Cambridge International A and AS Level Mathematics
Pure Mathematics 1 Practice Book

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Answers

1 Algebra

Exercise 1.1

1 (i) $11a - 9b$

(ii) $-17cd^3 - 14cd - 15c^4d^2$

(iii) $5a + 3ab$

(iv) $2f^3g$

(v) $\frac{6x}{y}$

(vi) $\frac{8}{x}$

(vii) $\frac{149}{60x}$

(viii) $\frac{x+15}{20}$

2 (i) $3mn^2(4 + 3n)$

(ii) $(p - 4)(p + 3)$

(iii) $(3q - 1)(q + 2)$

(iv) $(t - 2u)(s + p)$

3 (i) $x = -17$

(ii) $x = \frac{2}{-12} = -\frac{1}{6}$

4 $x = 168$ km

5 (i) $e = \frac{bd}{v - bc}$

(ii) $k = \frac{p - n}{m^2 + w}$

(iii) $v = \frac{fu}{u - f}$

(iv) $e = \frac{d}{3} - \frac{p}{12\pi^2 w}$

Exercise 1.2

1 (i) $x = 0$ or -5

(ii) $x = 5$ or -5

(iii) $x = 4$ or -2

(iv) $x = -7$ or 2

(v) $x = 8$ or -5

(vi) $x = -\frac{1}{2}$ or 3

(vii) $x = \frac{3}{2}$ or -1

(viii) $x = -\frac{1}{3}$ or 2

(ix) $x = \frac{2}{5}$ or -3

(x) $x = 1$

(xi) $x = \pm\frac{1}{3}$

(xii) $x = \frac{1}{3}$ or $-\frac{3}{2}$

(xiii) $x = 0$ or 2

(xiv) $x = -\frac{4}{3}$ or $\frac{1}{2}$

2 (i) $x = \pm 1$

(ii) $x = \frac{1}{2}$ or 2

(iii) $x = 4$

(iv) $x = 1$ or $x = 2$

(v) $x = \pm\frac{1}{2}$

(vi) $x = 4$

Exercise 1.3

1 (i) $(x - 3)^2 - 8$

(ii) $(x + 2)^2 - 4$

(iii) $\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$

(iv) $(x + 1)^2 + 4$

2 (i) $x = 3 \pm \sqrt{8}$

(ii) $x = 0$ or -4

(iii) $x = 2$ or 1

3 (i) $2(x - 1)^2 + 5$

(ii) $2(x + 3)^2 - 7$

(iii) $3(x + 2)^2 - 16$

(iv) $5(x - 4)^2 - 8$

(v) $4(x + 3)^2 - 52$

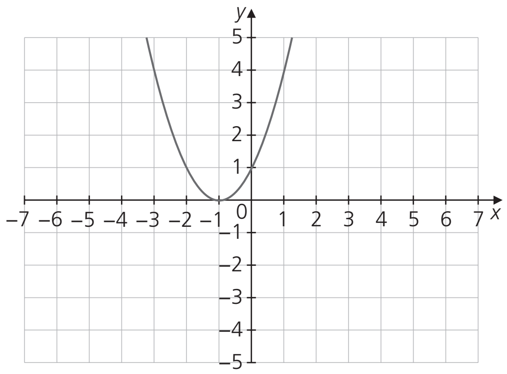
(vi) $9\left(x - \frac{1}{3}\right)^2 - 1$

4 (i) $a = 19, b = 4$

(ii) $a = 2, b = 1$

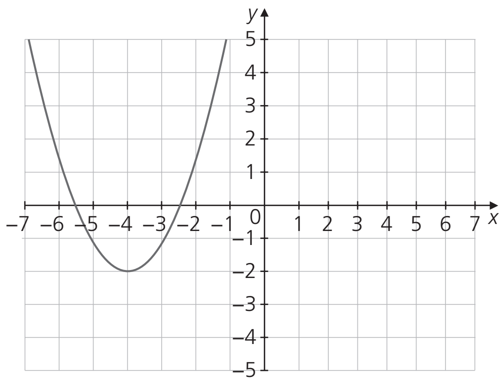
5 $a = 15, b = 2, c = 2$

6 (i)



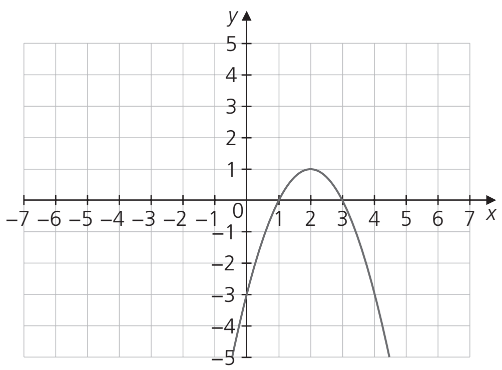
Vertex (–1, 0)

(ii)



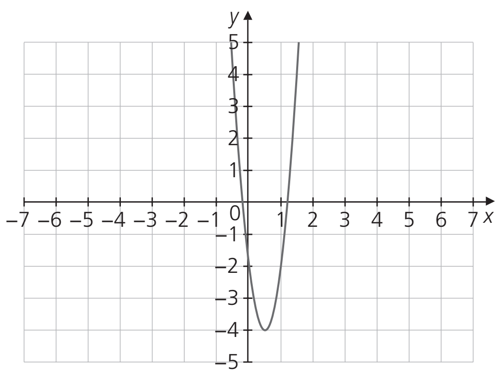
Vertex (–4, –2)

(iii)



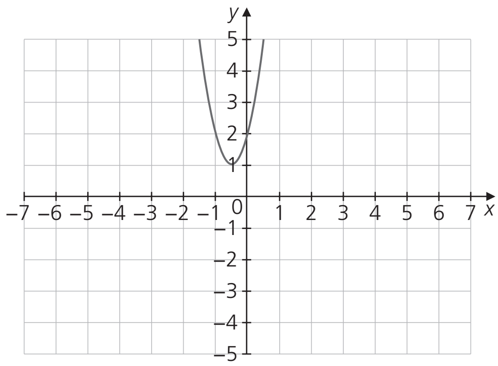
Vertex (2, 1)

(iv)



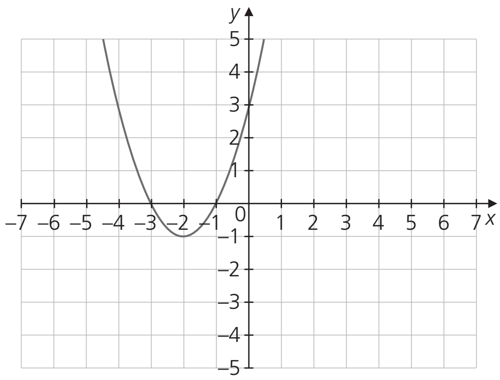
Vertex $(\frac{1}{2}, -4)$

(v)



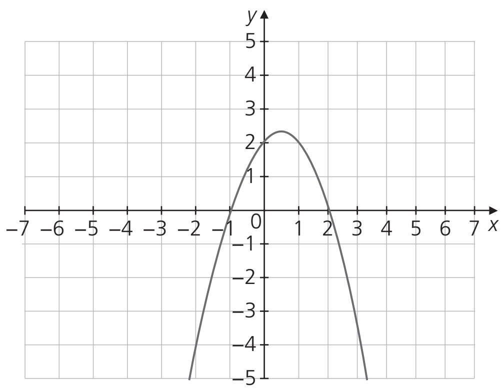
Vertex $(-\frac{1}{2}, 1)$

(vi)



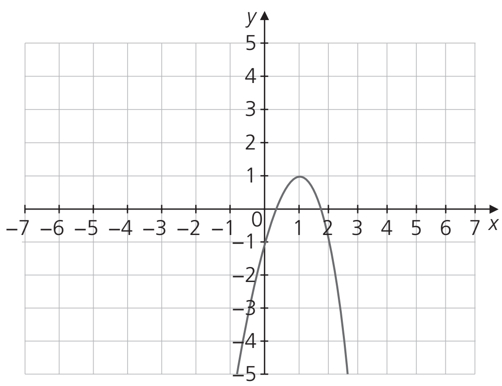
Vertex (– 2, –1)

(vii)



Vertex (0.5, 2.25)

(viii)



Vertex (1, 1)

7 (i) $y = (x - 4)^2 - 2$

(ii) $y = -(x - 1)^2 + 5$

(iii) $y = x(x + 2) = (x + 1)^2 - 1$

(iv) $y = -(x - 3)^2 + 4$

Exercise 1.4

1 (i) $x = 1.79$ or -2.79 (3 s.f.)

(ii) $x = 6.14$ or -1.14 (3 s.f.)

(iii) $x = 0.180$ or -1.85 (3 s.f.)

(iv) $x = -0.354$ or 2.35

2 (i) $3\sqrt{2}$

(ii) $5\sqrt{3}$

(iii) $3\sqrt{5}$

3 $a = 3$.

4 Using Δ to be the value of the discriminant:

(i) No solutions

(ii) No solutions

(iii) Two solutions

(iv) One solution

(v) No solutions

(vi) Two solutions

5 (i) $k = -4$

(ii) $k = 4$ or -4

(iii) $k = 2$

(iv) $k = 0$ or -4

6 (i) $1 > k$
$k < 1$

(ii) $k < -6$ or $k > 6$

(iii) $k < 0$ or $k > 4$

(iv) $k > \frac{-36}{24}$

$k > -\frac{3}{2}$

7 (i) $-\sqrt{2} < k < \sqrt{2}$

(ii) $\frac{1}{36} < k$

$k > \frac{1}{36}$

(iii) $k > \sqrt{2}$ or $k < -\sqrt{2}$

(iv) $k < -1$

8 (i) $m = -4, n = -12$

(ii) $p = -16$

Exercise 1.5

1 (i) $x = 3, y = -1$

(ii) $x = -2, y = 5$

(iii) $x = 3, y = 12$

(iv) $x = 6, y = -3$

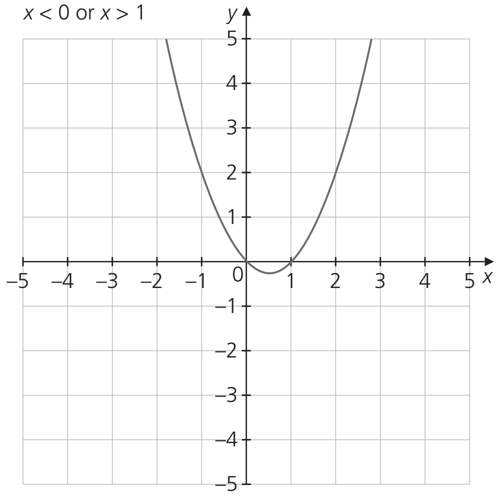
Exercise 1.6

1 (i) $x \leqslant -9$

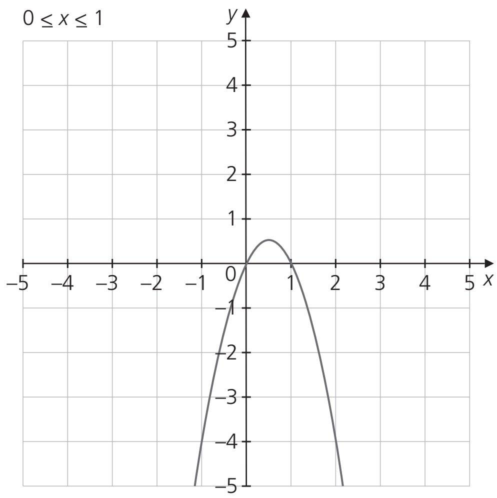
(ii) $x < -8$

(iii) $x > -2$ or $-2 < x$

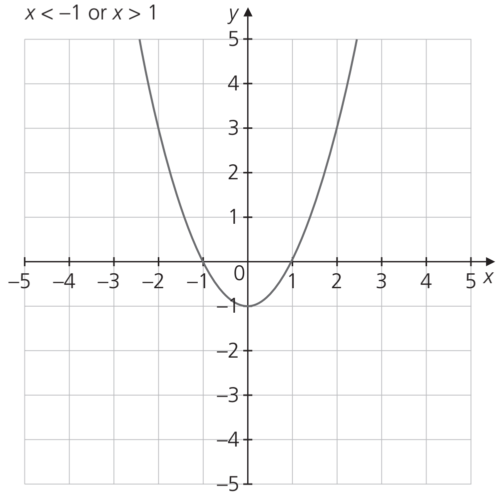
2 (i) $x < 0$ or $x > 1$



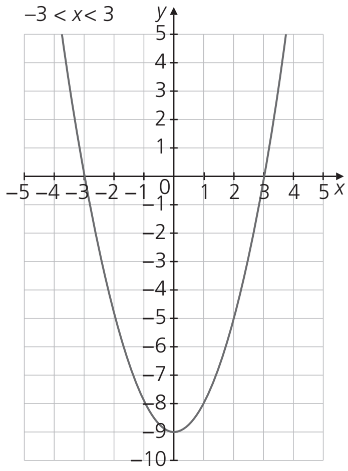
(ii) $0 \leq x \leq 1$



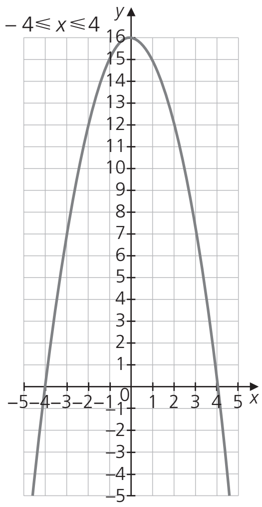
(iii) $x < -1$ or $x > 1$



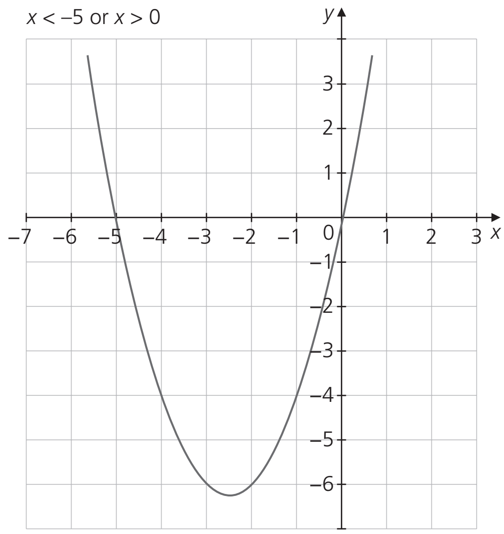
(iv) $-3 < x < 3$



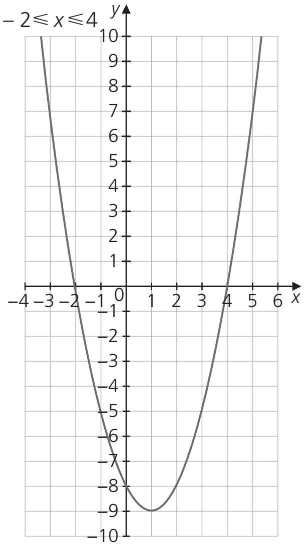
(v) $-4 \leqslant x \leqslant 4$



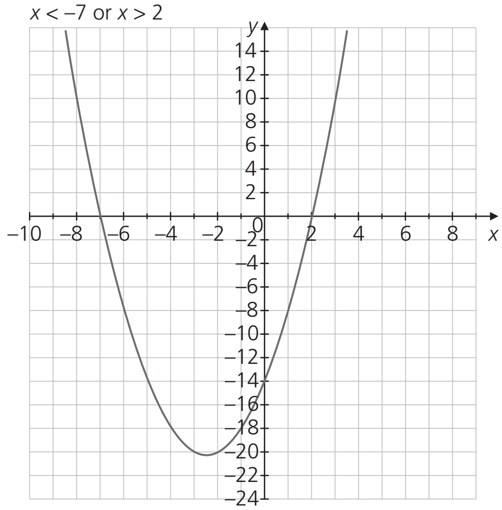
(vi) $x < -5$ or $x > 0$



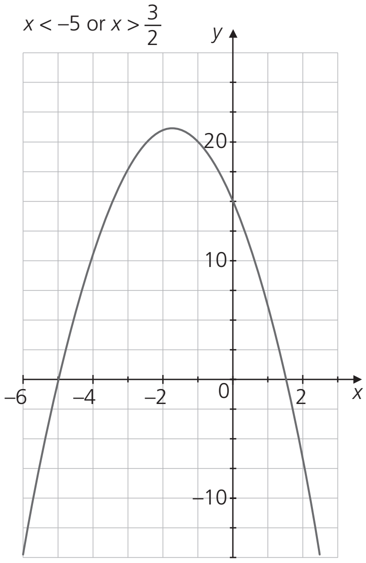
(vii) $-2 \leqslant x \leqslant 4$



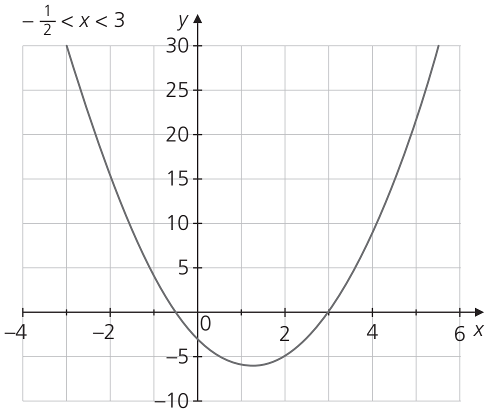
(viii) $x < -7$ or $x > 2$



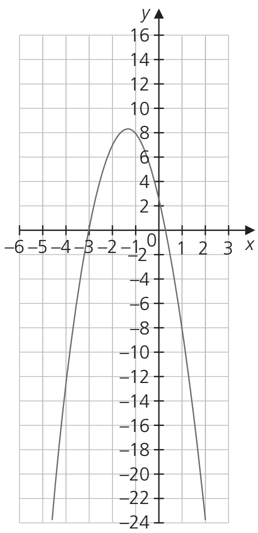
(ix) $x < -5$ or $x > \frac{3}{2}$



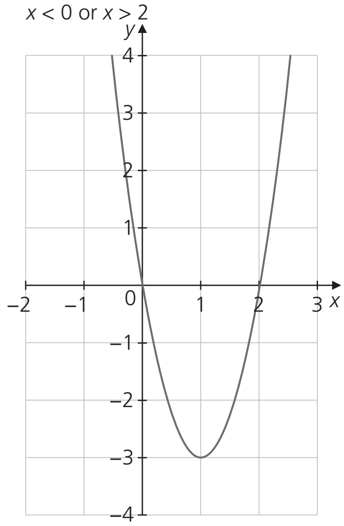
(x) $-\frac{1}{2} < x < 3$



(xi) $-3 \leqslant x \leqslant \frac{1}{3}$



(xii) $x < 0$ or $x > 2$



Stretch and challenge

1 (i) The equation could be thought of in the form $y = -a(x-b)^2 + c$.

Since the vertex is at approximately (9, 15) the equation becomes $y = -a(x-9)^2 + 15$.

Substituting the point (0, 8) we get:

$$8 = -a(0-9)^2 + 15$$
$$-7 = -a \times 81$$
$$a = \frac{7}{81}$$

Equation is $y = 15 - \frac{7}{81}(x-9)^2$

(Answers will vary depending on vertex chosen.)

(ii) Many answers are possible.
One approach is to determine the co-ordinates of the centre of the hoop and ensure these co-ordinates fit the equation.

2 (i) $ax^2 + bx + c = 0 \Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Given the roots we can write the equation as:

$$(x-\alpha)(x-\beta) = 0$$
$$x^2 - \alpha x - \beta x + \alpha\beta = 0$$
$$x^2 - (\alpha+\beta)x + \alpha\beta = 0$$

Equating, we get

$$-(\alpha+\beta) = \frac{b}{a}$$
$$\Rightarrow \alpha+\beta = -\frac{b}{a}$$
$$\alpha\beta = \frac{c}{a}$$

(ii) $4x^2 + (k+2)x + 72 = 0$

The roots are α and 2α.

$\alpha + 2\alpha = -\frac{k+2}{4}$ and $\alpha \times 2\alpha = \frac{72}{4}$

Solving the second equation,

$2\alpha^2 = 18$

$\alpha^2 = 9$

$\alpha = \pm 3$

Substituting into the first equation we get:

$3 + 2 \times 3 = -\frac{k+2}{4} \Rightarrow k = -38$

or $(-3) + 2 \times (-3) = -\frac{k+2}{4} \Rightarrow k = 34$

(iii) $\alpha + \beta = \frac{4}{3}$ and $\alpha\beta = \frac{7}{3}$

$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{\frac{4}{3}}{\frac{7}{3}} = \frac{4}{7}$ and $\frac{1}{\alpha\beta} = \frac{3}{7}$

So the equation is:

$x^2 - \frac{4}{7}x + \frac{3}{7} = 0$ or $7x^2 - 4x + 3 = 0$

(iv) $\alpha + \beta = 2$ and $\alpha\beta = 3$

$(\alpha+\beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$

$= \alpha^3 + \beta^3 + 3\alpha\beta(\alpha+\beta)$

Rearranging, we get

$\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$= 2^3 - 3(3)(2) = -10$

$(\alpha\beta)^3 = 3^3 = 27$

The equation is $x^2 + 10x + 27 = 0$

3 $9^x - 3^{x+1} - 54 = 0$

$(3^x)^2 - 3 \times 3^x - 54 = 0$

Let $t = 3^x$

$t^2 - 3t - 54 = 0$

$(t+6)(t-9) = 0$

$t = -6$ or 9

$3^x = -6$ or 9

$3^x = -6$ has no solutions so $3^x = 9 \Rightarrow x = 2$

4 $k2^x + 2^{-x} = 8$

Let $t = 2^x$, the equation can then be written as

$kt + \frac{1}{t} = 8$

$kt^2 + 1 = 8t$

$kt^2 - 8t + 1 = 0$

For a single solution $b^2 - 4ac = 0$

$(-8)^2 - 4 \times k \times 1 = 0 \Rightarrow k = 16$

$16t^2 - 8t + 1 = 0$

$(4t-1)^2 = 0 \Rightarrow t = \frac{1}{4}$

$2^x = \frac{1}{4} \Rightarrow x = -2$

Exam focus

1 $a = 2$, $b = 2$, $c = -20$

2 $a = 4$ and $b = -2$

3 $x \leqslant -1$ or $x \geqslant \frac{1}{2}$

4 **(i)** Vertex is $(-4, 6)$

(ii) $-6 < x < -2$

2 Co-ordinate geometry

Exercise 2.1

1 (i) $m = 3$

(ii) $m = 0$

(iii) $m = -\frac{1}{4}$

2 (i) Length $\sqrt{20} \approx 4.47$

Mid-point = (1, 0)

Gradient = $\frac{1}{2}$

(ii) Length = $\sqrt{116} \approx 10.8$

Mid-point = (10, 2)

Gradient = –2.5

(iii) Length = $\sqrt{80} \approx 8.94$

Mid-point = (–1, 1)

Gradient = $-\frac{1}{2}$

(iv) Length = $\sqrt{180} \approx 13.4$

Mid-point = (–7, –3)

Gradient = 2

3 $m = 5$

$n = 5$

4 (i) $x = 0$

(ii) $x = -4$

(iii) $x = 1$

5 (i) $m_1 = 26.6^\circ$

$m_2 = 63.4^\circ$

$m_3 = 135^\circ$

(ii) 36.9° (1 d.p.)

Exercise 2.2

1 (i) Gradient: 3, *y* intercept: –1

(ii) Gradient: –2, *y* intercept: 3

(iii) Gradient: $\frac{3}{2}$, *y* intercept: 1

(iv) Gradient: $-\frac{1}{2}$, *y* intercept: –2

2 (i) Equation: $y = -x + 2$, Gradient: –1, *y* intercept: 2

(ii) Equation: $y = 3x - 2$, Gradient: 3, *y* intercept: –2

(iii) Equation: $y = -\frac{1}{2}x + \frac{9}{4}$, Gradient: $-\frac{1}{2}$, *y* intercept: $\frac{9}{4}$

(iv) Equation: $y = \frac{3}{2}x - 4$, Gradient: $\frac{3}{2}$, *y* intercept: –4

3 (i) $3y = -x - 6$

$x + 3y + 6 = 0$

(ii) $15y = 12x + 5$

$12x - 15y + 5 = 0$

Exercise 2.3

1 A: $y = -x + 1$ or $x + y = 1$

B: $y = 2x - 3$

C: $y = \frac{1}{3}x$

D: $y = -3$

E: $x = 6$

F: $y = -\frac{2}{3}x - 2$

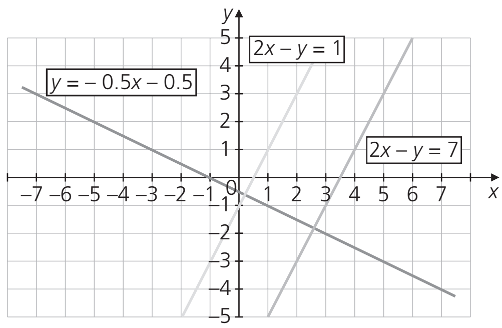
2

Gradient of line	Gradient of perpendicular
1	–1
$\frac{1}{4}$	–4
$-2\frac{1}{3} = -\frac{7}{3}$	$\frac{3}{7}$
$-\frac{3}{2}$	$\frac{2}{3}$
$0.3 = \frac{3}{10}$	$-\frac{10}{3}$

3 (i) $y = 2x - 7$

(ii) $y = -\frac{1}{2}x - \frac{1}{2}$

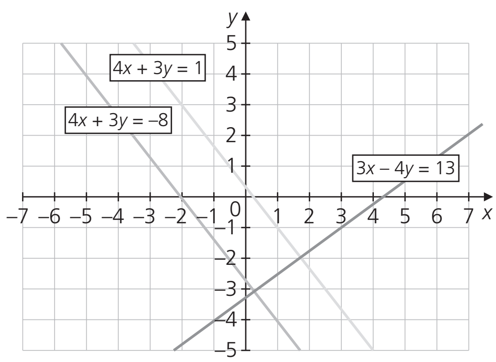
(iii)



4 (i) $4x + 3y + 8 = 0$

(ii) $3x - 4y - 13 = 0$

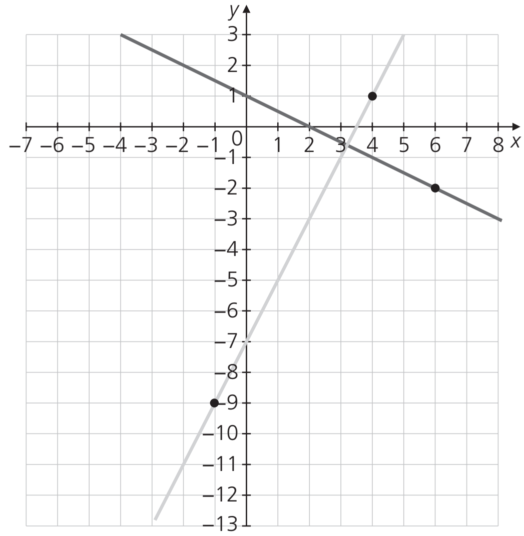
(iii)



5 (i) $\left(\frac{3}{2}, -4\right)$

(ii) $x + 2y - 2 = 0$

(iii)



Exercise 2.4

1 Area $= \frac{1}{2} \times \frac{8}{3} \times 4 = \frac{16}{3} = 5\frac{1}{3}$

2 $y = -2x - 2$

3 (i) $y = -3x - 7$

(ii) $d = \sqrt{\left(-3 - -\frac{3}{2}\right)^2 + \left(2 - -\frac{5}{2}\right)^2} = 4.74$ (3 s.f.)

4 $y = -\frac{3}{8}x + \frac{1}{8}$

5 $a : b = 3 : 2$

6 (i) $b = 10$

(ii) $y = \frac{3}{4}x + \frac{31}{4}$ or $3x - 4y + 31 = 0$

(iii) $y = -\frac{4}{3}x - \frac{8}{3}$ or $4x + 3y + 8 = 0$

(iv) D is (–5, 4).

Exercise 2.5

1 (i) (–1, –3) or (–2, –4)

(ii) (0, –2) or (–6, 10)

(iii) (2, 13)

(iv) $\left(\frac{1}{2}, 6\right)$ or (3, 1)

(v) $\left(\frac{7}{3}, -\frac{4}{3}\right)$ or (–1, 2)

(vi) (4, 1)

2 k = 8

3 $k > 2$ or $k < -2$

4 $k > \frac{9}{8}$

5 $k = 2$ or -1

6 $k = -7$ or 1

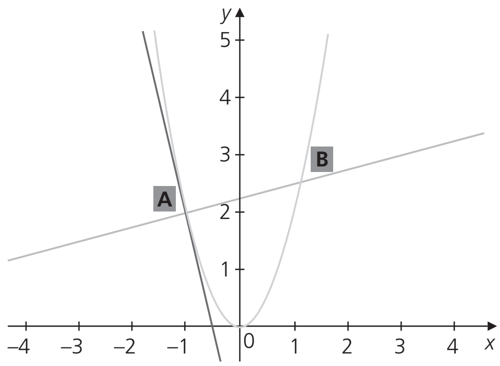
7 (i) $0 < k < 4$

(ii) Point is $\left(\frac{1}{2}, 3\right)$

8 Area of triangle ABC $= \frac{1}{2} \times 7\frac{1}{2} \times 3 = 11\frac{1}{4}$.

Stretch and challenge

1 The diagram shows the relationship between the tangent and normal at the point A.



We need to find the co-ordinates of both A and B.

To find where the line meets the curve, solve simultaneously:

$$4x + y + 2 = 0 \Rightarrow y = -4x - 2$$
$$2x^2 = -4x - 2$$
$$2x^2 + 4x + 2 = 0$$
$$x^2 + 2x + 1 = 0$$
$$(x + 1)(x + 1) = 0$$
$$x = -1$$
$$y = 2(-1)^2 = 2$$

So A is (–1, 2).

The gradient of the tangent to the curve is –4 so the gradient of the normal is $\frac{1}{4}$.

The equation of the normal is:

$$y = mx + c \Rightarrow 2 = \frac{1}{4} \times -1 + c \Rightarrow c = \frac{9}{4}$$

$$y = \frac{1}{4}x + \frac{9}{4}$$

To find B, solve simultaneously:

$$2x^2 = \frac{1}{4}x + \frac{9}{4}$$
$$8x^2 = x + 9 \Rightarrow 8x^2 - x - 9 = 0$$
$$(8x - 9)(x + 1) = 0$$
$$x = \frac{9}{8} \text{ or } -1$$

Substitute $x = \frac{9}{8}$ into the equation of the curve to find the y co-ordinate:

$$y = 2\left(\frac{9}{8}\right)^2 = 2 \times \frac{81}{64} = \frac{81}{32}$$

So B is $\left(\frac{9}{8}, \frac{81}{32}\right)$.

Finally, use Pythagoras' theorem to find the length of AB.

$$AB = \sqrt{\left(-1 - \frac{9}{8}\right)^2 + \left(2 - \frac{81}{32}\right)^2}$$
$$= 2.19 \text{ (3 s.f.)}$$

2 $\frac{x}{a} - \frac{y}{b} = 4$ multiplying every term by ab:

$bx - ay = 4ab$

$bx - 4ab = ay$

$y = \frac{bx - 4ab}{a}$

$y = \frac{b}{a}x - 4b$

From this form the gradient of the line is $\frac{b}{a}$,

so $\frac{b}{a} = \frac{1}{2} \Rightarrow 2b = a$

The x intercept is when $y = 0$ so

$\frac{x}{a} - \frac{0}{b} = 4 \Rightarrow \frac{x}{a} = 4 \Rightarrow x = 4a$, so M is $(4a, 0)$.

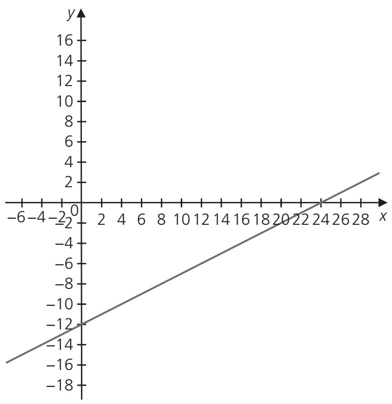
The y intercept is when $x = 0$ so

$\frac{0}{a} - \frac{y}{b} = 4 \Rightarrow -\frac{y}{b} = 4 \Rightarrow y = -4b$, so N is $(0, -4b)$.

By Pythagoras' theorem,

$MN = \sqrt{(4a)^2 + (4b)^2} = \sqrt{16a^2 + 16b^2}$

$$\begin{aligned}
\sqrt{16a^2 + 16b^2} &= \sqrt{720} \\
16a^2 + 16b^2 &= 720 \\
16(2b)^2 + 16b^2 &= 720 \\
16 \times 4b^2 + 16b^2 &= 720 \\
64b^2 + 16b^2 &= 720 \\
80b^2 &= 720 \\
b^2 &= 9 \\
b &= 3 \\
a &= 2b = 6
\end{aligned}$$



3 Mid-point of A(–6, 1) and B(k, –3) is

$\left(\frac{-6+k}{2}, \frac{1+-3}{2}\right) = \left(\frac{k-6}{2}, -1\right)$.

Gradient of AB is $\frac{1-(-3)}{-6-k} = \frac{4}{-6-k}$.

Gradient of perpendicular is $-\frac{-6-k}{4} = \frac{6+k}{4}$.

Equation of perpendicular bisector is

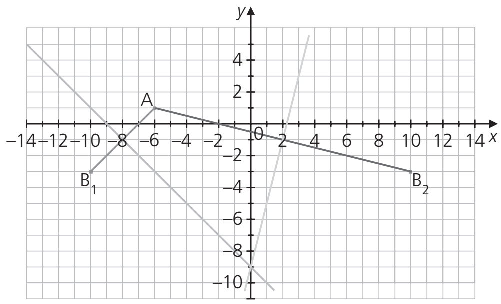
$y = mx + c \Rightarrow -1 = \frac{6+k}{4} \times \frac{k-6}{2} + c$

$-1 = \frac{k^2 - 36}{8} + c$

$c = -\frac{k^2 - 36}{8} - 1$

Since the y intercept is –9,

$$\begin{aligned}
-\frac{k^2 - 36}{8} - 1 &= -9 \\
-\frac{k^2 - 36}{8} &= -8 \\
\frac{k^2 - 36}{8} &= 8 \\
k^2 - 36 &= 64 \\
k^2 &= 100 \\
k &= \pm 10
\end{aligned}$$



4 (i) $x^2 - 4x + 3 = (x-1)(x-3)$ so the x intercepts are $x = 1$ or 3.

So the line must pass through (1, 0) or (3, 0).

$y = kx + 1 \Rightarrow 0 = k \times 1 + 1 \Rightarrow k = -1$

or $y = kx + 1 \Rightarrow 0 = k \times 3 + 1 \Rightarrow k = -\frac{1}{3}$

(ii) The x intercepts are $x = a$ or b.

So the line must pass through (a, 0) or (b, 0).

$y = kx + 1 \Rightarrow 0 = k \times a + 1 \Rightarrow k = -\frac{1}{a}$

$y = kx + 1 \Rightarrow 0 = k \times b + 1 \Rightarrow k = -\frac{1}{b}$

(iii) The x intercepts are $x = a$ or b.

So the line must pass through (a, 0) or (b, 0).

$y = mx + c \Rightarrow 0 = k \times a + c \Rightarrow k = -\frac{c}{a}$

$y = mx + c \Rightarrow 0 = k \times b + c \Rightarrow k = -\frac{c}{b}$

5 $2y = kx + 1 \Rightarrow y = \frac{k}{2}x + \frac{1}{2}$

$$\begin{aligned}
2x^2 + x + 1 &= \frac{k}{2}x + \frac{1}{2} \\
4x^2 + 2x + 2 &= kx + 1 \\
4x^2 + (2-k)x + 1 &= 0
\end{aligned}$$

For two points of intersection, $b^2 - 4ac > 0$

$$\begin{aligned}
(2-k)^2 - 4 \times 4 \times 1 &> 0 \\
4 - 4k + k^2 - 16 &> 0 \\
k^2 - 4k - 12 &> 0 \\
(k+2)(k-6) &> 0 \\
k < -2 \text{ or } k &> 6
\end{aligned}$$

Exam focus

1 The co-ordinates of D are (0, 1).

2 (i) The equation of CD is $y = -\frac{1}{2}x + 11$.

(ii) The equation of BD is $y = 2x - 14$.

(iii) The co-ordinates of D are (10, 6).

3 $k = -5$

3 Sequences and series

Exercise 3.1

1 **(i)** 71

(ii) –73

(iii) $\frac{58}{15}$

(iv) –2.15

2 **(i)** –27

(ii) 1080

(iii) 1591

(iv) 84.05

3 878

4 $10m + 9n$

5 78

6 \$60

7 –290

8 **(i)** Solving simultaneously, $d = 1.5$, $a = 23$

(ii) $n = 23$

9 –6

Exercise 3.2

1 **(i)** 256

(ii) $\frac{1}{8}$

(iii) 1×10^{-9}

(iv) 519 (3 s.f.)

2 **(i)** 765

(ii) 121.5

(iii) –213000 (3 s.f.)

(iv) –200

3 Parts **(ii)** and **(iv)** have a sum to infinity.

4 $x = \pm 8$

5 $u_5 = 128 \times 0.75^2 = 72$

$S_\infty = \dfrac{227.\dot{5}}{1 - 0.75} = 910.\dot{2}$

6 $S_\infty = 12 \Rightarrow \dfrac{8}{1 - r} = 12$

$u_5 = 8 \times \left(\frac{1}{3}\right)^{5-1} = \frac{8}{81}$

7 $-\frac{1}{2} < x < \frac{1}{2}$

8 $r = \dfrac{2}{\sqrt{2}} = \dfrac{\sqrt{8}}{2} = \sqrt{2}$

$u_6 = \sqrt{2} \times \left(\sqrt{2}\right)^{6-1} = 8$

9 \$29900 (3 s.f.)

10 **(i)** $x = -1$ or 16

(ii) 121.5

11 **(i)** 1260 minutes (3 s.f.)

(ii) $t = 6.53$ minutes (3 s.f.)

12 **(i)** For the A.P., $a = 18$

$u_4 = a + (4 - 1)d = 18 + 3d$

$u_6 = a + (6 - 1)d = 18 + 5d$

First three terms of the G.P. are

$18,\ 18 + 3d,\ 18 + 5d$

So $r = \dfrac{18 + 3d}{18} = \dfrac{18 + 5d}{18 + 3d}$

$$(18 + 3d)^2 = 18(18 + 5d)$$
$$324 + 108d + 9d^2 = 324 + 90d$$
$$9d^2 + 18d = 0$$
$$9d(d + 2) = 0$$
$$d = 0 \text{ or } -2$$

Since $d \neq 0$, $d = -2$

When $d = -2$,

$$r = \frac{18 + 3d}{18} = \frac{18 + 3 \times -2}{18} = \frac{12}{18} = \frac{2}{3}$$

(ii) 54

(iii) $n = 19$

Exercise 3.3

1 **(i)** $x^4 + 8x^3 + 24x^2 + 32x + 16$

(ii) $1 - 9x + 27x^2 - 27x^3$

2 **(i)** $256x^8 - 3072x^7 + 16128x^6 - \ldots$

(ii) $\dfrac{1}{x^6} + \dfrac{12}{x^4} + \dfrac{60}{x^2} + \ldots$

(iii) $2187a^7 - 5103a^6b + 5103a^5b^2 - \ldots$

(iv) $\dfrac{1}{x^{14}} + \dfrac{7}{x^9} + \dfrac{21}{x^4} + \ldots$

3 The x^2 term is $\dbinom{10}{4}(2x)^6\left(\dfrac{1}{x}\right)^4 = 13440x^2$

4 The x^4 term is $\dbinom{8}{2}\left(\dfrac{x}{2}\right)^6\left(-\dfrac{3}{x}\right)^2 = \dfrac{63}{16}x^4$

5 The term independent of x is

$$\binom{15}{6}\left(x^2\right)^9\left(\frac{5}{x^3}\right)^6 = 78203125$$

6 **(i)** The coefficient is 2160.

(ii) The coefficient is 5328.

7 $k = \pm 3$

8 $b = -\frac{1}{2}$

9 (i) $729 + 1458u + 1215u^2 + \ldots$

(ii) The coefficient of the x^2 term is -243

10 The coefficient of the x^2 term is 720

11 (i) $2187 - 10206x^2 + 20412x^4$

(ii) The coefficient of the x^4 term is 10 206

12 (i) $x^8 + 24x^6 + 252x^4 + \ldots$

(ii) The coefficient of the x^6 term is -228

13 $k = -4$

14 $a = 2.5$

Stretch and challenge

1 $u_7 = 400 \Rightarrow a + 6d = 400$

$$S_{30} = 1800 \Rightarrow \frac{30}{2}\left[2a + (30-1)\times d\right] = 1800$$

$$15\left[2a + 29d\right] = 1800 \Rightarrow 30a + 435d = 1800$$

$$2a + 29d = 120$$

Solving simultaneously,

$d = -40$ and $a = 640$

So Anna used Facebook for 640 minutes in week 1 of her plan.

2 General term for the first expansion is

$$\binom{6}{r}(kx^3)^{6-r}\left(-\frac{7}{x^3}\right)^r$$

Term independent of x is when $r = 3$

$$\binom{6}{3}(kx^3)^{6-3}\left(-\frac{7}{x^3}\right)^3 = 20k^3x^9 \times \frac{-343}{x^9}$$

$$= -6860k^3$$

General term for the second expansion is

$$\binom{8}{r}(kx^4)^{8-r}\left(\frac{m}{x^4}\right)^r$$

Term independent of x is when $r = 4$

$$\binom{8}{4}(kx^4)^{8-4}\left(\frac{m}{x^4}\right)^4 = 70k^4x^{16} \times \frac{m^4}{x^{16}} = 70k^4m^4$$

So $70k^4m^4 = -6860k^3$

$$k = -\frac{98}{m^4}$$

3 $a + ar = -3 \Rightarrow a(1+r) = -3$

$$ar^5 + ar^6 = 729 \Rightarrow ar^5(1+r) = 729$$

$$\frac{ar^5(1+r)}{a(1+r)} = \frac{729}{-3}$$

$$r^5 = -243$$

$$r = \sqrt[5]{-243} = -3$$

$$a(1 + (-3)) = -3 \Rightarrow -2a = -3 \Rightarrow a = 1.5$$

4 $u_{12} = 3 \times u_6 \Rightarrow a + 11d = 3(a + 5d)$

$$a + 11d = 3a + 15d$$

$$2a + 4d = 0$$

$$a + 2d = 0$$

$$S_{30} = 450 \Rightarrow \frac{30}{2}\left[2a + 29d\right] = 450$$

$$30a + 435d = 450$$

$$6a + 87d = 90$$

Solving simultaneously:

$a + 2d = 0$

$6a + 87d = 90$

$a = -2.4$, $d = 1.2$

5 Value of stamp 1: 55 000, 52 600, …

A.P. with $a = 55\,000$, $d = -2400$

Value of stamp 2: G.P. with $r = 0.96$

For Stamp 1,

$$u_{10} = 55\,000 + (10-1)\times -2400 = 33\,400$$

For stamp 2:

$$a \times 0.96^{10} = 33\,400$$

$$a = \frac{33\,400}{0.96^{10}} = \$50\,238$$

So the second stamp was bought for \$50 238

6 (i) The general term is $\binom{10}{r}(x)^{10-r}(a)^r$

The expansion is:

$$x^{10} + 10ax^9 + 45a^2x^8 + 120a^3x^7 + 210a^4x^6 + 252a^5x^5 + 210a^6x^4 + 120a^7x^3 + 45a^8x^2 + 10a^9x + a^{10}$$

Comparing the coefficients of the terms either side of the x^3 term:

$$120a^7 > 210a^6 \Rightarrow a > \frac{7}{4}$$

$$120a^7 > 45a^8 \Rightarrow a < \frac{8}{3}$$

$$\frac{7}{4} < a < \frac{8}{3}$$

(ii) The terms we are interested in are:

$$\ldots\binom{100}{56}(x)^{44}(a)^{56} + \binom{100}{57}(x)^{43}(a)^{57} + \binom{100}{58}(x)^{42}(a)^{58}\ldots$$

We want $\binom{100}{57}a^{57} > \binom{100}{56}a^{56} \Rightarrow a > \frac{57}{44}$

and $\binom{100}{57}a^{57} > \binom{100}{58}a^{58} \Rightarrow a < \frac{58}{43}$

$$\frac{57}{44} < a < \frac{58}{43}$$

Exam focus

1 **(i)** The coefficient of the x^3 term is –241 920.

(ii) The coefficient of the x^3 term is –435 456.

2 $\binom{12}{4}\left(\frac{2}{x}\right)^{12-4}\left(x^2\right)^4 = 495 \times \frac{2^8}{x^8} \times x^8 = 126\,720$

3 $135a^4 = 2160$

$a^4 = 16$

$a = 2$

4 $u_{21} = -15$

5 $n = 29$.

6 36

4 Functions

Exercise 4.1

1 (i) $f(x) = x + 3$

(ii) $f(x) = x^2 + 1$

(iii) $f(x) = -2x + 1 = 1 - 2x$

(iv) $f(x) = 10x + 2$

2 All are functions.

(ii) Is *not* one-to-one the rest are one-to-one.

3 (i) Is a function. Is not one-to-one.

(ii) Is not a function. Is not one-to-one.

(iii) Is a one-to-one function.

(iv) Is a function. Is not one-to-one.

4 (i) $f(-2) = 9$

(ii) $f(x + 1) = x^2$

5 (i) Is not a function. Is not one-to-one.

(ii) Is a function. Is not one-to-one.

(iii) Is a one-to-one function.

(iv) Is a function. Is not one-to-one.

6 (i) Domain: $x \in \mathbb{R}$

Range: $f(x) \geqslant -2$

(ii) Domain: $x \in \mathbb{R}$

Range: $f(x) \leqslant 3$

(iii) Domain: $x \geqslant 1$

Range: $f(x) \geqslant 2$

(iv) Domain: $x \in \mathbb{R}$

Range: $f(x) \in \mathbb{R}$

7 Range is $f(x) \geqslant -2$

8 Domain: $0 \leqslant x \leqslant 4$

Range: $0 \leqslant f(x) \leqslant \sqrt{2}$

9 Domain: $x \in \mathbb{R}$

Range: $f(x) \leqslant 1$

10 Given that $x \geqslant -1$, the range is $f(x) \geqslant -3$

Exercise 4.2

1 (i) $f(-2) = 3$

(ii) $g(-2) = -3$

(iii) $fg(-2) = 4$

(iv) $gf(-2) = -8$

(v) $fg(x) = x^2$

(vi) $gf(x) = 2x - x^2$

(vii) $ff(x) = x$

(viii) $gg(x) = 2x^2 - x^4$

2 $a = 3, b = -1$ or $a = -3, b = 2$

3 (i) $fg(x) = 14 + \frac{6}{x}$

(ii)
$$gf(x) = g(3x + 2) = 4 + \frac{2}{3x + 2} = \frac{4(3x + 2) + 2}{3x + 2} = \frac{12x + 8 + 2}{3x + 2} = \frac{12x + 10}{3x + 2}$$

(iii) $x = -\frac{11}{9}$

(iv) $k > -2$

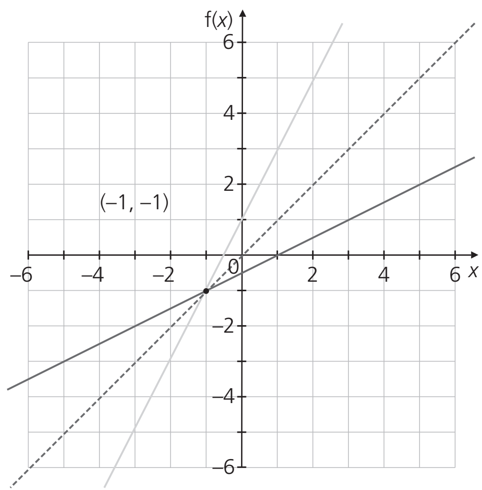
4 $a = 3, b = 2$

5 (i) $x \leqslant -3$ or $x \geqslant -1$

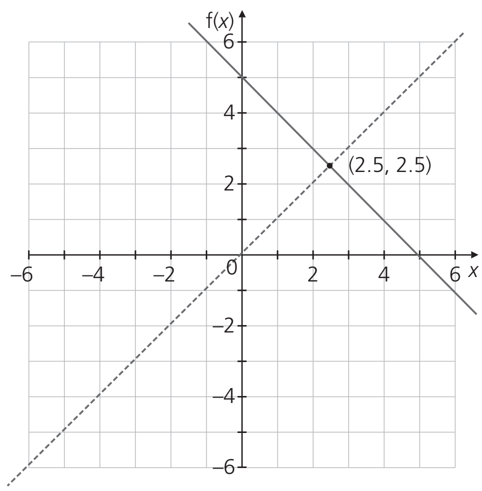
(ii) $-5 \leqslant x \leqslant -\frac{1}{3}$

Exercise 4.3

1 (i) $f^{-1}(x) = \frac{x - 1}{2}$

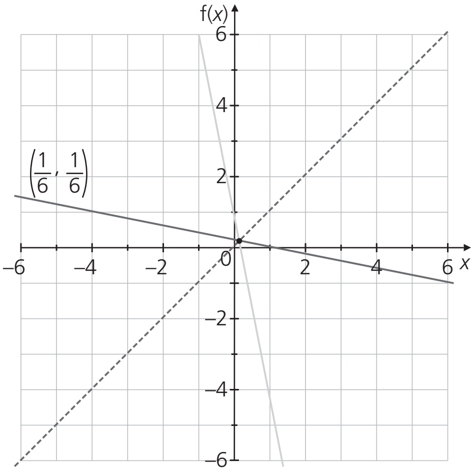


(ii) $f^{-1}(x) = 5 - x$

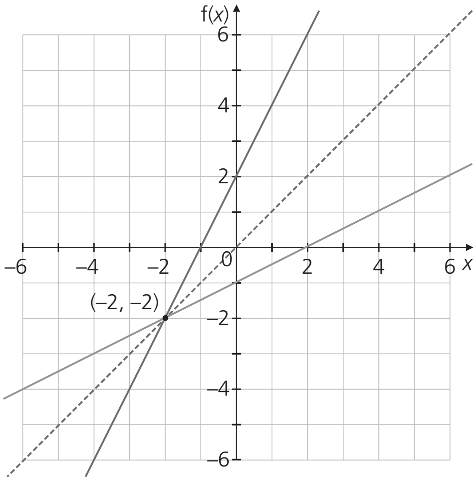


(iii) $f(x) = 1 - 5x = -5x + 1$

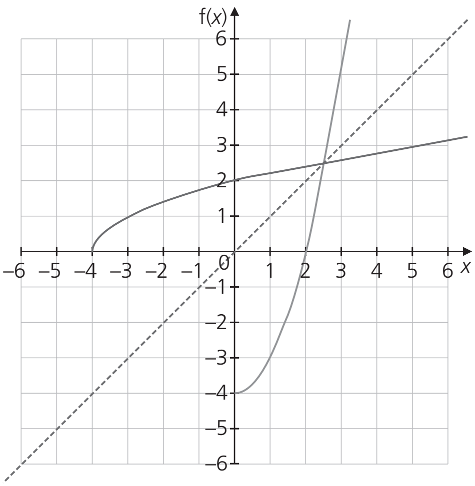
$$f^{-1}(x) = \frac{x-1}{-5} = \frac{1-x}{5}$$



(iv) $f^{-1}(x) = 2(x + 1) = 2x + 2$



2 (i)



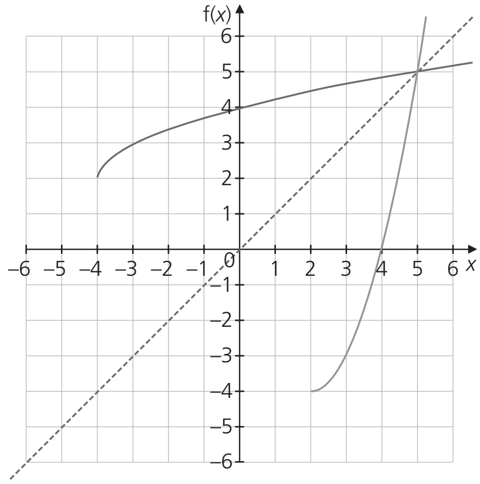
Domain $f(x)$: $x \geqslant 0$

Range $f(x)$: $f(x) \geqslant -4$

Domain $f^{-1}(x)$: $x \geqslant -4$

Range $f^{-1}(x)$: $f^{-1}(x) \geqslant 0$

(ii)



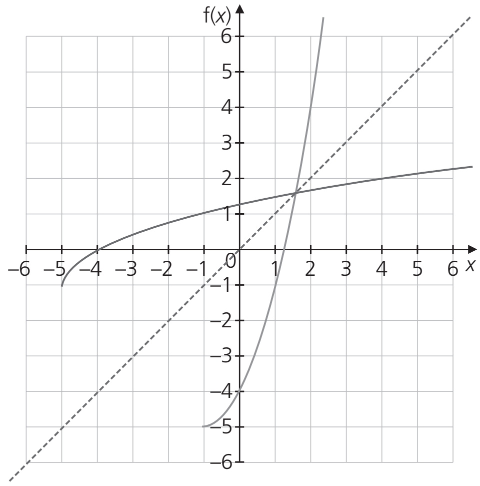
Domain $f(x)$: $x \geqslant 2$

Range $f(x)$: $f(x) \geqslant -4$

Domain $f^{-1}(x)$: $x \geqslant -4$

Range $f^{-1}(x)$: $f^{-1}(x) \geqslant 2$

(iii)



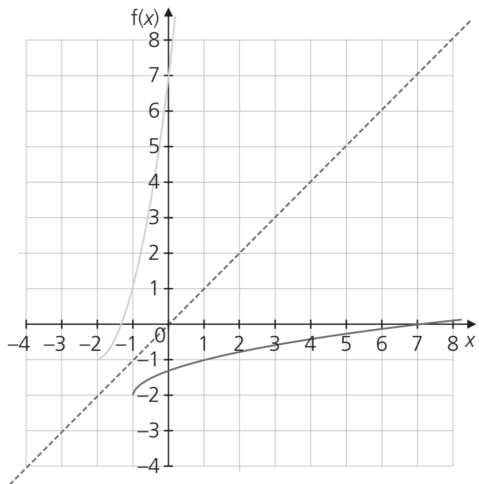
Domain $f(x)$: $x \geqslant -1$

Range $f(x)$: $f(x) \geqslant -5$

Domain $f^{-1}(x)$: $x \geqslant -5$

Range $f^{-1}(x)$: $f^{-1}(x) \geqslant -1$

(iv)



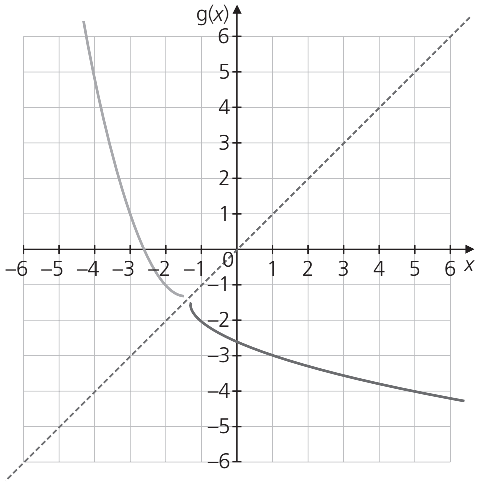
Domain $f(x)$: $x \geqslant -2$

Range $f(x)$: $f(x) \geqslant -1$

Domain $f^{-1}(x)$: $x \geqslant -1$

Range $f^{-1}(x)$: $f^{-1}(x) \geqslant -2$

3 The largest possible value of k is $-\frac{3}{2}$.



4 $j^{-1}(x) = 3 - \frac{2}{x} = \frac{3x-2}{x}$

5 $f^{-1}(x) = \sqrt[3]{\frac{x+1}{2}} - 1$

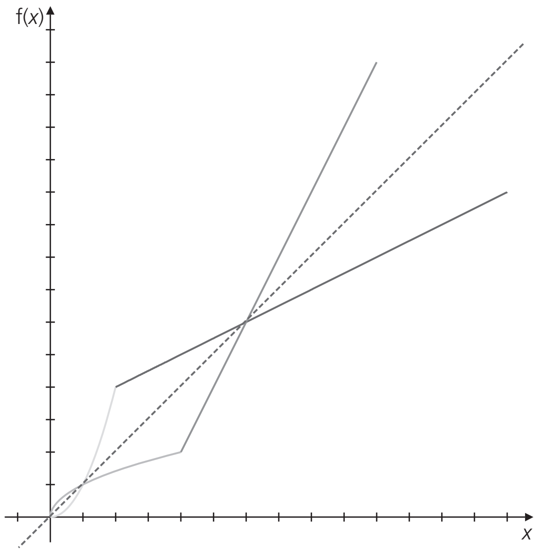
6 $g^{-1}(x) = \sqrt{(x-2)^4 - 1}$

Domain $g(x)$: $x \geqslant 0$
Range $g(x)$: $g(x) \geqslant 3$
Domain $g^{-1}(x)$: $x \geqslant 3$
Range $g^{-1}(x)$: $g^{-1}(x) \geqslant 0$

7 $k = \pi$

8 (i) Range is $0 \leqslant f \leqslant 10$

(ii)



(iii) $4 \leqslant x \leqslant 10$

9 (i) $ff(x) = f\left(\frac{x+2}{4x-1}\right) = \frac{\left(\frac{x+2}{4x-1}\right)+2}{4\left(\frac{x+2}{4x-1}\right)-1}$

$$= \frac{\frac{x+2+2(4x-1)}{4x-1}}{\frac{4(x+2)-(4x-1)}{4x-1}}$$

$$= \frac{\frac{9x}{4x-1}}{\frac{9}{4x-1}}$$

$$= \frac{9x}{4x-1} \times \frac{4x-1}{9}$$

$$= x$$

(ii) $f^{-1}(x) = \frac{x+2}{4x-1},\ x \in \mathbb{R},\ x \neq \frac{1}{4}.$

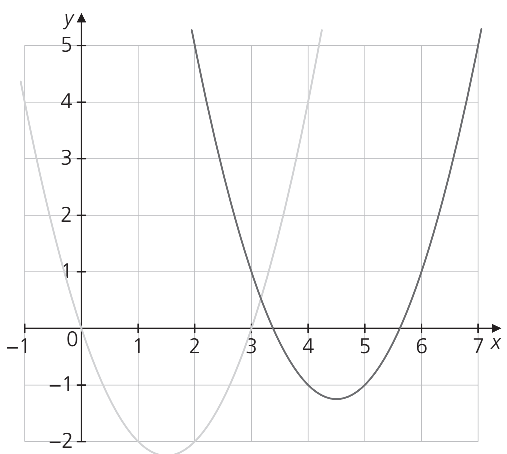
Stretch and challenge

1 (i) $y = af(b(x+c)) + d$

Constant	**Effect on graph**
a	$\lvert a\rvert > 1$ Vertical expansion $0 < \lvert a\rvert < 1$ Vertical compression $a < 0$ Reflects the graph in x axis
b	$b > 1$ Horizontal compression $0 < b < 1$ Horizontal expansion
c	Shift left ($c > 0$) or right ($c < 0$)
d	Shift d units up ($d > 0$) or down ($d < 0$)

(ii) The translation of $f(x) = x^2 - 3x$ 3 units right and one up is given by

$f(x-3)+1$
$= (x-3)^2 - 3(x-3) + 1$
$= x^2 - 6x + 9 - 3x + 9 + 1$
$= x^2 - 9x + 19$



2 $g(1) = f(0) = 1$
$g(2) = fg(1) = f(1) = 3$
$g(3) = fg(2) = f(3) = 7$
$g(4) = fg(3) = f(7) = 15$

We can see the pattern here, every number is one less than a power of 2 so $g(n) = 2^n - 1$.

$h(0) = g(2) = 3$
$h(1) = gh(0) = g(3) = 7$
$h(2) = gh(1) = g(7) = 2^7 - 1 = 127$

3 $f(x+1) + f(x-1) = af(x)$

$$3^{x+1} + 3^{x-1} = a3^x$$

$$a = \frac{3^{x+1} + 3^{x-1}}{3^x} = \frac{3^x(3 + 3^{-1})}{3^x} = 3 + 3^{-1} = \frac{10}{3}$$

4 $f(2n-1) = a(2n-1) + b = 2an - a + b$

$f(2n) - 1 = a(2n) + b - 1 = 2an + b - 1$

$2f(n) - 1 = 2(an + b) - 1 = 2an + 2b - 1$

$2an + b$ is common to all three expression so the three expressions can be simplified to $-a$, -1 and $b - 1$·

Since -1 is one of the terms, the possible consecutive terms are

(i) $-3, -2, -1$

(ii) $-2, -1, 0$

(iii) $-1, 0, 1$

(i) $a = 3, b = -1 \Rightarrow f(n) = 3n - 1$

$a = 2, b = -2 \Rightarrow f(n) = 2n - 2$

(ii) $a = 2, b = 1 \Rightarrow f(n) = 2n + 1$

(iii) $a = -1, b = 1 \Rightarrow f(n) = -n + 1$

Exam focus

1 The domain of $f(x)$ is $x \geqslant -2$

The range of $f(x)$ is $y \geqslant -2$

The domain of $f^{-1}(x)$ is $x \geqslant -2$

The range of $f^{-1}(x)$ is $y \geqslant -2$

2 (i) $a = 2 \quad b = 3$

(ii) $x = -2$

3 (i) $k = -3.$

(ii) $g^{-1}(x) = \sqrt{x+7} - 3$

4 (i) $x = \frac{1}{3}$ or $x = 2$

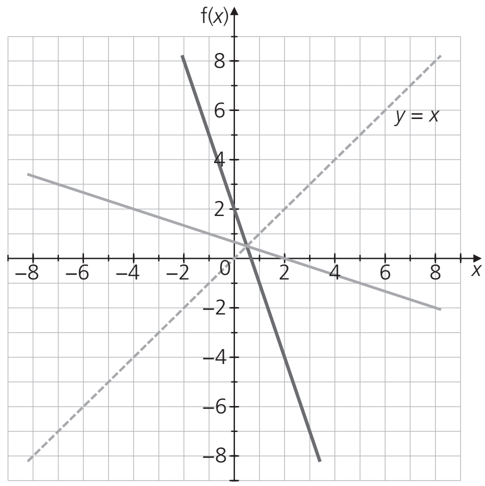
(ii) $g^{-1}(x) = \dfrac{2 - 7x}{2x - 1}$

(iii) Since $b^2 - 4ac = (-1)^2 - 4 \times 1 \times 1 = -3$ the equation has no real solutions.

(iv) $f^{-1}g(x) = \dfrac{x + 4}{2x + 7}$

(v)

Point on $y = f(x)$	**Point on $y = f^{-1}(x)$**
(0, 2)	(2, 0)
(1, –1)	(–1, 1)
(–1, 5)	(5, –1)



5 Differentiation

Exercise 5.1

1 (i) $\frac{dy}{dx} = 20x - 3$

(ii) $\frac{dy}{dx} = 1 - 20x^3$

(iii) $\frac{dy}{dx} = -10x^{-3} = -\frac{10}{x^3}$

(iv) $\frac{dy}{dx} = \frac{6x^2}{3} = 2x^2$

(v) $y = \frac{4}{x} - \frac{x}{4} = 4x^{-1} - \frac{1}{4}x$

$\Rightarrow \frac{dy}{dx} = -4x^{-2} - \frac{1}{4} = -\frac{4}{x^2} - \frac{1}{4}$

(vi) $y = \frac{1}{4x^3} = \frac{x^{-3}}{4} \Rightarrow \frac{dy}{dx} = \frac{-3x^{-4}}{4} = -\frac{3}{4x^4}$

(vii) $y = 2\sqrt{x} - 3x = 2x^{\frac{1}{2}} - 3x$

$\Rightarrow \frac{dy}{dx} = x^{-\frac{1}{2}} - 3 = \frac{1}{\sqrt{x}} - 3$

(viii) $y = \sqrt[3]{x} + \frac{2}{3x} = x^{\frac{1}{3}} + \frac{2x^{-1}}{3}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}} + \frac{-2x^{-2}}{3} = \frac{1}{3\sqrt[3]{x^2}} - \frac{2}{3x^2}$

2 (i) $f'(x) = -2x^{-2} - 2x^{-3} = -\frac{2}{x^2} - \frac{2}{x^3}$

(ii) $f'(x) = 3x^{\frac{1}{2}} - 3x^{-\frac{1}{2}} = 3\sqrt{x} - \frac{3}{\sqrt{x}}$

(iii) $f'(x) = \frac{\pi}{2} - \frac{2x^{-2}}{\pi} = \frac{\pi}{2} - \frac{2}{\pi x^2}$

(iv) $f'(x) = 6 \times \frac{5}{3}x^{\frac{2}{3}} = 10\sqrt[3]{x^2}$

3 $f'(9) = -\frac{3}{\sqrt{9^3}} + 5 = -\frac{3}{27} + 5 = 5 - \frac{1}{9} = 4\frac{8}{9}$

4 Point is (–5, 18).

5 Points are $\left(\frac{1}{3}, -\frac{5}{27}\right)$ or (–1, –3).

Exercise 5.2

1 Equation of the tangent is: $y = -10x - 8$

2 $y = 4x - 17$

3 (i) $PR = \sqrt{1^2 + 1^2} = \sqrt{2}$

(ii) S is (–1, 0)

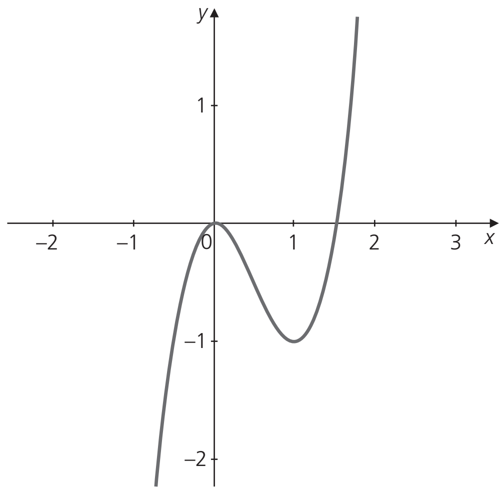
4 (i) $y = 2x + 4 \Rightarrow 2x - y + 4 = 0$

(ii) Q is $\left(4\frac{1}{2}, 6\frac{3}{4}\right)$

5 $k = 5$

Exercise 5.3

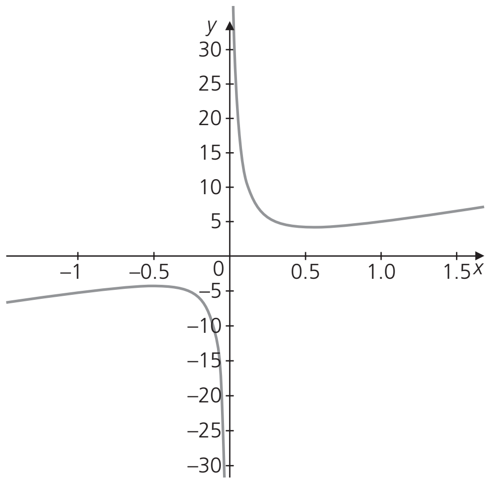
1 (i) Stationary points are (0, 0) and (1, –1).



(0, 0) is a local minimum

(1, –1) is a local minimum

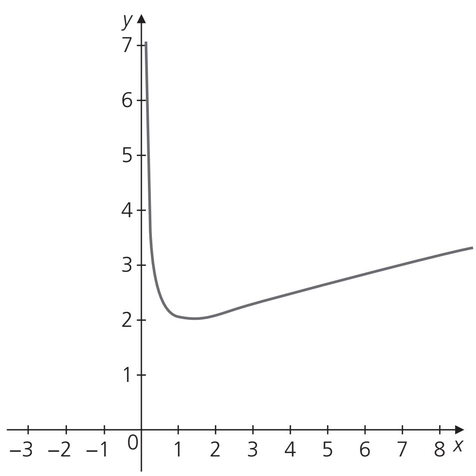
(ii) Points are $\left(\frac{1}{2}, 4\right)$ and $\left(-\frac{1}{2}, -4\right)$



$f''\left(\frac{1}{2}\right) > 0 \Rightarrow \left(\frac{1}{2}, 4\right)$ is a local minimum

$f''\left(-\frac{1}{2}\right) < 0 \Rightarrow \left(-\frac{1}{2}, -4\right)$ is a local maximum

(iii) Point is (1, 2)



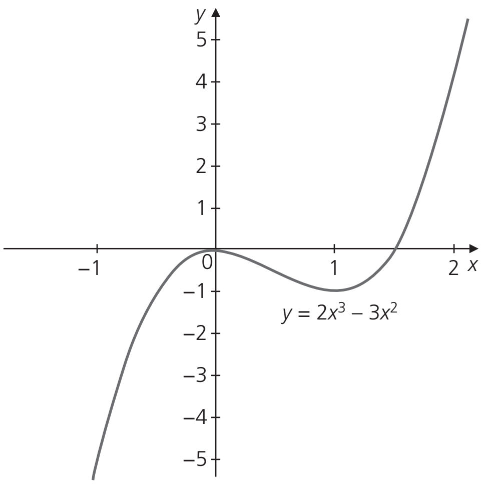
$x = 1$ is a local minimum

2 $0 < x < 1$

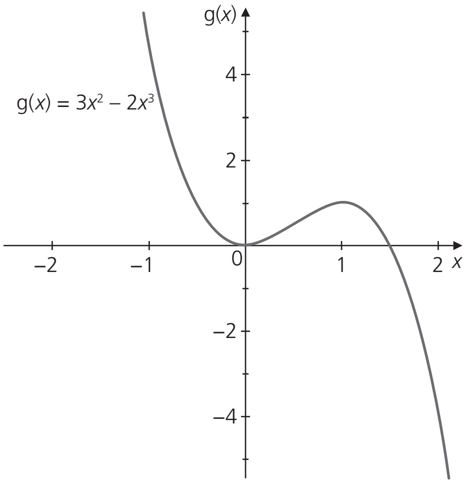
3 Since the discriminant is < 0, there are no solutions when $-1 + 2x - 3x^2 = 0$

So $-1 + 2x - 3x^2 = 0$ is always < 0

4 (i)



(ii)



5 (i) $k = 5$

(ii) The x co-ordinate of the other stationary point is $x = -\frac{8}{3} = -2\frac{2}{3}$.

(iii) $f''(1) = 6 \times 1 + 5 = 11 > 0$

$\Rightarrow x = 1$ is a minimum

$f''\left(-\frac{8}{3}\right) = 6 \times \left(-\frac{8}{3}\right) + 5 = -11 < 0$

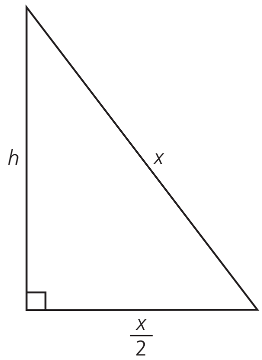
$\Rightarrow x = -\frac{8}{3}$ is a maximum

Exercise 5.4

1 Maximum value of P is $9 \times 9 = 81$

2 Dimensions are 3 m by 4.5 m.

3 (i)



$$h^2 + \left(\frac{x}{2}\right)^2 = x^2$$

$$h^2 = x^2 - \frac{x^2}{4}$$

$$h^2 = \frac{3x^2}{4}$$

$$h = \frac{\sqrt{3}}{2}x$$

(ii) Perimeter = 10

$$3x + 2y = 10 \Rightarrow y = \frac{10 - 3x}{2}$$

$$A = xy + \frac{1}{2}x \times \frac{\sqrt{3}}{2}x$$

$$= x\left(\frac{10 - 3x}{2}\right) + \frac{\sqrt{3}}{4}x^2$$

$$= \frac{10x - 3x^2}{2} + \frac{\sqrt{3}}{4}x^2$$

$$= 5x - \frac{3}{2}x^2 + \frac{\sqrt{3}}{4}x^2$$

$$= 5x + \frac{\sqrt{3} - 6}{4}x^2$$

(iii) Stationary value when $\frac{dA}{dx} = 0$

= 2.34 m (3 s.f.)

(iv) $x = 2.34$ m is a maximum

4 (i) $V = \pi r^2 h = 5000 \Rightarrow h = \frac{5000}{\pi r^2}$

$$S = 2\pi r^2 + 2\pi r h$$

$$= 2\pi r^2 + 2\pi r\left(\frac{5000}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{10\,000}{r}$$

(ii) Stationary value when $\frac{dS}{dr} = 0$

$$r = \sqrt[3]{\frac{10\,000}{4\pi}} = 9.27\text{ cm}$$

(iii) $r = 9.27$ is a minimum value

5 **(i)** $2b + 2\pi r = 400$

$2b = 400 - 2\pi r$

$b = 200 - \pi r$

(ii) $r = \dfrac{400}{4\pi} = 31.8\,\text{m}$ (3 s.f.)

$b = 200 - \pi \times 31.8... = 100\,\text{m}$

6 Let the length of the end be x

Let the length be y

$4x + y = 120$ so $y = 120 - 4x$

$V = x^2 y$

$= x^2(120 - 4x)$

$= 120x^2 - 4x^3$

$\dfrac{dV}{dx} = 240x - 12x^2$

Maximum volume when $\dfrac{dV}{dx} = 0$

$240x - 12x^2 = 0$

$12x(20 - x) = 0$

$x = 0\,\text{cm}$ or $20\,\text{cm}$

Since $x \neq 0$, $x = 20\,\text{cm}$

$y = 120 - 4 \times 20 = 40\,\text{cm}$

Volume is $16\,000\,\text{cm}^3$

7 The largest area is 0.5 units^2

8 $1241.123... = 1240\,\text{cm}^3$ (3 s.f.)

9 Dimensions are:

length = 1.87 m, width = 7.47 m, height = 3.56 m

Exercise 5.5

1 **(i)** $y = (x-4)^7 \Rightarrow \dfrac{dy}{dx} = 7(x-4)^6$

(ii) $y = (3x+2)^8 \Rightarrow \dfrac{dy}{dx} = 8(3x+2)^7 \times 3$

$= 24(3x+2)^7$

(iii) $y = (5-2x)^9 \Rightarrow \dfrac{dy}{dx} = 9(5-2x)^8 \times -2$

$= -18(5-2x)^8$

(iv) $y = \sqrt{3+4x} = (3+4x)^{\frac{1}{2}}$

$\Rightarrow \dfrac{dy}{dx} = \tfrac{1}{2}(3+4x)^{-\frac{1}{2}} \times 4$

$= \dfrac{2}{\sqrt{3+4x}}$

(v) $y = \sqrt[3]{1-9x} = (1-9x)^{\frac{1}{3}}$

$\Rightarrow \dfrac{dy}{dx} = \tfrac{1}{3}(1-9x)^{-\frac{2}{3}} \times -9$

$= -\dfrac{3}{\sqrt[3]{(1-9x)^2}}$

(vi) $y = \dfrac{2}{x+3} = 2(x+3)^{-1}$

$\Rightarrow \dfrac{dy}{dx} = -2(x+3)^{-2} = -\dfrac{2}{(x+3)^2}$

(vii) $y = \dfrac{4}{(1-2x)^2} = 4(1-2x)^{-2}$

$\Rightarrow \dfrac{dy}{dx} = -8(1-2x)^{-3} \times -2 = \dfrac{16}{(1-2x)^3}$

(viii) $y = \dfrac{5}{\sqrt{5x+3}} = 5(5x+3)^{-\frac{1}{2}}$

$\Rightarrow \dfrac{dy}{dx} = -\tfrac{5}{2}(5x+3)^{-\frac{3}{2}} \times 5 = -\dfrac{25}{2\sqrt{(5x+3)^3}}$

2 Equation of the tangent is

$y = mx + c \Rightarrow 3 = 2 \times 12 + c \Rightarrow c = -21$

$y = 2x - 21$

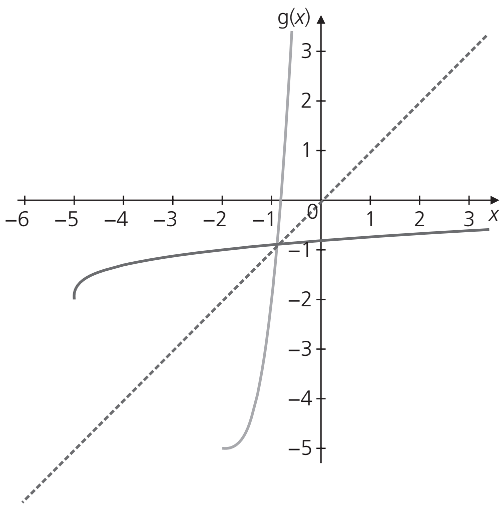
3 Point is (2, –2)

4 $g'(x) = 9(x+2)^2$

Since $g'(x) > 0$ for all values of x, the curve is always increasing so there are no stationary points.

Hence the function is one-to-one so it has an inverse.

The graph of $y = g(x)$ for $x > -2$ and its inverse are shown on the next page.



5 **(i)** $k = -\frac{3}{2}$

(ii) $x < -\frac{5}{2}$ or $x > -\frac{1}{2}$

6 **(i)** $\dfrac{dy}{dx} = -2(x-1)^{-2} + 2 = -\dfrac{2}{(x-1)^2} + 2$

$\dfrac{d^2y}{dx^2} = 4(x-1)^{-3} = \dfrac{4}{(x-1)^3}$

(ii) maximum point A is (0, –2), minimum point B is (2, 6)

7 0.2 cm/s

8 0.111 m/min (3 s.f.)
= 11.1 cm/min (3 s.f.)

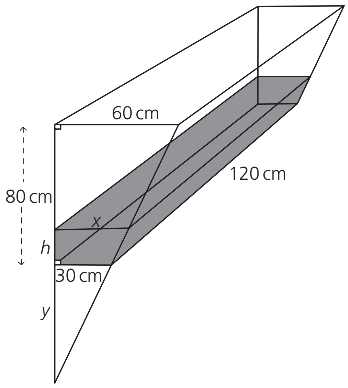
9 0.16 units/s

10 (i) 0.810 cm/s (3 s.f.)

(ii) 247 cm^2/s (3 s.f.)

11 0.05 rad/s

12 (i) Using similar triangles



$$\frac{y+80}{60} = \frac{y}{30} \Rightarrow y = 80\,\text{cm}$$

$$\frac{h+y}{x} = \frac{h+80}{x} \Rightarrow 3h + 240 = 8x \Rightarrow x = \frac{3}{8}h + 30$$

Area of trapezium cross-section

$$= \frac{\left(\frac{3}{8}h + 30\right) + 30}{2} \times h = \frac{\frac{3}{8}h^2 + 60h}{2} = \frac{3}{16}h^2 + 30h$$

$$\text{Volume} = 120 \times \left(\frac{3}{16}h^2 + 30h\right)$$

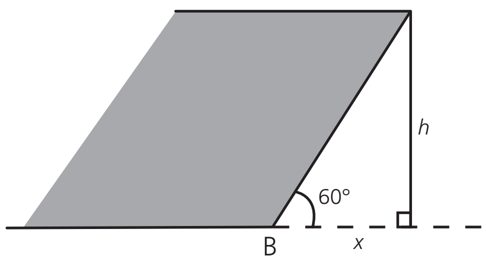
$$= \frac{45}{2}h^2 + 3600h\,\text{cm}^3$$

(ii) $h = 10.4$ cm or -170.4 cm

Since h cannot be negative, $h = 10.4$ cm

(iii) 0.983 cm/min

13 (i)



In the right-angled triangle

$$\tan 60° = \frac{h}{x} \Rightarrow \sqrt{3} = \frac{h}{x} \Rightarrow x = \frac{h}{\sqrt{3}}$$

Length of CD is $60 + 2x = 60 + 2 \times \frac{h}{\sqrt{3}}$

Volume = area of trapezium × length

$$= \frac{h}{2}(a+b) \times 200$$

$$= \frac{h}{2}\left(60 + 60 + 2 \times \frac{h}{\sqrt{3}}\right) \times 200$$

$$= 100h\left(120 + \frac{2h}{\sqrt{3}}\right)$$

$$= 200h\left(60 + \frac{h}{\sqrt{3}}\right)$$

(ii) 0.0818 cm/s

Stretch and challenge

1 (i) $g'(x) = k - 2x$ so if the gradient of the normal is $\frac{1}{2}$, the gradient of the tangent is -2.

$$k - 2x = -2$$
$$k - 2 \times 2 = -2$$
$$k - 4 = -2$$
$$k = 2$$

(ii) The y value when $x = 2$ is $g(2) = 2 \times 2 - 2^2 = 0$.

Equation of the normal is $y = mx + c \Rightarrow$

$0 = \frac{1}{2} \times 2 + c \Rightarrow c = -1$ so $y = \frac{1}{2}x - 1$

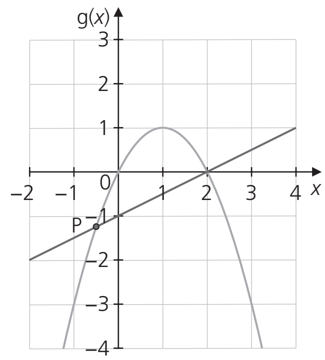
To find the other point, solve simultaneously

$$\frac{1}{2}x - 1 = 2x - x^2$$
$$x - 2 = 4x - 2x^2$$
$$2x^2 - 3x - 2 = 0$$
$$(2x+1)(x-2) = 0$$
$$x = -\frac{1}{2} \text{ or } 2$$

So the x co-ordinate of the other point is $x = -\frac{1}{2}$.

$$g\left(-\tfrac{1}{2}\right) = 2 \times -\tfrac{1}{2} - \left(-\tfrac{1}{2}\right)^2 = -1\tfrac{1}{4}$$

The co-ordinates of the point P are $\left(-\frac{1}{2}, -1\frac{1}{4}\right)$.



2 (i) $SP = \sqrt{x^2 + 10^2}$ so $\cos\alpha = \dfrac{x}{\sqrt{x^2 + 10^2}}$

$$PW = \sqrt{(30 - x)^2 + 5^2}$$
$$= \sqrt{(900 - 60x + x^2) + 25}$$
$$= \sqrt{925 - 60x + x^2}$$

$$\cos\beta = \frac{30 - x}{\sqrt{925 - 60x + x^2}}$$

(ii) $$\text{Time } T = \frac{\sqrt{x^2+10^2}}{10} + \frac{\sqrt{925-60x+x^2}}{5}$$

$$= \frac{\sqrt{x^2+10^2} + 2\sqrt{925-60x+x^2}}{10}$$

$$T' = \frac{1}{10}\left[\frac{1}{2}(x^2+100)^{-\frac{1}{2}} \times 2x + 2 \times \frac{1}{2}(925-60x+x^2)^{-\frac{1}{2}} \times (2x-60)\right]$$

$$= \frac{1}{10}\left[\frac{x}{\sqrt{x^2+100}} + \frac{2(x-30)}{\sqrt{925-60x+x^2}}\right]$$

$$= \frac{1}{10}\left[\frac{x}{\sqrt{x^2+100}} - 2\frac{30-x}{\sqrt{925-60x+x^2}}\right]$$

$$= \frac{1}{10}[\cos\alpha - 2\cos\beta]$$

$$T' = 0 \text{ when } \frac{1}{10}[\cos\alpha - 2\cos\beta] = 0$$

$$\cos\alpha = 2\cos\beta$$

3 (i) $$V = \pi r^2 h$$

$$R^2 = r^2 + \left(\frac{h}{2}\right)^2 \Rightarrow r^2 = R^2 - \frac{h^2}{4}$$

$$V = \pi\left(R^2 - \frac{h^2}{4}\right)h$$

$$= \pi R^2 h - \frac{\pi h^3}{4}$$

$$\frac{dV}{dh} = \pi R^2 - \frac{3\pi h^2}{4}$$

$$\frac{dV}{dh} = 0 \text{ when } \pi R^2 - \frac{3\pi h^2}{4} = 0$$

$$\pi R^2 = \frac{3\pi h^2}{4}$$

$$\frac{4\pi R^2}{3\pi} = h^2$$

$$h = \frac{2R}{\sqrt{3}}$$

$$r^2 = R^2 - \frac{h^2}{4} = R^2 - \frac{\left(\frac{2R}{\sqrt{3}}\right)^2}{4} = R^2 - \frac{R^2}{3} = \frac{2R^2}{3}$$

$$r = \sqrt{\frac{2}{3}}R$$

(ii) $$\frac{V_{cylinder}}{V_{sphere}} = \frac{\pi r^2 h}{\frac{4}{3}\pi R^3}$$

$$= \frac{3r^2 h}{4R^3}$$

$$= \frac{3\left(\sqrt{\frac{2}{3}}R\right)^2\left(\frac{2R}{\sqrt{3}}\right)}{4R^3}$$

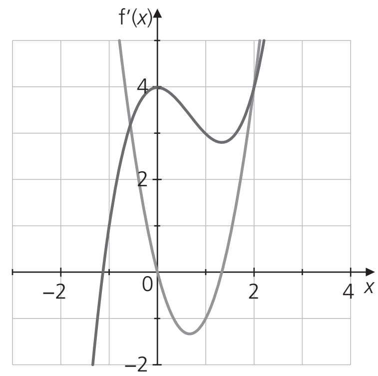
$$= \frac{3\left(\frac{2R^2}{3}\right)\left(\frac{2R}{\sqrt{3}}\right)}{4R^3}$$

$$= \frac{\frac{4R^3}{\sqrt{3}}}{4R^3}$$

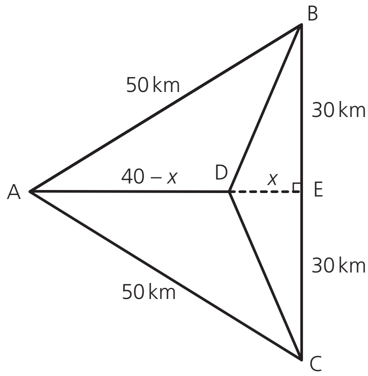
$$= \frac{1}{\sqrt{3}}$$

So ratio is 1: $\sqrt{3}$ or $\sqrt{3}$:3

4



5 $AE^2 + 30^2 = 50^2 \Rightarrow AE = 40\,\text{km}$



In triangle BDE, $BD^2 = x^2 + 30^2 \Rightarrow BD = \sqrt{x^2+900}$

Total length of road, L, is:

$$L = AD + BD + CD$$

$$= (40-x) + \sqrt{x^2+900} + \sqrt{x^2+900}$$

$$= (40-x) + 2\sqrt{x^2+900}$$

$$= (40-x) + 2(x^2+900)^{\frac{1}{2}}$$

$$\frac{dL}{dx} = -1 + 2 \times \frac{1}{2}(x^2+900)^{-\frac{1}{2}} \times 2x$$

$$= -1 + \frac{2x}{\sqrt{x^2+900}}$$

Minimum length is when

$$-1 + \frac{2x}{\sqrt{x^2+900}} = 0$$

$$\frac{2x}{\sqrt{x^2+900}} = 1$$

$$2x = \sqrt{x^2+900}$$

$$4x^2 = x^2 + 900$$

$$3x^2 = 900$$

$$x^2 = 300$$

$$x = \sqrt{300} \approx 17.3\,\text{km}$$

$$L = (40-x) + 2\sqrt{x^2+900}$$

$$= (40 - 17.3\ldots) + 2\sqrt{(17.3\ldots)^2 + 900}$$

$$= 91.9615\ldots\,\text{km}$$

$$= 92\,\text{km} \quad (3\text{ s.f.})$$

Exam focus

1 The points are $\left(1, -\frac{13}{6}\right)$ and $\left(-2, \frac{16}{3}\right)$.

2 The equation of the tangent is

$$y = -3x + 8 \text{ or } 3x + y - 8 = 0$$

The slope of the normal is $\frac{1}{3}$

$$y = mx + c \Rightarrow 5 = \frac{1}{3} \times 1 + c \Rightarrow c = \frac{14}{3}$$

The equation of the normal is

$$y = \frac{1}{3}x + \frac{14}{3} \text{ or } x - 3y + 14 = 0$$

3 The stationary points are (0, 2), (2, –14) and (–2, –14).

$$\frac{d^2y}{dx^2} = 12x^2 - 16$$

At $x = 0$, $\frac{d^2y}{dx^2} = -16$ so (0, 2) is a maximum.

At $x = 2$, $\frac{d^2y}{dx^2} = 12 \times 2^2 - 16 = 32$ so (2, –14) is a minimum.

At $x = -2$, $\frac{d^2y}{dx^2} = 12 \times (-2)^2 - 16 = 32$ so (–2, –14) is a minimum.

4 The function is increasing for $0 < x < 4$.

5 1200 mm²/minute

6 0.012 units/second

7 (i) Volume $= \pi r^2 h + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \pi r^2 h + \frac{2}{3}\pi r^3 = 100$

$$\pi r^2 h = 100 - \frac{2}{3}\pi r^3$$

$$h = \frac{100 - \frac{2}{3}\pi r^3}{\pi r^2}$$

$$= \frac{100}{\pi r^2} - \frac{\frac{2}{3}\pi r^3}{\pi r^2}$$

$$= \frac{100}{\pi r^2} - \frac{2r}{3}$$

(ii) $S = 2\pi r h + \pi r^2 + \frac{1}{2}(4\pi r^2)$

$$S = 2\pi r\left(\frac{100}{\pi r^2} - \frac{2r}{3}\right) + \pi r^2 + 2\pi r^2$$

$$= \frac{200}{r} - \frac{4\pi r^2}{3} + 3\pi r^2$$

$$= \frac{200}{r} + \left(3\pi - \frac{4\pi}{3}\right)r^2$$

$$= \frac{200}{r} + \frac{5\pi}{3}r^2$$

(iii) Stationary point is when

$$-\frac{200}{r^2} + \frac{10\pi}{3}r = 0$$

$$-200 + \frac{10\pi}{3}r^3 = 0$$

$$r^3 = \frac{200}{\frac{10\pi}{3}} = 19.098\ldots$$

$r = 2.67$ cm (3 s.f.)

(iv) $r = 2.67$ cm is a minimum value.

6 Integration

Exercise 6.1

1 (i) $\dfrac{x^4}{2} + \dfrac{x^2}{2} + c$

(ii) $x + \dfrac{3x^2}{2} + c$

(iii) $x^2 - x^5 + c$

(iv) $-\dfrac{2}{x^2} + c$

(v) $\dfrac{x^2}{4} + \dfrac{x^6}{3} + c$

(vi) $-\dfrac{1}{2x} + 3x^2 + c$

2 (i) $\dfrac{9\sqrt[3]{x^4}}{2} + c$

(ii) $\dfrac{2x^3}{3} - \dfrac{6\sqrt{x^5}}{5} + c$

(iii) $4\sqrt{x} + c$

(iv) $4\sqrt{x^5} + c$

3 $-\dfrac{5}{x} + \dfrac{1}{3x^3} + c$

4 $y = -\dfrac{1}{x^3} + x + 2$

5 $\mathrm{f}(x) = 2\sqrt{x^3} - x^2 + 30$

6 The equation is $y = x + \dfrac{3}{x} - 2$

Exercise 6.2

1 (i)

$$\begin{aligned} A &= \int_1^2 (3x+1)\,dx \\ &= \left[\frac{3x^2}{2} + x\right]_1^2 \\ &= \left(\frac{3\times 2^2}{2} + 2\right) - \left(\frac{3\times 1^2}{2} + 1\right) \\ &= 8 - 2.5 \\ &= 5.5 \end{aligned}$$

(ii) $A = \dfrac{4+7}{2} \times 1 = 5.5$

2 The area is $1\frac{1}{3}$.

3 Area between the curve and the x axis:

$2 \times \frac{16}{3} = 10\frac{2}{3}$

Area between the curve and the y axis:

$= 4 \times 4 - 10\frac{2}{3} = 5\frac{1}{3}$

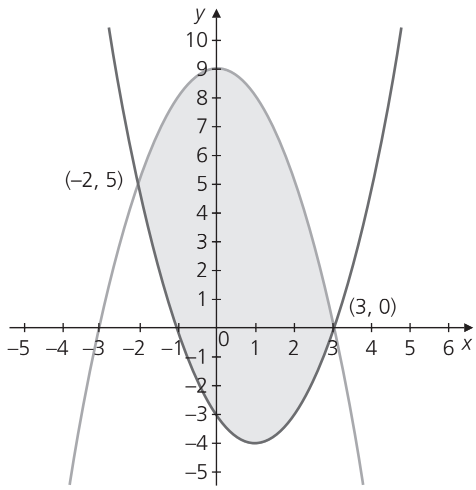
4 Area $= 2 \times 5\frac{1}{3} = 10\frac{2}{3}$

5 Area $= 10 - (-2) = 12$

6 Total area $= 5 + 1 = 6$

7 Area = 4.5

8



Area $= 27 - \left(-14\frac{2}{3}\right) = 41\frac{2}{3}$

9 Area between curve and the x axis:

$7\frac{1}{3} - \left(1\frac{2}{3}\right) = 5\frac{2}{3}$

Area between curve and the y axis:

$= 9 \times 2 - 5\frac{2}{3} - 3 = 9\frac{1}{3}$

10 $k = 4$

Exercise 6.3

1 (i) $\dfrac{(x-2)^4}{4} + c$

(ii) $\dfrac{(3x+1)^6}{18} + c$

(iii) $-\dfrac{(1-6x)^{10}}{30} + c$

(iv) $\left(\dfrac{x}{4} + 3\right)^4 + c$

2 $\dfrac{(2x-1)^{\frac{3}{2}}}{3} + c$

3 (i) $-\dfrac{2\sqrt{(1-x)^3}}{3} + c$

(ii) $-\dfrac{1}{(2x+1)^2} + c$

(iii) $-\dfrac{1}{6(x-4)^4} + c$

(iv) $\dfrac{2}{5(7x+2)^5} + c$

(v) $-3\sqrt{5-2x} + c$

(vi) $36\sqrt[3]{\left(1+\dfrac{x}{2}\right)^2} + c$

Exercise 6.4

1 $-\frac{1}{36}-\left(-\frac{1}{4}\right)=\frac{2}{9}$

2 4

3 3

Exercise 6.5

1 (i) $8\pi = 25.1(3\text{ s.f.})$

(ii) $\frac{4}{3}\pi = 4.19(3\text{ s.f.})$

2 $75.4(3\text{ s.f.})$

$V(\text{solid}) = \frac{1}{3}\pi\times 6^2\times 2 = 24\pi$

3 $\frac{27}{4}\pi$

4 15π

V (solid) = $24\pi - 15\pi = 9\pi = 28.3$ (3 s.f.)

5 Points of intersection:

$$x(x-2) = -\frac{1}{2}x$$
$$2x(x-2) = -x$$
$$2x^2 - 3x = 0$$
$$x(2x-3) = 0$$
$$x = 0 \text{ or } x = \frac{3}{2}$$

V(solid)

$$= \int_0^{1.5}\pi\left[x(x-2)^2 - \left(-\frac{1}{2}x\right)^2\right]dx$$
$$= \pi\int_0^{1.5}\left[(x^4-4x^3+4x^2)-\left(\frac{1}{4}x^2\right)\right]dx$$
$$= \pi\int_0^{1.5}\left(x^4-4x^3+\frac{15}{4}x^2\right)dx$$
$$= \pi\left[\frac{x^5}{5}-x^4+\frac{5x^3}{4}\right]_0^{1.5}$$
$$= \pi\left[\left(\frac{1.5^5}{5}-1.5^4+\frac{5\times1.5^3}{4}\right)-\left(\frac{0^5}{5}-0^4+\frac{5\times0^3}{4}\right)\right]$$
$$= \frac{27}{40}\pi$$

6 $a = 8$ (since $a > 0$)

7 (i) A is $(\frac{1}{2}, 10)$, B is (2, 10)

M is (1, 8)

(ii) Final volume = $150\pi - 66\pi = 84\pi \approx 264$ units^3

Stretch and challenge

1 $\int_{-2}^{0}(2x+k)-(x+2)^2\,dx = \frac{10}{3}$

$$\int_{-2}^{0}(2x+k)-(x^2+4x+4)\,dx = \frac{10}{3}$$
$$\int_{-2}^{0}(-x^2-2x+k-4)\,dx = \frac{10}{3}$$
$$\left[-\frac{x^3}{3}-x^2+(k-4)x\right]_{-2}^{0} = \frac{10}{3}$$
$$\left[0-\left(-\frac{(-2)^3}{3}-(-2)^2+(k-4)\times-2\right)\right] = \frac{10}{3}$$
$$-\frac{8}{3}+4+2k-8 = \frac{10}{3}$$
$$-\frac{20}{3}+2k = \frac{10}{3}$$
$$2k = 10$$
$$k = 5$$

2 Gradient of the sloping line is $m = \frac{h}{R-r}$

Equation is

$$y = mx + c \Rightarrow 0 = \frac{h}{R-r}\times r + c \Rightarrow c = -\frac{hr}{R-r}$$
$$y = \frac{h}{R-r}x - \frac{hr}{R-r}$$
$$y + \frac{hr}{R-r} = \frac{h}{R-r}x$$
$$x = \frac{R-r}{h}y + r$$
$$x^2 = \left(\frac{R-r}{h}\right)^2 y^2 + \frac{2r(R-r)}{h}y + r^2$$
$$V = \int_0^h \pi x^2 dy$$
$$= \pi\int_0^h\left(\left(\frac{R-r}{h}\right)^2y^2+\frac{2r(R-r)}{h}y+r^2\right)dy$$
$$= \pi\left[\left(\frac{R-r}{h}\right)^2\frac{y^3}{3}+\frac{2r(R-r)}{h}\frac{y^2}{2}+r^2y\right]_0^h$$
$$= \pi\left[\left(\frac{R-r}{h}\right)^2\frac{h^3}{3}+\frac{2r(R-r)}{h}\frac{h^2}{2}+r^2h\right]$$
$$= \pi\left[\frac{h(R-r)^2}{3}+hr(R-r)+r^2h\right]$$
$$= \pi\left[\frac{h(R^2-2Rr+r^2)}{3}+hr(R-r)+r^2h\right]$$
$$= \frac{\pi h}{3}\left[R^2-2Rr+r^2+3rR-3r^2+3r^2\right]$$
$$= \frac{\pi h}{3}\left[R^2+Rr+r^2\right]$$
$$= \frac{1}{3}\pi h\left(R^2+rR+r^2\right)$$

3 Area = $\int_k^{k+1}2x^2\,dx = \frac{109}{6}$

$$\left[\frac{2x^3}{3}\right]_k^{k+1} = \frac{109}{6}$$
$$\left[\frac{2(k+1)^3}{3}-\frac{2k^3}{3}\right] = \frac{109}{6}$$
$$\left[\frac{2(k^3+3k^2+3k+1)-2k^3}{3}\right] = \frac{109}{6}$$

$$\left[\frac{2k^3+6k^2+6k+2-2k^3}{3}\right]=\frac{109}{6}$$

$$\left[\frac{6k^2+6k+2}{3}\right]=\frac{109}{6}$$

$$2(6k^2+6k+2)=109$$

$$12k^2+12k+4=109$$

$$12k^2+12k-105=0$$

$$4k^2+4k-35=0$$

$$(2k+7)(2k-5)=0$$

$$k=\frac{5}{2}\text{ or }-\frac{7}{2}$$

Since $x>0,\ k=\frac{5}{2}$

4 $x^2+y^2=R^2 \Rightarrow x^2=R^2-y^2$

Also $\left(\frac{w}{2}\right)^2+r^2=R^2 \Rightarrow r^2=R^2-\frac{w^2}{4}$

$$V=\int_{-w/2}^{w/2}\pi x^2\,dy-V(\text{cylinder})$$

$$=2\int_0^{w/2}\pi x^2\,dy-\pi r^2 w$$

$$=2\int_0^{w/2}\pi\left[R^2-y^2\right]dy-\pi r^2 w$$

$$=2\pi\left[R^2y-\frac{y^3}{3}\right]_0^{w/2}-\pi r^2 w$$

$$=2\pi\left[R^2\left(\frac{w}{2}\right)-\frac{\left(\frac{w}{2}\right)^3}{3}\right]-\pi r^2 w$$

$$=2\pi\left[\frac{R^2w}{2}-\frac{w^3}{24}\right]-\pi r^2 w$$

$$=\pi R^2w-\frac{\pi w^3}{12}-\pi r^2 w$$

$$=\pi R^2w-\frac{\pi w^3}{12}-\pi\left(R^2-\frac{w^2}{4}\right)w$$

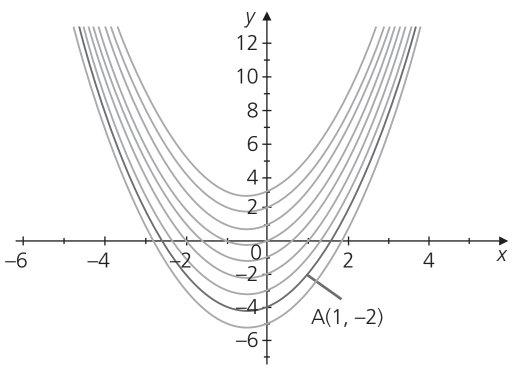
$$=\pi R^2w-\frac{\pi w^3}{12}-\pi R^2w+\frac{\pi w^3}{4}$$

$$=\frac{1}{6}\pi w^3$$

Since the volume does not depend on R, the claim is true.

Exam focus

1 The curve is $y=x^2+x-4$



2 $-\frac{3}{4(4x+1)}+c$

3 2

4 1

5 Area of DFGE = area of OBGE – area of ABGF – area of OAFD

$$=(4\times 5)-12\tfrac{1}{3}-(3\times 1)$$

$$=4\tfrac{2}{3}$$

6 Shaded Area $=42\frac{2}{3}$

7 Volume $=\frac{18}{5}\pi$ units^3

8 **(i)** Area under line $=\frac{1}{2}\times 1\times 1=\frac{1}{2}$

Shaded area = Area under curve – area of Δ

$$=\frac{2}{3}-\frac{1}{2}=\frac{1}{6}$$

(ii) **(a)** Volume of curve rotated around the x axis:

Final volume $=\frac{1}{2}\pi-\frac{1}{3}\pi=\frac{1}{6}\pi$

or

$=\frac{1}{6}\pi$

(b) Final volume $=\frac{8}{15}\pi-\frac{1}{3}\pi=\frac{1}{5}\pi$

or

$=\frac{1}{5}\pi$

9 **(i)** $\frac{4}{21}\pi$

or

Final volume $=\frac{1}{3}\pi-\frac{1}{7}\pi=\frac{4}{21}\pi$

(ii) Volume $=\frac{4\pi}{15}$ units^3 (0.838 units^3 to 3 s.f.)

or

Final volume $=\frac{3}{5}\pi-\frac{1}{3}\pi=\frac{4}{15}\pi$ units

7 Trigonometry

Exercise 7.1

1 (i) $a = \sqrt{32.65} \approx 5.71\,\text{cm}$ (3 s.f.)

(ii) $b = \sqrt{7.2225} \approx 2.69\,\text{m}$ (3 s.f.)

(iii) $c = 8.40\,\text{cm}$

(iv) $\theta = 40.5°$

2 (i) $d = 11.1\,\text{cm}$

(ii) $\beta = 60.9°$
$\theta = 180 - 60.9 - 53 = 66.1°$

(iii) $x = 17.9\,\text{cm}$

(iv) $\theta = 107°$

3 (Using rounded answers from question 2)

(i) Area = 27.1 cm

(ii) Area = 512 cm

(iii) Area = $89.1\,\text{cm}^2$

(iv) Area = 5.49 m

4 (i) 11.2 cm

(ii) $y = 6\,\text{cm}$

Exercise 7.2

1

Angle	Other angle that has the same value	Exact value
sin 30°	sin 150°	$\frac{1}{2}$
sin 210°	sin 330°	$-\frac{1}{2}$
cos 60°	cos 300°	$\frac{1}{2}$
tan 30°	tan 210°	$\frac{1}{\sqrt{3}}$
cos 150°	cos 210°	$-\frac{\sqrt{3}}{2}$
tan 120°	tan 300°	$-\sqrt{3}$

2 (i) $\sin x \tan x \equiv \frac{1}{\cos x} - \cos x$

$$\begin{aligned}\text{LHS} &= \sin x \times \frac{\sin x}{\cos x}\\ &= \frac{\sin^2 x}{\cos x}\\ &= \frac{1 - \cos^2 x}{\cos x}\\ &= \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}\\ &= \frac{1}{\cos x} - \cos x\\ &= \text{RHS}\end{aligned}$$

(ii) $\tan x\left(1 - \sin^2 x\right) \equiv \sin x \cos x$

$$\begin{aligned}\text{LHS} &= \tan x \times \cos^2 x\\ &= \frac{\sin x}{\cos x} \times \cos^2 x\\ &= \sin x \cos x\\ &= \text{RHS}\end{aligned}$$

(iii) $\frac{1 + \cos x}{\sin x} \equiv \frac{\sin x}{1 - \cos x}$

$$\begin{aligned}\text{RHS} &= \frac{\sin x}{1 - \cos x} \times \frac{1 + \cos x}{1 + \cos x}\\ &= \frac{\sin x(1 + \cos x)}{1 - \cos^2 x}\\ &= \frac{\sin x(1 + \cos x)}{\sin^2 x}\\ &= \frac{1 + \cos x}{\sin x}\\ &= \text{LHS}\end{aligned}$$

(iv) $\tan^2 x - \sin^2 x \equiv \tan^2 x \sin^2 x$

$$\begin{aligned}\text{LHS} &= \frac{\sin^2 x}{\cos^2 x} - \sin^2 x\\ &= \frac{\sin^2 x - \sin^2 x \cos^2 x}{\cos^2 x}\\ &= \frac{\sin^2 x\left(1 - \cos^2 x\right)}{\cos^2 x}\\ &= \frac{\sin^2 x \times \sin^2 x}{\cos^2 x}\\ &= \frac{\sin^2 x}{\cos^2 x} \times \sin^2 x\\ &= \tan^2 x \sin^2 x\\ &= \text{RHS}\end{aligned}$$

3 (i) $\sin^4 x - \cos^4 x \equiv \sin^2 x - \cos^2 x$

$$\begin{aligned}\text{LHS} &= \left(\sin^2 x - \cos^2 x\right)\left(\sin^2 x + \cos^2 x\right)\\ &= \left(\sin^2 x - \cos^2 x\right)(1)\\ &= \sin^2 x - \cos^2 x\\ &= \text{RHS}\end{aligned}$$

(ii) $\tan x + \frac{1}{\cos x} \equiv \frac{\cos x}{1-\sin x}$

$$
\begin{aligned}
\text{LHS} &= \frac{\sin x}{\cos x} + \frac{1}{\cos x} \\
&= \frac{\sin x + 1}{\cos x} \\
&= \frac{\sin x + 1}{\cos x} \times \frac{\cos x}{\cos x} \\
&= \frac{(\sin x + 1)\cos x}{1 - \sin^2 x} \\
&= \frac{(\sin x + 1)\cos x}{(1-\sin x)(1+\sin x)} \\
&= \frac{\cos x}{1-\sin x} \\
&= \text{RHS}
\end{aligned}
$$

(iii) $\frac{1}{1-\cos x} + \frac{1}{1+\cos x} \equiv \frac{2}{\sin^2 x}$

$$
\begin{aligned}
\text{LHS} &= \frac{(1+\cos x)+(1-\cos x)}{(1-\cos x)(1+\cos x)} \\
&= \frac{2}{1-\cos^2 x} \\
&= \frac{2}{\sin^2 x} \\
&= \text{RHS}
\end{aligned}
$$

(iv) $\frac{1+\tan^2 x}{1-\tan^2 x} \equiv \frac{1}{2\cos^2 x - 1}$

$$
\begin{aligned}
\text{LHS} &= \frac{1+\frac{\sin^2 x}{\cos^2 x}}{1-\frac{\sin^2 x}{\cos^2 x}} \\
&= \frac{\frac{\cos^2 x + \sin^2 x}{\cos^2 x}}{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}} \\
&= \frac{\frac{1}{\cos^2 x}}{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}} \\
&= \frac{1}{\cos^2 x} \times \frac{\cos^2 x}{\cos^2 x - \sin^2 x} \\
&= \frac{1}{\cos^2 x - \sin^2 x} \\
&= \frac{1}{\cos^2 x - \left(1-\cos^2 x\right)} \\
&= \frac{1}{2\cos^2 x - 1} \\
&= \text{RHS}
\end{aligned}
$$

4 (i)

$\frac{\cos\theta}{\tan\theta(1+\sin\theta)} \equiv \frac{1}{\sin\theta} - 1$

$$
\begin{aligned}
\text{LHS} &= \frac{\cos\theta}{\frac{\sin\theta}{\cos\theta}(1+\sin\theta)} \\
&= \frac{\cos^2\theta}{\sin\theta(1+\sin\theta)} \\
&= \frac{\left(1-\sin^2\theta\right)}{\sin\theta(1+\sin\theta)} \\
&= \frac{(1-\sin\theta)(1+\sin\theta)}{\sin\theta(1+\sin\theta)} \\
&= \frac{1-\sin\theta}{\sin\theta} \\
&= \frac{1}{\sin\theta} - \frac{\sin\theta}{\sin\theta} \\
&= \frac{1}{\sin\theta} - 1 \\
&= \text{RHS}
\end{aligned}
$$

(ii) $\frac{\cos x}{1-\cos x} - \frac{\cos x}{1+\cos x} \equiv \frac{2}{\tan^2 x}$

$$
\begin{aligned}
\text{LHS} &= \frac{\cos x}{1-\cos x} - \frac{\cos x}{1+\cos x} \\
&= \frac{\cos x(1+\cos x) - \cos x(1-\cos x)}{(1-\cos x)(1+\cos x)} \\
&= \frac{\cos x + \cos^2 x - \cos x + \cos^2 x}{1-\cos^2 x} \\
&= \frac{2\cos^2 x}{\sin^2 x} \\
&= \frac{2}{\frac{\sin^2 x}{\cos^2 x}} \\
&= \frac{2}{\tan^2 x} \\
&= \text{RHS}
\end{aligned}
$$

Exercise 7.3

1 (i) $\sin x = \frac{\sqrt{15}}{4}$

(ii) $\tan^2 x = 15$

2 (i) $\cos x = -\frac{7}{\sqrt{53}}$

(ii) $\sin^2 x = \frac{4}{53}$

3 (i) $x = 210°$ or $330°$

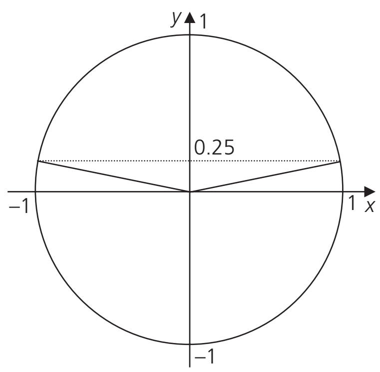
(ii) $x = 30°$ or $150°$ or $210°$ or $330°$

(iii) $x = 63.4°$ or $243.4°$ (1 d.p.)

(iv) $x = 120°$

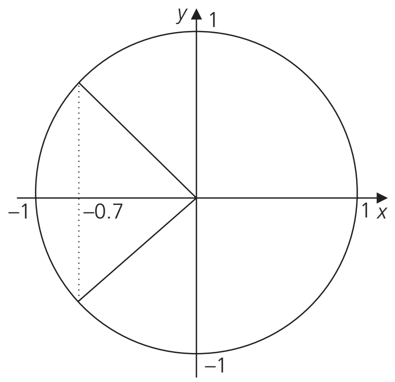
4 $x = 21.1°$ or $81.1°$ or $141.1°$ (1 d.p.)

5 (i)



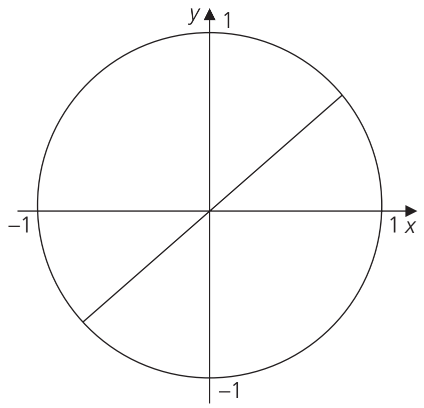
$x = 14.5°$ or $165.5°$ (1 d.p.)

(ii)



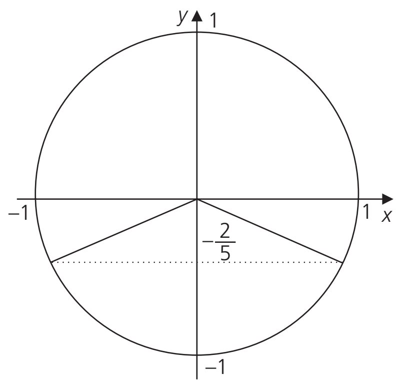
$x = 134.4°$ or $225.6°$

(iii)



$x = 41.2°$ or $221.2°$

(iv)



$x = 203.6°$ or $336.4°$

6 (i)

$$\left(\frac{1}{\cos\theta} + \tan\theta\right)^2 \equiv \frac{1+\sin\theta}{1-\sin\theta}$$

$$\text{LHS} = \left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)^2$$

$$= \left(\frac{1+\sin\theta}{\cos\theta}\right)^2$$

$$= \frac{(1+\sin\theta)^2}{\cos^2\theta}$$

$$= \frac{(1+\sin\theta)^2}{(1-\sin^2\theta)}$$

$$= \frac{(1+\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)}$$

$$= \frac{1+\sin\theta}{1-\sin\theta}$$

$$= \text{RHS}$$

(ii) $\theta = 203.6°$ or $336.4°$

7 (i) $x = 90°, 270°, 60°, 300°$

(ii) $x = 120°, -120°$

(iii) $x = 180°$

8 (i)

$$4\cos^2 x + 7\sin x - 2 = 0$$

$$4(1-\sin^2 x) + 7\sin x - 2 = 0$$

$$4 - 4\sin^2 x + 7\sin x - 2 = 0$$

$$-4\sin^2 x + 7\sin x + 2 = 0$$

$$4\sin^2 x - 7\sin x - 2 = 0$$

$$(4\sin x + 1)(\sin x - 2) = 0$$

$$\sin x = -\frac{1}{4} \text{ or } \sin x = 2$$

Since $\sin x$ is never greater than 1, $\sin x = -\frac{1}{4}$

(ii) $\theta = 5.5°, 214.5°$

Exercise 7.4

1

Degrees	Radians
120°	$\frac{2\pi}{3}$
36°	$\frac{\pi}{5}$
330°	$\frac{11\pi}{6}$
150°	$\frac{5\pi}{6}$
240°	$\frac{4\pi}{3}$
54°	$\frac{3\pi}{10}$
225°	$\frac{5\pi}{4}$
315°	$\frac{7\pi}{4}$

2

Degrees	Radians
12°	0.209
10.9°	0.190
145°	2.53
169.0°	2.95
235°	4.10
288.8°	5.04
342.5°	5.98
78.5°	1.37

3

Angle	Other angle that has the same value	Exact value
$\sin\frac{\pi}{4}$	$\sin\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
$\cos\frac{\pi}{6}$	$\cos\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\tan\frac{7\pi}{6}$	$\tan\frac{\pi}{6}$	$\frac{1}{\sqrt{3}}$
$\cos\frac{3\pi}{4}$	$\cos\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$
$\sin\frac{5\pi}{3}$	$\sin\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\tan\frac{\pi}{4}$	$\tan\frac{5\pi}{4}$	1

4 (i) $\tan x = 0$ or $\tan x = 2$

$x = 0, \pi, 2\pi, 1.11, 4.25$

(ii) $\sin x = -\frac{3}{2}$ or $\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

(iii) $\sin x = 3.30$ or -0.303

$x = -0.308$ or -2.83

(iv) $\cos x = \frac{1}{2}$ or $\cos x = 1$

$x = \frac{\pi}{3}, \frac{5\pi}{3}, 0, 2\pi$

(v) $\tan x = \frac{4}{3}$ or $\tan x = -1$

$x = 0.927, -2.21, -0.785, 2.36$

(vi) $\sin x = \frac{1}{2}$ or $\sin x = -5$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

Exercise 7.5

1 (i) $P = 22.4\,\text{cm}$

$A = 29.4\,\text{cm}^2$

(ii) $P = 75.2\,\text{cm}\ (3\text{ s.f.})$

$A = 314\,\text{cm}^2\ (3\text{ s.f.})$

2 $\theta = \frac{12}{7} = 1.71 = 98.2°$

3 Perimeter: $\frac{10\pi}{3} + 10\sqrt{3} = 27.8\,\text{cm}\ (3\text{ s.f.})$

Area: $17.1\,\text{cm}^2$

4 (i) Perimeter = 18.2 cm (3 s.f.)

(ii) Area of shaded region = $15.0\,\text{cm}^2\ (3\text{ s.f.})$

5 (i) $\frac{1}{2}6^2 \times \theta - \frac{1}{2} \times 6 \times 6 \times \sin\theta = 36$

$18\theta - 18\sin\theta = 36$

$18(\theta - \sin\theta) = 36$

$\theta - \sin\theta = 2$

$\theta = \sin\theta + 2$

(ii) $\theta = 2.55$

6 Total length is: $49.1\,\text{cm}\ (3\text{ s.f.})$

7 (i) $\tan\theta = \frac{AC}{8} \Rightarrow AC = 8\tan\theta$

Shaded area = Area of triangle OAC – area of sector OAB

$= \frac{1}{2} \times 8\tan\theta \times 8 - \frac{1}{2} \times 8^2 \times \theta$

$= 32\tan\theta - 32\theta$

$= 32(\tan\theta - \theta)$

(ii) Perimeter = $\frac{8\pi}{3} + 8 + 8\sqrt{3} \approx 30.2\,\text{cm}\ (3\text{ s.f.})$

8 (i) Since triangle OCD is equilateral, angle OCD = $\frac{\pi}{3}$

So angle ACD = $\pi - \frac{\pi}{3} = \frac{2}{3}\pi$

(ii) Perimeter = AD + BD + AB

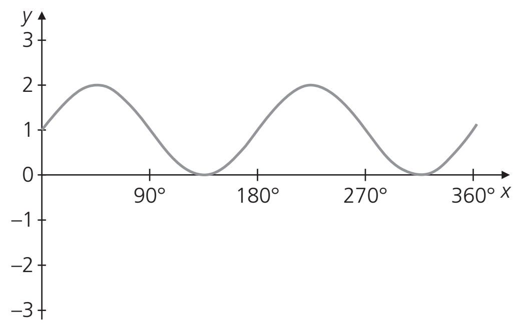
$= 8 \times \frac{2}{3}\pi + 8 + 16 \times \frac{1}{3}\pi$

$= \frac{32}{3}\pi + 8\,\text{cm}$

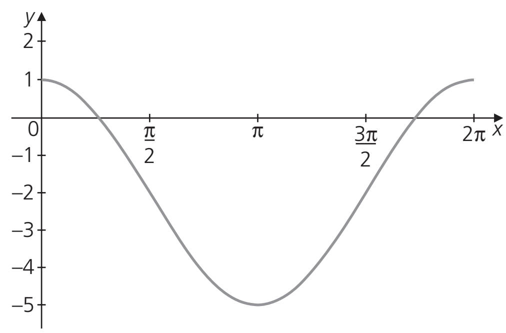
(iii) Shaded area $= \frac{64}{3}\pi - 16\sqrt{3}$

Exercise 7.6

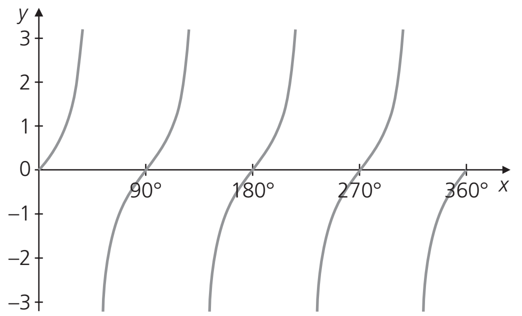
1 (i)



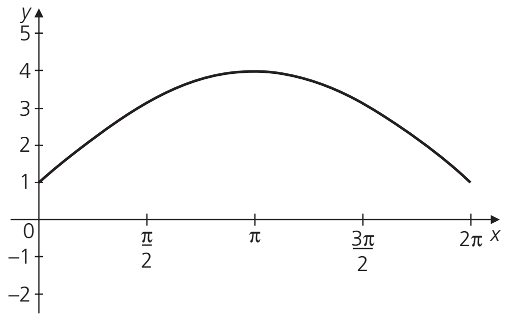
(ii)



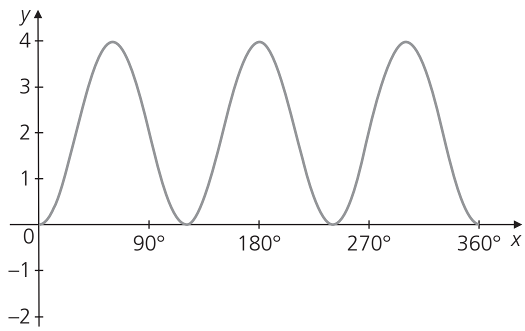
(iii)



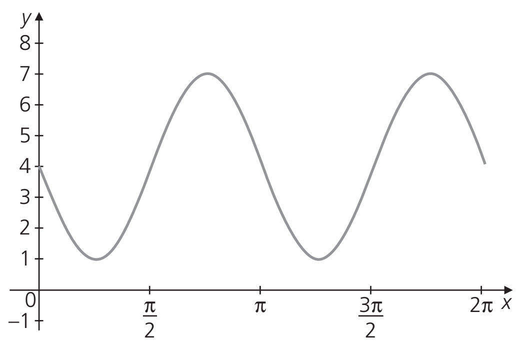
(iv)



(v)



(vi)



2 $A = 4$, $B = 1$, $C = -2$

3 $A = 5$, $B = 2$, $C = -1$

4 $A = 2$, $B = 3$, $C = -2$

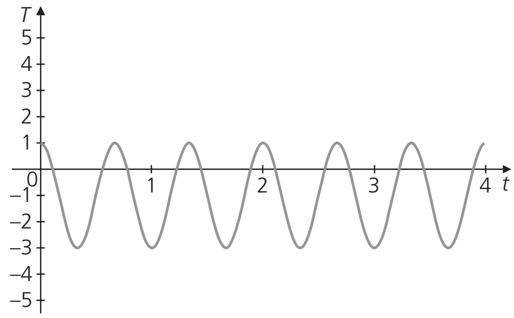
5 $A = -2$, $B = \frac{1}{2}$, $C = 3$

6 $A = 2$, $B = 3$, $C = 2$

7 $A = 2$, $B = -1$

8 (i) period $= \frac{2}{3}$

(ii)

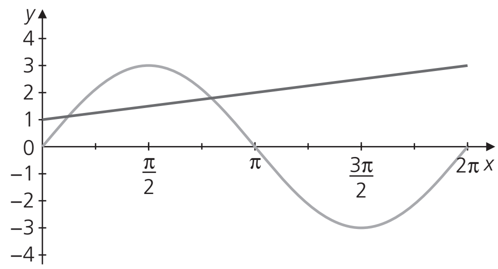


(iii) From the graph, the highest points occur at

$t = 0.7$, 1.3, 2, 2.7, 3.3 and 4 (1 d.p.)

$t =$ 12.40 pm, 1.20 pm, 2 pm, 2.40 pm, 3.20 pm, 4 pm

9 (i)



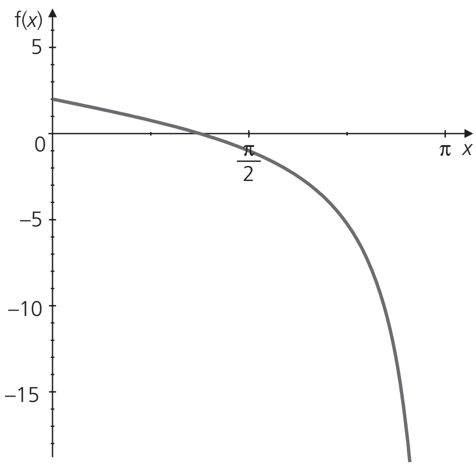
(ii) The line we need to draw is:

$$y = \frac{\pi + x}{\pi} = \frac{\pi}{\pi} + \frac{x}{\pi} = 1 + \frac{1}{\pi}x = 1 + 0.318x$$

10 (i) $f(x) \leqslant 2$

(ii) $f\left(\frac{1}{3}\pi\right) = 2 - \sqrt{3}$

(iii)

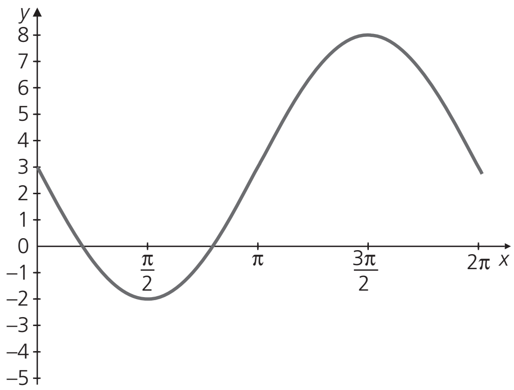


(iv) $f^{-1}(x) = 2\tan^{-1}\left(\frac{2 - x}{3}\right)$

11 (i) $b = 5$.

$a = 3$.

(ii)



12 (i) $\cos(-x) = k$

(ii) $\cos(\pi - x) = -k$

(iii) $\sin x = \sqrt{1-k^2}$

(iv) $\cos(\pi + x) = -k$

(v) $\tan x = \dfrac{\sqrt{1-k^2}}{k}$

13 (i) $\tan(-x) = k$

(ii) $\tan(\pi - x) = k$

(iii) $\cos x = -\dfrac{1}{\sqrt{1+k^2}}$

(iv) $\sin(\pi + x) = -\dfrac{k}{\sqrt{1+k^2}}$

(v) $\tan\left(\frac{1}{2}\pi + x\right) = \dfrac{1}{k}$

14 (i) $x = \dfrac{4\pi}{3}$ or $\dfrac{5\pi}{3}$

(ii) $x = \dfrac{\pi}{3}$ or π or $\dfrac{5\pi}{3}$

(iii) $x = \tan^{-1}(1) = \dfrac{\pi}{4}$ or $\dfrac{5\pi}{4}$

(iv) $x = 0.970$ or 2.74 or 4.11 or 5.88 (3 s.f.)

15 (i) Period = 40 seconds

(ii) Maximum height = 107 m

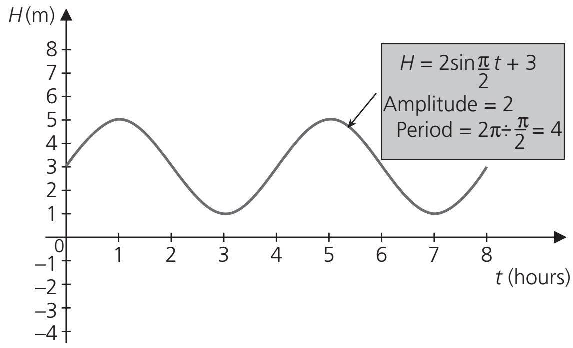
(iii) $t = 23$ seconds

Stretch and challenge

1

$$3\sin 3\theta + 3 = 2\cos^2 3\theta$$
$$3\sin 3\theta + 3 = 2\left(1 - \sin^2 3\theta\right)$$
$$3\sin 3\theta + 3 = 2 - 2\sin^2 3\theta$$
$$2\sin^2 3\theta + 3\sin 3\theta + 1 = 0$$
$$(2\sin 3\theta + 1)(\sin 3\theta + 1) = 0$$
$$\sin 3\theta = -\tfrac{1}{2} \text{ or } \sin 3\theta = -1$$
$$3\theta = -30°, -150°, -390°, -510°, 210°, 330°, -90°, 270°, -450°$$
$$\theta = -170°, -150°, -130°, -50°, -30°, -10°, 70°, 90°, 110°$$

2 (i)



$$H = 2\sin\frac{\pi}{2}t + 3$$

(ii) Period $= \dfrac{2\pi}{\frac{\pi}{2}} = 4$ hours

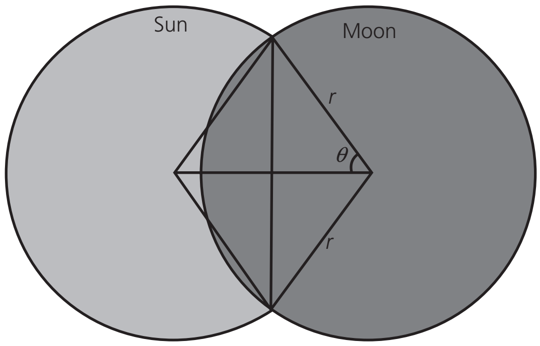
(iii) 1 am

3

$$10.5 = 9 - 3\cos\left(\frac{\pi}{8}t\right)$$
$$3\cos\left(\frac{\pi}{8}t\right) = -1.5$$
$$\cos\left(\frac{\pi}{8}t\right) = -\tfrac{1}{2}$$
$$\left(\frac{\pi}{8}t\right) = \cos^{-1}\left(-\tfrac{1}{2}\right)$$
$$\left(\frac{\pi}{8}t\right) = \frac{2\pi}{3}, \frac{4\pi}{3}$$
$$t = \tfrac{16}{3}, \tfrac{32}{3}$$
$$t = 5\tfrac{1}{3}, 10\tfrac{2}{3}$$

Since t is hours after 9.30 am, refloating can occur between $5\frac{1}{3}$ and $10\frac{2}{3}$ hours after 9.30 am, i.e. between 2.50 pm and 8.10 pm.

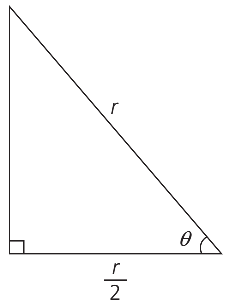
4



(i)

$$A = 2\left(\tfrac{1}{2}r^2 2\theta - \tfrac{1}{2}r^2\sin 2\theta\right)$$
$$\tfrac{1}{2}\pi r^2 = r^2 2\theta - r^2\sin 2\theta$$
$$\pi = 4\theta - 2\sin 2\theta$$
$$2\sin 2\theta = 4\theta - \pi$$

(ii)



$$\cos\theta = \frac{\frac{r}{2}}{r} \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = 60° = \frac{\pi}{3}$$

(iii)

$$\begin{aligned} A &= 2\left(\tfrac{1}{2}r^2 2\theta - \tfrac{1}{2}r^2\sin 2\theta\right) \\ &= 2r^2\theta - r^2\sin 2\theta \\ &= 2r^2\frac{\pi}{3} - r^2\sin\frac{2\pi}{3} \\ &= \frac{2\pi}{3}r^2 - \frac{\sqrt{3}}{2}r^2 \\ &= \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)r^2 \end{aligned}$$

$$\begin{aligned} \text{\% covered} &= \frac{\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)r^2}{\pi r^2} \times 100\% \\ &= \frac{\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)}{\pi} \times 100\% \\ &= 39.1\% \end{aligned}$$

Exam focus

1 **(i)** The equation is $y = 2\sin 3x$; i.e. $a = 2$, $b = 3$.

(ii) $c = 2$.

$b = 4$.

$a = 2$.

The equation is $y = 2 - 4\cos 2x$.

2 **(i)** $\sin x = \sin(\pi - x) = k$.

(ii) $\cos x = \sqrt{1 - k^2}$

(iii) $\tan\left(\frac{\pi}{2} - x\right) = \frac{\sqrt{1-k^2}}{k}$

3
$$\begin{aligned} \text{LHS} &= 1 + \frac{\sin^2\theta}{\cos^2\theta} \\ &= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} \\ &= \frac{1}{\cos^2\theta} \\ &= \text{RHS} \end{aligned}$$

4
$$\begin{aligned} \text{LHS} &= \frac{\sin x}{\cos x} + \frac{1}{\frac{\sin x}{\cos x}} \\ &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \\ &= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} \\ &= \frac{1}{\sin x \cos x} \\ &= \text{RHS} \end{aligned}$$

5
$$\begin{aligned} \text{LHS} &= \frac{(1-\cos x)(1-\cos x) + \sin^2 x}{\sin x(1-\cos x)} \\ &= \frac{\left(1 - 2\cos x + \cos^2 x\right) + \sin^2 x}{\sin x(1-\cos x)} \\ &= \frac{2 - 2\cos x}{\sin x(1-\cos x)} \\ &= \frac{2(1-\cos x)}{\sin x(1-\cos x)} \\ &= \frac{2}{\sin x} \\ &= \text{RHS} \end{aligned}$$

6 $x = 15°$ or $165°$ or $195°$ or $345°$

7 $x = -\frac{\pi}{12}$ or $\frac{5\pi}{12}$ or $\frac{11\pi}{12}$ or $-\frac{7\pi}{12}$

8 $\sin x = 0$ or $\sin x = -1$

$x = 0$ or $x = -\frac{\pi}{2}$

9 $\cos x = \frac{1}{2}$ or -2

$x = \frac{\pi}{3}, -\frac{\pi}{3}$

10 Area of segment $= 5.80\ \text{cm}^2$ (3 s.f.)

11 Perimeter of shaded region

$= \frac{10\pi}{3} + 10\sqrt{3} + 10$

12 **(i)** $\theta = \frac{5}{3}$

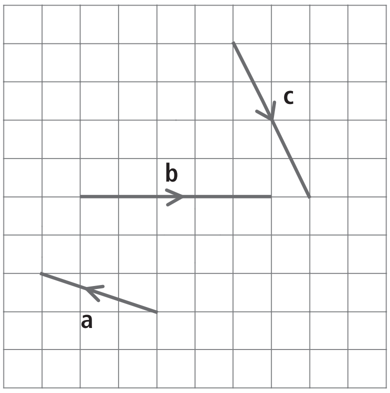
(ii) DE = 6.60 = (3 s.f.)

(iii) Shaded area = 9.63 cm^2 (3 s.f.)

8 Vectors

Exercise 8.1

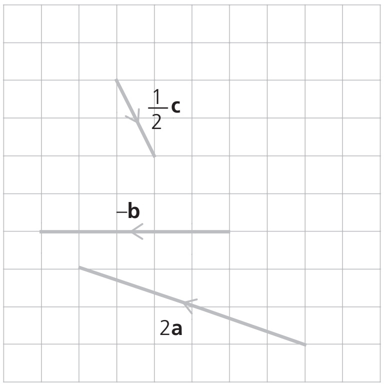
1 (i)



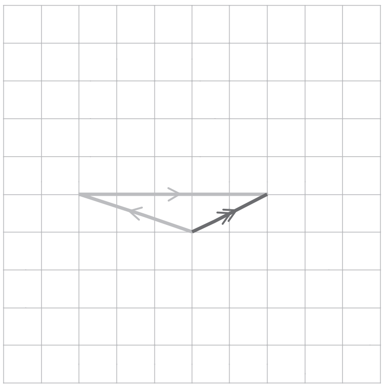
(ii) $2\mathbf{a} = \begin{pmatrix} -6 \\ 2 \end{pmatrix}$

$-\mathbf{b} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$

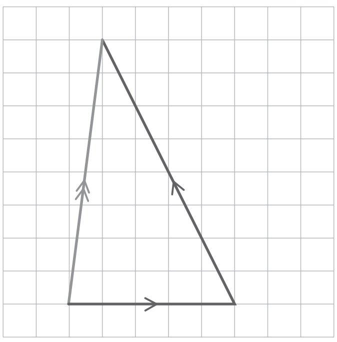
$\frac{1}{2}\mathbf{c} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$



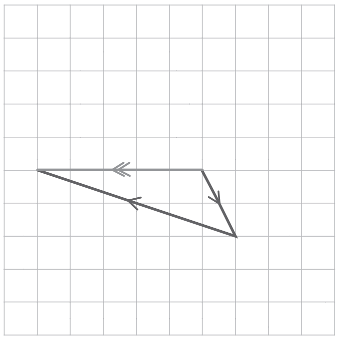
(iii) $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



$\mathbf{b} - 2\mathbf{c} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$



$\frac{1}{2}\mathbf{c} + 2\mathbf{a} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$



2 (i) 4i – 5j

(ii) –**i** + 2**j**

(iii) **g** = –6**j**

h = 4**i** – 5**j**

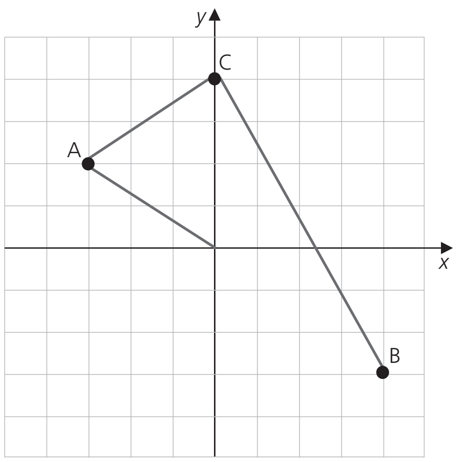
g + **h** = –6**j** + 4**i** – 5**j** = 4**i** – 11**j**

3 (i) $l = 3.61$ (3 s.f.)

(ii) $l = 2.24$ (3 s.f.)

(iii) $\overrightarrow{AB} = 6.08$ (3 s.f.)

4



(i) $\overrightarrow{OA} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

(ii) $\overrightarrow{BC} = \begin{pmatrix} -4 \\ 7 \end{pmatrix}$

(iii) $\overrightarrow{CA} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$

(iv) D is (–15, 23)

5 (i) $\overrightarrow{OB} = 2\mathbf{m} + \frac{1}{2}\mathbf{n}$

(ii) $\overrightarrow{OE} = \frac{5}{2}\mathbf{m} + \frac{5}{2}\mathbf{n}$

(iii) $\overrightarrow{BD} = 2\mathbf{m} + \frac{3}{2}\mathbf{n}$

(iv) $\overrightarrow{EC} = \frac{1}{2}\mathbf{m} - \frac{3}{2}\mathbf{n}$

6 (i) Unit vector = $\begin{pmatrix} -0.8 \\ 0.6 \end{pmatrix}$

(ii) Unit vector = $\begin{pmatrix} \frac{4}{\sqrt{17}} \\ -\frac{1}{\sqrt{17}} \end{pmatrix}$

or $\frac{1}{\sqrt{17}}(4\mathbf{i} - \mathbf{j})$

7 (i) $\begin{pmatrix} 2 \\ 8 \\ -18 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 3 \\ -18 \\ 13 \end{pmatrix}$

8 $\overrightarrow{AB} = -\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$

9 (i) $3\mathbf{i} + 4\mathbf{k}$

(ii) $5\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}$

(iii) $5\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$

(iv) $-3\mathbf{i} + 4\mathbf{k}$

(v) $5\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}$

(vi) $2\mathbf{i} + 6\mathbf{j}$

10 $\overrightarrow{CD} = 9.64$ (3 s.f.)

11 (i) $\overrightarrow{PQ} = \begin{pmatrix} 0 \\ 3 \\ -6 \end{pmatrix}$

(ii) $\overrightarrow{QR} = \begin{pmatrix} 0 \\ -6 \\ 12 \end{pmatrix}$

(iii) Since $\overrightarrow{QR} = -2\overrightarrow{PQ}$,
they are parallel
$\overrightarrow{QR}$ is twice the length of $\overrightarrow{PQ}$
$\overrightarrow{QR}$ goes in the opposite direction to $\overrightarrow{PQ}$.

(iv) S is (1, 2, –3).

12 (i) $\overrightarrow{CD} = \begin{pmatrix} 8 \\ 4 \\ -21 \end{pmatrix}$

(ii) $\overrightarrow{EC} = \begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix}$

13 G is (8, –13, –6)

14 (i) Unit vector = $-\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$

(ii) Unit vector is $\frac{1}{13}\begin{pmatrix} 5 \\ 0 \\ -12 \end{pmatrix} = \begin{pmatrix} \frac{5}{13} \\ 0 \\ -\frac{12}{13} \end{pmatrix}$

or $\frac{1}{13}(5\mathbf{i} - 12\mathbf{k})$

15 The unit vector is $\begin{pmatrix} 0.521 \\ -0.047 \\ 0.852 \end{pmatrix}$

Exercise 8.2

1 (i) $\theta = 169.7^\circ$ (1 d.p.)

(ii) $\theta = 147.3^\circ$ (1 d.p.)

2 $\theta = 130.2^\circ$ (1 d.p.)

3 (i) $\theta = 97.9^\circ$ (1 d.p.)

(ii) $\theta = 121.9^\circ$ (1 d.p.)

(iii) $a = \frac{9}{19}$

4 $\theta = 77.5^\circ$ (1 d.p.)

5 $c = -\frac{2}{5}$

6 (i) AB = 10

(ii) $\overrightarrow{AF} = 4\mathbf{i} + 10\mathbf{j} - 3\mathbf{k}$
$\overrightarrow{BF} = -4\mathbf{i} + 10\mathbf{j} + 3\mathbf{k}$

(iii) $\theta = 53.1^\circ$ (1 d.p.)

7 (i) $\overrightarrow{OM} = 14\mathbf{i} + 20\mathbf{k}$
$\overrightarrow{C'M} = 30\mathbf{i} - 20\mathbf{k}$
$\overrightarrow{MB'} = -14\mathbf{i} + 16\mathbf{j} + 20\mathbf{k}$
$\overrightarrow{CB'} = 12\mathbf{i} + 16\mathbf{j} + 40\mathbf{k}$

(ii) $\theta = 47.1^\circ$ (1 d.p.)

Stretch and challenge

1 $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = 0 \Rightarrow -a + b + 3c = 0$ (1)

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} = 0 \Rightarrow 2a + 5c = 0$ (2)

$$-2a + 2b + 6c = 0 \qquad (1)\times 2\ \ (3)$$

$$2b + 11c = 0 \qquad (2)+(3)$$

There are infinitely many solutions to this system of two equations with three unknowns.

Choosing the whole number solutions, $b = 11$, $c = -2$, $a = 5$, the vector is $\begin{pmatrix} 5 \\ 11 \\ -2 \end{pmatrix}$ (or any scalar multiple).

2 $|\mathbf{p}| = |\mathbf{q}| \Rightarrow \sqrt{a^2 + b^2 + 3^2} = \sqrt{4^2 + a^2 + 3^2}$

$$a^2 + b^2 + 9 = 16 + a^2 + 9$$

$$b^2 = 16$$

$$b = \pm 4$$

$$\begin{pmatrix} a \\ b \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ a \\ 3 \end{pmatrix} = 0 \Rightarrow 4a + ab + 9 = 0$$

If $b = 4$, $4a + 4a + 9 = 0 \Rightarrow a = -\frac{9}{8}$

If $b = -4$, there are no solutions for a.

So $a = -\frac{9}{8}$ and $b = 4$

3 $|\mathbf{v}_b| = \sqrt{120^2 + (-60)^2 + 40^2} = 140$ m/s

$|\mathbf{v}_a| = 140 + 35 = 175$ m/s

$$|\mathbf{v}_a| = k|\mathbf{v}_b| = k\begin{pmatrix} 120 \\ -60 \\ 40 \end{pmatrix} = \begin{pmatrix} 120k \\ -60k \\ 40k \end{pmatrix}$$

$$|\mathbf{v}_a| = \sqrt{(120k)^2 + (-60k)^2 + (40k)^2} = 175$$

$$19600k^2 = 175^2$$

$$k^2 = \frac{25}{16}$$

$$k = \frac{5}{4}$$

$$\mathbf{v}_a = \frac{5}{4}\begin{pmatrix} 120 \\ -60 \\ 40 \end{pmatrix} = \begin{pmatrix} 150 \\ -75 \\ 50 \end{pmatrix} = 150\mathbf{i} - 75\mathbf{j} + 50\mathbf{k}$$

Exam focus

1 So the unit vector in the direction of

$\overrightarrow{OR}$ is $\frac{1}{10}\begin{pmatrix} 6 \\ 0 \\ 8 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0 \\ 0.8 \end{pmatrix}$

2 $p = -6$

3 $\theta = 51.2°$ (1 d.p.)

Past examination questions

1 Algebra

1 (i) $16-(x-4)^2$
$a=16, b=-4$

(ii) $-2 \leqslant x \leqslant 10$

2 (i) $x<-1$ or $x>4$

(ii) $y=\left(x-\frac{3}{2}\right)^2-\frac{9}{4}$
$a=\frac{3}{2}, b=\frac{9}{4}$

2 Co-ordinate geometry

1 (i) Equation of *BC* is
$y=\frac{1}{2}x+\frac{11}{2}$ or $x-2y+11=0$

(ii) C is $(13, 12)$

(iii) Perimeter $=2\sqrt{20}+2\sqrt{180} \approx 35.8$

2 (i) Equation of *BC* is:
$y=\frac{1}{2}x+4$
Equation of CD is:
$y=-2x+29$

(ii) C is $(10, 9)$

3 (i) Points of intersection are (1.5, 8) and (4, 3)

(ii) $-\sqrt{96}<k<\sqrt{96}$
or $|k|<\sqrt{96}$

4 (i) Equation of *CD* is:
$y=-\frac{3}{2}x+24$ or $3x+2y-48=0$

(ii) D is $(10, 9)$

5 $m<-10$ or $m>2$

6 (i) M is $(5, 2)$
D is $(7, -2)$

(ii) $6:4=3:2$

7 (i) Points are $(6, 1)$ and $(2, 3)$

(ii) $k=8.5$

3 Sequences and series

1 (i) $a=\frac{18}{\frac{2}{3}}=27$

(ii) $S_\infty=81$

2 (i) \$61.50

(ii) $S_\infty=18$

3 Coefficient of x is 1080

4 (i) $S_{10}=239$ (3 s.f.)

(ii) $S_{32}=3280$

5 (i) \$369 000

(ii) \$3 140 000

(iii) \$14 300

6 (i) 10 836

(ii) (a) $x=96$

(b) $S_\infty=432$

7 (i) $32+80u+80u^2+\ldots$

(ii) The coefficient of the x^2 term is 160.

8 (i) $32+80x^2+80x^4+\ldots$

(ii) Coefficient of x^4 term is 272

9 $a=5$

10 (i) (a) $d=3$

(b) 57

(c) 570

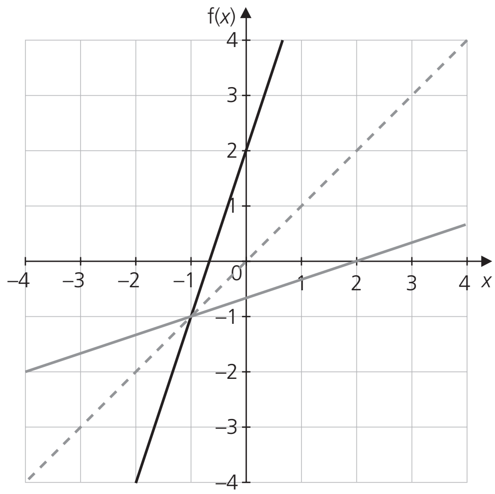
(ii) (a) \$6515.58 or \$6516

(b) \$56 827 or \$56 800

4 Functions

1 (i) $x=7\frac{1}{2}$

(ii)



(iii) $x=-\frac{9}{2}$ or 2

2 (i) $a=2, b=-3$

(ii) $x=\frac{9}{4}$

3 (i) The domain of $g^{-1}(x)$ is $x \leqslant 16$ and range is $y \geqslant 4$.

(ii) $g^{-1}(x)=\sqrt{16-x}+4$

4 (i) $x < -3$ and $x > 5$

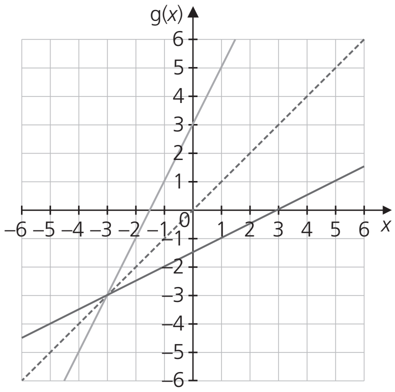
(ii) Range is $f(x) \geqslant -1$.

No inverse as f(x) is not one-to-one.

(iii) $gf(x) = g(x^2 - 2x) = 2(x^2 - 2x) + 3$
$= 2x^2 - 4x + 3$

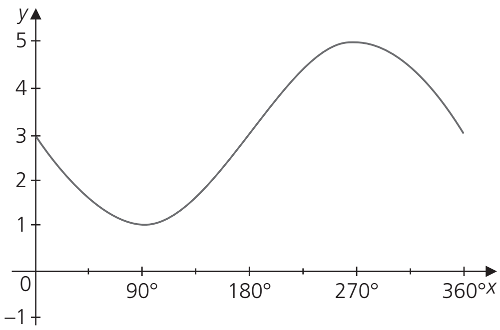
$2x^2 - 4x + 3 = 0$ has $b^2 - 4ac = -8$
so has no real solutions.

(iv)



5 (i) Range is $1 \leqslant f(x) \leqslant 5$.

(ii)



(iii) Largest value is when $A = 90°$.

(iv) $g^{-1}(x) = \sin^{-1}\left(\dfrac{3 - x}{2}\right)$

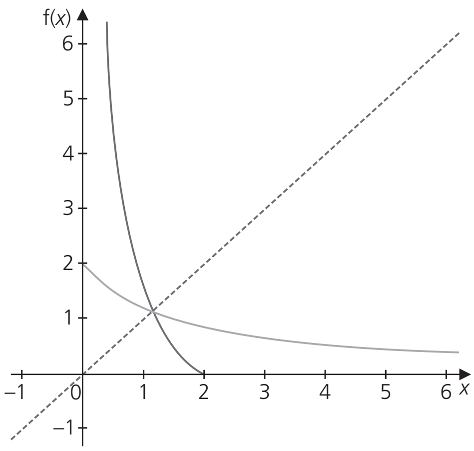
6 (i) $f(x) \geqslant -\frac{9}{4}$

(ii) f does not have an inverse as the function is not one-to-one.

(iii) $x = 25$

7 (i) The domain of f^{-1} is $0 < x \leqslant 2$

(ii)

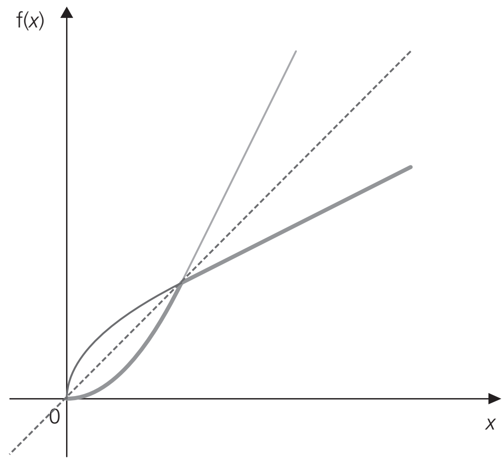


(iii) $x = 1$

8 The domain of f^{-1} is $x \geqslant 3$

9 (i) Range of f is $0 \leqslant f \leqslant 4$.

(ii)



(iii) $f^{-1}(x) = \begin{cases} \sqrt{2x} & \text{for } 0 \leqslant x \leqslant 2 \\ 2x - 2 & \text{for } 2 < x \leqslant 4 \end{cases}$

10 (i) $a = -2 \quad b = 8$

(ii) $fg(x) = 22 - 6x$

5 Differentiation

1 (i) $V = \frac{1}{2}\pi(192r - r^3)$

(ii) $r = 8$ cm

(iii) $r = 8$ cm is a maximum

2 (i) $h = 4 - r - \dfrac{\pi}{2}r$

(ii) $A = 2r \times h + \frac{1}{2}\pi r^2$
$= 2r\left(4 - r - \frac{\pi}{2}r\right) + \frac{1}{2}\pi r^2$
$= 8r - 2r^2 - \pi r^2 + \frac{1}{2}\pi r^2$
$= 8r - 2r^2 - \frac{1}{2}\pi r^2$

(iii) $r = 1.12$ cm (3 s.f.)

(iv) $r = 1.12$ cm is a maximum

3 The point is $\left(\frac{1}{2}, 5\frac{1}{4}\right)$.

4 $y = -2\frac{2}{3}$

5 Angle between the two lines is 8.2° (1 d.p.).

6 $f'(x) = -\dfrac{12}{(2x + 3)^2}$

Since $(2x + 3)^2 > 0$ for all values of x, then $-\dfrac{12}{(2x + 3)^2}$ is always negative.

Since $f'(x)$ is always negative, the curve is always decreasing.

7 $f'(x) = 9(3x+2)^2$

Since $(3x+2)^2 > 0$ for all values of x, $f'(x) > 0$ for all values of x, hence the function is always increasing.

8 (i) $\frac{dy}{dx} = -\frac{24x}{(x^2+3)^2}$

(ii) $y = \frac{2}{3}x + \frac{7}{3}$ or $2x - 3y + 7 = 0$

(iii) -0.018 units/second

9 $x = 4$

$x = 4$ is a minimum point

6 Integration

1 (i) Equation is $y = \frac{\sqrt{(1+2x)^3}}{3} + 2$

(ii) The y intercept is $\left(0, 2\frac{1}{3}\right)$

2 (i) $y = -2x + 12$

(ii) Area $= 9\frac{1}{3}$

3 (i) $x = 1, \frac{dy}{dx} = \frac{5}{6}$

(ii) At $x = 1 = 0.025$ units/second

(iii) Area $= 2\frac{8}{15}$

4 (i) Equation of curve is $y = x^3 - 2x^2 + x + 5$

(ii) $x < \frac{1}{3}$ or $x > 1$

5 (i) $y = \frac{8}{3x+2} = 8(3x+2)^{-1}$

$\frac{dy}{dx} = -8(3x+2)^{-2} \times 3 = -\frac{24}{(3x+2)^2}$

At $x = 2, \frac{dy}{dx} = -\frac{24}{(3\times 2+2)^2} = -\frac{3}{8}$

Equation of tangent at $(2, 1)$ is:

$y = mx + c \Rightarrow 1 = -\frac{3}{8} \times 2 + c \Rightarrow c = \frac{7}{4}$

$y = -\frac{3}{8}x + \frac{7}{4}$

Co-ordinates of C is when $y = 0$

$0 = -\frac{3}{8}x + \frac{7}{4} \Rightarrow x = \frac{14}{3} = 4\frac{2}{3}$

$DC = 4\frac{2}{3} - 2 = 2\frac{2}{3}$

Area of triangle $BDC = \frac{1}{2} \times 2\frac{2}{3} \times 1 = \frac{4}{3}$

(ii) $V = \int_0^2 \pi\left(\frac{8}{3x+2}\right)^2 dx$

$= 64\pi \int_0^2 (3x+2)^{-2} dx$

$= 64\pi \left[\frac{(3x+2)^{-1}}{-1} \times \frac{1}{3}\right]_0^2$

$= 64\pi \left[-\frac{1}{3(3x+2)}\right]_0^2$

$= 64\pi \left[\left(-\frac{1}{3(3\times 2+2)}\right) - \left(-\frac{1}{3(3\times 0+2)}\right)\right]$

$= 64\pi \left[-\frac{1}{24} + \frac{1}{6}\right]$

$= 64\pi \left[\frac{1}{8}\right]$

$= 8\pi$

6 (i) Equation of normal at P is:

$y = -\frac{1}{2}x + \frac{9}{2} \Rightarrow 2y = -x + 9 \Rightarrow x + 2y = 9$

(ii) Equation of curve is $y = 3\sqrt{4x-3} - 6$

7 (i) $k = 27$

(ii) Maximum point is $(-1, 32)$

(iii) The function is decreasing for $-1 < x < 3$

(iv) $A = 33.75$

8 (i) When $x = 1, \frac{dy}{dx} = \frac{4}{3}$

(ii) 0.015 units/second

(iii) $V = \int_0^1 \pi\left(\frac{6}{5-2x}\right)^2 dx$

$= 36\pi \int_0^1 (5-2x)^{-2} dx$

$= 36\pi \left[\frac{(5-2x)^{-1}}{-1} \times \frac{1}{-2}\right]_0^1$

$= 36\pi \left[\frac{1}{2(5-2x)}\right]_0^1$

$= 36\pi \left[\left(\frac{1}{2(5-2\times 1)}\right) - \left(\frac{1}{2(5-2\times 0)}\right)\right]$

$= 36\pi \left[\frac{1}{6} - \frac{1}{10}\right]$

$= 36\pi \left[\frac{1}{15}\right]$

$= \frac{12}{5}\pi$

9 (i) Shaded area $= 2 \times 1 - \frac{14}{9} = \frac{4}{9}$

(ii) $V = \frac{3}{2}\pi = 4.71$ (3 s.f.)

(iii) Angle between the tangents $= 19.4°$

10 (i) $k = 5$

(ii) Equation is $y = 5x - x^2 + 5$

11 (i) $a = 1, b = 2$

(ii) $A = 2.25$

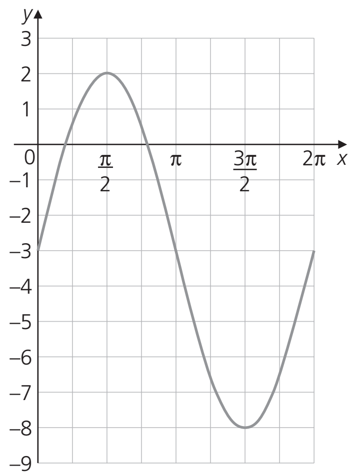
(iii) $c = \sqrt[6]{8} = \sqrt{2} \approx 1.41$

7 Trigonometry

1 (i) $b = -3$
$a = 5$

(ii) $x = 0.64, 2.50 (2 \text{ d.p.})$

(iii)



2 (i) Cosine rule:

$$\cos AOB = \frac{20^2 + 20^2 - 32^2}{2 \times 20 \times 20}$$
$$\cos AOB = -0.28$$
$$AOB = 1.855 (3 \text{ s.f.})$$
or
$$\sin\left(\tfrac{1}{2}AOB\right) = \frac{16}{20}$$
$$\left(\tfrac{1}{2}AOB\right) = \sin^{-1}\frac{16}{20}$$
$$AOB = 1.855 \ (3 \text{ s.f.})$$

(ii) $A = 371 \text{ cm}^2$

(iii) Area of cross-section $= 502 \text{ cm}^2 (3 \text{ s.f.})$

3 (i)
$$3\tan\theta = 2\cos\theta$$
$$\frac{3\sin\theta}{\cos\theta} = 2\cos\theta$$
$$3\sin\theta = 2\cos^2\theta$$
$$3\sin\theta = 2\left(1 - \sin^2\theta\right)$$
$$3\sin\theta = 2 - 2\sin^2\theta$$
$$2\sin^2\theta + 3\sin\theta - 2 = 0$$

(ii) $\theta = 30°$ or $150°$

4 (i)
$$\cos 30° = \frac{AC}{l} \Rightarrow AC = \cos 30° \times l = \frac{\sqrt{3}}{2}l$$
$$\sin 30° = \frac{DC}{l} \Rightarrow DC = \sin 30° \times l = \tfrac{1}{2}l$$
$$BC = 2 \times DC = l$$

Using Pythagoras' theorem,

$$AB^2 = AC^2 + BC^2$$
$$= \left(\frac{\sqrt{3}}{2}l\right)^2 + l^2$$
$$= \tfrac{3}{4}l^2 + l^2$$
$$= \tfrac{7}{4}l^2$$
$$AB = \sqrt{\tfrac{7}{4}l^2} = \frac{\sqrt{7}}{2}l = \tfrac{1}{2}l\sqrt{7}$$

(ii) In triangle ABC

$$\tan(x + 30°) = \frac{l}{\frac{\sqrt{3}}{2}l}$$
$$\tan(x + 30°) = \frac{2}{\sqrt{3}}$$
$$(x + 30°) = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$$
$$x = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right) - 30°$$

5 (i) Area of sector $= 68.5 \text{ cm}^2 (3 \text{ s.f.})$

(ii) $\theta = \tfrac{1}{3}(\pi - 2)$

(iii)
$$AB^2 = 8^2 + 8^2 - 2 \times 8 \times 8 \times \cos\frac{\pi}{3}$$
$$AB^2 = 64$$
$$AB = 8 \text{ cm}$$
or

Since angle $AOB = 60°$, and $AO = BO$, then angle OAB = angle $OBA = 60°$

So triangle OAB is equilateral so AB = 8 cm

$$BC^2 = 8^2 + 8^2 - 2 \times 8 \times 8 \times \cos\frac{2\pi}{3}$$
$$BC^2 = 192$$
$$BC = \sqrt{192} = 8\sqrt{3}$$
$$\text{Perimeter} = 8 + 8\sqrt{3} + 16$$
$$= \left(24 + 8\sqrt{3}\right) \text{cm}$$

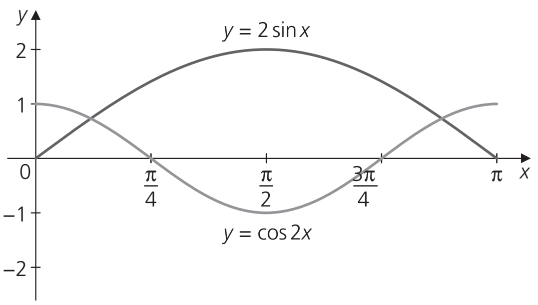
6 (i)
$$4\sin^4\theta + 5 = 7\cos^2\theta$$
$$4\sin^4\theta + 5 = 7\left(1 - \sin^2\theta\right)$$
$$4\sin^4\theta + 5 = 7 - 7\sin^2\theta$$
$$4\sin^4\theta + 7\sin^2\theta - 2 = 0$$
With $x = \sin^2\theta$ we have $4x^2 + 7x - 2 = 0$

(ii) $\theta = 30°, 150°, 210°, 330°$

7 (i) Shaded area $= 21.5 \text{ cm}^2 (3 \text{ s.f.})$

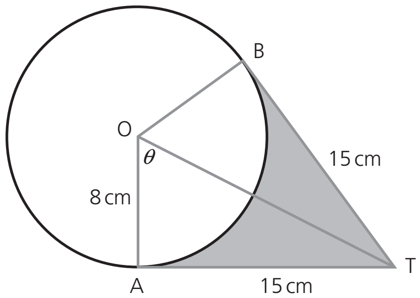
(ii) Perimeter $= 20.6 \text{ cm} (3 \text{ s.f.})$

8 (i)



(ii) Two solutions

9 (i)



$\tan\theta = \frac{15}{8} \Rightarrow \theta = \tan^{-1}\left(\frac{15}{8}\right) = 1.08 \text{(3 s.f.)}$

So $AOB = 2 \times 1.08 = 2.16$ (3 s.f.)

(ii) Perimeter $= 47.3\text{ cm}$ (3 s.f.)

(iii) Shaded area $= 50.8\text{ cm}^2$ (3 s.f.)

10 (i) $\cos^2 x = \frac{21}{25}$

(ii) $\tan^2 x = \frac{4}{21}$

11 (i) $3\sin x \tan x = 8$

$3\sin x \times \frac{\sin x}{\cos x} = 8$

$\frac{3\sin^2 x}{\cos x} = 8$

$3\sin^2 x = 8\cos x$

$3(1 - \cos^2 x) = 8\cos x$

$3 - 3\cos^2 x = 8\cos x$

$0 = 3\cos^2 x + 8\cos x - 3$

$3\cos^2 x + 8\cos x - 3 = 0$

(ii) $x = 70.5°$ or $289.5°$

12 (i) Perimeter of shaded region $= 25.9\text{ cm}$ (3 s.f.)

(ii) Shaded area $= 15.3\text{ cm}^2$ (3 s.f.)

13 (i) $2\tan^2\theta\cos\theta = 3$

$2\left(\frac{\sin^2\theta}{\cos^2\theta}\right) \times \cos\theta = 3$

$\frac{2\sin^2\theta}{\cos\theta} = 3$

$2\sin^2\theta = 3\cos\theta$

$2(1 - \cos^2\theta) = 3\cos\theta$

$2 - 2\cos^2\theta = 3\cos\theta$

$0 = 2\cos^2\theta + 3\cos\theta - 2$

$2\cos^2\theta + 3\cos\theta - 2 = 0$

(ii) $\theta = 60°$ or $300°$

14 (i) $BD = 6.66\text{ cm}$

(ii) Shaded area $= 10.3\text{ cm}^2$ (or $9\pi - 18$)

15 (i) $2 - 5\cos^2 x$

$a = 2, b = -5$

(ii) Greatest value is when $\cos^2 x = 0$, $f(x) = 2$

Least value is when $\cos^2 x = 1$, $f(x) = -3$

$-3 \leqslant f(x) \leqslant 2$

(iii) $x = 0.685$ or 2.46

16 $x = 113.6°$ or $70.5°$

17 (i) $\text{LHS} = \left(\frac{1}{\sin\theta} - \frac{1}{\tan\theta}\right)^2$

$= \left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)^2$

$= \left(\frac{1-\cos\theta}{\sin\theta}\right)^2$

$= \frac{(1-\cos\theta)^2}{\sin^2\theta}$

$= \frac{(1-\cos\theta)^2}{(1-\cos^2\theta)}$

$= \frac{\cancel{(1-\cos\theta)}(1-\cos\theta)}{\cancel{(1-\cos\theta)}(1+\cos\theta)}$

$= \frac{1-\cos\theta}{1+\cos\theta}$

$= \text{RHS}$

(ii) $\theta = 64.6°$ or $295.4°$

18 (i) Distance from D to AB is $6\sin\frac{\pi}{3} = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$

Distance from E to AB is $10\sin\theta$

Equating these distances gives:

$10\sin\theta = 3\sqrt{3}$

$\sin\theta = \frac{3\sqrt{3}}{10}$

$\theta = \sin^{-1}\left(\frac{3\sqrt{3}}{10}\right)$

(ii) $P = 16.20\text{ cm}$ (4 s.f.)

8 Vectors

1 (i) $\overrightarrow{MO} = \overrightarrow{MB} + \overrightarrow{BO} = 4\mathbf{i} - 6\mathbf{k} = \begin{pmatrix} 4 \\ 0 \\ -6 \end{pmatrix}$

$\overrightarrow{MC'} = \overrightarrow{MO} + \overrightarrow{OC} + \overrightarrow{CC'}$

$= (4\mathbf{i} - 6\mathbf{k}) + 4\mathbf{j} + 12\mathbf{k}$

$= 4\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}$

(ii) $\theta = 109.7°$ (1 d.p.)

2 (i) $\overrightarrow{BA} = \mathbf{a} - \mathbf{b} = (3\mathbf{i} + 2\mathbf{k}) - (2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k})$
$= \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

$\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = (2\mathbf{j} + 7\mathbf{k}) - (2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k})$
$= -2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} = 1 \times -2 + 2 \times 4 + -3 \times 2$$
$$= 0$$

Since $\overrightarrow{BA} \bullet \overrightarrow{BC} = 0$ the vectors are perpendicular.

(ii) $\overrightarrow{AD} = \mathbf{d} - \mathbf{a} = (-2\mathbf{i} + 10\mathbf{j} + 7\mathbf{k}) - (3\mathbf{i} + 2\mathbf{k})$
$= -5\mathbf{i} + 10\mathbf{j} + 5\mathbf{k}$
$= 5(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

$\overrightarrow{BC} = -2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} = 2(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$
Since both vectors are multiples of
$-\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ they are parallel
$|\overrightarrow{BC}| : |\overrightarrow{AD}| = 2 : 5$

3 (i) The unit vector is $\frac{1}{6}\begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$

(ii) $p = 10$

(iii) $q = -7$ or 5

4 (i) $\theta = 160.5°$

(ii) The vector must be $5 \times \overrightarrow{AC} = 5\begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -20 \\ 10 \\ 20 \end{pmatrix}$

(iii) $p = \frac{1}{2}$

5 (i) The unit vector is $\frac{1}{6}\begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$ or $\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$

(ii) Acute angle is $180° - 105.8° = 74.2°$

(iii) Perimeter = 15.4

6 (i) $\theta = 66.6°$

(ii) $\begin{pmatrix} 3 + 2p \\ -2 + p \\ 4 - 3p \end{pmatrix}$ or $(3 + 2p)\mathbf{i} + (-2 + p)\mathbf{j} + (4 - 3p)\mathbf{k}$

(iii) $p = \frac{8}{14} = \frac{4}{7}$

7 (i) $\overrightarrow{PQ} = 3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$
$\overrightarrow{RQ} = -3\mathbf{i} + 8\mathbf{j} + 3\mathbf{k}$

(ii) $\theta = 63.2°$