

3. Is the number 2.2 rational or irrational? Explain your answer.

4. Find the value of:
a) 7^2 b) 12^2 c) $(0.1)^2$ d) $(0.3)^2$

5. Draw a square of side 4.1 units. Use it to find $(4.1)^2$.

6. Find the value of:
a) $(2.5)^2$ b) $(3.3)^2$ c) $(0.25)^2$

7. Copy and complete the table below for $y = x^2$.

x	1	2	3	4	5	6	7	8
y								

Plot the graph of $y = x^2$. Use your graph to find the value of:
a) $(3.6)^2$ b) $(6.5)^2$ c) $(7.5)^2$ d) $(5.3)^2$

8. Calculate the following without using a calculator:
a) $\sqrt{49}$ b) $\sqrt{144}$ c) $\sqrt{0.64}$
d) $\sqrt{6\frac{1}{4}}$ e) $\sqrt{\frac{81}{100}}$ f) $\sqrt{2\frac{28}{36}}$

9. Without using a calculator, find:
a) 2^3 b) 5^3 c) 10^3

10. Without using a calculator, find:
a) $\sqrt[3]{64}$ b) $\sqrt[3]{125}$ c) $\sqrt[3]{1000}$

Student assessment 2

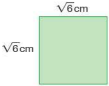
1. List the prime factors of the following numbers:

a) 28 b) 38

2. Find the lowest common multiple of the following numbers:

a) 6, 10 b) 7, 14, 28

3. The diagram shows a square with a side length of $\sqrt{6}$ cm.



Explain, giving reasons, whether the following are rational or irrational:

a) The perimeter of the square
b) The area of the square

4. Find the value of:

a) 9^2 b) 15^2 c) $(0.2)^2$ d) $(0.7)^2$

5. Draw a square of side 2.5 units. Use it to find $(2.5)^2$.

6. Calculate:

a) $(3.5)^2$ b) $(4.1)^2$ c) $(0.15)^2$

7. Copy and complete the table below for $y = \sqrt{x}$.

x	0	1	4	9	16	25	36	49
y								

Plot the graph of $y = \sqrt{x}$. Use your graph to find:

a) $\sqrt{7}$ b) $\sqrt{30}$ c) $\sqrt{45}$

8. Without using a calculator, find:

a) $\sqrt{225}$ b) $\sqrt{0.01}$ c) $\sqrt{0.81}$
d) $\sqrt{\frac{9}{25}}$ e) $\sqrt{5\frac{4}{9}}$ f) $\sqrt{2\frac{23}{49}}$

9. Without using a calculator, find:

a) 4^3 b) $(0.1)^3$ c) $(\frac{2}{3})^3$

10. Without using a calculator, find:

a) $\sqrt[3]{27}$ b) $\sqrt[3]{1\,000\,000}$ c) $\sqrt[3]{\frac{64}{125}}$

Student assessment 3

Date	Event
2900BCE	Great Pyramid built
1650BCE	Rhind Papyrus written
540BCE	Pythagoras born
300BCE	Euclid born
AD290	Lui Chih calculated π as 3.14
AD1500	Leonardo da Vinci born
AD1900	Albert Einstein born
AD1998	Fermat's last theorem proved

The table above shows dates of some significance to mathematics. Use the table to answer Q.1–6 below.

1. How many years before Einstein was born was the Great Pyramid built?

2. How many years before Leonardo was born was Pythagoras born?

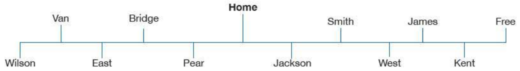
3. How many years after Lui Chih's calculation of π was Fermat's last theorem proved?

4. How many years were there between the births of Euclid and Einstein?

5. How long before Fermat's last theorem was proved was the Rhind Papyrus written?

6. How old was the Great Pyramid when Leonardo was born?

7. A bus route runs past Danny's house. Each stop is given the name of a street. From Home to Smith Street is the positive direction.



Find where Danny is after the stages of these journeys from Home:

a) $+4-3$
b) $+2-5$
c) $+2-7$
d) $-3-2$
e) $-1-1$
f) $+6-8+1$
g) $-1+3-5$
h) $-2-2+8$
i) $+1-3+5$
j) $-5+8-1$

8. Using the diagram from Q.7, and starting from Home each time, find the missing stages in these journeys if they end at the stop given:

a) $+3+?$	Pear	b) $+6+?$	Jackson
c) $-1+?$	Van	d) $-5+?$	James
e) $+5+?$	Home	f) $?-2$	Smith
g) $?+2$	East	h) $?-5$	Van
i) $?-1$	East	j) $?+4$	Pear

Student assessment 4

Date	Event
1200BCE	End of the Hittite Empire in Turkey
300BCE	Ptolemy rules Egypt
AD969	Fatimids found Cairo
AD1258	Mongols destroy Baghdad
AD1870	Suez Canal opens

Some significant dates in the history of the Middle East are shown in the table above. Use the table to answer Q.1–4 below.

1. How long after the end of the Hittite Empire did Ptolemy rule Egypt?

2. How many years before the destruction of Baghdad by Mongols was Cairo founded?

3. How many years were there between Ptolemy's rule and the opening of the Suez Canal?

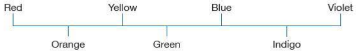
4. How many years were there between the end of the Hittite Empire and the founding of Cairo?

5. My bank statement for seven days in October is shown below:

Date	Payments ($)	Receipts ($)	Balance ($)
01/10			200
02/10	284		(a)
03/10		175	(b)
04/10	(c)		46
05/10		(d)	120
06/10	163		(e)
07/10		28	(f)

Copy and complete the statement by entering the amounts (a) to (f).

6. Stops on an underground railway are given the colours of the rainbow:



If I start at Green Station, where do I finish up if I make the following moves? (Positive moves are towards Violet and negative moves are towards Red.)

a) $+2-3$ b) $+3-2$ c) $+1-4$
d) $-1-2$ e) $+1-3$ f) $-3+5$
g) $-1-1+3$ h) $-2+5-2$ i) $+3-6+1$
j) $-1+4-2$ k) $-3+6-1$ l) $+3-6+3$

7. The noon and midnight temperatures, in degrees Celsius, in the Sahara during one week in January are shown below:

	Noon	Midnight	Range
Sunday	+30	0	(a)
Monday	+28	(b)	32
Tuesday	+26	–4	(c)
Wednesday	+32	(d)	34
Thursday	+33	+3	(e)
Friday	+34	(f)	30
Saturday	+28	–1	(g)

Copy and complete the above chart by putting the correct values for (a) to (g).

2 Accuracy

Approximation

In many instances exact numbers are not necessary or even desirable. In those circumstances approximations are given. The approximations can take several forms. The common types of approximation are dealt with below.

Rounding

If 28 617 people attend a gymnastics competition, this figure can be reported to various levels of accuracy.

To the nearest 10 000 this figure would be rounded up to 30 000.
To the nearest 1000 the figure would be rounded up to 29 000.
To the nearest 100 the figure would be rounded down to 28 600.

In this type of situation it is unlikely that the exact number would be reported.

Exercise 2.1

1. Round the following numbers to the nearest 1000:

a) 68 786	b) 74 245	c) 89 000
d) 4020	e) 99 500	f) 999 999

2. Round the following numbers to the nearest 100:

a) 78 540	b) 6858	c) 14 099
d) 8084	e) 950	f) 2984

3. Round the following numbers to the nearest 10:

a) 485	b) 692	c) 8847
d) 83	e) 4	f) 997

Decimal places

A number can also be approximated to a given number of decimal places (d.p.). This refers to the number of digits written after a decimal point.

Worked examples

a) Write 7.864 to 1 d.p.
The answer needs to be written with one digit after the decimal point. However, to do this, the second digit after the decimal point also needs to be considered. If it is 5 or more then the first digit is rounded up.

i.e. 7.864 is written as 7.9 to 1 d.p.

b) Write 5.574 to 2 d.p.
The answer here is to be given with two digits after the decimal point. In this case the third digit after the decimal point needs to be considered. As the third digit after the decimal point is less than 5, the second digit is not rounded up.

i.e. 5.574 is written as 5.57 to 2 d.p.

Exercise 2.2

1. Give the following to 1 d.p.

a) 5.58	b) 0.73	c) 11.86
d) 157.39	e) 4.04	f) 15.045
g) 2.95	h) 0.98	i) 12.049

2. Give the following to 2 d.p.

a) 6.473	b) 9.587	c) 16.476
d) 0.088	e) 0.014	f) 9.3048
g) 99.996	h) 0.0048	i) 3.0037

Significant figures

Numbers can also be approximated to a given number of significant figures (s.f.). In the number 43.25 the 4 is the most significant figure as it has a value of 40. In contrast, the 5 is the least significant as it only has a value of 5 hundredths.

Worked examples

a) Write 43.25 to 3 s.f.
Only the three most significant digits are written, however the fourth digit needs to be considered to see whether the third digit is to be rounded up or not.

i.e. 43.25 is written as 43.3 to 3 s.f.

b) Write 0.0043 to 1 s.f.
In this example only two digits have any significance, the 4 and the 3. The 4 is the most significant and therefore is the only one of the two to be written in the answer.

i.e. 0.0043 is written as 0.004 to 1 s.f.

Exercise 2.3

1. Write the following to the number of significant figures written in brackets:

a) 48 599 (1 s.f.)	b) 48 599 (3 s.f.)	c) 6841 (1 s.f.)
d) 7538 (2 s.f.)	e) 483.7 (1 s.f.)	f) 2.5728 (3 s.f.)
g) 990 (1 s.f.)	h) 2045 (2 s.f.)	i) 14.952 (3 s.f.)

2. Write the following to the number of significant figures written in brackets:

a) 0.085 62 (1 s.f.)	b) 0.5932 (1 s.f.)	c) 0.942 (2 s.f.)
d) 0.954 (1 s.f.)	e) 0.954 (2 s.f.)	f) 0.003 05 (1 s.f.)
g) 0.003 05 (2 s.f.)	h) 0.009 73 (2 s.f.)	i) 0.009 73 (1 s.f.)

Appropriate accuracy

In many instances calculations carried out using a calculator produce answers which are not whole numbers. A calculator will give the answer to as many decimal places as will fit on its screen. In most cases this degree of accuracy is neither desirable nor necessary. Unless specifically asked for, answers should not be given to more than two decimal places. Indeed, one decimal place is usually sufficient.

Worked example

Calculate $4.64 \div 2.3$ giving your answer to an appropriate degree of accuracy.

The calculator will give the answer to $4.64 \div 2.3$ as 2.0173913. However the answer given to 1 d.p. is sufficient.
Therefore $4.64 \div 2.3 = 2.0$ (1 d.p.).

Exercise 2.4

Calculate the following, giving your answer to an appropriate degree of accuracy:

a) 23.456×17.89 b) 0.4×12.62 c) 18×9.24
d) $76.24 \div 3.2$ e) 7.6^2 f) 16.42^3
g) $\frac{2.3 \times 3.37}{4}$ h) $\frac{8.31}{2.02}$ i) $9.2 \div 4^2$

Limits of accuracy

Numbers can be written to different degrees of accuracy. For example 4.5, 4.50 and 4.500, although appearing to represent the same number, do not. This is because they are written to different degrees of accuracy.

4.5 is written to one decimal place and therefore could represent any number from 4.45 up to but not including 4.55. On a number line this would be represented as:

As an inequality where x represents the number, 4.5 would be expressed as

$$4.45 \leqslant x < 4.55$$

4.45 is known as the **lower bound** of 4.5, whilst 4.55 is known as the **upper bound**.

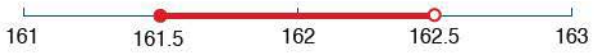
4.50 on the other hand is written to two decimal places and only numbers from 4.495 up to but not including 4.505 would be rounded to 4.50. This therefore represents a much smaller range of numbers than those which would be rounded to 4.5. Similarly the range of numbers being rounded to 4.500 would be even smaller.

Worked example A girl's height is given as 162 cm to the nearest centimetre.

i) Work out the lower and upper bounds within which her height can lie.

Lower bound = 161.5 cm
Upper bound = 162.5 cm

ii) Represent this range of numbers on a number line.



iii) If the girl's height is h cm, express this range as an inequality.

$161.5 \leqslant h < 162.5$

Exercise 2.5

1. Each of the following numbers is expressed to the nearest whole number.
i) Give the upper and lower bounds of each.
ii) Using x as the number, express the range in which the number lies as an inequality.
a) 6 b) 83 c) 152
d) 1000 e) 100

2. Each of the following numbers is correct to one decimal place.
i) Give the upper and lower bounds of each.
ii) Using x as the number, express the range in which the number lies as an inequality.
a) 3.8 b) 15.6 c) 1.0
d) 10.0 e) 0.3

3. Each of the following numbers is correct to two significant figures.
i) Give the upper and lower bounds of each.
ii) Using x as the number, express the range in which the number lies as an inequality.
a) 4.2 b) 0.84 c) 420
d) 5000 e) 0.045 f) 25 000

4. The mass of a sack of vegetables is given as 5.4 kg.
a) Illustrate the lower and upper bounds of the mass on a number line.
b) Using M kg for the mass, express the range of values in which M must lie as an inequality.

5. At a school sports day, the winning time for the 100 m race was given as 11.8 seconds.
a) Illustrate the lower and upper bounds of the time on a number line.
b) Using T seconds for the time, express the range of values in which T must lie as an inequality.

6. The capacity of a swimming pool is given as 620 m^3 correct to two significant figures.
   a) Calculate the lower and upper bounds of the pool's capacity.
   b) Using x cubic metres for the capacity, express the range of values in which x must lie as an inequality.

7. A farmer measures the dimensions of his rectangular field to the nearest 10 m. The length is recorded as 630 m and the width is recorded as 400 m.
   a) Calculate the lower and upper bounds of the length.
   b) Using W metres for the width, express the range of values in which W must lie as an inequality.

Exercise 2.6

1. Each of the following numbers is expressed to the nearest whole number.
   i) Give the upper and lower bounds of each.
   ii) Using x as the number, express the range in which the number lies as an inequality.
   a) 8 b) 71 c) 146
   d) 200 e) 1

2. Each of the following numbers is correct to one decimal place.
   i) Give the upper and lower bounds of each.
   ii) Using x as the number, express the range in which the number lies as an inequality.
   a) 2.5 b) 14.1 c) 2.0
   d) 20.0 e) 0.5

3. Each of the following numbers is correct to two significant figures.
   i) Give the upper and lower bounds of each.
   ii) Using x as the number, express the range in which the number lies as an inequality.
   a) 5.4 b) 0.75 c) 550
   d) 6000 e) 0.012 f) 10 000

4. The mass of a sack of vegetables is given as 7.8 kg.
   a) Illustrate the lower and upper bounds of the mass on a number line.
   b) Using M kg for the mass, express the range of values in which M must lie as an inequality.

5. At a school sports day, the winning time for the 100 m race was given as 12.1 s.
   a) Illustrate the lower and upper bounds of the time on a number line.
   b) Using T seconds for the time, express the range of values in which T must lie as an inequality.

6. The capacity of a swimming pool is given as 740 m^3 correct to two significant figures.
   a) Calculate the lower and upper bounds of the pool's capacity.
   b) Using x cubic metres for the capacity, express the range of values in which x must lie as an inequality.

7. A farmer measures the dimensions of his rectangular field to the nearest 10 m. The length is recorded as 570 m and the width is recorded as 340 m.
   a) Calculate the lower and upper bounds of the length.
   b) Using W metres for the width, express the range of values in which W must lie as an inequality.

Student assessment 1

1. Round the following numbers to the degree of accuracy shown in brackets:
   a) 2841 (nearest 100)
   b) 7286 (nearest 10)
   c) 48 756 (nearest 1000)
   d) 951 (nearest 100)

2. Round the following numbers to the number of decimal places shown in brackets:
   a) 3.84 (1 d.p.)
   b) 6.792 (1 d.p.)
   c) 0.8526 (2 d.p.)
   d) 1.5849 (2 d.p.)
   e) 9.954 (1 d.p.)
   f) 0.0077 (3 d.p.)

3. Round the following numbers to the number of significant figures shown in brackets:
   a) 3.84 (1 s.f.)
   b) 6.792 (2 s.f.)
   c) 0.7765 (1 s.f.)
   d) 9.624 (1 s.f.)
   e) 834.97 (2 s.f.)
   f) 0.004 51 (1 s.f.)

4. A cuboid's dimensions are given as 12.32 cm by 1.8 cm by 4.16 cm. Calculate its volume, giving your answer to an appropriate degree of accuracy.

Student assessment 2

1. Round the following numbers to the degree of accuracy shown in brackets:
   a) 6472 (nearest 10)
   b) 88 465 (nearest 100)
   c) 64 785 (nearest 1000)
   d) 6.7 (nearest 10)

2. Round the following numbers to the number of decimal places shown in brackets:
   a) 6.78 (1 d.p.)
   b) 4.438 (2 d.p.)
   c) 7.975 (1 d.p.)
   d) 63.084 (2 d.p.)
   e) 0.0567 (3 d.p.)
   f) 3.95 (2 d.p.)

3. Round the following numbers to the number of significant figures shown in brackets:
   a) 42.6 (1 s.f.) b) 5.432 (2 s.f.)
   c) 0.0574 (1 s.f.) d) 48 572 (2 s.f.)
   e) 687 453 (1 s.f.) f) 687 453 (3 s.f.)

4. A cuboid's dimensions are given as 3.973 m by 2.4 m by 3.16 m. Calculate its volume, giving your answer to an appropriate degree of accuracy.

Student assessment 3

1. The following numbers are expressed to the nearest whole number. Illustrate on a number line the range in which each must lie.
   a) 7 b) 40
   c) 300

2. The following numbers are expressed correct to two significant figures. Representing each number by the letter x, express the range in which each must lie using an inequality.
   a) 210 b) 64
   c) 300

3. A school measures the dimensions of its rectangular playing field to the nearest metre. The length was recorded as 350 m and the width as 200 m. Express the ranges in which the length and width lie using inequalities.

4. A boy's mass was measured to the nearest 0.1 kg. If his mass was recorded as 58.9 kg, illustrate on a number line the range within which it must lie.

5. An electronic clock is accurate to $\frac{1}{1000}$ of a second. The duration of a flash from a camera is timed at 0.004 second. Express the upper and lower bounds of the duration of the flash using inequalities.

6. The following numbers are rounded to the degree of accuracy shown in brackets. Express the lower and upper bounds of these numbers as an inequality.
   a) $x = 4.83$ (2 d.p.) b) $y = 5.05$ (2 d.p.)
   c) $z = 10.0$ (1 d.p.)

Student assessment 4

1. The following numbers are rounded to the nearest 100. Illustrate on a number line the range in which they must lie.
   a) 500 b) 7000
   c) 0

2. The following numbers are expressed correct to three significant figures. Represent the limits of each number using inequalities.
   a) 254 b) 40.5
   c) 0.410 d) 100

3. The dimensions of a rectangular courtyard are given to the nearest 0.5 m. The length is recorded as 20.5 m and the width as 10.0 m. Represent the limits of these dimensions using inequalities.

4. The circumference c of a tree is measured to the nearest 2 mm. If its circumference is measured as 245.6 cm, illustrate on a number line the range within which it must lie.

5. The time it takes Earth to rotate around the Sun is given as 365.25 days to five significant figures. What are the upper and lower bounds of this time?

6. The following numbers are rounded to the degree of accuracy shown in brackets. Express the lower and upper bounds of these numbers as an inequality.
   a) 10.90 (2 d.p.) b) 3.00 (2 d.p.)
   c) 0.5 (1 d.p.)

3 Calculations and order

Ordering

The following symbols have a specific meaning in mathematics:

- $=$ is equal to
- $\neq$ is not equal to
- $>$ is greater than
- $\geqslant$ is greater than or equal to
- $<$ is less than
- $\leqslant$ is less than or equal to

$x \geqslant 3$ states that x is greater than or equal to 3, i.e. x can be 3, 4, 4.2, 5, 5.6, etc.

$3 \leqslant x$ states that 3 is less than or equal to x, i.e. x can be 3, 4, 4.2, 5, 5.6, etc.

Therefore:

$5 > x$ can be rewritten as $x < 5$, i.e. x can be 4, 3.2, 3, 2.8, 2, 1, etc.

$-7 \leqslant x$ can be rewritten as $x \geqslant -7$, i.e. x can be -7, -6, -5, etc.

These inequalities can also be represented on a number line:

$x < 5$

$x \geqslant -7$

Note that ○⟶ implies that the number is not included in the solution whilst ●⟶ implies that the number is included in the solution.

Worked examples

a) Write $a > 3$ in words:

a is greater than 3.

b) Write 'x is greater than or equal to 8' using appropriate symbols:

$x \geqslant 8$

c) Write 'V is greater than 5, but less than or equal to 12' using the appropriate symbols:

$5 < V \leqslant 12$

Exercise 3.1

1. Write the following in words:

   a) $a < 7$ b) $b > 4$ c) $c \neq 8$
   d) $d \leqslant 3$ e) $e \geqslant 9$ f) $f \leqslant 11$

2. Rewrite the following, using the appropriate symbols:

   a) *a* is less than 4
   b) *b* is greater than 7
   c) *c* is equal to or less than 9
   d) *d* is equal to or greater than 5
   e) *e* is not equal to 3
   f) *f* is not more than 6
   g) *g* is not less than 9
   h) *h* is at least 6
   i) *i* is not 7
   j) *j* is not greater than 20

3. Write the following in words:

   a) $5 < n < 10$ b) $6 \leqslant n \leqslant 15$
   c) $3 \leqslant n < 9$ d) $8 < n \leqslant 12$

4. Write the following using the appropriate symbols:

   a) *p* is more than 7, but less than 10
   b) *q* is less than 12, but more than 3
   c) *r* is at least 5, but less than 9
   d) *s* is greater than 8, but not more than 15

Worked examples

a) The maximum number of players from one football team allowed on the pitch at any one time is eleven. Represent this information:
i) as an inequality,
ii) on a number line.

i) Let the number of players be represented by the letter *n*. *n* must be less than or equal to 11. Therefore $n \leqslant 11$.

ii) 

b) The maximum number of players from one football team allowed on the pitch at any one time is eleven. The minimum allowed is seven players. Represent this information:
i) as an inequality,
ii) on a number line.

i) Let the number of players be represented by the letter *n*. *n* must be greater than or equal to 7, but less than or equal to 11.

Therefore $7 \leqslant n \leqslant 11$.

ii) 

Exercise 3.2

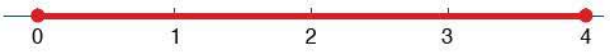
1. Copy each of the following statements, and insert one of the symbols =, >, < into the space to make the statement correct:
   a) $7 \times 2 \ldots 8 + 7$ b) $6^2 \ldots 9 \times 4$
   c) $5 \times 10 \ldots 7^2$ d) 80 cm … 1 m
   e) 1000 litres … 1 m^3 f) $48 \div 6 \ldots 54 \div 9$

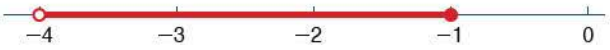
2. Represent each of the following inequalities on a number line, where x is a real number:
   a) $x < 2$ b) $x \geqslant 3$
   c) $x \leqslant -4$ d) $x \geqslant -2$
   e) $2 < x < 5$ f) $-3 < x < 0$
   g) $-2 \leqslant x < 2$ h) $2 \geqslant x \geqslant -1$

3. Write down the inequalities which correspond to the following number lines:

   a) 

   b) 

   c) 

   d) 

4. Write the following sentences using inequality signs.
   a) The maximum capacity of an athletics stadium is 20000 people.
   b) In a class the tallest student is 180 cm and the shortest is 135 cm.
   c) Five times a number plus 3 is less than 20.
   d) The maximum temperature in May was 25 °C.
   e) A farmer has between 350 and 400 apples on each tree in his orchard.
   f) In December temperatures in Kenya were between 11 °C and 28 °C.

Exercise 3.3

1. Write the following decimals in order of magnitude, starting with the largest:

   0.45 0.405 0.045 4.05 4.5

2. Write the following decimals in order of magnitude, starting with the smallest:

   6.0 0.6 0.66 0.606 0.06 6.6 6.606

3. Write the following decimals in order of magnitude, starting with the largest:

   0.906 0.96 0.096 9.06 0.609 0.690

4. Write the following fractions in order of magnitude, starting with the smallest:

$\frac{1}{3}$ $\frac{1}{4}$ $\frac{1}{2}$ $\frac{2}{5}$ $\frac{3}{10}$ $\frac{3}{4}$

5. Write the following fractions in order of magnitude, starting with the largest:

$\frac{1}{2}$ $\frac{1}{3}$ $\frac{6}{13}$ $\frac{4}{5}$ $\frac{7}{18}$ $\frac{2}{19}$

6. Write the following fractions in order of magnitude, starting with the smallest:

$\frac{3}{4}$ $\frac{3}{5}$ $\frac{2}{3}$ $\frac{4}{7}$ $\frac{5}{9}$ $\frac{1}{2}$

Exercise 3.4

1. Write the following lengths in order of magnitude, starting with the smallest:

0.5 km 5000 m 15 000 cm $\frac{2}{5}$ km 750 m

2. Write the following lengths in order of magnitude, starting with the smallest:

2 m 60 cm 800 mm 180 cm 0.75 m

3. Write the following masses in order of magnitude, starting with the largest:

4 kg 3500 g $\frac{3}{4}$ kg 700 g 1 kg

4. Write the following volumes in order of magnitude, starting with the smallest:

1 litre 430 ml 800 cm^3 120 cl 150 cm^3

Use of an electronic calculator

There are many different types of calculator available today. These include basic calculators, scientific calculators and the latest graphical calculators. However, these are all useless unless you make use of their potential. The following sections are aimed at familiarising you with some of the basic operations.

The four basic operations

Worked examples

a) Using a calculator, work out the answer to the following:

$12.3 + 14.9 =$

[1] [2] [.] [3] [+] [1] [4] [.] [9] [=] 27.2

b) Using a calculator, work out the answer to the following:

$16.3 \times 10.8 =$

[1] [6] [.] [3] [×] [1] [0] [.] [8] [=] 176.04

c) Using a calculator, work out the answer to the following:

$4.1 \times -3.3 =$

[4] [.] [1] [×] [3] [.] [3] [+/−] [=] −13.53

Exercise 3.5

1. Using a calculator, work out the answers to the following:
   a) $9.7 + 15.3$ b) $13.6 + 9.08$
   c) $12.9 + 4.92$ d) $115.0 + 6.24$
   e) $86.13 + 48.2$ f) $108.9 + 47.2$

2. Using a calculator, work out the answers to the following:
   a) $15.2 - 2.9$ b) $12.4 - 0.5$
   c) $19.06 - 20.3$ d) $4.32 - 4.33$
   e) $-9.1 - 21.2$ f) $-6.3 - 2.1$
   g) $-28 - -15$ h) $-2.41 - -2.41$

3. Using a calculator, work out the answers to the following:
   a) 9.2×8.7 b) 14.6×8.1
   c) $4.1 \times 3.7 \times 6$ d) $9.3 \div 3.1$
   e) 14.2×-3 f) $15.5 \div -5$
   g) -2.2×-2.2 h) $-20 \div -4.5$

The order of operations

When carrying out calculations, care must be taken to ensure that they are carried out in the correct order.

Worked examples

a) Use a scientific calculator to work out the answer to the following:

$2 + 3 \times 4 =$

[2] [+] [3] [×] [4] [=] 14

b) Use a scientific calculator to work out the answer to the following:

$(2 + 3) \times 4 =$

[(] [2] [+] [3] [)] [×] [4] [=] 20

The reason why different answers are obtained is because, by convention, the operations have different priorities. These are as follows:

(1) brackets
(2) multiplication/division
(3) addition/subtraction

Therefore in ***Worked example* a)** 3×4 is evaluated first, and then the 2 is added, whilst in ***Worked example* b)** $(2 + 3)$ is evaluated first, followed by multiplication by 4.

Exercise 3.6

In the following questions, evaluate the answers:
i) in your head,
ii) using a scientific calculator.

1. a) $8 \times 3 + 2$ b) $4 \div 2 + 8$
   c) $12 \times 4 - 6$ d) $4 + 6 \times 2$
   e) $10 - 6 \div 3$ f) $6 - 3 \times 4$

2. a) $7 \times 2 + 3 \times 2$ b) $12 \div 3 + 6 \times 5$
   c) $9 + 3 \times 8 - 1$ d) $36 - 9 \div 3 - 2$
   e) $14 \times 2 - 16 \div 2$ f) $4 + 3 \times 7 - 6 \div 3$

3. a) $(4 + 5) \times 3$ b) $8 \times (12 - 4)$
   c) $3 \times (8 + 3) - 3$ d) $(4 + 11) \div (7 - 2)$
   e) $4 \times 3 \times (7 + 5)$ f) $24 \div 3 \div (10 - 5)$

Exercise 3.7

In each of the following questions:
i) Copy the calculation and put in any brackets which are needed to make it correct.
ii) Check your answer using a scientific calculator.

1. a) $6 \times 2 + 1 = 18$ b) $1 + 3 \times 5 = 16$
   c) $8 + 6 \div 2 = 7$ d) $9 + 2 \times 4 = 44$
   e) $9 \div 3 \times 4 + 1 = 13$ f) $3 + 2 \times 4 - 1 = 15$

2. a) $12 \div 4 - 2 + 6 = 7$ b) $12 \div 4 - 2 + 6 = 12$
   c) $12 \div 4 - 2 + 6 = -5$ d) $12 \div 4 - 2 + 6 = 1.5$
   e) $4 + 5 \times 6 - 1 = 33$ f) $4 + 5 \times 6 - 1 = 29$
   g) $4 + 5 \times 6 - 1 = 53$ h) $4 + 5 \times 6 - 1 = 45$

It is important to use brackets when dealing with more complex calculations.

Worked examples

a) Evaluate the following using a scientific calculator:

$$\frac{12+9}{10-3} =$$

[( ] [1] [2] [+] [9] [ )] [÷] [( ] [1] [0] [−] [3] [ )] [=] 3

b) Evaluate the following using a scientific calculator:

$$\frac{20+12}{4^2} =$$

[( ] [2] [0] [+] [1] [2] [ )] [÷] [4] [x^2] [=] 2

c) Evaluate the following using a scientific calculator:

$$\frac{90+38}{4^3} =$$

[( ] [9] [0] [+] [3] [8] [ )] [÷] [4] [x^y] [3] [=] 2

Note: Different types of calculator have different 'to the power of' buttons.

Exercise 3.8 Using a scientific calculator, evaluate the following:

1. a) $\frac{9+3}{6}$ b) $\frac{30-6}{5+3}$

 c) $\frac{40+9}{12-5}$ d) $\frac{15\times 2}{7+8}+2$

 e) $\frac{100+21}{11}+4\times 3$ f) $\frac{7+2\times 4}{7-2}-3$

2. a) $\frac{4^2-6}{2+8}$ b) $\frac{3^2+4^2}{5}$

 c) $\frac{6^3-4^2}{4\times 25}$ d) $\frac{3^3\times 4^4}{12^2}+2$

 e) $\frac{3+3^3}{5}+\frac{4^2-2^3}{8}$ f) $\frac{(6+3)\times 4}{2^3}-2\times 3$

Checking answers to calculations

Even though many calculations can be done quickly and effectively on a calculator, often an estimate for an answer can be a useful check. This is found by rounding each of the numbers in such a way that the calculation becomes relatively straightforward.

Worked examples

a) Estimate the answer to 57 × 246.

Here are two possibilities:
i) 60 × 200 = 12 000,
ii) 50 × 250 = 12 500.

b) Estimate the answer to 6386 ÷ 27.

6000 ÷ 30 = 200

Exercise 3.9

1. Without using a calculator, estimate the answers to the following:
 a) 62 × 19 b) 270 × 12 c) 55 × 60
 d) 4950 × 28 e) 0.8 × 0.95 f) 0.184 × 475

2. Without using a calculator, estimate the answers to the following:
 a) 3946 ÷ 18 b) 8287 ÷ 42 c) 906 ÷ 27
 d) 5520 ÷ 13 e) 48 ÷ 0.12 f) 610 ÷ 0.22

3. Without using a calculator, estimate the answers to the following:

a) $78.45 + 51.02$ b) $168.3 - 87.09$ c) 2.93×3.14

d) $84.2 \div 19.5$ e) $\dfrac{4.3 \times 752}{15.6}$ f) $\dfrac{(9.8)^3}{(2.2)^2}$

4. Using estimation, identify which of the following are definitely incorrect. Explain your reasoning clearly.

a) $95 \times 212 = 20\,140$ b) $44 \times 17 = 748$

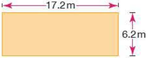
c) $689 \times 413 = 28\,457$ d) $142\,656 \div 8 = 17\,832$

e) $77.9 \times 22.6 = 2512.54$

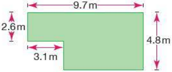
f) $\dfrac{8.42 \times 46}{0.2} = 19\,366$

5. Estimate the shaded areas of the following shapes. Do *not* work out an exact answer.

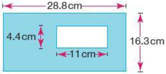
a)



b)

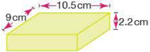


c)

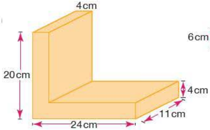


6. Estimate the volume of each of the solids below. Do *not* work out an exact answer.

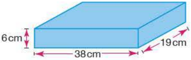
a)



b)



c)



Student assessment 1

1. Write the following in words:

a) $m = 7$ b) $n \neq 10$

c) $e < 4$ d) $f \leqslant 6$

e) $x \geqslant 12$ f) $y > -1$

2. Rewrite the following using the appropriate symbols:
   a) a is greater than 5
   b) b is greater than or equal to 6
   c) c is at least 4
   d) d is not equal to 14

3. Illustrate each of the following inequalities on a number line:
   a) $d < 4$
   b) $e \geqslant 6$
   c) $2 < f < 5$
   d) $-3 \leqslant g < 1$

4. Illustrate the information in each of the following statements on a number line:
   a) The temperature in a greenhouse must be kept between 18 °C and 28 °C.
   b) The pressure in a car tyre should be less than 2.3 kPa, and equal to or more than 1.9 kPa.

5. Write the following decimals in order of magnitude, starting with the smallest:

   0.99 9.99 9.09 9 0.09 0.9

Student assessment 2

1. Write the following in words:
   a) $p \neq 2$
   b) $q > 0$
   c) $r \leqslant 3$
   d) $s = 2$
   e) $t \geqslant 1$
   f) $u < -5$

2. Rewrite the following using the appropriate symbols:
   a) a is less than 2
   b) b is less than or equal to 4
   c) c is equal to 8
   d) d is not greater than 0

3. Illustrate each of the following inequalities on a number line:
   a) $j > 2$
   b) $k \leqslant 16$
   c) $-2 \leqslant L \leqslant 5$
   d) $3 \leqslant m < 7$

4. Illustrate the information in each of the following statements on a number line:
   a) A ferry can carry no more than 280 cars.
   b) The minimum temperature overnight was 4 °C.

5. Write the following masses in order of magnitude, starting with the largest:

   900 g 1 kg 1800 g 1.09 kg 9 g

Student assessment 3

1. Insert one of the symbols =, > or < into the space to make the following statements correct:
   a) $8 \times 5 \ldots 5 \times 8$
   b) $9^2 \ldots 100 - 21$
   c) 45 cm … 0.5 m
   d) Days in June … 31 days

2. Illustrate the following information on a number line:
   a) The greatest number of days in a month is 31.
   b) A month has at least 28 days.
   c) A cake takes $1\frac{1}{2}$ to $1\frac{3}{4}$ hours to bake.
   d) An aeroplane will land between 1440 and 1445.

3. Write the following sentences using inequalities:
   a) No more than 52 people can be carried on a bus.
   b) The minimum temperature tonight will be 11 °C.
   c) There are between 24 and 38 pupils in a class.
   d) Three times a number plus six is less than 50.
   e) The minimum reaction time for an alarm system is 0.03 second.

4. Illustrate the following inequalities on a number line:
   a) $x > 5$ b) $y \leqslant -3$
   c) $-3 < x < -1$ d) $-2 \leqslant y < 1$

5. Write the following fractions in order of magnitude, starting with the largest:

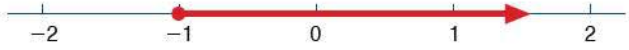
   $\frac{1}{6}$ $\frac{2}{3}$ $\frac{7}{12}$ $\frac{13}{18}$ $\frac{6}{7}$

Student assessment 4

1. Insert one of the symbols =, > or < into the space to make the following statements correct:
   a) $4 \times 2 \ldots 2^3$ b) $6^2 \ldots 2^6$
   c) 850 ml . . . 0.5 litre d) Days in May . . . 30 days

2. Illustrate the following information on a number line:
   a) The temperature during the day reached a maximum of 35 °C.
   b) There were 20 to 25 pupils in a class.
   c) The world record for the 100 m sprint is under 10 seconds.
   d) Doubling a number and subtracting four gives an answer greater than 16.

3. Write the information on the following number lines as inequalities:

   a) 

   b) 

   c) 

   d) 

4. Illustrate each of the following inequalities on a number line:

a) $x \geqslant 3$ b) $x < 4$
c) $0 < x < 4$ d) $-3 \leqslant x < 1$

5. Write the following fractions in order of magnitude, starting with the smallest:

$\frac{4}{7}$ $\frac{3}{14}$ $\frac{9}{10}$ $\frac{1}{2}$ $\frac{2}{5}$

Student assessment 5

1. Using a calculator, work out the answers to the following:

a) $6.9 + 18.2$ b) $12.2 - 4.9$
c) $-5.1 + 10$ d) $4.8 - -8.2$
e) 3.3×-4.2 f) $-15 \div -4$

2. Evaluate the following:

a) $6 \times 8 - 4$ b) $3 + 5 \times 2$
c) $3 \times 3 + 4 \times 4$ d) $3 + 3 \times 4 + 4$
e) $(5 + 2) \times 7$ f) $18 \div 2 \div (5 - 2)$

3. Copy the following, if necessary putting in brackets to make the statement correct:

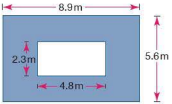
a) $7 - 4 \times 2 = 6$ b) $12 + 3 \times 3 + 4 = 33$
c) $5 + 5 \times 6 - 4 = 20$ d) $5 + 5 \times 6 - 4 = 56$

4. Evaluate the following using a calculator:

a) $\frac{2^4 - 3^2}{2}$ b) $\frac{(8-3)\times 3}{5} + 7$

5. 1 mile is 1760 yards. Estimate the number of yards in 11.5 miles.

6. Estimate the shaded area of the figure below:



7. Estimate the answers to the following. Do *not* work out exact answers.

a) $\frac{5.3\times 11.2}{2.1}$ b) $\frac{(9.8)^2}{(4.7)^2}$ c) $\frac{18.8\times(7.1)^2}{(3.1)^2\times(4.9)^2}$

Student assessment 6

1. Using a calculator, work out the answers to the following:
   a) $7.1 + 8.02$ b) $2.2 - 5.8$
   c) $-6.1 + 4$ d) $4.2 - -5.2$
   e) -3.6×4.1 f) $-18 \div -2.5$

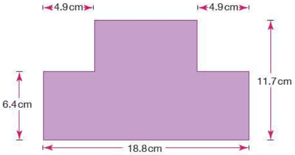
2. Evaluate the following:
   a) $3 \times 9 - 7$ b) $12 + 6 \div 2$
   c) $3 + 4 \div 2 \times 4$ d) $6 + 3 \times 4 - 5$
   e) $(5 + 2) \div 7$ f) $14 \times 2 \div (9 - 2)$

3. Copy the following, if necessary putting in brackets to make the statement correct:
   a) $7 - 5 \times 3 = 6$ b) $16 + 4 \times 2 + 4 = 40$
   c) $4 + 5 \times 6 - 1 = 45$ d) $1 + 5 \times 6 - 6 = 30$

4. Using a calculator, evaluate the following:
   a) $\dfrac{3^3 - 4^2}{2}$ b) $\dfrac{(15-3) \div 3}{2} + 7$

5. 1 mile is 1760 yards. Estimate the number of yards in 19 miles.

6. Estimate the area of the figure below:



7. Estimate the answers to the following. Do *not* work out exact answers.
   a) $\dfrac{3.9 \times 26.4}{4.85}$ b) $\dfrac{(3.2)^3}{(5.4)^2}$ c) $\dfrac{2.8 \times (7.3)^2}{(3.2)^2 \times 6.2}$

4 Integers, fractions, decimals and percentages

Fractions

A single unit can be broken into equal parts called fractions, e.g. $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$.

If, for example, the unit is broken into ten equal parts and three parts are then taken, the fraction is written as $\frac{3}{10}$. That is, three parts out of ten parts.

In the fraction $\frac{3}{10}$:

The ten is called the **denominator**.

The three is called the **numerator**.

A **proper fraction** has its numerator less than its denominator, e.g. $\frac{3}{4}$.

An **improper fraction** has its numerator more than its denominator, e.g. $\frac{9}{2}$.

A **mixed number** is made up of a whole number and a proper fraction, e.g. $4\frac{1}{5}$.

Exercise 4.1

1. Copy the following fractions and indicate which is the numerator and which the denominator.

   a) $\frac{2}{3}$ b) $\frac{15}{22}$ c) $\frac{4}{3}$ d) $\frac{5}{2}$

2. Draw up a table with three columns headed 'proper fraction', 'improper fraction' and 'mixed number'. Put each of the numbers below into the appropriate column.

   a) $\frac{2}{3}$ b) $\frac{15}{22}$ c) $\frac{4}{3}$ d) $\frac{5}{2}$
   e) $1\frac{1}{2}$ f) $2\frac{3}{4}$ g) $\frac{7}{4}$ h) $\frac{7}{11}$
   i) $7\frac{1}{4}$ j) $\frac{5}{6}$ k) $\frac{6}{5}$ l) $1\frac{1}{5}$
   m) $\frac{1}{10}$ n) $2\frac{7}{8}$ o) $\frac{5}{3}$

A fraction of an amount

Worked examples

a) Find $\frac{1}{5}$ of 35.

This means 'divide 35 into 5 equal parts'.

$\frac{1}{5}$ of 35 is $\frac{35}{5} = 7$.

b) Find $\frac{3}{5}$ of 35.

Since $\frac{1}{5}$ of 35 is 7, $\frac{3}{5}$ of 35 is 7×3.

That is, 21.

Exercise 4.2

1. Evaluate the following:

a) $\frac{1}{5}$ of 40 b) $\frac{3}{5}$ of 40 c) $\frac{1}{9}$ of 36 d) $\frac{5}{9}$ of 36
e) $\frac{1}{8}$ of 72 f) $\frac{7}{8}$ of 72 g) $\frac{1}{12}$ of 60 h) $\frac{5}{12}$ of 60
i) $\frac{1}{4}$ of 8 j) $\frac{3}{4}$ of 8

2. Evaluate the following:

a) $\frac{3}{4}$ of 12 b) $\frac{4}{5}$ of 20 c) $\frac{4}{9}$ of 45 d) $\frac{5}{8}$ of 64
e) $\frac{3}{11}$ of 66 f) $\frac{9}{10}$ of 80 g) $\frac{5}{7}$ of 42 h) $\frac{8}{9}$ of 54
i) $\frac{7}{8}$ of 240 j) $\frac{4}{5}$ of 65

Changing a mixed number to an improper fraction

Worked examples

a) Change $2\frac{3}{4}$ to an improper fraction.

$1 = \frac{4}{4}$

$2 = \frac{8}{4}$

$2\frac{3}{4} = \frac{8}{4} + \frac{3}{4}$

$= \frac{11}{4}$

b) Change $3\frac{5}{8}$ to an improper fraction.

$3\frac{5}{8} = \frac{24}{8} + \frac{5}{8}$

$= \frac{24+5}{8}$

$= \frac{29}{8}$

Exercise 4.3

Change the following mixed numbers to improper fractions:

a) $4\frac{2}{3}$ b) $3\frac{3}{5}$ c) $5\frac{7}{8}$ d) $2\frac{5}{6}$
e) $8\frac{1}{2}$ f) $9\frac{5}{7}$ g) $6\frac{4}{9}$ h) $4\frac{1}{4}$
i) $5\frac{4}{11}$ j) $7\frac{6}{7}$ k) $4\frac{3}{10}$ l) $11\frac{3}{13}$

Changing an improper fraction to a mixed number

Worked example

Change $\frac{27}{4}$ to a mixed number.

$\frac{27}{4} = \frac{24+3}{4}$

$= \frac{24}{4} + \frac{3}{4}$

$= 6\frac{3}{4}$

Exercise 4.4

Change the following improper fractions to mixed numbers:

a) $\frac{29}{4}$ b) $\frac{33}{5}$ c) $\frac{41}{6}$ d) $\frac{53}{8}$
e) $\frac{49}{9}$ f) $\frac{17}{12}$ g) $\frac{66}{7}$ h) $\frac{33}{10}$
i) $\frac{19}{2}$ j) $\frac{73}{12}$

Decimals

H	T	U	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
		3	2	7	
		0	0	3	8

3.27 is 3 units, 2 tenths and 7 hundredths

i.e. $3.27 = 3 + \frac{2}{10} + \frac{7}{100}$

0.038 is 3 hundredths and 8 thousandths

i.e. $0.038 = \frac{3}{100} + \frac{8}{1000}$

Note that 2 tenths and 7 hundredths is equivalent to 27 hundredths

i.e. $\frac{2}{10} + \frac{7}{100} = \frac{27}{100}$

and that 3 hundredths and 8 thousandths is equivalent to 38 thousandths

i.e. $\frac{3}{100} + \frac{8}{1000} = \frac{38}{1000}$

A **decimal fraction** is a fraction between 0 and 1 in which the denominator is a power of 10 and the numerator is an integer.

$\frac{3}{10}, \frac{23}{100}, \frac{17}{1000}$ are all examples of decimal fractions.

$\frac{1}{20}$ is not a decimal fraction because the denominator is not a power of 10.

$\frac{11}{10}$ is not a decimal fraction because its value is greater than 1.

Exercise 4.5

1. Make a table similar to the one above. List the digits in the following numbers in their correct position:
   a) 6.023 b) 5.94 c) 18.3
   d) 0.071 e) 2.001 f) 3.56

2. Write the following fractions as decimals:
   a) $4\frac{5}{10}$ b) $6\frac{3}{10}$ c) $17\frac{8}{10}$ d) $3\frac{7}{100}$
   e) $9\frac{27}{100}$ f) $11\frac{36}{100}$ g) $4\frac{6}{1000}$ h) $5\frac{27}{1000}$
   i) $4\frac{356}{1000}$ j) $9\frac{204}{1000}$

3. Evaluate the following without using a calculator:
   a) $2.7 + 0.35 + 16.09$ b) $1.44 + 0.072 + 82.3$
   c) $23.8 - 17.2$ d) $16.9 - 5.74$
   e) $121.3 - 85.49$ f) $6.03 + 0.5 - 1.21$
   g) $72.5 - 9.08 + 3.72$ h) $100 - 32.74 - 61.2$
   i) $16.0 - 9.24 - 5.36$ j) $1.1 - 0.92 - 0.005$

Percentages

A fraction whose denominator is 100 can be expressed as a percentage.

$\frac{29}{100}$ can be written as 29%

$\frac{45}{100}$ can be written as 45%

Exercise 4.6 Write the following fractions as percentages:

a) $\frac{39}{100}$ b) $\frac{42}{100}$ c) $\frac{63}{100}$ d) $\frac{5}{100}$

Changing a fraction to a percentage

By using equivalent fractions to change the denominator to 100, other fractions can be written as percentages.

Worked example Change $\frac{3}{5}$ to a percentage.

$\frac{3}{5} = \frac{3}{5} \times \frac{20}{20} = \frac{60}{100}$

$\frac{60}{100}$ can be written as 60%

Exercise 4.7

1. Express each of the following as a fraction with denominator 100, then write them as percentages:

a) $\frac{29}{50}$ b) $\frac{17}{25}$ c) $\frac{11}{20}$ d) $\frac{3}{10}$

e) $\frac{23}{25}$ f) $\frac{19}{50}$ g) $\frac{3}{4}$ h) $\frac{2}{5}$

2. Copy and complete the table of equivalents below.

Fraction	Decimal	Percentage
$\frac{1}{10}$		
	0.2	
		30%
$\frac{4}{10}$		
	0.5	
		60%
	0.7	
$\frac{4}{5}$		
	0.9	
$\frac{1}{4}$		
		75%

The four rules

Calculations with whole numbers

Addition, subtraction, multiplication and division are mathematical operations.

Long multiplication

When carrying out long multiplication, it is important to remember place value.

Worked example

184×37

$$\begin{array}{rl} 1\;8\;4 & \\ \times \quad 3\;7 & \\ \hline 1\;2\;8\;8 & (184 \times 7) \\ 5\;5\;2\;0 & (184 \times 30) \\ \hline 6\;8\;0\;8 & (184 \times 37) \end{array}$$

Short division

Worked example

$453 \div 6$

$$\begin{array}{r|l} & 7\;\;5\;\text{r}3 \\ \hline 6 & 4\;\;5\;\;{}^{3}3 \end{array}$$

It is usual, however, to give the final answer in decimal form rather than with a remainder. The division should therefore be continued:

$453 \div 6$

$$\begin{array}{r|l} & 7\;\;5\;.\;5 \\ \hline 6 & 4\;\;5\;\;{}^{3}3\;.\;{}^{3}0 \end{array}$$

Long division

Worked example Calculate $7184 \div 23$ to one decimal place (1 d.p.).

$$\begin{array}{r|l} & 3\;\;1\;\;2\;.\;3\;\;4 \\ \hline 23 & 7\;\;1\;\;8\;\;4\;.\;0\;\;0 \\ & \underline{6\;\;9} \\ & \;\;\;2\;\;8 \\ & \;\;\;\underline{2\;\;3} \\ & \;\;\;\;\;\;5\;\;4 \\ & \;\;\;\;\;\;\underline{4\;\;6} \\ & \;\;\;\;\;\;\;\;\;8\;\;0 \\ & \;\;\;\;\;\;\;\;\;\underline{6\;\;9} \\ & \;\;\;\;\;\;\;\;\;1\;\;1\;\;0 \\ & \;\;\;\;\;\;\;\;\;\;\;\;\;\underline{9\;\;2} \\ & \;\;\;\;\;\;\;\;\;\;\;\;\;\underline{1\;\;8} \end{array}$$

Therefore $7184 \div 23 = 312.3$ to 1 d.p.

Mixed operations

When a calculation involves a mixture of operations, the order of the operations is important. Multiplications and divisions are done first, whilst additions and subtractions are done afterwards. To override this, brackets need to be used.

Worked examples

a) $3 + 7 \times 2 - 4$
$= 3 + 14 - 4$
$= 13$

b) $(3 + 7) \times 2 - 4$
$= 10 \times 2 - 4$
$= 20 - 4$
$= 16$

c) $3 + 7 \times (2 - 4)$
$= 3 + 7 \times (-2)$
$= 3 - 14$
$= -11$

d) $(3 + 7) \times (2 - 4)$
$= 10 \times (-2)$
$= -20$

Exercise 4.8

1. Evaluate the answer to each of the following:
a) $3 + 5 \times 2 - 4$
b) $6 + 4 \times 7 - 12$
c) $3 \times 2 + 4 \times 6$
d) $4 \times 5 - 3 \times 6$
e) $8 \div 2 + 18 \div 6$
f) $12 \div 8 + 6 \div 4$

2. Copy these equations and put brackets in the correct places to make them correct:
a) $6 \times 4 + 6 \div 3 = 20$
b) $6 \times 4 + 6 \div 3 = 36$
c) $8 + 2 \times 4 - 2 = 12$
d) $8 + 2 \times 4 - 2 = 20$
e) $9 - 3 \times 7 + 2 = 44$
f) $9 - 3 \times 7 + 2 = 54$

3. Without using a calculator, work out the solutions to the following multiplications:
a) 63×24
b) 531×64
c) 785×38
d) 164×253
e) 144×144
f) 170×240

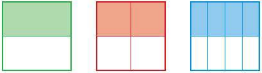
4. Work out the remainders in the following divisions:
a) $33 \div 7$
b) $68 \div 5$
c) $72 \div 7$
d) $430 \div 9$
e) $156 \div 5$
f) $687 \div 10$

5. a) The sum of two numbers is 16, their product is 63. What are the two numbers?
b) When a number is divided by 7 the result is 14 remainder 2. What is the number?
c) The difference between two numbers is 5, their product is 176. What are the numbers?
d) How many 9s can be added to 40 before the total exceeds 100?
e) A length of rail track is 9 m long. How many complete lengths will be needed to lay 1 km of track?
f) How many 35 cent stamps can be bought for 10 dollars?

6. Work out the following long divisions to 1 d.p.
a) $7892 \div 7$
b) $45\,623 \div 6$
c) $9452 \div 8$
d) $4564 \div 4$
e) $7892 \div 15$
f) $79\,876 \div 24$

Fractions

Equivalent fractions



$\frac{1}{2}$ $\frac{2}{4}$ $\frac{4}{8}$

It should be apparent that $\frac{1}{2}$, $\frac{2}{4}$ and $\frac{4}{8}$ are equivalent fractions.

Similarly, $\frac{1}{3}$, $\frac{2}{6}$, $\frac{3}{9}$ and $\frac{4}{12}$ are equivalent, as are $\frac{1}{5}$, $\frac{10}{50}$ and $\frac{20}{100}$.

Equivalent fractions are mathematically the same as each other. In the diagrams above $\frac{1}{2}$ is mathematically the same as $\frac{4}{8}$. However $\frac{1}{2}$ is a simplified form of $\frac{4}{8}$.

When carrying out calculations involving fractions it is usual to give your answer in its **simplest form**. Another way of saying 'simplest form' is '**lowest terms**'.

Worked examples

a) Write $\frac{4}{22}$ in its simplest form.

Divide both the numerator and the denominator by their highest common factor.

The highest common factor of both 4 and 22 is 2.

Dividing both 4 and 22 by 2 gives $\frac{2}{11}$.

Therefore $\frac{2}{11}$ is $\frac{4}{22}$ written in its simplest form.

b) Write $\frac{12}{40}$ in its lowest terms.

Divide both the numerator and the denominator by their highest common factor.

The highest common factor of both 12 and 40 is 4.

Dividing both 12 and 40 by 4 gives $\frac{3}{10}$.

Therefore $\frac{3}{10}$ is $\frac{12}{40}$ written in its lowest terms.

Exercise 4.9

1. Copy the following sets of equivalent fractions and fill in the blanks:

a) $\frac{2}{5} = \frac{4}{} = \frac{}{20} = \frac{}{50} = \frac{16}{}$

b) $\frac{3}{8} = \frac{6}{} = \frac{}{24} = \frac{15}{} = \frac{}{72}$

c) $\frac{}{7} = \frac{8}{14} = \frac{12}{} = \frac{}{56} = \frac{36}{}$

d) $\frac{5}{} = \frac{}{27} = \frac{20}{36} = \frac{}{90} = \frac{55}{}$

2. Express the following fractions in their lowest terms.

a) $\frac{5}{10}$ b) $\frac{7}{21}$ c) $\frac{8}{12}$

d) $\frac{16}{36}$ e) $\frac{75}{100}$ f) $\frac{81}{90}$

3. Write the following improper fractions as mixed numbers.
e.g. $\frac{15}{4} = 3\frac{3}{4}$

a) $\frac{17}{4}$ b) $\frac{23}{5}$ c) $\frac{8}{3}$

d) $\frac{19}{3}$ e) $\frac{12}{3}$ f) $\frac{43}{12}$

4. Write the following mixed numbers as improper fractions.
e.g. $3\frac{4}{5} = \frac{19}{5}$

a) $6\frac{1}{2}$ b) $7\frac{1}{4}$ c) $3\frac{3}{8}$

d) $11\frac{1}{9}$ e) $6\frac{4}{5}$ f) $8\frac{9}{11}$

Addition and subtraction of fractions

For fractions to be either added or subtracted, the denominators need to be the same.

Worked examples

a) $\frac{3}{11} + \frac{5}{11} = \frac{8}{11}$

b) $\frac{7}{8} + \frac{5}{8} = \frac{12}{8} = 1\frac{1}{2}$

c) $\frac{1}{2} + \frac{1}{3}$
$= \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$

d) $\frac{4}{5} - \frac{1}{3}$
$= \frac{12}{15} - \frac{5}{15} = \frac{7}{15}$

When dealing with calculations involving mixed numbers, it is sometimes easier to change them to improper fractions first.

Worked examples

a) $5\frac{3}{4} - 2\frac{5}{8}$
$= \frac{23}{4} - \frac{21}{8}$
$= \frac{46}{8} - \frac{21}{8}$
$= \frac{25}{8} = 3\frac{1}{8}$

b) $1\frac{4}{7} + 3\frac{3}{4}$
$= \frac{11}{7} + \frac{15}{4}$
$= \frac{44}{28} + \frac{105}{28}$
$= \frac{149}{28} = 5\frac{9}{28}$

Exercise 4.10

Evaluate each of the following and write the answer as a fraction in its simplest form:

1. a) $\frac{3}{5} + \frac{4}{5}$ b) $\frac{3}{11} + \frac{7}{11}$

c) $\frac{2}{3} + \frac{1}{4}$ d) $\frac{3}{5} + \frac{4}{9}$

e) $\frac{8}{13} + \frac{2}{5}$ f) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4}$

2. a) $\frac{1}{8} + \frac{3}{8} + \frac{5}{8}$ b) $\frac{3}{7} + \frac{5}{7} + \frac{4}{7}$

c) $\frac{1}{3} + \frac{1}{2} + \frac{1}{4}$ d) $\frac{1}{5} + \frac{1}{3} + \frac{1}{4}$

e) $\frac{3}{8} + \frac{3}{5} + \frac{3}{4}$ f) $\frac{3}{13} + \frac{1}{4} + \frac{1}{2}$

3. a) $\frac{3}{7} - \frac{2}{7}$ b) $\frac{4}{5} - \frac{7}{10}$
c) $\frac{8}{9} - \frac{1}{3}$ d) $\frac{7}{12} - \frac{1}{2}$
e) $\frac{5}{8} - \frac{2}{5}$ f) $\frac{3}{4} - \frac{2}{5} + \frac{7}{10}$

4. a) $\frac{3}{4} + \frac{1}{5} - \frac{2}{3}$ b) $\frac{3}{8} + \frac{7}{11} - \frac{1}{2}$
c) $\frac{4}{5} - \frac{3}{10} + \frac{7}{20}$ d) $\frac{9}{13} + \frac{1}{3} - \frac{4}{5}$
e) $\frac{9}{10} - \frac{1}{5} - \frac{1}{4}$ f) $\frac{8}{9} - \frac{1}{3} - \frac{1}{2}$

5. a) $2\frac{1}{2} + 3\frac{1}{4}$ b) $3\frac{3}{5} + 1\frac{7}{10}$
c) $6\frac{1}{2} - 3\frac{2}{5}$ d) $8\frac{5}{8} - 2\frac{1}{3}$
e) $5\frac{7}{8} - 4\frac{3}{4}$ f) $3\frac{1}{4} - 2\frac{5}{9}$

6. a) $2\frac{1}{2} + 1\frac{1}{4} + 1\frac{3}{8}$ b) $2\frac{4}{5} + 3\frac{1}{8} + 1\frac{3}{10}$
c) $4\frac{1}{2} - 1\frac{1}{4} - 3\frac{5}{8}$ d) $6\frac{1}{2} - 2\frac{3}{4} - 3\frac{2}{5}$
e) $2\frac{4}{7} - 3\frac{1}{4} - 1\frac{3}{5}$ f) $4\frac{7}{20} - 5\frac{1}{2} + 2\frac{2}{5}$

Multiplication and division of fractions

Worked examples

a) $\frac{3}{4} \times \frac{2}{3}$
$= \frac{6}{12}$
$= \frac{1}{2}$

b) $3\frac{1}{2} \times 4\frac{4}{7}$
$= \frac{7}{2} \times \frac{32}{7}$
$= \frac{224}{14}$
$= 16$

The reciprocal of a number is obtained when 1 is divided by that number. The reciprocal of 5 is $\frac{1}{5}$, the reciprocal of $\frac{2}{5}$ is $\frac{5}{2}$, etc.

Worked examples

Dividing fractions is the same as multiplying by the reciprocal.

a) $\frac{3}{8} \div \frac{3}{4}$
$= \frac{3}{8} \times \frac{4}{3}$
$= \frac{12}{24}$
$= \frac{1}{2}$

b) $5\frac{1}{2} \div 3\frac{2}{3}$
$= \frac{11}{2} \div \frac{11}{3}$
$= \frac{11}{2} \times \frac{3}{11}$
$= \frac{3}{2}$

Exercise 4.11

1. Write the reciprocal of each of the following:

a) $\frac{3}{4}$ b) $\frac{5}{9}$ c) 7
d) $\frac{1}{9}$ e) $2\frac{3}{4}$ f) $4\frac{5}{8}$

2. Write the reciprocal of each of the following:

a) $\frac{1}{8}$ b) $\frac{7}{12}$ c) $\frac{3}{5}$
d) $1\frac{1}{2}$ e) $3\frac{3}{4}$ f) 6

3. Evaluate the following:

a) $\frac{3}{8} \times \frac{4}{9}$ b) $\frac{2}{3} \times \frac{9}{10}$ c) $\frac{5}{7} \times \frac{4}{15}$

d) $\frac{3}{4}$ of $\frac{8}{9}$ e) $\frac{5}{6}$ of $\frac{3}{10}$ f) $\frac{7}{8}$ of $\frac{2}{5}$

4. Evaluate the following:

a) $\frac{5}{8} \div \frac{3}{4}$ b) $\frac{5}{6} \div \frac{1}{3}$ c) $\frac{4}{5} \div \frac{7}{10}$

d) $1\frac{2}{3} \div \frac{2}{5}$ e) $\frac{3}{7} \div 2\frac{1}{7}$ f) $1\frac{1}{4} \div 1\frac{7}{8}$

5. Evaluate the following:

a) $\frac{3}{4} \times \frac{4}{5}$ b) $\frac{7}{8} \times \frac{2}{3}$

c) $\frac{3}{4} \times \frac{4}{7} \times \frac{3}{10}$ d) $\frac{4}{5} \div \frac{2}{3} \times \frac{7}{10}$

e) $\frac{1}{2}$ of $\frac{3}{4}$ f) $4\frac{1}{2} \div 3\frac{1}{9}$

6. Evaluate the following:

a) $\left(\frac{3}{8} \times \frac{4}{5}\right) + \left(\frac{1}{2} \text{ of } \frac{3}{5}\right)$ b) $\left(1\frac{1}{2} \times 3\frac{3}{4}\right) - \left(2\frac{3}{5} \div 1\frac{1}{2}\right)$

c) $\left(\frac{3}{5} \text{ of } \frac{4}{9}\right) + \left(\frac{4}{9} \text{ of } \frac{3}{5}\right)$ d) $\left(1\frac{1}{3} \times 2\frac{5}{8}\right)^2$

Changing a fraction to a decimal

To change a fraction to a decimal, divide the numerator by the denominator.

Worked examples

a) Change $\frac{5}{8}$ to a decimal.

$$\begin{array}{r} 0.6\;2\;5 \\ 8\;\overline{)\,5.0\;{}^{2}0\;{}^{4}0} \end{array}$$

b) Change $2\frac{3}{5}$ to a decimal.

This can be represented as $2 + \frac{3}{5}$.

$$\begin{array}{r} 0.6 \\ 5\;\overline{)\,3.0} \end{array}$$

Therefore $2\frac{3}{5} = 2.6$

Exercise 4.12

1. Change the following fractions to decimals:

a) $\frac{3}{4}$ b) $\frac{4}{5}$ c) $\frac{9}{20}$

d) $\frac{17}{50}$ e) $\frac{1}{3}$ f) $\frac{3}{8}$

g) $\frac{7}{16}$ h) $\frac{2}{9}$

2. Change the following mixed numbers to decimals:

a) $2\frac{3}{4}$ b) $3\frac{3}{5}$ c) $4\frac{7}{20}$

d) $6\frac{11}{50}$ e) $5\frac{2}{3}$ f) $6\frac{7}{8}$

g) $5\frac{9}{16}$ h) $4\frac{2}{9}$

Changing a decimal to a fraction

Changing a decimal to a fraction is done by knowing the 'value' of each of the digits in any decimal.

Worked examples

a) Change 0.45 from a decimal to a fraction.

units	.	tenths	hundredths
0	.	4	5

0.45 is therefore equivalent to 4 tenths and 5 hundredths, which in turn is the same as 45 hundredths.

Therefore $0.45 = \frac{45}{100} = \frac{9}{20}$

b) Change 2.325 from a decimal to a fraction.

units	.	tenths	hundredths	thousandths
2	.	3	2	5

Therefore $2.325 = 2\frac{325}{1000} = 2\frac{13}{40}$

Exercise 4.13

1. Change the following decimals to fractions:

a) 0.5 b) 0.7 c) 0.6
d) 0.75 e) 0.825 f) 0.05
g) 0.050 h) 0.402 i) 0.0002

2. Change the following decimals to mixed numbers:

a) 2.4 b) 6.5 c) 8.2
d) 3.75 e) 10.55 f) 9.204
g) 15.455 h) 30.001 i) 1.0205

Student assessment 1

1. Copy the following numbers. Circle improper fractions and underline mixed numbers:

a) $3\frac{1}{2}$ b) $\frac{2}{5}$ c) $\frac{7}{5}$ d) $\frac{8}{9}$ e) $1\frac{3}{4}$

2. Evaluate the following:

a) $\frac{1}{5}$ of 60 b) $\frac{3}{5}$ of 55 c) $\frac{2}{7}$ of 21 d) $\frac{3}{4}$ of 120

3. Change the following mixed numbers to improper fractions:

a) $2\frac{1}{2}$ b) $3\frac{6}{7}$ c) $5\frac{11}{12}$

4. Change the following improper fractions to mixed numbers:

a) $\frac{22}{7}$ b) $\frac{36}{5}$ c) $\frac{67}{9}$

5. Copy the following set of equivalent fractions and fill in the missing numerators:

$$\frac{5}{6} = \frac{}{12} = \frac{}{24} = \frac{}{60} = \frac{}{108}$$

6. Write the following fractions as decimals:

a) $\frac{27}{100}$ b) $\frac{105}{1000}$ c) $\frac{7}{100}$ d) $\frac{87}{1000}$

7. Write the following as percentages:

a) $\frac{3}{10}$ b) $\frac{29}{100}$ c) $\frac{1}{2}$ d) $\frac{7}{10}$
e) $\frac{4}{5}$ f) $2\frac{19}{100}$ g) $\frac{6}{100}$ h) $\frac{3}{4}$
i) 0.31 j) 0.07 k) 3.4 l) 2

Student assessment 2

1. Copy the following numbers. Circle improper fractions and underline mixed numbers:

a) $\frac{3}{11}$ b) $5\frac{3}{4}$ c) $\frac{27}{8}$ d) $\frac{3}{7}$

2. Evaluate the following:

a) $\frac{1}{3}$ of 63 b) $\frac{3}{8}$ of 72 c) $\frac{2}{5}$ of 55 d) $\frac{3}{13}$ of 169

3. Change the following mixed numbers to improper fractions:

a) $2\frac{3}{5}$ b) $3\frac{4}{9}$ c) $5\frac{5}{8}$

4. Change the following improper fractions to mixed numbers:

a) $\frac{33}{5}$ b) $\frac{47}{9}$ c) $\frac{67}{11}$

5. Copy the following set of equivalent fractions and fill in the missing numerators:

$$\frac{2}{3} = \frac{}{6} = \frac{}{12} = \frac{}{18} = \frac{}{27} = \frac{}{30}$$

6. Write the following fractions as decimals:

a) $\frac{35}{100}$ b) $\frac{275}{1000}$ c) $\frac{675}{100}$ d) $\frac{35}{1000}$

7. Write the following as percentages:

a) $\frac{3}{5}$ b) $\frac{49}{100}$ c) $\frac{1}{4}$ d) $\frac{9}{10}$
e) $1\frac{1}{2}$ f) $3\frac{27}{100}$ g) $\frac{5}{100}$ h) $\frac{7}{20}$
i) 0.77 j) 0.03 k) 2.9 l) 4

Student assessment 3

1. Evaluate the following:
a) $5 + 8 \times 3 - 6$ b) $15 + 45 \div 3 - 12$

2. The sum of two numbers is 21 and their product is 90. What are the numbers?

3. How many seconds are there in $2\frac{1}{2}$ hours?

4. Work out 851×27.

5. Work out $6843 \div 19$ giving your answer to 1 d.p.

6. Copy these equivalent fractions and fill in the blanks:

$$\frac{8}{18} = \frac{}{9} = \frac{16}{} = \frac{56}{} = \frac{}{90}$$

7. Evaluate the following:

a) $3\frac{3}{4} - 1\frac{11}{16}$ b) $4\frac{4}{5} \div \frac{8}{15}$

8. Change the following fractions to decimals:

a) $\frac{2}{5}$ b) $1\frac{3}{4}$

9. Change the following decimals to fractions. Give each fraction in its simplest form.

a) 4.2   b) 0.06
c) 1.85   d) 2.005

Student assessment 4

1. Evaluate the following:

a) $6 \times 4 - 3 \times 8$ b) $15 \div 3 + 2 \times 7$

2. The product of two numbers is 72, and their sum is 18. What are the two numbers?

3. How many days are there in 42 weeks?

4. Work out 368×49.

5. Work out $7835 \div 23$ giving your answer to 1 d.p.

6. Copy these equivalent fractions and fill in the blanks:

$$\frac{24}{36} = \frac{}{12} = \frac{4}{} = \frac{}{30} = \frac{60}{}$$

7. Evaluate the following:

a) $2\frac{1}{2} - \frac{4}{5}$ b) $3\frac{1}{2} \times \frac{4}{7}$

8. Change the following fractions to decimals:

a) $\frac{7}{8}$ b) $1\frac{2}{5}$

9. Change the following decimals to fractions. Give each fraction in its simplest form.

a) 6.5   b) 0.04
c) 3.65   d) 3.008

5 Further percentages

You should already be familiar with the percentage equivalents of simple fractions and decimals as outlined in the table below.

Fraction	Decimal	Percentage
$\frac{1}{2}$	0.5	50%
$\frac{1}{4}$	0.25	25%
$\frac{3}{4}$	0.75	75%
$\frac{1}{8}$	0.125	12.5%
$\frac{3}{8}$	0.375	37.5%
$\frac{5}{8}$	0.625	62.5%
$\frac{7}{8}$	0.875	87.5%
$\frac{1}{10}$	0.1	10%
$\frac{2}{10}$ or $\frac{1}{5}$	0.2	20%
$\frac{3}{10}$	0.3	30%
$\frac{4}{10}$ or $\frac{2}{5}$	0.4	40%
$\frac{6}{10}$ or $\frac{3}{5}$	0.6	60%
$\frac{7}{10}$	0.7	70%
$\frac{8}{10}$ or $\frac{4}{5}$	0.8	80%
$\frac{9}{10}$	0.9	90%

Simple percentages

Worked examples

a) Of 100 sheep in a field, 88 are ewes.

i) What percentage of the sheep are ewes?
88 out of 100 are ewes
= 88%

ii) What percentage are not ewes?
12 out of 100
= 12%

b) A gymnast scored marks out of 10 from five judges.
They were: 8.0, 8.2, 7.9, 8.3 and 7.6.
Express these marks as percentages.

$\frac{8.0}{10} = \frac{80}{100} = 80\%$ $\quad \frac{8.2}{10} = \frac{82}{100} = 82\%$ $\quad \frac{7.9}{10} = \frac{79}{100} = 79\%$

$\frac{8.3}{10} = \frac{83}{100} = 83\%$ $\quad \frac{7.6}{10} = \frac{76}{100} = 76\%$

c) Convert the following percentages into fractions and decimals:

i) 27% ii) 5%

$\frac{27}{100} = 0.27$ $\frac{5}{100} = \frac{1}{20} = 0.05$

Exercise 5.1

1. In a survey of 100 cars, 47 were white, 23 were blue and 30 were red. Express each of these numbers as a percentage of the total.

2. $\frac{7}{10}$ of the surface of the Earth is water. Express this as a percentage.

3. There are 200 birds in a flock. 120 of them are female. What percentage of the flock are:

a) female? b) male?

4. Write these percentages as fractions of 100:

a) 73% b) 28%
c) 10% d) 25%

5. Write these fractions as percentages:

a) $\frac{27}{100}$ b) $\frac{3}{10}$ c) $\frac{7}{50}$ d) $\frac{1}{4}$

6. Convert the following percentages to decimals:

a) 39% b) 47% c) 83%
d) 7% e) 2% f) 20%

7. Convert the following decimals to percentages:

a) 0.31 b) 0.67 c) 0.09
d) 0.05 e) 0.2 f) 0.75

Calculating a percentage of a quantity

Worked examples

a) Find 25% of 300 m.

25% can be written as 0.25.
$0.25 \times 300\text{ m} = 75\text{ m}$.

b) Find 35% of 280 m.

35% can be written as 0.35.
$0.35 \times 280\text{ m} = 98\text{ m}$.

Exercise 5.2

1. Write the percentage equivalent of each of the following fractions:

a) $\frac{1}{4}$ b) $\frac{2}{3}$ c) $\frac{5}{8}$
d) $1\frac{4}{5}$ e) $4\frac{9}{10}$ f) $3\frac{7}{8}$

2. Write the decimal equivalent of each of the following:

a) $\frac{3}{4}$ b) 80% c) $\frac{1}{5}$
d) 7% e) $1\frac{7}{8}$ f) $\frac{1}{6}$

3. Evaluate the following:

a) 25% of 80 b) 80% of 125 c) 62.5% of 80
d) 30% of 120 e) 90% of 5 f) 25% of 30

4. Evaluate the following:

a) 17% of 50 b) 50% of 17 c) 65% of 80
d) 80% of 65 e) 7% of 250 f) 250% of 7

5. In a class of 30 students, 20% have black hair, 10% have blonde hair and 70% have brown hair. Calculate the number of students with
a) black hair,
b) blonde hair,
c) brown hair.

6. A survey conducted among 120 schoolchildren looked at which type of meat they preferred. 55% said they preferred chicken, 20% said they preferred lamb, 15% preferred goat and 10% were vegetarian. Calculate the number of children in each category.

7. A survey was carried out in a school to see what nationality its students were. Of the 220 students in the school, 65% were Australian, 20% were Pakistani, 5% were Greek and 10% belonged to other nationalities. Calculate the number of students of each nationality.

8. A shopkeeper keeps a record of the numbers of items he sells in one day. Of the 150 items he sold, 46% were newspapers, 24% were pens, 12% were books whilst the remaining 18% were other items. Calculate the number of each item he sold.

Expressing one quantity as a percentage of another

To express one quantity as a percentage of another, first write the first quantity as a fraction of the second and then multiply by 100.

Worked example In an examination a girl obtains 69 marks out of 75. Express this result as a percentage.

$$\frac{69}{75} \times 100\% = 92\%$$

Exercise 5.3

1. For each of the following express the first quantity as a percentage of the second.
   a) 24 out of 50 b) 46 out of 125
   c) 7 out of 20 d) 45 out of 90
   e) 9 out of 20 f) 16 out of 40
   g) 13 out of 39 h) 20 out of 35

2. A hockey team plays 42 matches. It wins 21, draws 14 and loses the rest. Express each of these results as a percentage of the total number of games played.

3. Four candidates stood in an election:

   A received 24 500 votes
   B received 18 200 votes
   C received 16 300 votes
   D received 12 000 votes

   Express each of these as a percentage of the total votes cast.

4. A car manufacturer produces 155 000 cars a year. The cars are available for sale in six different colours. The numbers sold of each colour were:

Red	55 000
Blue	48 000
White	27 500
Silver	10 200
Green	9300
Black	5000

   Express each of these as a percentage of the total number of cars produced. Give your answers to 1 d.p.

Percentage increases and decreases

Worked examples

a) A doctor in Thailand has a salary of 18 000 baht per month. If her salary increases by 8%, calculate:
   i) the amount extra she receives per month,
   ii) her new monthly salary.

   i) Increase = 8% of 18 000 baht
   = 0.08 × 18 000 baht = 1440 baht

   ii) New salary = old salary + increase
   = 18 000 baht + 1440 baht per month
   = 19 440 baht per month

b) A garage increases the price of a truck by 12%. If the original price was \$14 500, calculate its new price.
The original price represents 100%, therefore the increase can be represented as 112%.

$$\begin{aligned}\text{New price} &= 112\% \text{ of } \$14\,500\\ &= 1.12 \times \$14\,500\\ &= \$16\,240\end{aligned}$$

c) A shop is having a sale. It sells a set of tools costing \$130 at a 15% discount. Calculate the sale price of the tools.
The old price represents 100%, therefore the new price can be represented as $(100 - 15)\% = 85\%$.

$$\begin{aligned}85\% \text{ of } \$130 &= 0.85 \times \$130\\ &= \$110.50\end{aligned}$$

Exercise 5.4

1. Increase the following by the given percentage:
a) 150 by 25% b) 230 by 40% c) 7000 by 2%
d) 70 by 250% e) 80 by 12.5% f) 75 by 62%

2. Decrease the following by the given percentage:
a) 120 by 25% b) 40 by 5% c) 90 by 90%
d) 1000 by 10% e) 80 by 37.5% f) 75 by 42%

3. In the following questions the first number is increased to become the second number. Calculate the percentage increase in each case.
a) $50 \to 60$ b) $75 \to 135$ c) $40 \to 84$
d) $30 \to 31.5$ e) $18 \to 33.3$ f) $4 \to 13$

4. In the following questions the first number is decreased to become the second number. Calculate the percentage decrease in each case.
a) $50 \to 25$ b) $80 \to 56$ c) $150 \to 142.5$
d) $3 \to 0$ e) $550 \to 352$ f) $20 \to 19$

5. A farmer increases the yield on his farm by 15%. If his previous yield was 6500 tonnes, what is his present yield?

6. The cost of a computer in a computer store is reduced by 12.5% in a sale. If the computer was priced at \$7800, what is its price in the sale?

7. A winter coat is priced at \$100. In the sale its price is reduced by 25%.
a) Calculate the sale price of the coat.
b) After the sale its price is increased by 25% again. Calculate the coat's price after the sale.

8. A farmer takes 250 chickens to be sold at a market. In the first hour he sells 8% of his chickens. In the second hour he sells 10% of those that were left.
   a) How many chickens has he sold in total?
   b) What percentage of the original number did he manage to sell in the two hours?

9. The number of fish on a fish farm increases by approximately 10% each month. If there were originally 350 fish, calculate to the nearest 100 how many fish there would be after 12 months.

Student assessment 1

1. Copy the table below and fill in the missing values:

Fraction	Decimal	Percentage
$\frac{3}{4}$		
	0.8	
$\frac{5}{8}$		
	1.5	

2. Find 40% of 1600 m.

3. A shop increases the price of a television set by 8%. If the original price was \$320, what is the new price?

4. A car loses 55% of its value after four years. If it cost \$22 500 when new, what is its value after the four years?

5. Express the first quantity as a percentage of the second.
   a) 40 cm, 2 m   b) 25 mins, 1 hour   c) 450 g, 2 kg
   d) 3 m, 3.5 m   e) 70 kg, 1 tonne   f) 75 cl, 2.5 litres

6. A house is bought for 75 000 rand, then resold for 87 000 rand. Calculate the percentage profit.

7. A pair of shoes is priced at \$45. During a sale the price is reduced by 20%.
   a) Calculate the sale price of the shoes.
   b) What is the percentage increase in the price if after the sale it is once again restored to \$45?

8. The population of a town increases by 5% each year. If in 2007 the population was 86 000, in which year is the population expected to exceed 100 000 for the first time?

Student assessment 2

1. Copy the table below and fill in the missing values:

Fraction	Decimal	Percentage
	0.25	
$\frac{3}{5}$		
		$62\frac{1}{2}\%$
$2\frac{1}{4}$		

2. Find 30% of 2500 m.
3. In a sale a shop reduces its prices by 12.5%. What is the sale price of a desk previously costing 2400 Hong Kong dollars?
4. In the last six years the value of a house has increased by 35%. If it cost $72 000 six years ago, what is its value now?
5. Express the first quantity as a percentage of the second.
   a) 35 mins, 2 hours
   b) 650 g, 3 kg
   c) 5 m, 4 m
   d) 15 s, 3 mins
   e) 600 kg, 3 tonnes
   f) 35 cl, 3.5 litres
6. Shares in a company are bought for $600. After a year, the same shares are sold for $550. Calculate the percentage depreciation.
7. In a sale the price of a jacket originally costing $850 is reduced by $200. Any item not sold by the last day of the sale is reduced by a further 50%. If the jacket is sold on the last day of the sale:
   a) calculate the price it is finally sold for,
   b) calculate the overall percentage reduction in price.
8. Each day the population of a type of insect increases by approximately 10%. How many days will it take for the population to double?

6 Ratio and proportion

Direct proportion

Workers in a pottery factory are paid according to how many plates they produce. The wage paid to them is said to be in **direct proportion** to the number of plates made. As the number of plates made increases so does their wage. Other workers are paid for the number of hours worked. For them the wage paid is in **direct proportion** to the number of hours worked.

There are two main methods for solving problems involving direct proportion: the ratio method and the unitary method.

Worked example A bottling machine fills 500 bottles in 15 minutes. How many bottles will it fill in $1\frac{1}{2}$ hours?

Note: The time units must be the same, so for either method the $1\frac{1}{2}$ hours must be changed to 90 minutes.

The ratio method

Let x be the number of bottles filled. Then:

$$\frac{x}{90} = \frac{500}{15}$$

$$\text{so } x = \frac{500 \times 90}{15} = 3000$$

3000 bottles are filled in $1\frac{1}{2}$ hours.

The unitary method

In 15 minutes 500 bottles are filled.

Therefore in 1 minute $\frac{500}{15}$ bottles are filled.

So in 90 minutes $90 \times \frac{500}{15}$ bottles are filled.

In $1\frac{1}{2}$ hours, 3000 bottles are filled.

Exercise 6.1

Use either the ratio method or the unitary method to solve the problems below.

1. A machine prints four books in 10 minutes. How many will it print in 2 hours?
2. A farmer plants five apple trees in 25 minutes. If he continues to work at a constant rate, how long will it take him to plant 200 trees?
3. A television set uses 3 units of electricity in 2 hours. How many units will it use in 7 hours? Give your answer to the nearest unit.

4. A bricklayer lays 1500 bricks in an 8-hour day. Assuming he continues to work at the same rate, calculate:
   a) how many bricks he would expect to lay in a five-day week,
   b) how long to the nearest hour it would take him to lay 10 000 bricks.

5. A machine used to paint white lines on a road uses 250 litres of paint for each 8 km of road marked. Calculate:
   a) how many litres of paint would be needed for 200 km of road,
   b) what length of road could be marked with 4000 litres of paint.

6. An aircraft is cruising at 720 km/h and covers 1000 km. How far would it travel in the same period of time if the speed increased to 800 km/h?

7. A production line travelling at 2 m/s labels 150 tins. In the same period of time how many will it label at:
   a) 6 m/s　　b) 1 m/s　　c) 1.6 m/s?

Exercise 6.2

Use either the ratio method or unitary method to solve the problems below.

1. A production line produces 8 cars in 3 hours.
   a) Calculate how many it will produce in 48 hours.
   b) Calculate how long it will take to produce 1000 cars.

2. A machine produces six golf balls in fifteen seconds. Calculate how many are produced in:
   a) 5 minutes　　b) 1 hour　　c) 1 day.

3. An MP3 player uses 0.75 units of electricity in 90 minutes. Calculate:
   a) how many units it will use in 8 hours,
   b) how long it will operate for 15 units of electricity.

4. A combine harvester takes 2 hours to harvest a 3 hectare field. If it works at a constant rate, calculate:
   a) how many hectares it will harvest in 15 hours,
   b) how long it will take to harvest a 54 hectare field.

5. A road-surfacing machine can re-surface 8 m of road in 40 seconds. Calculate how long it will take to re-surface 18 km of road, at the same rate.

6. A sailing yacht is travelling at 1.5 km/h and covers 12 km. If its speed increased to 2.5 km/h, how far would it travel in the same period of time?

7. A plate-making machine produces 36 plates in 8 minutes.
   a) How many plates are produced in one hour?
   b) How long would it take to produce 2880 plates?

If the information is given in the form of a ratio, the method of solution is the same.

Worked example Tin and copper are mixed in the ratio 8 : 3. How much tin is needed to mix with 36 g of copper?

The ratio method

Let x grams be the mass of tin needed.

$$\frac{x}{36} = \frac{8}{3}$$

Therefore $x = \frac{8 \times 36}{3}$

$= 96$

So 96 g of tin is needed.

The unitary method

3 g of copper mixes with 8 g of tin.

1 g of copper mixes with $\frac{8}{3}$ g of tin.

So 36 g of copper mixes with $36 \times \frac{8}{3}$ g of tin.

Therefore 36 g of copper mixes with 96 g of tin.

Exercise 6.3

1. Sand and gravel are mixed in the ratio 5 : 3 to form ballast.
   a) How much gravel is mixed with 750 kg of sand?
   b) How much sand is mixed with 750 kg of gravel?

2. A recipe uses 150 g butter, 500 g flour, 50 g sugar and 100 g currants to make 18 cakes.
   a) How much of each ingredient will be needed to make 72 cakes?
   b) How many whole cakes could be made with 1 kg of butter?

3. A paint mix uses red and white paint in a ratio of 1 : 12.
   a) How much white paint will be needed to mix with 1.4 litres of red paint?
   b) If a total of 15.5 litres of paint is mixed, calculate the amount of white paint and the amount of red paint used. Give your answers to the nearest 0.1 litre.

4. A tulip farmer sells sacks of mixed bulbs to local people. The bulbs develop into two different colours of tulips, red and yellow. The colours are packaged in a ratio of 8 : 5 respectively.
   a) If a sack contains 200 red bulbs, calculate the number of yellow bulbs.
   b) If a sack contains 351 bulbs in total, how many of each colour would you expect to find?
   c) One sack is packaged with a bulb mixture in the ratio 7 : 5 by mistake. If the sack contains 624 bulbs, how many more yellow bulbs would you expect to have compared with a normal sack of 624 bulbs?

5. A pure fruit juice is made by mixing the juices of oranges and mangoes in the ratio of 9 : 2.
   a) If 189 litres of orange juice are used, calculate the number of litres of mango juice needed.
   b) If 605 litres of the juice are made, calculate the numbers of litres of orange juice and mango juice used.

● Divide a quantity in a given ratio

Worked examples

a) Divide 20 m in the ratio 3 : 2.

The ratio method

3 : 2 gives 5 parts.

$\frac{3}{5} \times 20 \text{ m} = 12 \text{ m}$

$\frac{2}{5} \times 20 \text{ m} = 8 \text{ m}$

20 m divided in the ratio 3 : 2 is 12 m : 8 m.

The unitary method

3 : 2 gives 5 parts.

5 parts is equivalent to 20 m.

1 part is equivalent to $\frac{20}{5}$ m.

Therefore 3 parts is $3 \times \frac{20}{5}$ m; that is 12 m.

Therefore 2 parts is $2 \times \frac{20}{5}$ m; that is 8 m.

b) A factory produces cars in red, blue, white and green in the ratio 7 : 5 : 3 : 1. Out of a production of 48 000 cars how many are white?

$7 + 5 + 3 + 1$ gives a total of 16 parts.
Therefore the total number of white cars is
$\frac{3}{16} \times 48\,000 = 9000$.

Exercise 6.4

1. Divide 150 in the ratio 2 : 3.
2. Divide 72 in the ratio 2 : 3 : 4.
3. Divide 5 kg in the ratio 13 : 7.
4. Divide 45 minutes in the ratio 2 : 3.
5. Divide 1 hour in the ratio 1 : 5.
6. $\frac{7}{8}$ of a can of drink is water, the rest is syrup. What is the ratio of water to syrup?
7. $\frac{5}{9}$ of a litre carton of orange is pure orange juice, the rest is water. How many millilitres of each are in the carton?
8. 55% of students in a school are boys.
   a) What is the ratio of boys to girls?
   b) How many boys and how many girls are there if the school has 800 students?
9. A piece of wood is cut in the ratio 2 : 3. What fraction of the length is the longer piece?
10. If the original piece of wood in Q.9 is 80 cm long, how long is the shorter piece?
11. A gas pipe is 7 km long. A valve is positioned in such a way that it divides the length of the pipe in the ratio 4 : 3. Calculate the distance of the valve from each end of the pipe.
12. The sizes of the angles of a quadrilateral are in the ratio 1 : 2 : 3 : 3. Calculate the size of each angle.
13. The angles of a triangle are in the ratio 3 : 5 : 4. Calculate the size of each angle.
14. A millionaire leaves 1.4 million dollars in his will to be shared between his three children in the ratio of their ages. If they are 24, 28 and 32 years old, calculate to the nearest dollar the amount they will each receive.
15. A small company makes a profit of $8000. This is divided between the directors in the ratio of their initial investments. If Alex put $20 000 into the firm, Maria $35 000 and Ahmet $25 000, calculate the amount of the profit they will each receive.

Inverse proportion

Sometimes an increase in one quantity causes a decrease in another quantity. For example, if fruit is to be picked by hand, the more people there are picking the fruit, the less time it will take. The time taken is said to be **inversely proportional** to the number of people picking the fruit.

Worked examples

a) If 8 people can pick the apples from the trees in 6 days, how long will it take 12 people?

8 people take 6 days.
1 person will take 6×8 days.

Therefore 12 people will take $\frac{6 \times 8}{12}$ days, i.e. 4 days.

b) A cyclist averages a speed of 27 km/h for 4 hours. At what average speed would she need to cycle to cover the same distance in 3 hours?

Completing it in 1 hour would require cycling at 27×4 km/h.
Completing it in 3 hours requires cycling at

$\frac{27 \times 4}{3}$ km/h; that is 36 km/h.

Exercise 6.5

1. A teacher shares sweets among 8 students so that they get 6 each. How many sweets would they each have got if there had been 12 students?

2. The table below represents the relationship between the speed and the time taken for a train to travel between two stations.

Speed (km/h)	60			120	90	50	10
Time (h)	2	3	4				

Copy and complete the table.

3. Six people can dig a trench in 8 hours.
a) How long would it take:
i) 4 people ii) 12 people iii) 1 person?
b) How many people would it take to dig the trench in:
i) 3 hours ii) 16 hours iii) 1 hour?

4. Chairs in a hall are arranged in 35 rows of 18.
a) How many rows would there be with 21 chairs to a row?
b) How many chairs would there be in each row if there were 15 rows?

5. A train travelling at 100 km/h takes 4 hours for a journey. How long would it take a train travelling at 60 km/h?

6. A worker in a sugar factory packs 24 cardboard boxes with 15 bags of sugar in each. If he had boxes which held 18 bags of sugar each, how many fewer boxes would be needed?

7. A swimming pool is filled in 30 hours by two identical pumps. How much quicker would it be filled if five similar pumps were used instead?

Student assessment 1

1. A ruler 30 cm long is broken into two parts in the ratio 8 : 7. How long are the two parts?
2. A recipe needs 400 g of flour to make 8 cakes. How much flour would be needed in order to make 24 cakes?
3. To make 6 jam tarts, 120 g of jam is needed. How much jam is needed to make 10 tarts?
4. The scale of a map is 1 : 25 000.
   a) Two villages are 8 cm apart on the map. How far apart are they in real life? Give your answer in kilometres.
   b) The distance from a village to the edge of a lake is 12 km in real life. How far apart would they be on the map? Give your answer in centimetres.
5. A motorbike uses petrol and oil mixed in the ratio 13 : 2.
   a) How much of each is there in 30 litres of mixture?
   b) How much petrol would be mixed with 500 ml of oil?
6. a) A model car is a $\frac{1}{40}$ scale model. Express this as a ratio.
   b) If the length of the real car is 5.5 m, what is the length of the model car?
7. An aunt gives a brother and sister $2000 to be divided in the ratio of their ages. If the girl is 13 years old and the boy 12 years old, how much will each get?
8. The angles of a triangle are in the ratio 2 : 5 : 8. Find the size of each of the angles.
9. A photocopying machine is capable of making 50 copies each minute.
   a) If four identical copiers are used simultaneously how long would it take to make a total of 50 copies?
   b) How many of these copiers would be needed to make 6000 copies in 15 minutes?
10. It takes 16 hours for three bricklayers to build a wall. Calculate how long it would take for eight bricklayers to build a similar wall.

Student assessment 2

1. A piece of wood is cut in the ratio 3 : 7.
   a) What fraction of the whole is the longer piece?
   b) If the wood is 1.5 m long, how long is the shorter piece?

NB: All diagrams are not drawn to scale.

2. A recipe for two people requires $\frac{1}{4}$ kg of rice to 150 g of meat.
   a) How much meat would be needed for five people?
   b) How much rice would there be in 1 kg of the final dish?

3. The scale of a map is 1 : 10 000.
   a) Two rivers are 4.5 cm apart on the map. How far apart are they in real life? Give your answer in metres.
   b) Two towns are 8 km apart in real life. How far apart are they on the map? Give your answer in centimetres.

4. a) A model train is a $\frac{1}{25}$ scale model. Express this as a ratio.
   b) If the length of the model engine is 7 cm, what is the true length of the engine?

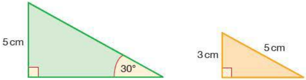
5. Divide 3 tonnes in the ratio 2 : 5 : 13.

6. The ratio of the angles of a quadrilateral is 2 : 3 : 3 : 4. The angles of a quadrilateral add up to 360°. Calculate the size of each of the angles.

7. The ratio of the interior angles of a pentagon is 2 : 3 : 4 : 4 : 5. The angles of a pentagon add up to 540°. Calculate the size of the largest angle.

8. A large swimming pool takes 36 hours to fill using three identical pumps.
   a) How long would it take to fill using eight identical pumps?
   b) If the pool needs to be filled in 9 hours, how many of these pumps will be needed?

9. The first triangle is an enlargement of the second. Calculate the size of the missing sides and angles.



10. A tap issuing water at a rate of 1.2 litres per minute fills a container in 4 minutes.
   a) How long would it take to fill the same container if the rate was decreased to 1 litre per minute? Give your answer in minutes and seconds.
   b) If the container is to be filled in 3 minutes, calculate the rate at which the water should flow.

7 Indices and standard form

Indices

The index refers to the power to which a number is raised. In the example 5^3 the number 5 is raised to the power 3. The 3 is known as the **index**. Indices is the plural of index.

Worked examples

a) $5^3 = 5 \times 5 \times 5$
$= 125$

b) $7^4 = 7 \times 7 \times 7 \times 7$
$= 2401$

c) $3^1 = 3$

Exercise 7.1

1. Using indices, simplify the following expressions:

a) $3 \times 3 \times 3$
b) $2 \times 2 \times 2 \times 2 \times 2$
c) 4×4
d) $6 \times 6 \times 6 \times 6$
e) $8 \times 8 \times 8 \times 8 \times 8 \times 8$
f) 5

2. Simplify the following using indices:

a) $2 \times 2 \times 2 \times 3 \times 3$
b) $4 \times 4 \times 4 \times 4 \times 4 \times 5 \times 5$
c) $3 \times 3 \times 4 \times 4 \times 4 \times 5 \times 5$
d) $2 \times 7 \times 7 \times 7 \times 7$
e) $1 \times 1 \times 6 \times 6$
f) $3 \times 3 \times 3 \times 4 \times 4 \times 6 \times 6 \times 6 \times 6 \times 6$

3. Write out the following in full:

a) 4^2
b) 5^7
c) 3^5
d) $4^3 \times 6^3$
e) $7^2 \times 2^7$
f) $3^2 \times 4^3 \times 2^4$

4. Without a calculator work out the value of the following:

a) 2^5
b) 3^4
c) 8^2
d) 6^3
e) 10^6
f) 4^4
g) $2^3 \times 3^2$
h) $10^3 \times 5^3$

Laws of indices

When working with numbers involving indices there are three basic laws which can be applied. These are:

(1) $a^m \times a^n = a^{m+n}$

(2) $a^m \div a^n$ or $\dfrac{a^m}{a^n} = a^{m-n}$

(3) $(a^m)^n = a^{mn}$

Positive indices

Worked examples

a) Simplify $4^3 \times 4^2$.

$$4^3 \times 4^2 = 4^{(3+2)} = 4^5$$

b) Simplify $2^5 \div 2^3$.

$$2^5 \div 2^3 = 2^{(5-3)} = 2^2$$

c) Evaluate $3^3 \times 3^4$.

$$3^3 \times 3^4 = 3^{(3+4)} = 3^7 = 2187$$

d) Evaluate $(4^2)^3$.

$$(4^2)^3 = 4^{(2\times3)} = 4^6 = 4096$$

Exercise 7.2

1. Simplify the following using indices:

a) $3^2 \times 3^4$
b) $8^5 \times 8^2$
c) $5^2 \times 5^4 \times 5^3$
d) $4^3 \times 4^5 \times 4^2$
e) $2^1 \times 2^3$
f) $6^2 \times 3^2 \times 3^3 \times 6^4$
g) $4^5 \times 4^3 \times 5^5 \times 5^4 \times 6^2$
h) $2^4 \times 5^7 \times 5^3 \times 6^2 \times 6^6$

2. Simplify the following:

a) $4^6 \div 4^2$
b) $5^7 \div 5^4$
c) $2^5 \div 2^4$
d) $6^5 \div 6^2$
e) $\frac{6^5}{6^2}$
f) $\frac{8^6}{8^5}$
g) $\frac{4^8}{4^5}$
h) $\frac{3^9}{3^2}$

3. Simplify the following:

a) $(5^2)^2$
b) $(4^3)^4$
c) $(10^2)^5$
d) $(3^3)^5$
e) $(6^2)^4$
f) $(8^2)^3$

4. Simplify the following:

a) $\frac{2^2\times2^4}{2^3}$
b) $\frac{3^4\times3^2}{3^5}$
c) $\frac{5^6\times5^7}{5^2\times5^8}$
d) $\frac{(4^2)^5\times4^2}{4^7}$
e) $\frac{4^4\times2^5\times4^2}{4^3\times2^3}$
f) $\frac{6^3\times6^3\times8^5\times8^6}{8^6\times6^2}$
g) $\frac{(5^2)^2\times(4^4)^3}{5^8\times4^9}$
h) $\frac{(6^3)^4\times6^3\times4^9}{6^8\times(4^2)^4}$

The zero index

The zero index indicates that a number is raised to the power 0. A number raised to the power 0 is equal to 1. This can be explained by applying the laws of indices.

$a^m \div a^n = a^{m-n}$ therefore $\frac{a^m}{a^m} = a^{m-m}$

$= a^0$

However, $\frac{a^m}{a^m} = 1$

therefore $a^0 = 1$

Exercise 7.3 Without using a calculator, evaluate the following:

a) $2^3 \times 2^0$
b) $5^2 \div 6^0$
c) $5^2 \times 5^{-2}$
d) $6^3 \times 6^{-3}$
e) $(4^0)^2$
f) $4^0 \div 2^2$

Negative indices

A negative index indicates that a number is being raised to a negative power, e.g. 4^{-3}.

Another law of indices states that $a^{-m} = \frac{1}{a^m}$. This can be proved as follows.

$$a^{-m} = a^{0-m}$$

$$= \frac{a^0}{a^m} \text{ (from the second law of indices)}$$

$$= \frac{1}{a^m}$$

therefore $a^{-m} = \frac{1}{a^m}$

Exercise 7.4 Without using a calculator, evaluate the following:

1. a) 4^{-1}
b) 3^{-2}
c) 6×10^{-2}
d) 5×10^{-3}
e) 100×10^{-2}
f) 10^{-3}

2. a) 9×3^{-2}
b) 16×2^{-3}
c) 64×2^{-4}
d) 4×2^{-3}
e) 36×6^{-3}
f) 100×10^{-1}

3. a) $\frac{3}{2^{-2}}$
b) $\frac{4}{2^{-3}}$
c) $\frac{9}{5^{-2}}$
d) $\frac{5}{4^{-2}}$
e) $\frac{7^{-3}}{7^{-4}}$
f) $\frac{8^{-6}}{8^{-8}}$

Standard form

Standard form is also known as standard index form or sometimes as scientific notation. It involves writing large numbers or very small numbers in terms of powers of 10.

Positive indices and large numbers

$$100 = 1 \times 10^2$$
$$1000 = 1 \times 10^3$$
$$10\,000 = 1 \times 10^4$$
$$3000 = 3 \times 10^3$$

For a number to be in standard form it must take the form $A \times 10^n$ where the index n is a positive or negative integer and A must lie in the range $1 \leqslant A < 10$.

e.g. 3100 can be written in many different ways:

3.1×10^3 31×10^2 0.31×10^4 etc.

However, only 3.1×10^3 satisfies the above conditions and therefore is the only one which is written in standard form.

Worked examples

a) Write 72 000 in standard form.

7.2×10^4

b) Write 4×10^4 as an ordinary number.

$4 \times 10^4 = 4 \times 10000$
$= 40\,000$

c) Multiply the following and write your answer in standard form:

600×4000
$= 2\,400\,000$
$= 2.4 \times 10^6$

Exercise 7.5

1. Deduce the value of n in the following:

a) $79\,000 = 7.9 \times 10^n$ b) $53\,000 = 5.3 \times 10^n$
c) $4\,160\,000 = 4.16 \times 10^n$ d) 8 million $= 8 \times 10^n$
e) 247 million $= 2.47 \times 10^n$ f) $24\,000\,000 = 2.4 \times 10^n$

2. Write the following numbers in standard form:

a) 65 000 b) 41 000 c) 723 000
d) 18 million e) 950 000 f) 760 million
g) 720 000 h) $\frac{1}{4}$ million

3. Write the numbers below which are written in standard form:

26.3×10^5 2.6×10^7 0.5×10^3 8×10^8
0.85×10^9 8.3×10^{10} 1.8×10^7 18×10^5
3.6×10^6 6.0×10^1

4. Write the following as ordinary numbers:
a) 3.8×10^3 b) 4.25×10^6
c) 9.003×10^7 d) 1.01×10^5

5. Multiply the following and write your answers in standard form:
a) 400×2000 b) 6000×5000
c) $75\,000 \times 200$ d) $33\,000 \times 6000$
e) 8 million $\times$ 250 f) $95\,000 \times 3000$
g) 7.5 million $\times$ 2 h) 8.2 million $\times$ 50
i) $300 \times 200 \times 400$ j) $(7000)^2$

6. Which of the following are not in standard form?
a) 6.2×10^5 b) 7.834×10^{16}
c) 8.0×10^5 d) 0.46×10^7
e) 82.3×10^6 f) 6.75×10^1

7. Write the following numbers in standard form:
a) 600 000 b) 48 000 000
c) 784 000 000 000 d) 534 000
e) 7 million f) 8.5 million

8. Write the following in standard form:
a) 68×10^5 b) 720×10^6
c) 8×10^5 d) 0.75×10^8
e) 0.4×10^{10} f) 50×10^6

9. Multiply the following and write your answers in standard form:
a) 200×3000 b) 6000×4000
c) 7 million $\times$ 20 d) 500 $\times$ 6 million
e) 3 million $\times$ 4 million f) 4500×4000

10. Light from the Sun takes approximately 8 minutes to reach Earth. If light travels at a speed of 3×10^8 m/s, calculate to three significant figures (s.f.) the distance from the Sun to the Earth.

Multiplying and dividing numbers in standard form

When you multiply or divide numbers in standard form, you work with the numbers and the powers of 10 separately. You use the laws of indices when working with the powers of 10.

Worked examples

a) Multiply the following and write your answer in standard form:

$(2.4 \times 10^4) \times (5 \times 10^7)$
$= 12 \times 10^{11}$
$= 1.2 \times 10^{12}$ when written in standard form

b) Divide the following and write your answer in standard form:

$(6.4 \times 10^7) \div (1.6 \times 10^3)$
$= 4 \times 10^4$

Exercise 7.6

1. Multiply the following and write your answers in standard form:
a) $(4 \times 10^3) \times (2 \times 10^5)$
b) $(2.5 \times 10^4) \times (3 \times 10^4)$
c) $(1.8 \times 10^7) \times (5 \times 10^6)$
d) $(2.1 \times 10^4) \times (4 \times 10^7)$
e) $(3.5 \times 10^4) \times (4 \times 10^7)$
f) $(4.2 \times 10^5) \times (3 \times 10^4)$
g) $(2 \times 10^4)^2$
h) $(4 \times 10^8)^2$

2. Find the value of the following and write your answers in standard form:
a) $(8 \times 10^6) \div (2 \times 10^3)$
b) $(8.8 \times 10^9) \div (2.2 \times 10^3)$
c) $(7.6 \times 10^8) \div (4 \times 10^7)$
d) $(6.5 \times 10^{14}) \div (1.3 \times 10^7)$
e) $(5.2 \times 10^6) \div (1.3 \times 10^6)$
f) $(3.8 \times 10^{11}) \div (1.9 \times 10^3)$

3. Find the value of the following and write your answers in standard form:
a) $(3 \times 10^4) \times (6 \times 10^5) \div (9 \times 10^5)$
b) $(6.5 \times 10^8) \div (1.3 \times 10^4) \times (5 \times 10^3)$
c) $(18 \times 10^3) \div 900 \times 250$
d) $27\,000 \div 3000 \times 8000$
e) $4000 \times 8000 \div 640$
f) $2500 \times 2500 \div 1250$

4. Find the value of the following and write your answers in standard form:
a) $(4.4 \times 10^3) \times (2 \times 10^5)$
b) $(6.8 \times 10^7) \times (3 \times 10^3)$
c) $(4 \times 10^5) \times (8.3 \times 10^5)$
d) $(5 \times 10^9) \times (8.4 \times 10^{12})$
e) $(8.5 \times 10^6) \times (6 \times 10^{15})$
f) $(5.0 \times 10^{12})^2$

5. Find the value of the following and write your answers in standard form:
   a) $(3.8 \times 10^8) \div (1.9 \times 10^6)$
   b) $(6.75 \times 10^9) \div (2.25 \times 10^4)$
   c) $(9.6 \times 10^{11}) \div (2.4 \times 10^5)$
   d) $(1.8 \times 10^{12}) \div (9.0 \times 10^7)$
   e) $(2.3 \times 10^{11}) \div (9.2 \times 10^4)$
   f) $(2.4 \times 10^8) \div (6.0 \times 10^3)$

Adding and subtracting numbers in standard form

You can only add and subtract numbers in standard form if the indices are the same. If the indices are different, you can change one of the numbers so that it has the same index as the other. It will not then be in standard form and you may need to change your answer back to standard form after doing the calculation.

Worked examples

a) Add the following and write your answer in standard form:

$(3.8 \times 10^6) + (8.7 \times 10^4)$

Changing the indices to the same value gives the sum:

$(380 \times 10^4) + (8.7 \times 10^4)$
$= 388.7 \times 10^4$
$= 3.887 \times 10^6$ when written in standard form

b) Subtract the following and write your answer in standard form:

$(6.5 \times 10^7) - (9.2 \times 10^5)$

Changing the indices to the same value gives
$(650 \times 10^5) - (9.2 \times 10^5)$
$= 640.8 \times 10^5$
$= 6.408 \times 10^7$ when written in standard form

Exercise 7.7

Find the value of the following and write your answers in standard form:
a) $(3.8 \times 10^5) + (4.6 \times 10^4)$
b) $(7.9 \times 10^7) + (5.8 \times 10^8)$
c) $(6.3 \times 10^7) + (8.8 \times 10^5)$
d) $(3.15 \times 10^9) + (7.0 \times 10^6)$
e) $(5.3 \times 10^8) - (8.0 \times 10^7)$
f) $(6.5 \times 10^7) - (4.9 \times 10^6)$
g) $(8.93 \times 10^{10}) - (7.8 \times 10^9)$
h) $(4.07 \times 10^7) - (5.1 \times 10^6)$

Negative indices and small numbers

A negative index is used when writing a number between 0 and 1 in standard form.

e.g. $100 = 1 \times 10^2$
$10 = 1 \times 10^1$
$1 = 1 \times 10^0$
$0.1 = 1 \times 10^{-1}$
$0.01 = 1 \times 10^{-2}$
$0.001 = 1 \times 10^{-3}$
$0.0001 = 1 \times 10^{-4}$

Note that A must still lie within the range $1 \leqslant A < 10$.

Worked examples

a) Write 0.0032 in standard form.

3.2×10^{-3}

b) Write 1.8×10^{-4} as an ordinary number.

$$1.8 \times 10^{-4} = 1.8 \div 10^4$$
$$= 1.8 \div 10000$$
$$= 0.00018$$

c) Write the following numbers in order of magnitude, starting with the largest:

3.6×10^{-3} 5.2×10^{-5} 1×10^{-2} 8.35×10^{-2} 6.08×10^{-8}

8.35×10^{-2} 1×10^{-2} 3.6×10^{-3} 5.2×10^{-5} 6.08×10^{-8}

Exercise 7.8

1. Copy and complete the following so that the answers are correct (the first question is done for you):

a) 0.0048 $= 4.8 \times 10^{-3}$
b) 0.0079 $= 7.9 \times \ldots$
c) 0.000 81 $= 8.1 \times \ldots$
d) 0.000 009 $= 9 \times \ldots$
e) 0.000 000 45 $= 4.5 \times \ldots$
f) 0.000 000 003 24 $= 3.24 \times \ldots$
g) 0.000 008 42 $= 8.42 \times \ldots$
h) 0.000 000 000 403 $= 4.03 \times \ldots$

2. Write the following numbers in standard form:

a) 0.0006
b) 0.000 053
c) 0.000 864
d) 0.000 000 088
e) 0.000 000 7
f) 0.000 414 5

3. Write the following as ordinary numbers:

a) 8×10^{-3}
b) 4.2×10^{-4}
c) 9.03×10^{-2}
d) 1.01×10^{-5}

4. Write the following numbers in standard form:
a) 68×10^{-5}
b) 750×10^{-9}
c) 42×10^{-11}
d) 0.08×10^{-7}
e) 0.057×10^{-9}
f) 0.4×10^{-10}

5. Deduce the value of n in each of the following cases:
a) $0.000\,25 = 2.5 \times 10^{n}$
b) $0.003\,57 = 3.57 \times 10^{n}$
c) $0.000\,000\,06 = 6 \times 10^{n}$
d) $0.004^{2} = 1.6 \times 10^{n}$
e) $0.000\,65^{2} = 4.225 \times 10^{n}$
f) $0.0002^{n} = 8 \times 10^{-12}$

6. Write these numbers in order of magnitude, starting with the largest:

3.2×10^{-4} 6.8×10^{5} 5.57×10^{-9} 6.2×10^{3}
5.8×10^{-7} 6.741×10^{-4} 8.414×10^{2}

Student assessment 1

1. Simplify the following using indices:
a) $2 \times 2 \times 2 \times 5 \times 5$ b) $2 \times 2 \times 3 \times 3 \times 3 \times 3 \times 3$

2. Write the following out in full:
a) 4^{3} b) 6^{4}

3. Work out the value of the following without using a calculator:
a) $2^{3} \times 10^{2}$ b) $1^{4} \times 3^{3}$

4. Simplify the following using indices:
a) $3^{4} \times 3^{3}$ b) $6^{3} \times 6^{2} \times 3^{4} \times 3^{5}$
c) $\frac{4^{5}}{2^{3}}$ d) $\frac{(6^{2})^{3}}{6^{5}}$
e) $\frac{3^{5} \times 4^{2}}{3^{3} \times 4^{0}}$ f) $\frac{4^{-2} \times 2^{6}}{2^{2}}$

5. Without using a calculator, evaluate the following:
a) $2^{4} \times 2^{-2}$ b) $\frac{3^{5}}{3^{3}}$
c) $\frac{5^{-5}}{5^{-6}}$ d) $\frac{2^{5} \times 4^{-3}}{2^{-1}}$

Student assessment 2

1. Simplify the following using indices:
   a) $3 \times 2 \times 2 \times 3 \times 27$
   b) $2 \times 2 \times 4 \times 4 \times 4 \times 2 \times 32$

2. Write the following out in full:
   a) 6^5 b) 2^{-5}

3. Work out the value of the following without using a calculator:
   a) $3^3 \times 10^3$ b) $1^{-4} \times 5^3$

4. Simplify the following using indices:
   a) $2^4 \times 2^3$ b) $7^5 \times 7^2 \times 3^4 \times 3^8$
   c) $\frac{4^8}{2^{10}}$ d) $\frac{(3^3)^4}{27^3}$
   e) $\frac{7^6 \times 4^2}{4^3 \times 7^6}$ f) $\frac{8^{-2} \times 2^6}{2^{-2}}$

5. Without using a calculator, evaluate the following:
   a) $5^2 \times 5^{-1}$ b) $\frac{4^5}{4^3}$ c) $\frac{7^{-5}}{7^{-7}}$ d) $\frac{3^{-5} \times 4^2}{3^{-6}}$

Student assessment 3

1. Write the following numbers in standard form:
   a) 8 million b) 0.000 72
   c) 75 000 000 000 d) 0.0004
   e) 4.75 billion f) 0.000 000 64

2. Write the following as ordinary numbers:
   a) 2.07×10^4 b) 1.45×10^{-3}

3. Write the following numbers in order of magnitude, starting with the smallest:

   6.2×10^7 5.5×10^{-3} 4.21×10^7 4.9×10^8
   3.6×10^{-5} 7.41×10^{-9}

4. Write the following numbers:
   a) in standard form,
   b) in order of magnitude, starting with the largest.

   6 million 820 000 0.0044 0.8 52 000

5. Deduce the value of n in each of the following:
   a) $620 = 6.2 \times 10^n$ b) $555\,000\,000 = 5.55 \times 10^n$
   c) $0.000\,45 = 4.5 \times 10^n$ d) $500^2 = 2.5 \times 10^n$
   e) $0.0035^2 = 1.225 \times 10^n$ f) $0.04^3 = 6.4 \times 10^n$

6. Write the answers to the following calculations in standard form:
   a) $4000 \times 30\,000$ b) $(2.8 \times 10^5) \times (2.0 \times 10^3)$
   c) $(3.2 \times 10^9) \div (1.6 \times 10^4)$ d) $(2.4 \times 10^8) \div (9.6 \times 10^2)$

7. The speed of light is 3×10^8 m/s. Venus is 108 million km from the Sun. Calculate the number of minutes it takes for sunlight to reach Venus.

8. A star system is 500 light years away from Earth. The speed of light is 3×10^5 km/s. Calculate the distance the star system is from Earth. Give your answer in kilometres and written in standard form.

Student assessment 4

1. Write the following numbers in standard form:
   a) 6 million b) 0.0045
   c) 3 800 000 000 d) 0.000 000 361
   e) 460 million f) 3

2. Write the following as ordinary numbers:
   a) 8.112×10^6 b) 3.05×10^{-4}

3. Write the following numbers in order of magnitude, starting with the largest:

   3.6×10^2 2.1×10^{-3} 9×10^1 4.05×10^8
   1.5×10^{-2} 7.2×10^{-3}

4. Write the following numbers:
   a) in standard form,
   b) in order of magnitude, starting with the smallest.

   15 million 430 000 0.000 435 4.8 0.0085

5. Deduce the value of n in each of the following:
   a) $4750 = 4.75 \times 10^n$ b) $6\,440\,000\,000 = 6.44 \times 10^n$
   c) $0.0040 = 4.0 \times 10^n$ d) $1000^2 = 1 \times 10^n$
   e) $0.9^3 = 7.29 \times 10^n$ f) $800^3 = 5.12 \times 10^n$

6. Write the answers to the following calculations in standard form:
   a) $50\,000 \times 2400$ b) $(3.7 \times 10^6) \times (4.0 \times 10^4)$
   c) $(5.8 \times 10^7) + (9.3 \times 10^6)$ d) $(4.7 \times 10^6) - (8.2 \times 10^5)$

7. The speed of light is 3×10^8 m/s. Jupiter is 778 million km from the Sun. Calculate the number of minutes it takes for sunlight to reach Jupiter.

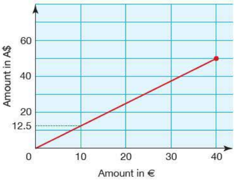
8. A star is 300 light years away from Earth. The speed of light is 3×10^5 km/s. Calculate the distance from the star to Earth. Give your answer in kilometres and written in standard form.

8 Money and finance

Currency conversion

In 2012, 1 euro could be exchanged for 1.25 Australian dollars (A$).

A graph to enable conversion from euro to dollars and from dollars to euro can be seen below.



Exercise 8.1

1. Use the conversion graph above to convert the following to Australian dollars:

   a) €20 b) €30 c) €5
   d) €25 e) €35 f) €15

2. Use the graph above to convert the following to euro:

   a) A$20 b) A$30 c) A$40
   d) A$35 e) A$25 f) A$48

3. 1 euro could be exchanged for 70 Indian rupees. Draw a conversion graph. Use an appropriate scale with the horizontal scale up to €100. Use your graph to convert the following to rupees:

   a) €10 b) €40 c) €50
   d) €90 e) €25 f) €1000

4. Use your graph from Q.3 to convert the following to euro:

   a) 140 rupees b) 770 rupees c) 630 rupees
   d) 490 rupees e) 5600 rupees f) 2730 rupees

The table (below) shows the exchange rate for €1 into various currencies.

Draw conversion graphs for the exchange rates shown to answer Q.5–10.

Zimbabwe	470 Zimbabwe dollars
South Africa	11 rand
Turkey	2.3 Turkish lira
Japan	103 yen
Kuwait	0.4 dinar
U.S.A	1.3 dollars

5. How many Zimbabwe dollars would you receive for the following?
a) €20 b) €50 c) €75
d) €30 e) €25

6. How many euro would you receive for the following numbers of South African rand?
a) 220 rand b) 550 rand c) 1100 rand d) 100 rand

7. In the Grand Bazaar in Istanbul, a visitor sees three carpets priced at 120 Turkish lira, 400 Turkish lira and 88 Turkish lira. Draw and use a conversion graph to find the prices in euro.

8. €1 can be exchanged for US$1.3.
€1 can also be exchanged for 103 yen.
Draw a conversion graph for US dollars to Japanese yen, and answer the questions below.
a) How many yen would you receive for:
i) $300 ii) $750 iii) $1000?
b) How many US dollars would you receive for:
i) 5000 yen ii) 8500 yen iii) 100 yen?

9. Use the currency table above to draw a conversion graph for Kuwaiti dinars to Zimbabwe dollars. Use the graph to find the number of dollars you would receive for:
a) 50 dinars b) 200 dinars
c) 120 dinars d) 600 dinars

10. Use the currency table above to draw a conversion graph for Turkish lira to Zimbabwe dollars. Use the graph to find the number of lira you would receive for:
a) 1000 dollars b) 7000 dollars
c) 12 000 dollars

Personal and household finance

Net pay is what is left after deductions such as tax, insurance and pension contributions are taken from **gross earnings**. That is,

Net pay = Gross pay − Deductions

A **bonus** is an extra payment sometimes added to an employee's basic pay.

In many companies there is a fixed number of hours that an employee is expected to work. Any work done in excess of this **basic week** is paid at a higher rate, referred to as **overtime**. Overtime may be 1.5 times basic pay, called **time and a half**, or twice basic pay, called **double time**.

Exercise 8.2

1. Copy the table below and find the net pay for the following employees:

	Gross pay ($)	Deductions ($)	Net pay ($)
a) A Ahmet	162.00	23.50	
b) B Martinez	205.50	41.36	
c) C Stein	188.25	33.43	
d) D Wong	225.18	60.12	

2. Copy and complete the table below for the following employees:

	Basic pay ($)	Overtime ($)	Bonus ($)	Gross pay ($)
a) P Small	144	62	23	
b) B Smith	152		31	208
c) A Chang		38	12	173
d) U Zafer	115	43		213
e) M Said	128	36	18	

3. Copy and complete the table below for the following employees:

	Gross pay ($)	Tax ($)	Pension ($)	Net pay ($)
a) A Hafar	203	54	18	
b) K Zyeb		65	23	218
c) H Such	345		41	232
d) K Donald	185	23		147

4. Find the basic pay in each of the cases below. Copy and complete the table.

	No. of hours worked	Basic rate per hour ($)	Basic pay ($)
a)	40	3.15	
b)	44	4.88	
c)	38	5.02	
d)	35	8.30	
e)	48	7.25	

5. Copy and complete the table below, which shows basic pay and overtime at time and a half.

	Basic hours worked	Rate per hour ($)	Basic pay ($)	Overtime hours worked	Overtime pay ($)	Total gross pay ($)
a)	40	3.60		8		
b)	35		203.00	4		
c)	38	4.15		6		
d)		6.10	256.20	5		
e)	44	5.25		4		
f)		4.87	180.19	3		
g)	36	6.68		6		
h)	45	7.10	319.50	7		

6. In Q.5, deductions amount to 32% of the total gross pay. Calculate the net pay for each employee.

Piece work is a method of payment where an employee is paid for the number of articles made, not for time taken.

Exercise 8.3

1. Four people help to pick grapes in a vineyard. They are paid \$5.50 for each basket of grapes. Copy and complete the table below.

	Mon	Tue	Wed	Thur	Fri	Total	Gross pay
Pepe	4	5	7	6	6		
Felicia	3	4	4	5	5		
Delores	5	6	6	5	6		
Juan	3	4	6	6	6		

2. Five people work in a pottery factory, making plates. They are paid \$5 for every 12 plates made. Copy and complete the following table, which shows the number of plates that each person produces.

	Mon	Tue	Wed	Thur	Fri	Total	Gross pay
Maria	240	360	288	192	180		
Ben	168	192	312	180	168		
Joe	288	156	192	204	180		
Bianca	228	144	108	180	120		
Selina	192	204	156	228	144		

3. A group of five people work at home making clothes. The patterns and material are provided by the company, and for each article produced they are paid:

   Jacket \$25   Trousers \$11
   Shirt \$13   Dress \$12

   The five people make the numbers of articles of clothing shown in the table below. Find each person's gross pay. If the deductions amount to 15% of gross earnings, calculate each person's net pay.

	Jackets	Shirts	Trousers	Dresses
Neo	3	12	7	0
Keletso	8	5	2	9
Ditshele	0	14	12	2
Mpho	6	8	3	12
Kefilwe	4	9	16	5

4. A school organises a sponsored walk. Below is a list of how far students walked, the amount they were sponsored per mile, and the total each raised.

a) Copy and complete the table.

Distance walked (km)	Amount per km ($)	Total raised ($)
10	0.80	
	0.65	9.10
18	0.38	
	0.72	7.31
12		7.92
	1.20	15.60
15	1	
	0.88	15.84
18		10.44
17		16.15

b) How much was raised in total?

c) This total was divided between three children's charities in the ratio of 2 : 3 : 5. How much did each charity receive?

Simple interest

Interest can be defined as money added by a bank to sums deposited by customers. The money deposited is called the **principal**. The **percentage interest** is the given rate and the money is left for a fixed period of time.

A formula can be obtained for **simple interest**:

$$SI = \frac{Ptr}{100}$$

where SI = simple interest, i.e. the interest paid
P = the principal
t = time in years
r = rate per cent

Worked example Find the simple interest earned on \$250 deposited for 6 years at 8% p.a.

$$SI = \frac{Ptr}{100}$$

$$SI = \frac{250 \times 6 \times 8}{100}$$

$$SI = 120$$

So the interest paid is \$120.

Exercise 8.4

All rates of interest given here are annual rates.

Find the simple interest paid in the following cases:

a)	Principal \$300	rate 6%	time 4 years
b)	Principal \$750	rate 8%	time 7 years
c)	Principal \$425	rate 6%	time 4 years
d)	Principal \$2800	rate 4.5%	time 2 years
e)	Principal \$6500	rate 6.25%	time 8 years
f)	Principal \$880	rate 6%	time 7 years

Worked example How long will it take for a sum of \$250 invested at 8% to earn interest of \$80?

$$SI = \frac{Ptr}{100}$$

$$80 = \frac{250 \times t \times 8}{100}$$

$$80 = 20t$$

$$4 = t$$

It will take 4 years.

Exercise 8.5

Calculate how long it will take for the following amounts of interest to be earned at the given rate.

a)	$P = \$500$	$r = 6\%$	$SI = \$150$
b)	$P = \$5800$	$r = 4\%$	$SI = \$96$
c)	$P = \$4000$	$r = 7.5\%$	$SI = \$1500$
d)	$P = \$2800$	$r = 8.5\%$	$SI = \$1904$
e)	$P = \$900$	$r = 4.5\%$	$SI = \$243$
f)	$P = \$400$	$r = 9\%$	$SI = \$252$

Worked example What rate per year must be paid for a principal of \$750 to earn interest of \$180 in 4 years?

$$SI = \frac{Ptr}{100}$$

$$180 = \frac{750 \times 4 \times r}{100}$$

$$180 = 30r$$

$$6 = r$$

The rate must be 6% per year.

Exercise 8.6

Calculate the rate of interest per year which will earn the given amount of interest:

a)	Principal \$400	time 4 years	interest \$112
b)	Principal \$800	time 7 years	interest \$224
c)	Principal \$2000	time 3 years	interest \$210
d)	Principal \$1500	time 6 years	interest \$675
e)	Principal \$850	time 5 years	interest \$340
f)	Principal \$1250	time 2 years	interest \$275

Worked example Find the principal which will earn interest of \$120 in 6 years at 4%.

$$SI = \frac{Ptr}{100}$$

$$120 = \frac{P \times 6 \times 4}{100}$$

$$120 = \frac{24P}{100}$$

$$12\,000 = 24P$$

$$500 = P$$

So the principal is \$500.

Exercise 8.7

1. Calculate the principal which will earn the interest below in the given number of years at the given rate:

a)	SI = \$80	time = 4 years	rate = 5%
b)	SI = \$36	time = 3 years	rate = 6%
c)	SI = \$340	time = 5 years	rate = 8%
d)	SI = \$540	time = 6 years	rate = 7.5%
e)	SI = \$540	time = 3 years	rate = 4.5%
f)	SI = \$348	time = 4 years	rate = 7.25%

2. What rate of interest is paid on a deposit of \$2000 which earns \$400 interest in 5 years?

3. How long will it take a principal of \$350 to earn \$56 interest at 8% per year?

4. A principal of \$480 earns \$108 interest in 5 years. What rate of interest was being paid?

5. A principal of \$750 becomes a total of \$1320 in 8 years. What rate of interest was being paid?

6. \$1500 is invested for 6 years at 3.5% per year. What is the interest earned?

7. \$500 is invested for 11 years and becomes \$830 in total. What rate of interest was being paid?

Compound interest

Compound interest means that not only is interest paid on the principal amount but interest is paid on the interest. It is compounded (or added to).

This sounds complicated but the example below will make it clear.

A builder is going to build six houses on a plot of land in Spain. He borrows 500 000 euro at 10% and will pay off the loan in full after three years.

At the end of the first year he will owe
€500 000 + 10% of €500 000
i.e. 550 000 euro

At the end of the second year he will owe
€550 000 + 10% of €550 000
i.e. €550 000 + €55 000
= 605 000 euro

At the end of the third year he will owe
€605 000 + 10% of €605 000
i.e. €605 000 + €60 500
= 665 500 euro

His interest will be 665 500 – 500 000 euro
i.e. 165 500 euro

At simple interest he would pay 50 000 euro a year
i.e. 150 000 euro in total.

The extra 15 500 euro was the compound interest.

Exercise 8.8

1. A shipping company borrows $70 million at 5% compound interest to build a new cruise ship. If it repays the debt after 3 years, how much interest will the company pay?
2. A woman borrows $100 000 to improve her house. She borrows the money at 15% interest and repays it in full after 3 years. What interest will she pay?
3. A man owes $5000 on his credit cards. The annual percentage rate (APR) is 20%. If he lets the debt grow how much will he owe in 4 years?
4. A school increases its intake by 10% each year. If it starts with 1000 students, how many will it have at the beginning of the fourth year of expansion?
5. 8 million tonnes of fish were caught in the North Sea in 2005. If the catch is reduced by 20% each year for 4 years, what amount can be caught at the end of this time?

It is possible to calculate the time taken for a debt to grow using compound interest.

Worked example How many years will it take for a debt, D, to double at 27% compound interest?
After 1 year the debt will be $D \times (1 + 27\%)$ or $1.27D$.
After 2 years the debt will be $D \times (1.27 \times 1.27)$ or $1.61D$.
After 3 years the debt will be $D \times (1.27 \times 1.27 \times 1.27)$ or $2.05D$.

So the debt will have doubled after 3 years.

Note that the amount of the debt does not affect this calculation. Note also that if the debt were reducing it would take the same number of years to halve.

Exercise 8.9

1. How many years would it take a debt to double at a compound interest rate of 42%?
2. How many years would it take a debt to double at a compound interest rate of 15%?
3. A car loses value at 27% compound interest each year. How many years will it take to halve in value?
4. The value of a house increases by 20% each year compound increase. How many years before it doubles in value?
5. A boat loses value at a rate of 15% compound per year. How many years before its value has halved?

Profit and loss

Foodstuffs and manufactured goods are produced at a cost, known as the **cost price**, and sold at the **selling price**. If the selling price is greater than the cost price, a profit is made.

Worked example A market trader buys oranges in boxes of 144 for \$14.40 per box. He buys three boxes and sells all the oranges for 12c each. What is his profit or loss?

Cost price: $3 \times \$14.40 = \$43.20$
Selling price: $3 \times 144 \times 12\text{c} = \$51.84$
In this case he makes a profit of $\$51.84 - \$43.20$
His profit is \$8.64.

A second way of solving this problem would be:
\$14.40 for a box of 144 oranges is 10c each.
So cost price of each orange is 10c, and selling price of each orange is 12c. The profit is 2c per orange.
So 3 boxes would give a profit of $3 \times 144 \times 2\text{c}$.
That is, \$8.64.

Exercise 8.10

1. A market trader buys peaches in boxes of 120. He buys 4 boxes at a cost price of \$13.20 per box. He sells 425 peaches at 12c each – the rest are ruined. How much profit or loss does he make?

2. A shopkeeper buys 72 bars of chocolate for \$5.76. What is his profit if he sells them for 12c each?

3. A holiday company charters an aircraft to fly to Malta at a cost of \$22 000. It then sells 195 seats at \$185 each. Calculate the profit made per seat if the plane has 200 seats.

4. A car is priced at \$7200. The car dealer allows a customer to pay a one-third deposit and 12 payments of \$420 per month. How much extra does it cost the customer?

5. At an auction a company sells 150 television sets for an average of \$65 each. The production cost was \$10 000. How much loss did the company make?

6. A market trader sells tools and small electrical goods. Find his profit or loss at the end of a day in which he sells each of the following:
   a) 15 torches: cost price \$2 each, selling price \$2.30 each
   b) 60 plugs: cost price \$10 for 12, selling price \$1.10 each
   c) 200 CDs: cost price \$9 for 10, selling price \$1.30 each
   d) 5 MP3 players: cost price \$82, selling price \$19 each
   e) 96 batteries costing \$1 for 6, selling price 59c for 3
   f) 3 clock radios costing \$65, sold for \$14 each

Percentage profit and loss

Most profits or losses are expressed as a percentage.

$$\text{Percentage profit or loss} = \frac{\text{Profit or Loss}}{\text{Cost price}} \times 100$$

Worked example A woman buys a car for \$7500 and sells it two years later for \$4500. Calculate her loss over two years as a percentage of the cost price.

Cost price = \$7500   Selling price = \$4500   Loss = \$3000

$$\text{Percentage loss} = \frac{3000}{7500} \times 100 = 40$$

Her loss is 40%.

When something becomes worth less over a period of time, it is said to **depreciate**.

Exercise 8.11

1. Find the depreciation of the following cars as a percentage of the cost price. (C.P. = Cost price, S.P. = Selling price)

a) VW	C.P. $4500	S.P. $4005
b) Rover	C.P. $9200	S.P. $6900
c) Mercedes	C.P. $11 000	S.P. $5500
d) Toyota	C.P. $4350	S.P. $3480
e) Fiat	C.P. $6850	S.P. $4795
f) Ford	C.P. $7800	S.P. $2600

2. A company manufactures electrical items for the kitchen. Find the percentage profit on each of the following:

a) Cooker	C.P. $240	S.P. $300
b) Fridge	C.P. $50	S.P. $65
c) Freezer	C.P. $80	S.P. $96
d) Microwave	C.P. $120	S.P. $180
e) Washing machine	C.P. $260	S.P. $340
f) Dryer	C.P. $70	S.P. $91

3. A developer builds a number of different kinds of house on a village site. Given the cost prices and the selling prices below, which type of house gives the developer the largest percentage profit?

	Cost price ($)	Selling price ($)
Type A	40000	52000
Type B	65000	75000
Type C	81000	108000
Type D	110000	144000
Type E	132000	196000

4. Students in a school organise a disco. The disco company charges $350 hire charge. The students sell 280 tickets at $2.25. What is the percentage profit?

5. A shop sells secondhand sailing yachts. Calculate the percentage profit on each of the following:

	Cost price ($)	Selling price ($)
Mirror	400	520
Wayfarer	1100	1540
Laser	900	1305
Fireball	1250	1720

Student assessment 1

A visitor from Hong Kong receives 12 Pakistan rupees for each Hong Kong dollar.

Draw a conversion graph for Hong Kong dollars and Pakistan rupees, and use it to answer the questions below.

1. How many Pakistan rupees would you get for:
a) HK$30 b) HK$240 c) HK$5000?

2. How many Hong Kong dollars would you get for:
a) 120 rupees b) 450 rupees c) 1000 rupees?

Below is a currency conversion table showing the amounts of foreign currency received for 1 euro. Draw the appropriate conversion graphs to enable you to answer Q.3–5.

New Zealand	1.6 dollars
Brazil	2.6 reals
India	70 Indian rupees

3. Convert the following numbers of New Zealand dollars into euro:
a) 320 dollars b) 72 dollars c) 960 dollars

4. Convert the following numbers of euro into New Zealand dollars:
a) 100 euro b) 35 euro c) 450 euro

5. Convert the following numbers of Brazilian reals into Indian rupees:
a) 60 reals b) 300 reals c) 100 reals

Student assessment 2

1. 1 Australian dollar can be exchanged for €0.8. Draw a conversion graph to find the number of Australian dollars you would get for:
a) €50 b) €80 c) €70

2. Use your graph from Q.1 to find the number of euro you could receive for:
a) A$54 b) A$81 c) A$320

Below is a currency conversion table showing the amounts of foreign currency received for €1. Draw the appropriate conversion graphs to answer Q.3–5.

Nigeria	203 nairas
Malaysia	3.9 ringgits
Jordan	0.9 Jordanian dinars

3. Convert the following numbers of Malaysian ringgits into Jordanian dinars:
a) 100 ringgits b) 1200 ringgits c) 150 ringgits

4. Convert the following numbers of Jordanian dinars into Nigerian nairas:
a) 1 dinar b) 6 dinars c) 4 dinars

5. Convert the following numbers of Nigerian nairas into Malaysian ringgitss:
a) 1000 nairas b) 5000 nairas c) 7500 nairas

Student assessment 3

1. A boy worked 3 hours a day each weekday for $3.75 per hour. What was his 4-weekly gross payment?

2. A woman works at home making curtains. In one week she makes 4 pairs of long curtains and 13 pairs of short curtains. What is her gross pay if she receives $2.10 for each long curtain, and $1.85 for each short curtain?

3. Calculate the missing numbers from the simple interest table below:

Principal ($)	Rate (%)	Time (years)	Interest ($)
200	9	3	a)
350	7	b)	98
520	c)	5	169
d)	3.75	6	189

4. A car cost $7200 new and sold for $5400 after two years. What was the percentage average annual depreciation?

5. A farmer sold eight cows at market at an average sale price of $48 each. If his total costs for rearing all the animals were $432, what was his percentage loss on each animal?

Student assessment 4

1. A girl works in a shop on Saturdays for 8.5 hours. She is paid \$3.60 per hour. What is her gross pay for 4 weeks' work?

2. A potter makes cups and saucers in a factory. He is paid \$1.44 per batch of cups and \$1.20 per batch of saucers. What is his gross pay if he makes 9 batches of cups and 11 batches of saucers in one day?

3. Calculate the missing numbers from the simple interest table below:

Principal ($)	Rate (%)	Time (years)	Interest ($)
300	6	4	a)
250	b)	3	60
480	5	c)	96
650	d)	8	390
e)	3.75	4	187.50

4. A family house was bought for \$48 000 twelve years ago. It is now valued at \$120 000. What is the average annual increase in the value of the house?

5. An electrician bought five broken washing machines for \$550. He repaired them and sold them for \$143 each. What was his percentage profit?

9 Time

Times may be given in terms of the 12-hour clock. We tend to say, 'I get up at seven o'clock in the morning, play football at half past two in the afternoon, and go to bed before eleven o'clock'.

These times can be written as 7 a.m., 2.30 p.m. and 11 p.m.

In order to save confusion, most timetables are written using the 24-hour clock.

7 a.m. is written as 07 00
2.30 p.m. is written as 14 30
11 p.m. is written as 23 00

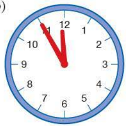
To change p.m. times to 24-hour clock times, add 12 hours.

To change 24-hour clock times later than 12.00 noon to 12-hour clock times, subtract 12 hours.

Exercise 9.1

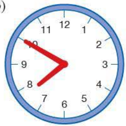
1. The clocks below shows times in the morning. Write down the times using both the 12-hour and the 24-hour clock.

a)


b)


2. The clocks below shows times in the afternoon. Write down the times using both the 12-hour and the 24-hour clock.

a)


b)


3. Change these times into the 24-hour clock:

a) 2.30 p.m.	b) 9 p.m.	c) 8.45 a.m.	d) 6 a.m.
e) midday	f) 10.55 p.m.	g) 7.30 a.m.	h) 7.30 p.m.
i) 1 a.m.	j) midnight		

4. Change these times into the 24-hour clock:
a) A quarter past seven in the morning
b) Eight o'clock at night
c) Ten past nine in the morning
d) A quarter to nine in the morning
e) A quarter to three in the afternoon
f) Twenty to eight in the evening

5. These times are written for the 24-hour clock. Rewrite them using a.m. and p.m.

a) 07 20	b) 09 00	c) 14 30	d) 18 25
e) 23 40	f) 01 15	g) 00 05	h) 11 35
i) 17 50	j) 23 59	k) 04 10	l) 05 45

6. A journey to work takes a woman three quarters of an hour. If she catches the bus at the following times, when does she arrive?
a) 07 20 b) 07 55 c) 08 20 d) 08 45

7. The same woman catches buses home at the times shown below. The journey takes 55 minutes. If she catches the bus at the following times, when does she arrive?
a) 17 25 b) 17 50 c) 18 05 d) 18 20

8. A boy cycles to school each day. His journey takes 70 minutes. When will he arrive if he leaves home at:
a) 07 15 b) 08 25 c) 08 40 d) 08 55?

9. The train into the city from a village takes 1 hour and 40 minutes. Copy and complete the train timetable below.

Depart	Arrive
06 15	
	08 10
09 25	
	12 00
13 18	
	16 28
18 54	
	21 05

10. The same journey by bus takes 2 hours and 5 minutes. Copy and complete the bus timetable on the next page.

Depart	Arrive
0600	
	0850
0855	
	1114
1348	
	1622
2125	
	0010

11. A coach runs from Cambridge to the airports at Stansted, Gatwick and Heathrow. The time taken for the journey remains constant. Copy and complete the timetables below for the outward and return journeys.

Cambridge	0400	0835	1250	1945	2110
Stansted	0515				
Gatwick	0650				
Heathrow	0735				

Heathrow	0625	0940	1435	1810	2215
Gatwick	0812				
Stansted	1003				
Cambridge	1100				

12. The flight time from London to Johannesburg is 11 hours and 20 minutes. Copy and complete the timetable below.

	London	Jo'burg	London	Jo'burg
Sunday	0615		1420	
Monday		1845		0525
Tuesday	0720		1513	
Wednesday		1912		0730
Thursday	0610		1627	
Friday		1725		0815
Saturday	0955		1850	

13. The flight time from London to Kuala Lumpur is 13 hours and 45 minutes. Copy and complete the timetable below.

	London	Kuala Lumpur	London	Kuala Lumpur	London	Kuala Lumpur
Sunday	08 28		14 00		18 30	
Monday		22 00		03 15		09 50
Tuesday	09 15		15 25		17 55	
Wednesday		21 35		04 00		08 22
Thursday	07 00		13 45		18 40	
Friday		00 10		04 45		07 38
Saturday	10 12		14 20		19 08	

Average speed

Worked example A train covers the 480 km journey from Paris to Lyon at an average speed of 100 km/h. If the train leaves Paris at 08 35, when does it arrive in Lyon?

Time taken $= \frac{\text{distance}}{\text{speed}}$

Paris to Lyon: $\frac{480}{100}$ hours, that is, 4.8 hours.

4.8 hours is 4 hours and (0.8 × 60 minutes), that is, 4 hours and 48 minutes.

Departure 08 35; arrival 08 35 + 04 48

Arrival time is 13 23.

Exercise 9.2

Find the time in hours and minutes for the following journeys of the given distance at the average speed stated:

1. a) 240 km at 60 km/h b) 340 km at 40 km/h
   c) 270 km at 80 km/h d) 100 km at 60 km/h
   e) 70 km at 30 km/h f) 560 km at 90 km/h
   g) 230 km at 100 km/h h) 70 km at 50 km/h
   i) 4500 km at 750 km/h j) 6000 km at 800 km/h

2. Grand Prix racing cars cover a 120 km race at the following average speeds. How long do the first five cars take to complete the race? Give your answer in minutes and seconds.

   First 240 km/h Second 220 km/h Third 210 km/h
   Fourth 205 km/h Fifth 200 km/h

3. A train covers the 1500 km distance from Amsterdam to Barcelona at an average speed of 100 km/h. If the train leaves Amsterdam at 09 30, when does it arrive in Barcelona?

4. A plane takes off at 16 25 for the 3200 km journey from Moscow to Athens. If the plane flies at an average speed of 600 km/h, when will it land in Athens?

5. A plane leaves London for Boston, a distance of 5200 km, at 09 45. The plane travels at an average speed of 800 km/h. If Boston time is five hours behind British time, what is the time in Boston when the aircraft lands?

Student assessment 1

1. The clock shows a time in afternoon. Write down the time using
   a) the 12-hour clock
   b) the 24-hour clock.

2. Change the times below into the 24-hour clock:
   a) 4.35 a.m.
   b) 6.30 p.m.
   c) a quarter to eight in the morning
   d) half past seven in the evening

3. The times below are written for the 24-hour clock. Rewrite them using a.m. and p.m.
   a) 08 45 b) 18 35 c) 21 12 d) 00 15

4. A journey to school takes a girl 25 minutes. What time does she arrive if she leaves home at the following times?
   a) 07 45 b) 08 15 c) 08 38

5. A bus service visits the towns on the timetable below. Copy the timetable and fill in the missing times, given the following information:

   The journey from:
   Alphaville to Betatown takes 37 minutes
   Betatown to Gammatown takes 18 minutes
   Gammatown to Deltaville takes 42 minutes.

Alphaville	07 50		
Betatown		11 38	
Gammatown			16 48
Deltaville			

6. Find the times for the following journeys of given distance at the average speed stated. Give your answers in hours and minutes.
   a) 250 km at 50 km/h b) 375 km at 100 km/h
   c) 80 km at 60 km/h d) 200 km at 120 km/h
   e) 70 km at 30 km/h f) 300 km at 80 km/h

Student assessment 2

1. The clock shows a time in the morning. Write down the time using
   a) the 12-hour clock
   b) the 24-hour clock.

2. Change the times below to the 24-hour clock:
   a) 5.20 a.m. b) 8.15 p.m.
   c) ten to nine in the morning d) half past eleven at night

3. The times below are written for the 24-hour clock. Rewrite them using a.m. and p.m.
   a) 0715 b) 1643 c) 1930 d) 0035

4. A journey to school takes a boy 22 minutes. When does he arrive if he leaves home at the following times?
   a) 0748 b) 0817 c) 0838

5. A train stops at the following stations. Copy the timetable and fill in the times, given the following information:

   The journey from:
   Apple to Peach is 1 hr 38 minutes
   Peach to Pear is 2 hrs 4 minutes
   Pear to Plum is 1 hr 53 minutes.

Apple	1014		
Peach		1720	
Pear			2315
Plum			

6. Find the time for the following journeys of given distance at the average speed stated. Give your answers in hours and minutes.
   a) 350 km at 70 km/h b) 425 km at 100 km/h
   c) 160 km at 60 km/h d) 450 km at 120 km/h
   e) 600 km at 160 km/h

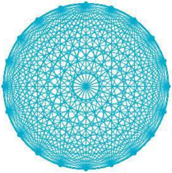
Topic 1 Mathematical investigations and ICT

Investigations are an important part of mathematical learning. All mathematical discoveries stem from an idea that a mathematician has and then investigates.

Sometimes when faced with a mathematical investigation, it can seem difficult to know how to start. The structure and example below may help you.

1. Read the question carefully and start with simple cases.
2. Draw simple diagrams to help.
3. Put the results from simple cases in a table.
4. Look for a pattern in your results.
5. Try to find a general rule in words.
6. Express your rule algebraically.
7. Test the rule for a new example.
8. Check that the original question has been answered.

Worked example



A mystic rose is created by placing a number of points evenly spaced on the circumference of a circle. Straight lines are then drawn from each point to every other point. The diagram (left) shows a mystic rose with 20 points.

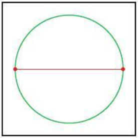
i) How many straight lines are there?
ii) How many straight lines would there be on a mystic rose with 100 points?

To answer these questions, you are not expected to draw either of the shapes and count the number of lines.

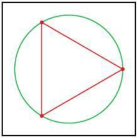
1/2. Try simple cases:

By drawing some simple cases and counting the lines, some results can be found:

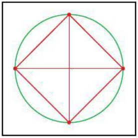
Mystic rose with 2 points
Number of lines = 1



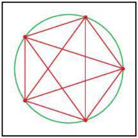
Mystic rose with 3 points
Number of lines = 3



Mystic rose with 4 points
Number of lines = 6



Mystic rose with 5 points
Number of lines = 10



3. Enter the results in an ordered table:

Number of points	2	3	4	5
Number of lines	1	3	6	10

4/5. Look for a pattern in the results:

There are two patterns.

The first pattern shows how the values change.

1 → 3 → 6 → 10

+2 +3 +4

It can be seen that the difference between successive terms is increasing by one each time.

The problem with this pattern is that to find the 20th or 100th term, it would be necessary to continue this pattern and find all the terms leading up to the 20th or 100th term.

The second pattern is the relationship between the number of points and the number of lines.

Number of points	2	3	4	5
Number of lines	1	3	6	10

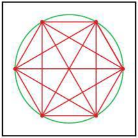
It is important to find a relationship that works for all values, for example subtracting one from the number of points gives the number of lines in the first example only, so is not useful. However, halving the number of points and multiplying this by one less than the number of points works each time,

i.e. Number of lines = (half the number of points) × (one less than the number of points).

6. **Express the rule algebraically:**
The rule expressed in words above can be written more elegantly using algebra. Let the number of lines be l and the number of points be p.

$$l = \frac{1}{2}p(p - 1)$$

Note: Any letters can be used to represent the number of lines and the number of points, not just l and p.



7. **Test the rule:**
The rule was derived from the original results. It can be tested by generating a further result.

If the number of points $p = 6$, then the number of lines l is:

$$\begin{aligned} l &= \tfrac{1}{2} \times 6(6 - 1) \\ &= 3 \times 5 \\ &= 15 \end{aligned}$$

From the diagram to the left, the number of lines can also be counted as 15.

8. **Check that the original questions have been answered:**
Using the formula, the number of lines in a mystic rose with 20 points is:

$$\begin{aligned} l &= \tfrac{1}{2} \times 20(20 - 1) \\ &= 10 \times 19 \\ &= 190 \end{aligned}$$

The number of lines in a mystic rose with 100 points is:

$$\begin{aligned} l &= \tfrac{1}{2} \times 100(100 - 1) \\ &= 50 \times 99 \\ &= 4950 \end{aligned}$$

● Primes and squares

13, 41 and 73 are prime numbers.

Two different square numbers can be added together to make these prime numbers, e.g. $3^2 + 8^2 = 73$.

1. Find the two square numbers that can be added to make 13 and 41.
2. List the prime numbers less than 100.
3. Which of the prime numbers less than 100 can be shown to be the sum of two different square numbers?
4. Is there a rule to the numbers in Q.3?
5. Your rule is a predictive rule not a formula. Discuss the difference.

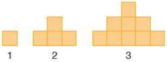
Football leagues

There are 18 teams in a football league.

1. If each team plays the other teams twice, once at home and once away, then how many matches are played in a season?
2. If there are t teams in a league, how many matches are played in a season?

ICT activity 1

The step patterns below follow a rule.



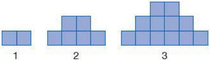
1. On squared paper, draw the next two patterns in this sequence.
2. Count the number of squares used in each of the first five patterns. Enter the results into a table on a spreadsheet, similar to the one shown below.

	A	B
1	Pattern	Number of squares
2	1	
3	2	
4	3	
5	4	
6	5	
7	10	
8	20	
9	50	

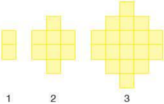
3. The number of squares needed for each pattern follows a rule. Describe the rule.
4. By writing a formula in cell B7 and copying it down to B9, use the spreadsheet to generate the results for the 10th, 20th and 50th patterns.

5. Repeat Q.1–4 for the following patterns:

a)



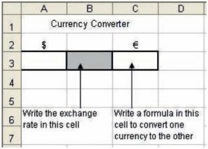
b)



ICT activity 2

In this activity, you will be using both the internet and a spreadsheet in order to produce currency conversions.

1. Log on to the internet and search for a website that shows the exchange rates between different currencies.
2. Compare your own currency with another currency of your choice. Write down the exchange rate, e.g. $1 = €1.29.
3. Using a spreadsheet construct a currency converter. An example is shown below:



4. By entering different amounts of your own currency, use the currency converter to calculate the correct conversion. Record your answers in a table.
5. Repeat Q.1–4 for five different currencies of your choice.

Topic 2

Algebra and graphs

Syllabus

C2.1

Use letters to express generalised numbers and express basic arithmetic processes algebraically.
Substitute numbers for words and letters in formulae.
Transform simple formulae.
Construct simple expressions and set up simple equations.

C2.2

Manipulate directed numbers.
Use brackets and extract common factors.

C2.3

Extended curriculum only.

C2.4

Use and interpret positive, negative and zero indices.
Use the rules of indices.

C2.5

Solve simple linear equations in one unknown.
Solve simultaneous linear equations in two unknowns.

C2.6

Extended curriculum only.

C2.7

Continue a given number sequence.
Recognise patterns in sequences and relationships between different sequences.
Find the *n*th term of sequences.

C2.8

Extended curriculum only.

C2.9

Interpret and use graphs in practical situations including travel graphs and conversion graphs, draw graphs from given data.

C2.10

Construct tables of values for functions of the form $ax + b$, $\pm x^2 + ax + b$, $\frac{a}{x}$ $(x \neq 0)$ where a and b are integral constants.
Draw and interpret such graphs.
Solve linear and quadratic equations approximately by graphical methods.

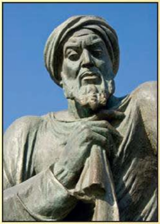
C2.11

Extended curriculum only.

C2.12

Extended curriculum only.

Contents


al-Khwarizmi

The development of algebra

The roots of algebra can be traced to the ancient Babylonians, who used formulae for solving problems. However, the word *algebra* comes from the Arabic language. Muhammad ibn Musa al-Khwarizmi (AD790–850) wrote *Kitab al-Jabr* (*The Compendious Book on Calculation by Completion and Balancing*), which established algebra as a mathematical subject. He is known as the father of algebra.

The Persian mathematician Omar Khayyam (1048–1131), who studied in Bukhara (now in Uzbekistan), discovered algebraic geometry and found the general solution of a cubic equation.

In 1545, the Italian mathematician Girolamo Cardano published *Ars Magna* (*The Great Art*) a 40-chapter book in which he gave for the first time a method for solving a quartic equation.

10 Algebraic representation and manipulation

Expanding brackets

When removing brackets, every term inside the bracket must be multiplied by whatever is outside the bracket.

Worked examples

a) $3(x + 4)$
$= 3x + 12$

b) $5x(2y + 3)$
$= 10xy + 15x$

c) $2a(3a + 2b - 3c)$
$= 6a^2 + 4ab - 6ac$

d) $-4p(2p - q + r^2)$
$= -8p^2 + 4pq - 4pr^2$

e) $-2x^2\left(x + 3y - \frac{1}{x}\right)$
$= -2x^3 - 6x^2y + 2x$

f) $\frac{-2}{x}\left(-x + 4y + \frac{1}{x}\right)$
$= 2 - \frac{8y}{x} - \frac{2}{x^2}$

Exercise 10.1

Expand the following:

1. a) $2(a + 3)$ b) $4(b + 7)$
c) $5(2c + 8)$ d) $7(3d + 9)$
e) $9(8e - 7)$ f) $6(4f - 3)$

2. a) $3a(a + 2b)$ b) $4b(2a + 3b)$
c) $2c(a + b + c)$ d) $3d(2b + 3c + 4d)$
e) $e(3c - 3d - e)$ f) $f(3d - e - 2f)$

3. a) $2(2a^2 + 3b^2)$ b) $4(3a^2 + 4b^2)$
c) $-3(2c + 3d)$ d) $-(2c + 3d)$
e) $-4(c^2 - 2d^2 + 3e^2)$ f) $-5(2e - 3f^2)$

4. a) $2a(a + b)$ b) $3b(a - b)$
c) $4c(b^2 - c^2)$ d) $3d^2(a^2 - 2b^2 + c^2)$
e) $-3e^2(4d - e)$ f) $-2f(2d - 3e^2 - 2f)$

Exercise 10.2

Expand the following:

1. a) $4(x - 3)$ b) $5(2p - 4)$
c) $-6(7x - 4y)$ d) $3(2a - 3b - 4c)$
e) $-7(2m - 3n)$ f) $-2(8x - 3y)$

2. a) $3x(x - 3y)$ b) $a(a + b + c)$
c) $4m(2m - n)$ d) $-5a(3a - 4b)$
e) $-4x(-x + y)$ f) $-8p(-3p + q)$

3. a) $-(2x^2 - 3y^2)$ b) $-(-a + b)$
c) $-(-7p + 2q)$ d) $\frac{1}{2}(6x - 8y + 4z)$
e) $\frac{3}{4}(4x - 2y)$ f) $\frac{1}{5}x(10x - 15y)$

4. a) $3r(4r^2 - 5s + 2t)$ b) $a^2(a + b + c)$
c) $3a^2(2a - 3b)$ d) $pq(p + q - pq)$
e) $m^2(m - n + nm)$ f) $a^3(a^3 + a^2b)$

Exercise 10.3 Expand and simplify the following:

1. a) $2a + 2(3a + 2)$ b) $4(3b - 2) - 5b$
   c) $6(2c - 1) - 7c$ d) $-4(d + 2) + 5d$
   e) $-3e + (e - 1)$ f) $5f - (2f - 7)$

2. a) $2(a + 1) + 3(b + 2)$ b) $4(a + 5) - 3(2b + 6)$
   c) $3(c - 1) + 2(c - 2)$ d) $4(d - 1) - 3(d - 2)$
   e) $-2(e - 3) - (e - 1)$ f) $2(3f - 3) + 4(1 - 2f)$

3. a) $2a(a + 3) + 2b(b - 1)$
   b) $3a(a - 4) - 2b(b - 3)$
   c) $2a(a + b + c) - 2b(a + b - c)$
   d) $a^2(c^2 + d^2) - c^2(a^2 + d^2)$
   e) $a(b + c) - b(a - c)$
   f) $a(2d + 3e) - 2e(a - c)$

Exercise 10.4 Expand and simplify the following:

1. a) $3a - 2(2a + 4)$ b) $8x - 4(x + 5)$
   c) $3(p - 4) - 4$ d) $7(3m - 2n) + 8n$
   e) $6x - 3(2x - 1)$ f) $5p - 3p(p + 2)$

2. a) $7m(m + 4) + m^2 + 2$ b) $3(x - 4) + 2(4 - x)$
   c) $6(p + 3) - 4(p - 1)$ d) $5(m - 8) - 4(m - 7)$
   e) $3a(a + 2) - 2(a^2 - 1)$ f) $7a(b - 2c) - c(2a - 3)$

3. a) $\frac{1}{2}(6x + 4) + \frac{1}{3}(3x + 6)$
   b) $\frac{1}{4}(2x + 6y) + \frac{3}{4}(6x - 4y)$
   c) $\frac{1}{8}(6x - 12y) + \frac{1}{2}(3x - 2y)$
   d) $\frac{1}{5}(15x + 10y) + \frac{3}{10}(5x - 5y)$
   e) $\frac{2}{3}(6x - 9y) + \frac{1}{3}(9x + 6y)$
   f) $\frac{x}{7}(14x - 21y) - \frac{x}{2}(4x - 6y)$

Factorising

When factorising, the largest possible factor is removed from each of the terms and placed outside the brackets.

Worked examples Factorise the following expressions:

a) $10x + 15$
$= 5(2x + 3)$

b) $8p - 6q + 10r$
$= 2(4p - 3q + 5r)$

c) $-2q - 6p + 12$
$= 2(-q - 3p + 6)$

d) $2a^2 + 3ab - 5ac$
$= a(2a + 3b - 5c)$

e) $6ax - 12ay - 18a^2$
$= 6a(x - 2y - 3a)$

f) $3b + 9ba - 6bd$
$= 3b(1 + 3a - 2d)$

Exercise 10.5

Factorise the following:

1. a) $4x - 6$ b) $18 - 12p$
   c) $6y - 3$ d) $4a + 6b$
   e) $3p - 3q$ f) $8m + 12n + 16r$

2. a) $3ab + 4ac - 5ad$ b) $8pq + 6pr - 4ps$
   c) $a^2 - ab$ d) $4x^2 - 6xy$
   e) $abc + abd + fab$ f) $3m^2 + 9m$

3. a) $3pqr - 9pqs$ b) $5m^2 - 10mn$
   c) $8x^2y - 4xy^2$ d) $2a^2b^2 - 3b^2c^2$
   e) $12p - 36$ f) $42x - 54$

4. a) $18 + 12y$ b) $14a - 21b$
   c) $11x + 11xy$ d) $4s - 16t + 20r$
   e) $5pq - 10qr + 15qs$ f) $4xy + 8y^2$

5. a) $m^2 + mn$ b) $3p^2 - 6pq$
   c) $pqr + qrs$ d) $ab + a^2b + ab^2$
   e) $3p^3 - 4p^4$ f) $7b^3c + b^2c^2$

6. a) $m^3 - m^2n + mn^2$ b) $4r^3 - 6r^2 + 8r^2s$
   c) $56x^2y - 28xy^2$ d) $72m^2n + 36mn^2 - 18m^2n^2$

7. a) $3a^2 - 2ab + 4ac$ b) $2ab - 3b^2 + 4bc$
   c) $2a^2c - 4b^2c + 6bc^2$ d) $39cd^2 + 52c^2d$

8. a) $12ac - 8ac^2 + 4a^2c$ b) $34a^2b - 51ab^2$
   c) $33ac^2 + 121c^3 - 11b^2c^2$ d) $38c^3d^2 - 57c^2d^3 + 95c^2d^2$

9. a) $15\frac{a}{c} - 25\frac{b}{c} + 10\frac{d}{c}$ b) $46\frac{a}{c^2} - 23\frac{b}{c^2}$
   c) $\frac{1}{2a} - \frac{1}{4a}$ d) $\frac{3}{5d} - \frac{1}{10d} + \frac{4}{15d}$

10. a) $\frac{5}{a^2} - \frac{3}{a}$ b) $\frac{6}{b^2} - \frac{3}{b}$
    c) $\frac{2}{3a} - \frac{3}{3a^2}$ d) $\frac{3}{5d^2} - \frac{4}{5d}$

Substitution

Worked examples Evaluate the expressions below if $a = 3$, $b = 4$, $c = -5$:

a) $2a + 3b - c$
$= 2 \times 3 + 3 \times 4 - (-5)$
$= 6 + 12 + 5$
$= 23$

b) $3a - 4b + 2c$
$= 3 \times 3 - 4 \times 4 + 2 \times (-5)$
$= 9 - 16 - 10$
$= -17$

c) $-2a + 2b - 3c$
$= -2 \times 3 + 2 \times 4 - 3 \times (-5)$
$= -6 + 8 + 15$
$= 17$

d) $a^2 + b^2 + c^2$
$= 3^2 + 4^2 + (-5)^2$
$= 9 + 16 + 25$
$= 50$

e) $3a(2b - 3c)$
$= 3 \times 3 \times (2 \times 4 - 3 \times (-5))$
$= 9 \times (8 + 15)$
$= 9 \times 23$
$= 207$

f) $-2c(-a + 2b)$
$= -2 \times (-5) \times (-3 + 2 \times 4)$
$= 10 \times (-3 + 8)$
$= 10 \times 5$
$= 50$

Exercise 10.6

Evaluate the following expressions if $a = 2$, $b = 3$ and $c = 5$:

1. a) $3a + 2b$ b) $4a - 3b$
c) $a - b - c$ d) $3a - 2b + c$

2. a) $-b(a + b)$ b) $-2c(a - b)$
c) $-3a(a - 3c)$ d) $-4b(b - c)$

3. a) $a^2 + b^2$ b) $b^2 + c^2$
c) $2a^2 - 3b^2$ d) $3c^2 - 2b^2$

4. a) $-a^2$ b) $(-a)^2$
c) $-b^3$ d) $(-b)^3$

5. a) $-c^3$ b) $(-c)^3$
c) $(-ac)^2$ d) $-(ac)^2$

Exercise 10.7

Evaluate the following expressions if $p = 4$, $q = -2$, $r = 3$ and $s = -5$:

1. a) $2p + 4q$ b) $5r - 3s$
c) $3q - 4s$ d) $6p - 8q + 4s$
e) $3r - 3p + 5q$ f) $-p - q + r + s$

2. a) $2p - 3q - 4r + s$ b) $3s - 4p + r + q$
c) $p^2 + q^2$ d) $r^2 - s^2$
e) $p(q - r + s)$ f) $r(2p - 3q)$

3. a) $2s(3p - 2q)$ b) $pq + rs$
c) $2pr - 3rq$ d) $q^3 - r^2$
e) $s^3 - p^3$ f) $r^4 - q^5$

4. a) $-2pqr$ b) $-2p(q + r)$
c) $-2rq + r$ d) $(p + q)(r - s)$
e) $(p + s)(r - q)$ f) $(r + q)(p - s)$

5. a) $(2p + 3q)(p - q)$ b) $(q + r)(q - r)$
c) $q^2 - r^2$ d) $p^2 - r^2$
e) $(p + r)(p - r)$ f) $(-s + p)q^2$

Transformation of formulae

In the formula $a = 2b + c$, 'a' is the subject. In order to make either b or c the subject, the formula has to be rearranged.

Worked examples Rearrange the following formulae to make the **bold** letter the subject:

a) $a = 2b + \mathbf{c}$
$a - 2b = \mathbf{c}$
$\mathbf{c} = a - 2b$

b) $2r + \mathbf{p} = q$
$\mathbf{p} = q - 2r$

c) $ab = \mathbf{c}d$
$\frac{ab}{d} = \mathbf{c}$
$\mathbf{c} = \frac{ab}{d}$

d) $\frac{a}{b} = \frac{c}{\mathbf{d}}$
$a\mathbf{d} = cb$
$\mathbf{d} = \frac{cb}{a}$

Exercise 10.8

In the following questions, make the letter in **bold** the subject of the formula:

1. a) $a + \mathbf{b} = c$ b) $\mathbf{b} + 2c = d$ c) $2b + \mathbf{c} = 4a$
d) $3d + \mathbf{b} = 2a$

2. a) $\mathbf{a}b = c$ b) $\mathbf{a}c = bd$ c) $\mathbf{a}b = c + 3$
d) $a\mathbf{c} = b - 4$

3. a) $m + \mathbf{n} = r$ b) $\mathbf{m} + n = p$ c) $2m + \mathbf{n} = 3p$
d) $3x = 2p + \mathbf{q}$ e) $\mathbf{a}b = cd$ f) $ab = c\mathbf{d}$

4. a) $3\mathbf{x}y = 4m$ b) $7pq = 5\mathbf{r}$ c) $3\mathbf{x} = c$
d) $3\mathbf{x} + 7 = y$ e) $5\mathbf{y} - 9 = 3r$ f) $5y - 9 = 3\mathbf{x}$

5. a) $6\mathbf{b} = 2a - 5$ b) $6b = 2\mathbf{a} - 5$ c) $3x - 7y = 4\mathbf{z}$
d) $3\mathbf{x} - 7y = 4z$ e) $3x - 7\mathbf{y} = 4z$ f) $2\mathbf{p}r - q = 8$

6. a) $\frac{\mathbf{p}}{4} = r$ b) $\frac{4}{\mathbf{p}} = 3r$ c) $\frac{1}{5}n = 2\mathbf{p}$
d) $\frac{1}{5}\mathbf{n} = 2p$ e) $\mathbf{p}(q + r) = 2t$ f) $p(\mathbf{q} + r) = 2t$

7. a) $3m - n = \mathbf{r}t(p + q)$ b) $3m - n = r\mathbf{t}(p + q)$
c) $3\mathbf{m} - n = rt(p + q)$ d) $3m - \mathbf{n} = rt(p + q)$
e) $3m - n = rt(\mathbf{p} + q)$ f) $3m - n = rt(p + \mathbf{q})$

8. a) $\frac{ab}{c} = \mathbf{d}e$ b) $\frac{\mathbf{a}b}{c} = de$ c) $\frac{ab}{\mathbf{c}} = de$
d) $\frac{\mathbf{a}+b}{c} = d$ e) $\frac{a}{c} + \mathbf{b} = d$ f) $\frac{a}{\mathbf{c}} + b = d$

Student assessment 1

1. Expand the following:

a) $3(a + 4)$
b) $7(4b - 2)$
c) $3c(c + 4d)$
d) $5d(3c - 2d)$
e) $-6(2e - 3f)$
f) $-(2f - 3g)$

2. Expand the following and simplify where possible:

a) $3a + 3(a + 4)$
b) $2(3b - 2) - 5b$
c) $-3c - (c - 1)$
d) $2(d - 1) + 3(d + 1)$
e) $-2e(e + 3) + 4(e^2 + 2)$
f) $2f(d + e + f) - 2e(d + e + f)$

3. Factorise the following:

a) $3a + 12$
b) $15b^2 + 25b$
c) $4cf - 6df + 8ef$
d) $4d^3 - 2d^4$

4. If $a = 3$, $b = 4$ and $c = 5$, evaluate the following:

a) $a + b + c$
b) $4a - 3b$
c) $a^2 + b^2 + c^2$
d) $(a + c)(a - c)$

5. Rearrange the following formulae to make the **bold** letter the subject:

a) $\boldsymbol{a} + b = c$
b) $c = \boldsymbol{b} - d$
c) $a\boldsymbol{c} = bd$
d) $3\boldsymbol{d} - e = 2c$
e) $\boldsymbol{e}(f - g) = 3a$
f) $e(\boldsymbol{f} - g) = 3a$

Student assessment 2

1. Expand the following:

a) $4(a + 2)$
b) $5(2b - 3)$
c) $2c(c + 2d)$
d) $3d(2c - 4d)$
e) $-5(3e - f)$
f) $-(-f + 2g)$

2. Expand the following and simplify where possible:

a) $2a + 5(a + 2)$
b) $3(2b - 3) - b$
c) $-4c - (4 - 2c)$
d) $3(d + 2) - 2(d + 4)$
e) $-e(2e + 3) + 3(2 + e^2)$
f) $f(d - e - f) - e(e + f)$

3. Factorise the following:

a) $7a + 14$
b) $26b^2 + 39b$
c) $3cf - 6df + 9gf$
d) $5d^2 - 10d^3$

4. If $a = 2$, $b = 3$ and $c = 5$, evaluate the following:

a) $a - b - c$
b) $2b - c$
c) $a^2 - b^2 + c^2$
d) $(a + c)^2$

5. Rearrange the following formulae to make the **bold** letter the subject:

a) $\boldsymbol{a} - b = c$
b) $2c = \boldsymbol{b} - 3d$
c) $ad = b\boldsymbol{c}$
d) $e = 5\boldsymbol{d} - 3c$
e) $4a = \boldsymbol{e}(f + g)$
f) $4a = e(\boldsymbol{f} + g)$

Student assessment 3

1. Expand the following and simplify where possible:

a) $5(2a - 6b + 3c)$ b) $3x(5x - 9)$
c) $-5y(3xy + y^2)$ d) $3x^2(5xy + 3y^2 - x^3)$
e) $5p - 3(2p - 4)$
f) $4m(2m - 3) + 2(3m^2 - m)$
g) $\frac{1}{3}(6x - 9) + \frac{1}{4}(8x + 24)$
h) $\frac{m}{4}(6m - 8) + \frac{m}{2}(10m - 2)$

2. Factorise the following:

a) $12a - 4b$ b) $x^2 - 4xy$
c) $8p^3 - 4p^2q$ d) $24xy - 16x^2y + 8xy^2$

3. If $x = 2$, $y = -3$ and $z = 4$, evaluate the following:

a) $2x + 3y - 4z$ b) $10x + 2y^2 - 3z$
c) $z^2 - y^3$ d) $(x + y)(y - z)$
e) $z^2 - x^2$ f) $(z + x)(z - x)$

4. Rearrange the following formulae to make the **bold** letter the subject:

a) $x = 3p + \boldsymbol{q}$ b) $3m - 5\boldsymbol{n} = 8r$ c) $2m = \frac{3\boldsymbol{y}}{t}$
d) $x(\boldsymbol{w} + y) = 2y$ e) $\frac{xy}{2\boldsymbol{p}} = \frac{rs}{t}$ f) $\frac{\boldsymbol{x} + y}{w} = m + n$

Student assessment 4

1. Expand the following and simplify where possible:

a) $3(2x - 3y + 5z)$ b) $4p(2m - 7)$
c) $-4m(2mn - n^2)$ d) $4p^2(5pq - 2q^2 - 2p)$
e) $4x - 2(3x + 1)$ f) $4x(3x - 2) + 2(5x^2 - 3x)$
g) $\frac{1}{5}(15x - 10) - \frac{1}{3}(9x - 12)$ h) $\frac{x}{2}(4x - 6) + \frac{x}{4}(2x + 8)$

2. Factorise the following:

a) $16p - 8q$ b) $p^2 - 6pq$
c) $5p^2q - 10pq^2$ d) $9pq - 6p^2q + 12q^2p$

3. If $a = 4$, $b = 3$ and $c = -2$, evaluate the following:

a) $3a - 2b + 3c$ b) $5a - 3b^2$
c) $a^2 + b^2 + c^2$ d) $(a + b)(a - b)$
e) $a^2 - b^2$ f) $b^3 - c^3$

4. Rearrange the following formulae to make the **bold** letter the subject:

a) $p = 4m + \boldsymbol{n}$ b) $4x - 3\boldsymbol{y} = 5z$
c) $2x = \frac{3\boldsymbol{y}}{5p}$ d) $m(x + \boldsymbol{y}) = 3w$
e) $\frac{pq}{4\boldsymbol{r}} = \frac{mn}{t}$ f) $\frac{p + \boldsymbol{q}}{r} = m - n$

11 Algebraic indices

In Chapter 7 you saw how numbers can be expressed using indices. For example, $5 \times 5 \times 5 = 125$, therefore $125 = 5^3$. The 3 is called the **index**. **Indices** is the plural of index.

Three laws of indices were introduced:

(1) $a^m \times a^n = a^{m+n}$

(2) $a^m \div a^n$ or $\frac{a^m}{a^n} = a^{m-n}$

(3) $(a^m)^n = a^{mn}$

Positive indices

Worked examples

a) Simplify $d^3 \times d^4$.

$$d^3 \times d^4 = d^{(3+4)}$$
$$= d^7$$

b) Simplify $\frac{(p^2)^4}{p^2 \times p^4}$.

$$\frac{(p^2)^4}{p^2 \times p^4} = \frac{p^{(2\times 4)}}{p^{(2+4)}}$$
$$= \frac{p^8}{p^6}$$
$$= p^{(8-6)}$$
$$= p^2$$

Exercise 11.1

1. Simplify the following:

a) $c^5 \times c^3$

b) $m^4 \div m^2$

c) $(b^3)^5 \div b^6$

d) $\frac{m^4n^9}{mn^3}$

e) $\frac{6a^6b^4}{3a^2b^3}$

f) $\frac{12x^5y^7}{4x^2y^5}$

g) $\frac{4u^3v^6}{8u^2v^3}$

h) $\frac{3x^6y^5z^3}{9x^4y^2z}$

2. Simplify the following:

a) $4a^2 \times 3a^3$

b) $2a^2b \times 4a^3b^2$

c) $(2p^2)^3$

d) $(4m^2n^3)^2$

e) $(5p^2)^2 \times (2p^3)^3$

f) $(4m^2n^2) \times (2mn^3)^3$

g) $\frac{(6x^2y^4)^2 \times (2xy)^3}{12x^6y^8}$

h) $(ab)^d \times (ab)^e$

The zero index

As shown in Chapter 7, the zero index indicates that a number or algebraic term is raised to the power of zero. A term raised to the power of zero is always equal to 1. This is shown below.

$$a^m \div a^n = a^{m-n} \qquad \text{therefore } \frac{a^m}{a^m} = a^{m-m}$$
$$= a^0$$
$$\text{However,} \qquad \frac{a^m}{a^m} = 1$$
$$\text{therefore } a^0 = 1$$

Exercise 11.2 Simplify the following:

a) $c^3 \times c^0$
b) $g^{-2} \times g^3 \div g^0$
c) $(p^0)^3(q^2)^{-1}$
d) $(m^3)^3(m^{-2})^5$

Negative indices

A negative index indicates that a number or an algebraic term is being raised to a negative power, e.g. a^{-4}.

As shown in Chapter 7, one law of indices states that $a^{-m} = \frac{1}{a^m}$. This is proved as follows.

$$a^{-m} = a^{0-m}$$
$$= \frac{a^0}{a^m} \text{ (from the second law of indices)}$$
$$= \frac{1}{a^m}$$
$$\text{therefore } a^{-m} = \frac{1}{a^m}$$

Exercise 11.3 Simplify the following:

a) $\frac{a^{-3} \times a^5}{(a^2)^0}$
b) $\frac{(r^3)^{-2}}{(p^{-2})^3}$
c) $(t^3 \div t^{-5})^2$
d) $\frac{m^0 \div m^{-6}}{(m^{-1})^3}$

Student assessment 1

1. Simplify the following using indices:
   a) $a \times a \times a \times b \times b$
   b) $d \times d \times e \times e \times e \times e \times e$

2. Write the following out in full:
   a) m^3
   b) r^4

3. Simplify the following using indices:
   a) $a^4 \times a^3$
   b) $p^3 \times p^2 \times q^4 \times q^5$
   c) $\frac{b^7}{b^4}$
   d) $\frac{(e^4)^5}{e^{14}}$

4. Simplify the following:
   a) $r^4 \times t^0$
   b) $\frac{(a^3)^0}{b^2}$
   c) $\frac{(m^0)^5}{n^{-3}}$

5. Simplify the following:
   a) $\frac{(p^2 \times p^{-5})^2}{p^3}$
   b) $\frac{(h^{-2} \times h^{-5})^{-1}}{h^0}$

12 Equations

An equation is formed when the value of an unknown quantity is needed.

Solving linear equations

Worked examples Solve the following linear equations:

a)
$$3x + 8 = 14$$
$$3x = 6$$
$$x = 2$$

b)
$$12 = 20 + 2x$$
$$-8 = 2x$$
$$-4 = x$$

c)
$$3(p + 4) = 21$$
$$3p + 12 = 21$$
$$3p = 9$$
$$p = 3$$

d)
$$4(x - 5) = 7(2x - 5)$$
$$4x - 20 = 14x - 35$$
$$4x + 15 = 14x$$
$$15 = 10x$$
$$1.5 = x$$

Exercise 12.1

Solve the following linear equations:

1. a) $5a - 2 = 18$ b) $7b + 3 = 17$
c) $9c - 12 = 60$ d) $6d + 8 = 56$
e) $4e - 7 = 33$ f) $12f + 4 = 76$

2. a) $4a = 3a + 7$ b) $8b = 7b - 9$
c) $7c + 5 = 8c$ d) $5d - 8 = 6d$

3. a) $3a - 4 = 2a + 7$ b) $5b + 3 = 4b - 9$
c) $8c - 9 = 7c + 4$ d) $3d - 7 = 2d - 4$

4. a) $6a - 3 = 4a + 7$ b) $5b - 9 = 2b + 6$
c) $7c - 8 = 3c + 4$ d) $11d - 10 = 6d - 15$

5. a) $\frac{a}{4} = 3$ b) $\frac{1}{4}b = 2$
c) $\frac{c}{5} = 2$ d) $\frac{1}{5}d = 3$
e) $4 = \frac{e}{3}$ f) $-2 = \frac{1}{8}f$

6. a) $\frac{a}{3} + 1 = 4$ b) $\frac{b}{5} + 2 = 6$
c) $8 = 2 + \frac{c}{3}$ d) $-4 = 3 + \frac{d}{5}$
e) $9 = 5 + \frac{2e}{3}$ f) $-7 = \frac{3f}{2} - 1$

7. a) $\frac{2a}{3} = 3$ b) $5 = \frac{3b}{2}$
c) $\frac{4c}{5} = 2$ d) $7 = \frac{5d}{8}$
e) $1 + \frac{3e}{8} = -5$ f) $2 = \frac{5f}{7} - 8$

8. a) $\frac{a+3}{2} = 4$ b) $\frac{b+5}{3} = 2$
c) $5 = \frac{c-2}{3}$ d) $2 = \frac{d-5}{3}$
e) $3 = \frac{2e-1}{5}$ f) $6 = \frac{4f-2}{5}$

9. a) $3(a+1) = 9$ b) $5(b-2) = 25$
c) $8 = 2(c-3)$ d) $14 = 4(3-d)$
e) $21 = 3(5-e)$ f) $36 = 9(5-2f)$

10. a) $\frac{a+2}{3} = \frac{a-3}{2}$ b) $\frac{b-1}{4} = \frac{b+5}{3}$
c) $\frac{2-c}{5} = \frac{7-c}{4}$ d) $\frac{8+d}{7} = \frac{7+d}{6}$
e) $\frac{3-e}{4} = \frac{5-e}{2}$ f) $\frac{10+f}{3} = \frac{5-f}{2}$

Exercise 12.2

Solve the following linear equations:

1. a) $3x = 2x - 4$ b) $5y = 3y + 10$
c) $2y - 5 = 3y$ d) $p - 8 = 3p$
e) $3y - 8 = 2y$ f) $7x + 11 = 5x$

2. a) $3x - 9 = 4$ b) $4 = 3x - 11$
c) $6x - 15 = 3x + 3$ d) $4y + 5 = 3y - 3$
e) $8y - 31 = 13 - 3y$ f) $4m + 2 = 5m - 8$

3. a) $7m - 1 = 5m + 1$ b) $5p - 3 = 3 + 3p$
c) $12 - 2k = 16 + 2k$ d) $6x + 9 = 3x - 54$
e) $8 - 3x = 18 - 8x$ f) $2 - y = y - 4$

4. a) $\frac{x}{2} = 3$ b) $\frac{1}{2}y = 7$
c) $\frac{x}{4} = 1$ d) $\frac{1}{4}m = 3$
e) $7 = \frac{x}{5}$ f) $4 = \frac{1}{5}p$

5. a) $\frac{x}{3} - 1 = 4$ b) $\frac{x}{5} + 2 = 1$
c) $\frac{2}{3}x = 5$ d) $\frac{3}{4}x = 6$
e) $\frac{1}{5}x = \frac{1}{2}$ f) $\frac{2x}{5} = 4$

6. a) $\frac{x+1}{2} = 3$ b) $4 = \frac{x-2}{3}$
c) $\frac{x-10}{3} = 4$ d) $8 = \frac{5x-1}{3}$
e) $\frac{2(x-5)}{3} = 2$ f) $\frac{3(x-2)}{4} = 4x - 8$

7. a) $6 = \frac{2(y-1)}{3}$ b) $2(x + 1) = 3(x - 5)$

c) $5(x - 4) = 3(x + 2)$ d) $\frac{3+y}{2} = \frac{y+1}{4}$

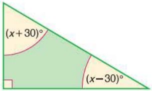
e) $\frac{7+2x}{3} = \frac{9x-1}{7}$ f) $\frac{2x+3}{4} = \frac{4x-2}{6}$

Constructing equations

NB: All diagrams are not drawn to scale.

In many cases, when dealing with the practical applications of mathematics, equations need to be constructed first before they can be solved. Often the information is either given within the context of a problem or in a diagram.

Worked examples



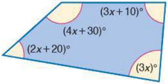
a) Find the size of each of the angles in the triangle (left) by constructing an equation and solving it to find the value of x.

The sum of the angles of a triangle is 180°.

$$\begin{aligned}(x + 30) + (x - 30) + 90 &= 180\\ 2x + 90 &= 180\\ 2x &= 90\\ x &= 45\end{aligned}$$

The three angles are therefore: 90°, $x + 30 = 75°$ and $x - 30 = 15°$.

Check: 90° + 75° + 15° = 180°.



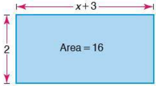
b) Find the size of each of the angles in the quadrilateral (left) by constructing an equation and solving it to find the value of x.

The sum of the angles of a quadrilateral is 360°.

$$\begin{aligned}4x + 30 + 3x + 10 + 3x + 2x + 20 &= 360\\ 12x + 60 &= 360\\ 12x &= 300\\ x &= 25\end{aligned}$$

The angles are:

$$\begin{aligned}4x + 30 &= (4 \times 25) + 30 = 130°\\ 3x + 10 &= (3 \times 25) + 10 = 85°\\ 3x &= 3 \times 25 = 75°\\ 2x + 20 &= (2 \times 25) + 20 = \underline{70°}\\ &\text{Total} = 360°\end{aligned}$$



c) Construct an equation and solve it to find the value of x in the diagram (left).

Area of rectangle = base × height

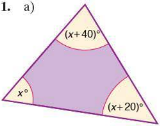
$$\begin{aligned}2(x + 3) &= 16\\ 2x + 6 &= 16\\ 2x &= 10\\ x &= 5\end{aligned}$$

Exercise 12.3

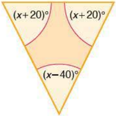
In Q.1–3:

i) construct an equation in terms of x,
ii) solve the equation,
iii) calculate the size of each of the angles,
iv) check your answers.

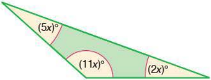
1. a)



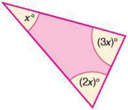
b)



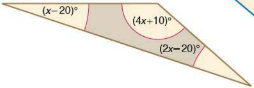
c)



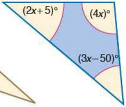
d)



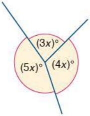
e)



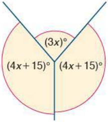
f)



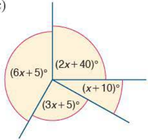
2. a)



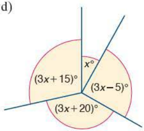
b)



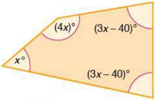
c)



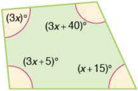
d)



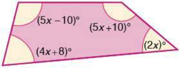
3. a)



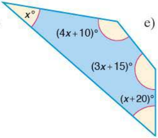
b)



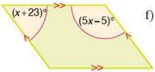
c)



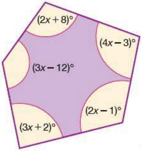
d)



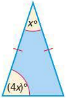
e)

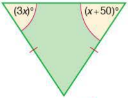


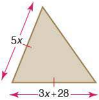
f)

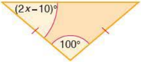


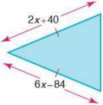
4. By constructing an equation and solving it, find the value of x in each of these isosceles triangles:

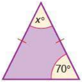
a) 

b) 

c) 

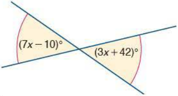
d) 

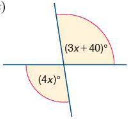
e) 

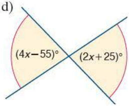
f) 

5. Using angle properties, calculate the value of x in each of these questions:

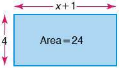
a) 

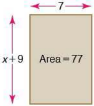
b) 

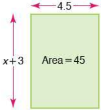
c) 

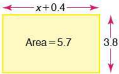
d) 

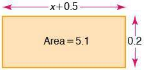
6. Calculate the value of x:

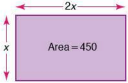
a) 

b) 

c) 

d) 

e) 

f) 

Sometimes the question is put into words and an equation has to be formed from those words.

Worked example If I multiply a number by 4 and then add 6 the total is 26. What is the number?

Let the number be n

Then $4n + 6 = 26$

$4n = 26 - 6$

$4n = 20$

$n = 5$

So the number is 5.

Exercise 12.4

Find the number in each case by forming an equation.

1. a) If I multiply a number by 7 and add 1 the total is 22.
   b) If I multiply a number by 9 and add 7 the total is 70.

2. a) If I multiply a number by 8 and add 12 the total is 92.
   b) If I add 2 to a number and then multiply by 3 the answer is 18.

3. a) If I add 12 to a number and then multiply by 5 the answer is 100.
   b) If I add 6 to a number and divide it by 4 the answer is 5.

4. a) If I add 3 to a number and multiply by 4 the answer is the same as multiplying by 6 and adding 8.
   b) If I add 1 to a number and then multiply by 3 the answer is the same as multiplying by 5 and substracting 3.

Simultaneous equations

When the values of two unknowns are needed, two equations need to be formed and solved. The process of solving two equations and finding a common solution is known as solving equations simultaneously.

The two most common ways of solving simultaneous equations algebraically are by **elimination** and by **substitution**.

By elimination

The aim of this method is to eliminate one of the unknowns by either adding or subtracting the two equations.

Worked examples Solve the following simultaneous equations by finding the values of x and y which satisfy both equations.

a) $3x + y = 9$ (1)
$5x - y = 7$ (2)

By adding equations (1) + (2) we eliminate the variable y:

$8x = 16$

$x = 2$

To find the value of y we substitute $x = 2$ into either equation (1) or (2).
Substituting $x = 2$ into equation (1):

$$\begin{aligned} 3x + y &= 9 \\ 6 + y &= 9 \\ y &= 3 \end{aligned}$$

To check that the solution is correct, the values of x and y are substituted into equation (2). If it is correct then the left-hand side of the equation will equal the right-hand side.

$$\begin{aligned} 5x - y &= 7 \\ \text{LHS} = 10 - 3 &= 7 \\ &= \text{RHS}\checkmark \end{aligned}$$

b) $4x + y = 23$ (1)
$x + y = 8$ (2)

By subtracting the equations, i.e. (1) − (2), we eliminate the variable y:

$$\begin{aligned} 3x &= 15 \\ x &= 5 \end{aligned}$$

By substituting $x = 5$ into equation (2), y can be calculated:

$$\begin{aligned} x + y &= 8 \\ 5 + y &= 8 \\ y &= 3 \end{aligned}$$

Check by substituting both values into equation (1):

$$\begin{aligned} 4x + y &= 23 \\ \text{LHS} = 20 + 3 &= 23 \\ &= \text{RHS}\checkmark \end{aligned}$$

By substitution

The same equations can also be solved by the method known as **substitution**.

Worked examples **a)** $3x + y = 9$ (1)
$5x - y = 7$ (2)

Equation (2) can be rearranged to give: $y = 5x - 7$
This can now be substituted into equation (1):

$$\begin{aligned} 3x + (5x - 7) &= 9 \\ 3x + 5x - 7 &= 9 \\ 8x - 7 &= 9 \\ 8x &= 16 \\ x &= 2 \end{aligned}$$

To find the value of y, $x = 2$ is substituted into either equation (1) or (2) as before, giving $y = 3$.

b) $4x + y = 23$ (1)
$x + y = 8$ (2)

Equation (2) can be rearranged to give $y = 8 - x$.
This can be substituted into equation (1):

$$\begin{aligned} 4x + (8 - x) &= 23 \\ 4x + 8 - x &= 23 \\ 3x + 8 &= 23 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

y can be found as before, giving the result $y = 3$.

Exercise 12.5

Solve the following simultaneous equations either by elimination or by substitution:

1. a) $x + y = 6$, $x - y = 2$ b) $x + y = 11$, $x - y - 1 = 0$ c) $x + y = 5$, $x - y = 7$
d) $2x + y = 12$, $2x - y = 8$ e) $3x + y = 17$, $3x - y = 13$ f) $5x + y = 29$, $5x - y = 11$

2. a) $3x + 2y = 13$, $4x = 2y + 8$ b) $6x + 5y = 62$, $4x - 5y = 8$ c) $x + 2y = 3$, $8x - 2y = 6$
d) $9x + 3y = 24$, $x - 3y = -14$ e) $7x - y = -3$, $4x + y = 14$ f) $3x = 5y + 14$, $6x + 5y = 58$

3. a) $2x + y = 14$, $x + y = 9$ b) $5x + 3y = 29$, $x + 3y = 13$ c) $4x + 2y = 50$, $x + 2y = 20$
d) $x + y = 10$, $3x = -y + 22$ e) $2x + 5y = 28$, $4x + 5y = 36$ f) $x + 6y = -2$, $3x + 6y = 18$

4. a) $x - y = 1$, $2x - y = 6$ b) $3x - 2y = 8$, $2x - 2y = 4$ c) $7x - 3y = 26$, $2x - 3y = 1$
d) $x = y + 7$, $3x - y = 17$ e) $8x - 2y = -2$, $3x - 2y = -7$ f) $4x - y = -9$, $7x - y = -18$

5. a) $x + y = -7$, $x - y = -3$ b) $2x + 3y = -18$, $2x = 3y + 6$
c) $5x - 3y = 9$, $2x + 3y = 19$ d) $7x + 4y = 42$, $9x - 4y = -10$
e) $4x - 4y = 0$, $8x + 4y = 12$ f) $x - 3y = -25$, $5x - 3y = -17$

6. a) $2x + 3y = 13$, $2x - 4y + 8 = 0$ b) $2x + 4y = 50$, $2x + y = 20$
c) $x + y = 10$, $3y = 22 - x$ d) $5x + 2y = 28$, $5x + 4y = 36$
e) $2x - 8y = 2$, $2x - 3y = 7$ f) $x - 4y = 9$, $x - 7y = 18$

7. a) $-4x = 4y$
$4x - 8y = 12$

b) $3x = 19 + 2y$
$-3x + 5y = 5$

c) $3x + 2y = 12$
$-3x + 9y = -12$

d) $3x + 5y = 29$
$3x + y = 13$

e) $-5x + 3y = 14$
$5x + 6y = 58$

f) $-2x + 8y = 6$
$2x = 3 - y$

Further simultaneous equations

If neither x nor y can be eliminated by simply adding or subtracting the two equations then it is necessary to multiply one or both of the equations. The equations are multiplied by a number in order to make the coefficients of x (or y) numerically equal.

Worked examples

a) $3x + 2y = 22$ (1)
$x + y = 9$ (2)

To eliminate y, equation (2) is multiplied by 2:

$3x + 2y = 22$ (1)
$2x + 2y = 18$ (3)

By subtracting (3) from (1), the variable y is eliminated:

$x = 4$

Substituting $x = 4$ into equation (2), we have:

$x + y = 9$
$4 + y = 9$
$y = 5$

Check by substituting both values into equation (1):

$3x + 2y = 22$
$\text{LHS} = 12 + 10 = 22$
$= \text{RHS}$ ✓

b) $5x - 3y = 1$ (1)
$3x + 4y = 18$ (2)

To eliminate the variable y, equation (1) is multiplied by 4, and equation (2) is multiplied by 3.

$20x - 12y = 4$ (3)
$9x + 12y = 54$ (4)

By adding equations (3) and (4) the variable y is eliminated:

$29x = 58$
$x = 2$

Substituting $x = 2$ into equation (2) gives:

$$\begin{aligned} 3x + 4y &= 18 \\ 6 + 4y &= 18 \\ 4y &= 12 \\ y &= 3 \end{aligned}$$

Check by substituting both values into equation (1):

$$\begin{aligned} 5x - 3y &= 1 \\ \text{LHS} = 10 - 9 &= 1 \\ &= \text{RHS}\checkmark \end{aligned}$$

Exercise 12.6

Solve the following simultaneous equations either by elimination or by substitution.

1. a) $2a + b = 5$, $3a - 2b = 4$
b) $3b + 2c = 18$, $2b - c = 5$
c) $4c - d = 18$, $2c + 2d = 14$
d) $d + 5e = 17$, $2d - 8e = -2$
e) $3e - f = 5$, $e + 2f = 11$
f) $f + 3g = 5$, $2f - g = 3$

2. a) $a + 2b = 8$, $3a - 5b = -9$
b) $4b - 3c = 17$, $b + 5c = 10$
c) $6c - 4d = -2$, $5c + d = 7$
d) $5d + e = 18$, $2d + 3e = 15$
e) $e + 2f = 14$, $3e - f = 7$
f) $7f - 5g = 9$, $f + g = 3$

3. a) $3a - 2b = -5$, $a + 5b = 4$
b) $b + 2c = 3$, $3b - 5c = -13$
c) $c - d = 4$, $3c + 4d = 5$
d) $2d + 3e = 2$, $3d - e = -8$
e) $e - 2f = -7$, $3e + 3f = -3$
f) $f + g = -2$, $3f - 4g = 1$

4. a) $2x + y = 7$, $3x + 2y = 12$
b) $5x + 4y = 21$, $x + 2y = 9$
c) $x + y = 7$, $3x + 4y = 23$
d) $2x - 3y = -3$, $3x + 2y = 15$
e) $4x = 4y + 8$, $x + 3y = 10$
f) $x + 5y = 11$, $2x - 2y = 10$

5. a) $x + y = 5$, $3x - 2y + 5 = 0$
b) $2x - 2y = 6$, $x - 5y = -5$
c) $2x + 3y = 15$, $2y = 15 - 3x$
d) $x - 6y = 0$, $3x - 3y = 15$
e) $2x - 5y = -11$, $3x + 4y = 18$
f) $x + y = 5$, $2x - 2y = -2$

6. a) $3y = 9 + 2x$
   $3x + 2y = 6$
   b) $x + 4y = 13$
   $3x - 3y = 9$
   c) $2x = 3y - 19$
   $3x + 2y = 17$
   d) $2x - 5y = -8$
   $-3x - 2y = -26$
   e) $5x - 2y = 0$
   $2x + 5y = 29$
   f) $8y = 3 - x$
   $3x - 2y = 9$

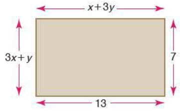
7. a) $4x + 2y = 5$
   $3x + 6y = 6$
   b) $4x + y = 14$
   $6x - 3y = 3$
   c) $10x - y = -2$
   $-15x + 3y = 9$
   d) $-2y = 0.5 - 2x$
   $6x + 3y = 6$
   e) $x + 3y = 6$
   $2x - 9y = 7$
   f) $5x - 3y = -0.5$
   $3x + 2y = 3.5$

Exercise 12.7

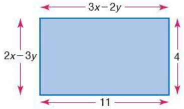
1. The sum of two numbers is 17 and their difference is 3. Find the two numbers by forming two equations and solving them simultaneously.

2. The difference between two numbers is 7. If their sum is 25, find the two numbers by forming two equations and solving them simultaneously.

3. Find the values of x and y.



4. Find the values of x and y.



5. A man's age is three times his son's age. Ten years ago he was five times his son's age. By forming two equations and solving them simultaneously, find both of their ages.

6. A grandfather is ten times as old as his granddaughter. He is also 54 years older than her. How old is each of them?

Student assessment 1

Solve the following equations:

1. a) $a + 9 = 15$ b) $3b + 7 = -14$
   c) $3 - 5c = 18$ d) $4 - 7d = -24$

2. a) $5a + 7 = 4a - 3$ b) $8 - 3b = 4 - 2b$
   c) $6 - 3c = c + 8$ d) $4d - 3 = d + 9$

3. a) $\frac{a}{5} = 2$ b) $\frac{b}{7} = 3$
   c) $4 = c - 2$ d) $6 = \frac{1}{3}d$

4. a) $\frac{a}{2} + 1 = 5$ b) $\frac{b}{3} - 2 = 2$
   c) $7 = \frac{c}{3} - 1$ d) $1 = \frac{1}{3}d - 2$

5. a) $\frac{a-2}{3} = \frac{a+2}{2}$ b) $\frac{b+5}{4} = \frac{2+b}{3}$
   c) $4(c - 5) = 3(c + 1)$ d) $6(2 + 3d) = 5(4d - 2)$

Solve the following simultaneous equations:

6. a) $a + 2b = 4$ b) $b - 2c = -2$
   $3a + b = 7$ $3b + c = 15$
   c) $2c - 3d = -5$ d) $4d + 5e = 0$
   $4c + d = -3$ $d + e = -1$

Student assessment 2

Solve the following equations:

1. a) $a + 5 = 7$ b) $4b - 3 = 13$
   c) $2 - 3c = 11$ d) $9 - 6d = 39$

2. a) $a - 4 = 3a + 4$ b) $3 - b = 4b - 7$
   c) $3c + 1 = 8 - 4c$ d) $6d - 9 = 3 - 4d$

3. a) $\frac{a}{7} = 3$ b) $\frac{b}{4} = -2$
   c) $4 = \frac{c}{-3}$ d) $-5 = \frac{d}{-3}$

4. a) $\frac{1}{2}a - 2 = 3$ b) $\frac{1}{3}b = \frac{2}{3}b - 5$
   c) $\frac{1}{5}c = 2 - \frac{4}{5}c$ d) $1 = \frac{3}{7}d + 2$

5. a) $\frac{a-4}{4} = \frac{2+a}{3}$ b) $\frac{1+b}{5} = \frac{2+b}{3}$
   c) $2(c - 2) = 2(5c - 3)$ d) $4(d + 1) - 7(3 - d) = 5$

Solve the following simultaneous equations:

6. a) $a + 4b = 9$
$4a + b = 21$
b) $b + 2c = -3$
$4b - 3c = 10$
c) $3c + 4d = 8$
$c + 6d = 5$
d) $4d - 3e = 1$
$d + e = -5$

Student assessment 3

Solve the following equations:

1. a) $x + 7 = 16$
b) $2x - 9 = 13$
c) $8 - 4x = 24$
d) $5 - 3x = -13$

2. a) $7 - m = 4 + m$
b) $5m - 3 = 3m + 11$
c) $6m - 1 = 9m - 13$
d) $18 - 3p = 6 + p$

3. a) $\frac{x}{-5} = 2$
b) $4 = \frac{1}{3}x$
c) $\frac{x+2}{3} = 4$
d) $\frac{2x-5}{7} = \frac{5}{2}$

4. a) $\frac{2}{3}(x - 4) = 8$
b) $4(x - 3) = 7(x + 2)$
c) $4 = \frac{2}{7}(3x + 8)$
d) $\frac{3}{4}(x - 1) = \frac{5}{8}(2x - 4)$

Solve the following simultaneous equations:

5. a) $2x + 3y = 16$
$2x - 3y = 4$
b) $4x + 2y = 22$
$-2x + 2y = 2$
c) $x + y = 9$
$2x + 4y = 26$
d) $2x - 3y = 7$
$-3x + 4y = -11$

Student assessment 4

Solve the following equations:

1. a) $y + 9 = 3$
b) $3x - 5 = 13$
c) $12 - 5p = -8$
d) $2.5y + 1.5 = 7.5$

2. a) $5 - p = 4 + p$
b) $8m - 9 = 5m + 3$
c) $11p - 4 = 9p + 15$
d) $27 - 5r = r - 3$

3. a) $\frac{p}{-2} = -3$
b) $6 = \frac{2}{5}x$
c) $\frac{m-7}{5} = 3$
d) $\frac{4t-3}{3} = 7$

4. a) $\frac{2}{5}(t - 1) = 3$
b) $5(3 - m) = 4(m - 6)$
c) $5 = \frac{2}{3}(x - 1)$
d) $\frac{4}{5}(t - 2) = \frac{1}{4}(2t + 8)$

Solve the following simultaneous equations:

5. a) $x + y = 11$
$x - y = 3$
b) $5p - 3q = -1$
$-2p - 3q = -8$
c) $3x + 5y = 26$
$x - y = 6$
d) $2m - 3n = -9$
$3m + 2n = 19$

13 Sequences

Sequences

A **sequence** is an ordered set of numbers. Each number in a sequence is known as a **term**. The terms of a sequence form a pattern. For the sequence of numbers

2, 5, 8, 11, 14, 17, ...

the difference between successive terms is +3. The term-to-term rule is therefore + 3.

Worked examples

a) Below is a sequence of numbers.

5, 9, 13, 17, 21, 25, ...

i) What is the term-to-term rule for the sequence?

The term-to-term rule is + 4.

ii) Calculate the 10th term of the sequence.
Continuing the pattern gives:

5, 9, 13, 17, 21, 25, 29, 33, 37, 41, ...

Therefore the 10th term is 41.

b) Below is a sequence of numbers.

1, 2, 4, 8, 16, ...

i) What is the term-to-term rule for the sequence?

The term-to-term rule is × 2.

ii) Calculate the 10th term of the sequence.
Continuing the pattern gives:

1, 2, 4, 8, 16, 32, 64, 128, 256, 512, ...

Therefore the 10th term is 512.

Exercise 13.1

For each of the sequences given below:

i) State a rule to describe the sequence.

ii) Calculate the 10th term.

a) 3, 6, 9, 12, 15, ...

b) 8, 18, 28, 38, 48, ...

c) 11, 33, 55, 77, 99, ...

d) 0.7, 0.5, 0.3, 0.1, ...

e) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

f) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

g) 1, 4, 9, 16, 25, ...

h) 4, 7, 12, 19, 28, ...

i) 1, 8, 27, 64, ...

j) 5, 25, 125, 625, ...

Sometimes the pattern in a sequence of numbers is not obvious. By looking at the differences between successive terms a pattern is often found.

Worked examples

a) Calculate the 8th term in the sequence

8, 12, 20, 32, 48, ...

The pattern in this sequence is not immediately obvious, so a row for the differences between successive terms can be constructed.

	8		12		20		32		48
1st differences		4		8		12		16	

The pattern in the differences row is + 4 and this can be continued to complete the sequence to the 8th term.

	8		12		20		32		48		68		92		120
1st differences		4		8		12		16		20		24		28	

So the 8th term is 120.

b) Calculate the 8th term in the sequence

	3		6		13		28		55
1st differences		3		7		15		27	

The row of first differences is not sufficient to spot the pattern, so a row of 2nd differences is constructed.

	3		6		13		28		55
1st differences		3		7		15		27	
2nd differences			4		8		12		

The pattern in the 2nd differences row can be seen to be + 4. This can now be used to complete the sequence.

	3		6		13		28		55		98		161		248
1st differences		3		7		15		27		43		63		87	
2nd differences			4		8		12		16		20		24		

So the 8th term is 248.

Exercise 13.2

For each of the sequences given below calculate the next two terms:

a) 8, 11, 17, 26, 38, ...
b) 5, 7, 11, 19, 35, ...
c) 9, 3, 3, 9, 21, ...
d) −2, 5, 21, 51, 100, ...
e) 11, 9, 10, 17, 36, 79, ...
f) 4, 7, 11, 19, 36, 69, ...
g) −3, 3, 8, 13, 17, 21, 24, ...

The nth term

So far the method used for generating a sequence relies on knowing the previous term in order to work out the next one. This method works but can be a little cumbersome if the 100th term is needed and only the first five terms are given! A more efficient rule is one which is related to a term's position in a sequence.

Worked examples

a) For the sequence shown below give an expression for the nth term.

Position	1	2	3	4	5	n
Term	3	6	9	12	15	?

By looking at the sequence it can be seen that the term is always 3 × position.

Therefore the nth term can be given by the expression $3n$.

b) For the sequence shown below give an expression for the nth term.

Position	1	2	3	4	5	n
Term	2	5	8	11	14	?

You will need to spot similarities between sequences. The terms of the above sequence are the same as the terms in example a) above but with 1 subtracted each time.

The expression for the nth term is therefore $3n - 1$.

Exercise 13.3

1. For each of the following sequences:

i) Write down the next two terms.

ii) Give an expression for the nth term.

a) 5, 8, 11, 14, 17, ...
b) 5, 9, 13, 17, 21, ...
c) 4, 9, 14, 19, 24, ...
d) 8, 10, 12, 14, 16, ...
e) 1, 8, 15, 22, 29, ...
f) 0, 4, 8, 12, 16, 20, ...
g) 1, 10, 19, 28, 37, ...
h) 15, 25, 35, 45, 55, ...
i) 9, 20, 31, 42, 53, ...
j) 1.5, 3.5, 5.5, 7.5, 9.5, 11.5, ...
k) 0.25, 1.25, 2.25, 3.25, 4.25, ...
l) 0, 1, 2, 3, 4, 5, ...

2. For each of the following sequences:
   i) Write down the next two terms.
   ii) Give an expression for the nth term.
      a) 2, 5, 10, 17, 26, 37, ...
      b) 8, 11, 16, 23, 32, ...
      c) 0, 3, 8, 15, 24, 35, ...
      d) 1, 8, 27, 64, 125, ...
      e) 2, 9, 28, 65, 126, ...
      f) 11, 18, 37, 74, 135, ...
      g) −2, 5, 24, 61, 122, ...
      h) 2, 6, 12, 20, 30, 42, ...

Student assessment 1

1. For each of the sequences given below:
   i) Calculate the next two terms.
   ii) Explain the pattern in words.
      a) 9, 18, 27, 36, ...
      b) 54, 48, 42, 36, ...
      c) 18, 9, 4.5, ...
      d) 12, 6, 0, −6, ...
      e) 216, 125, 64, ...
      f) 1, 3, 9, 27, ...

2. For each of the sequences given below:
   i) Calculate the next two terms.
   ii) Explain the pattern in words.
      a) 6, 12, 18, 24, ...
      b) 24, 21, 18, 15, ...
      c) 10, 5, 0, ...
      d) 16, 25, 36, 49, ...
      e) 1, 10, 100, ...
      f) 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ...

3. For each of the sequences shown below, give an expression for the nth term:
   a) 6, 10, 14, 18, 22, ...
   b) 13, 19, 25, 31, ...
   c) 3, 9, 15, 21, 27, ...
   d) 4, 7, 12, 19, 28, ...
   e) 0, 10, 20, 30, 40, ...
   f) 0, 7, 26, 63, 124, ...

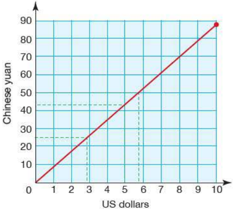
4. For each of the sequences shown below, give an expression for the nth term:
   a) 3, 5, 7, 9, 11, ...
   b) 7, 13, 19, 25, 31, ...
   c) 8, 18, 28, 38, ...
   d) 1, 9, 17, 25, ...
   e) −4, 4, 12, 20, ...
   f) 2, 5, 10, 17, 26, ...

14 Graphs in practical situations

Conversion graphs

A straight line graph can be used to convert one set of units to another. Examples include converting from one currency to another, converting distance in miles to kilometres and converting temperature from degrees Celsius to degrees Fahrenheit.

Worked example The graph below converts US dollars into Chinese yuan based on an exchange rate of $1 = 8.80 yuan.



i) Using the graph estimate the number of yuan equivalent to $5.

A line is drawn up from $5 until it reaches the plotted line, then across to the *y*-axis.
From the graph it can be seen that $5 ≈ 44 yuan.
(≈ is the symbol for 'is approximately equal to')

ii) Using the graph, what would be the cost in dollars of a drink costing 25 yuan?

A line is drawn across from 25 yuan until it reaches the plotted line, then down to the *x*-axis.
From the graph it can be seen that the cost of the drink ≈ $2.80.

iii) If a meal costs 200 yuan, use the graph to estimate its cost in US dollars.

The graph does not go up to 200 yuan, therefore a factor of 200 needs to be used, e.g. 50 yuan.
From the graph 50 yuan ≈ $5.70, therefore it can be deduced that 200 yuan ≈ $22.80 (i.e. 4 × $5.70).

Exercise 14.1

1. Given that 80 km = 50 miles, draw a conversion graph up to 100 km. Using your graph, estimate:
   a) how many miles is 50 km,
   b) how many kilometres is 80 miles,
   c) the speed in miles per hour (mph) equivalent to 100 km/h,
   d) the speed in km/h equivalent to 40 mph.

2. You can roughly convert temperature in degrees Celsius to degrees Fahrenheit by doubling the degrees Celsius and adding 30.
   Draw a conversion graph up to 50 °C. Use your graph to estimate the following:
   a) the temperature in °F equivalent to 25 °C,
   b) the temperature in °C equivalent to 100 °F,
   c) the temperature in °F equivalent to 0 °C,

3. Given that 0 °C = 32 °F and 50 °C = 122 °F, on the graph you drew for Q.2, draw a true conversion graph.
   i) Use the true graph to calculate the conversions in Q.2.
   ii) Where would you say the rough conversion is most useful?

4. Long-distance calls from New York to Harare are priced at 85 cents/min off peak and $1.20/min at peak times.
   a) Draw, on the same axes, conversion graphs for the two different rates.
   b) From your graph estimate the cost of an 8-minute call made off peak.
   c) Estimate the cost of the same call made at peak rate.
   d) A call is to be made from a telephone box. If the caller has only $4 to spend, estimate how much more time he can talk for if he rings at off peak instead of at peak times.

5. A maths exam is marked out of 120. Draw a conversion graph and use it to change the following marks to percentages.

   a) 80 b) 110 c) 54 d) 72

Speed, distance and time

You may already be aware of the following formula:

$$\text{distance} = \text{speed} \times \text{time}$$

Rearranging the formula gives:

$$\text{time} = \frac{\text{distance}}{\text{speed}} \text{ and speed} = \frac{\text{distance}}{\text{time}}$$

Where the speed is not constant:

$$\text{average speed} = \frac{\text{total distance}}{\text{total time}}$$

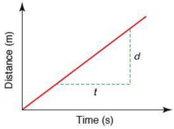
Exercise 14.2

1. Find the average speed of an object moving:
   a) 30 m in 5 s b) 48 m in 12 s
   c) 78 km in 2 h d) 50 km in 2.5 h
   e) 400 km in 2 h 30 min f) 110 km in 2 h 12 min

2. How far will an object travel during:
   a) 10 s at 40 m/s b) 7 s at 26 m/s
   c) 3 hours at 70 km/h d) 4 h 15 min at 60 km/h
   e) 10 min at 60 km/h f) 1 h 6 min at 20 m/s?

3. How long will it take to travel:
   a) 50 m at 10 m/s b) 1 km at 20 m/s
   c) 2 km at 30 km/h d) 5 km at 70 m/s
   e) 200 cm at 0.4 m/s f) 1 km at 15 km/h?

Travel graphs

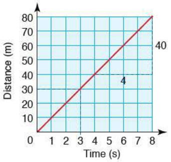


The graph of an object travelling at a constant speed is a straight line as shown (left).

$$\text{Gradient} = \frac{d}{t}$$

The units of the gradient are m/s, hence the gradient of a distance–time graph represents the speed at which the object is travelling.

Worked example



The graph (left) represents an object travelling at constant speed.

i) From the graph calculate how long it took to cover a distance of 30 m.

The time taken to travel 30 m is 3 seconds.

ii) Calculate the gradient of the graph.

Taking two points on the line,

$$\text{gradient} = \frac{40\,\text{m}}{4\,\text{s}} = 10\,\text{m/s}.$$

iii) Calculate the speed at which the object was travelling.

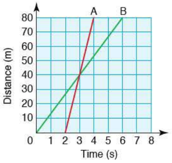
Gradient of a distance–time graph = speed.
Therefore the speed is 10 m/s.

Exercise 14.3

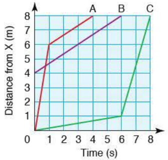
1. Draw a distance–time graph for the first 10 seconds of an object travelling at 6 m/s.

2. Draw a distance–time graph for the first 10 seconds of an object travelling at 5 m/s. Use your graph to estimate:

    a) the time taken to travel 25 m,
    b) how far the object travels in 3.5 seconds.



3. Two objects A and B set off from the same point and move in the same straight line. B sets off first, whilst A sets off 2 seconds later. Using the distance–time graph (left) estimate:
    a) the speed of each of the objects,
    b) how far apart the objects would be 20 seconds after the start.

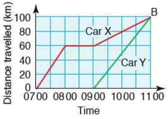


4. Three objects A, B and C move in the same straight line away from a point X. Both A and C change their speed during the journey, whilst B travels at a constant speed throughout. From the distance–time graph (left) estimate:

    a) the speed of object B,
    b) the two speeds of object A,
    c) the average speed of object C,
    d) how far object C is from X, 3 seconds from the start,
    e) how far apart objects A and C are 4 seconds from the start.

The graphs of two or more journeys can be shown on the same axes. The shape of the graph gives a clear picture of the movement of each of the objects.

Worked example



The journeys of two cars, X and Y, travelling between A and B, are represented on the distance–time graph (left). Car X and Car Y both reach point B 100 km from A at 1100.

i) Calculate the speed of Car X between 0700 and 0800.

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

$$= \frac{60}{1} \text{ km/h}$$

$$= 60 \text{ km/h}$$

ii) Calculate the speed of Car Y between 0900 and 1100.

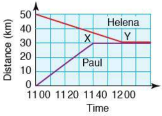
$$\text{speed} = \frac{100}{2} \text{ km/h}$$

$$= 50 \text{ km/h}$$

iii) Explain what is happening to Car X between 0800 and 0900.

No distance has been travelled, therefore Car X is stationary.

Exercise 14.4



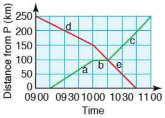
1. Two friends Paul and Helena arrange to meet for lunch at noon. They live 50 km apart and the restaurant is 30 km from Paul's home. The travel graph (left) illustrates their journeys.
   a) What is Paul's average speed between 11 00 and 11 40?
   b) What is Helena's average speed between 11 00 and 12 00 ?
   c) What does the line XY represent?

2. A car travels at a speed of 60 km/h for 1 hour. It stops for 30 minutes, then continues at a constant speed of 80 km/h for a further 1.5 hours. Draw a distance–time graph for this journey.

3. A girl cycles for 1.5 hours at 10 km/h. She stops for an hour, then travels for a further 15 km in 1 hour. Draw a distance–time graph of the girl's journey.

4. Two friends leave their houses at 16 00. The houses are 4 km apart and the friends travel towards each other on the same road. Fyodor walks at 7 km/h and Yin walks at 5 km/h.
   a) On the same axes, draw a distance–time graph of their journeys.
   b) From your graph estimate the time at which they meet.
   c) Estimate the distance from Fyodor's house to the point where they meet.

5. A train leaves a station P at 18 00 and travels to station Q 150 km away. It travels at a steady speed of 75 km/h. At 18 10 another train leaves Q for P at a steady speed of 100 km/h.
   a) On the same axes draw a distance–time graph to show both journeys.
   b) From the graph estimate the time at which the trains pass each other.
   c) At what distance from station Q do the trains pass each other?
   d) Which train arrives at its destination first?



6. A train sets off from town P at 09 15 and heads towards town Q 250 km away. Its journey is split into the three stages, a, b and c. At 09 00 a second train leaves town Q heading for town P. Its journey is split into the two stages, d and e. Using the graph (left), calculate the following:
   a) the speed of the first train during stages a, b and c,
   b) the speed of the second train during stages d and e.

Student assessment 1

1. In 2012, 1 euro had an exchange rate of 11 South African rand and 1.6 Singapore dollars.
   a) Draw a conversion graph for rand to Singapore dollars up to 80 rand.
   b) Estimate from your graph how many dollars you would get for 50 rand.
   c) Estimate from your graph how many rand you would get for 12 dollars.

2. A South African taxi driver has a fixed charge of 20 rand and then charges 6 rand per km.
   a) Draw a conversion graph to enable you to estimate the cost of the following taxi rides:
      i) 5 km,
      ii) 8.5 km.
   b) If a trip costs 80 rand, estimate from your graph the distance travelled.

3. An electricity account can work in two ways:
   - account A which involves a fixed charge of \$5 and then a rate of 7c per unit,
   - account B which involves no fixed charge but a rate of 9.5c per unit.

   a) On the same axes draw a graph up to 400 units for each type of account, converting units used to cost.
   b) Use your graph to advise a customer on which account to use.

4. A car travels at 60 km/h for 1 hour. The driver then takes a 30-minute break. After her break, she continues at 80 km/h for 90 minutes.
   a) Draw a distance–time graph for her journey.
   b) Calculate the total distance travelled.

5. Two trains depart at the same time from cities M and N, which are 200 km apart. One train travels from M to N, the other from N to M. The train departing from M travels a distance of 60 km in the first hour, 120 km in the next 1.5 hours and then the rest of the journey at 40 km/h. The train departing from N travels the whole distance at a speed of 100 km/h. Assuming all speeds are constant:
   a) draw a travel graph to show both journeys,
   b) estimate how far from city M the trains are when they pass each other,
   c) estimate how long after the start of the journey the trains pass each other.

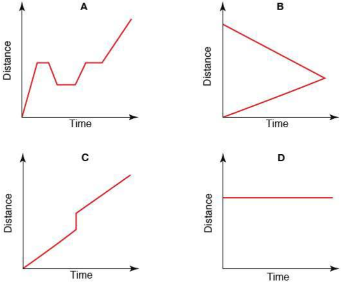
Student assessment 2

1. Absolute zero (0 K) is equivalent to −273 °C and 0 °C is equivalent to 273 K. Draw a conversion graph which will convert K into °C. Use your graph to estimate:
   a) the temperature in K equivalent to −40 °C,
   b) the temperature in °C equivalent to 100 K.

2. A German plumber has a call-out charge of €70 and then charges a rate of €50 per hour.
   a) Draw a conversion graph and estimate the cost of the following:
      i) a job lasting $4\frac{1}{2}$ hours,
      ii) a job lasting $6\frac{3}{4}$ hours.
   b) If a job cost €245, estimate from your graph how long it took to complete.

3. A boy lives 3.5 km from his school. He walks home at a constant speed of 9 km/h for the first 10 minutes. He then stops and talks to his friends for 5 minutes. He finally runs the rest of his journey home at a constant speed of 12 km/h.
   a) Illustrate this information on a distance–time graph.
   b) Use your graph to estimate the total time it took the boy to get home that day.

4. Below are four distance–time graphs A, B, C and D. Two of them are not possible.
   a) Which two graphs are impossible?
   b) Explain why the two you have chosen are not possible.



15 Graphs of functions

Linear functions

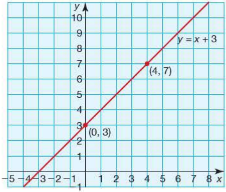
A linear function when plotted produces a straight line. A straight line consists of an infinite number of points. However, to plot a linear function, only two points on the line are needed. Once these have been plotted, the line can be drawn through them and extended if necessary at both ends.

Worked examples **a)** Plot the line $y = x + 3$.

To identify two points simply choose two values of x, substitute these into the equation and calculate the corresponding y-values. Sometimes a small table of results is clearer.

x	y
0	3
4	7

Therefore two of the points on the line are (0, 3) and (4, 7). Plot the points on a pair of axes and draw a line through them.



It is good practice to check with a third point. Substituting $x = 2$ into the equation gives $y = 5$. As the point (2, 5) lies on the line, the line is drawn correctly.

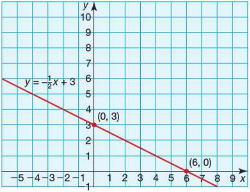
b) Plot the line $2y + x = 6$.
It is often easier to plot a line if the function is first written with y as the subject.

$$2y + x = 6$$
$$2y = -x + 6$$
$$y = -\tfrac{1}{2}x + 3$$

Choose two values of x and find the corresponding values of y.

x	y
0	3
6	0

Therefore two of the points on the line are (0, 3) and (6, 0).
Plot the points on a pair of axes and draw a line through them.



Check with a third point. Substituting $x = 4$ into the equation gives $y = 1$. As the point (4, 1) lies on the line, the line is drawn correctly.

Exercise 15.1

1. Plot the following straight lines:
   a) $y = 2x + 4$ b) $y = 2x + 3$ c) $y = 2x - 1$
   d) $y = x - 4$ e) $y = x + 1$ f) $y = x + 3$
   g) $y = 1 - x$ h) $y = 3 - x$ i) $y = -(x + 2)$

2. Plot the following straight lines:
   a) $y = 2x + 3$ b) $y = x - 4$ c) $y = 3x - 2$
   d) $y = -2x$ e) $y = -x - 1$ f) $-y = x + 1$
   g) $-y = 3x - 3$ h) $2y = 4x - 2$ i) $y - 4 = 3x$

3. Plot the following straight lines:
   a) $-2x + y = 4$ b) $-4x + 2y = 12$
   c) $3y = 6x - 3$ d) $2x = x + 1$
   e) $3y - 6x = 9$ f) $2y + x = 8$
   g) $x + y + 2 = 0$ h) $3x + 2y - 4 = 0$
   i) $4 = 4y - 2x$

Graphical solution of simultaneous equations

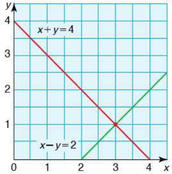
When solving two equations simultaneously the aim is to find a solution which works for both equations. In Chapter 12 it was shown how to arrive at the solution algebraically. It is, however, possible to arrive at the same solution graphically.

Worked example

i) By plotting both of the following equations on the same axes, find a common solution.

$$x + y = 4$$
$$x - y = 2$$

When both lines are plotted, the point at which they cross gives the common solution as it is the only point which lies on both lines.



Therefore the common solution is the point (3, 1).

ii) Check the result obtained above by solving the equations algebraically.

$$x + y = 4 \qquad (1)$$
$$x - y = 2 \qquad (2)$$

$$\text{eq.}(1) + \text{eq.}(2) \rightarrow 2x = 6$$
$$x = 3$$

Substituting $x = 3$ into equation (1) we have:

$$3 + y = 4$$
$$y = 1$$

Therefore the common solution occurs at (3, 1).

Exercise 15.2

Solve the simultaneous equations below:
i) by graphical means,
ii) by algebraic means.

1. a) $x + y = 5$
$x - y = 1$
b) $x + y = 7$
$x - y = 3$
c) $2x + y = 5$
$x - y = 1$
d) $2x + 2y = 6$
$2x - y = 3$
e) $x + 3y = -1$
$x - 2y = -6$
f) $x - y = 6$
$x + y = 2$

2. a) $3x - 2y = 13$
$2x + y = 4$
b) $4x - 5y = 1$
$2x + y = -3$
c) $x + 5 = y$
$2x + 3y - 5 = 0$
d) $x = y$
$x + y + 6 = 0$
e) $2x + y = 4$
$4x + 2y = 8$
f) $y - 3x = 1$
$y = 3x - 3$

Quadratic functions

The general expression for a quadratic function takes the form $ax^2 + bx + c$ where a, b and c are constants. Some examples of quadratic functions are given below.

$$y = 2x^2 + 3x - 12 \qquad y = x^2 - 5x + 6 \qquad y = 3x^2 + 2x - 3$$

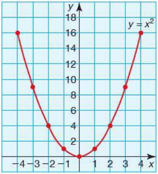
If a graph of a quadratic function is plotted, the smooth curve produced is called a **parabola**.

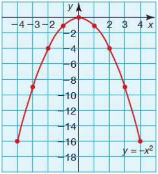
e.g. $y = x^2$

x	−4	−3	−2	−1	0	1	2	3	4
y	16	9	4	1	0	1	4	9	16

$y = -x^2$

x	−4	−3	−2	−1	0	1	2	3	4
y	−16	−9	−4	−1	0	−1	−4	−9	−16



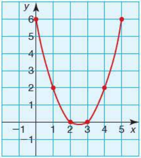


Worked examples **a)** Plot a graph of the function $y = x^2 - 5x + 6$ for $0 \leqslant x \leqslant 5$.

A table of values for x and y is given below:

x	0	1	2	3	4	5
y	6	2	0	0	2	6

These can then be plotted to give the graph:

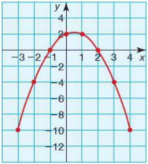


b) Plot a graph of the function $y = -x^2 + x + 2$ for $-3 \leqslant x \leqslant 4$.

Drawing up a table of values gives:

x	−3	−2	−1	0	1	2	3	4
y	−10	−4	0	2	2	0	−4	−10

The graph of the function is given below:



Exercise 15.3

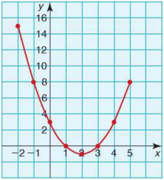
For each of the following quadratic functions, construct a table of values and then draw the graph.

1. $y = x^2 + x - 2, \quad -4 \leqslant x \leqslant 3$
2. $y = -x^2 + 2x + 3, \quad -3 \leqslant x \leqslant 5$
3. $y = x^2 - 4x + 4, \quad -1 \leqslant x \leqslant 5$
4. $y = -x^2 - 2x - 1, \quad -4 \leqslant x \leqslant 2$
5. $y = x^2 - 2x - 15, \quad -4 \leqslant x \leqslant 6$

Graphical solution of a quadratic equation

Worked example **i)** Draw a graph of $y = x^2 - 4x + 3$ for $-2 \leqslant x \leqslant 5$.

x	−2	−1	0	1	2	3	4	5
y	15	8	3	0	−1	0	3	8



ii) Use the graph to solve the equation $x^2 - 4x + 3 = 0$.

To solve the equation it is necessary to find the values of x when $y = 0$, i.e. where the graph crosses the x-axis. These points occur when $x = 1$ and $x = 3$, therefore these are the solutions.

Exercise 15.4

Solve each of the quadratic equations below by plotting a graph for the ranges of x stated.

1. $x^2 - x - 6 = 0$, $-4 \leqslant x \leqslant 4$

2. $-x^2 + 1 = 0$, $-4 \leqslant x \leqslant 4$

3. $x^2 - 6x + 9 = 0$, $0 \leqslant x \leqslant 6$

4. $-x^2 - x + 12 = 0$, $-5 \leqslant x \leqslant 4$

5. $x^2 - 4x + 4 = 0$, $-2 \leqslant x \leqslant 6$

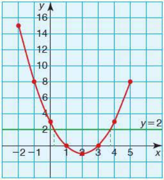
In the previous worked example, as $y = x^2 - 4x + 3$, a solution could be found to the equation $x^2 - 4x + 3 = 0$ by reading off where the graph crossed the x-axis. The graph can, however, also be used to solve other quadratic equations.

Worked example Use the graph of $y = x^2 - 4x + 3$ to solve the equation $x^2 - 4x + 1 = 0$.

$x^2 - 4x + 1 = 0$ can be rearranged to give:

$x^2 - 4x + 3 = 2$

Using the graph of $y = x^2 - 4x + 3$ and plotting the line $y = 2$ on the same axes gives the graph shown below.



Where the curve and the line cross gives the solution to $x^2 - 4x + 3 = 2$ and hence also $x^2 - 4x + 1 = 0$.

Therefore the solutions to $x^2 - 4x + 1 = 0$ are $x \approx 0.3$ and $x \approx 3.7$.

Exercise 15.5

Using the graphs which you drew in Exercise 15.4, solve the following quadratic equations. Show your method clearly.

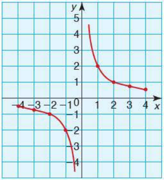
1. $x^2 - x - 4 = 0$
2. $-x^2 - 1 = 0$
3. $x^2 - 6x + 8 = 0$
4. $-x^2 - x + 9 = 0$
5. $x^2 - 4x + 1 = 0$

The reciprocal function

Worked example Draw the graph of $y = \frac{2}{x}$ for $-4 \leqslant x \leqslant 4$.

x	−4	−3	−2	−1	0	1	2	3	4
y	−0.5	−0.7	−1	−2	–	2	1	0.7	0.5

This is a reciprocal function. If the graph is plotted, the curves produced are called a **hyperbola**.



Exercise 15.6

1. Plot the graph of the function $y = \frac{1}{x}$ for $-4 \leqslant x \leqslant 4$.

2. Plot the graph of the function $y = \frac{3}{x}$ for $-4 \leqslant x \leqslant 4$.

3. Plot the graph of the function $y = \frac{5}{2x}$ for $-4 \leqslant x \leqslant 4$.

Student assessment 1

1. Plot the following lines on the same pair of axes, labelling each one clearly:

a) $x = -2$ b) $y = 3$

c) $y = 2x$ d) $y = -\frac{x}{2}$

2. Plot the graphs of the following linear equations:

a) $y = x + 1$ b) $y = 3 - 3x$

c) $2x - y = 4$ d) $2y - 5x = 8$

3. Solve the following pairs of simultaneous equations graphically:

a) $x + y = 4$
$x - y = 0$

b) $3x + y = 2$
$x - y = 2$

c) $y + 4x + 4 = 0$
$x + y = 2$

d) $x - y = -2$
$3x + 2y + 6 = 0$

4. a) Copy and complete the table below for the function $y = x^2 + 8x + 15$.

x	−7	−6	−5	−4	−3	−2	−1	0	1	2
y		3				3				

b) Plot a graph of the function.

5. Plot the graphs of the following functions between the given limits of x:

a) $y = x^2 - 3,\quad -4 \leqslant x \leqslant 4$

b) $y = 3 - x^2,\quad -4 \leqslant x \leqslant 4$

6. Plot a graph of each of the functions below between the given limits of x:

a) $y = -x^2 - 2x - 1,\quad -3 \leqslant x \leqslant 3$

b) $y = x^2 + 2x - 7,\quad -5 \leqslant x \leqslant 3$

7. a) Plot the graph of the quadratic function $y = x^2 + 9x + 20$ for $-7 \leqslant x \leqslant -2$.

b) Showing your method clearly, use your graph to solve the equation $x^2 = -9x - 14$.

8. Plot the graph of $y = \frac{1}{x}$ for $-4 \leqslant x \leqslant 4$.

Student assessment 2

1. Plot the following lines on the same axes, labelling each one clearly:

a) $x = 3$ b) $y = -2$

c) $y = -3x$ d) $y = \frac{x}{4} + 4$

2. Plot the graphs of the following linear equations:

a) $y = 2x + 3$ b) $y = 4 - x$

c) $2x - y = 3$ d) $-3x + 2y = 5$

3. Solve the following pairs of simultaneous equations graphically:

a) $x + y = 6$
$x - y = 0$

b) $x + 2y = 8$
$x - y = -1$

c) $2x - y = -5$
$x - 3y = 0$

d) $4x - 2y = -2$
$3x - y + 2 = 0$

4. a) Copy and complete the table below for the function $y = -x^2 - 7x - 12$.

x	−7	−6	−5	−4	−3	−2	−1	0	1	2
y		−6				−2				

b) Plot a graph of the function.

5. Plot the following functions between the given limits of x:

a) $y = x^2 - 5$, $-4 \leqslant x \leqslant 4$

b) $y = 1 - x^2$, $-4 \leqslant x \leqslant 4$

6. Plot a graph of each of the functions below between the given limits of x:

a) $y = x^2 - 3x - 10$, $-3 \leqslant x \leqslant 6$

b) $y = -x^2 - 4x - 4$, $-5 \leqslant x \leqslant 1$

7. a) Plot the graph of the quadratic equation $y = -x^2 - x + 15$ for $-6 \leqslant x \leqslant 4$.

b) Showing your method clearly, use your graph to solve the following equations:

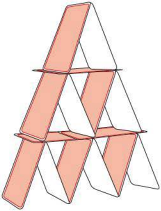
i) $10 = x^2 + x$ ii) $x^2 = -x + 5$

8. Plot the graph of $y = \frac{2}{x}$ for $-4 \leqslant x \leqslant 4$.

Topic 2 Mathematical investigations and ICT

House of cards

The drawing shows a house of cards 3 layers high. 15 cards are needed to construct it.

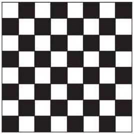


1. How many cards are needed to construct a house 10 layers high?
2. The world record is for a house 75 layers high. How many cards are needed to construct this house of cards?
3. Show that the general formula for a house n layers high requiring c cards is:

$$c = \frac{1}{2}n(3n + 1)$$

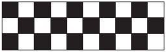
Chequered boards

A chessboard is an 8×8 square grid consisting of alternating black and white squares as shown:



There are 64 unit squares of which 32 are black and 32 are white.

Consider boards of different sizes. The examples below show rectangular boards, each consisting of alternating black and white unit squares.



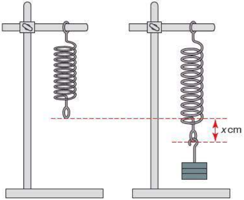
Total number of unit squares is 30
Number of black squares is 15
Number of white squares is 15

Total number of unit squares is 21
Number of black squares is 10
Number of white squares is 11

1. Investigate the number of black and white unit squares on different rectangular boards. Note: For consistency you may find it helpful to always keep the bottom right-hand square the same colour.
2. What are the numbers of black and white squares on a board $m \times n$ units?

Modelling: Stretching a spring

A spring is attached to a clamp stand as shown below.



Different weights are attached to the end of the spring. The mass (m) in grams is noted as is the amount by which the spring stretches (x) in centimetres, as shown on the right.

The data collected is shown in the table below:

Mass (g)	50	100	150	200	250	300	350	400	450	500
Extension (cm)	3.1	6.3	9.5	12.8	15.4	18.9	21.7	25.0	28.2	31.2

1. Plot a graph of mass against extension.
2. Describe the approximate relationship between the mass and the extension.
3. Draw a line of best fit through the data.
4. Calculate the equation of the line of best fit.
5. Use your equation to predict what the length of the spring would be for a mass of 275 g.
6. Explain why it is unlikely that the equation would be useful to find the extension if a mass of 5 kg was added to the spring.

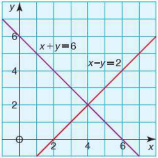
ICT activity 1

You have seen that the solution of two simultaneous equations gives the coordinates of the point which satisfies both equations.

If the simultaneous equations were plotted, then the point at which the lines cross would correspond to the solution of the simultaneous equations.

e.g. Solving $x + y = 6$ and $x - y = 2$ produces the result $x = 4$ and $y = 2$, i.e. coordinates (4, 2).

A plot of both lines is shown below.
From the graph it can be seen that the point of intersection of the two lines occurs at (4, 2).



1. Using a graphing package, solve the following simultaneous equations graphically:
   a) $y = x$ and $x + y = 4$
   b) $y = 2x$ and $x + y = 3$
   c) $y = 2x$ and $y = 3$
   d) $y - x = 2$ and $y + \frac{1}{2}x = 5$
   e) $y + x = 5$ and $y + \frac{1}{2}x = 3$

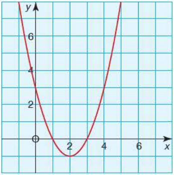
2. Check your answers to Q.1 by solving the simultaneous equations algebraically.

ICT activity 2

In this activity you will be using a graphing package or graphical calculator to find the solution to quadratic and reciprocal functions.

You will know that if a quadratic equation is plotted, its solution is represented by where the graph intersects the x-axis.

e.g. When plotting $y = x^2 - 4x + 3$, the solution of $x^2 - 4x + 3 = 0$ occurs where the graph crosses the x-axis as shown in the graph below.
i.e. $x = 1$ and $x = 3$.



Using a graphing package or a graphical calculator solve the following equations graphically:

a) $x^2 + x - 2 = 0$
b) $x^2 - 7x + 6 = 0$
c) $x^2 + x - 12 = 0$
d) $2x^2 + 5x - 3 = 0$
e) $\frac{2}{x} - 2 = 0$
f) $\frac{2}{x} + 1 = 0$

Topic 3

Geometry

Syllabus

C3.1

Use and interpret the geometrical terms: point, line, parallel, bearing, right angle, acute, obtuse and reflex angles, perpendicular, similarity and congruence.
Use and interpret vocabulary of triangles, quadrilaterals, circles, polygons and simple solid figures including nets.

C3.2

Measure lines and angles.
Construct a triangle given the three sides using ruler and pair of compasses only.
Construct other simple geometrical figures from given data using ruler and protractor as necessary.
Construct angle bisectors and perpendicular bisectors using straight edge and pair of compasses only.

C3.3

Read and make scale drawings.

C3.4

Calculate lengths of similar figures.

C3.5

Recognise rotational and line symmetry (including order of rotational symmetry) in two dimensions.

C3.6

Calculate unknown angles using the following geometrical properties:

- angles at a point
- angles at a point on a straight line and intersecting straight lines
- angles formed within parallel lines
- angle properties of triangles and quadrilaterals
- angle properties of regular polygons
- angle in a semi-circle
- angle between tangent and radius of a circle.

C3.7

Use the following loci and the method of intersecting loci for sets of points in two dimensions which are:

- at a given distance from a given point
- at a given distance from a given straight line
- equidistant from two given points
- equidistant from two given intersecting straight lines.

Contents

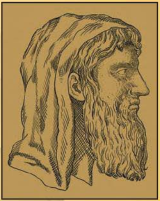
The development of geometry

The beginnings of geometry can be traced back to around 2000BCE in ancient Mesopotamia and Egypt. Early geometry was a practical subject concerning lengths, angles, areas and volumes and was used in surveying, construction, astronomy and various crafts.

The earliest known texts on geometry are the Egyptian Rhind Papyrus (c.1650BCE), the Moscow Papyrus (c.1890BCE) and Babylonian clay tablets such as Plimpton 322 (c.1900BCE). For example, the Moscow Papyrus gives a formula for calculating the volume of a truncated pyramid, or frustum.

In the 7th century BCE, the Greek mathematician Thales of Miletus (which is now in Turkey) used geometry to solve problems such as calculating the height of pyramids and the distance of ships from the shore.

In around 300BCE, Euclid wrote his book *Elements*, perhaps the most successful textbook of all time. It introduced the concepts of definition, theorem and proof. Its contents are still taught in geometry classes today.



Euclid

16 Geometrical vocabulary

NB: All diagrams are not drawn to scale.

Angle

Different types of angle have different names:

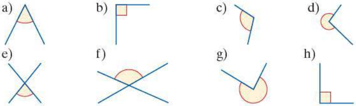
acute angles lie between 0° and 90°
right angles are exactly 90°
obtuse angles lie between 90° and 180°
reflex angles lie between 180° and 360°

Exercise 16.1

1. Draw and label one example of each of the following types of angle:

   right acute obtuse reflex

2. Copy the angles below and write beneath each drawing the type of angle it shows:



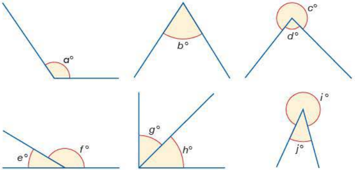
Two angles which add together to total 180° are called **supplementary** angles.
Two angles which add together to total 90° are called **complementary** angles.

Exercise 16.2

1. State whether the following pairs of angles are supplementary, complementary or neither:

   a) 70°, 20° b) 90°, 90° c) 40°, 50° d) 80°, 30°
   e) 15°, 75° f) 145°, 35° g) 133°, 57° h) 33°, 67°
   i) 45°, 45° j) 140°, 40°

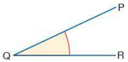
2. Measure the angles in the diagrams below:



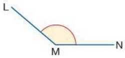
3. Use a protractor, ruler and pencil to draw the following angles accurately:

a) 45°	b) 72°	c) 90°
d) 142°	e) 260°	f) 318°

Angles can be named either by labelling them as shown in Q.2, or by indicating the lines which meet to form the angle as shown below.



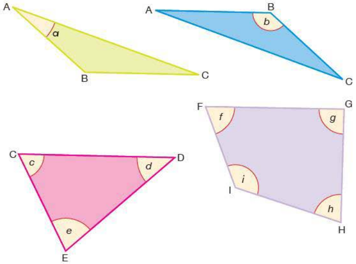
∠PQR



∠LMN

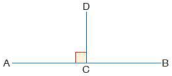
Exercise 16.3

1. Sketch the shapes below. Name the angles marked with a single letter, in terms of the lines which meet to form the angles.



2. Measure accurately the angles marked in the sketches you made for Q.1 above.

Perpendicular lines



To find the shortest distance between two points, you measure the length of the **straight line** which joins them.

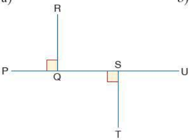
Two lines which meet at right angles are **perpendicular** to each other.

So in this diagram (left) CD is **perpendicular** to AB, and AB is perpendicular to CD.

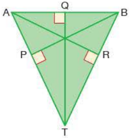
If the lines AD and BD are drawn to form a triangle, the line CD can be called the **height** or **altitude** of the triangle ABD.

Exercise 16.4 In the diagrams below, state which pairs of lines are perpendicular to each other.

a)



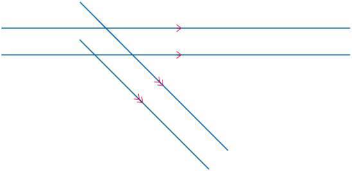
b)



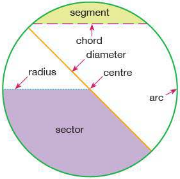
Parallel lines

Parallel lines are straight lines which can be continued to infinity in either direction without meeting.

Railway lines are an example of parallel lines. Parallel lines are marked with arrows as shown:

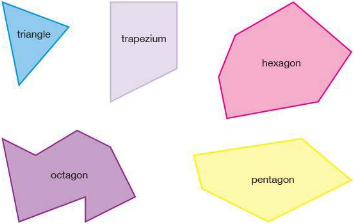


Vocabulary of the circle

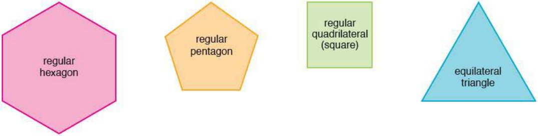


Polygons

A **polygon** is a closed two-dimensional shape bounded by straight lines. Examples of polygons include triangles, quadrilaterals, pentagons and hexagons. Hence the shapes below all belong to the polygon family:



A **regular polygon** is distinctive in that all its sides are of equal length and all its angles are of equal size. Below are some examples of regular polygons.



The name of each polygon is derived from the number of angles it contains. The following list identifies some of these polygons.

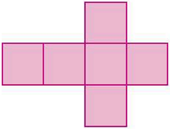
3 angles = **tri**angle
4 angles = **quad**rilateral (tetragon)
5 angles = **penta**gon
6 angles = **hexa**gon
7 angles = **hepta**gon
8 angles = **octa**gon
9 angles = **nona**gon
10 angles = **deca**gon
12 angles = **dodeca**gon

Exercise 16.5

1. Draw a sketch of each of the shapes listed on the previous page.
2. Draw accurately a regular hexagon, a regular pentagon, and a regular octagon.

Nets

The diagram below is the **net** of a cube. It shows the faces of the cube opened out into a two-dimensional plan. The net of a three-dimensional shape can be folded up to make that shape.



Exercise 16.6

Draw the following on squared paper:

1. Two other possible nets of a cube
2. The net of a cuboid (rectangular prism)
3. The net of a triangular prism
4. The net of a cylinder
5. The net of a square-based pyramid
6. The net of a tetrahedron

Congruence

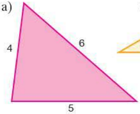
Congruent triangles are **identical**. They have corresponding sides of the same length, and corresponding angles which are equal.

Triangles are congruent if any of the following can be proved:

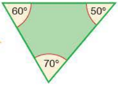
- Three corresponding sides are equal (S S S);
- Two corresponding sides and the included angle are equal (S A S);
- Two angles and the corresponding side are equal (A S A);
- Each triangle has a right angle, and the hypotenuse and a corresponding side are equal in length.

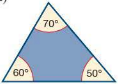
Exercise 16.7

1. Which of the triangles below are congruent?

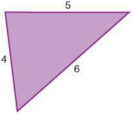
a)


b)


c)


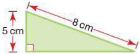
d)


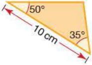
e)

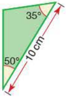

f)


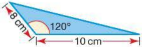
2. Make accurate drawings of pairs of triangles to illustrate the proofs of congruence given above.

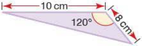
3. In the diagrams below, identify pairs of congruent triangles. Give reasons for your answers.

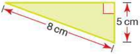
a)


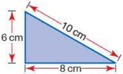
b)


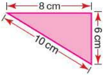
c)


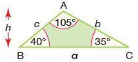
d)


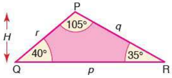
e)


f)


g)


h)






Similarity

Two polygons are said to be **similar** if their angles are the same and corresponding sides are in proportion.

For triangles, having equal angles implies that corresponding sides are in proportion. The converse is also true.

In the diagrams (left) triangle ABC and triangle PQR are similar.

For similar figures the ratios of the lengths of the sides are the same and represent the **scale factor**, i.e.

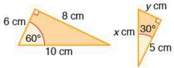
$$\frac{p}{a} = \frac{q}{b} = \frac{r}{c} = k \text{ (where } k \text{ is the scale factor of enlargement)}$$

The heights of similar triangles are proportional also:

$$\frac{H}{h} = \frac{p}{a} = \frac{q}{b} = \frac{r}{c} = k$$

Exercise 16.8

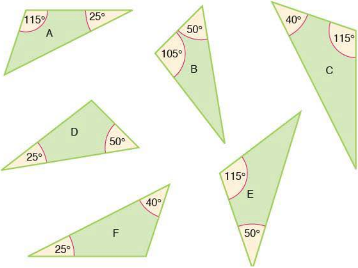
1. a) Explain why the two triangles below are similar.



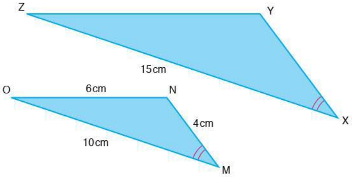
b) Calculate the scale factor which reduces the larger triangle to the smaller one.

c) Calculate the value of x and the value of y.

2. Which of the triangles below are similar?

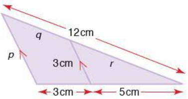


3. The triangles below are similar.

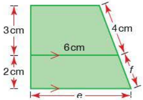


a) Calculate the length XY.
b) Calculate the length YZ.

4. In the triangle to the right calculate the lengths of sides p, q and r.



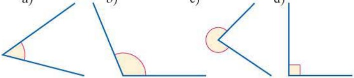
5. In the trapezium (below) calculate the lengths of sides e and f.



Student assessment 1

1. Are the angles below acute, obtuse, reflex or right angles?

a) b) c) d)

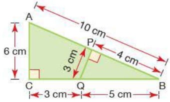


2. Use a ruler, pencil and protractor to draw angles of 135° and 72°.

3. Draw a circle of radius 3 cm. Mark on it:
   a) a diameter b) a chord c) a sector.

4. Draw two congruent isosceles triangles with base angles of 45°.

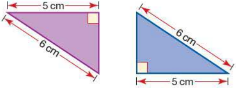
5. Using the triangle (below), explain fully why △ABC and △PBQ are similar.



Student assessment 2

1. Draw and label two pairs of intersecting parallel lines.

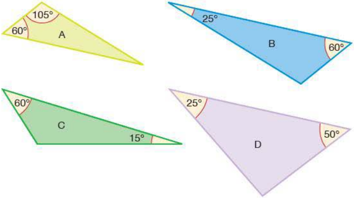
2. Make two statements about these two triangles:



3. Draw a circle of radius 5 cm. Construct a regular hexagon within the circle.

4. On squared paper, draw the net of a triangular prism.

5. Which of the triangles below are similar?



17 Geometrical constructions and scale drawings

Lines

A straight line can be both drawn and measured accurately using a ruler.

Exercise 17.1

1. Using a ruler, measure the length of the following lines to the nearest mm:

2. Draw lines of the following lengths using a ruler:

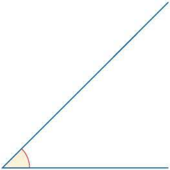
a) 3 cm b) 8 cm
c) 4.6 cm d) 94 mm
e) 38 mm f) 61 mm

Angles

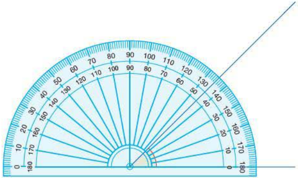
An angle is a measure of turn. When drawn it can be measured using either a protractor or an angle measurer. The units of turn are degrees (°). Measuring with a protractor needs care, as there are two scales marked on it – an inner one and an outer one.

Worked examples

a) Measure the angle drawn below:



- Place the protractor over the angle so that the cross lies on the point where the two arms meet.
- Align the 0° with one of the arms:

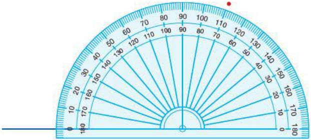


- Decide which scale is appropriate. In this case, it is the inner scale as it starts at 0°.
- Measure the angle using the inner scale.

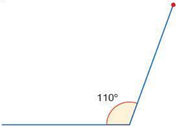
The angle is 45°.

b) Draw an angle of 110°.

- Start by drawing a straight line.
- Place the protractor on the line so that the cross is on one of the end points of the line. Ensure that the line is aligned with the 0° on the protractor:

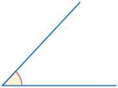


- Decide which scale to use. In this case, it is the outer scale as it starts at 0°.
- Mark where the protractor reads 110°.
- Join the mark made to the end point of the original line.

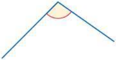


Exercise 17.2

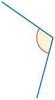
1. Measure each of the following angles:

a) 

b) 

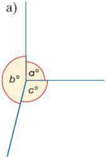
c) 

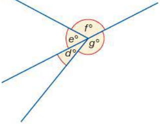
d) 

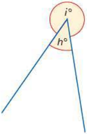
e) 

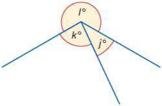
f) 

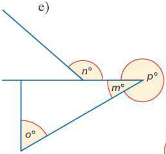
2. Measure each of the following angles:

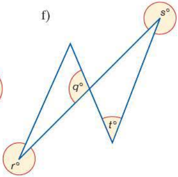
a) 

b) 

c) 

d) 

e) 

f) 

3. Draw angles of the following sizes:

a) 20°	b) 45°	c) 90°
d) 120°	e) 157°	f) 172°
g) 14°	h) 205°	i) 311°
j) 283°	k) 198°	l) 352°

Constructing triangles

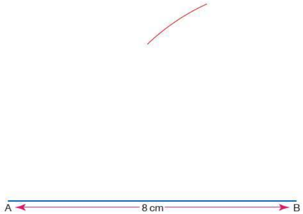
Triangles can be drawn accurately by using a ruler and a pair of compasses. This is called **constructing** a triangle.

Worked example Construct the triangle ABC given that:

AB = 8 cm, BC = 6 cm and AC = 7 cm

- Draw the line AB using a ruler:

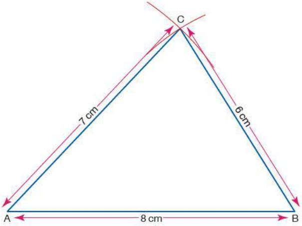
- Open up a pair of compasses to 6 cm. Place the compass point on B and draw an arc:



Note that every point on the arc is 6 cm away from B.

- Open up the pair of compasses to 7 cm. Place the compass point on A and draw another arc, with centre A and radius 7 cm, ensuring that it intersects with the first arc.
  Every point on the second arc is 7 cm from A.
  Where the two arcs intersect is point C, as it is both 6 cm from B and 7 cm from A.

- Join C to A and C to B:



Exercise 17.3 Using only a ruler and a pair of compasses, construct the following triangles:

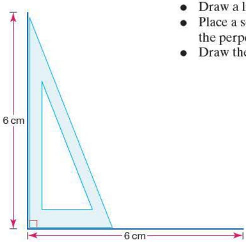
1. $\triangle$ABC where AB = 10 cm, AC = 7 cm and BC = 9 cm
2. $\triangle$LMN where LM = 4 cm, LN = 8 cm and MN = 5 cm
3. $\triangle$PQR, an equilateral triangle of side length 7 cm
4. a) $\triangle$ABC where AB = 8 cm, AC = 4 cm and BC = 3 cm
   b) Is this triangle possible? Explain your answer.

Constructing simple geometric figures

Squares and rectangles can be constructed using only a ruler and a set square. The set square is used because it provides a right angle with which to draw lines perpendicular to each other.

Worked example Construct a square of side length 6 cm.

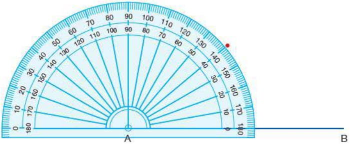
- Draw a line 6 cm long.
- Place a set square at the end of the line, ensuring that one of the perpendicular sides rests on the line drawn.
- Draw the perpendicular line 6 cm long.



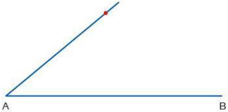
- Repeat this process for the remaining two sides.

Worked example Using a ruler and a protractor only, construct the parallelogram ABCD in which AB = 6 cm, AD = 3 cm and angle DAB = 40°.

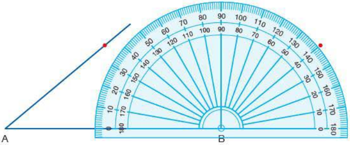
- Draw a line 6 cm long and label it AB.
- Place the protractor on A and mark an angle of 40°.



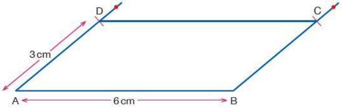
- Draw a line from A through the marked point.



- Place the protractor on B and mark an angle of 40° reading from the inner scale.



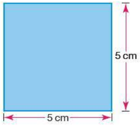
- Measure 3 cm from A and mark the point D.
- Measure 3 cm from B and mark the point C.
- Join D to C.



Exercise 17.4

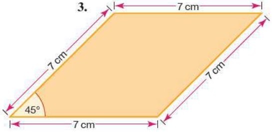
Use appropriate geometrical instruments to construct the following shapes:

1.

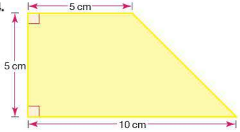


2.

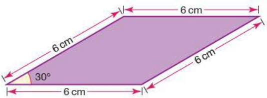
3.



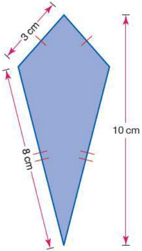
4.



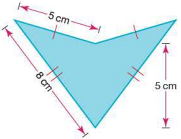
5.



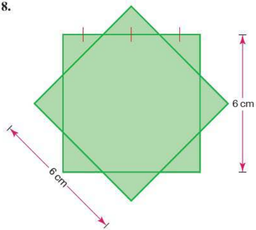
6.



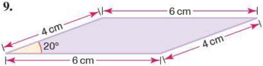
7.



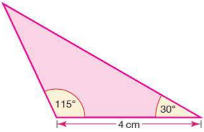
8.



9.



10.

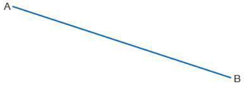


Bisecting a line

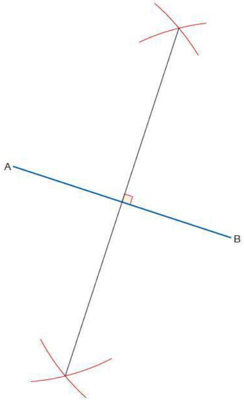
The word **bisect** means 'to divide in half'. Therefore, to bisect an angle means to divide an angle in half. Similarly, to bisect a line means to divide a line in half. A **perpendicular bisector** of a line is another line which divides it in half and meets the original line at right angles.

To bisect either a line or an angle involves the use of a pair of compasses.

Worked example A line AB is drawn below. Construct the perpendicular bisector of AB.



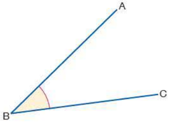
- Open a pair of compasses to more than half the distance AB.
- Place the compass point on A and draw arcs above and below AB.
- With the same radius, place the compass point on B and draw arcs above and below AB. Note that the two pairs of arcs should intersect (see diagram below).
- Draw a line through the two points where the arcs intersect:



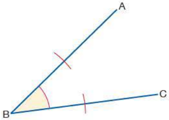
The line drawn is known as the perpendicular bisector of AB, as it divides AB in half and also meets it at right angles.

Bisecting an angle

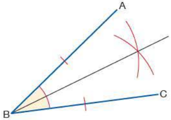
Worked example Using a pair of compasses, bisect the angle ABC below:



- Open a pair of compasses and place the point on B. Draw two arcs such that they intersect the arms of the angle:



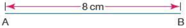
- Place the compass point on the points of intersection in turn, and draw another pair of arcs of the same radius. Ensure that they intersect.
- Draw a line through B and the point of intersection of the two arcs. This line bisects angle ABC.



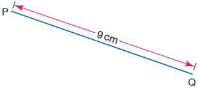
Exercise 17.5

1. Copy each of the lines drawn below on to plain paper, and construct the perpendicular bisectors.

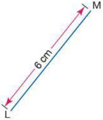
a)



b)



c)

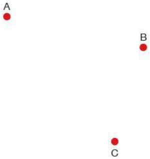


d)

2. For each of the following:
   i) draw the angle,
   ii) bisect the angle using a pair of compasses.

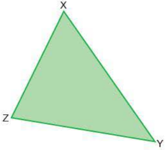
   a) 45° b) 70° c) 130°
   d) 173° e) 210° f) 312°

3. Copy the diagram below.



a) Construct the perpendicular bisector of AB.
b) Construct the perpendicular bisector of BC.
c) What can be said about the point of intersection of the two perpendicular bisectors?

4. Draw a triangle similar to the one shown below:

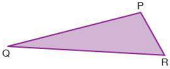


Construct the perpendicular bisector of each of the sides of your triangle.

Use a pair of compasses to draw a circle using the point where the three perpendicular bisectors cross as the centre and passing through the points X, Y and Z.

This is called the **circumcircle** of the triangle.

5. Draw a triangle similar to the one shown below:



By construction, draw a circle to pass through points P, Q and R.

Scale drawings

Scale drawings are used when an accurate diagram, drawn in proportion, is needed. Common uses of scale drawings include maps and plans. The use of scale drawings involves understanding how to scale measurements.

Worked examples

a) A map is drawn to a scale of 1 : 10 000. If two objects are 1 cm apart on the map, how far apart are they in real life? Give your answer in metres.

A scale of 1 : 10 000 means that 1 cm on the map represents 10 000 cm in real life.

Therefore the distance = 10 000 cm
= 100 m

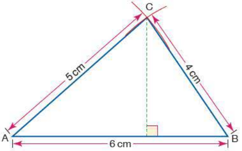
b) A model boat is built to a scale of 1 : 50. If the length of the real boat is 12 m, calculate the length of the model boat in cm.

A scale of 1 : 50 means that 50 cm on the real boat is 1 cm on the model boat.

12 m = 1200 cm

Therefore the length of the model boat = 1200 ÷ 50 cm
= 24 cm

c) i) Construct, to a scale of 1 : 1, a triangle ABC such that AB = 6 cm, AC = 5 cm and BC = 4 cm.



ii) Measure the perpendicular length of C from AB. Perpendicular length is 3.3 cm.

iii) Calculate the area of the triangle.

$$\text{Area} = \frac{\text{base length} \times \text{perpendicular height}}{2}$$

$$\text{Area} = \frac{6 \times 3.3}{2}\ \text{cm} = 9.9\ \text{cm}^2$$

Exercise 17.6

1. In the following questions, both the scale to which a map is drawn and the distance between two objects on the map are given.

Find the real distance between the two objects, giving your answer in metres.

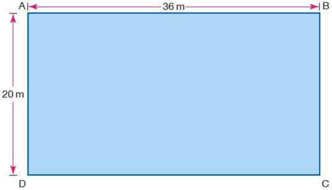
a)	1 : 10 000	3 cm	b)	1 : 10 000	2.5 cm
c)	1 : 20 000	1.5 cm	d)	1 : 8000	5.2 cm

2. In the following questions, both the scale to which a map is drawn and the true distance between two objects are given.

Find the distance between the two objects on the map, giving your answer in cm.

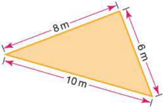
a)	1 : 15 000	1.5 km	b)	1 : 50 000	4 km
c)	1 : 10 000	600 m	d)	1 : 25 000	1.7 km

3. A rectangular pool measures 20 m by 36 m as shown below:



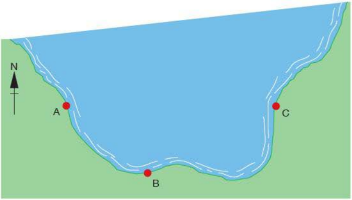
a) Construct a scale drawing of the pool, using 1 cm for every 4 m.
b) A boy swims across the pool in such a way that his path is the perpendicular bisector of BD. Show, by construction, the path that he takes.
c) Work out the distance the boy swam.

4. A triangular enclosure is shown in the diagram below:



a) Using a scale of 1 cm for each metre, construct a scale drawing of the enclosure.
b) Calculate the true area of the enclosure.

5. Three radar stations A, B and C, pick up a distress signal from a boat at sea.

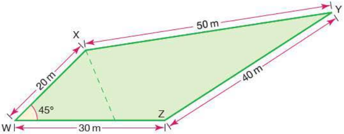


C is 24 km due East of A, AB = 12 km and BC = 18 km. The signal indicates that the boat is equidistant from all three radar stations.

a) By construction and using a scale of 1 cm for every 3 km, locate the position of the boat.

b) What is the boat's true distance from each radar station?

6. A plan view of a field is shown below:



a) Using a scale of 1 cm for every 5 m, construct a scale drawing of the field.

b) A farmer divides the field by running a fence from X in such a way that it bisects $\angle WXY$. By construction, show the position of the fence on your diagram.

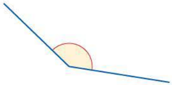
c) Work out the length of fencing used.

Student assessment 1

1. a) Using a ruler, measure the length of the line below:

   b) Draw a line 4.7 cm long.

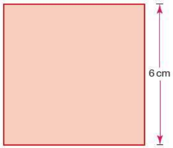
2. a) Using a protractor, measure the angle below:



   b) Draw an angle of 300°.

3. Construct △ABC such that AB = 8 cm, AC = 6 cm and BC = 12 cm.

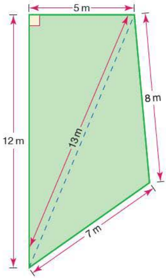
4. Construct the following square:



5. Three players, P, Q and R, are approaching a football. Their positions relative to each other are shown below:

   The ball is equidistant from all three players. Copy the diagram and show, by construction, the position of the ball.

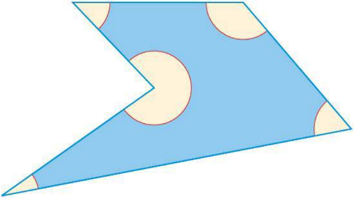
6. A plan of a living room is shown below:



a) Using a pair of compasses, construct a scale drawing of the room using 1 cm for every metre.
b) Using a set square if necessary, calculate the total area of the actual living room.

Student assessment 2

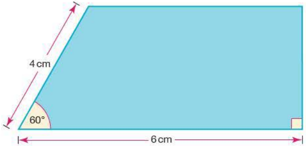
1. Measure each of the five angles of the pentagon below:



2. Draw, using a ruler and a protractor, a triangle with angles of 40°, 60° and 80°.

3. a) Draw an angle of 320°.
b) Using a pair of compasses, bisect the angle.

4. Construct the following trapezium:



5. In the following questions, both the scale to which a map is drawn and the true distance between two objects are given. Find the distance between the two objects on the map, giving your answer in cm.

a) 1 : 20 000 4.4 km

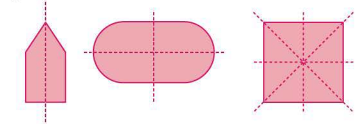
b) 1 : 50 000 12.2 km

18 Symmetry

Line symmetry

A **line of symmetry** divides a two-dimensional (flat) shape into two congruent (identical) shapes.

e.g.



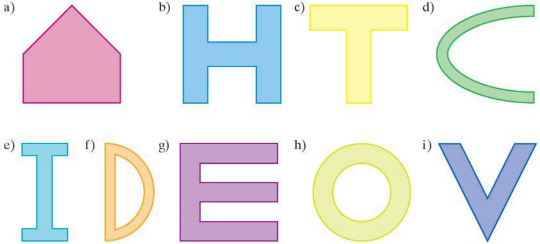
1 line of symmetry　　2 lines of symmetry　　4 lines of symmetry

Exercise 18.1

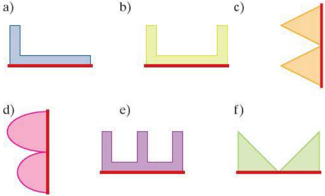
1. Draw the following shapes and, where possible, show all their lines of symmetry:

a) square	b) rectangle
c) equilateral triangle	d) isosceles triangle
e) kite	f) regular hexagon
g) regular octagon	h) regular pentagon
i) isosceles trapezium	j) circle

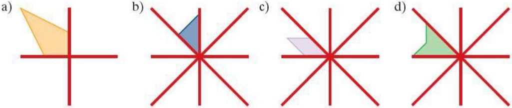
2. Copy the shapes below and, where possible, show all their lines of symmetry:



3. Copy the shapes below and complete them so that the **bold** line becomes a line of symmetry:



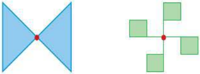
4. Copy the shapes below and complete them so that the **bold** lines become lines of symmetry:



Rotational symmetry

A two-dimensional shape has **rotational symmetry** if, when rotated about a central point, it fits its outline. The number of times it fits its outline during a complete revolution is called the **order of rotational symmetry**.

e.g.



rotational symmetry of order 2   rotational symmetry of order 4

Exercise 18.2

1. Draw the following shapes. Identify the centre of rotation, and state the order of rotational symmetry:
   a) square
   b) equilateral triangle
   c) regular pentagon
   d) parallelogram
   e) rectangle
   f) rhombus
   g) regular hexagon
   h) regular octagon
   i) circle

2. Copy the shapes below. Indicate the centre of rotation, and state the order of rotational symmetry:

a) b) c) d) e)

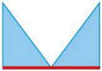
Student assessment 1

1. Draw a shape with exactly:
   a) one line of symmetry,
   b) two lines of symmetry,
   c) three lines of symmetry.
   Mark the lines of symmetry on each diagram.

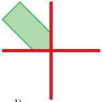
2. Draw a shape with:
   a) rotational symmetry of order exactly 2,
   b) rotational symmetry of order exactly 3.
   Mark the position of the centre of rotation on each diagram.

3. Copy and complete the following shapes so that the **bold** lines become lines of symmetry:

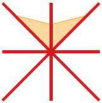
a)



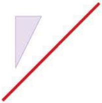
b)



c)



d)



4. State the order of rotational symmetry for the completed drawings in Q.3.

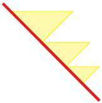
Student assessment 2

1. Draw a two-dimensional shape with:
   a) rotational symmetry of order exactly 4,
   b) rotational symmetry of order exactly 8.
   Mark the position of the centre of rotation on each diagram.

2. Draw and name a shape with each of the following numbers of lines of reflective symmetry:
   a) infinite b) six

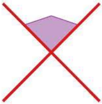
3. Copy and complete the following shapes so that the **bold** lines become lines of symmetry:

   a)

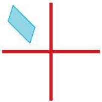


   b)

   c)



   d)



4. State the order of rotational symmetry for the completed drawings in Q.3.

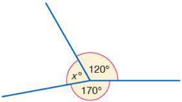
19 Angle properties

NB: All diagrams are not drawn to scale.

Angles at a point and on a line

One complete revolution is equivalent to a rotation of 360° about a point. Similarly, half a complete revolution is equivalent to a rotation of 180° about a point. These facts can be seen clearly by looking at either a circular angle measurer or a semi-circular protractor.

Worked examples **a)** Calculate the size of the angle x in the diagram below:

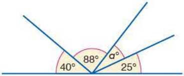


The sum of all the angles around a point is 360°. Therefore:

$$120 + 170 + x = 360$$
$$x = 360 - 120 - 170$$
$$x = 70$$

Therefore angle x is 70°.
Note that the size of the angle is **calculated** and **not measured**.

b) Calculate the size of angle a in the diagram below:



The sum of all the angles at a point on a straight line is 180°. Therefore:

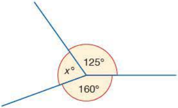
$$40 + 88 + a + 25 = 180$$
$$a = 180 - 40 - 88 - 25$$
$$a = 27$$

Therefore angle a is 27°.

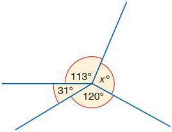
Exercise 19.1

1. In the following questions, calculate the size of angle x:

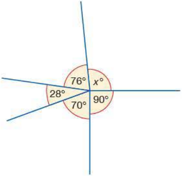
a)



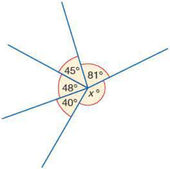
b)



c)

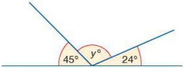


d)

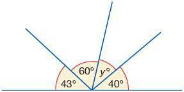


2. In the following questions, the angles lie about a point on a straight line. Calculate the size of angle y in each case:

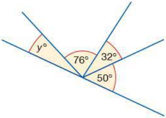
a)



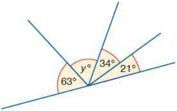
b)



c)

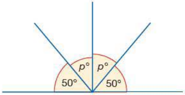


d)

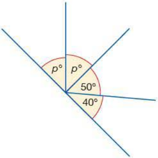


3. Calculate the size of angle p in each of the following:

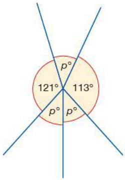
a)



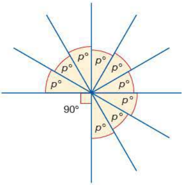
b)



c)



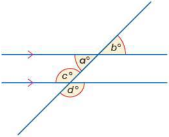
d)



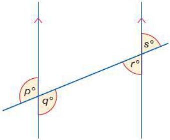
Angles formed by intersecting lines

Exercise 19.2

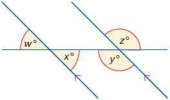
1. Draw a similar diagram to the one shown below. Measure carefully each of the labelled angles and write them down.



2. Draw a similar diagram to the one shown below. Measure carefully each of the labelled angles and write them down.

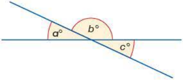


3. Draw a similar diagram to the one shown below. Measure carefully each of the labelled angles and write them down.



4. Write down what you have noticed about the angles you measured in Q.1–3.

When two straight lines cross, it is found that the angles opposite each other are the same size. They are known as **vertically opposite angles**. By using the fact that angles at a point on a straight line add up to 180°, it can be shown why vertically opposite angles must always be equal in size.

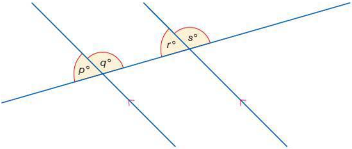


$$a + b = 180°$$
$$c + b = 180°$$

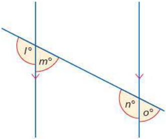
Therefore, a is equal to c.

Exercise 19.3

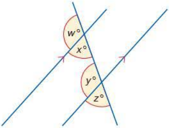
1. Draw a similar diagram to the one shown below. Measure carefully each of the labelled angles and write them down.



2. Draw a similar diagram to the one shown below. Measure carefully each of the labelled angles and write them down.



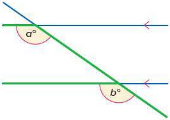
3. Draw a similar diagram to the one shown below. Measure carefully each of the labelled angles and write them down.



4. Write down what you have noticed about the angles you measured in Q.1–3.

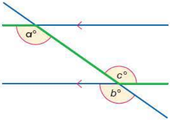
Angles formed within parallel lines

When a line intersects two parallel lines, as in the diagram below, it is found that certain angles are the same size.



The angles a and b are equal and are known as **corresponding angles**. Corresponding angles can be found by looking for an 'F' formation in a diagram.

A line intersecting two parallel lines also produces another pair of equal angles, known as **alternate angles**. These can be shown to be equal by using the fact that both vertically opposite and corresponding angles are equal.



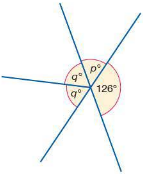
In the diagram above, $a = b$ (corresponding angles). But $b = c$ (vertically opposite). It can therefore be deduced that $a = c$.

Angles a and c are alternate angles. These can be found by looking for a 'Z' formation in a diagram.

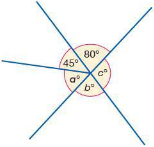
Exercise 19.4

In each of the following questions, some of the angles are given. Deduce, giving your reasons, the size of the other labelled angles.

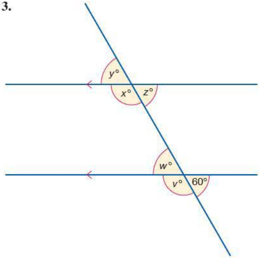
1.



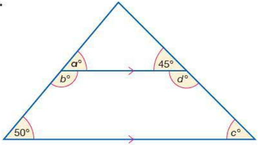
2.



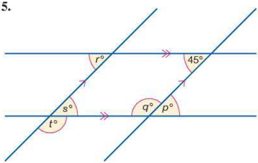
3.



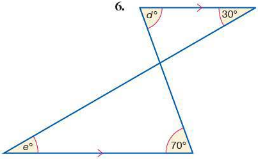
4.



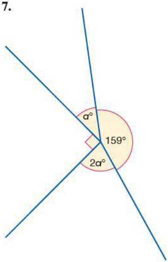
5.



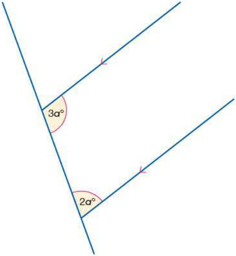
6.



7.



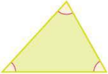
8.



Angle properties of triangles

A **triangle** is a plane (two-dimensional) shape consisting of three angles and three sides. There are six main types of triangle. Their names refer to the sizes of their angles and/or the lengths of their sides, and are as follows:

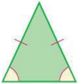
An **acute-angled** triangle has all its angles less than 90°.



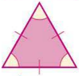
A **right-angled** triangle has an angle of 90°.

An **obtuse-angled** triangle has one angle greater than 90°.

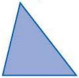
An **isosceles** triangle has two sides of equal length, and the angles opposite the equal sides are equal.



An **equilateral** triangle has three sides of equal length and three equal angles.

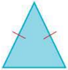


A **scalene** triangle has three sides of different lengths and all three angles are different.

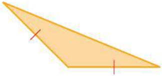


Exercise 19.5

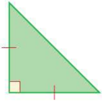
1. Describe the triangles below in two ways. The example on the right shows an **acute-angled isosceles** triangle.



a)

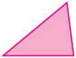


b)



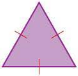
c)

d)



e)

f)



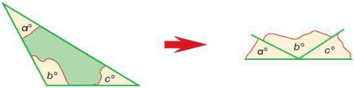
2. Draw the following triangles using a ruler and compasses:
   a) an acute-angled isosceles triangle of sides 5 cm, 5 cm and 6 cm, and altitude 4 cm,
   b) a right-angled scalene triangle of sides 6 cm, 8 cm and 10 cm,
   c) an equilateral triangle of side 7.5 cm,
   d) an obtuse-angled isosceles triangle of sides 13 cm, 13 cm and 24 cm, and altitude 5 cm.

Exercise 19.6

1. a) Draw five different triangles. Label their angles x, y and z. As accurately as you can, measure the three angles of each triangle and add them together.

b) What do you notice about the sum of the three angles of each of your triangles?

2. a) Draw a triangle on a piece of paper and label the angles a, b and c. Tear off the corners of the triangle and arrange them as shown below:

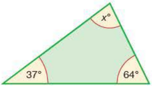


b) What do you notice about the total angle that a, b and c make?

The sum of the interior angles of a triangle

It can be seen from the questions above that triangles of any shape have one thing in common. That is, that the sum of their three angles is constant: 180°.

Worked example Calculate the size of the angle x in the triangle below:



$37 + 64 + x = 180$

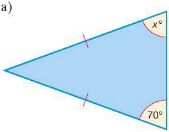
$x = 180 - 37 - 64$

Therefore angle x is 79°.

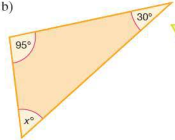
Exercise 19.7

1. For each of the triangles below, use the information given to calculate the size of angle x.

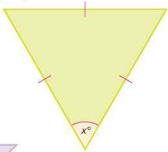
a)



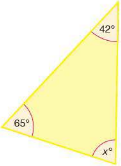
b)



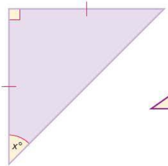
c)



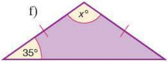
d)



e)

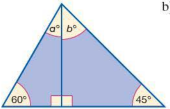


f)

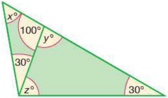


2. In each of the diagrams below, calculate the size of the labelled angles.

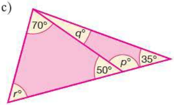
a)



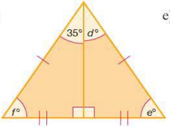
b)



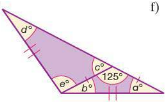
c)



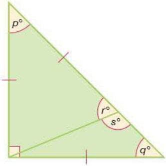
d)



e)



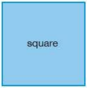
f)



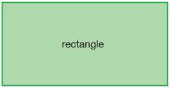
Angle properties of quadrilaterals

A **quadrilateral** is a plane shape consisting of four angles and four sides. There are several types of quadrilateral. The main ones, and their properties, are described below.

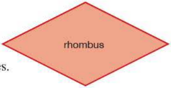
Two pairs of parallel sides.
All sides are equal.
All angles are equal.
Diagonals intersect at right angles.



Two pairs of parallel sides.
Opposite sides are equal.
All angles are equal.



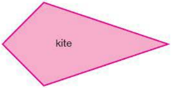
Two pairs of parallel sides.
All sides are equal.
Opposite angles are equal.
Diagonals intersect at right angles.



Two pairs of parallel sides.
Opposite sides are equal.
Opposite angles are equal.

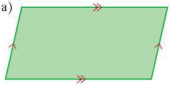
One pair of parallel sides.
An isosceles trapezium has one pair of parallel sides and the other pair of sides are equal in length.

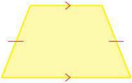
Two pairs of equal sides.
One pair of equal angles.
Diagonals intersect at right angles.

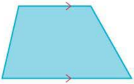


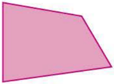
Exercise 19.8

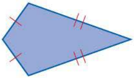
1. Copy the diagrams and name each shape according to the definitions given above.

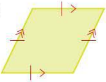
a) 

b) 

c) 

d) 

e) 

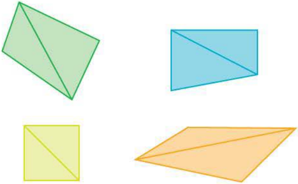
f) 

2. Copy and complete the following table. The first line has been started for you.

	Rectangle	Square	Parallelogram	Kite	Rhombus	Equilateral triangle
Opposite sides equal in length	Yes		Yes			
All sides equal in length						
All angles right angles						
Both pairs of opposite sides parallel						
Diagonals equal in length						
Diagonals intersect at right angles						
All angles equal						

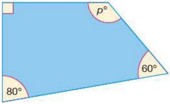
The sum of the interior angles of a quadrilateral

In the quadrilaterals below, a straight line is drawn from one of the corners (vertices) to the opposite corner. The result is to split each quadrilateral into two triangles.



As already shown earlier in the chapter, the sum of the angles of a triangle is 180°. Therefore, as a quadrilateral can be drawn as two triangles, the sum of the four angles of any quadrilateral must be 360°.

Worked example Calculate the size of angle p in the quadrilateral (below):



$90 + 80 + 60 + p = 360$

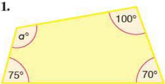
$p = 360 - 90 - 80 - 60$

Therefore angle p is 130°.

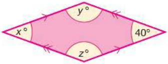
Exercise 19.9

In each of the diagrams below, calculate the size of the labelled angles.

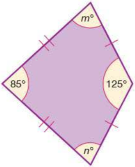
1.



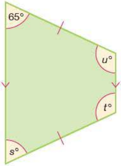
2.



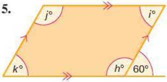
3.



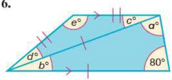
4.



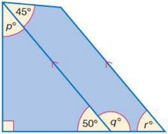
5.



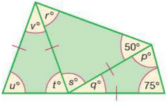
6.



7.



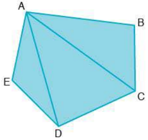
8.

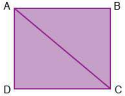


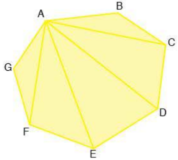
Angle properties of polygons

The sum of the interior angles of a polygon

In the polygons below a straight line is drawn from each vertex to vertex A.







As can be seen, the number of triangles is always two less than the number of sides the polygon has, i.e. if there are n sides, there will be $(n - 2)$ triangles.

Since the angles of a triangle add up to 180°, the sum of the interior angles of a polygon is therefore $180(n - 2)°$.

Worked example Find the sum of the interior angles of a regular pentagon and hence the size of each interior angle.

For a pentagon, $n = 5$.

$$\text{Therefore the sum of the interior angles} = 180(5 - 2)° = 180 \times 3° = 540°$$

For a regular pentagon the interior angles are of equal size.

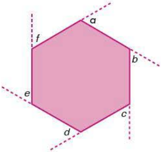
Therefore each angle $= \dfrac{540°}{5} = 108°$.

The sum of the exterior angles of a polygon

The angles marked a, b, c, d, e and f (right) represent the exterior angles of the regular hexagon drawn.

For any convex polygon the sum of the exterior angles is 360°.

If the polygon is regular and has n sides, then each exterior angle $= \dfrac{360°}{n}$.



Worked examples

a) Find the size of an exterior angle of a regular nonagon.

$$\frac{360^\circ}{9} = 40^\circ$$

b) Calculate the number of sides a regular polygon has if each exterior angle is 15°.

$$n = \frac{360}{15}$$
$$= 24$$

The polygon has 24 sides.

Exercise 19.10

1. Find the sum of the interior angles of the following polygons:
a) a hexagon
b) a nonagon
c) a heptagon

2. Find the value of each interior angle of the following regular polygons:
a) an octagon
b) a square
c) a decagon
d) a dodecagon

3. Find the size of each exterior angle of the following regular polygons:
a) a pentagon
b) a dodecagon
c) a heptagon

4. The exterior angles of regular polygons are given below. In each case calculate the number of sides the polygon has.
a) 20°
b) 36°
c) 10°
d) 45°
e) 18°
f) 3°

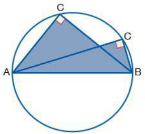
5. The interior angles of regular polygons are given below. In each case calculate the number of sides the polygon has.
a) 108°
b) 150°
c) 162°
d) 156°
e) 171°
f) 179°

6. Calculate the number of sides a regular polygon has if an interior angle is five times the size of an exterior angle.

7. Copy and complete the table below for regular polygons:

Number of sides	Name	Sum of exterior angles	Size of an exterior angle	Sum of interior angles	Size of an interior angle
3					
4					
5					
6					
7					
8					
9					
10					
12					

The angle in a semi-circle

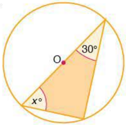


If AB represents the diameter of the circle, then the angle at C is 90°.

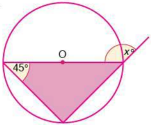
Exercise 19.11

In each of the following diagrams, O marks the centre of the circle. Calculate the value of x in each case.

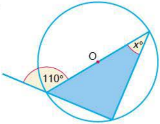
1.



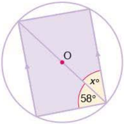
2.



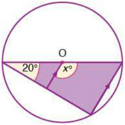
3.



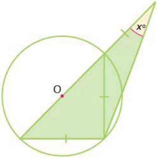
4.



5.



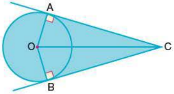
6.



The angle between a tangent and a radius of a circle

The angle between a tangent at a point and the radius to the same point on the circle is a right angle.

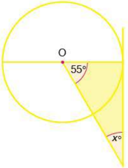
Triangles OAC and OBC (below) are congruent as angle OAC and angle OBC are right angles, OA = OB because they are both radii and OC is common to both triangles.



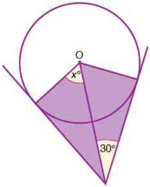
Exercise 19.12

In each of the following diagrams, O marks the centre of the circle. Calculate the value of x in each case.

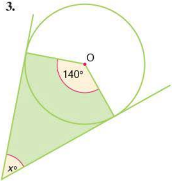
1.



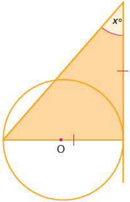
2.



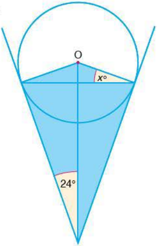
3.



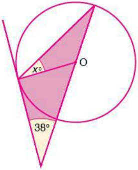
4.



5.

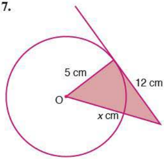


6.

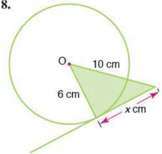


In the following diagrams, calculate the value of x.

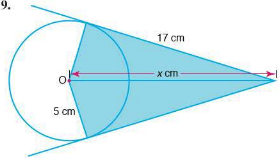
7.



8.



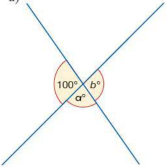
9.



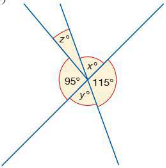
Student assessment 1

1. For the diagrams below, calculate the size of the labelled angles.

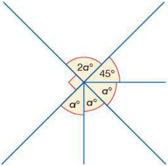
a)



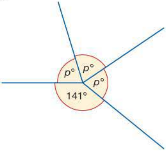
b)



c)

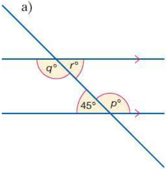


d)

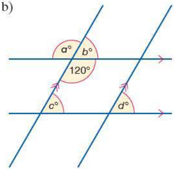


2. For the diagrams below, calculate the size of the labelled angles.

a)

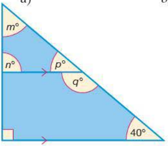


b)

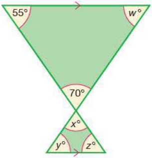


3. For the diagrams below, calculate the size of the labelled angles.

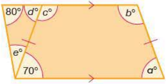
a)



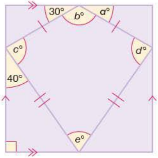
b)



c)



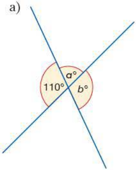
d)



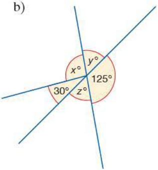
Student assessment 2

1. For the diagrams below, calculate the size of the labelled angles.

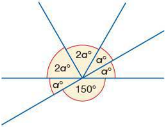
a)



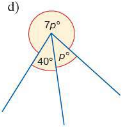
b)



c)

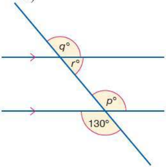


d)

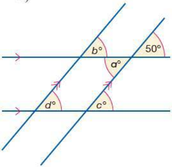


2. For the diagrams below, calculate the size of the labelled angles.

a)

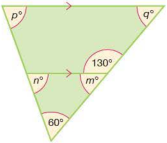


b)

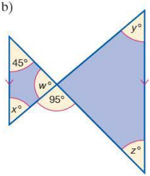


3. For the diagrams below, calculate the size of the labelled angles.

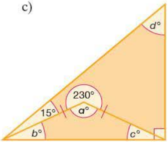
a)



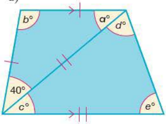
b)



c)



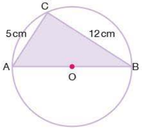
d)



Student assessment 3

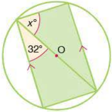
1. Draw a diagram of an octagon to help illustrate the fact that the sum of the internal angles of an octagon is given by $180 \times (8 - 2)°$.
2. Find the size of each interior angle of a twenty-sided regular polygon.
3. What is the sum of the interior angles of a nonagon?
4. What is the sum of the exterior angles of a polygon?
5. What is the size of the exterior angle of a regular pentagon?

6. If AB is the diameter of the circle (right) and AC = 5 cm and BC = 12 cm, calculate:
   a) the size of angle ACB,
   b) the length of the radius of the circle.

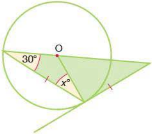


In Q.7–10, O marks the centre of the circle. Calculate the size of the angle marked x in each case.

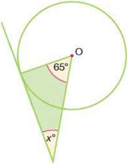
7.



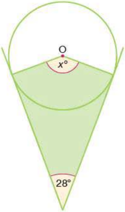
8.



9.



10.

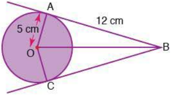


Student assessment 4

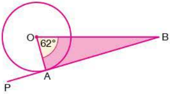
1. Draw a diagram of a hexagon to help illustrate the fact that the sum of the internal angles of a hexagon is given by $180 \times (6 - 2)°$.
2. Find the size of each interior angle of a regular polygon with 24 sides.
3. What is the sum of the interior angles of a regular dodecagon?

4. What is the size of an exterior angle of a regular dodecagon?

5. AB and BC are both tangents to the circle centre O (below). If OA = 5 cm and AB = 12 cm, calculate:
a) the size of angle OAB,
b) the length OB.

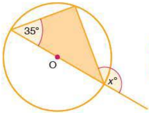


6. If OA is a radius of the circle and PB the tangent to the circle at A, calculate angle ABO.

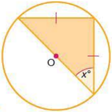


In Q.7–10, O marks the centre of the circle. Calculate the size of the angle marked x in each case.

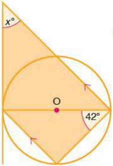
7.



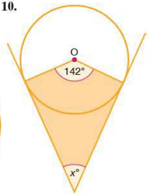
8.



9.



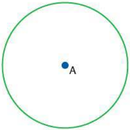
10.



20 Loci

NB: All diagrams are not drawn to scale.

A **locus** (plural **loci)** refers to all the points which fit a particular description. These points can belong to either a region or a line, or both. The principal types of loci are explained below.

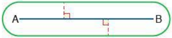


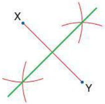
The locus of the points which are at a given distance from a given point

In the diagram (left) it can be seen that the locus of all the points equidistant from a point A is the circumference of a circle centre A. This is due to the fact that all points on the circumference of a circle are equidistant from the centre.

The locus of the points which are at a given distance from a given straight line

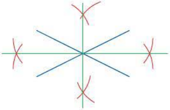
In the diagram (left) it can be seen that the locus of the points equidistant from a straight line AB runs parallel to that straight line. It is important to note that the distance of the locus from the straight line is measured at right angles to the line. This diagram, however, excludes the ends of the line. If these two points are taken into consideration then the locus takes the form shown in the diagram below.





The locus of the points which are equidistant from two given points

The locus of the points equidistant from points X and Y lies on the perpendicular bisector of the line XY.

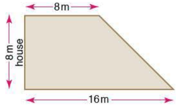


The locus of the points which are equidistant from two given intersecting straight lines

The locus in this case lies on the bisectors of both pairs of opposite angles as shown (left).

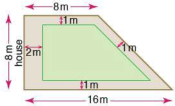
The application of the above cases will enable you to tackle problems involving loci at this level.

Worked example The diagram (below) shows a trapezoidal garden. Three of its sides are enclosed by a fence, and the fourth is adjacent to a house.

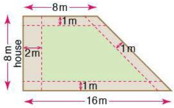


i) Grass is to be planted in the garden. However, it must be at least 2 m away from the house and at least 1 m away from the fence. Shade the region in which the grass can be planted.

The shaded region is therefore the locus of all the points which are both at least 2 m away from the house and at least 1 m away from the surrounding fence. Note that the boundary of the region also forms part of the locus of the points.



ii) Using the same garden as before, grass must now be planted according to the following conditions: it must be **more than** 2 m away from the house and **more than** 1 m away from the fence. Shade the region in which the grass can be planted.



The shape of the region is the same as in the first case; however, in this instance the boundary is not included in the locus of the points as the grass cannot be exactly 2 m away from the house or exactly 1 m away from the fence.

Note: If the boundary is included in the locus points, it is represented by a **solid** line. If it is not included then it is represented by a **dashed** (broken) line.

Exercise 20.1

Q.1–4 are about a rectangular garden measuring 8 m by 6 m. For each question draw a scale diagram of the garden and identify the locus of the points which fit the criteria.

1. Draw the locus of all the points at least 1 m from the edge of the garden.

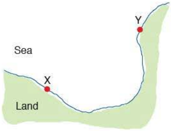
2. Draw the locus of all the points at least 2 m from each corner of the garden.

3. Draw the locus of all the points more than 3 m from the centre of the garden.

4. Draw the locus of all the points equidistant from the longer sides of the garden.

5. A port has two radar stations at P and Q which are 20 km apart. The radar at P is set to a range of 20 km, whilst the radar at Q is set to a range of 15 km.
   a) Draw a scale diagram to show the above information.
   b) Shade the region in which a ship must be sailing if it is only picked up by radar P. Label this region 'a'.
   c) Shade the region in which a ship must be sailing if it is only picked up by radar Q. Label this region 'b'.
   d) Identify the region in which a ship must be sailing if it is picked up by both radars. Label this region 'c'.

6. X and Y are two ship-to-shore radio receivers. They are 25 km apart. Y is North East of X.



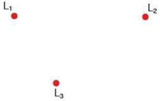
A ship sends out a distress signal. The signal is picked up by both X and Y. The radio receiver at X indicates that the ship is within a 30 km radius of X, whilst the radio receiver at Y indicates that the ship is within 20 km of Y. Draw a scale diagram and identify the region in which the ship must lie.

7. a) Mark three points L, M and N not in a straight line. By construction find the point which is equidistant from L, M and N.
   b) What would happen if L, M and N were on the same straight line?

8. Draw a line AB 8 cm long. What is the locus of a point C such that the angle ACB is always a right angle?

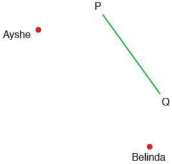
9. Draw a circle by drawing round a circular object (do not use a pair of compasses). By construction determine the position of the centre of the circle.

10. Three lionesses L_1, L_2 and L_3 have surrounded a gazelle. The three lionesses are equidistant from the gazelle.



Draw a diagram with the lionesses in similar positions to those shown (above) and by construction determine the position (G) of the gazelle.

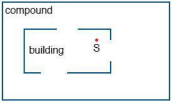
Exercise 20.2



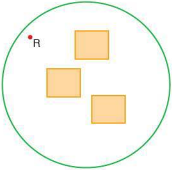
1. Three girls are playing hide and seek. Ayshe and Belinda are at the positions shown (left) and are trying to find Cristina. Cristina is on the opposite side of a wall PQ to her two friends.

   Assuming Ayshe and Belinda cannot see over the wall identify, by copying the diagram, the locus of points where Cristina could be if:
   a) Cristina can only be seen by Ayshe,
   b) Cristina can only be seen by Belinda,
   c) Cristina can not be seen by either of her two friends,
   d) Cristina can be seen by both of her friends.

2. A security guard S is inside a building in the position shown. The building is inside a rectangular compound. If the building has three windows as shown, identify the locus of points in the compound which can be seen by the security guard.



3. The circular cage shown (below) houses a snake. Inside the cage are three obstacles. A rodent is placed inside the cage at R. From where it is lying, the snake can see the rodent.

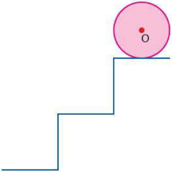


Trace the diagram and identify the regions in which the snake could be lying.

Exercise 20.3

1. A coin is rolled in a straight line on a flat surface as shown below.

Draw the locus of the centre of the coin O as the coin rolls along the surface.



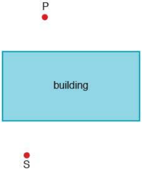
2. The diameter of the disc (left) is the same as the width and height of each of the steps shown. Copy the diagram and draw the locus of the centre of the disc as it rolls down the steps.

3. X and Y are two fixed posts in the ground. The ends of a rope are tied to X and Y. A goat is attached to the rope by a ring on its collar which enables it to move freely along the rope's length.

Copy the diagram and sketch the locus of points in which the goat is able to graze.

Student assessment 1

1. Pedro and Sara are on opposite sides of a building as shown.

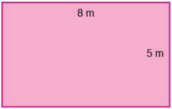


Their friend Raul is standing in a place such that he cannot be seen by either Pedro or Sara. Copy the diagram and identify the locus of points at which Raul could be standing.

2. A rectangular garden measures 10 m by 6 m. A tree stands in the centre of the garden. Grass is to be planted according to the following conditions:

- it must be at least 1 m from the edge of the garden,
- it must be more than 2 m away from the centre of the tree.

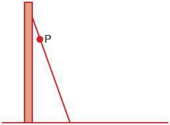
a) Make a scale drawing of the garden.
b) Draw the locus of points in which the grass can be planted.



3. A rectangular flower bed in a park measures 8 m by 5 m as shown (left).

The park keeper puts a low fence around the flower bed. The fence is at a constant distance of 2 m from the flower bed.

a) Make a scale drawing of the flower bed.
b) Draw the position of the fence.



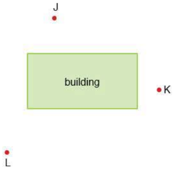
4. A ladder 10 m long is propped up against a wall as shown. (left). A point P on the ladder is 2 m from the top.

Make a scale drawing to show the locus of point P if the ladder were to slide down the wall. Note: several positions of the ladder will need to be shown.

5. The equilateral triangle PQR is rolled along the line shown (below). At first, corner Q acts as the pivot point until P reaches the line, then P acts as the pivot point until R reaches the line, and so on.

Showing your method clearly, draw the locus of point P as the triangle makes one full rotation, assuming there is no slipping.

Student assessment 2



1. Jose, Katrina and Luis are standing at different points around a building as shown (left).
   Trace the diagram and show whether any of the three friends can see each other or not.
2. A rectangular courtyard measures 20 m by 12 m. A horse is tethered in the centre with a rope 7 m long. Another horse is tethered, by a rope 5 m long, to a rail which runs along the whole of the left-hand (shorter) side of the courtyard. This rope is able to run freely along the length of the rail.
   Draw a scale diagram of the courtyard and draw the locus of points which can be reached by both horses.
3. A ball is rolling along the line shown in the diagram (right).
   Copy the diagram and draw the locus of the centre, O, of the ball as it rolls.

4. A square ABCD is 'rolled' along the flat surface shown below. Initially corner C acts as a pivot point until B touches the surface, then B acts as a pivot point until A touches the surface, and so on.

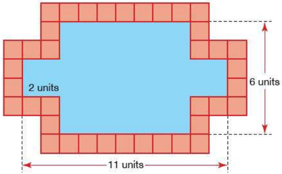
Assuming there is no slipping, draw the locus of point A as the square makes one complete rotation. Show your method clearly.

Topic 3

Mathematical investigations and ICT

Fountain borders

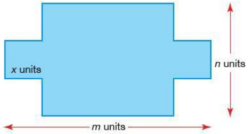
The Alhambra Palace in Granada, Spain has many fountains which pour water into pools. Many of the pools are surrounded by beautiful ceramic tiles. This investigation looks at the number of square tiles needed to surround a particular shape of pool.



The diagram above shows a rectangular pool 11 × 6 units, in which a square of dimension 2 × 2 units is taken from each corner.

The total number of unit square tiles needed to surround the pool is 38.

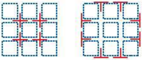
The shape of the pools can be generalised as shown below:



1. Investigate the number of unit square tiles needed for different sized pools. Record your results in an ordered table.
2. From your results write an algebraic rule in terms of m, n and x (if necessary) for the number of tiles T needed to surround a pool.
3. Justify, in words and using diagrams, why your rule works.

Tiled walls

Many cultures have used tiles to decorate buildings. Putting tiles on a wall takes skill. These days, to make sure that each tile is in the correct position, 'spacers' are used between the tiles.



You can see from the diagram that there are + shaped and T shaped spacers.

1. Draw other sized squares and rectangles, and investigate the relationship between the dimensions of the shape (length and width) and the number of + shaped and T shaped spacers.
2. Record your results in an ordered table.
3. Write an algebraic rule for the number of + shaped spacers c in a rectangle l tiles long by w tiles wide.
4. Write an algebraic rule for the number of T shaped spacers t in a rectangle l tiles long by w tiles wide.

ICT activity

In this activity, you will use a spreadsheet to calculate the sizes of interior and exterior angles of regular polygons.

Set up a spreadsheet as shown below:

	A	B	C	D	E	F
1	Regular Polygons					
2	Number of sides	Name	Sum of exterior angles	Size of an exterior angle	Size of an interior angle	Sum of interior angles
3	3					
4	4					
5	5					
6	6					
7	7					
8	8					
9	9					
10	10					
11	12		↑	↑	↑	↑
12	20					
13						
14						
15						
16			Use formulae to generate the results in these columns			

1. By using formulae, use the spreadsheet to generate the results for the sizes of the interior and exterior angles.
2. Write down the general formulae you would use to calculate the sizes of the interior and exterior angles of an n-sided regular polygon.

Topic 4

Mensuration

Syllabus

C4.1

Use current units of mass, length, area, volume and capacity in practical situations and express quantities in terms of larger or smaller units.

C4.2

Carry out calculations involving the perimeter and area of a rectangle, triangle, parallelogram and trapezium and compound shapes derived from these.

C4.3

Carry out calculations involving the circumference and area of a circle.

C4.4

Carry out calculations involving the volume of a cuboid, prism and cylinder and the surface area of a cuboid and a cylinder.

C4.5

Carry out calculations involving the areas and volumes of compound shapes.

Contents

Measurement

A measurement is the ratio of a physical quantity, such as a length, time or temperature, to a unit of measurement, such as the metre, the second or the degree Celsius. So if someone is 1.68 m tall they are 1.68 times bigger than the standard measure called a metre.

The International System of Units (or SI units from the French language name *Système International d'Unités*) is the world's most widely used system of units. The SI units for the seven basic physical quantities are:

- the metre (m) – the SI unit of length
- the kilogram (kg) – the SI unit of mass
- the second (s) – the SI unit of time
- the ampere (A) – the SI unit of electric current
- the kelvin (K) – the SI unit of temperature
- the mole (mol) – the SI unit of amount of substance
- the candela (cd) – the SI unit of luminous intensity.

This system was a development of the metric system which was first used in the 1790s during the French Revolution. This early system used just the metre and the kilogram and was intended to give fair and consistent measures in France.

21 Measures

Metric units

The metric system uses a variety of units for length, mass and capacity.

- The common units of length are: kilometre (km), metre (m), centimetre (cm) and millimetre (mm).
- The common units of mass are: tonne (t), kilogram (kg), gram (g) and milligram (mg).
- The common units of capacity are: litre (L or l) and millilitre (ml).

Note: 'centi' comes from the Latin *centum* meaning hundred (a centimetre is one hundredth of a metre);
'milli' comes from the Latin *mille* meaning thousand (a millimetre is one thousandth of a metre);
'kilo' comes from the Greek *khilloi* meaning thousand (a kilometre is one thousand metres).

It may be useful to have some practical experience of estimating lengths, volumes and capacities before starting the following exercises.

Exercise 21.1

Copy and complete the sentences below:

1. a) There are ... centimetres in one metre.
b) There are ... millimetres in one metre.
c) One metre is one ... of a kilometre.
d) There are ... kilograms in one tonne.
e) There are ... grams in one kilogram.
f) One milligram is one ... of a gram.
g) One thousand kilograms is one ....
h) One thousandth of a gram is one ....
i) One thousand millilitres is one ....
j) One thousandth of a litre is one ....

2. Which of the units below would be used to measure the following?

mm, cm, m, km, mg, g, kg, t, ml, litres

a) your height
b) the length of your finger
c) the mass of a shoe
d) the amount of liquid in a cup
e) the height of a van
f) the mass of a ship
g) the capacity of a swimming pool
h) the length of a highway
i) the mass of an elephant
j) the capacity of the petrol tank of a car

3. Use a ruler to draw lines of the following lengths:
   a) 6 cm
   b) 18 cm
   c) 41 mm
   d) 8.7 cm
   e) 67 mm

4. Draw four lines in your exercise book and label them A, B, C and D.
   a) Estimate their lengths in mm.
   b) Measure them to the nearest mm.

5. Copy the sentences below and put in the correct unit:
   a) A tree in the school grounds is 28 . . . tall.
   b) The distance to the nearest big city is 45 . . . .
   c) The depth of a lake is 18 . . . .
   d) A woman's mass is about 60 . . . .
   e) The capacity of a bowl is 5 . . . .
   f) The distance Ahmet can run in 10 seconds is about 70 . . . .
   g) The mass of my car is about 1.2 . . . .
   h) Ayse walks about 1700 . . . to school.
   i) A melon has a mass of 650 . . . .
   j) The amount of blood in your body is 5 . . . .

Converting from one unit to another

Length

1 km = 1000 m

Therefore $1\text{ m} = \frac{1}{1000}\text{ km}$

1 m = 1000 mm

Therefore $1\text{ mm} = \frac{1}{1000}\text{ m}$

1 m = 100 cm

Therefore $1\text{ cm} = \frac{1}{100}\text{ m}$

1 cm = 10 mm

Therefore $1\text{ mm} = \frac{1}{10}\text{ cm}$

Worked examples

a) Change 5.8 km into m.
Since 1 km = 1000 m,
5.8 km is 5.8 × 1000 m

5.8 km = 5800 m

b) Change 4700 mm to m.
Since 1 m is 1000 mm,
4700 mm is 4700 ÷ 1000 m

4700 mm = 4.7 m

c) Convert 2.3 km into cm.
2.3 km is 2.3×1000 m = 2300 m
2300 m is 2300×100 cm

2.3 km = 230 000 cm

Exercise 21.2

1. Put in the missing unit to make the following statements correct:

a) 3 cm = 30 . . .
b) 25 . . . = 2.5 cm
c) 3200 m = 3.2 . . .
d) 7.5 km = 7500 . . .
e) 300 . . . = 30 cm
f) 6000 mm = 6 . . .
g) 3.2 m = 3200 . . .
h) 4.2 . . . = 4200 mm
i) 1 million mm = 1 . . .
j) 2.5 km = 2500 . . .

2. Convert the following to millimetres:

a) 2 cm
b) 8.5 cm
c) 23 cm
d) 1.2 m
e) 0.83 m
f) 0.05 m
g) 62.5 cm
h) 0.087 m
i) 0.004 m
j) 2 m

3. Convert the following to metres:

a) 3 km
b) 4700 mm
c) 560 cm
d) 6.4 km
e) 0.8 km
f) 96 cm
g) 62.5 cm
h) 0.087 km
i) 0.004 km
j) 12 mm

4. Convert the following to kilometres:

a) 5000 m
b) 6300 m
c) 1150 m
d) 2535 m
e) 250 000 m
f) 500 m
g) 70 m
h) 8 m
i) 1 million m
j) 700 million m

Mass

1 tonne is 1000 kg

Therefore 1 kg = $\frac{1}{1000}$ tonne

1 kilogram is 1000 g

Therefore 1 g = $\frac{1}{1000}$ kg

1 g is 1000 mg

Therefore 1 mg = $\frac{1}{1000}$ g

Worked examples

a) Convert 8300 kg to tonnes.
Since 1000 kg = 1 tonne, 8300 kg is $8300 \div 1000$ tonnes

8300 kg = 8.3 tonnes

b) Convert 2.5 g to mg.

Since 1 g is 1000 mg, 2.5 g is 2.5×1000 mg

2.5 g = 2500 mg

Exercise 21.3

1. Convert the following:
   a) 3.8 g to mg
   b) 28 500 kg to tonnes
   c) 4.28 tonnes to kg
   d) 320 mg to g
   e) 0.5 tonnes to kg

2. One item has a mass of 630 g, another item has a mass of 720 g. Express the total mass in kg.

3. a) Express the total of the following in kg:

   1.2 tonne, 760 kg, 0.93 tonne, 640 kg

   b) Express the total of the following in g:

   460 mg, 1.3 g, 1260 mg, 0.75 g

   c) A cat weighs 2800 g and a dog weighs 6.5 kg. What is the total weight in kg of the two animals?
   d) In one bag of shopping Imran has items of total mass 1350 g. In another bag there are items of total mass 3.8 kg. What is the mass in kg of both bags of shopping?
   e) What is the total mass in kg of the fruit listed below?

   apples 3.8 kg, peaches 1400 g, bananas 0.5 kg, oranges 7500 g, grapes 0.8 kg

Capacity

1 litre is 1000 millilitres

Therefore 1 ml $= \frac{1}{1000}$ litre

Exercise 21.4

1. Convert the following to litres:
   a) 8400 ml
   b) 650 ml
   c) 87 500 ml
   d) 50 ml
   e) 2500 ml

2. Convert the following to ml:
   a) 3.2 litres
   b) 0.75 litre
   c) 0.087 litre
   d) 8 litres
   e) 0.008 litre
   f) 0.3 litre

3. Calculate the following and give the totals in ml:
   a) 3 litres + 1500 ml
   b) 0.88 litre + 650 ml
   c) 0.75 litre + 6300 ml
   d) 450 ml + 0.55 litre

4. Calculate the following and give the totals in litres:
   a) 0.75 litre + 450 ml
   b) 850 ml + 490 ml
   c) 0.6 litre + 0.8 litre
   d) 80 ml + 620 ml + 0.7 litre

Student assessment 1

1. Convert the following lengths into the units indicated:

a) 2.6 cm to mm
b) 62.5 cm to mm
c) 0.88 m to cm
d) 0.007 m to mm
e) 4800 mm to m
f) 7.81 km to m
g) 6800 m to km
h) 0.875 km to m
i) 2 m to mm
j) 0.085 m to mm

2. Convert the following masses into the units indicated:

a) 4.2 g to mg
b) 750 mg to g
c) 3940 g to kg
d) 4.1 kg to g
e) 0.72 tonnes to kg
f) 4100 kg to tonnes
g) 6 280 000 mg to kg
h) 0.83 tonnes to g
i) 47 million kg to tonnes
j) 1 kg to mg

3. Add the following masses, giving your answer in kg:

3.1 tonnes, 4860 kg, 0.37 tonnes

4. Convert the following liquid measures into the units indicated:

a) 1800 ml to litres
b) 3.2 litres to ml
c) 0.083 litre to ml
d) 250 000 ml to litres

Student assessment 2

1. Convert the following lengths into the units indicated:

a) 4.7 cm to mm
b) 0.003 m to mm
c) 3100 mm to cm
d) 6.4 km to m
e) 49 000 m to km
f) 4 m to mm
g) 0.4 cm to mm
h) 0.034 m to mm
i) 460 mm to cm
j) 50 000 m to km

2. Convert the following masses into the units indicated:

a) 3.6 mg to g
b) 550 mg to g
c) 6500 g to kg
d) 6.7 kg to g
e) 0.37 tonnes to kg
f) 1510 kg to tonnes
g) 380 000 kg to tonnes
h) 0.077 kg to g
i) 6 million mg to kg
j) 2 kg to mg

3. Subtract 1570 kg from 2 tonnes.

4. Convert the following measures of capacity to the units indicated:

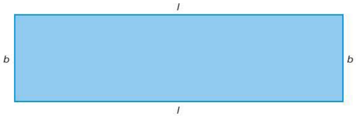
a) 3400 ml to litres
b) 6.7 litres to ml
c) 0.73 litre to ml
d) 300 000 ml to litres

22 Perimeter, area and volume

NB: All diagrams are not drawn to scale.

The perimeter and area of a rectangle

The **perimeter** of a shape is the distance around the outside of the shape. Perimeter can be measured in mm, cm, m, km, etc.



The perimeter of the rectangle above of length l and breadth b is therefore:

$$\text{Perimeter} = l + b + l + b$$

This can be rearranged to give:

$$\text{Perimeter} = 2l + 2b$$

This in turn can be factorised to give:

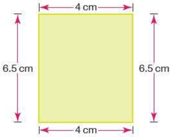
$$\text{Perimeter} = 2(l + b)$$

The **area** of a shape is the amount of surface that it covers. Area is measured in mm^2, cm^2, m^2, km^2, etc.

The area A of the rectangle above is given by the formula:

$$A = lb$$

Worked example Find the area of the rectangle below.



$A = lb$
$A = 6.5 \times 4$
$A = 26$
Area is $26\ cm^2$.

Exercise 22.1

Calculate the area and perimeter of the rectangles described below.

	Length	Breadth	Area	Perimeter
a)	6 cm	4 cm		
b)	5 cm	9 cm		
c)	4.5 cm	6 cm		
d)	3.8 m	10 m		
e)	5 m	4.2 m		
f)	3.75 cm	6 cm		
g)	3.2 cm	4.7 cm		
h)	18.7 m	5.5 cm		
i)	85 cm	1.2 m		
j)	3.3 m	75 cm		

Worked example Calculate the breadth of a rectangle of area 200 cm^2 and length 25 cm.

$A = lb$

$200 = 25b$

$b = 8$

So the breadth is 8 cm.

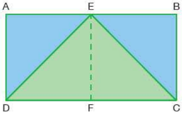
Exercise 22.2

1. Use the formula for the area of a rectangle to find the value of A, l or b as indicated in the table below.

	Length	Breadth	Area
a)	8.5 cm	7.2 cm	A cm^2
b)	25 cm	b cm	250 cm^2
c)	l cm	25 cm	400 cm^2
d)	7.8 m	b m	78 m^2
e)	l cm	8.5 cm	102 cm^2
f)	22 cm	b cm	330 cm^2
g)	l cm	7.5 cm	187.5 cm^2

2. Find the area and perimeter of each of the following squares or rectangles:
   a) the floor of a room which is 8 m long by 7.5 m wide
   b) a stamp which is 35 mm long by 25 mm wide
   c) a wall which is 8.2 m long by 2.5 m high
   d) a field which is 130 m long by 85 m wide
   e) a chessboard of side 45 cm
   f) a book which is 25 cm wide by 35 cm long
   g) an airport runway which is 3.5 km long by 800 m wide
   h) a street which is 1.2 km long by 25 m wide
   i) a sports hall 65 m long by 45 m wide
   j) a tile which is a square of side 125 mm

The area of a triangle



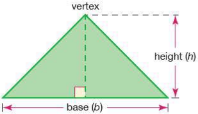
Rectangle ABCD has a triangle CDE drawn inside it.

Point E is said to be a **vertex** of the triangle.
EF is the **height** or **altitude** of the triangle.
CD is the **length** of the rectangle, but is called the **base** of the triangle.

It can be seen from the diagram that triangle DEF is half the area of the rectangle AEFD.

Also triangle CFE is half the area of rectangle EBCF.

It follows that **triangle CDE is half the area of rectangle ABCD**.



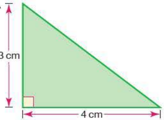
Area of a triangle $A = \frac{1}{2}bh$, where b is the base and h is the height.

Note: it does not matter which side is called the base, but the height must be measured at right angles from the base to the opposite vertex.

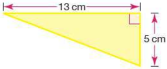
Exercise 22.3

Calculate the areas of the triangles below:

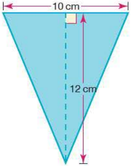
1.



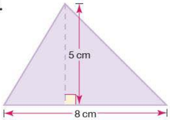
2.



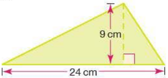
3.



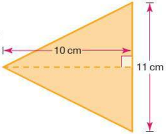
4.



5.



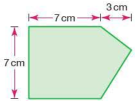
6.



Compound shapes

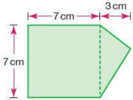
Sometimes being asked to work out the perimeter and area of a shape seems difficult. However, the calculations can often be made easier by splitting it up into simpler shapes. A shape that can be split into simpler ones is known as a compound shape.

Worked example The diagram below shows a pentagon and its dimensions.



Calculate the area of the shape.

The area of the pentagon is easier to calculate if it is split into two simpler shapes. In this case, a square and a triangle, as shown.



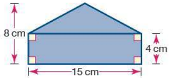
The area of the square is $7 \times 7 = 49\,\text{cm}^2$.

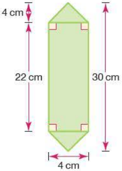
The area of the triangle is $\frac{1}{2} \times 7 \times 3 = 10.5\,\text{cm}^2$.

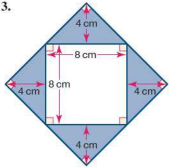
Therefore the total area of the pentagon is $49 + 10.5 = 59.5\,\text{cm}^2$.

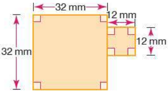
Exercise 22.4

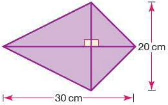
Calculate the areas of the shapes below:

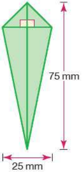
1. 

2. 

3. 

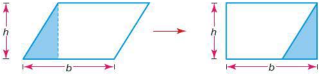
4. 

5. 

6. 

The area of a parallelogram and a trapezium

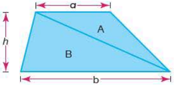
A **parallelogram** can be rearranged to form a rectangle in the following way:



Therefore:

area of parallelogram = base length × perpendicular height.

A **trapezium** can be visualised as being split into two triangles as shown on the left:



Area of triangle A $= \frac{1}{2} \times a \times h$

Area of triangle B $= \frac{1}{2} \times b \times h$

Area of the trapezium

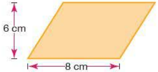
$= $ area of triangle A + area of triangle B

$= \frac{1}{2}ah + \frac{1}{2}bh$

$= \frac{1}{2}h(a + b)$

Worked examples

a) Calculate the area of the parallelogram (left):

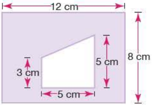


Area = base length × perpendicular height

$= 8 \times 6$

$= 48\ \text{cm}^2$

b) Calculate the shaded area in the shape (left):



Area of rectangle $= 12 \times 8$

$= 96\ \text{cm}^2$

Area of trapezium $= \frac{1}{2} \times 5(3 + 5)$

$= 2.5 \times 8$

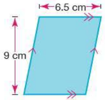
$= 20\ \text{cm}^2$

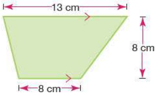
Shaded area $= 96 - 20$

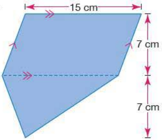
$= 76\ \text{cm}^2$

Exercise 22.5

Find the area of each of the following shapes:

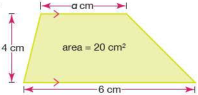
1.


2.


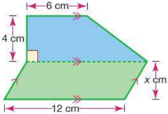
3.


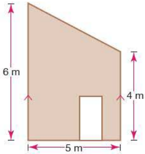
Exercise 22.6

1. Calculate the value of *a*.

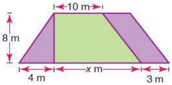


2. If the areas of the trapezium and parallelogram in the diagram (below) are equal, calculate the value of *x*.



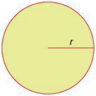


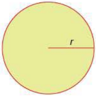
3. The end view of a house is as shown in the diagram (left). If the door has a width and height of 0.75 m and 2 m, respectively, calculate the area of brickwork.



4. A garden in the shape of a trapezium (left) is split into three parts: flower beds in the shape of a triangle and a parallelogram, and a section of grass in the shape of a trapezium. The area of the grass is two and a half times the total area of flower beds. Calculate:
   a) the area of each flower bed,
   b) the area of grass,
   c) the value of *x*.

The circumference and area of a circle





The circumference is $2\pi r$.

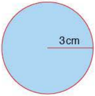
$C = 2\pi r$

The area is πr^2.

$A = \pi r^2$

Worked examples

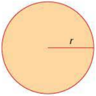
a) Calculate the circumference of this circle, giving your answer to 3 s.f.



$$\begin{aligned} C &= 2\pi r \\ &= 2\pi \times 3 \\ &= 18.8 \end{aligned}$$

The circumference is 18.8 cm.

b) If the circumference of this circle is 12 cm, calculate the radius, giving your answer to 3 s.f.



$$C = 2\pi r$$

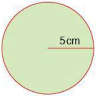
$$r = \frac{C}{2\pi}$$

$$r = \frac{12}{2\pi}$$

$$= 1.91$$

The radius is 1.91 cm.

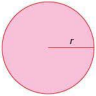
c) Calculate the area of this circle, giving your answer to 3 s.f.



$$\begin{aligned} A &= \pi r^2 \\ &= \pi \times 5^2 \\ &= 78.5 \end{aligned}$$

The area is 78.5 cm^2.

d) If the area of this circle is 34 cm^2, calculate the radius, giving your answer to 3 s.f.



$$A = \pi r^2$$

$$r = \sqrt{\frac{A}{\pi}}$$

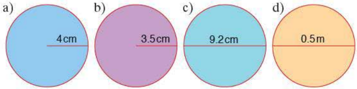
$$r = \sqrt{\frac{34}{\pi}}$$

$$= 3.29$$

The radius is 3.29 cm.

Exercise 22.7

1. Calculate the circumference of each circle below, giving your answers to 2 d.p.



2. Calculate the area of each of the circles in Q.1. Give your answers to 2 d.p.

3. Calculate the radius of a circle when the circumference is:
   a) 15 cm b) π cm
   c) 4 m d) 8 mm

4. Calculate the diameter of a circle when the area is:
   a) 16 cm^2 b) 9π cm^2
   c) 8.2 m^2 d) 14.6 mm^2

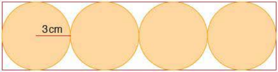
Exercise 22.8

1. The wheel of a car has an outer radius of 25 cm. Calculate:
   a) how far the car has travelled after one complete turn of the wheel,
   b) how many times the wheel turns for a journey of 1 km.

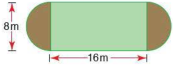
2. If the wheel of a bicycle has a diameter of 60 cm, calculate how far a cyclist will have travelled after the wheel has rotated 100 times.

3. A circular ring has a cross-section as shown (left). If the outer radius is 22 mm and the inner radius 20 mm, calculate the cross-sectional area of the ring.

4. Four circles are drawn in a line and enclosed by a rectangle as shown.



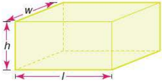
If the radius of each circle is 3 cm, calculate:
   a) the area of the rectangle,
   b) the area of each circle,
   c) the unshaded area within the rectangle.



5. A garden is made up of a rectangular patch of grass and two semi-circular vegetable patches. If the length and width of the rectangular patch are 16 m and 8 m respectively, calculate:
   a) the perimeter of the garden,
   b) the total area of the garden.

The surface area of a cuboid and a cylinder

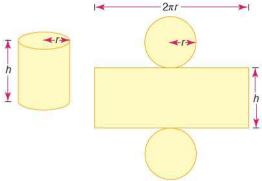
To calculate the surface area of a **cuboid** start by looking at its individual faces. These are either squares or rectangles. The surface area of a cuboid is the sum of the areas of its faces.



Area of top $= wl$ Area of bottom $= wl$
Area of front $= lh$ Area of back $= lh$
Area of one side $= wh$ Area of other side $= wh$

Total surface area
$= 2wl + 2lh + 2wh$
$= 2(wl + lh + wh)$

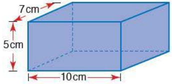
For the surface area of a **cylinder** it is best to visualise the net of the solid: it is made up of one rectangular piece and two circular pieces.



Area of circular pieces $= 2 \times \pi r^2$
Area of rectangular piece $= 2\pi r \times h$
Total surface area $= 2\pi r^2 + 2\pi rh$
$= 2\pi r(r + h)$

Worked examples

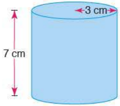
a) Calculate the surface area of the cuboid shown (left).



Total area of top and bottom $= 2 \times 7 \times 10 = 140\ \text{cm}^2$
Total area of front and back $= 2 \times 5 \times 10 = 100\ \text{cm}^2$
Total area of both sides $= 2 \times 5 \times 7 = 70\ \text{cm}^2$

Total surface area $= 140 + 100 + 70$
$= 310\ \text{cm}^2$

b) If the height of a cylinder is 7 cm and the radius of its circular top is 3 cm, calculate its surface area.



Total surface area $= 2\pi r(r + h)$
$= 2\pi \times 3 \times (3 + 7)$
$= 6\pi \times 10$
$= 60\pi$
$= 188.50\ \text{cm}^2$ (2 d.p.)

The total surface area is 188.50 cm^2.

Exercise 22.9

1. Calculate the surface area of each of the following cuboids:
   a) $l = 12$ cm, $w = 10$ cm, $h = 5$ cm
   b) $l = 4$ cm, $w = 6$ cm, $h = 8$ cm
   c) $l = 4.2$ cm, $w = 7.1$ cm, $h = 3.9$ cm
   d) $l = 5.2$ cm, $w = 2.1$ cm, $h = 0.8$ cm

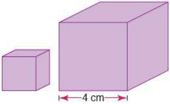
2. Calculate the height of each of the following cuboids:
   a) $l = 5$ cm, $w = 6$ cm, surface area $= 104\text{ cm}^2$
   b) $l = 2$ cm, $w = 8$ cm, surface area $= 112\text{ cm}^2$
   c) $l = 3.5$ cm, $w = 4$ cm, surface area $= 118\text{ cm}^2$
   d) $l = 4.2$ cm, $w = 10$ cm, surface area $= 226\text{ cm}^2$

3. Calculate the surface area of each of the following cylinders:
   a) $r = 2$ cm, $h = 6$ cm b) $r = 4$ cm, $h = 7$ cm
   c) $r = 3.5$ cm, $h = 9.2$ cm d) $r = 0.8$ cm, $h = 4.3$ cm

4. Calculate the height of each of the following cylinders. Give your answers to 1 d.p.
   a) $r = 2.0$ cm, surface area $= 40\text{ cm}^2$
   b) $r = 3.5$ cm, surface area $= 88\text{ cm}^2$
   c) $r = 5.5$ cm, surface area $= 250\text{ cm}^2$
   d) $r = 3.0$ cm, surface area $= 189\text{ cm}^2$

Exercise 22.10

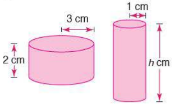
1. Two cubes (below) are placed next to each other. The length of each of the edges of the larger cube is 4 cm.



   If the ratio of their surface areas is 1 : 4, calculate:
   a) the surface area of the small cube,
   b) the length of an edge of the small cube.

2. A cube and a cylinder have the same surface area. If the cube has an edge length of 6 cm and the cylinder a radius of 2 cm, calculate:
   a) the surface area of the cube,
   b) the height of the cylinder.

3. Two cylinders (below) have the same surface area. The shorter of the two has a radius of 3 cm and a height of 2 cm, and the taller cylinder has a radius of 1 cm.



Calculate:

a) the surface area of one of the cylinders,

b) the height of the taller cylinder.

4. Two cuboids have the same surface area. The dimensions of one of them are: length = 3 cm, width = 4 cm and height = 2 cm.

Calculate the height of the other cuboid if its length is 1 cm and its width is 4 cm.

The volume of a prism

A **prism** is any three-dimensional object which has a constant cross-sectional area.

Below are a few examples of some of the more common types of prism:

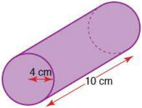
Rectangular prism (cuboid) Circular prism (cylinder) Triangular prism

When each of the shapes is cut parallel to the shaded face, the cross-section is constant and the shape is therefore classified as a prism.

Volume of a prism = area of cross-section × length

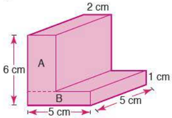
Worked examples

a) Calculate the volume of the cylinder shown in the diagram (below).



$$\text{Volume} = \text{cross-sectional area} \times \text{length}$$
$$= \pi \times 4^2 \times 10$$
$$\text{Volume} = 502.7\text{ cm}^3 \text{ (1 d.p.)}$$

b) Calculate the volume of the 'L' shaped prism shown in the diagram (below).



The cross-sectional area can be split into two rectangles:

Area of rectangle A $= 5 \times 2$
$= 10\text{ cm}^2$

Area of rectangle B $= 5 \times 1$
$= 5\text{ cm}^2$

Total cross-sectional area $= 10\text{ cm}^2 + 5\text{ cm}^2 = 15\text{ cm}^2$

Volume of prism $= 15 \times 5$
$= 75\text{ cm}^3$

Exercise 22.11

1. Calculate the volume of each of the following cuboids, where w, l and h represent the width, length and height, respectively.

a) $w = 2$ cm, $l = 3$ cm, $h = 4$ cm
b) $w = 6$ cm, $l = 1$ cm, $h = 3$ cm
c) $w = 6$ cm, $l = 23$ mm, $h = 2$ cm
d) $w = 42$ mm, $l = 3$ cm, $h = 0.007$ m

2. Calculate the volume of each of the following cylinders where r represents the radius of the circular face and h the height of the cylinder.

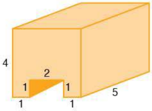
a) $r = 4$ cm, $h = 9$ cm
b) $r = 3.5$ cm, $h = 7.2$ cm
c) $r = 25$ mm, $h = 10$ cm
d) $r = 0.3$ cm, $h = 17$ mm

3. Calculate the volume of each of the following triangular prisms where b represents the base length of the triangular face, h its perpendicular height and l the length of the prism.

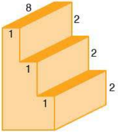
a) $b = 6$ cm, $h = 3$ cm, $l = 12$ cm
b) $b = 4$ cm, $h = 7$ cm, $l = 10$ cm
c) $b = 5$ cm, $h = 24$ mm, $l = 7$ cm
d) $b = 62$ mm, $h = 2$ cm, $l = 0.01$ m

4. Calculate the volume of each of the following prisms. All dimensions are given in centimetres.

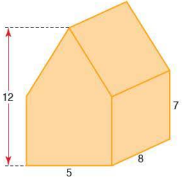
a)



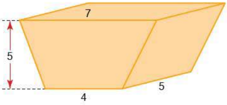
b)



c)

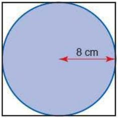


d)



Exercise 22.12

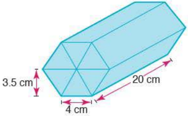
1. The diagram (below) shows a plan view of a cylinder inside a box the shape of a cube.



If the radius of the cylinder is 8 cm, calculate:

a) the height of the cube,
b) the volume of the cube,
c) the volume of the cylinder,
d) the percentage volume of the cube not occupied by the cylinder.

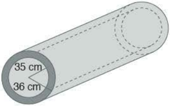
2. A chocolate bar is made in the shape of a triangular prism. The triangular face of the prism is equilateral and has an edge length of 4 cm and a perpendicular height of 3.5 cm. The manufacturer also sells these in special packs of six bars arranged as a hexagonal prism.



If the prisms are 20 cm long, calculate:
a) the cross-sectional area of the pack,
b) the volume of the pack.

3. A cuboid and a cylinder have the same volume. The radius and height of the cylinder are 2.5 cm and 8 cm, respectively. If the length and width of the cuboid are each 5 cm, calculate its height to 1 d.p.

4. A section of steel pipe is shown in the diagram below. The inner radius is 35 cm and the outer radius 36 cm. Calculate the volume of steel used in making the pipe if it has a length of 130 m.



Student assessment 1

1. A swimming pool is 50 m long by 20 m wide. Find:
a) the surface area,
b) the perimeter of the pool.
Square tiles of side 25 cm are to be placed around the edge of the pool.
c) How many tiles will it take to fit around it?

2. A floor measures 8 m by 6 m and is to be covered in square tiles of side 50 cm. How many tiles are needed?

3. A carpet is 2 m 40 cm by 3 m 80 cm. Calculate its area and perimeter.

4. A set square has a base of 40 cm and a height of 25 cm. Calculate its area.

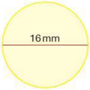
5. A drawing of a rocket shows a rectangle 70 cm by 50 cm with a triangular nose cone of height 8 cm. Calculate the area of the drawing of the rocket.

Student assessment 2

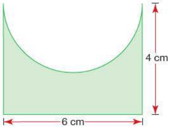
1. Calculate the circumference and area of each of the following circles. Give your answers to 1 d.p.

a)

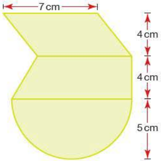
b)



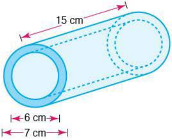
2. A semi-circular shape is cut out of the side of a rectangle as shown. Calculate the shaded area to 1 d.p.



3. For the shape shown in the diagram (right), calculate the area of:
a) the semi-circle,
b) the trapezium,
c) the whole shape.

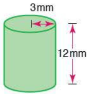


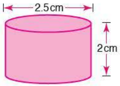
4. A cylindrical tube has an inner diameter of 6 cm, an outer diameter of 7 cm and a length of 15 cm.



Calculate the following to 1 d.p.:
a) the surface area of the shaded end,
b) the inside surface area of the tube,
c) the total surface area of the tube.

5. Calculate the volume of each of the following cylinders:

a)


b)


Student assessment 3

1. A rowing lake, rectangular in shape, is 2.5 km long by 500 m wide. Calculate the surface area of the water in km^2.

2. A rectangular floor 12 m long by 8 m wide is to be covered in ceramic tiles 40 cm long by 20 cm wide.
   a) Calculate the number of tiles required to cover the floor.
   b) The tiles are bought in boxes of 24 at a cost of $70 per box. What is the cost of the tiles needed to cover the floor?

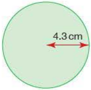
3. A flower bed is in the shape of a right-angled triangle of sides 3 m, 4 m and 5 m. Sketch the flower bed, and calculate its area and perimeter.

4. A drawing of a building shows a rectangle 50 cm high and 10 cm wide with a triangular tower 20 cm high and 10 cm wide at the base on top of it. Find the area of the drawing of the building.

5. The squares of a chessboard are each of side 7.5 cm. What is the area of the chessboard?

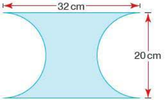
Student assessment 4

1. Calculate the circumference and area of each of the following circles. Give your answers to 1 d.p.

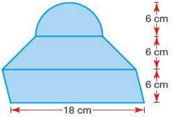
a) 

b) 

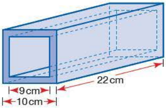
2. A rectangle of length 32 cm and width 20 cm has a semi-circle cut out of two of its sides as shown (right). Calculate the shaded area to 1 d.p.



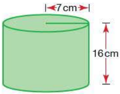
3. In the diagram (right), calculate the area of:
   a) the semi-circle,
   b) the parallelogram,
   c) the whole shape.

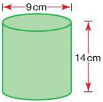


4. A prism in the shape of a hollowed-out cuboid has dimensions as shown below. If the end is square, calculate the volume of the prism.



5. Calculate the surface area of each of the following cylinders:

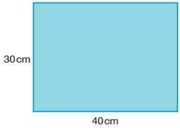
a) 

b) 

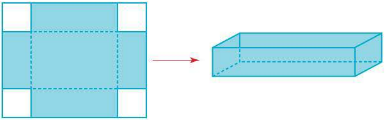
Mathematical investigations and ICT

Metal trays

A rectangular sheet of metal measures 30 cm by 40 cm.



The sheet has squares of equal size cut from each corner. It is then folded to form a metal tray as shown.



1. a) Calculate the length, width and height of the tray if a square of side length 1 cm is cut from each corner of the sheet of metal.
b) Calculate the volume of this tray.

2. a) Calculate the length, width and height of the tray if a square of side length 2 cm is cut from each corner of the sheet of metal.
b) Calculate the volume of this tray.

3. Using a spreadsheet if necessary, investigate the relationship between the volume of the tray and the size of the square cut from each corner. Enter your results in an ordered table.

4. Calculate, to 1 d.p., the side length of the square that produces the tray with the greatest volume.

5. State the greatest volume to the nearest whole number.

Topic 5

Coordinate geometry

Syllabus

C5.1

Demonstrate familiarity with Cartesian co-ordinates in two dimensions.

C5.2

Find the gradient of a straight line.

C5.3

Extended curriculum only.

C5.4

Interpret and obtain the equation of a straight line graph in the form $y = mx + c$.

C5.5

Determine the equation of a straight line parallel to a given line.

C5.6

Extended curriculum only.

Contents

Position fixing

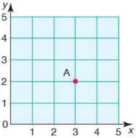
On a flat surface two points at right angles will fix a position. These two points are called coordinates. On the surface of the Earth, position is fixed using latitude and longitude. Latitude is the distance from the equator. Longitude is measured at right angles to latitude.

In the early 18th century, however, only latitude could be calculated. There were several disasters caused by errors in determining a position at sea. One such disaster was the loss of 1400 lives and four ships of the English fleet of Sir Cloudesley Shovell in 1707. In 1714 the British government established the Board of Longitude and a large cash prize was offered for a practical method of finding the longitude of a ship at sea.

John Harrison, a self-educated English clockmaker, then invented the marine chronometer. This instrument allowed longitude to be calculated. It worked by comparing local noon with the time at a place given the longitude zero. Each hour was $360 \div 24 = 15$ degrees of longitude. Unlike latitude, which has the equator as a natural starting position, there is no natural starting position for longitude. The Greenwich meridian in London was chosen.

John Harrison (1693–1776)

23 Coordinate geometry



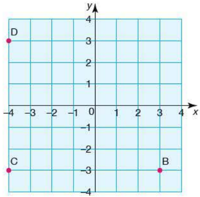
Coordinates

To fix a point in two dimensions (2D), its position is given in relation to a point called the **origin**. Through the origin, axes are drawn perpendicular to each other. The horizontal axis is known as the ***x*-axis**, and the vertical axis is known as the ***y*-axis**.

The x-axis is numbered from left to right. The y-axis is numbered from bottom to top.

The position of point A is given by two coordinates: the x-coordinate first, followed by the y-coordinate. So the coordinates of point A are (3, 2).

A number line can extend in both directions by extending the x- and y-axes below zero, as shown in the grid below:



Points B, C, and D can be described by their coordinates:

Point B is at (3, –3)
Point C is at (–4, –3)
Point D is at (–4, 3)

Exercise 23.1

1. Draw a pair of axes with both x and y from –8 to +8. Mark each of the following points on your grid:

   a) A = (5, 2) b) B = (7, 3) c) C = (2, 4)
   d) D = (–8, 5) e) E = (–5, –8) f) F = (3, –7)
   g) G = (7, –3) h) H = (6, –6)

Draw a separate grid for each of Q.2–4 with x- and y-axes from –6 to +6. Plot and join the points in order to name each shape drawn.

2. A = (3, 2) B = (3, –4) C = (–2, –4) D = (–2, 2)
3. E = (1, 3) F = (4, –5) G = (–2, –5)
4. H = (–6, 4) I = (0, –4) J = (4, –2) K = (–2, 6)

Exercise 23.2

Draw a pair of axes with both x and y from −10 to +10.

1. Plot the points P = (−6, 4), Q = (6, 4) and R = (8, −2).

   Plot point S such that PQRS when drawn is a parallelogram.

   a) Draw diagonals PR and QS. What are the coordinates of their point of intersection?
   b) What is the area of PQRS?

2. On the same axes, plot point M at (−8, 4) and point N at (4, 4).
   a) Join points MNRS. What shape is formed?
   b) What is the area of MNRS?
   c) Explain your answer to Q.2(b).

3. On the same axes, plot point J where point J has y-coordinate +10 and JRS, when joined, forms an isosceles triangle.
   What is the x-coordinate of all points on the line of symmetry of triangle JRS?

Exercise 23.3

1. a) On a grid with axes numbered from −10 to +10 draw a hexagon ABCDEF with centre (0, 0), points A(0, 8) and B(7, 4) and two lines of symmetry.
   b) Write down the coordinates of points C, D, E and F.

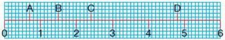
2. a) On a similar grid to Q.1, draw an octagon PQRSTUVW which has point P(2, −8), point Q(−6, −8) and point R(−7, −5).
   PQ = RS = TU = VW and QR = ST = UV = WP.
   b) List the coordinates of points S, T, U, V, and W.
   c) What are the coordinates of the centre of rotational symmetry of the octagon?

Reading scales

Exercise 23.4

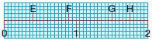
1. The points A, B, C and D are not at whole number points on the number line. Point A is at 0.7

   What are the positions of points B, C and D?

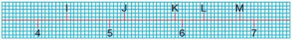


2. On this number line point E is at 0.4 (2 small squares represents 0.1)

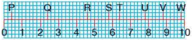
   What are the positions of points F, G and H?



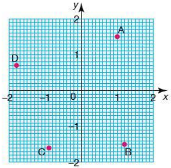
3. What are the positions of points I, J, K, L and M? (Each small square is 0.05, i.e. 2 squares is 0.1)

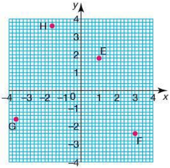


4. Point P is at position 0.4 and point W is at position 9.8 (Each small square is 0.2)
What are the positions of points Q, R, S, T, U, and V?



Exercise 23.5 Give the coordinates of points A, B, C, D, E, F, G and H.

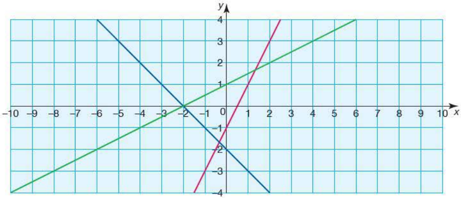




The gradient of a straight line

Lines are made of an infinite number of points. This chapter looks at those whose points form a straight line.

The graph below shows three straight lines.

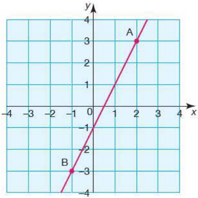


The lines have some properties in common (i.e. they are straight), but also have differences. One of their differences is that they have different slopes. The slope of a line is called its **gradient**.

Gradient

The gradient of a straight line is constant, i.e. it does not change. The gradient of a straight line can be calculated by considering the coordinates of any two points on the line.

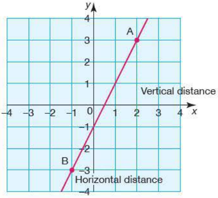
On the line below two points A and B have been chosen.



The coordinates of the points are A(2, 3) and B(−1, −3). The gradient is calculated using the following formula:

$$\text{Gradient} = \frac{\text{vertical distance between two points}}{\text{horizontal distance between two points}}$$

Graphically this can be represented as follows:

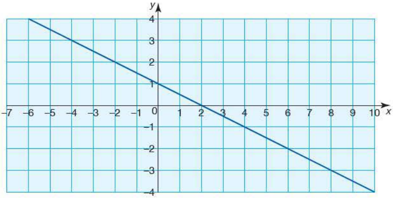


$$\text{Therefore gradient} = \frac{3-(-3)}{2-(-1)} = \frac{6}{3} = 2$$

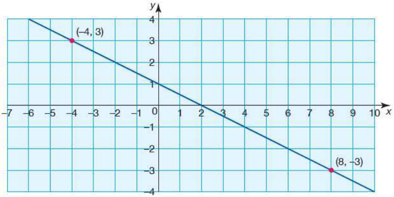
In general, therefore, if the two points chosen have coordinates (x_1, y_1) and (x_2, y_2) the gradient is calculated as:

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

Worked example Calculate the gradient of the line shown below.



Choose two points on the line, e.g. (–4, 3) and (8, –3).



Let point 1 be (–4, 3) and point 2 be (8, –3).

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-3}{8-(-4)}$$

$$= \frac{-6}{12} = -\frac{1}{2}$$

Note: The gradient is not affected by which point is chosen as point 1 and which is chosen as point 2. In the example above, if point 1 was $(8, -3)$ and point 2 was $(-4, 3)$, the gradient would be calculated as:

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-(-3)}{-4-8}$$

$$= \frac{6}{-12} = -\frac{1}{2}$$

To check whether or not the sign of the gradient is correct, the following guideline is useful.

A line sloping this way will have a positive gradient.

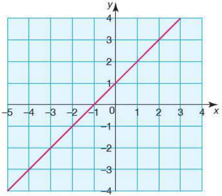
A line sloping this way will have a negative gradient.

A large value for the gradient implies that the line is steep. The line on the right below will have a greater value for the gradient than the line on the left, as it is steeper.

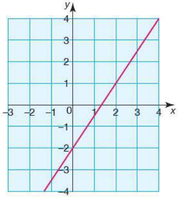
Exercise 23.6

1. For each of the following lines, select two points on the line and then calculate its gradient.

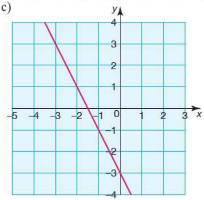
a)



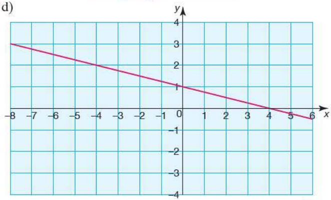
b)



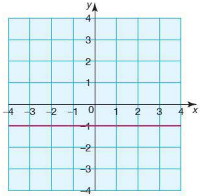
c)



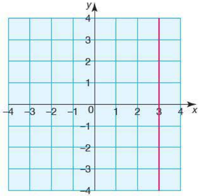
d)



e)



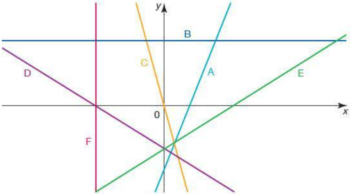
f)



2. From your answers to Q.1, what conclusion can you make about the gradient of any horizontal line?

3. From your answers to Q.1, what conclusion can you make about the gradient of any vertical line?

4. The graph below shows six straight lines labelled A–F.



Six gradients are given below. Deduce which line has which gradient.

Gradient = $\frac{1}{2}$ Gradient is infinite Gradient = 2

Gradient = -3 Gradient = 0 Gradient = $-\frac{1}{2}$

The equation of a straight line

The coordinates of every point on a straight line all have a common relationship. This relationship when expressed algebraically as an equation in terms of x and/or y is known as the equation of the straight line.

Worked examples

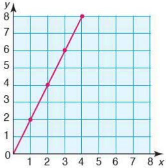
a) By looking at the coordinates of some of the points on the line below, establish the equation of the straight line.

x	y
1	4
2	4
3	4
4	4
5	4
6	4

Some of the points on the line have been identified and their coordinates entered in a table above. By looking at the table it can be seen that the only rule all the points have in common is that $y = 4$.

Hence the equation of the straight line is $y = 4$.

b) By looking at the coordinates of some of the points on the line (below), establish the equation of the straight line.



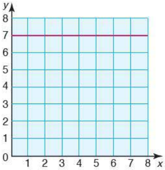
x	y
1	2
2	4
3	6
4	8

Once again, by looking at the table it can be seen that the relationship between the x- and y-coordinates is that each y-coordinate is twice the corresponding x-coordinate.

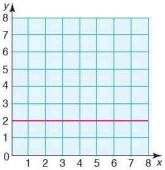
Hence the equation of the straight line is $y = 2x$.

Exercise 23.7 In each of the following identify the coordinates of some of the points on the line and use these to find the equation of the straight line.

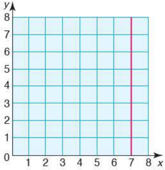
a)



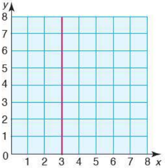
b)



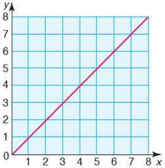
c)



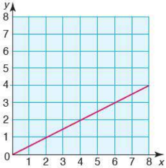
d)



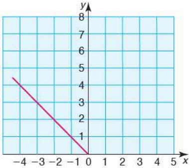
e)



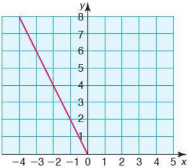
f)



g)

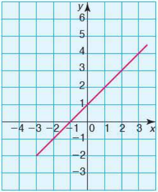


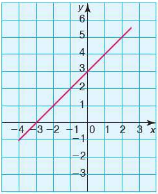
h)

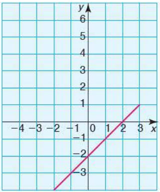


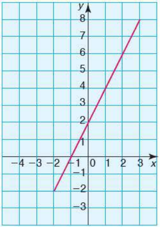
Exercise 23.8

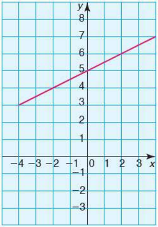
1. In each of the following identify the coordinates of some of the points on the line and use these to find the equation of the straight line.

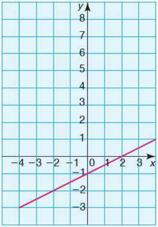
a)


b)


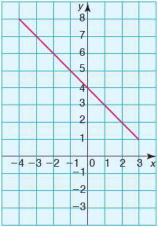
c)


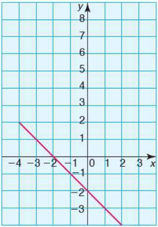
d)


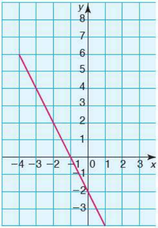
e)


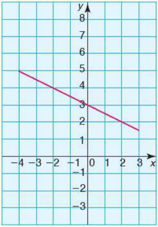
f)


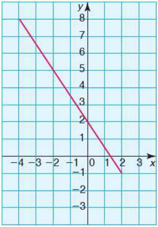
2. In each of the following identify the coordinates of some of the points on the line and use these to find the equation of the straight line.

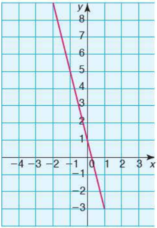
a)


b)


c)


d)


e)


f)


3. a) For each of the graphs in Q.1 and 2 calculate the gradient of the straight line.

b) What do you notice about the gradient of each line and its equation?

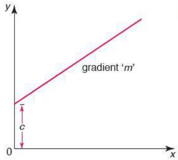
c) What do you notice about the equation of the straight line and where the line intersects the y-axis?

4. Copy the diagrams in Q.1. Draw two lines on each diagram parallel to the given line.

a) Write the equation of these new lines in the form $y = mx + c$.

b) What do you notice about the equations of these new parallel lines?

5. In Q.2 you have an equation for these lines in the form $y = mx + c$. Copy the diagrams in Q.2. Change the value of the intercept c and then draw the new line.
What do you notice about this new line and the first line?



The general equation of a straight line

In general, the equation of any straight line can be written in the form:

$$y = mx + c$$

where 'm' represents the gradient of the straight line and 'c' the intercept with the y-axis. This is shown in the diagram (left).

By looking at the equation of a straight line written in the form $y = mx + c$, it is therefore possible to deduce the line's gradient and intercept with the y-axis without having to draw it.

Worked examples

a) Calculate the gradient and y-intercept of the following straight lines:

i) $y = 3x - 2$ gradient $= 3$
y-intercept $= -2$

ii) $y = -2x + 6$ gradient $= -2$
y-intercept $= 6$

b) Calculate the gradient and y-intercept of the following straight lines:

i) $2y = 4x + 2$

This needs to be rearranged into **gradient–intercept** form (i.e. $y = mx + c$).

$y = 2x + 1$ gradient $= 2$
y-intercept $= 1$

ii) $y - 2x = -4$

Rearranging into gradient–intercept form, we have:

$y = 2x - 4$ gradient $= 2$
y-intercept $= -4$

iii) $-4y + 2x = 4$

Rearranging into gradient–intercept form, we have:

$y = \frac{1}{2}x - 1$ gradient $= \frac{1}{2}$
y-intercept $= -1$

iv) $\frac{y+3}{4} = -x + 2$

Rearranging into gradient–intercept form, we have:

$y + 3 = -4x + 8$
$y = -4x + 5$ gradient $= -4$
y-intercept $= 5$

Exercise 23.9

For the following linear equations, calculate both the gradient and y-intercept in each case.

1. a) $y = 2x + 1$ b) $y = 3x + 5$ c) $y = x - 2$
   d) $y = \frac{1}{2}x + 4$ e) $y = -3x + 6$ f) $y = -\frac{2}{3}x + 1$
   g) $y = -x$ h) $y = -x - 2$ i) $y = -(2x - 2)$

2. a) $y - 3x = 1$ b) $y + \frac{1}{2}x - 2 = 0$
   c) $y + 3 = -2x$ d) $y + 2x + 4 = 0$
   e) $y - \frac{1}{4}x - 6 = 0$ f) $-3x + y = 2$
   g) $2 + y = x$ h) $8x - 6 + y = 0$
   i) $-(3x + 1) + y = 0$

3. a) $2y = 4x - 6$ b) $2y = x + 8$
   c) $\frac{1}{2}y = x - 2$ d) $\frac{1}{4}y = -2x + 3$
   e) $3y - 6x = 0$ f) $\frac{1}{3}y + x = 1$
   g) $6y - 6 = 12x$ h) $4y - 8 + 2x = 0$
   i) $2y - (4x - 1) = 0$

4. a) $2x - y = 4$ b) $x - y + 6 = 0$
   c) $-2y = 6x + 2$ d) $12 - 3y = 3x$
   e) $5x - \frac{1}{2}y = 1$ f) $-\frac{2}{3}y + 1 = 2x$
   g) $9x - 2 = -y$ h) $-3x + 7 = -\frac{1}{2}y$
   i) $-(4x - 3) = -2y$

5. a) $\frac{y+2}{4} = \frac{1}{2}x$ b) $\frac{y-3}{x} = 2$ c) $\frac{y-x}{8} = 0$
   d) $\frac{2y-3x}{2} = 6$ e) $\frac{3y-2}{x} = -3$ f) $\frac{\frac{1}{2}y-1}{x} = -2$
   g) $\frac{3x-y}{2} = 6$ h) $\frac{6-2y}{3} = 2$ i) $\frac{-(x+2y)}{5x} = 1$

6. a) $\frac{3x-y}{y} = 2$ b) $\frac{-x+2y}{4} = y+1$
   c) $\frac{y-x}{x+y} = 2$ d) $\frac{1}{y} = \frac{1}{x}$
   e) $\frac{-(6x+y)}{2} = y+1$ f) $\frac{2x-3y+4}{4} = 4$
   g) $\frac{y+1}{x} + \frac{3y-2}{2x} = -1$ h) $\frac{x}{y+1} + \frac{1}{2y+2} = 3$
   i) $\frac{-(-y+3x)}{-(6x-2y)} = 1$
   j) $\frac{-(x-2y)-(-x-2y)}{4+x-y} = -2$

Parallel lines and their equations

Lines that are parallel, by their very definition must have the same gradient. Similarly, lines with the same gradient must be parallel. So a straight line with equation $y = -3x + 4$ must be parallel to a line with equation $y = -3x - 2$ as both have a gradient of -3.

Worked example A straight line has equation $4x - 2y + 1 = 0$.

Another straight line has equation $\frac{2x-4}{y} = 1$.

Explain, giving reasons, whether the two lines are parallel to each other or not.

Rearranging the equations into gradient–intercept form gives:

$$4x - 2y + 1 = 0 \qquad\qquad \frac{2x-4}{y} = 1$$
$$2y = 4x + 1 \qquad\qquad y = 2x - 4$$
$$y = 2x + \tfrac{1}{2}$$

With both equations written in gradient–intercept form it is possible to see that both lines have a gradient of 2 and are therefore parallel.

Exercise 23.10

1. A straight line has equation $3y - 3x = 4$. Write down the equation of another straight line parallel to it.
2. A straight line has equation $y = -x + 6$. Which of the following lines is/are parallel to it?

   a) $2(y + x) = -5$ b) $-3x - 3y + 7 = 0$

   c) $2y = -x + 12$ d) $y + x = \frac{1}{10}$

Worked example A straight line A has equation $y = -3x + 6$. A second line B is parallel to line A and passes through the point with coordinates $(-4, 10)$.

Calculate the equation of line B.

As line B is a straight line it must take the form $y = mx + c$.
As it is parallel to line A, its gradient must be -3.
Because line B passes through the point $(-4, 10)$ these values can be substituted into the general equation of the straight line to give:

$$10 = -3 \times (-4) + c$$

Rearranging to find c gives: $c = -2$
The equation of line B is therefore $y = -3x - 2$.

Exercise 23.11

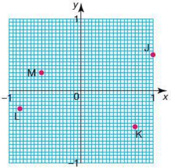
1. Find the equation of the line parallel to $y = 4x - 1$ that passes through $(0, 0)$.
2. Find the equations of lines parallel to $y = -3x + 1$ that pass through each of the following points:

   a) $(0, 4)$ b) $(-2, 4)$ c) $(-\frac{5}{2}, 4)$
3. Find the equations of lines parallel to $x - 2y = 6$ that pass through each of the following points:

   a) $(-4, 1)$ b) $(\frac{1}{2}, 0)$

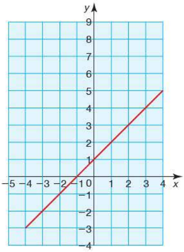
Student assessment 1

1. Give the coordinates of points J, K, L and M.

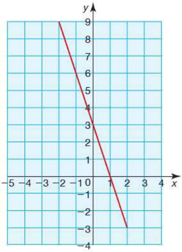


2. For each of the following lines, select two points on the line and then calculate its gradient.

a)

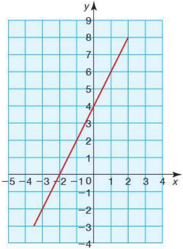


b)

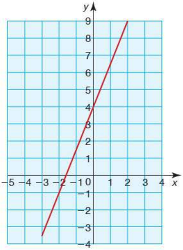


3. Find the equation of the straight line for each of the following:

a)



b)



4. Calculate the gradient and y-intercept for each of the following linear equations:

a) $y = -3x + 4$

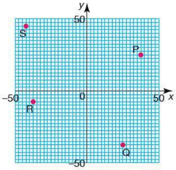
b) $\frac{1}{3}y - x = 2$

c) $2x + 4y - 6 = 0$

5. Write down the equation of the line parallel to the line $y = \frac{2}{3}x + 4$ which passes through the point (6, 2).

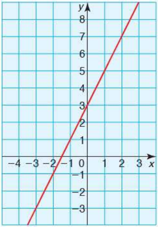
Student assessment 2

1. Give the coordinates of points P, Q, R and S.

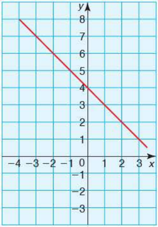


2. For each of the following lines, select two points on the line and then calculate its gradient.

a)

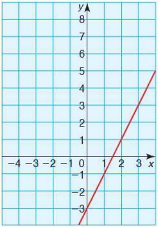


b)

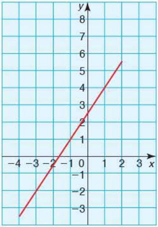


3. Find the equation of the straight line for each of the following:

a)



b)



4. Calculate the gradient and y-intercept for each of the following linear equations:

a) $y = \frac{1}{2}x$

b) $-4x + y = 6$

c) $2y - (5 - 3x) = 0$

5. Write down the equation of the line parallel to the line $y = 5x + 6$ which passes through the origin.

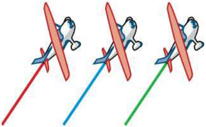
Topic 5

Mathematical investigations and ICT

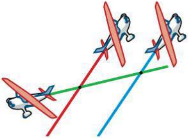
Plane trails

In an aircraft show, planes are made to fly with a coloured smoke trail. Depending on the formation of the planes, the trails can intersect in different ways.

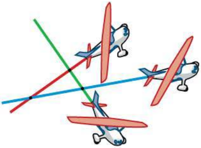
In the diagram below the three smoke trails do not cross, as they are parallel.



In the following diagram there are two crossing points.



By flying differently, the three planes can produce trails that cross at three points.



1. Investigate the connection between the maximum number of crossing points and the number of planes.
2. Record the results of your investigation in an ordered table.
3. Write an algebraic rule linking the number of planes (p) and the maximum number of crossing points (n).

Topic 6
Trigonometry

Syllabus

C6.1
Interpret and use three-figure bearings.

C6.2
Apply Pythagoras' theorem and the sine, cosine and tangent ratios for acute angles to the calculation of a side or of an angle of a right-angled triangle.

C6.3
Extended curriculum only.

C6.4
Extended curriculum only.

Contents

The development of trigonometry

In about 2000BCE, astronomers in Sumer in ancient Mesopotamia introduced angle measure. They divided the circle into 360 degrees. They and the ancient Babylonians studied the ratios of the sides of similar triangles. They discovered some properties of these ratios. However, they did not develop these into a method for finding sides and angles of triangles, what we now call trigonometry.

The ancients Greeks, among them Euclid and Archimedes, developed trigonometry further. They studied the properties of chords in circles and produced proofs of the trigonometric formulae we use today.

The modern sine function was first defined in an ancient Hindu text, the *Surya Siddhanta*, and further work was done by the Indian mathematician and astronomer Aryabhata in the 5th century.

Aryabhata (476–550)

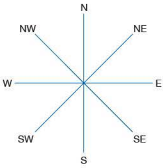
By the 10th century, Islamic mathematicians were using all six trigonometric functions (sine, cosine, tangent and their reciprocals). They made tables of trigonometric values and were applying them to problems in the geometry of the sphere.

As late as the 16th century, trigonometry was not well known in Europe. Nicolaus Copernicus decided it was necessary to explain the basic concepts of trigonometry in his book to enable people to understand his theory that the Earth went around the Sun.

Soon after, however, the need for accurate maps of large areas for navigation meant that trigonometry grew into a major branch of mathematics.

24 Bearings

NB: All diagrams are not drawn to scale.

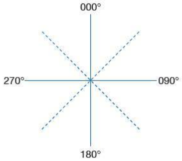


Bearings

In the days when sailing ships travelled the oceans of the world, compass bearings like the ones in the diagram (left) were used.

As the need for more accurate direction arose, extra points were added to N, S, E, W, NE, SE, SW and NW. Midway between North and North East was North North East, and midway between North East and East was East North East, and so on. This gave sixteen points of the compass. This was later extended even further, eventually to sixty four points.

As the speed of travel increased, a new system was required. The new system was the **three-figure bearing** system. North was given the bearing zero. 360° in a clockwise direction was one full rotation.



Exercise 24.1

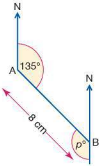
1. Copy the three-figure bearing diagram (above). Mark on your diagram the bearings for the compass points North East, South East, South West and North West.

2. Draw diagrams to show the following compass bearings and journeys. Use a scale of 1 cm : 1 km. North can be taken to be a line vertically up the page.
   a) Start at point A. Travel a distance of 7 km on a bearing of 135° to point B. From B, travel 12 km on a bearing of 250° to point C. Measure the distance and bearing of A from C.
   b) Start at point P. Travel a distance of 6.5 km on a bearing of 225° to point Q. From Q, travel 7.8 km on a bearing of 105° to point R. From R, travel 8.5 km on a bearing of 090° to point S. What are the distance and bearing of P from S?

c) Start from point M. Travel a distance of 11.2 km on a bearing of 270° to point N. From point N, travel 5.8 km on a bearing of 170° to point O. What are the bearing and distance of M from O?

Back bearings

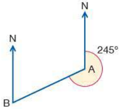
Worked examples



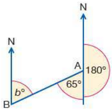
a) The bearing of B from A is 135° and the distance from A to B is 8 cm, as shown (left). The bearing of A from B is called the **back bearing**.

Since the two North lines are parallel:
$p = 135°$ (alternate angles), so the back bearing is $(180 + 135)°$.
That is, 315°.
(There are a number of methods of solving this type of problem.)

b) The bearing of B from A is 245°.
What is the bearing of A from B?



Since the two North lines are parallel:
$b = (245 - 180)° = 65°$
(alternate angles), so the bearing is 065°.



Exercise 24.2

1. Given the following bearings of point B from point A, draw diagrams and use them to calculate the bearing of A from B.

a) bearing 130°	b) bearing 145°
c) bearing 220°	d) bearing 200°
e) bearing 152°	f) bearing 234°
g) bearing 163°	h) bearing 214°

2. Given the following bearings of point D from point C, draw diagrams and use them to calculate the bearing of C from D.

a) bearing 300°	b) bearing 320°
c) bearing 290°	d) bearing 282°

Student assessment 1

1. From the top of a tall building in a town it is possible to see five towns. The bearing and distance of each one are given below.

Bourn 8 km	bearing 070°
Catania 12 km	bearing 135°
Deltaville 9 km	bearing 185°
Etta 7.5 km	bearing 250°
Freetown 11 km	bearing 310°

   Choose an appropriate scale and draw a diagram to show the position of each town. What are the distance and bearing of the following?
   a) Bourn from Deltaville
   b) Etta from Catania

2. A coastal radar station picks up a distress call from a ship. It is 50 km away on a bearing of 345°. The radar station contacts a lifeboat at sea which is 20 km away on a bearing of 220°.
   Make a scale drawing and use it to find the distance and bearing of the ship from the lifeboat.

3. A climber gets to the top of Mont Blanc. He can see in the distance a number of ski resorts. He uses his map to find the bearing and distance of the resorts, and records them as shown below:

Val d'Isère 30 km	bearing 082°
Les Arcs 40 km	bearing 135°
La Plagne 45 km	bearing 205°
Méribel 35 km	bearing 320°

   Choose an appropriate scale and draw a diagram to show the position of each resort. What are the distance and bearing of the following?
   a) Val d'Isère from La Plagne
   b) Méribel from Les Arcs

4. An aircraft is seen on radar at Milan airport. The aircraft is 210 km away from the airport on a bearing of 065°. The aircraft is diverted to Rome airport, which is 130 km away from Milan on a bearing of 215°. Use an appropriate scale and make a scale drawing to find the distance and bearing of Rome airport from the aircraft.

25 Right-angled triangles

NB: All diagrams are not drawn to scale.

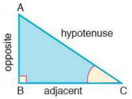
Trigonometric ratios

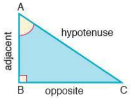
There are three basic trigonometric ratios: sine, cosine and tangent.

Each of these relates an angle of a right-angled triangle to a ratio of the lengths of two of its sides.

The sides of the triangle have names, two of which are dependent on their position in relation to a specific angle.

The longest side (always opposite the right angle) is called the **hypotenuse**. The side opposite the angle is called the **opposite** side and the side next to the angle is called the **adjacent** side.

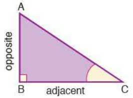




Note that, when the chosen angle is at A, the sides labelled opposite and adjacent change (above right).

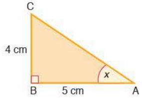
Tangent

$$\tan C = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$$



Worked examples

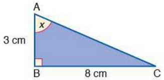
a) Calculate the size of angle BAC in each of the following triangles.



i) $\tan x = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{4}{5}$

$x = \tan^{-1}\left(\dfrac{4}{5}\right)$

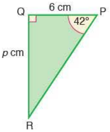
$x = 38.7$ (3 s.f.)
$\angle BAC = 38.7°$ (3 s.f.)



ii) $\tan x = \dfrac{8}{3}$

$x = \tan^{-1}\left(\dfrac{8}{3}\right)$

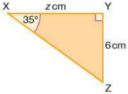
$x = 69.4$ (3 s.f.)
$\angle BAC = 69.4°$ (3 s.f.)



b) Calculate the length of the opposite side QR (right).

$$\tan 42° = \frac{p}{6}$$

$$6 \times \tan 42° = p$$
$$p = 5.40 \text{ (3 s.f.)}$$
$$\text{QR} = 5.40 \text{ cm (3 s.f.)}$$



c) Calculate the length of the adjacent side XY.

$$\tan 35° = \frac{6}{z}$$

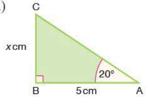
$$z \times \tan 35° = 6$$

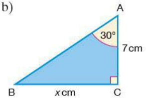
$$z = \frac{6}{\tan 35°}$$

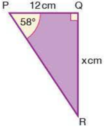
$$z = 8.57 \text{ (3 s.f.)}$$
$$\text{XY} = 8.57 \text{ cm (3 s.f.)}$$

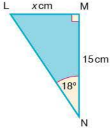
Exercise 25.1

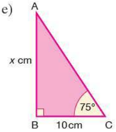
Calculate the length of the side marked x cm in each of the diagrams in Q.1 and 2. Give your answers to 1 d.p.

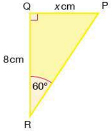
1. a) 

b) 

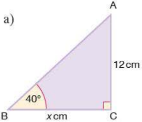
c) 

d) 

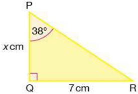
e) 

f) 

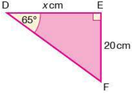
2. a)



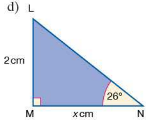
b)



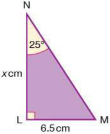
c)



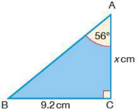
d)



e)

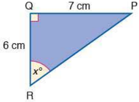


f)

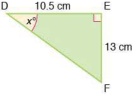


3. Calculate the size of the marked angle $x°$ in each of the following diagrams. Give your answers to 1 d.p.

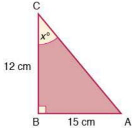
a)



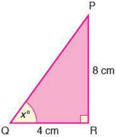
b)



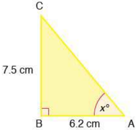
c)



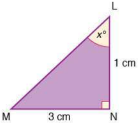
d)



e)

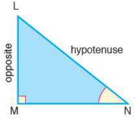


f)



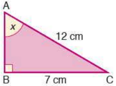
Sine

$$\sin N = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$$



Worked examples

a) Calculate the size of angle BAC.



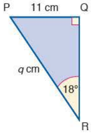
$$\sin x = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{7}{12}$$

$$x = \sin^{-1}\left(\frac{7}{12}\right)$$

$x = 35.7$ (1 d.p.)

$\angle BAC = 35.7°$ (1 d.p.)

b) Calculate the length of the hypotenuse PR.



$$\sin 18° = \frac{11}{q}$$

$q \times \sin 18° = 11$

$$q = \frac{11}{\sin 18°}$$

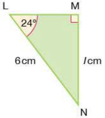
$q = 35.6$ (3 s.f.)

PR = 35.6 cm (3 s.f.)

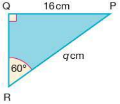
Exercise 25.2

1. Calculate the length of the marked side in each of the following diagrams. Give your answers to 1 d.p.

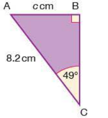
a)



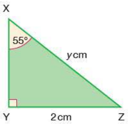
b)



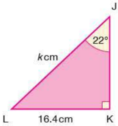
c)



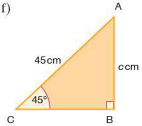
d)



e)

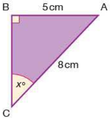


f)

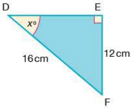


2. Calculate the size of the angle marked $x°$ in each of the following diagrams. Give your answers to 1 d.p.

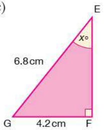
a)



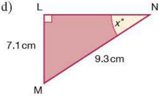
b)



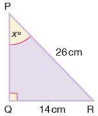
c)



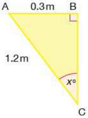
d)



e)

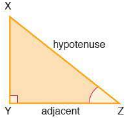


f)

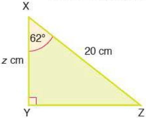


Cosine

$$\cos Z = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$$



Worked examples



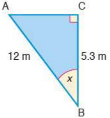
a) Calculate the length XY.

$$\cos 62° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{z}{20}$$

$z = 20 \times \cos 62°$

$z = 9.39$ (3 s.f.)

XY = 9.39 cm (3 s.f.)



b) Calculate the size of angle ABC.

$$\cos x = \frac{5.3}{12}$$

$$x = \cos^{-1}\left(\frac{5.3}{12}\right)$$

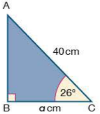
$x = 63.8$ (1 d.p.)

$\angle ABC = 63.8°$ (1 d.p.)

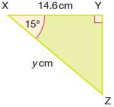
Exercise 25.3

Calculate the marked side or angle in each of the following diagrams. Give your answers to 1 d.p.

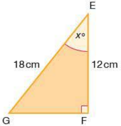
a)



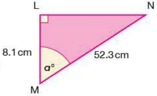
b)



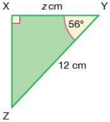
c)



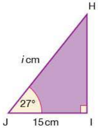
d)



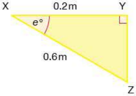
e)



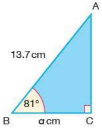
f)

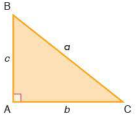


g)



h)





Pythagoras' theorem

Pythagoras' theorem states the relationship between the lengths of the three sides of a right-angled triangle.

Pythagoras' theorem states that:

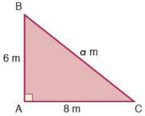
$$a^2 = b^2 + c^2$$

Worked examples

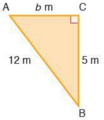
a) Calculate the length of the side BC.

Using Pythagoras:

$$a^2 = b^2 + c^2$$
$$a^2 = 8^2 + 6^2$$
$$a^2 = 64 + 36 = 100$$
$$a = \sqrt{100}$$
$$a = 10$$
$$\text{BC} = 10\ \text{m}$$



b) Calculate the length of the side AC.



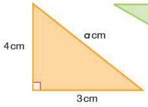
Using Pythagoras:

$$a^2 = b^2 + c^2$$
$$a^2 - c^2 = b^2$$
$$b^2 = 144 - 25$$
$$= 119$$
$$b = \sqrt{119}$$
$$b = 10.9 \text{ (3 s.f.)}$$
$$\text{AC} = 10.9 \text{ m (3 s.f.)}$$

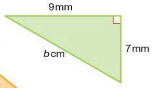
Exercise 25.4

In each of the diagrams in Q.1 and 2, use Pythagoras' theorem to calculate the length of the marked side.

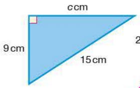
1. a)



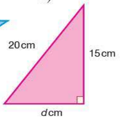
b)



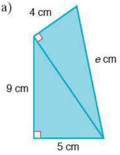
c)



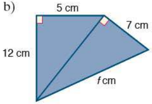
d)



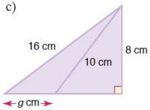
2. a)



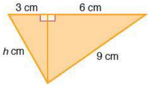
b)



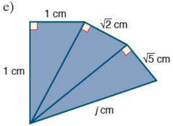
c)



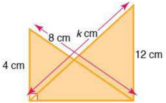
d)

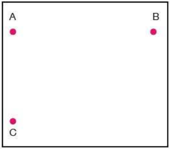


e)



f)



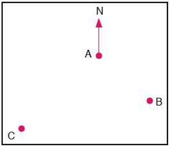


3. Villages A, B and C (left) lie on the edge of the Namib desert. Village A is 30 km due North of village C. Village B is 65 km due East of A.

   Calculate the shortest distance between villages C and B, giving your answer to the nearest 0.1 km.

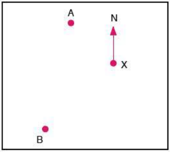
4. Town X is 54 km due West of town Y. The shortest distance between town Y and town Z is 86 km. If town Z is due South of X calculate the distance between X and Z, giving your answer to the nearest kilometre.

5. Village B (below) is on a bearing of 135° and at a distance of 40 km from village A. Village C is on a bearing of 225° and a distance of 62 km from village A.



   a) Show that triangle ABC is right-angled.
   b) Calculate the distance from B to C, giving your answer to the nearest 0.1 km.

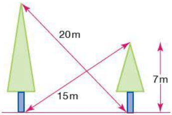
6. Two boats set off from X at the same time. Boat A sets off on a bearing of 325° and with a velocity of 14 km/h. Boat B sets off on a bearing of 235° with a velocity of 18 km/h.



   Calculate the distance between the boats after they have been travelling for 2.5 hours. Give your answer to the nearest kilometre.

7. A boat sets off on a trip from S. It heads towards B, a point 6 km away and due North. At B it changes direction and heads towards point C, also 6 km away and due East of B. At C it changes direction once again and heads on a bearing of 135° towards D which is 13 km from C.
   a) Calculate the distance between S and C to the nearest 0.1 km.
   b) Calculate the distance the boat will have to travel if it is to return to S from D.

8. Two trees are standing on flat ground. The height of the smaller tree is 7 m. The distance between the top of the smaller tree and the base of the taller tree is 15 m.
   The distance between the top of the taller tree and the base of the smaller tree is 20 m.

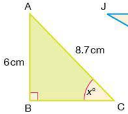


   a) Calculate the horizontal distance between the two trees.
   b) Calculate the height of the taller tree.

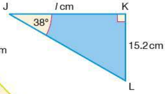
Exercise 25.5

1. By using Pythagoras' theorem, trigonometry or both, calculate the marked value in each of the following diagrams. In each case give your answer to 1 d.p.

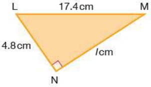
a)



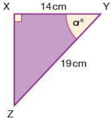
b)



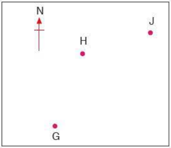
c)



d)

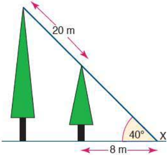


2. A sailing boat sets off from a point X and heads towards Y, a point 17 km North. At point Y it changes direction and heads towards point Z, a point 12 km away on a bearing of 090°. Once at Z the crew want to sail back to X. Calculate:
   a) the distance ZX,
   b) the bearing of X from Z.



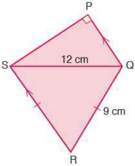
3. An aeroplane sets off from G (left) on a bearing of 024° towards H, a point 250 km away. At H it changes course and heads towards J on a bearing of 055° and a distance of 180 km away.
   a) How far is H to the North of G?
   b) How far is H to the East of G?
   c) How far is J to the North of H?
   d) How far is J to the East of H?
   e) What is the shortest distance between G and J?
   f) What is the bearing of G from J?

4. Two trees are standing on flat ground. The angle of elevation of their tops from a point X on the ground is 40°.



If the horizontal distance between X and the small tree is 8 m and the distance between the tops of the two trees is 20 m, calculate:
   a) the height of the small tree,
   b) the height of the tall tree,
   c) the horizontal distance between the trees.

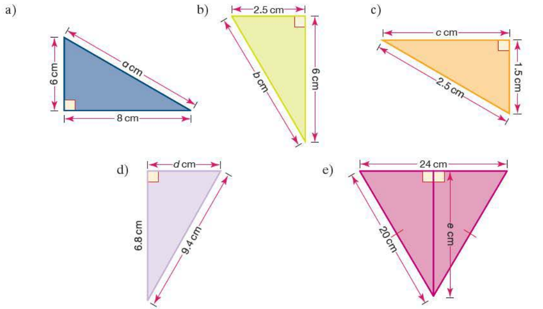
5. PQRS is a quadrilateral. The sides RS and QR are the same length. The sides QP and RS are parallel.



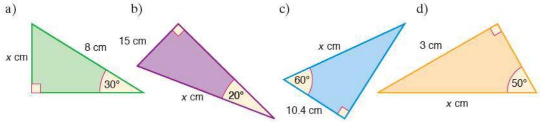
Calculate:
   a) angle SQR,
   b) angle PSQ,
   c) length PQ,
   d) length PS,
   e) the area of PQRS.

Student assessment 1

1. Calculate the length of the side marked with a letter in each of the following diagrams. Give your answers correct to 1 d.p.



2. Calculate the length of the side marked *x* cm in each of the following. Give your answers correct to 1 d.p.

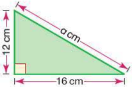


3. A rectangular swimming pool measures 50 m long by 15 m wide. Calculate the length of the diagonal of the pool. Give your answer correct to 1 d.p.

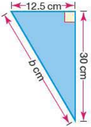
Student assessment 2

1. Calculate the length of the side marked with a letter in each of the following diagrams. Give your answers correct to 1 d.p.

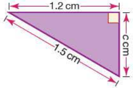
a)



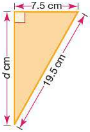
b)



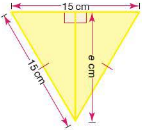
c)



d)

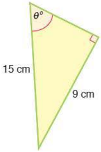


e)

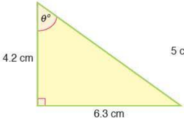


2. Calculate the size of the angle marked $\theta°$ in each of the following. Give your answers correct to the nearest degree.

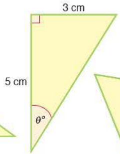
a)



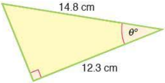
b)



c)

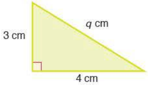


d)

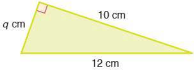


3. Calculate the length of the side marked q cm in each of the following. Give your answers correct to 1 d.p.

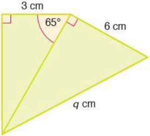
a)



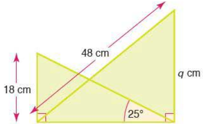
b)



c)



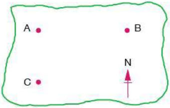
d)



4. A table measures 3.9 m by 2.4 m. Calculate the distance between the opposite corners. Give your answer correct to 1 d.p.

Student assessment 3

1. A map shows three towns A, B and C. Town A is due North of C. Town B is due East of A. The distance AC is 75 km and the bearing of C from B is 245°. Calculate, giving your answers to the nearest 100 m:
   a) the distance AB,
   b) the distance BC.



2. Two trees stand 16 m apart. Their tops make an angle of $\theta°$ at point A on the ground.

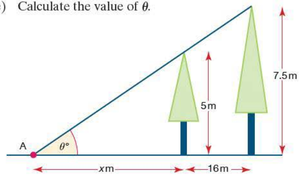
a) Express $\theta°$ in terms of the height of the shorter tree and its distance x metres from point A.

b) Express $\theta°$ in terms of the height of the taller tree and its distance from A.

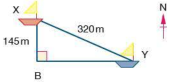
c) Form an equation in terms of x.

d) Calculate the value of x.

e) Calculate the value of θ.



3. Two boats X and Y, sailing in a race, are shown in the diagram (below).



Boat X is 145 m due North of a buoy B.
Boat Y is due East of buoy B. Boats X and Y are 320 m apart. Calculate:

a) the distance BY,

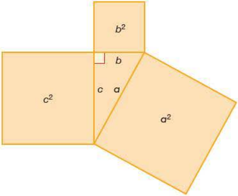
b) the bearing of Y from X,

c) the bearing of X from Y.

Topic 6 Mathematical investigations and ICT

Pythagoras and circles

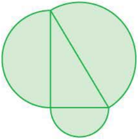
The explanation for Pythagoras' theorem usually shows a right-angled triangle with squares drawn on each of its three sides, as in the diagram below.



In this example, the area of the square on the hypotenuse, a^2, is equal to the sum of the areas of the squares on the other two sides, $b^2 + c^2$.

This gives the formula $a^2 = b^2 + c^2$.

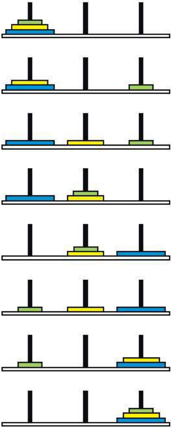
1. Draw a right-angled triangle.
2. Using a pair of compasses, construct a semi-circle off each side of the triangle. Your diagram should look similar to the one shown below.



3. By measuring the diameter of each semi-circle, calculate their areas.
4. Is the area of the semi-circle on the hypotenuse the sum of the areas of the semi-circles drawn on the other two sides? Does Pythagoras' theorem still hold for semi-circles?

5. Does Pythagoras' theorem still hold if equilateral triangles are drawn on each side?

6. Investigate for other regular polygons.



Towers of Hanoi

This investigation is based on an old Vietnamese legend. The legend is as follows:

> At the beginning of time a temple was created by the Gods. Inside the temple stood three giant rods. On one of these rods, 64 gold discs, all of different diameters, were stacked in descending order of size, i.e the largest at the bottom rising to the smallest at the top. Priests at the temple were responsible for moving the discs onto the remaining two rods until all 64 discs were stacked in the same order on one of the other rods. When this task was completed, time would cease and the world would come to an end.

The discs however could only be moved according to certain rules. These were:

- Only one disc could be moved at a time.
- A disc could only be placed on top of a larger one.

The diagram (left) shows the smallest number of moves required to transfer three discs from the rod on the left to the rod on the right.

With three discs, the smallest number of moves is seven.

1. What is the smallest number of moves needed for 2 discs?

2. What is the smallest number of moves needed for 4 discs?

3. Investigate the smallest number of moves needed to move different numbers of discs.

4. Display the results of your investigation in an ordered table.

5. Describe any patterns you see in your results.

6. Predict, from your results, the smallest number of moves needed to move 10 discs.

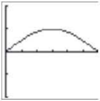
7. Determine a formula for the smallest number of moves for n discs.

8. Assume the priests have been transferring the discs at the rate of one per second and assume the Earth is approximately 4.54 billion years old (4.54×10^9 years). According to the legend, is the world coming to an end soon? Justify your answer with relevant calculations.

ICT activity

In this activity you will need to use a graphical calculator to investigate the relationship between different trigonometric ratios.

1. a) Using the graphical calculator or graphing software, plot the graph of $y = \sin x$ for $0° \leqslant x \leqslant 180°$.
The graph should look similar to the one shown below:



b) Using the equation solving facility evaluate sin 70°.
c) Referring to the graph explain why $\sin x = 0.7$ has two solutions between 0° and 180°.
d) Use the graph to solve the equation $\sin x = 0.5$.

2. a) On the same axes as before plot $y = \cos x$.
b) How many solutions are there to the equation $\sin x = \cos x$ between 0° and 180°?
c) What is the solution to the equation $\sin x = \cos x$ between 0° and 180°?

3. By plotting appropriate graphs solve the following for $0° \leqslant x \leqslant 180°$.
a) $\sin x = \tan x$
b) $\cos x = \tan x$

Topic 7

Vectors and transformations

Syllabus

C7.1

Describe a translation by using a vector represented by e.g. $\binom{x}{y}$, $\overrightarrow{AB}$ or $\mathbf{a}$.
Add and subtract vectors.
Multiply a vector by a scalar.

C7.2

Reflect simple plane figures in horizontal or vertical lines.
Rotate simple plane figures about the origin, vertices or midpoints of edges of the figures, through multiples of 90°.
Construct given translations and enlargements of simple plane figures.
Recognise and describe reflections, rotations, translations and enlargements.

C7.3

Extended curriculum only.

C7.4

Extended curriculum only.

C7.5

Extended curriculum only.

Contents

The development of vectors

The study of vectors arose from coordinates in two dimensions. Around 1636, René Descartes and Pierre de Fermat founded analytic geometry by linking the solutions to an equation with two variables with points on a curve.

In 1804 the Czech mathematician Bernhard Bolzano worked on the mathematics of points, lines and planes and this work formed the beginnings of work on vectors. Later in the 19th century, this work was further developed by the German mathematician August Möbius and the Italian mathematician Giusto Bellavitis.

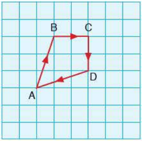
Bernhard Bolzano (1781–1848)

26 Vectors

Translations

A **translation** (a sliding movement) can be described using a **column vector**. A column vector describes the movement of the object in both the x direction and the y direction.

Worked example i) Describe the translation from A to B in the diagram (below) in terms of a column vector.



$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

i.e. 1 unit in the x direction, 3 units in the y direction

ii) Describe $\overrightarrow{BC}$ in terms of a column vector.

$$\overrightarrow{BC} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

iii) Describe $\overrightarrow{CD}$ in terms of a column vector.

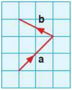
$$\overrightarrow{CD} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

iv) Describe $\overrightarrow{DA}$ in terms of a column vector.

$$\overrightarrow{DA} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$$

Translations can also be named by a single letter. The direction of the arrow indicates the direction of the translation.

Worked example Describe **a** and **b** in the diagram (right) using column vectors.

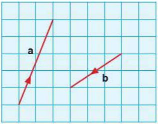


$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

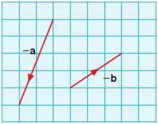
$\mathbf{b} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Note: When you represent vectors by single letters, e.g. **a**, in handwritten work, you should write them as $\underline{a}$.

If $\mathbf{a} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$, they can be represented diagrammatically as shown (below).



The diagrammatic representation of $-\mathbf{a}$ and $-\mathbf{b}$ is shown below.



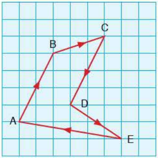
It can be seen from the diagram above that

$-\mathbf{a} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$ and $-\mathbf{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

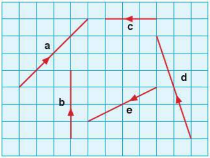
Exercise 26.1

In Q.1 and 2 describe each translation using a column vector.

1. a) $\overrightarrow{AB}$
b) $\overrightarrow{BC}$
c) $\overrightarrow{CD}$
d) $\overrightarrow{DE}$
e) $\overrightarrow{EA}$
f) $\overrightarrow{AE}$
g) $\overrightarrow{DA}$
h) $\overrightarrow{CA}$
i) $\overrightarrow{DB}$



2. a) **a**
b) **b**
c) **c**
d) **d**
e) **e**
f) −**b**
g) −**c**
h) −**d**
i) −**a**



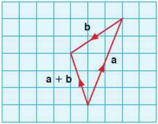
3. Draw and label the following vectors on a square grid:

a) $\mathbf{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ b) $\mathbf{b} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$ c) $\mathbf{c} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

d) $\mathbf{d} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$ e) $\mathbf{e} = \begin{pmatrix} 0 \\ -6 \end{pmatrix}$ f) $\mathbf{f} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$

g) $\mathbf{g} = -\mathbf{c}$ h) $\mathbf{h} = -\mathbf{b}$ i) $\mathbf{i} = -\mathbf{f}$

Addition of vectors



Vectors can be added together and represented diagrammatically as shown (left).

The translation represented by **a** followed by **b** can be written as a single transformation **a** + **b**:

i.e. $\begin{pmatrix} 2 \\ 5 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

Exercise 26.2 In the following questions,

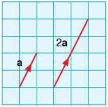
$$\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \quad \mathbf{d} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

1. Draw vector diagrams to represent the following:
   a) $\mathbf{a} + \mathbf{b}$ b) $\mathbf{b} + \mathbf{a}$ c) $\mathbf{a} + \mathbf{d}$
   d) $\mathbf{d} + \mathbf{a}$ e) $\mathbf{b} + \mathbf{c}$ f) $\mathbf{c} + \mathbf{b}$

2. What conclusions can you draw from your answers to Q.1 above?

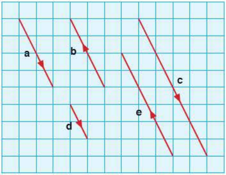
Multiplying a vector by a scalar

Look at the two vectors in the diagram.



$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad 2\mathbf{a} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Worked example If $\mathbf{a} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$, express the vectors **b**, **c**, **d** and **e** in terms of **a**.

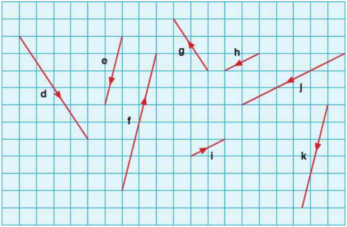


$$\mathbf{b} = -\mathbf{a} \quad \mathbf{c} = 2\mathbf{a} \quad \mathbf{d} = \tfrac{1}{2}\mathbf{a} \quad \mathbf{e} = -\tfrac{3}{2}\mathbf{a}$$

Exercise 26.3

1. $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$ $\mathbf{c} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$

Express the following vectors in terms of either **a**, **b** or **c**.



2. $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$ $\mathbf{c} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

Represent each of the following as a single column vector:

a) $2\mathbf{a}$ b) $3\mathbf{b}$ c) $-\mathbf{c}$
d) $\mathbf{a} + \mathbf{b}$ e) $\mathbf{b} - \mathbf{c}$ f) $3\mathbf{c} - \mathbf{a}$
g) $2\mathbf{b} - \mathbf{a}$ h) $\frac{1}{2}(\mathbf{a} + \mathbf{b})$ i) $2\mathbf{a} + 3\mathbf{c}$

3. $\mathbf{a} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$ $\mathbf{c} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

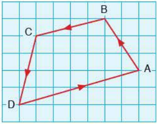
Express each of the following vectors in terms of **a**, **b** and **c**:

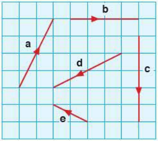
a) $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$ b) $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ c) $\begin{pmatrix} 4 \\ -4 \end{pmatrix}$ d) $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$

Student assessment 1

1. Using the diagram (below), describe the following translations using column vectors.

a) $\overrightarrow{AB}$
b) $\overrightarrow{DA}$
c) $\overrightarrow{CA}$





2. Describe each of the translations **a** to **e** shown in the diagram (left) using column vectors.

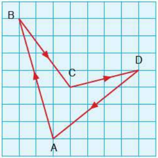
3. Using the vectors from Q.2 above, draw diagrams to represent:
   a) **a** + **b** b) **e** − **d** c) **c** − **e** d) 2**e** + **b**

4. In the following,

$$\mathbf{a} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -3 \\ -1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

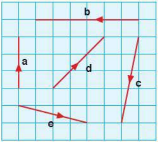
   Calculate:
   a) **a** + **b** b) **c** + **b** c) 2**a** + **b** d) 3**c** + 2**b**

Student assessment 2



1. Using this diagram (left), describe the following translations using column vectors.
   a) $\overrightarrow{AB}$ b) $\overrightarrow{DA}$ c) $\overrightarrow{CA}$

2. Describe each of the translations **a** to **e** in the diagram (below) using column vectors.



3. Using the vectors from Q.2 above, draw diagrams to represent:
   a) **a** + **e** b) **c** + **d** c) −**c** + **e** d) −**b** + 2**a**

4. In the following,

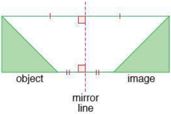
$$\mathbf{a} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$$

   Calculate:
   a) **a** + **c** b) **b** + **a** c) 2**a** + **b** d) 3**c** + 2**a**

27 Transformations

An object undergoing a transformation changes in either position or shape. In its simplest form this change can occur as a result of either a **reflection**, **rotation**, **translation** or **enlargement**. If an object undergoes a transformation, then its new position or shape is known as the **image**.

Reflection



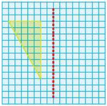
If an object is reflected it undergoes a 'flip' movement about a dashed (broken) line known as the **mirror line**, as shown in the diagram (left).

A point on the object and its equivalent point on the image are equidistant from the mirror line. This distance is measured at right angles to the mirror line. The line joining the point to its image is perpendicular to the mirror line.

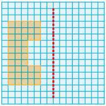
Exercise 27.1

In each of the following, copy the diagram and draw the object's image under reflection in the dashed line(s).

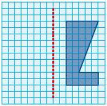
1.



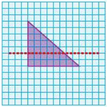
2.



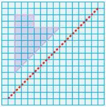
3.



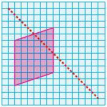
4.



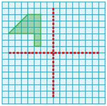
5.



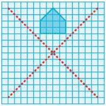
6.



7.



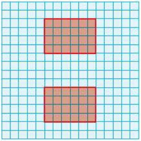
8.



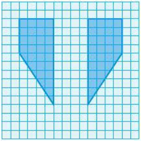
Exercise 27.2

Copy the following objects and images and in each case draw in the position of the mirror line(s).

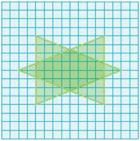
1.



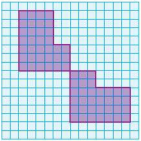
2.



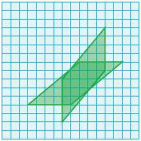
3.



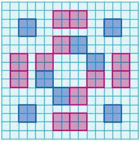
4.

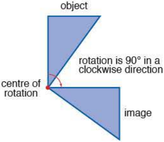


5.



6.





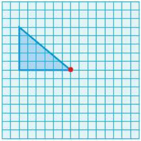
Rotation

If an object is rotated it undergoes a 'turning' movement about a specific point known as the **centre of rotation**. When describing a rotation it is necessary to identify not only the position of the centre of rotation, but also the angle and direction of the turn, as shown in the diagram (left).

Exercise 27.3

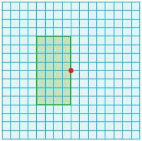
In the following, the object and centre of rotation have both been given. Copy each diagram and draw the object's image under the stated rotation about the marked point.

1.



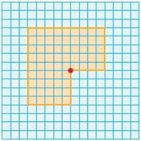
rotation 180°

2.



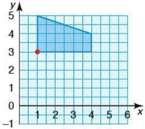
rotation 90° clockwise

3.



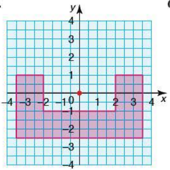
rotation 180°

4.



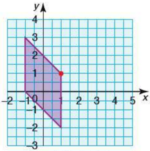
rotation 90° clockwise about (1, 3)

5.



rotation 90° anti-clockwise about (0, 0)

6.



rotation 90° clockwise about (1, 1)

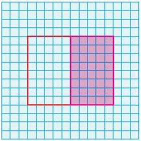
Exercise 27.4

In the following, the object (unshaded) and image (shaded) have been drawn. Copy each diagram.

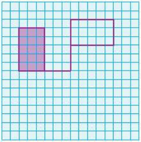
a) Mark the centre of rotation.

b) Calculate the angle and direction of rotation.

1.

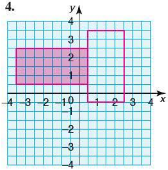


2.

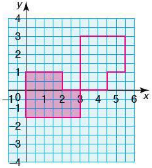


3.

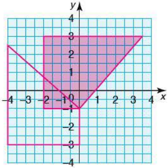
4.



5.

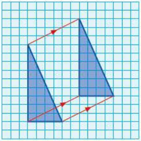


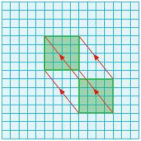
6.



Translation

If an object is translated, it undergoes a 'straight sliding' movement. When describing a translation it is necessary to give the **translation vector**. As no rotation is involved, each point on the object moves in the same way to its corresponding point on the image, e.g.

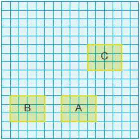




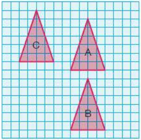
Vector $= \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ Vector $= \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

Exercise 27.5 In the following diagrams, object A has been translated to each of images B and C. Give the translation vectors in each case.

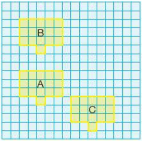
1.



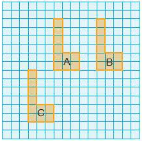
2.



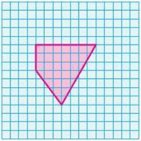
3.



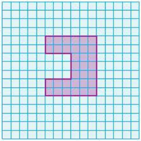
4.



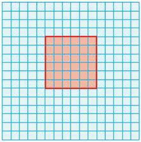
Exercise 27.6 Copy each of the following diagrams and draw the object. Translate the object by the vector given in each case and draw the image in its position. (Note that a bigger grid than the one shown may be needed.)

1. 

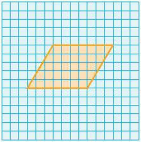
Vector $= \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

2. 

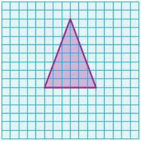
Vector $= \begin{pmatrix} 5 \\ -4 \end{pmatrix}$

3. 

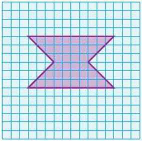
Vector $= \begin{pmatrix} -4 \\ 6 \end{pmatrix}$

4. 

Vector $= \begin{pmatrix} -2 \\ -5 \end{pmatrix}$

5. 

Vector $= \begin{pmatrix} -6 \\ 0 \end{pmatrix}$

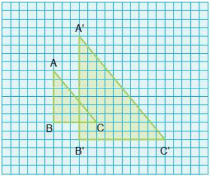
6. 

Vector $= \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Enlargement

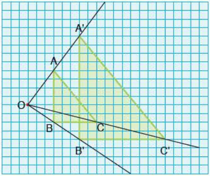
If an object is enlarged, the result is an image which is mathematically similar to the object but of a different size. The image can be either larger or smaller than the original object. When describing an enlargement two pieces of information need to be given, the position of the **centre of enlargement** and the **scale factor of enlargement**.

Worked examples **a)** In the diagram below, triangle ABC is enlarged to form triangle A'B'C'.



i) Find the centre of enlargement.

The centre of enlargement is found by joining corresponding points on the object and image with a straight line. These lines are then extended until they meet. The point at which they meet is the centre of enlargement O.



ii) Calculate the scale factor of enlargement.

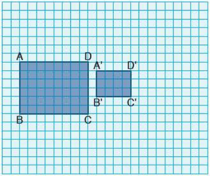
The scale factor of enlargement can be calculated in two ways. From the diagram above it can be seen that the distance OA' is twice the distance OA. Similarly OC' and OB' are twice OC and OB respectively, hence the scale factor of enlargement is 2.

Alternatively the scale factor can be found by considering the ratio of the length of a side on the image to the length of the corresponding side on the object, i.e.

$$\frac{A'B'}{AB} = \frac{12}{6} = 2$$

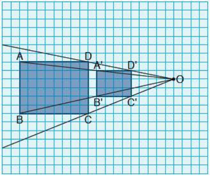
Hence the scale factor of enlargement is 2.

b) In the diagram below, the rectangle ABCD undergoes a transformation to form rectangle A'B'C'D'.



i) Find the centre of enlargement.

By joining corresponding points on both the object and the image the centre of enlargement is found at O.



ii) Calculate the scale factor of enlargement.

The scale factor of enlargement $= \dfrac{A'B'}{AB} = \dfrac{3}{6} = \dfrac{1}{2}$

Note: If the scale factor of enlargement is greater than 1, then the image is larger than the object. If the scale factor lies between 0 and 1, then the resulting image is smaller than the object. In these cases, although the image is smaller than the object, the transformation is still known as an enlargement.

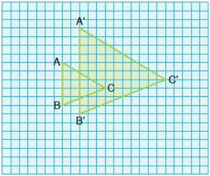
Exercise 27.7

Copy the following diagrams and find:

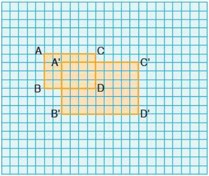
a) the centre of enlargement,

b) the scale factor of enlargement.

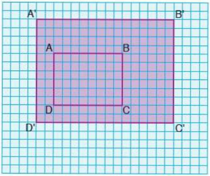
1.



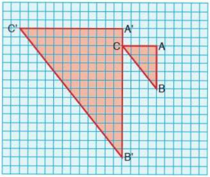
2.



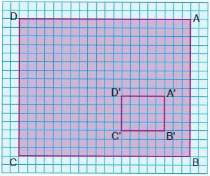
3.



4.



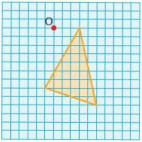
5.



Exercise 27.8

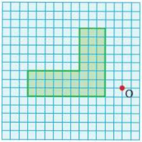
Copy the following diagrams and enlarge the objects by the scale factor given and from the centre of enlargement shown. Grids larger than those shown may be needed.

1.



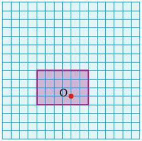
scale factor 2

2.



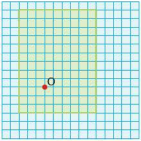
scale factor 2

3.



scale factor 3

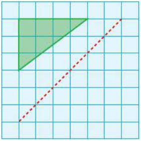
4.



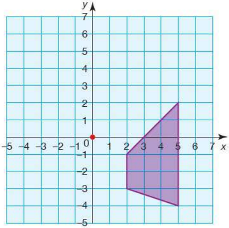
scale factor $\frac{1}{3}$

Student assessment 1

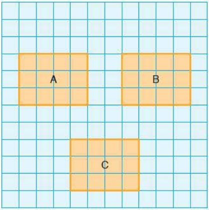
1. Reflect the object below in the mirror line shown.



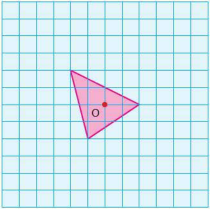
2. Rotate the object below 90° anti-clockwise about the origin.



3. Write down the column vector of the translation which maps:
a) rectangle A to rectangle B,
b) rectangle B to rectangle C.

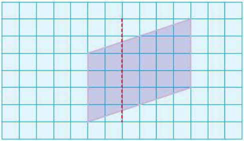


4. Enlarge the triangle below by a scale factor 2 and from the centre of enlargement O.

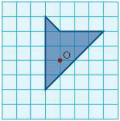


Student assessment 2

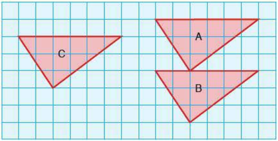
1. Reflect the object below in the mirror line shown.



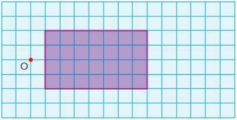
2. Rotate the object below 180° about the centre of rotation O.



3. Write down the column vector of the translation which maps:
   a) triangle A to triangle B,
   b) triangle B to triangle C.



4. Enlarge the rectangle below by a scale factor 1.5 and from the centre of enlargement O.

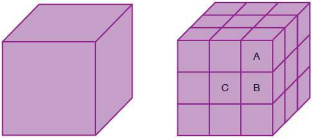


Topic 7

Mathematical investigations and ICT

A painted cube

A $3 \times 3 \times 3$ cm cube is painted on the outside as shown in the left-hand diagram below:



The large cube is then cut up into 27 smaller cubes, each $1\text{cm} \times 1\text{cm} \times 1\text{cm}$ as shown on the right.

$1 \times 1 \times 1$ cm cubes with 3 painted faces are labelled type A.
$1 \times 1 \times 1$ cm cubes with 2 painted faces are labelled type B.
$1 \times 1 \times 1$ cm cubes with 1 face painted are labelled type C.
$1 \times 1 \times 1$ cm cubes with no faces painted are labelled type D.

1. a) How many of the 27 cubes are type A?
b) How many of the 27 cubes are type B?
c) How many of the 27 cubes are type C?
d) How many of the 27 cubes are type D?

2. Consider a $4 \times 4 \times 4$ cm cube cut into $1 \times 1 \times 1$ cm cubes. How many of the cubes are type A, B, C and D?

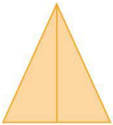
3. How many type A, B, C and D cubes are there when a $10 \times 10 \times 10$ cm cube is cut into $1 \times 1 \times 1$ cm cubes?

4. Generalise for the number of type A, B, C and D cubes in an $n \times n \times n$ cube.

5. Generalise for the number of type A, B, C and D cubes in a cuboid l cm long, w cm wide and h cm high.

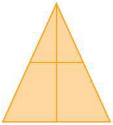
Triangle count

The diagram below shows an isosceles triangle with a vertical line drawn from its apex to its base.



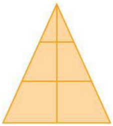
There is a total of 3 triangles in this diagram.

If a horizontal line is drawn across the triangle, it will look as shown:



There is a total of 6 triangles in this diagram.

When one more horizontal line is added, the number of triangles increases further:

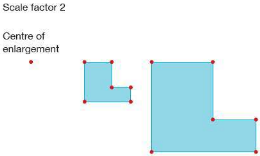


1. Calculate the total number of triangles in the diagram above with the two inner horizontal lines.
2. Investigate the relationship between the total number of triangles (t) and the number of inner horizontal lines (h). Enter your results in an ordered table.
3. Write an algebraic rule linking the total number of triangles and the number of inner horizontal lines.

ICT activity

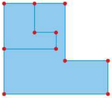
In this activity, you will be using a geometry package to investigate enlargements.

1. Using a geometry package, draw an object and enlarge it by a scale factor of 2. An example is shown below:

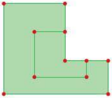


2. Describe the position of the centre of enlargement used to produce the following diagrams:

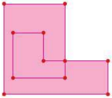
a)



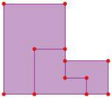
b)



c)



d)



3. Move the position of the centre of enlargement to test your answers to Q.2 above.

Topic 8 Probability

Syllabus

C8.1

Calculate the probability of a single event as either a fraction, decimal or percentage.

C8.2

Understand and use the probability scale from 0 to 1.

C8.3

Understand that the probability of an event occurring = 1 – the probability of the event not occurring.

C8.4

Understand relative frequency as an estimate of probability.

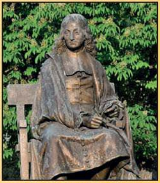
C8.5

Extended curriculum only.

Contents

The development of probability

Probability was first discussed in letters between the French mathematicians Pierre de Fermat and Blaise Pascal in 1654. Christiaan Huygens, a Dutch mathematician, gave the earliest known scientific treatment of the subject in 1657. By the early 18th century probability was being treated as a branch of mathematics. The Swiss mathematician Jakob Bernoulli published his book *Ars Conjectandi* (*The Art of Conjecturing*) in 1713, which included work on permutations and combinations and other important concepts, and the French mathematician Abraham de Moivre published his book *The Doctrine of Chances* in 1718, in which he explains probability theory.



Blaise Pascal (1623–1662)

28 Probability

Probability is the study of chance, or the likelihood of an event happening. However, because probability is based on chance, what theory predicts does not necessarily happen in practice.

Theoretical probability

A **favourable outcome** refers to the event in question actually happening. The **total number of possible outcomes** refers to all the different types of outcome one can get in a particular situation. In general:

$$\text{Probability of an event} = \frac{\text{number of favourable outcomes}}{\text{total number of equally likely outcomes}}$$

If the probability = 0, the event is impossible.
If the probability = 1, the event is certain to happen.

If an event can either happen or not happen then:

Probability of the event not occurring
= 1 – the probability of the event occurring.

Worked examples

a) An ordinary, fair dice is rolled. Calculate the probability of getting a six.

Number of favourable outcomes = 1 (i.e. getting a 6)
Total number of possible outcomes = 6
(i.e. getting a 1, 2, 3, 4, 5 or 6)

Probability of getting a six $= \frac{1}{6}$

Probability of not getting a six $= 1 - \frac{1}{6} = \frac{5}{6}$

b) An ordinary, fair dice is rolled. Calculate the probability of getting an even number.

Number of favourable outcomes = 3
(i.e. getting a 2, 4 or 6)
Total number of possible outcomes = 6
(i.e. getting a 1, 2, 3, 4, 5 or 6)

Probability of getting an even number $= \frac{3}{6} = \frac{1}{2}$

c) Thirty students are asked to choose their favourite subject out of Maths, English and Art. The results are shown in the table below:

	Maths	English	Art
Girls	7	4	5
Boys	5	3	6

A student is chosen at random.

i) What is the probability that it is a girl?
Total number of girls is 16.
Probability of choosing a girl is $\frac{16}{30} = \frac{8}{15}$.

ii) What is the probability that it is a boy whose favourite subject is Art?
Number of boys whose favourite subject is Art is 6.
Probability is therefore $\frac{6}{30} = \frac{1}{5}$.

iii) What is the probability of **not** choosing a girl whose favourite subject is English?
There are two ways of approaching this:

Method 1:
Total number of students who are not girls whose favourite subject is English is $7 + 5 + 5 + 3 + 6 = 26$.
Therefore probability is $\frac{26}{30} = \frac{13}{15}$.

Method 2:
Total number of girls whose favourite subject is English is 4.
Probability of choosing a girl whose favourite subject is English is $\frac{4}{30}$.
Therefore the probability of **not** choosing a girl whose favourite subject is English is:

$$1 - \frac{4}{30} = \frac{26}{30} = \frac{13}{15}$$

The probability scale

Exercise 28.1

Draw a line in your book about 15 cm long. At the extreme left put 0, and at the extreme right put 1. These represent impossible and certain events, respectively. Try to estimate the chance of the following events happening and place them on your line.

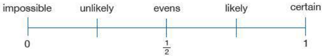
a) You will watch TV tonight.
b) You will play sport tomorrow.
c) You will miss school one day this month.
d) You will be on a plane next year.
e) You will learn a new language one day.
f) You will have a visitor at school today.

The likelihood of the events in Exercise 28.1 vary from person to person. Therefore, the probability of each event is not constant. However, the probability of some events, such as the result of throwing dice, spinning a coin or dealing cards, can be found by experiment or calculation.

The line drawn in Exercise 28.1 is called a probability scale. The scale goes from 0 to 1.
A probability of 0 means the event is impossible.
A probability of 1 means it is certain.

Exercise 28.2

1. Copy the probability scale below.



Mark on the probability scale the probability that:

a) a day chosen at random is a Saturday,
b) a coin will show tails when spun,
c) the sun will rise tomorrow,
d) a woman will run a marathon in two hours,
e) the next car you see will be silver.

2. Express your answers to Q.1 as fractions, decimals and percentages.

Exercise 28.3

1. Calculate the theoretical probability, when rolling an ordinary, fair dice, of getting each of the following:

a) a score of 1   b) a score of 2, 3, 4, 5 or 6
c) an odd number   d) a score less than 6
e) a score of 7   f) a score less than 7

2. a) Calculate the probability of:
   i) being born on a Wednesday,
   ii) not being born on a Wednesday.
   b) Explain the result of adding the answers to a) i) and ii) together.

3. 250 balls are numbered from 1 to 250 and placed in a box. A ball is picked at random. Find the probability of picking a ball with:

a) the number 1   b) an even number
c) a three-digit number   d) a number less than 300

4. In a class there are 25 girls and 15 boys. The teacher takes in all of their books in a random order. Calculate the probability that the teacher will:
   a) mark a book belonging to a girl first,
   b) mark a book belonging to a boy first.

5. Tiles, each lettered with one different letter of the alphabet, are put into a bag. If one tile is taken out at random, calculate the probability that it is:

a) an A or P   b) a vowel
c) a consonant   d) an X, Y or Z
e) a letter in your first name.

6. A boy was late for school 5 times in the previous 30 school days. If tomorrow is a school day, calculate the probability that he will arrive late.

7. a) Three red, 10 white, 5 blue and 2 green counters are put into a bag. If one is picked at random, calculate the probability that it is:
i) a green counter ii) a blue counter.
b) If the first counter taken out is green and it is not put back into the bag, calculate the probability that the second counter picked is:
i) a green counter ii) a red counter.

8. A circular spinner has the numbers 0 to 36 equally spaced around its edge. Assuming that it is unbiased, calculate the probability on spinning it of getting:
a) the number 5 b) not 5
c) an odd number d) zero
e) a number greater than 15 f) a multiple of 3
g) a multiple of 3 or 5 h) a prime number.

9. The letters R, C and A can be combined in several different ways.
a) Write the letters in as many different orders as possible.
If a computer writes these three letters at random, calculate the probability that:
b) the letters will be written in alphabetical order,
c) the letter R is written before both the letters A and C,
d) the letter C is written after the letter A,
e) the computer will spell the word CART if the letter T is added.

10. A normal pack of playing cards contains 52 cards. These are made up of four suits (hearts, diamonds, clubs and spades). Each suit consists of 13 cards. These are labelled Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen and King. The hearts and diamonds are red; the clubs and spades are black.
If a card is picked at random from a normal pack of cards, calculate the probability of picking:
a) a heart b) not a heart
c) a 4 d) a red King
e) a Jack, Queen or King f) the Ace of spades
g) an even numbered card h) a 7 or a club.

Exercise 28.4

1. A student conducts a survey on the types of vehicle that pass his house. The results are shown below.

Vehicle type	Car	Lorry	Van	Bicycle	Motorbike	Other
Frequency	28	6	20	48	32	6

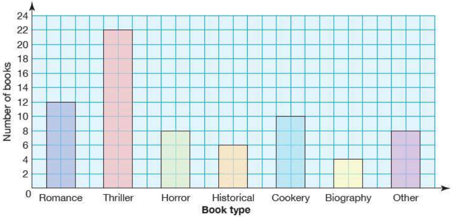
   a) How many vehicles passed the student's house?
   b) A vehicle is chosen at random from the results. Calculate the probability that it is:
      i) a car
      ii) a lorry
      iii) not a van.

2. In a class, data is collected about whether each student is right-handed or left-handed. The results are shown below.

	Left-handed	Right-handed
Boys	2	12
Girls	3	15

   a) How many students are in the class?
   b) A student is chosen at random. Calculate the probability that the student is:
      i) a girl
      ii) left-handed
      iii) a right-handed boy
      iv) not a right-handed boy.

3. A library keeps a record of the books that are borrowed during one day. The results are shown in the chart below.



a) How many books were borrowed that day?
b) A book is chosen at random from the ones borrowed. Calculate the probability that it is:
   i) a thriller
   ii) a horror or a romance
   iii) not a horror or romance
   iv) not a biography.

Relative frequency

A football referee always used a special coin. He noticed that out of the last twenty matches the coin had come down heads far more often than tails. He wanted to know if the coin was fair, that is, if it was as likely to come down heads as tails.

He decided to do a simple experiment by spinning the coin lots of times. His results are shown below:

Number of trials	Number of heads	Relative frequency
100	40	0.4
200	90	0.45
300	142	
400	210	
500	260	
600	290	
700	345	
800	404	
900	451	
1000	499	

$$\text{The relative frequency} = \frac{\text{number of successful trials}}{\text{total number of trials}}$$

In the 'long run', that is after a large number of trials, did the coin appear to be fair?

Notice that the greater the number of trials the better the estimated probability or relative frequency is likely to be. The key idea is that increasing the number of trials gives a better estimate of the probability and the closer the result obtained by experiment will be to that obtained by calculation.

Exercise 28.5

1. Copy and complete the table on page 321. Draw a graph with Relative frequency as the y-axis and Number of trials as the x-axis. What do you notice?
2. Conduct a similar experiment using a dice to see if the number of sixes you get is the same as the theory of probability would make you expect.
3. Make a hexagonal spinner. Conduct an experiment to see if it is fair.
4. Ask a friend to put some coloured beads in a bag. Explain how you could use relative frequency in an experiment to find out how many beads of each colour are in the bag.

Worked examples

a) There is a group of 250 people in a hall. A girl calculates that the probability of randomly picking someone that she knows from the group is 0.032. Calculate the number of people in the group that the girl knows.

$$\text{Probability} = \frac{\text{number of favourable results } (F)}{\text{number of possible results}}$$

$$0.032 = \frac{F}{250}$$

$$250 \times 0.032 = F$$

$$8 = F$$

The girl knows 8 people in the group.

b) A boy enters 8 short stories into a writing competition. His father knows how many short stories have been entered into the competition, and tells his son that he has a probability of 0.016 of winning the first prize (assuming all the entries have an equal chance). How many short stories were entered into the competition?

$$\text{Probability} = \frac{\text{number of favourable results}}{\text{number of possible results } (T)}$$

$$0.016 = \frac{8}{T}$$

$$T = \frac{8}{0.016}$$

$$T = 500$$

So 500 short stories were entered into the competition.

Exercise 28.6

1. A boy calculates that he has a probability of 0.004 of winning the first prize in a photography competition if the selection is made at random. If 500 photographs are entered into the competition, how many photographs did the boy enter?

2. The probability of getting any particular number on a spinner game is given as 0.04. How many numbers are there on the spinner?

3. A bag contains 7 red counters, 5 blue, 3 green and 1 yellow. If one counter is picked at random, what is the probability that it is:
   a) yellow b) red c) blue or green
   d) red, blue or green e) not blue?

4. A boy collects marbles. He has the following colours in a bag: 28 red, 14 blue, 25 yellow, 17 green and 6 purple. If he picks one marble from the bag at random, what is the probability that it is:
   a) red b) blue c) yellow or blue
   d) purple e) not purple?

5. The probability of a boy randomly picking a marble of one of the following colours from another bag of marbles is:

   blue 0.25 red 0.2 yellow 0.15 green 0.35 white 0.05

   If there are 49 green marbles, how many of each other colour does he have in his bag?

6. There are six red sweets in a bag. If the probability of randomly picking a red sweet is 0.02, calculate the number of sweets in the bag.

7. The probability of getting a bad egg in a batch of 400 is 0.035. How many bad eggs are there likely to be in a batch?

8. A sports arena has 25 000 seats, some of which are VIP seats. For a charity event all the seats are allocated randomly. The probability of getting a VIP seat is 0.008. How many VIP seats are there?

9. The probability of Juan's favourite football team winning 4–0 is 0.05. How many times are they likely to win by this score in a season of 40 matches?

Student assessment 1

1. What is the probability of throwing the following numbers with a fair dice?
   a) a 2 b) not a 2
   c) less than 5 d) a 7

2. If you have a normal pack of 52 cards, what is the probability of drawing:
   a) a diamond b) a 6 c) a black card
   d) a picture card e) a card less than 5?

3. 250 coins, one of which is gold, are placed in a bag. What is the probability of getting the gold coin if I take, without looking, the following numbers of coins?
   a) 1 b) 5 c) 20
   d) 75 e) 250

4. A bag contains 11 blue, 8 red, 6 white, 5 green and 10 yellow counters. If one counter is taken from the bag at random, what is the probability that it is:
   a) blue b) green c) yellow d) not red?

5. The probability of randomly picking a red, blue or green marble from a bag containing 320 marbles is:

   red 0.5 blue 0.3 green 0.2

   How many marbles of each colour are there?

6. In a small town there are a number of sports clubs. The clubs have 750 members in total. The table below shows the types of sports club and the number of members each has.

	Tennis	Football	Golf	Hockey	Athletics
Men	30	110	40	15	10
Women	15	25	20	45	30
Boys	10	200	5	10	40
Girls	20	35	0	30	60

   A sports club member is chosen at random from the town. Calculate the probability that the member is:
   a) a man
   b) a girl
   c) a woman who does athletics
   d) a boy who plays football
   e) not a boy who plays football
   f) not a golf player
   g) a male who plays hockey.

7. A dice is thought to be biased. In order to test it, a boy rolls it 12 times and gets the following results:

Number	1	2	3	4	5	6
Frequency	2	2	2	2	2	2

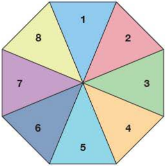
A girl decides to test the same dice and rolls it 60 times. The table below shows her results:

Number	1	2	3	4	5	6
Frequency	3	3	47	3	2	2

a) Which results are likely to be more reliable? Justify your answer.
b) What conclusion can you make about whether the dice is biased?

Student assessment 2

1. An octagonal spinner has the numbers 1 to 8 on it as shown (below).



What is the probability of spinning:
a) a 7 b) not a 7
c) a factor of 12 d) a 9?

2. A game requires the use of all the playing cards in a normal pack from 6 to King inclusive.
a) How many cards are used in the game?
b) What is the probability of randomly picking:
i) a 6 ii) a picture iii) a club
iv) a prime number v) an 8 or a spade?

3. 180 students in a school are offered a chance to attend a football match for free. If the students are chosen at random, what is the chance of being picked to go if the following numbers of tickets are available?

a) 1 b) 9 c) 15
d) 40 e) 180

4. A bag contains 11 white, 9 blue, 7 green and 5 red counters. What is the probability that a single counter drawn will be:

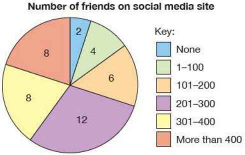
a) blue b) red or green c) not white?

5. The probability of randomly picking a red, blue or green marble from a bag containing 320 marbles is:

red 0.4 blue 0.25 green 0.35

If there are no other colours in the bag, how many marbles of each colour are there?

6. Students in a class conduct a survey to see how many friends they have on a social media site. The results were grouped and are shown in the pie chart below.



A student is chosen at random. What is the probability that he/she:

a) has 101–200 friends on the site
b) uses the site
c) has more than 200 friends on the site?

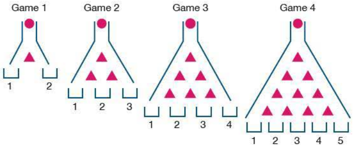
7. a) If I enter a competition and have a 0.000 02 probability of winning, how many people entered the competition?
b) What assumption do you have to make in order to answer part a)?

Topic 8

Mathematical investigations and ICT

Probability drop

A game involves dropping a red marble down a chute. On hitting a triangle divider, the marble can bounce either left or right. On completing the drop, the marble lands in one of the trays along the bottom. The trays are numbered from left to right. Different sizes of game exist; the four smallest versions are shown below:



To land in tray 2 in the second game above, the ball can travel in one of two ways. These are: Left – Right or Right – Left. This can be abbreviated to LR or RL.

1. State the different routes the marble can take to land in each of the trays in the third game.
2. State the different routes the marble can take to land in each of the trays in the fourth game.
3. State, giving reasons, the probability of a marble landing in tray 1 in the fourth game.
4. State, giving reasons, the probability of a marble landing in each of the other trays in the fourth game.
5. Investigate the probability of the marble landing in each of the different trays in larger games.
6. Using your findings from your investigation, predict the probability of a marble landing in tray 7 in the tenth game (11 trays at the bottom).

The following question is beyond the scope of the syllabus but is an interesting extension.

7. Investigate the links between this game and the sequence of numbers generated in Pascal's triangle.

Dice sum

Two ordinary dice are rolled and their scores added together.
Below is an incomplete table showing the possible outcomes:

		Dice 1					
		1	2	3	4	5	6
Dice 2	1	2			5		
	2						
	3				7		
	4				8		
	5				9	10	11
	6						12

1. Copy and complete the table to show all possible outcomes.

2. How many possible outcomes are there?

3. What is the most likely total when two dice are rolled?

4. What is the probability of getting a total score of 4?

5. What is the probability of getting the most likely total?

6. How many times more likely is a total score of 5 compared with a total score of 2?

Now consider rolling two four-sided dice, each numbered 1–4. Their scores are also added together.

7. Draw a table to show all the possible outcomes when the two four-sided dice are rolled and their scores added together.

8. How many possible outcomes are there?

9. What is the most likely total?

10. What is the probability of getting the most likely total?

11. Investigate the number of possible outcomes, the most likely total and its probability when two identical dice are rolled together and their scores added, i.e. consider 8-sided dice, 10-sided dice, etc.

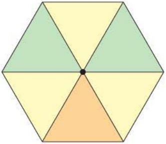
12. Consider two m-sided dice rolled together and their scores added.
a) What is the total number of outcomes in terms of m?
b) What is the most likely total, in terms of m?
c) What, in terms of m, is the probability of the most likely total?

13. Consider an m-sided and n-sided dice rolled together, where $m > n$.
   a) In terms of m and n, deduce the total number of outcomes.
   b) In terms of m and/or n, deduce the most likely total(s).
   c) In terms of m and/or n, deduce the probability of getting the most likely total.

ICT activity

For this activity, you will be testing the fairness of a spinner that you have constructed.

1. Using card, a pair of compasses and a ruler, construct a regular hexagon.
2. Divide your regular hexagon into six equal parts.
3. Colour the six parts using three different colours, as shown below:



4. Calculate the theoretical probability of each colour. Record these probabilities as percentages.
5. Carefully insert a small pencil through the centre of the hexagon to form a spinner.
6. Spin the spinner 60 times, recording your results in a spreadsheet.
7. Using the spreadsheet, produce a percentage pie chart of your results.
8. Compare the actual probabilities with the theoretical ones calculated in Q.4. What conclusions can you make about the fairness of your spinner?

Topic 9 Statistics

Syllabus

C9.1

Collect, classify and tabulate statistical data.
Read, interpret and draw simple inferences from tables and statistical diagrams.

C9.2

Construct and read bar charts, pie charts, pictograms, simple frequency distributions, histograms with equal intervals and scatter diagrams.

C9.3

Calculate the mean, median, mode and range for individual and discrete data and distinguish between the purposes for which they are used.

C9.4

Extended curriculum only.

C9.5

Extended curriculum only.

C9.6

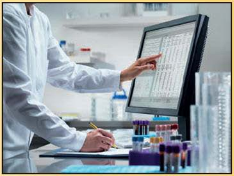
Understand what is meant by positive, negative and zero correlation with reference to a scatter diagram.

C9.7

Draw a straight line of best fit by eye.

Contents

The development of statistics



The earliest writing on statistics was found in a 9th-century book entitled *Manuscript on Deciphering Cryptographic Messages*, written by the Arab philosopher al-Kindi (801–873), who lived in Baghdad. In his book, he gave a detailed description of how to use statistics to unlock coded messages.

The *Nuova Cronica*, a 14th-century history of Florence by the Italian banker Giovanni Villani, includes much statistical information on population, commerce, trade and education.

Early statistics served the needs of states – *state -istics*. By the early 19th century, statistics included the collection and analysis of data in general. Today, statistics is widely employed in government, business, and natural and social sciences. The use of modern computers has enabled large-scale statistical computation and has also made possible new methods that are impractical to perform manually.

29 Mean, median, mode and range

Average

'Average' is a word which in general use is taken to mean somewhere in the middle. For example, a woman may describe herself as being of average height. A student may think he or she is of average ability in maths. Mathematics is more exact and uses three principal methods to measure average.

- The **mode** is the value occurring the most often.
- The **median** is the middle value when all the data is arranged in order of size.
- The **mean** is found by adding together all the values of the data and then dividing that total by the number of data values.

Spread

It is often useful to know how spread out the data is. It is possible for two sets of data to have the same mean and median but very different spreads.

The simplest measure of spread is the **range**. The range is simply the difference between the largest and smallest values in the data.

Worked examples **a)** i) Find the mean, median and mode of the data listed below.

1, 0, 2, 4, 1, 2, 1, 1, 2, 5, 5, 0, 1, 2, 3

$$\text{Mean} = \frac{1+0+2+4+1+2+1+1+2+5+5+0+1+2+3}{15}$$

$$= \frac{30}{15}$$

$$= 2$$

Arranging all the data in order and then picking out the middle number gives the median:

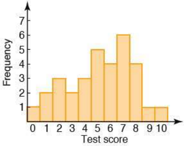
0, 0, 1, 1, 1, 1, 1, (2,) 2, 2, 2, 3, 4, 5, 5

The mode is the number which appeared most often. Therefore the mode is 1.

ii) Calculate the range of the data.

Largest value = 5
Smallest value = 0
Therefore the range = 5 – 0
= 5

b) i) The frequency chart (below) shows the score out of 10 achieved by a class in a maths test.



Calculate the mean, median and mode for this data. Transferring the results to a frequency table gives:

Test score	0	1	2	3	4	5	6	7	8	9	10	Total
Frequency	1	2	3	2	3	5	4	6	4	1	1	32
Frequency × score	0	2	6	6	12	25	24	42	32	9	10	168

In the total column we can see the number of students taking the test, i.e. 32, and also the total number of marks obtained by all the students, i.e. 168.

Therefore the mean score $= \frac{168}{32} = 5.25$

Arranging all the scores in order gives:

0, 1, 1, 2, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 5, (5, 6,) 6, 6, 6, 7, 7, 7, 7, 7, 7, 8, 8, 8, 8, 9, 10

Because there is an even number of students there isn't one middle number. There is a middle pair.

$$\text{Median} = \frac{(5+6)}{2}$$
$$= 5.5$$

The mode is 7 as it is the score which occurs most often.

ii) Calculate the range of the data.

Largest value = 10
Smallest value = 0
Therefore the range = 10 – 0
= 10

Exercise 29.1

1. Calculate the mean and range of each of the following sets of numbers:
   a) 6 7 8 10 11 12 13
   b) 4 4 6 6 6 7 8 10
   c) 36 38 40 43 47 50 55
   d) 7 6 8 9 5 4 10 11 12
   e) 12 24 36 48 60
   f) 17.5 16.3 18.6 19.1 24.5 27.8

2. Find the median and range of each of the following sets of numbers:
   a) 3 4 5 6 7
   b) 7 8 8 9 10 12 15
   c) 8 8 8 9 9 10 10 10 10
   d) 6 4 7 3 8 9 9 4 5
   e) 2 4 6 8
   f) 7 8 8 9 10 11 12 14
   g) 3.2 7.5 8.4 9.3 5.4 4.1 5.2 6.3
   h) 18 32 63 16 97 46 83

3. Find the mode and range of each of the following sets of numbers:
   a) 6 7 8 8 9 10 11
   b) 3 4 4 5 5 6 6 6 7 8 8
   c) 3 5 3 4 6 3 3 5 4 6 8
   d) 4 3 4 5 3 4 5 4
   e) 60 65 70 75 80 75
   f) 8 7 6 5 8 7 6 5 8

Exercise 29.2

In Q.1–5, find the mean, median, mode and range for each set of data.

1. A hockey team plays 15 matches. Below is a list of the numbers of goals scored in these matches.

   1, 0, 2, 4, 0, 1, 1, 1, 2, 5, 3, 0, 1, 2, 2

2. The total scores when two dice are thrown 20 times are:

   7, 4, 5, 7, 3, 2, 8, 6, 8, 7, 6, 5, 11, 9, 7, 3, 8, 7, 6, 5

3. The ages of a group of girls are:

   14 years 3 months, 14 years 5 months,
   13 years 11 months, 14 years 3 months,
   14 years 7 months, 14 years 3 months,
   14 years 1 month

4. The numbers of students present in a class over a three-week period are:

   28, 24, 25, 28, 23, 28, 27, 26, 27, 25, 28, 28, 28, 26, 25

5. An athlete keeps a record in seconds of her training times for the 100 m race:

   14.0, 14.3, 14.1, 14.3, 14.2, 14.0, 13.9, 13.8, 13.9, 13.8, 13.8, 13.7, 13.8, 13.8, 13.8

6. The mean mass of the 11 players in a football team is 80.3 kg. The mean mass of the team plus a substitute is 81.2 kg. Calculate the mass of the substitute.

7. After eight matches a basketball player had scored a mean of 27 points. After three more matches his mean was 29. Calculate the total number of points he scored in the last three games.

Exercise 29.3

1. An ordinary dice was rolled 60 times. The results are shown in the table below. Calculate the mean, median, mode and range of the scores.

Score	1	2	3	4	5	6
Frequency	12	11	8	12	7	10

2. Two dice were thrown 100 times. Each time their combined score was recorded. Below is a table of the results. Calculate the mean score.

Score	2	3	4	5	6	7	8	9	10	11	12
Frequency	5	6	7	9	14	16	13	11	9	7	3

3. Sixty flowering bushes are planted. At their flowering peak, the number of flowers per bush is counted and recorded. The results are shown in the table below.

Flowers per bush	0	1	2	3	4	5	6	7	8
Frequency	0	0	0	6	4	6	10	16	18

   a) Calculate the mean, median, mode and range of the number of flowers per bush.
   b) Which of the mean, median and mode would be most useful when advertising the bush to potential buyers?

Student assessment 1

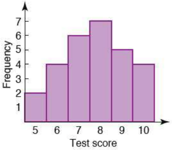
1. Find the mean, median, mode and range of each of the following sets of numbers:
   a) 63 72 72 84 86
   b) 6 6 6 12 18 24
   c) 5 10 5 15 5 20 5 25 15 10

2. The mean mass of the 15 players in a rugby team is 85 kg. The mean mass of the team plus a substitute is 83.5 kg. Calculate the mass of the substitute.

3. An ordinary dice was rolled 50 times. The results are shown below. Calculate the mean, median and mode of the scores.

Score	1	2	3	4	5	6
Frequency	8	11	5	9	7	10

4. The bar chart shows the marks out of 10 for an English test taken by a class of students.



   a) Calculate the number of students who took the test.
   b) Calculate for the class:
      i) the mean test result,
      ii) the median test result,
      iii) the modal test result.

5. A javelin thrower keeps a record of her best throws over ten competitions. These are shown in the table below.

Competition	1	2	3	4	5	6	7	8	9	10
Distance (m)	77	75	78	86	92	93	93	93	92	89

   Find the mean, median, mode and range of her throws.

30 Collecting and displaying data

Tally charts and frequency tables

Worked example The figures in the list below are the numbers of chocolate buttons in each of twenty packets of buttons.

35 36 38 37 35 36 38 36 37 35
36 36 38 36 35 38 37 38 36 38

The figures can be shown on a tally chart:

Number	Tally	Frequency
35	IIII	4
36	~~IIII~~ II	7
37	III	3
38	~~IIII~~ I	6

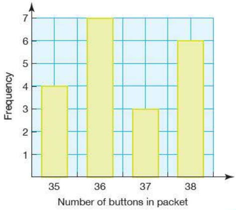
When the tallies are added up to get the frequency, the chart is usually called a **frequency table**. The information can then be displayed in a variety of ways.

Pictograms

@ = 1 packet of chocolate buttons

Buttons per packet	
35	@@@@
36	@@@@@@@
37	@@@
38	@@@@@@

Bar charts



Exercise 30.1

1. A group of students were asked to say which animal they would most like to have as a pet. The results are listed below:

   rabbit cat cat fish cat hamster rabbit cat
   hamster horse cat rabbit rabbit rabbit cat
   snake fish hamster horse cat rabbit cat cat
   rabbit hamster cat fish rabbit

   Make a frequency table of the data collected, then display the information on either a bar chart or a pictogram.

2. The colours of cars parked in a station car park are listed below:

   red blue green white red white red blue white
   red white green white red black red white blue
   red blue blue red white white white red white
   red white white red blue green

   Complete a frequency table of the data, and then display the information on a pictogram.

3. A survey is carried out among a group of students to find their favourite subject at school. The results are listed below:

   Art Maths Science English Maths Art English
   Maths English Art Science Science Science
   Maths Art English Art Science Maths English
   Art

   Complete a frequency table for this data, and illustrate it on a bar chart.

Grouped frequency tables

If there is a big range in the data it is easier to group the data in a **grouped frequency table**.

The groups are arranged so that no score can appear in two groups.

Worked example The scores for the first round of a golf competition are shown below.

71	75	82	96	83	75
76	82	103	85	79	77
83	85	88	104	76	77
79	83	84	86	88	102
95	96	99	102	75	72

This data can be grouped as shown in the table (right):

Score	Frequency
71–75	5
76–80	6
81–85	8
86–90	3
91–95	1
96–100	3
101–105	4
Total	30

Note: it is not possible to score 70.5 or 77.3 at golf. The scores are said to be **discrete**. If the data is **continuous**, for example when measuring time, the intervals can be shown as 0–, 10–, 20–, 30– and so on.

Histograms

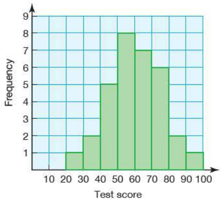
A histogram displays the frequency of either continuous or grouped discrete data in the form of bars. There are several important features of a histogram which distinguish it from a bar chart.

- The bars are joined together.
- The bars can be of varying width.
- The frequency of the data is represented by the area of the bar and not the height (though in the case of bars of equal width, the area is directly proportional to the height of the bar and so the height is usually used as the measure of frequency).

Worked example The table (below left) shows the marks out of 100 in a maths test for a class of 32 students. Draw a histogram representing this data.

All the class intervals are the same. As a result, the bars of the histogram will all be of equal width, and the frequency can be plotted on the vertical axis.

Test marks	Frequency
1–10	0
11–20	0
21–30	1
31–40	2
41–50	5
51–60	8
61–70	7
71–80	6
81–90	2
91–100	1



Exercise 30.2

1. The following data are the percentage scores obtained by students in an IGCSE Mathematics examination.

   35 42 48 53 67 69 52 54 73 46 62 59 71 65
   41 55 62 78 69 56 46 53 63 67 72 83 61 53
   92 65 45 52 61 59 75 53 57 68 72 64 63 85
   69 64

   Make a grouped frequency table with values 30–39, 40–49, etc., and illustrate the results on a histogram.

2. The numbers of mangoes collected from fifty trees are recorded below:

   35 78 15 65 69 32 12 9 89 110 112 148 98
   67 45 25 18 23 56 71 62 46 128 7 133 96 24
   38 73 82 142 15 98 6 123 49 85 63 19 111
   52 84 63 78 12 55 138 102 53 80

   Make a grouped frequency table with the values 0–9, 10–19, 20–29, etc. Illustrate this data on a histogram.

3. The number of cars coming off an assembly line per day is recorded over a thirty-day period. The results are below:

   15 26 34 53 64 9 32 49 21 15 7 58 63 48
   36 29 41 20 27 30 51 63 39 43 60 38 19 8
   35 10

   Make a grouped frequency table with class intervals 0–9, 10–19, 20–29, etc. Illustrate this data on a histogram.

4. The heights of 50 basketball players attending a tournament are recorded in the grouped frequency table below. Note that '1.8–' means $1.8 \leqslant H < 1.9$.

Height (m)	Frequency
1.8–	2
1.9–	5
2.0–	10
2.1–	22
2.2–	7
2.3–2.4	4

   Illustrate this data on a histogram.

5. The numbers of hours of overtime worked by employees at a factory over a period of a month are given in the table below.

Hours of overtime	Frequency
0–	12
10–	18
20–	22
30–	64
40–	32
50–60	20

Illustrate this data on a histogram.

6. The lengths of the index fingers of 30 students were measured. The results were recorded and are shown in a grouped frequency table below.

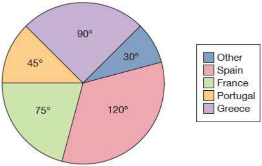
Length (cm)	Frequency
5.0–	3
5.5–	8
6.0–	10
6.5–	7
7.0–7.5	2

Illustrate this data on a histogram.

Pie charts

Data can be displayed on a **pie chart** – a circle divided into sectors. The size of the sector is in direct proportion to the frequency of the data. The sector size does not show the actual frequency. The actual frequency can be calculated easily from the size of the sector.

Worked examples **a)**



In a survey, 240 children were asked to vote for their favourite holiday destination. The results are shown on the pie chart above. Calculate the actual number of votes for each destination.

The total 240 votes are represented by 360°.
It follows that if 360° represents 240 votes:

There were $240 \times \frac{120}{360}$ votes for Spain
so, 80 votes for Spain.

There were $240 \times \frac{75}{360}$ votes for France
so, 50 votes for France.

There were $240 \times \frac{45}{360}$ votes for Portugal
so, 30 votes for Portugal.

There were $240 \times \frac{90}{360}$ votes for Greece
so, 60 votes for Greece.

Other destinations received $240 \times \frac{30}{360}$ votes
so, 20 votes for other destinations.

Note: it is worth checking your result by adding them:
$80 + 50 + 30 + 60 + 20 = 240$ total votes

b) The table shows the percentages of votes cast for various political parties in an election. If 5 million votes were cast in total, how many votes were cast for each party?

Party	Percentage of vote
Social Democrats	45%
Liberal Democrats	36%
Green Party	15%
Others	4%

The Social Democrats received $\frac{45}{100} \times 5$ million votes
so, 2.25 million votes.

The Liberal Democrats received $\frac{36}{100} \times 5$ million votes
so, 1.8 million votes.

The Green Party received $\frac{15}{100} \times 5$ million votes
so, 750 000 votes.

Other parties received $\frac{4}{100} \times 5$ million votes
so, 200 000 votes.

Check total:

$2.25 + 1.8 + 0.75 + 0.2 = 5$ (million votes)

c) The table (left) shows the results of a survey among 72 students to find their favourite sport. Display this data on a pie chart.

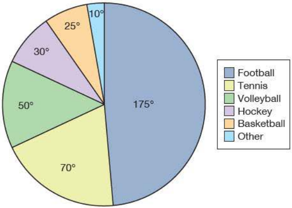
Sport	Frequency
Football	35
Tennis	14
Volleyball	10
Hockey	6
Basketball	5
Other	2

72 students are represented by 360°, so 1 student is represented by $\frac{360}{72}$ degrees. Therefore the size of each sector can be calculated as shown:

Football	$35 \times \frac{360}{72}$ degrees	i.e. 175°
Tennis	$14 \times \frac{360}{72}$ degrees	i.e. 70°
Volleyball	$10 \times \frac{360}{72}$ degrees	i.e. 50°
Hockey	$6 \times \frac{360}{72}$ degrees	i.e. 30°
Basketball	$5 \times \frac{360}{72}$ degrees	i.e. 25°
Other sports	$2 \times \frac{360}{72}$ degrees	i.e. 10°

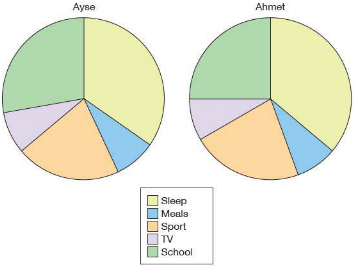
Check total:

$175 + 70 + 50 + 30 + 25 + 10 = 360$



Exercise 30.3

1. The pie charts below show how a girl and her brother spent one day. Calculate how many hours they spent on each activity. The diagrams are to scale.



2. A survey was carried out among a class of 40 students. The question asked was, 'How would you spend a gift of $15?'. The results are shown below:

Choice	Frequency
Music	14
Books	6
Clothes	18
Cinema	2

Illustrate these results on a pie chart.

3. A student works during the holidays. He earns a total of $2400. He estimates that the money earned has been used as follows: clothes $\frac{1}{3}$, transport $\frac{1}{5}$, entertainment $\frac{1}{4}$. He has saved the rest.

Calculate how much he has spent on each category, and illustrate this information on a pie chart.

4. The table below shows the numbers of passengers arriving at four British airports in one year. Show this information on a pie chart.

Airport	Number of passengers
Heathrow	27 million
Gatwick	6 million
Stansted	4.5 million
Luton	2.5 million

5. A girl completes a survey on the number of advertisements on each page of a 240-page magazine:

None – 15% One – 35% Two – 20%
Three – 22% More than three – 8%

a) Illustrate this information on a pie chart.
b) How many pages had no advertisements?

6. 300 students in a school of 1000 students play one musical instrument. Of these, 48% play the piano, 32% play the guitar, 12% play the violin and 5% play the drums. The rest play other instruments.
a) Illustrate this information on a pie chart.
b) Calculate how many students play the guitar.
c) Calculate the percentage of the students in the whole school who play the piano.

7. A research project looking at the careers of men and women in Spain produced the following results:

Career	Male (percentage)	Female (percentage)
Clerical	22	38
Professional	16	8
Skilled craft	24	16
Non-skilled craft	12	24
Social	8	10
Managerial	18	4

a) Illustrate this information on two pie charts, and make two statements that could be supported by the data.
b) If there are eight million women in employment in Spain, calculate the number in either professional or managerial employment.

Surveys

A survey requires data to be collected, organised, analysed and presented.

A survey may be carried out for interest's sake, for example to find out how many cars pass your school in an hour. A survey could be carried out to help future planning – information about traffic flow could lead to the building of new roads, or the placing of traffic lights or a pedestrian crossing.

Exercise 30.4

1. Below are a number of statements, some of which you may have heard or read before.

   Conduct a survey to collect data which will support or disprove one of the statements. Where possible, use pie charts to illustrate your results.

   a) Women's magazines are full of adverts.
   b) If you go to a football match you are lucky to see more than one goal scored.
   c) Every other car on the road is white.
   d) Girls are not interested in sport.
   e) Children today do nothing but watch TV.
   f) Newspapers have more sport than news in them.
   g) Most girls want to be nurses, teachers or secretaries.
   h) Nobody walks to school any more.
   i) Nearly everybody has a computer at home.
   j) Most of what is on TV comes from America.

2. Below are some instructions relating to a washing machine in English, French, German, Dutch and Italian.

   Analyse the data and write a report. You may wish to comment upon:

   a) the length of words in each language,
   b) the frequency of letters of the alphabet in different languages.

ENGLISH

ATTENTION
Do not interrupt drying during the programme.
This machine incorporates a temperature safety thermostat which will cut out the heating element and **switch the programme selector to an off position** in the event of a water blockage or power failure. In the event of this happening, reset the programme before selecting a further drying time.
For further instructions, consult the user manual.

FRENCH

ATTENTION
N'interrompez pas le séchage en cours de programme.
Une panne d'électricité ou un manque d'eau momentanés peuvent annuler le programme de séchage en cours. Dans ces cas arrêtez l'appareil, affichez de nouveau le programme et après remettez l'appareil en marche.

Pour d'ultérieures informations, rapportez-vous à la notice d'utilisation.

GERMAN

ACHTUNG
Die Trocknung soll nicht nach Anlaufen des Programms unterbrochen werden.
Ein kurzer Stromausfall bzw. Wassermangel kann das laufende Trocknungsprogramm annullieren. In diesem Falle Gerät ausschalten, Programm wieder einstellen und Gerät wieder einschalten.

Für nähere Angaben beziehen Sie sich auf die Bedienungsanleitung.

DUTCH

BELANGRIJK
Het droogprogramma niet onderbreken wanneer de machine in bedrijf is.
Door een korte stroom-of watertoevoeronderbreking kan het droogprogramma geannuleerd worden. Schakel in dit geval de machine uit, maak opnieuw uw programmakeuze en stel onmiddellijk weer in werking.

Verdere inlichtingen vindt u in de gebruiksaanwijzing.

ITALIAN

ATTENZIONE
Non interrompere l'asciugatura quando il programma è avviato.
La macchina è munita di un dispositivo di sicurezza che può annullare il programma di asciugaturea in corso quando si verifica una temporanea mancanza di acqua o di tensione.

In questi casi si dovrà spegnere la macchina, reimpostare il programma e poi riavviare la macchina.

Per ulteriori indicazioni, leggere il libretto istruzioni.

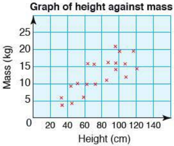
● Scatter diagrams

Scatter diagrams are particularly useful if we wish to see if there is a **correlation** (relationship) between two sets of data. The two values of data collected represent the coordinates of each point plotted. How the points lie when plotted indicates the type of relationship between the two sets of data.

Worked example The heights and masses of 20 children under the age of five were recorded. The heights were recorded in centimetres and the masses in kilograms. The data is shown below.

Height	32	34	45	46	52
Mass	5.8	3.8	9.0	4.2	10.1
Height	59	63	64	71	73
Mass	6.2	9.9	16.0	15.8	9.9
Height	86	87	95	96	96
Mass	11.1	16.4	20.9	16.2	14.0
Height	101	108	109	117	121
Mass	19.5	15.9	12.0	19.4	14.3

i) Plot a scatter diagram of the above data.

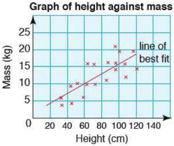


ii) Comment on any relationship you see.

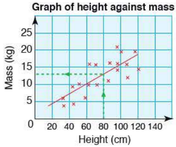
The points tend to lie in a diagonal direction from bottom left to top right. This suggests that as height increases then, in general, mass increases too. Therefore there is a **positive correlation** between height and mass.

iii) If another child was measured as having a height of 80 cm, approximately what mass would you expect him or her to be?

We assume that this child will follow the trend set by the other 20 children. To deduce an approximate value for the mass, we draw a **line of best fit**. This is a solid straight line which passes through the points as closely as possible, as shown (left).

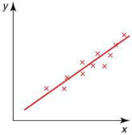


The line of best fit can now be used to give an approximate answer to the question. If a child has a height of 80 cm, you would expect his or her mass to be in the region of 13 kg.



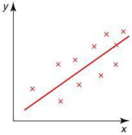
Types of correlation

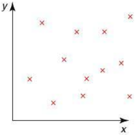
There are several types of correlation, depending on the arrangement of the points plotted on the scatter diagram. These are described below.



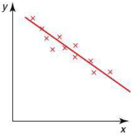
A **strong positive correlation** between the variables x and y. The points lie very close to the line of best fit.
As x increases, so does y.

A **weak positive correlation**. Although there is direction to the way the points are lying, they are not tightly packed around the line of best fit.
As x increases, y tends to increase too.

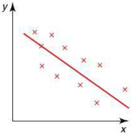




No correlation. As there is no pattern to the way in which the points are lying, there is no correlation between the variables x and y. As a result there can be no line of best fit.



A **strong negative correlation**. The points lie close around the line of best fit.
As x increases, y decreases.



A **weak negative correlation**.
The points are not tightly packed around the line of best fit.
As x increases, y tends to decrease.

Exercise 30.5

1. State what type of correlation you might expect, if any, if the following data was collected and plotted on a scatter diagram. Give reasons for your answer.
   a) A student's score in a maths exam and their score in a science exam.
   b) A student's hair colour and the distance they have to travel to school.
   c) The outdoor temperature and the number of cold drinks sold by a shop.
   d) The age of a motorcycle and its second-hand selling price.
   e) The number of people living in a house and the number of rooms the house has.
   f) The number of goals your opponents score and the number of times you win.
   g) A child's height and the child's age.
   h) A car's engine size and its fuel consumption.

2. A website gives average monthly readings for the number of hours of sunshine and the amount of rainfall in millimetres for several cities in Europe. The table below is a summary for July.

Place	Hours of sunshine	Rainfall (mm)
Athens	12	6
Belgrade	10	61
Copenhagen	8	71
Dubrovnik	12	26
Edinburgh	5	83
Frankfurt	7	70
Geneva	10	64
Helsinki	9	68
Innsbruck	7	134
Krakow	7	111
Lisbon	12	3
Marseilles	11	11
Naples	10	19
Oslo	7	82
Plovdiv	11	37
Reykjavik	6	50
Sofia	10	68
Tallinn	10	68
Valletta	12	0
York	6	62
Zurich	8	136

a) Plot a scatter diagram of the number of hours of sunshine against amount of rainfall. Use a spreadsheet if possible.

b) What type of correlation, if any, is there between the two variables? Comment on whether this is what you would expect.

3. The United Nations keeps an up-to-date database of statistical information on its member countries.
The table below shows some of the information available.

Country	Life expectancy at birth (years, 2005–2010)		Adult illiteracy rate (%, 2009)	Infant mortality rate (per 1000 births, 2005–2010)
	Female	Male		
Australia	84	79	1	5
Barbados	80	74	0.3	10
Brazil	76	69	10	24
Chad	50	47	68.2	130
China	75	71	6.7	23
Colombia	77	69	7.2	19
Congo	55	53	18.9	79
Cuba	81	77	0.2	5
Egypt	72	68	33	35
France	85	78	1	4
Germany	82	77	1	4
India	65	62	34	55
Israel	83	79	2.9	5
Japan	86	79	1	3
Kenya	55	54	26.4	64
Mexico	79	74	7.2	17
Nepal	67	66	43.5	42
Portugal	82	75	5.1	4
Russian Federation	73	60	0.5	12
Saudi Arabia	75	71	15	19
South Africa	53	50	12	49
United Kingdom	82	77	1	5
United States of America	81	77	1	6

a) By plotting a scatter diagram, decide if there is a correlation between the adult illiteracy rate and the infant mortality rate.

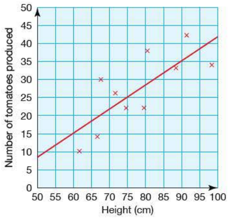
b) Are your findings in part a) what you expected? Explain your answer.

c) Without plotting a scatter diagram, decide if you think there is likely to be a correlation between male and female life expectancy at birth. Explain your reasons.

d) Plot a scatter diagram to test if your predictions for part c) were correct.

4. A gardener plants 10 tomato plants. He wants to see if there is a relationship between the number of tomatoes the plant produces and its height in centimetres.

The results are presented in the scatter diagram below. The line of best fit is also drawn.



a) Describe the correlation (if any) between the height of a plant and the number of tomatoes it produces.

b) A gardener has another plant grown in the same conditions as the others. If the height is 85 cm, estimate from the scatter diagram the number of tomatoes he can expect it to produce.

c) Another plant only produces 15 tomatoes. Deduce its height from the scatter diagram.

Student assessment 1

1. A supermarket manager notes the number of delivery vans which arrive each day:

Day	Mon	Tue	Wed	Thurs	Fri
Number of vans	6	12	9	33	36

Display this information on a suitable pictogram.

2. The areas of four countries are shown below. Illustrate this data as a bar chart.

Country	Nigeria	Republic of the Congo	Southern Sudan	Kenya
Area in 10 000 km^2	90	35	70	57

3. Display the data in Q.2 on a pie chart.

4. Fifty sacks of grain are weighed as they are unloaded from a truck. The mass of each is recorded in the grouped frequency table (below).

Mass (kg)	Frequency
$15 \leqslant M < 16$	0
$16 \leqslant M < 17$	3
$17 \leqslant M < 18$	6
$18 \leqslant M < 19$	14
$19 \leqslant M < 20$	18
$20 \leqslant M < 21$	8
$21 \leqslant M < 22$	1

Draw a histogram of this data.

5. Athletico football team keeps a record of attendance at their matches over a season. The attendances for the games are listed below:

18 418	16 161	15 988	13 417	12 004
11 932	8 461	10 841	19 000	19 214
16 645	14 782	17 935	8 874	19 023
19 875	16 472	14 840	18 450	16 875
13 012	17 858	19 098	6 972	8 452
7 882	11 972	16 461	11 311	19 458

a) Copy and complete the table below.

Attendance	Tally	Frequency
0–3999		
4000–7999		
8000–11 999		
12 000–15 999		
16 000–19 999		

b) Illustrate the above table using a histogram.

6. The table below gives the average time taken for 30 students in a class to get to school each morning, and the distance they live from the school.

Distance (km)	2	10	18	15	3	4	6	2	25	23	3	5	7	8	2
Time (min)	5	17	32	38	8	14	15	7	31	37	5	18	13	15	8
Distance (km)	19	15	11	9	2	3	4	3	14	14	4	12	12	7	1
Time (min)	27	40	23	30	10	10	8	9	15	23	9	20	27	18	4

a) Plot a scatter diagram of distance travelled against time taken.
b) Describe the correlation between the two variables.
c) Explain why some students who live further away may get to school more quickly than some of those who live nearer.
d) Draw a line of best fit on your scatter diagram.
e) A new student joins the class. Use your line of best fit to estimate how far away from school she might live if she takes, on average, 19 minutes to get to school each morning.

Student assessment 2

1. A building company spends four months working on a building site. The number of builders employed each month is as follows:

Month	July	August	September	October
Number of builders employed	40	55	85	20

Display this information on a suitable pictogram.

2. The table below shows the populations (in millions) of the continents:

Continent	Asia	Europe	America	Africa	Oceania
Population (millions)	4140	750	920	995	35

Display this information on a pie chart.

3. Display the data in Q.2 on a bar chart.

4. A hundred sacks of coffee with a nominal mass of 10 kg are unloaded from a train. The mass of each sack is checked and the results are presented in the table.

Mass (kg)	**Frequency**
$9.8 \leqslant M < 9.9$	14
$9.9 \leqslant M < 10.0$	22
$10.0 \leqslant M < 10.1$	36
$10.1 \leqslant M < 10.2$	20
$10.2 \leqslant M < 10.3$	8

Illustrate the data on a histogram.

5. 400 students sit an IGCSE exam. Their marks (as percentages) are shown in the table below.

Mark (%)	Frequency
31–40	21
41–50	55
51–60	125
61–70	74
71–80	52
81–90	45
91–100	28

Illustrate the data on a histogram.

6. A department store decides to investigate whether there is a correlation between the number of pairs of gloves it sells and the outside temperature. Over a one-year period the store records, every two weeks, how many pairs of gloves are sold and the mean daytime temperature during the same period. The results are given in the table below.

Mean temperature (°C)	3	6	8	10	10	11	12	14	16	16	17	18	18
Number of pairs of gloves	61	52	49	54	52	48	44	40	51	39	31	43	35
Mean temperature (°C)	19	19	20	21	22	22	24	25	25	26	26	27	28
Number of pairs of gloves	26	17	36	26	46	40	30	25	11	7	3	2	0

a) Plot a scatter diagram of mean temperature against number of pairs of gloves.

b) What type of correlation is there between the two variables?

c) How might this information be useful for the department store in the future?

d) The mean daytime temperature during the next two-week period is predicted to be 20°C. Draw a line of best fit on your scatter diagram and use it to estimate the number of pairs of gloves the department store can expect to sell.

Topic 9 Mathematical investigations and ICT

Reading age

Depending on their target audience, newspapers, magazines and books have different levels of readability. Some are easy to read and others more difficult.

1. Decide on some factors that you think would affect the readability of a text.
2. Write down the names of two newspapers which you think would have different reading ages. Give reasons for your answer.

There are established formulae for calculating the reading age of different texts.

One of these is the Gunning Fog Index. It calculates the reading age as follows:

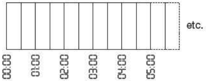
Reading age $= \frac{2}{5}\left(\frac{A}{n} + \frac{100L}{A}\right)$ where A = number of words
n = number of sentences
L = number of words with 3 or more syllables

3. Choose one article from each of the two newspapers you chose in Q.2. Use the Gunning Fog Index to calculate the reading ages for the articles. Do the results support your predictions?
4. Write down some factors which you think may affect the reliability of your results.

ICT activity

In this activity, you will be using the graphing facilities of a spreadsheet to compare the activities you do on a school day with the activities you do on a day at the weekend.

1. Prepare a 24-hour timeline for a weekday similar to the one shown below:



2. By using different colours for different activities, shade in the timeline to show what you did and when on a specific weekday, e.g. sleeping, eating, school, watching TV.

3. Add up the time spent on each activity and enter the results in a spreadsheet like the one below:

	A	B	C
1	Activity	Time spent (hrs)	
2	Sleeping		
3	Eating		
4	School		
5	TV		

4. Use the spreadsheet to produce a fully-labelled pie chart of this data.

5. Repeat steps 1–4 for a day at the weekend.

6. Comment on any differences and similarities between the pie charts for the two days.

Index